High-Order (Space and Time) Finite-Difference Modeling of the Elastic Wave Equation

SD1.2

Edward Crase, Montana Tech University/Institut de Physique du Globe de Paris, France

SUMMARY

Elastic finite-difference modeling can be very expensive with respect to its use of computer resources (memory and CPU time). To increase efficiency and accuracy of such modeling, an elastic finite-differencing scheme has been developed which is accurate to arbitrary order (user specified) in both space and time. For a given modeling application in which a predetermined degree of accuracy (minimal numerical dispersion) is desired, one can derive the optimal spatial and temporal differencing orders which give maximum computational effeciency. To determine the optimal differencing orders, equations which specify stability conditions and numerical dispersion are required and are given herein.

INTRODUCTION

Many methods have been proposed for modeling waves in hetrogeneous elastic media, and each has its own advantages and disadvantages for a given problem. Direct numerical solution of the wave equation via a finite-difference or finite-element approach will produce complete wavefields which, because of the nature of these methods, are known at all points over discrete subsurface grids. The accuracy of finite-difference and finite-element solutions of the elastic wave equation have been compared by Marfurt (1984). Since Marfurt's paper, several researchers have introduced finite-difference schemes which greatly improve versatility and accuracy as applied to elastic modeling. Virieux (1986) presents a 2nd-order (space and time) elastic finite-difference scheme which is stable for any value of Poisson's ratio. His introduction of staggered grids to wave-propagation modeling (Madariaga, 1976, uses a similar scheme for crack propagation modeling) has revolutionized the concept of elastic finite differencing as previously proposed in the classic paper of Kelly et al. (1976). Levander (1988) extends Virieux's method to a 4th-order spatial, 2nd-order temporal application. The pseudo-spectral method of finite differencing (spatial derivatives are calculated by Fourier transform) is basically an extension of traditional 2nd-order temporal methods in which the spatial differencing order can be considered as infinite. The advantage of higher spatial orders is that larger grid spacing (thus fewer grid points) can be used which reduces computing time. I do note, however, that any high-order spatial scheme (including the pseudo-spectral method), which employs 2nd-order time differencing, can have excessive numerical dispersion unless very small time steps are used.

The latest advance in finite-difference modeling is the Rapid Expansion Method (REM) of Tal-Ezer et al. (1987) which attacks the problem of temporal differencing (time extrapolation). Instead of the traditional 2nd-order Taylor expansion for time derivatives, a Chebyshev expansion is employed which allows large time steps and is (or can be made to be) accurate to within the numerical precission of the computer when combined with high-order or spectral spatial derivatives. The REM can be employed such that there is no numerical dispersion. In essence, the REM can be thought of as a scheme which is of infinite spatial and infinite temporal order. The main drawbacks of the method are (1) difficult absorbing and free-surface boundary conditions, and (2) extensive storage requirements if it is necessary to know the wavefield for small time intervals at many points of a subsurface grid. The advantages of this method are (1) its accuracy with respect to numerical dispersion, and (2) its computational efficiency when compared to schemes using 2nd-order temporal differencing.

The finite-difference algorithm which I present here is of arbitrary spatial and temporal order. The combination of arbitrary spatial and temporal differencing orders allows one to economically optimize the size of grid spacing and time steps to fit a particular modeling problem. For example, distinct interfaces (dipping or not) require small grid spacing if reflected waves are to be accurately modeled. For a given velocity distribution and a given maximum frequency of interest, it may not be economical to use high-order spatial derivative operators (or Fourier derivatives) when a 2nd-order spatial operator produces sufficient accuracy, which 2nd-order operators do for the case of small grid spacing relative to the wavelength of interest. On the other hand, if high-order spatial operators are appropriate for a given problem (e.g., see Dablain, 1986; Fornberg, 1988; Holberg, 1987), very small time steps must be taken when using 2nd-order temporal differencing to sufficiently suppress errors caused by numerical dispersion. In such cases, the maximum allowable time step that can be used is only 15-20% of the time step which satisfies stability conditions. Higher temporal-differencing orders greatly reduce numerical dispersion and allow modeling with time steps at (or very close to) those time steps specified by stability conditions.

