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Short Note

Modeling elastic fields across irregular boundaries

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INTRODUCTION

Geologists often see the earth as homogeneous blocks separated by smoothly curving boundaries. In contrast, computer modeling algorithms based on finite-difference schemes require elastic constants to be specified on the vertices of a regular rectangular grid. How can we convert a continuous geological model into a form suitable for a finite-difference grid? One common way is to lay the finite-difference grid down on the continuous geological model and use whatever elastic constants happen to lie beneath each of the grid points.

Unfortunately, this simple sampling scheme can cause artifacts. For example, a smooth gently-sloping interface in the geological model translates into a coarse staircase on sampling, with long horizontal runs punctuated by occasional vertical steps. These artificial steps generate unwanted diffractions.

Elastic parameter averaging scheme

Clearly, some other representation is called for, but what works best for elastic tensors? The Schoenberg-Muir calculus (S-M) shows how to correctly average a stack of flat elastic layers of arbitrary anisotropy, and so suggests itself as a starting point.

We apply S-M averaging to the gridding problem in a straightforward way. Lay the model grid down on the geological model as before, but this time consider not just each grid point, but the contents of a grid cell surrounding each point. Treat the contents of each cell as a stack of layers that can be averaged using the S-M calculus. This average set of elastic constants for the cell ought to be more representative than just whatever elastic constants happened to lie at the center. Also note that the average elastic constants calculated in this way depend not only on the contents of the cell but on the geometry of the contents as well.

Our model results show that this averaging method works surprisingly well. The price we pay is an increase in model complexity. For example, although the geological model itself may be isotropic everywhere, averaged grid cells at dipping layer boundaries will have transversely isotropic symmetry with the axis tilted normally to the layer boundary.

Accurate elastic modeling

Why should we bother averaging in this complex way?

Historically, the acoustic or elastic wave equation was solved on a discrete grid using first- or second-order differencing in time and space (Kelly et al., 1976). These methods were only accurate if the modeled frequencies remained well below the spatial and temporal Nyquist frequencies, in which case almost any interpolation method used to construct the model grid would work well enough. In recent years, the situation has changed. The pseudospectral method (Kosloff et al., 1985) can compute spatial derivatives accurate up to Nyquist; Edwards et al. (1987) show how a new time-update method due to Tal-Ezer (1986) can eliminate time-derivative discretization errors as well. These new high-accuracy modeling methods are efficient because they allow the wavefield to be coarsely sampled. Unfortunately, the velocity model is then coarsely sampled as well. The interpolation method used to construct the model grid becomes important.

There are ways of avoiding this drawback. One way is to use finite-element methods, which can discretize the model using elements that follow the geologic boundaries. Fornberg (1988) introduced another solution, a mapping method that converts a curved-grid model into a flat (or step) layered computational model that can be accurately represented on a rectangular grid.

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The main attraction of the explicit finite-difference methods on standard rectangular grids is their efficiency and simplicity. Finite-element methods or other distorted-grid methods require complicated setup steps to incorporate the local geometry into the equations, while the simpler rectangular-grid methods only need easily-specified two-dimensional (2-D) or three-dimensional (3-D) arrays of values. Our model representation allows for a better representation of dipping interfaces, without giving up the simplicity of a regular grid.

IMPLEMENTATION

We now describe the steps necessary to implement our sampling algorithm.

Smooth gridding

Figure 1 shows a medium consisting of two isotropic homogeneous regions separated by a gently dipping boundary. A 10 × 10 grid of cells has been laid down on the medium. We want to find the appropriate elastic constants to assign to each cell. There are three sorts of cells to consider; we have highlighted one of each kind.

The top highlighted cell is entirely within the top medium. Its elastic constants are therefore the same as that of the top (light-shaded) homogeneous medium. To the right of the blown-up cell, we show the isotropic *P-SV* impulse response for this medium. The bottom highlighted cell is entirely in the lower medium, and so its elastic constants are the same as that of the lower (dark-shaded) homogeneous medium. Again, to the right we show the isotropic *P-SV* impulse response. The middle highlighted cell is the interesting one, since it contains a mixture of both media. Intuitively, the elastic constants for this cell should be some sort of average of the top and bottom media. The question is, what sort of averaging is best?

