Project Euler - Problem 317

Problem statement

A firecracker explodes at a height of $100\,m$ above level ground. It breaks into a large number of very small fragments, which move in every direction; all of them have the same initial velocity of $20\,{}^{m}/{}_{s}$.

We assume that the fragments move without air resistance, in a uniform gravitational field with $g = 9.81\,{}^{m}/{}_{s^2}$.

Find the volume (in m^3) of the region through which the fragments move before reaching the ground. Give your answer rounded to four decimal places.

Solution

To analyze the problem let's work on the bidimensional trajectories on the (x, y) plane. The explosion takes place at point $(0, h)$; we can consider only the right side of the plane since the left one will behave symmetrically. A very large number of fragments will start following parabolic trajectories, with initial velocity v_0 and angle $-{}^{\pi}/{}_{2} \leq \theta \leq {}^{\pi}/{}_{2}$, $\theta = 0$ being the x axis direction. Therefore, we apply the projectile motion equations:

$$\begin{cases} v_{0x} = v_0 \cos\theta \\ v_{0y} = v_0 \sin\theta \\ x = v_{0x} t \\ y = h + v_{0y} t - \frac{1}{2} g t^2 \end{cases} \tag{1}$$

where v_{0x} and v_{0y} are the initial velocities along the x and y axes, respectively.

Depending on θ, these trajectories will vary in height and length and we're interested in understanding what their superposition will be like. We can see, in figure (1), four different plots with 5, 20, 50 and 70 trajectories, respectively. The result is, of course, a parabola.

Now, if we extend the figure by symmetry on the left, and rotate it for $180°$ along the y axis we'll have a paraboloid. Our task is to calculate its volume, so let's proceed.

First, we need to calculate the formula for the 2D-envelope of those planar trajectories and then we will calculate the volume of the revolution of that curve. To calculate the envelope we see that the first three eq.s in (1) can be plugged into the fourth giving us a new formula $F(\theta, x, y) = 0$.

$$F(\theta, x, y) = -y + h + \frac{v_{0y}}{v_{0x}} x - \frac{g x^2}{2 v_{0x}^2} = 0 \tag{2}$$

which becomes:

$$F(\theta, x, y) = -y + h + x \cdot \tan\theta - \frac{g x^2}{2 v_0^2 \cos^2\theta} = 0 \tag{3}$$

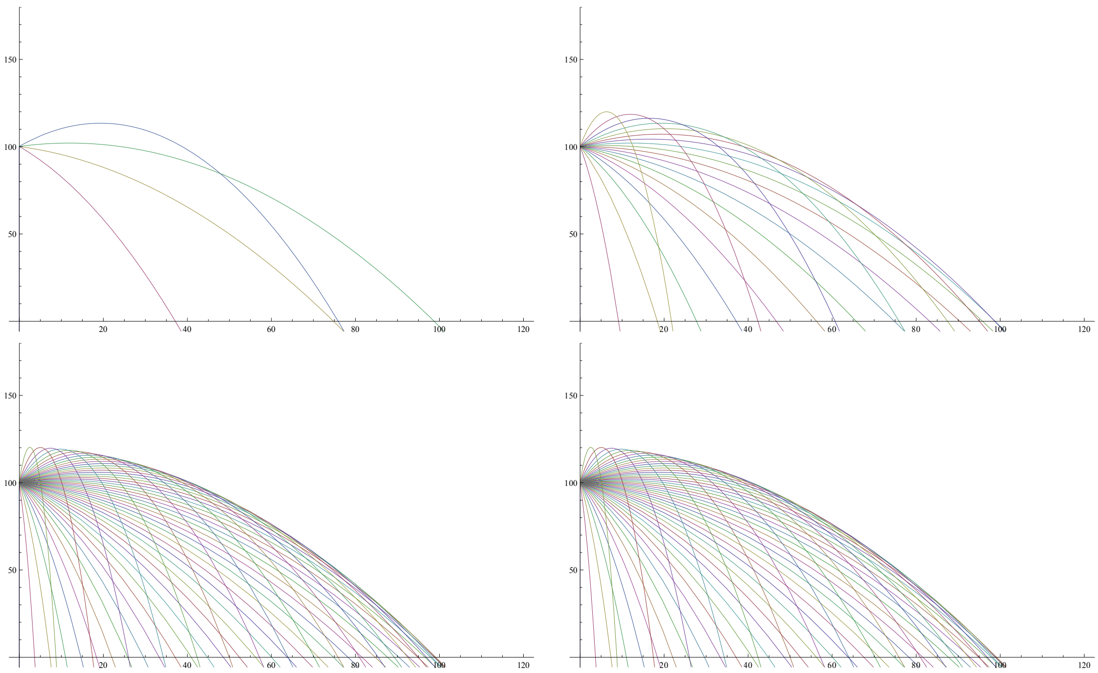

Figure 1 – From top left going clockwise: plot with 5, 20, 50 and 70 trajectories.

