The dynamical equilibrium of a fluid at rest in a gravitational field is called hydrostatic equilibrium. In hydrostatic equilibrium, two forces are in balance: gravity, which tries to accelerate the fluid downwards towards the center of the planet; and the net pressure, which counters gravity by trying to accelerate the fluid upward. The easiest way to think about this is to consider a virtual volume, that is a massless box but containing the fluid. And we'll consider that massless box in this example to be in a constant gravitational field whose acceleration g acts downward in the negative z direction.

Now, the dimensions of our virtual box will be delta x times delta y and delta z. And, of course, being part of a fluid, we can think of pressure as exerting force per unit area on each side of the box. So let's take stock of all the forces acting on our virtual box.

The first is the weight, which is the acceleration of gravity g times the fluid density here denoted by rho times the volume of the box delta x, delta y, delta z. So the density, which is just the mass per unit volume, multiplied by the volume of the box, is just the mass of the box. So the weight is minus gravity times the mass of the box.

Now, in addition, we have a pressure force acting both on the bottom surface and the top surface of the box. The pressure force acting on the bottom of the box is the pressure per unit area, p, times the area of the box: delta x times delta y. Whereas, the pressure force acting on the top of the box is p plus delta p, that is the pressure at the altitude of the top of the box, again, times the area of the box: delta x times delta y.

So now we're in a position to write Newton's first law for this box, F equals MA. So MA is just the mass of the box, or the density times the volume, times the acceleration, dw dt. w is the velocity of the box in the positive direction.

So w is called the vertical velocity. dw/ dt is its acceleration. So this is just the MA part of F equals MA. And over on the right-hand side, we have the sum of the weight and the pressure. And that sum is given by the forces that you see on the right-hand side of this equation.

Now, if we take this equation and divide it by the mass of the box, which is what I've underlined here on the left-hand side, we get the equation of motion for that virtual box which you see in red here. This simply says that the acceleration in the positive z-direction is minus the gravitational acceleration g minus alpha times the vertical gradient of pressure, partial of pressure with respect to z. We're allowing for the fact that pressure can vary in time and in x and then y.

Alpha is a new term which we call the specific volume, which is just 1 over the density. Specific volume is the volume per unit mass. So we have simply derived an equation of motion in the vertical for this virtual box.

Now, if the gas is at rest, it's not accelerating in the vertical. And we have a balance. And we can work out what that balance implies about the distribution of pressure in the atmosphere.

First, we're going to introduce the ideal gas law. The atmosphere, by the way, is very nearly an ideal gas. The ideal gas law simply states that a specific volume is a gas constant R times temperature divided by pressure. The gas constant here is the universal gas constant, R star, divided by a suitably defined average molecular weight of the constituents of air, here denoted by m bar.

So if we substitute that into the vertical equation of motion and set the vertical acceleration to 0, we get the force balance which can be written as 1 over pressure times the rate of change of pressure with altitude is equal to minus g over RT. So that is one form of the hydrostatic equation that is valid for an ideal gas, again, at rest in a gravitational field.

Now, if we happen to have an isothermal atmosphere, that is one with constant temperature, we can immediately integrate the hydrostatic equation in the vertical. And this would tell us that the pressure is equal to whatever its value is at z equals 0 or the surface, we'll call that p naught, times e to the minus z over H, where H is defined as RT over g and has the units of length. And we call that the scale height or sometimes the density scale height of the atmosphere.

In our particular atmosphere, H is about eight kilometers. This would tell us that if the atmosphere happened to be isothermal, pressure would decrease exponentially with altitude reaching a value of 1 over e of its surface value at a height of about eight kilometers. Now, of course, the real atmosphere is not isothermal. The temperature does vary, although it doesn't vary over a huge fraction. So we can assert that the distribution of pressure in the vertical in the atmosphere is approximately exponential.

Now, this allows us to progress to the next important concept that we'll discuss in this clip which is the idea of buoyancy. Let's talk about buoyancy. And the easiest way to do that is to go back to our virtual box that we talked about before. But we're going to make one difference here. We're going to allow the density of the box, rho sub b, to differ from the density of the fluid in the environment of this virtual box which we'll call rho sub e.

So now let's work out the force balance again but with these differences. Once again, we have the weight of the box. But here, we're going to use the actual density of the fluid in the box, rho sub b. The pressure acting on the bottom and top of the box is the same as before.

And here again, we're going to use the force balance F equals MA. The force balance F equals MA can be written this way where, once again, the densities that appear here are the density of the virtual box. So if I divide through by the mass of the box, I get the vertical acceleration is equal to minus g minus the specific volume of the box times dp by dz.

But now we're going to make an important assumption. It turns out to be correct. And that is that the pressure distribution around the box is given by the pressure distribution in the environment. In other words, we're assuming that the pressure at the top of the box is no different from the pressure at that altitude in the environment.

The same is true with the pressure at the bottom of the box. The reason that we can assume that is if there were a difference between the pressures, there would be a rapid acceleration of the fluid down the pressure gradient toward lower pressure that would tend to even out that pressure distribution. It turns out that we can make this assumption provided that the box, if it's moving at all, is moving at a velocity substantially smaller than the speed of sound.

So we're going to assume that the pressure in the environment is given by the hydrostatic law in the environment. And when we do that, we can write the equation of motion in the vertical as given at the bottom here, that the vertical acceleration is simply equal to gravity times the difference between the specific volume of the fluid in the box and the specific volume of the fluid in the environment. This is simply called the buoyancy.

So if the fluid in the box is lighter than the fluid in the environment, that is it's less dense or has a larger specific volume, it will accelerate upward. This, by the way, is probably the first equation ever developed in Western science. It is a version of a law developed by the Greek philosopher Archimedes more than 2000 years ago called Archimedes' Law.

Archimedes stated it somewhat differently. But it really is the same equation. Archimedes simply said that the force acting on a submerged body is proportional to the difference between the mass of that body and the mass of the fluid that it displaces. And that's precisely what this law says. The notion of buoyancy allows us to take a step toward assessing whether a particular distribution of density in a fluid at rest in a gravitational field is stable. And that idea of stability of hydrostatic equilibrium will be the subject of the next video.