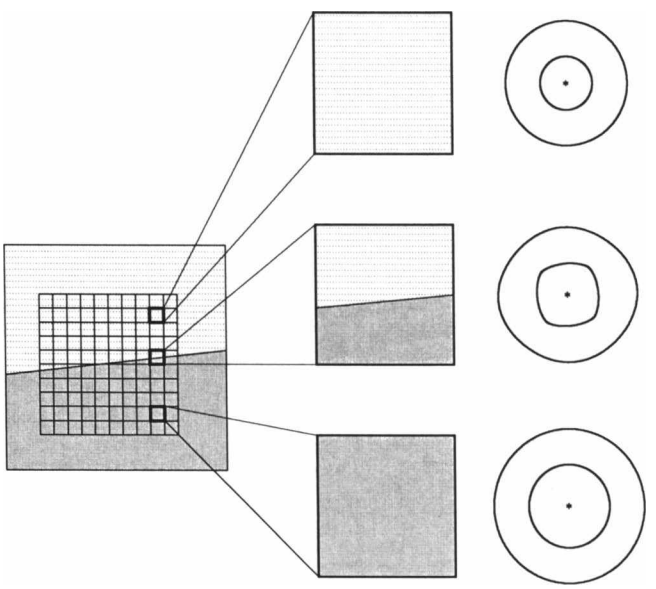

FIG. 1. Left: A sampling grid laid down on a medium made up of two homogeneous blocks. Three representative grid cells have been highlighted. Center column: Expanded views of the three representative grid cells. Right column: *P-SV* impulse responses of the homogeneous equivalent medium for each expanded grid cell.

One obvious way is to average each elastic constant independently. This method is somewhat better than simply choosing one component medium or the other, but it is still not very satisfactory. (It does not even get the kinematics right for the simplest of cases!)

A much better way is to average the slownesses, weighted according to the volume of each medium in the cell. This method at least gets the kinematics right for the obvious cases, and seems to work very well in practice if the two media being averaged are both isotropic and are not too different in their elastic properties. This slowness averaging method is still not the best. If the layers are not isotropic, the method becomes undefined. (What are "the" slownesses to average?) If the elastic properties of the media are too divergent, the assumption that the *P* and *S* properties can be averaged *independently* breaks down. (We will show an example of this later in Figure 3). Most importantly, this method cares nothing for the geometry of the layers within the cell, which we know must have some bearing on the answer.

We will perform the average a more sophisticated way, by considering the cell to be a bit of a tilted layered medium, with the proportions of each of the two media set in accordance with the area of each in the grid cell. We can then use the S-M calculus to find the homogeneous equivalent to this particular tilted layered medium. (Note that Backus and Postma averaging are just special cases of S-M averaging; since S-M also provides the most convenient formalism, there is no reason not to use it even in the standard isotropic case.) The middle rightmost part of Figure 1 shows the *P-SV* impulse response that results if this S-M averaging method is used for the sample middle cell. Note the resulting medium is *not* isotropic, which would have been the case if the *P* and *SV* slownesses had been averaged independently. It is transversely isotropic with an axis of symmetry normal to the tilted layer boundary within the cell. This anisotropy incorporates information about the geometry of the layer boundary within the cell.

AN EQUIVALENT MEDIUM TOOL KIT

We still have not specified exactly how the effective elastic constants of a layered-medium grid cell can be calculated using S-M theory. While this may seem like a complex undertaking, it is really only a trivial application of a set of basic utilities that all geophysicists in elastics should keep available.

In the S-M calculus (Schoenberg and Muir, 1989) layers are parameterized by their thickness, density, and a 6 × 6 (compressed subscript notation) stiffness tensor. This defines the fundamental unit to be operated on; it can be encoded in one of three ways. The symmetric 6 × 6 matrix can be rearranged to give three 3 × 3 submatrices; we call

the 6 × 6 form the *parameter* form and the three 3 × 3-matrix form the *model* form. (This change is merely a reordering of elements.) Layer parameters are mapped from the *model* form to a *group* domain using 3 × 3 matrix operations, as does the inverse mapping from group back to model. Layers can be combined by the simple addition of elements in the group domain. Therefore we need one pair of utilities to transform between the three basic representations, and one to add the units in the group domain element by element. The only other utility we need for our application is rotation of the coordinate frame of the elastic constants of a layer. This is done by expanding the 6 × 6 parameter form to its full 3 × 3 × 3 × 3 tensor form, applying a 3 × 3 rotation operator, and converting back to the standard parameter form.