To get the envelope of this family of curves, all we need to do is to solve the following system:

$$\begin{cases} F(\theta, x, y) = 0 \\ \dfrac{\partial F}{\partial \theta} = 0 \end{cases} \tag{4}$$

Calculation of $\dfrac{\partial F}{\partial \theta} = 0$ yields:

$$\frac{\partial F}{\partial \theta} = \frac{\partial}{\partial \theta}\left(-y + h + x \cdot \tan\theta - \frac{gx^2}{2v_0^2\cos^2\theta}\right) = 0$$

which, knowing that $\frac{\partial}{\partial \theta}(\tan\theta) = \frac{1}{\cos^2\theta}$ and $\frac{\partial}{\partial \theta}\left(\frac{1}{cos^2\theta}\right) = \frac{2\sin\theta}{\cos^3\theta}$, becomes:

$$\frac{\partial F}{\partial \theta} = \frac{x}{\cos^2\theta} - \frac{g \cdot x^2 \sin\theta}{v_0^2\cos^3\theta} = \frac{x}{\cos^2\theta}\left(1 - \frac{g \cdot x}{v_0^2}\tan\theta\right) = 0 \tag{5}$$

From equation (5) we get a nice formula for x:

$$x = \frac{v_0^2}{g\tan\theta} \tag{6}$$

If we now take eq. (3) and substitute x with the result given by eq. (6) we get:

$$\begin{aligned} 0 &= -y + h + \frac{v_0^2}{g} - \frac{g \cdot v_0^4}{2v_0^2\cos^2\theta \cdot g^2\tan^2\theta} \\ &= -y + h + \frac{v_0^2}{g} - \frac{v_0^2}{2g\sin^2\theta} \end{aligned}$$

$$= -y + h + \frac{v_0^2}{2g}\left(1 - \frac{1}{\tan^2\theta}\right)$$

which, combined again with eq. (6), finally yields:

$$= -y + h + \frac{v_0^2}{2g} - \frac{g \cdot x^2}{2v_0^2} = 0$$

or, expliciting on y:

$$y = h + \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2} \tag{7}$$

from which we can obtain an expression for x^2:[1]

$$x^2 = \left(h + \frac{v_0^2}{2g} - y\right)\frac{2v_0^2}{g} \tag{8}$$

This is exactly what we need for the final step. All that remains is to calculate the volume of the envelope curve revolution around the y axis. The formula we are going to use is:

$$V = \int_0^{y_m} \pi x^2 \mathrm{d}y \tag{9}$$

y_m being the highest point of the envelope surface, which is easily found by evaluating y from eq. (7) when $x = 0$, therefore: $y_m = h + \dfrac{v_0^2}{2g}$.

Solving the integral (9) yields:

$$\begin{aligned}
V &= \int_0^{y_m} \pi\left(h + \frac{v_0^2}{2g} - y\right)\frac{2v_0^2}{g}\,\mathrm{d}y \\
&= \pi\frac{2v_0^2}{g}\int_0^{y_m}\left(h + \frac{v_0^2}{2g} - y\right)\mathrm{d}y \\
&= \pi\frac{2v_0^2}{g}\left[\left(h + \frac{v_0^2}{2g}\right)y - \frac{y^2}{2}\right]_0^{y_m} \\
&= \pi\frac{2v_0^2}{g}\left[\left(h + \frac{v_0^2}{2g}\right)^2 - \frac{1}{2}\left(h + \frac{v_0^2}{2g}\right)^2 - (0 - 0)\right] \\
&= \pi\frac{v_0^2}{g}\left(h + \frac{v_0^2}{2g}\right)^2
\end{aligned}$$

and making the substitution for $h = 100$, $v_0 = 20$ and $g = 9.81$ yields $V = 1856532.8455$, which is our solution.

References

- Projectile motion equations on wikipedia.
- Envelope of a family of curves on mathworld and wikipedia.
- Solid of revolution on wikipedia.

[1] We will need x^2 for the final step of the calculation, so there's no need to get an equation for x.