THE FUNDAMENTAL EQUATIONS

In two dimensions, x horizontal and z vertical, the fundamental equations which describe elastic wave propagation in a hetrogeneous, isotropic media can be written as

$$\frac{\partial^2}{\partial t^2}u^x = \frac{1}{\rho}\left(\frac{\partial}{\partial x}\sigma^{xx} + \frac{\partial}{\partial z}\sigma^{xz} + \phi^x\right), \qquad (1)$$

$$\frac{\partial^2}{\partial t^2}u^z = \frac{1}{\rho}\left(\frac{\partial}{\partial z}\sigma^{zz} + \frac{\partial}{\partial x}\sigma^{xz} + \phi^z\right),$$

$$\sigma^{xx} = (\lambda+2\mu)\left(\frac{\partial}{\partial x}u^x + \frac{\partial}{\partial z}u^z\right) - 2\mu\frac{\partial}{\partial z}u^z + M^{xx},$$

$$\sigma^{zz} = (\lambda+2\mu)\left(\frac{\partial}{\partial x}u^x + \frac{\partial}{\partial z}u^z\right) - 2\mu\frac{\partial}{\partial x}u^x + M^{zz},$$

and

$$\sigma^{xz} = \mu\left(\frac{\partial}{\partial x}u^z + \frac{\partial}{\partial z}u^x\right) + M^{xz}.$$

with initial conditions such that

$$u^x = \frac{\partial}{\partial t}u^x = u^z = \frac{\partial}{\partial t}u^z = 0, \text{ for } t \le 0,$$

and free-surface conditions such that $\sigma^{zz} = \Gamma^{zz}$ and $\sigma^{xz} = \Gamma^{xz}$ at the free surface. In these equations, $u^x(x,z,t)$ and $u^z(x,z,t)$ are, respectively, the horizontal and vertical components of displacement and $\sigma^{xx}(x,z,t)$, $\sigma^{zz}(x,z,t)$, and $\sigma^{xz}(x,z,t)$ are the corresponding stresses. I have included source terms of force density, $\phi^x(x,z,t)$ and $\phi^z(x,z,t)$, moment density, $M^{xx}(x,z,t)$, $M^{zz}(x,z,t)$, and $M^{xz}(x,z,t)$, and surface tractions, $\Gamma^{zz}(x,0,t)$ and $\Gamma^{xz}(x,0,t)$. $\lambda(x,z)$ and $\mu(x,z)$ are Lamé coefficients and $\rho(x,z)$ is density. Given that all displacements, stresses, and source terms are zero for $t < 0$ and that the source functions are known for $t > 0$, we essentially want to extrapolate displacement and stress forward in time. Note that by taking a time derivative of each of the above equations, the same problem can readily be formulated in terms of particle velocity and time derivatives of stresses and sources.

TIME EXTRAPOLATION (TEMPORAL DIFFERENCING)

For time extrapolation, I use Taylor series expansions for time perturbations $+\Delta t$ and $-\Delta t$ of the displacement vector $\mathbf{u}(t) = (u^x, u^z)^T$ and sum them which gives

$$\mathbf{u}(t+\Delta t) = 2\mathbf{u}(t) - \mathbf{u}(t-\Delta t) + 2\sum_{\substack{k=2 \\ k \text{ even}}}^{\infty} \frac{\Delta t^k}{k!}\frac{\partial^k}{\partial t^k}\mathbf{u}(t). \qquad (2)$$

Reprinted from 60th Annual International Meeting, SEG, Expanded Abstracts, 987–991.

For the terminology used herein, I define *n*th-order temporal differencing as the approximation to u(t+Δt) when the summation of equation (2) is truncated at $k=n$. Note that, given u(t) and the appropriate source terms, approximation of spatial derivatives in equations (1) allows computation of the 2nd-order term of the summation; thus, if one is also given u(t-Δt), u(t+Δt) is readily computed (this is 2nd-order temporal differencing).

To implement higher temporal orders, I use a method described to me by G.H.F. Gardner (personal communication, 1985). A similar method has been published by Dablain (1986) for the case of 4th-order temporal differencing of the scalar wave equation. Taking the *k*th time derivatives (*k* even) of equations (1) and noting that density and Lamé coefficients are independent of time, any even-order time derivative needed in equation (2) can be readily computed from successive solutions of equation (3) by approximating the appropriate spatial derivatives of displacement and stress and by knowing (or approximating) the appropriate time derivatives of the source terms. In other words, given **u**(t), compute $\frac{\partial^2}{\partial t^2}\mathbf{u}$; from $\frac{\partial^2}{\partial t^2}\mathbf{u}(t)$, compute $\frac{\partial^4}{\partial t^4}\mathbf{u}$; from $\frac{\partial^4}{\partial t^4}\mathbf{u}(t)$, compute $\frac{\partial^6}{\partial t^6}\mathbf{u}$; etc. In this manner, the time extrapolation of equation (2) can be calculated up to *n*th-order accuracy.