To find the elastic constants for a boundary grid cell, rotate the coordinate frame so that the z- axis is normal to the boundary. Use the group domain to add the two bounding media in the appropriate volume ratio, and then rotate the result back to the original coordinates. (This leaves the problem of multiple layers with nonparallel dips untackled, but that is a refinement for the future.)

EXAMPLE

For our example we use a high-accuracy finite-difference technique. Spatial derivatives are calculated using a 2-D pseudospectral method; odd-order FFT's are used to obviate the need for special treatment of the spatial Nyquist frequency. We use the accurate time-integration method of Tal-Ezer (1986) to avoid squandering in the time derivatives the benefits of our expensive but highly accurate space derivatives. A nonstaggered grid is used to center all spatial derivatives at the grid points. This allows a complete set of 2-D elastic constants to be used (i.e., ρ, C_{11}, C_{13}, C_{15}, C_{33}, C_{35}, and C_{55}). This finite-difference method can take huge time steps; the wavefield can and does travel many gridpoint lengths in each ΔT time update. (It is even possible to do the entire calculation in a single massive time-update step. Although theoretically most efficient, this isn't usually done because of numerical precision limitations.) Most important for our application, the wavefield can be sampled quite closely to the spatial Nyquist frequency while still generating accurate model results.

The model consists of one homogeneous isotropic medium over a somewhat faster one (the model is just a scaled-up version of the one depicted in Figure 1). The model grid consists of 243 × 243 gridpoints, with $dx = dz = .75$. The bottom 53 gridpoints of the model are not shown. The top of the model is a free surface; strips 30 grid points wide at the left and right edges form absorbing boundaries. (The direct P can be seen to reflect slightly off the absorbing regions in the plots, and also to slightly wrap around). The source is a vertical (z) point force, located three gridpoints under top center. The vertical component of displacement is shown. The source emanates a second-derivative Gaussian wavelet with a fundamental frequency of 0.25; $dt = .06$. The upper medium has elastic constants $\rho = 1.$, $C_{11} = 4.41$, and $C_{44} = .81$ (isotropic). The lower medium has elastic constants $\rho = 1.25$, $C_{11} = 8.45$, and $C_{44} = 2.45$ (isotropic). The slope of the interface is 1/10. (Note all dimensions given above are in arbitrary but consistent units.)

The top part of Figure 2 shows the results if we follow the usual practice and fit a best-approximating set of stair steps to the interface. The step every 10 horizontal grid points causes very noticeable diffractions in the scattered wavefield. The lower left part of Figure 2 shows the same example using our Schoenberg-Muir averaging scheme. The S-M model nearly eliminates the stair-step scattering. The lower right part of Figure 2 shows the same example again, this time using isotropic slowness averaging. This method reduces the stair-step scattering almost as well as the S-M interpolation method.

A tougher example

Is there any reason to use the more costly S-M method? As mentioned before the S-M method does have the considerable advantage of being general enough to encompass the anisotropic case as formulated; isotropic slowness averaging does not. There are more subtle differences between the methods even for the isotropic case, however, although they are not usually obvious in "gentle" examples such as Figure 2. Figure 3 shows the results of a more severe isotropic test designed to show the differences. In this example the P-SV velocity ratio is substantially different across the boundary, although the P velocity remains the same.

The model is similar to that in the previous example, except the grid is 375 × 375 (80 grid points at the bottom and 45 grid points at the right edge are omitted from the plot), the source has a fundamental of 0.3, and the layer boundary has a slope of 1. The source is a point force directed toward the upper left (directly at the boundary). The upper right component of displacement (parallel to the boundary, perpendicular to the source direction) is shown. The upper left medium has elastic constants $\rho = .5$, $C_{11} = 8.$, and $C_{44} = .32$ (isotropic). The lower right medium has elastic constants $\rho = 1.$, $C_{11} = 16.$, and $C_{44} = 5.76$ (isotropic).

As can be seen in Figure 3, the S-M method of interpolation does a better job of reproducing the true amplitude-versus-offset behavior of the reflected S-S wave, correctly generating a clean sign change at the critical angle. (The isotropic averaging method instead performs a fuzzy sign change via an extended phase shift.) The isotropic slowness averaging method fails for this case because there is substantial interaction between the P and S components of the wavefield at the interface; the assumption that the P and S behaviors can be treated independently has broken down. While isotropic slowness averaging does conserve vertical traveltimes, the appeal of this method is based on ray-theoretic considerations that are inappropriate on scales that are small compared to the wavelength.

