In the last video, we developed a version of Archimedes' Law, written at the bottom of this page, which tells us that the upward acceleration of a sample of fluid otherwise in hydrostatic equilibrium is proportional to the acceleration of gravity times the difference between the specific volume of the fluid in the parcel and that of its environment, divided by the specific volume of the environment. We call that the buoyancy of the parcel. And now we're almost in a position to say something about the stability of a fluid at rest in a gravitational field.

The idea is to take a sample of fluid, like the box pictured above here. Displace it upward. And ask where the acceleration on that parcel is upward or downward.

If it's upward, we would say that that displacement is unstable since the parcel would continue to accelerate in the same direction it was pushed. Otherwise, it would be neutral or stable. Now the problem here is that when we displace the parcel, the specific volume of the parcel itself changes. It's not a conserved variable. So without knowing how it changes, we're not in a position to compare it to the specific volume of the environment.

So what we're going to do instead is to relate the specific volume of the fluid, or more particularly, the difference between it and its environment, to the difference between variables that are conserved during adiabatic displacements. An adiabatic displacement being one in which no heat is added or subtracted from the sample.

So the next step is to relate specific volume to something that is conserved. And that something is the entropy of the parcel. So let's try to develop this relationship. First of all, just to go back to the definition of the specific volume, it's the volume per unit mass. Which can also be written as the inverse density of the fluid. And here, in fact throughout this course, we will use the symbol s, lowercase s, for the specific entropy or entropy per unit mass of the gas.

Now if we have a homogeneous gas in which the mixing ratios of the various molecular constituents are held fixed, we can write that any particular state variable is a function of any other two state variables. So for example, we can write that the specific volume is a function of pressure and entropy. Now when we compare the specific volume of the sample to that of its environment, we're going to make that comparison at the same pressure. Since pressure decreases monotonically upward. That comparison is made of constant pressure.

So what we really need to know is how does the specific volume change with entropy. So we'll use the chain rule that says that the increment or difference in specific volume at constant pressure, which is what we see here on the left, is the partial derivative of the specific volume with respect to entropy at constant pressure times the difference in entropy. That's this term here. Now using a neat trick from the first law of thermodynamics, called Maxwell's Law-- one of Maxwell's laws-- we can write that the derivative of the specific volume with respect to entropy, holding the pressure fixed, is equivalent to the derivative of temperature with respect to pressure, holding entropy fixed.

Again, this is something we haven't derived here. It's one of Maxwell's Relations from the first law of thermodynamics. So what we're left with is an expression that tells us that the difference between the specific volume of a sample in its environment, for example, is proportional to the difference between its entropy and that of its environment, this delta s, multiplied by a coefficient, which is the rate of change of temperature with pressure, holding entropy constant. So this relationship tells us that the difference between the specific volume of the parcel and that of its environment at the same pressure, this term on the left, is proportional to the difference between its entropy and that of its environment, this term on the right, multiplied by a coefficient, which is the partial derivative of temperature with respect to pressure, holding entropy constant.

So at last, we can write the buoyancy this way. We start off with its original definition that the buoyancy is acceleration of gravity times the difference between the specific volume of the sample and that of its environment divided by specific volume of the environment. And using the Maxwell relation above, we can write this as g divided by alpha times the partial derivative of temperature with respect to pressure, holding entropy constant, times the difference between the sample's entropy and that of its environment.

Now using the hydrostatic relationship itself, we can write this underlined term here as minus the derivative of temperature with respect to altitude, holding entropy constant. That's just applying the hydrostatic relationship to this double underlined term. And that, we're going to denote by special symbol upper-case Gamma, which is something called the adiabatic lapse rate. So we have shown here that the buoyancy is proportional to the adiabatic lapse rate times the difference between the sample's entropy and that of its environment.

Now this gamma is defined to be positive, because the temperatures lapse rate, with altitude at constant entropy, is a negative number. So with a minus sign in front of it, gamma is a positive number. That means that the sample will be positively buoyant if its entropy exceeds that of its environment.

Now what we're going to do is to take a little bit of a side trip here and derive an expression for the adiabatic lapse rate itself, from the first law of thermodynamics. That law can be written this way-- the total heating, Q dot here, can also be written as temperature times the time rate of change of entropy. We'll call that reversible entropy. And that's equal to the heat capacity at constant volume times the time rate of change of temperature plus pressure times the time rate of change of specific volume.

Now we can rewrite this equation by collecting terms. So we can write the right hand side as cv times the time rate of change of temperature plus the time rate of change of the product of specific volume and pressure. But then we have to subtract off the product of the specific volume and the time rate of change of pressure. Now according to the ideal gas law, the product of specific volume and pressure is just the product of the gas constant, R, and temperature. So let's make use of that and collect the terms, multiplying the time rate of change of temperature, into cv plus R. And then we have a minus alpha, dp by dt.