In a general form, equation (2) can be written as

$$\mathbf{u}(t+\Delta t) \simeq 2\mathbf{u}(t) - \mathbf{u}(t-\Delta t) + 2\sum_{\substack{k=2 \\ k\ \text{even}}}^{n} b_k \mathbf{Z}^k \mathbf{u}(t) \qquad (3)$$

where, for a Taylor series expansion, the coefficients, b_k, are given by

$$b_k = \frac{\Delta t^k}{k!},$$

and where application of the operator $\mathbf{Z}^k$ implies the calculation of the spatial derivatives as discussed above. Time extrapolation in the REM can also be written in the form of equation (3) with the same operator $\mathbf{Z}^k$ but with different coefficients b_k. In the REM, b_k are given by combinations of scalars defined by Bessel functions and modified Chebychev polynomials. In essence, the REM coefficients are optimum (in a weighted least-sqares sense) and give the minimum total error over all frequencies for a given *n*. As *n* is increased, the error in all frequency components of **u**(t+Δt) can be reduced to below the numerical precission of computation. For large *n*, large time steps are stable and most economical. Furthermore, after a large time step is taken, smaller time steps in between the large step can be calculated by modifying the coefficients b_k. However, each term given by application $\mathbf{Z}^k\mathbf{u}(t)$ must be saved in order to compute the smaller time steps.

SPATIAL DERIVATIVE OPERATORS

High-order spatial derivative operators can be found by means of Taylor series expansions (e.g., see Dablain, 1986). Errors associated with these operators are easily calculated analytically and monotonically increase as spatial wavelength decreases. Another possibility for calculating spatial derivative operators is the method of Holberg (1987) in which optimum operators are derived such that errors in numerical dispersion over a given frequency band are minimized.

An easy method for calculating spatial derivative operators of the "Taylor-series" type is by taking the first derivative of an *m*th degree Lagrange interpolating polynomial (e.g., see Burden et al., 1978). Defining a_p as the coefficients of a centered, *m*th-order, first-derivative operator (for staggered grids as defined by Virieux, 1986), the coeficients are

$$a_p = \frac{1}{\Delta x}\sum_{\substack{j \\ j\neq p}} \frac{1}{p-j}\prod_{\substack{i \\ i\neq j,p}} \frac{i}{p-i} \qquad (4)$$

where *m* is even and

$$i, j, p = -\frac{m-1}{2}, -\frac{m-3}{2}, \ldots, \frac{m-3}{2}, \frac{m-1}{2}.$$

ERROR ANALYSIS AND STABILITY

For temporal and spatial differencing formulations of arbitrary order, I investigate stability conditions and numerical dispersion by extending to higher orders the spectral-analysis method used by Virieux (1986). I ignore computational roundoff errors. Although the analysis is carried out for two-dimensional wave propagation, it can readily be extended to three dimensions.

Spectral Analysis of spatial operators

In essence, we are approximating a spatial derivative such that

$$\frac{\partial}{\partial x} f(x) \simeq \frac{\delta}{\delta x} f(x)$$

where $\frac{\delta}{\delta x}$ implies convolution with an operator whose coefficients are given by equation (4). Letting

$$f(x) = e^{j(k_x x + k_z z - \omega t)}$$

and noting that both a_p and sin(x) are odd functions, it can be shown that

$$k_x \simeq \hat{k}_x = \frac{2}{\Delta x}\sum_{p=1/2}^{(m-1)/2} a_p \sin(k_x p\,\Delta x). \qquad (5)$$

Because the error in $\hat{k}_x$ monotonically increases as k_x increases (for "Taylor-type" operators), $\hat{k}_x$ has maximum error when k_x is at Nyquist wavenumber. Also, because the coefficients a_p are alternately + and - (again, for "Taylor-type" operators) and because the sine function at Nyquist wavenumber in equation (5) is alternately +1 and -1, k_x is approximated at Nyquist wavenumber as

$$k_{xN} \simeq \hat{k}_{xN} = \frac{2}{\Delta x}\sum_{p=1/2}^{(m-1)/2} |a_p|. \qquad (6)$$

A similar analysis can be carried out for $k_z \simeq \hat{k}_z$. In addition, equation (5) can be used to analyze any odd derivative operator applied on a staggered grid; however, if maximum error does not occur at Nyquist wavenumber or if the operator coefficients are not alternately + and -, maximum error is not easily calculated.

Spectral analysis of the total scheme

To analyze the numerical dispersion errors and stability conditions of schemes with arbitrary spatial and temporal differencing orders, I assume an infinite, isotropic, homogeneous medium for 2-D wave propagation. With these assumptions, only two distinct and independent modes, P waves and S waves, can be present. I note that, in a bounded and hetrogeneous medium, many modes will exist and analysis of errors and stability becomes extremely complicated. Marfurt (1984) gives some insight into the problems associated with such an analysis. To my knowledge, stability limits and error analysis of finite-difference schemes which include anisotropy and attenuation has never been performed. Obviously, further research is warranted.