DISCUSSION

The meaning of equivalent media

How could the S-M interpolation have worked better for *reflected* waves? The usual rule of thumb is that "high frequency parts of the model cause reflections, low frequency parts control propagation." Since S-M equivalent media theory is a *zero frequency* theory, the lowest possible,

shouldn't we expect it to get propagation right and reflections wrong? No. It is true that a slow variation in the model over a distance much longer than a wavelength will not create a noticeable reflection, because the reflections arriving from different parts of the varying model will smear out over many periods of the wavelet and nearly cancel. Changes in the model over distances much *shorter* than a wavelength, however, get "averaged out" *within* the wavelet; if there is no change in the perceived *wavelet-averaged* properties of the medium no reflection will result. Equivalent media theory deals precisely with this high-spatial-frequency regime, beyond where the "high frequency = reflection" rule of thumb applies. If the terms of the S-M approximation are met, a slab cut out of a heterogeneous medium can be replaced by its homogeneous equivalent without disturbing the overall transmission or reflection properties measured outside the exchanged slab.

A justification

Our usage of S-M averaging is somewhat ad hoc, because we do not have infinite plane layers. We would now like to suggest a basis for our method. The exact S-M solution for layers of infinite extent is based on conservation equations that derive from continuity conditions across boundaries. For boundaries that are vertical or horizontal, our method gives the exact long-wavelength solution (our method then fulfills the flatlayer assumptions underlying the standard Schoenberg-Muir approximation). For sloping boundaries that are locally smooth, the continuity conditions still hold. It is reasonable to use the same conserved quantities as the basis for averaging, although there is no longer the same long-wavelength argument to support the thesis. While it is an approximation, our method undeniably does two things that traditional sampling methods do not: it averages the elastic constants as a unified tensor quantity, and it takes into account the orientation of the layer boundaries within each cell.

The importance of this can be seen in Figure 1; C_{15} is zero in both of the isotropic homogeneous media, but it is far from zero in the tilted-axis transversely isotropic medium necessary to represent the border grid cells. No sampling method that considers each elastic constant independently could have produced this result. Any such method (for example, isotropic slowness averaging) would find a zero value for C_{15}.

