This final week of global warming science, we'll talk about climate models and the process of climate modeling. We'll begin by talking about the general philosophy and history behind global climate modeling, which is to simulate the large scale motions of the atmosphere, oceans, and, in more recent times, ice and other components of the climate system, to solve approximations to the full radiative transfer equations, as we've done in our simple, one dimensional climate model, parametrize processes like convection that are too small to resolve by such models, and in recent times, also try to simulate biogeochemical processes.

The very first general circulation models, or GCMs, were developed in the 1960's. The most fundamental aspect of any climate model is to solve partial differential equations governing the behavior of fluids. These in general are equations governing the conservation of momentum, the conservation of mass, the conservation of water in the atmosphere, conservation of certain chemical species, such as ozone, the first law of thermodynamics, which might be regarded as an equation for the thermodynamic component of energy, and the equation of state, which is quite different between the oceans and the atmospheres, and the radiative transfer equations.

Let's begin by having a look at the form of some of the key governing equations. We'll begin with the conservation of mass. The equation for the conservation of mass is actually an equation for the conservation of density or mass per unit volume, here. And what this equation says, is that the total time rate of change of rho, and we'll come back in a second to talk about what we mean by the total time rate of change, plus the density itself times the divergence of the three dimensional velocity is equal to 0. This simply says that the flux of mass across the sides of a particular volume will result in a change in the density of the fluid.

Now we have to be careful what we mean by a total time derivative, also sometimes called the material derivative, by the chain rule it's defined this way. It can be thought of the time rate of change following a fluid particle as it moves along. And by the chain rule, it's equal to the partial time rate of change, which is defined as a partial derivative in time at fixed points in space. That's what an observer, for example, would measure at a fixed point in space for a time rate of change, plus a term that's often called the advection term. It's the dot product of the three dimensional velocity with the gradient of that quantity.

So, for example, if we have a quantity that's conserved, that is, its material derivative vanishes, it can still change if there exists at some time, spatial gradients of that quantity. And then the local time rate of change would simply be equal to minus the dot product of the three dimensional velocity with that gradient. It's just a question of fluid motion bringing in larger or smaller values of that quantity from elsewhere.

Now the equivalent of the laws of motion for a fluid can be written this way. We have the density times the time rate of change of velocity, here, is equal to minus the gradient of the pressure, minus the divergence of a stress tensor tau, plus the density times gravity. Gravity is a vector. It's usually pointing downward. It's negative.

We've already talked about a very special form of this equation when we talked about hydrostatic balance, which basically ignores accelerations and fluid stresses, and says that the gradient of pressure is equal to density times gravity. So since gravity acts in the negative z direction, this simply says that minus the partial derivative of pressure, with respect to z, is equal to rho times the magnitude of the gravitational acceleration.

Now the stress tensor is given by the product of the coefficient of viscosity, mu, with a bunch of terms that are gradients of vectors, and and their transposes, and so forth. And this represents the effect of molecular motions in transporting momentum from one part of the fluid to another. The momentum equations written in this form, with the stress tensor included, are referred to as the Navier-Stokes equations.

We also have the conservation equation for thermodynamic energy, which in the atmosphere can be written this way. So the heat capacity at constant pressure, times the time rate of change of temperature, minus the specific volume, which is the inverse density, times the time rate of change of pressure, is equal to the heating. That's a particular form of the thermodynamic equation that's useful in the atmosphere, in which we have discussed in previous sections of this course. We have a similar equation, by the way, for the ocean. But the conservation laws are somewhat different because ocean water is nearly incompressible.

Another fundamental relation governing fluids is the equation of state. The atmosphere is treated as an ideal gas, so the product of the specific volume and the pressure is equal to the gas constant for air, times temperature. In the ocean we have quite a different equation of state, which I've only written symbolically here, which states that the specific volume, or if you like the density, is a function of pressure, temperature, and salinity, S.

In addition to the equations we reviewed here, we have equations for radiative transfer. And in the atmosphere, conservation of water substance, such as water vapor, cloud droplets, raindrops, snowflakes, and so forth, in the ocean we need to supplement this at the very least with an equation for the conservation of salt.

Given the set of partial differential equations that constitute a climate model, how do we actually go about solving them? Well there are various different techniques for doing this. Perhaps the easiest to understand is something called the finite difference method. So supposing we take this rather simple partial differential equation-- this is not one that we actually use in climate models, but it makes the point. Let's consider some variable, u, whose partial derivative in time, du by dt is equal to minus some constant, c, times the partial derivative of u with respect to x. OK.

So what we're going to do is to define u at discrete time steps, separated by time, delta t. And at discrete places in x, separated by a distance, delta x. So we'll write the time derivative as u at some time, minus u at the previous time, divided by delta t. And we'll write the space derivative likewise as u at some point in x minus u at some other point in x, divided by delta x. When we do that, we can write the resulting equation this way. This is an equation for u at the second time step, 2, at the i-th grid point.

So the superscripts represent the point in x, and the subscripts represent the time step. So this says that u at the second time step at the i-th point is equal to u at the first time step at the i-th point, minus the product of c delta t, times the difference at u at the first time step between the grid point in front of the one we're considering and the one behind the one we're considering, divided by 2 delta x. So this is an approximation to du by dx, this quantity in brackets here.

Notice that there is a nondimensional ratio, c times delta t divided by delta x, which turns out to be important in this problem. And so, this allows us, given the distribution in space of the quantity, u, at some particular time. We can solve this equation to find that quantity, u, at the next time step after delta t. Once we have that at all points, we can march the equation forward.

Now that sounds simple, but in practice, it is not so simple. We have to be very careful about how we discretize these equations. The differential equations, for example, may give perfectly well behaved solutions, but if we are not careful about just how we discretize the equations, the difference equations can turn out to be unstable. So there is a real art and science to integrating these partial differential equations.

Another method that is widely used in climate models is to represent the variation of the different variables in space as orthogonal functions. These are called spectral methods and we'll talk about them more in a little while.