For an infinite, isotropic, homogeneous medium, equations (1) can be transformed to the wavenumber domain, and spatial derivatives can be approximated by the above operators which gives

$$\frac{\partial^2}{\partial t^2}\begin{bmatrix} u^x \\ u^z \end{bmatrix} \simeq -\mathbf{B}\begin{bmatrix} u^x \\ u^z \end{bmatrix}$$

where **B** is a symmetric 2 x 2 matrix with elements

$$b_{11} = V_p^2\hat{k}_x^2 + V_s^2\hat{k}_z^2,$$
$$b_{22} = V_p^2\hat{k}_z^2 + V_s^2\hat{k}_x^2,$$
$$b_{12} = b_{21} = V_p^2\hat{k}_x\hat{k}_z - V_s^2\hat{k}_x\hat{k}_z.$$

B can be decomposed such that

$$\mathbf{B} = \mathbf{E}\,\mathbf{D}_\lambda\,\mathbf{E}^T$$

where $\mathbf{D}_\lambda$ is the diagonal matrix of eigenvalues and **E** is the corresponding matrix of eigenvectors. The eigenvalues of **B** are

$$\lambda_1 = V_p^2(\hat{k}_x^2 + \hat{k}_z^2) \quad \text{and} \quad \lambda_2 = V_s^2(\hat{k}_x^2 + \hat{k}_z^2),$$

while the matrix of eigenvectors is given by

$$E = \frac{1}{\sqrt{\hat{k}_x^2+\hat{k}_z^2}} \begin{pmatrix} \hat{k}_x & -\hat{k}_z \\ \hat{k}_z & \hat{k}_x \end{pmatrix}$$

For k even, higher-order time derivatives take the form

$$\frac{\partial^k}{\partial t^k}\begin{pmatrix} u^x \\ u^z \end{pmatrix} \simeq E\,(-D_\lambda)^{\frac{k}{2}}\,E^T \begin{pmatrix} u^x \\ u^z \end{pmatrix}.$$

Thus, the total mth-order spatial, nth-order temporal scheme can be written as

$$u(t+\Delta t) \simeq 2u(t) - u(t-\Delta t) + E\left[2\sum_{\substack{k=2 \\ k\ even}}^{n} \frac{\Delta t^k}{k!}(-D_\lambda)^{\frac{k}{2}}\right] E^T u(t).$$

A Fourier transform from t to ω yields

$$0 \simeq E\,Q\,E^T\,u$$

where Q is a diagonal matrix with elements

$$q_{11} = \sin^2\left(\frac{\hat{\omega}\Delta t}{2}\right) - \frac{1}{2}\sum_{\substack{k=2 \\ k\ even}}^{n} (-1)^{\frac{k}{2}-1}\frac{\Delta t^k}{k!}\left(V_p\sqrt{\hat{k}_x^2+\hat{k}_z^2}\right)^k, \quad (7)$$

and

$$q_{22} = \sin^2\left(\frac{\hat{\omega}\Delta t}{2}\right) - \frac{1}{2}\sum_{\substack{k=2 \\ k\ even}}^{n} (-1)^{\frac{k}{2}-1}\frac{\Delta t^k}{k!}\left(V_s\sqrt{\hat{k}_x^2+\hat{k}_z^2}\right)^k.$$

By setting the determinant of $E\,Q\,E^T$ equal to zero, the stability conditions and dispersion errors for a given scheme can be derived. Because $|E| = |E^T| = 1$, setting the determinant to zero, one has

$$|E\,Q\,E^T| = q_{11}q_{22} = 0. \quad (8)$$

Note that from the definitions of q_{11} and q_{22}, any root of equation (8) will contain either V_p or V_s but not both. This implies that stability and numerical dispersion are independent of the Poisson's ratio of the medium. This is not the case for elastic finite-difference algorithms not employing a staggered grid. The method proposed by Kelly et al. (1976) does not employ a staggered grid and the scheme begins to degenerate for Poisson ratios greater than 0.25. On the other hand, use of a staggered grid allows stability and minimal numerical dispersion even at a solid-liquid contact. No special treatment of solid-liquid contacts is necessary in the above formulation just as in the 2nd-order development of Virieux (1986). In practice, I have found, as does Virieux, that they can simply be considered as part of a general heterogeneous medium.