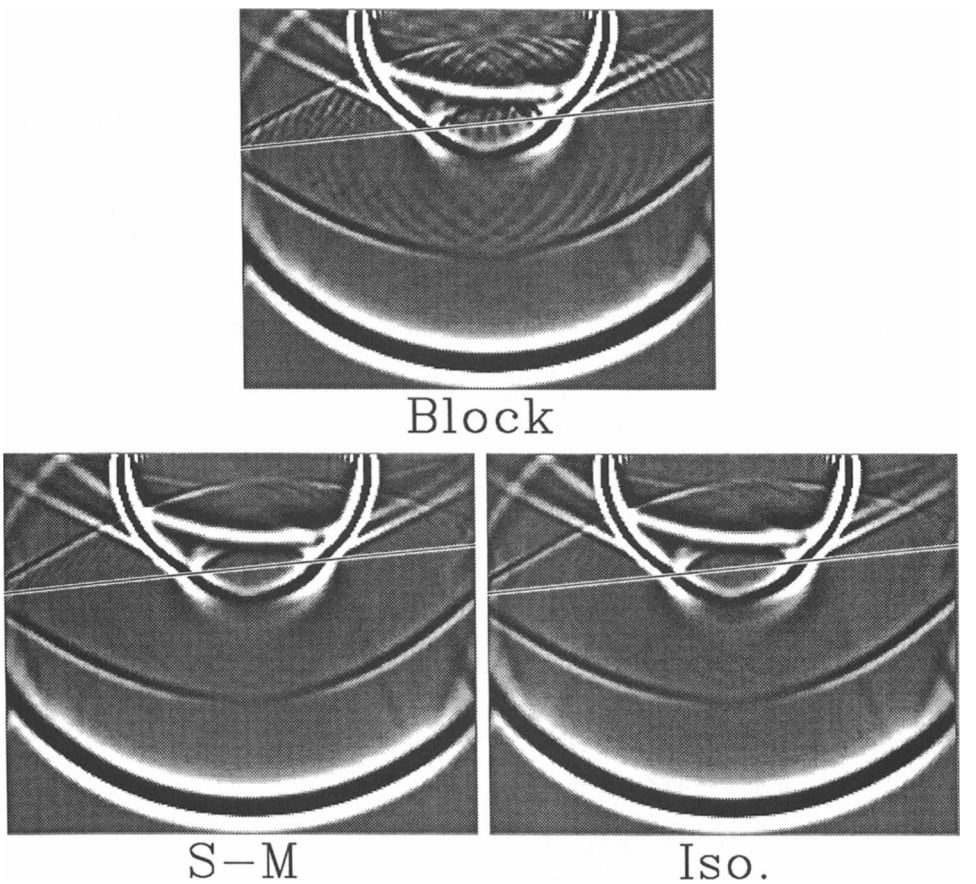

FIG. 2. Snapshots comparing the results of a wavefield hitting a gently sloping reflector for three similar models. The only difference is in how the sloped boundary was represented on the discrete computational grid. In the top plot, nearest neighbor interpolation was used; the boundary comes out as a series of flat stretches with a stair step every 10 grid cells. The steps cause noticeable diffractions that would not be present in an ideal continuum model. In the bottom left plot, S-M interpolation was used; in the bottom right plot, the isotropic P and S slownesses were interpolated independently. Both these methods mostly avoid the unwanted diffractions. (The primaries are strongly clipped so that the weaker converted waves and diffractions can be seen.)

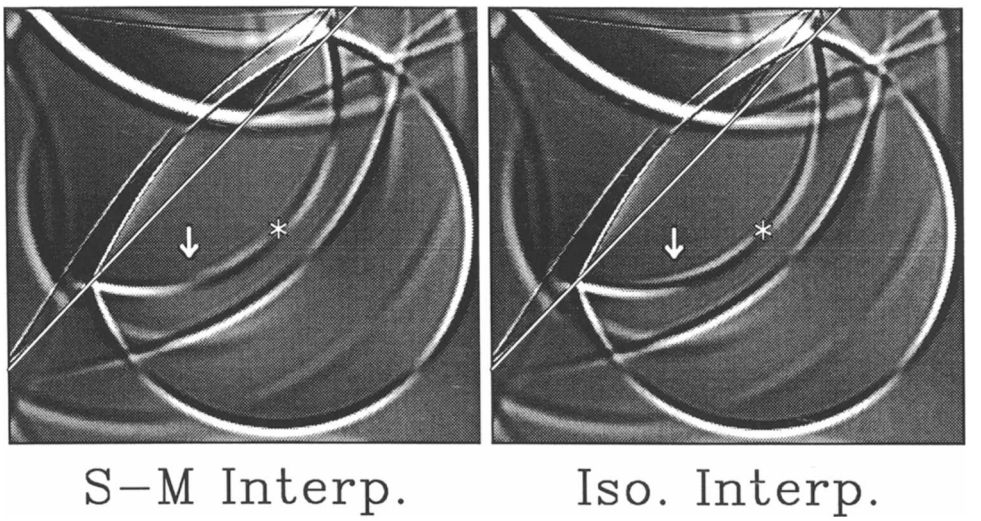

FIG. 3. Snapshots comparing the results of a wavefield hitting 45 degree-sloping reflector for two similar models. In the left plot, S-M interpolation was used; in the right plot, isotropic slowness averaging was used. The arrow points out where the reflected *S* wave would have a zero crossing in the ideal continuum case. S-M interpolation on the discrete grid correctly reproduces the zero crossing, while isotropic slowness averaging approximates the sign change with a gradual phase shift.

The strength of our method can also be viewed as a drawback: an averaged elastic tensor may be more anisotropic than those of the individual layers. In particular, as we mentioned in the previous paragraph, an average of two differing isotropic materials will not be isotropic. In the long term, as the use of anisotropic models becomes more common, this should not be a problem. If we always used modeling schemes that allowed for a general Hooke's law relation with a complete set of elastic constants, like the one we used to make Figures 2 and 3, no extra complexity would ever be added.

CONCLUSION

We have shown how our S-M interpolation method results in considerably cleaner wavefields than the direct scheme. One benefit is that the computational grid can be made somewhat coarser since there is no longer the same need to oversample as a partial cure for the diffracting staircase problem. In most cases, the choice of grid size becomes a matter of algorithmic stability and accuracy alone. While it is true that our method is informal, it does have the advantage of being simple to understand and relatively easy to implement, and it works.

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