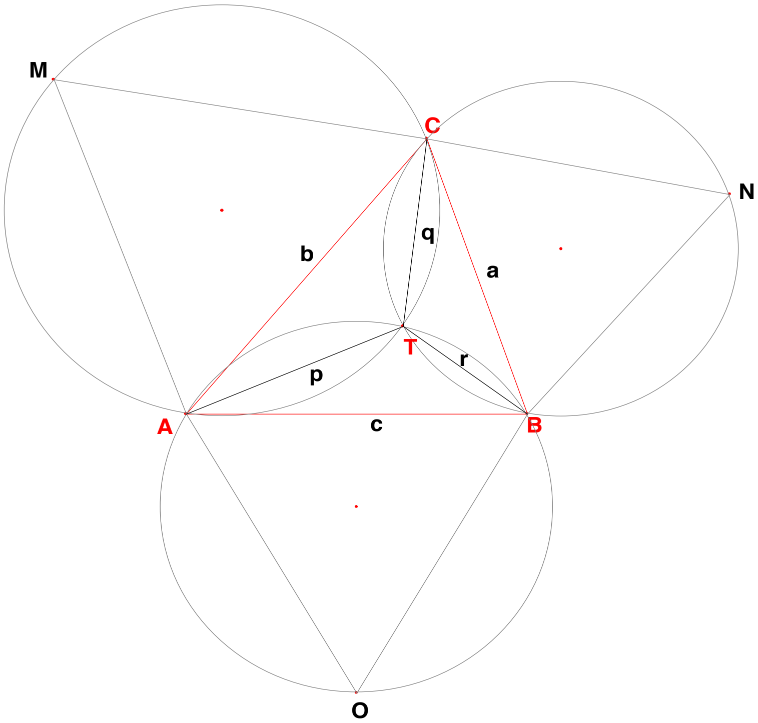
Problem 143

Investigating the Torricelli point of a triangle



As A, T , C and M are on the same circle ATCM is a quadrangle of chords.
From that follows that angle AMC+angle ATC=180 degrees. From that follows that angle ATC=120 degrees. So all angles at T are 120 degrees.

Deriving a parametrisation for 120 degree triangles.

Given a triangle with sides a, b and c, then if c is the side of the triangle opposite the 120 degree angle then, from the law of cosines:

$$a^2 + ab + b^2 = c^2$$

Putting $x = \frac{a}{c}$ and $y = \frac{b}{c}$ we get

$$x^2 + xy + y^2 = 1$$

This is an equation of an ellipse. It can be seen that the point A(0,-1) is a point of the ellipse.
The parametrisation of all the lines through A with slope t is $(x, tx-1)$.
The intersections of this line with the ellipse are then found from the equation

$$x^2 + tx^2 - x + (tx-1)^2 = 1$$

$$x^2 + tx^2 - x + t^2x^2 - 2tx + 1 = 1$$

Rearranging and dividing by x gives

$$(t^2 + t + 1)x = 2t + 1$$

$$x = \frac{2t+1}{t^2+t+1} \text{ and } y = \frac{2t^2 + t - t^2 - t - 1}{t^2+t+1} = \frac{t^2-1}{t^2+t+1}$$

It can be seen that x and y are rational if t is rational. Replacing t with $\frac{m}{n}$ we get:

$$x = \frac{2\frac{m}{n}+1}{\left(\frac{m}{n}\right)^2+\frac{m}{n}+1} = \frac{2mn+n^2}{m^2+mn+n^2} \text{ and } y = \frac{\left(\frac{m}{n}\right)^2-1}{\left(\frac{m}{n}\right)^2+\frac{m}{n}+1} = \frac{m^2-n^2}{m^2+mn+n^2}$$

If we now choose $m > n$ then the triple $(a,b,c) = (2mn+n^2, m^2-n^2, m^2+mn+n^2)$ will describe a triangle with an angle of 120 degrees, although not necessarily a primitive one. To be a primitive one a necessary condition is gcd(m,n)=1.
However this is not a sufficient condition:
This can be seen as follows:

$b = m^2 - n^2 = (m+n)(m-n)$ and

$c = m^2+mn+n^2 = m^2-2mn+n^2+3mn = (m-n)^2+3mn$

If now 3 divides m-n then both b and c will be divisible by 3, and

$a = 2mn+n^2 = n(2m+n) = n(n-m+3m)$ will be also divisble by 3.

The question now is: can we discard all cases with m-n divisible by 3 or are we losing then "families" of similar triangles and should we divide out the common factor 3 to get the primitive case. To show that that is not necessary we define $u = \frac{m+2n}{3}$ $v = \frac{m-n}{3}$ so that $m = u+2v,\ n = u-v$ then

$$2mn+n^2 = 2(u+2v)(u-v)+(u-v)^2 = 3u^2-3v^2 \text{ so } \frac{2mn+n^2}{3} = u^2-v^2$$

$$m^2-n^2 = (u+2v)^2-(u-v)^2 = 6uv+3v^2 \text{ so } \frac{m^2-n^2}{3} = 2uv+v^2$$

$$m^2+mn+n^2 = (u+2v)^2+(u+2v)(u-v)+(u-v)^2 = 3u^2+3uv+3v^2$$

$$\text{so } \frac{m^2+mn+n^2}{3} = u^2+uv+v^2$$

So it is possible to have another pair (u,v) (with u-v not divisible by 3) that describes the primitive cases we feared that we would lose when discarding the case m-n divisible by 3.

The complete parametrisation of all 120 degree triangles then is
$(a,b,c) = (k(2mn+n^2), k(m^2-n^2), k(m^2+mn+n^2))$ with gcd(m,n)=1 and m-n not divisible by 3.

The outline of an algoritm

It can be seen that the triangles formed by T and the three vertices of ABC pairwise have a side in common. So we are looking for pairs of triples (a,b,c) that have a common shorter side.
While generating the triples it is convenient to store them by the shortest of a and b.
So for each different shortest side we make a list of the corresponding larger side.
We then combine all values found in such a list generating the two shorter sides of the third triangle. We then check this third triangle to be a valid one. It is easiest to do so with the

cosine rule. If so, and the sum of the values of p,q and r is smaller than the given limit, we store this value in an array, checking that no duplicates occur.
Presorting each list and setting appropriate bounds greatly enhances speed.
This is left to the reader.