Stability limits

To derive stability limits, I first define a dispersion parameter γ which is given by

$$\gamma = V_p\Delta t\sqrt{\frac{1}{\Delta x^2}+\frac{1}{\Delta z^2}}. \quad (9)$$

I define the stability factor γ_{max} for a given differencing scheme as the maximum allowable value of the dispersion parameter for which the scheme is stable. Recalling that maximum errors associated with the spatial operators occur at $\hat{k}_{xN}$ and $\hat{k}_{zN}$ as given by equation (6), I also define

$$\alpha = 2\sum_{p=1/2}^{(m-1)/2} |a_p|$$

Setting q_{11} to zero when maximum spatial errors are present gives

$$\sin^2\left(\frac{\hat{\omega}\Delta t}{2}\right) = \frac{1}{2}\sum_{\substack{k=2 \\ k\ even}}^{n} \frac{(-1)^{\frac{k}{2}-1}}{k!}\alpha^k\gamma_{max}^k. \quad (10)$$

For stability, the right-hand side must remain within the limits of $\sin^2\left(\frac{\hat{\omega}\Delta t}{2}\right)$ such that

$$0 \le \frac{1}{2}\sum_{\substack{k=2 \\ k\ even}}^{n} \frac{(-1)^{\frac{k}{2}-1}}{k!}\alpha^k\gamma_{max}^k \le 1. \quad (11)$$

Thus, for a given spatial differencing order such that α is known, one has an nth degree polynomial in γ for nth-order temporal differencing. Depending on whether the last term in the summation is + or -, it can be shown that for temporal differencing orders 2, 6, 10, etc., the stability factor γ_{max} is the smallest real root of the polynomial

$$\frac{1}{2}\sum_{\substack{k=2 \\ k\ even}}^{n} \frac{(-1)^{\frac{k}{2}-1}}{k!}\alpha^k\gamma_{max}^k = 1.$$

For temporal differencing orders 4, 8, 12, etc., the stability factor γ_{max} is the smallest real root of the polyinomial

$$\frac{1}{2}\sum_{\substack{k=2 \\ k\ even}}^{n} \frac{(-1)^{\frac{k}{2}-1}}{k!}\alpha^k\gamma_{max}^k = 0.$$

Numerical dispersion

From equation (7), numerical dispersion errors can also be calculated which are independent for P waves and S waves. (This is because the roots associated with q_{11} and q_{22} are independent of Poisson's ratio.) For a given dispersion factor γ such that V_p, Δx, Δz, and Δt satisfy equation (11), one can chose a particular frequency ω to analyze. For a given propagation angle (with respect to a coordinate axis), the corresponding wavelength and wavenumbers are given by

$$\lambda = \frac{2\pi V_p}{\omega},$$

$$k_x = \frac{2\pi}{\lambda}\cos\theta,$$

$$k_z = \frac{2\pi}{\lambda}\sin\theta,$$

$$\hat{k}_x = \frac{2}{\Delta x}\sum_{p=1/2}^{(m-1)/2} a_p\sin(k_x p\,\Delta x),$$

$$\hat{k}_z = \frac{2}{\Delta z}\sum_{p=1/2}^{(m-1)/2} a_p\sin(k_z p\,\Delta z).$$

Setting q_{11} equal to zero and solving for $\hat{\omega}$ gives the actual frequency being propagated. Solution for $\hat{\omega}$ gives

$$\hat{\omega} = \frac{2}{\Delta t}\sin^{-1}\left(\left[\frac{1}{2}\sum_{\substack{k=2 \\ k\ even}}^{m} \frac{(-1)^{\frac{k}{2}-1}}{k!}\left(\Delta t V_p\sqrt{\hat{k}_x^2+\hat{k}_z^2}\right)^k\right]^{\frac{1}{2}}\right). \quad (12)$$

I define phase-velocity dispersion as the ratio of numerical propagation velocity and desired propagation velocity such that

$$\frac{\hat{V}_p}{V_p} = \frac{\frac{\hat{\omega}}{\sqrt{\hat{k}_x^2+\hat{k}_z^2}}}{\frac{\omega}{\sqrt{k_x^2+k_z^2}}} = \frac{\hat{\omega}}{\omega}. \quad (13)$$

Note that k_x and k_z are used to calculate both the numerical and desired velocities because errors associated with $\hat{k}_x$ $\hat{k}_z$ are already

accounted for in ω. A similar analysis of phase-velocity dispersion can be carried out for S waves. In figure 1, I show plots of P-wave phase-velocity dispersion versus the number of samples per wavelength for schemes of various spatial-differencing orders used in conjunction with (a) 2nd-order temporal differencing, and (b) 4th-order temporal differencing. (Similar plots can readily be calculated for higher temporal orders.) For these plots, propagation is assumed to be in the direction of a coordinate axis (θ=0) and the dispersion parameter for each scheme is set at 90% of the corresponding stability factor. Figure 2(a) shows dispersion curves for 2nd (lower-most curve), 4th, 6th, 8th, 10th, and 12th order spatial derivitives, and also a curve for spectral derivatives (top-most curve). Figure 2(b) is similar except that there is one less curve as the lower most is for 4th order spatial derivatives.