But the sum of the heat capacity at constant volume and the gas constant is equal to the heat capacity at constant pressure, c sub p. So an alternative form of the first law is that the heating is equal to heat capacity at constant pressure times time rate of change of temperature minus specific volume times the time rate of change of pressure.

Now in an adiabatic process, Q dot is zero. And it follows that cp dT minus alpha dp equals zero. And if we apply the hydrostatic equation to this last term, then it follows that cp dT plus gdz equals 0. From which we can immediately derive an expression for the adiabatic lapse rate, Gamma. It's just g over c sub p. Remarkably, it's a constant, since over the Earth's atmosphere and over the range of temperatures experienced in it, gravity and heat capacity at constant pressure are very nearly constant.

And it works out to be very nearly a rate of one degree centigrade per 100 meters. That's the adiabatic lapse rate. Remember the Gamma itself is positive. Now remember that buoyancy is proportional to Gamma times the difference between the entropy of the sample and the entropy of its environment. Let's go back now and use that to think about stability.

So Gamma is g over c sub p. That's about 1 degree per 100 meters. Let's think about stability. Let's start with an atmosphere in which the entropy decreases with height. So we measure the entropy at each altitude in the atmosphere and notice, in this particular example, that it's decreasing upward.

Now supposing we take a sample of fluid, and we lift it vertically and adiabatically. We note that, because its entropy is conserved, its entropy upon being lifted upward will be larger than the entropy of the fluid around it. Therefore it will experience positive buoyancy and be accelerated upward, like this. Because the sample is accelerating in the same direction it was pushed, just like the marble perched on the top of a hill accelerates downward when pushed, we would say that distribution is unstable.

Conversely, if we have a fluid in which the entropy increases with altitude, displacing a particle upward adiabatically results in an entropy which is less than its environment. It would be negatively buoyant, and therefore accelerate downward from which it came. So we would say that state is stable. And if the entropy doesn't change at all in the environment, then there's neutral stability.

Now it's a character of stable systems that when pushed like a pendulum, they'll oscillate. So atmospheres in which the entropy increases upward support a spectrum of internal oscillations which we call internal gravity waves. Not to be confused with gravitational waves that we see as part of the theory of gravity. If we were to observe a state of the fluid in which the entropy is decreasing upward, we would note that that is unstable.

Now as it turns out, the radiative equilibrium of the troposphere, if one calculates its entropy, one finds that it actually decreases upward through most of the troposphere. So one problem with radiative equilibrium in our atmosphere, and indeed in many atmospheres, is that it has decreasing entropy with altitude. It is unstable. And we tend not to observe unstable states.

The radiation is constantly driving the fluid toward a state of instability, and what happens when you have such an instability is something called convection, where relatively warm samples of fluid accelerate upward. And relatively cool or low entropy fluid accelerates downward. Now in this battle between radiation and convection-- radiation always trying to drive the atmosphere toward an unstable state, and convection trying to rearrange it towards stability-- most of the time, the convection wins that battle, in the sense that it succeeds in driving the atmosphere much closer to convective neutrality than to radiative equilibrium. And the reason for that is that convection time scales in our atmosphere are quite short. They can be measured in minutes to hours. Whereas radiative time scales are long, measured in tens of days, typically.

So convection is fast. And it succeeds in driving the atmosphere close to a neutral state. And that's revealed by the set of measurements shown in this picture, which were made by flying an instrumented model airplane in convecting air over a desert in New Mexico. And what you see here are individual measurements plotted on a diagram, in which virtual potential temperature is on the abscissa and altitude in meters on the ordinate.

Now virtual potential temperature is not a term we've discussed so far. But it's logarithm turns out to be proportional to the entropy of the air. You can see that, taken together, these measurements show entropy which is hardly changing with altitude. Even though the entropy of air, right down here in contact with the desert floor, has quite high values. This violently convecting desert air succeeds in driving its mean state toward a state of constant entropy.

It turns out to be nearly universal, whether we're talking about the Earth's atmosphere, or ocean, or planetary atmospheres, or even the sun, that constant entropy layers like this one almost invariably denote layers that are convecting. And even if we measure the temperature of water at 100 degrees Celsius, we could bet that it was boiling. This is such a universal principle that when we send probes into, for example, the Jovian atmosphere, and we observe layers in which the entropy is constant, we can assume that those layers are convecting.

And this brings us to the next topic, which we'll take up in the next video, which is the notion of radiative-convective equilibrium. What is the actual equilibrium state of an atmosphere in which both radiation and convection play critical roles in the distribution of temperature, humidity, and so on with altitude?