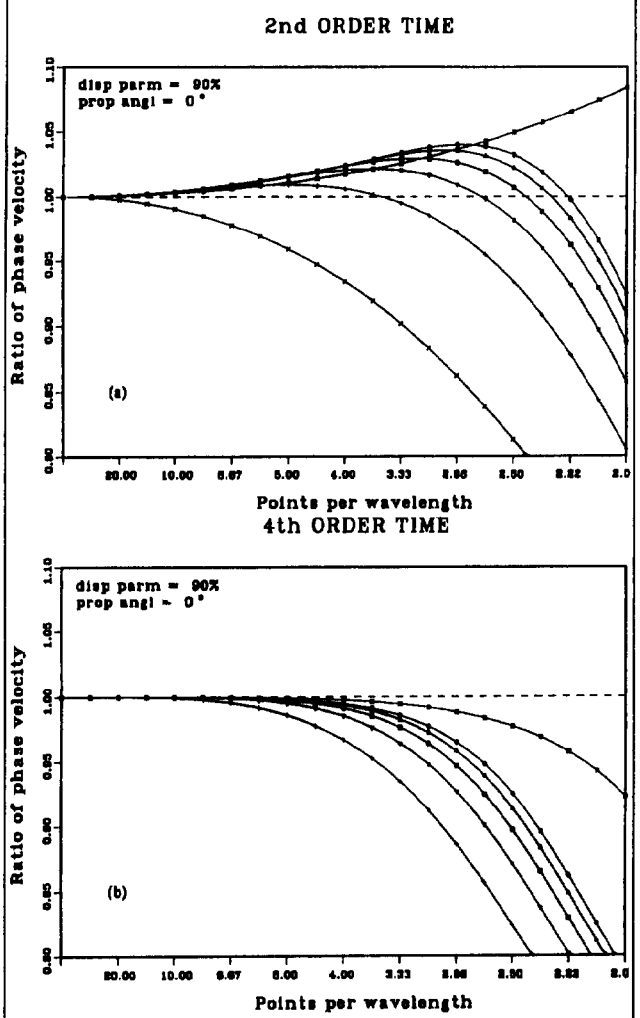

FIG 1. Curves showing modeling errors caused by numerical dispersion.

BOUNDARY CONDITIONS

For computer application of finite-difference modeling, grid sizes are finite; thus, explicit boundary conditions must be defined for the edges of the finite grids. Absorbing conditions (non-reflecting) are generally desired for the sides and bottom of the grids, while either absorbing or free-surface boundary conditions can be applied at the top of the grids. I would like to point out that, in finite-difference modeling, absorbing boundaries never perfectly absorb, and free-surface boundaries never perfectly emulate a free surface. I have included a brief discussion of boundary conditions, herein, *not* because the conditions I use are neccessarily novel, *but* because the conditions I employ took a great deal of time to implement and seem to work effectively.

In a traditional and correct approach, a free surface implies that stress normal to this surface is zero (unless a source is present) such that

$$\sigma^{zz} = \lambda \frac{\partial}{\partial x} u^x + \frac{(\lambda+2\mu)\partial}{\partial z} u^z = 0 , \qquad (14)$$

and

$$\sigma^{xz} = \mu \left(\frac{\partial}{\partial x} u^z + \frac{\partial}{\partial z} u^x \right) = 0 .$$

One way to approximate this condition for higher-order spatial schemes is given by Levander (1988) who uses the anti-symmetry properties of σ^{zz} and σ^{xz} and then solves a set of linear equations for the vertical derivatives of u^x and u^z. (If both σ^{zz} and σ^{xz} are zero at z=0, both must be anti-symmetric about z=0 for the length of the spatial derivative operator.)

The free-surface condition which I implement relies upon the same anti-symmetry properties of stress; however, instead of solving a set of linear equations to derive displacements, I approximate the true conditions from purely physical considerations. In essence, (1) displacement must always be continuous at any boundary, and (2) the free surface can neither add to nor resist displacement. Thus, if, in addition to the anti-symmetry of stress, one requires that u^x and u^z be symmetric about z=0, (1) and (2) are satisfied as are the free-surface conditions of equation (20), at least approximately. I have compared surface waves generated by this boundary condition to those shown by Virieux (1986). The phases of the surface waves are identical, and the amplitudes of my results appear to be slightly higher. This free-surface condition is also completely stable. Note that all boundary conditions, either free-surface or absorbing, can also be implemented on the higher-order time derivatives of stress as it is essentially the same conditions which must be satisfied. (e.g.,

$$\frac{\partial^2}{\partial t^2}\sigma^{zz} = \lambda \frac{\partial}{\partial x}\frac{\partial^2}{\partial t^2} u^x + \frac{(\lambda+2\mu)\partial}{\partial z}\frac{\partial^2}{\partial t^2} u^z = 0 ,$$

must also hold.)

Various methods for simulating absorbing boundaries have been presented in the literature. In practice, however, the most commonly used absorbing boundaries are those which are stable and can easily be converted to computer code. I have found that combinations of acoustic absorbing-boundary conditions work very well in the elastic case also. In essence, I combine (1) the acoustic A2 boundary conditions of Clayton and Engquist (1977), modified for angle dependence as proposed by Keys (1985), and (2) acoustic sponge boundaries as implemented by Dablain (1986).

For the A2 conditions, I use coefficients derived from P-wave velocity for application to normal stresses and coefficients derived from S-wave velocities for application to the shear stress. As for stability, it is important to note that, because I use only a stress field when applying the A2 conditions, I do not have the stability associated with a staggered grid. Thus, to insure stability, I apply a three-point triangular filter to these strips which supresses unstable higher wavenumbers. Typical coefficients for such a filter would be (.05, .90, .05) although other coefficients also work well. Because the A2 absorbing conditions are meant for acoustic applications, they do not absorb all the modes associated with elastic wave propagation. Furthermore, because they are only approximations to one-way wave propagation, they do not completely absorb pure P or S waves. For this reason, I also implement sponge boundaries as presented in Dablain (1986) who, in turn, credits their development to Israeli and Orszag (1981). I apply the sponge boundaries to the displacement fields.

DISCUSSION AND EXAMPLE

When considering the economics involved in finite-difference modeling, several factors must be considered. First, if one wants to accurately model reflections from complicated interfaces (pinchouts for example), either small grid spacing or special computer code for each model (Fornberg, 1988) will generally produce the best results.

(I do note, however, that I have had good results using Lagrange interpolating functions to shift interfaces and simulate curvature while retaining relatively large grid spacing. At this writing I am investigate this technique for complex subsurface models.)

Second, if large grid spacing is acceptable, use of high-order spatial differencing (including spectral derivatives) with 2nd-order temporal differencing can lead to excessive numerical dispersion unless very small time steps are used. Taking very small time steps is expensive. The dispersion problems are evident in the 2nd-order time, numerical-dispersion plots presented in figure 1. I would like to emphasize that numerical dispersion does not necessarily imply a trailing "tail" on the propagating wavelet. This is generally only true for very high frequencies and/or for 2nd-order spatial, 2nd-order time schemes. For high-order spatial operators combined with 2nd-order temporal differencing, numerical propagation velocities are most often faster than true velocities which implies a precursor to the wavelet.

In the event that minimal numerical dispersion is desired and large grid spacing is acceptable, one can set a limit on allowable numerical dispersion and calculate the minimum number of computations (inculding boundary conditions) that are required for various spatial and temporal schemes. In essence, one needs to know (1) the spatial dimensions of the subsurface to be modeled, (2) the minimum and maximum velocities, and (3) the maximum frequency of interest. I have made these calculations for various parameters and the optimal spatial and temporal differencing orders vary significantly depending upon the above factors. For example, if one requires that the maximum allowable error in phase velocity is less than 0.1%, then given a particular model in which (1) the subsurface is 1700 x 1700 m, (2) velocities range from 4600 m/s (maximum P velocity) to 1300 m/s (minimum S velocity), and (3) the maximum frequency of interest is 50 Hz, it turns out that the optimal scheme, in terms of minimum CPU time, is 10th-order in space and 4th-order in time. Figure 2 shows the vertical component (particle velocity) of a 60-channel shot record which was modeled using the above parameters. This particular modeling required 0.68% of the computations required by a 2nd-order space, 2nd -order time scheme, given the same allowable error in phase velocity. It also required only 1.6% of the memory required by the 2 x 2 scheme. I note that no attempt was made to analyse numerical dispersion errors associated with ground roll. I also emphasize that all testing was done under the assumption that large grid spacing is acceptable.

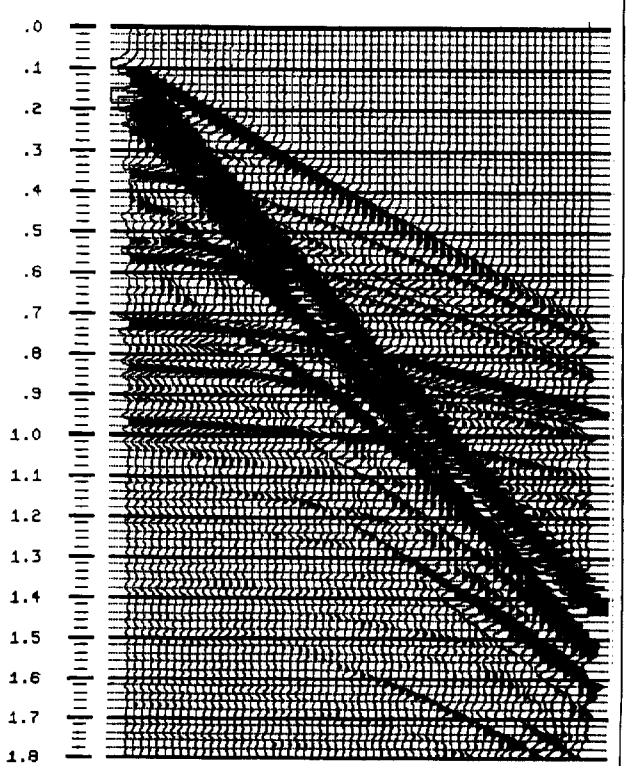

FIG 2. 60-channel shot record modeled with 10th-order spatial and 4th-order temporal finite-difference algorithm.

CONCLUSIONS

Elastic finite-difference modeling has been defined for arbitrary spatial and temporal differencing orders. Because a staggered grid is used when calculating spatial derivatives, modeling is stable and accurate for any value of Poisson's ratio. Thus, like the 2nd-order (space and time) scheme of Virieux (1986), solid-liquid interfaces are easily handled. Boundary conditions for the finite grids can also be devised which adequately simulate a free-surface and which absorb outward traveling waves. Stability limits and numerical dispersion errors in homogeneous media can be readily calculated; thus, optimal schemes can be devised for a given modeling application.

ACKNOWLEDGEMENTS

I would sincerely like to thank Cray Research, Inc., who donated countless hours of computer time while this modeling scheme was being developed. The Cray YMP is a fantastic machine. Much gratitude is also due the various people whose discussion and encouragement made this work a reality. I thank G.H.F. Gardner and Jean Virieux for many useful ideas. Thank you also to Albert Tarantola, Kurt Marfurt, Raul Madariaga, Antonio Pica, Mark Noble, Cao Di, Denis Trézéguet, and Hakim Jannane for some very stimulating discussions.

REFERENCES

Burden, R.L., Faires, J.D., and Reynolds, A.C., 1978, Numerical Analysis: Prindle, Weber, & Schmidt, Boston, pp 126-127.

Clayton, R., and Engquist, B., 1977, Absorbing boundary conditions for acoustic and elastic wave equations: Bull. Seism. Soc. Am., vol 67, pp 1529-1540.

Dablain, M.A., 1986, The application of high-order differencing to the scalar wave equation: Geophysics, vol 51, pp 54-66.

Fornberg, B., 1988, The pseudospectral method: accurate representation of interfaces in elastic wave calculations: Geophysics, vol 53, pp 625-637.

Holberg, O., 1987, Computational aspects of the choice of operator and sampling interval for numerical differentiation in large-scale simulation of wave phenomena: Geophys. Prosp., vol 35, pp 629-655.

Israeli, M., and Orszag, S.A., 1981, Approximation of radiation boundary conditions: J. Comp. Phys., vol 41, pp 115-135.

Kelly, K.R., Ward, R.W., Treitel, S., and Alford, R.M., 1976, Synthetic seismograms - A finite-difference approach: Geophysics, vol 41, pp 2-27.

Keys, R.G., 1985, Absorbing boundary conditions for acoustic media: Geophysics, vol 50, pp 892-902.

Levander, A.R., 1988, Fourth-order finite-difference P-SV seismograms: Geophysics, vol 53, pp 1425-1437.

Madariaga, R., 1976, Dynamics of an expanding circular fault: Bull. Seism. Soc. Am., vol 66, pp 163-182.

Marfurt, K.J., 1984, Accuracy of finite-difference and finite-element modeling of the scalar and elastic wave equations: Geophysics, vol 49, pp 533-549.

Tal-Ezar, H., Kosloff, D., and Koren, Z., 1987, An accurate scheme for seismic forward modeling: Geophys. Prosp., vol 35, pp 479-490.

Virieux, J., 1986, P-SV wave propagation in hetrogeneous media - Velocity-stress finite-difference method: Geophysics, vol 51, pp 889-901.