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Short Note

Wave-field separation in two-dimensional anisotropic media

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INTRODUCTION

Until recently, the term "elastic" usually implied two-dimensional (2-D) and isotropic. In this limited context, the divergence and curl operators have found wide use as wave separation operators. For example, Mora (1987) used them in his inversion method to allow separate correlation of P and S arrivals, although the separation is buried in the math and not obvious. Clayton (1981) used them explicitly in several modeling and inversion methods. Devaney and Oristaglio (1986) used closely related operators to separate P and S arrivals in elastic VSP data.

With the current widespread interest in anisotropy, it seems useful to extend the wave-type separation concept to anisotropic media. We give a simple geometrical explanation of why divergence and curl are wave-type separation operators in the isotropic case and then show how to construct wave-type separation operators for general 2-D anisotropic media. We demonstrate the method on a heterogeneous strongly anisotropic finite-difference example.

Extending existing isotropic 2-D algorithms based on wave-type separation to include anisotropy seems to be straightforward.

MODE SEPARATION IN TWO DIMENSIONS

Any arbitrary vector field $\mathbf{U}(x, y, z)$ can be expressed as

$$\mathbf{U} = \nabla P - \nabla \times \mathbf{S}, \tag{1}$$

where P is a scalar and $\mathbf{S}$ is a vector field with zero divergence. If $\mathbf{U}$ is an isotropic wave field, then P and $\mathbf{S}$ are called the P and S potentials, respectively (Aki and Richards, 1980). In two dimensions $\mathbf{S} = S\mathbf{1}_z$, where S is a scalar and $\mathbf{1}_z$ is the unit vector (0, 0, 1), so both P and $\mathbf{S}$ can be treated as scalars. Equation (1) is usually not used directly in practice. Instead, a 2-D isotropic wave field $\mathbf{U}$ is decomposed into P and S components by calculating $\nabla \cdot \mathbf{U}$ $(= \nabla^2 P)$ and $\nabla \times \mathbf{U}$ $(= \mathbf{1}_z \nabla^2 S)$, respectively. This wave separation property of $\nabla \cdot$ and $\nabla \times$ is commonly exploited by elastic inversion and migration algorithms to allow separate treatment of P-waves and S-waves.

The isotropic case

Any vector wave field, isotropic or not, can be split into two potentials via equation 1, but the resulting potentials only represent pure wavetypes in the isotropic case. Our goal is to separate quasi-P (abbreviated qP) and quasi-S (abbreviated qS) waves into *anisotropic* wave fields. We begin by showing how $\nabla \cdot$ extracts P-waves in the standard isotropic case.

The divergence of a 2-D elastic wave field $\mathbf{U}$ is

$$\nabla \cdot \mathbf{U} = \frac{\partial U_x(x, y)}{\partial x} + \frac{\partial U_y(x, y)}{\partial y}. \tag{2}$$

In the Fourier domain this is

$$\widehat{\nabla \cdot}\ \hat{\mathbf{U}} = ik_x \hat{U}_x(k_x, k_y) + ik_y \hat{U}_y(k_x, k_y), \tag{3}$$

where $\hat{\mathbf{U}}$ and $\widehat{\nabla \cdot}$ are the Fourier-domain representations of $\mathbf{U}$ and $\nabla \cdot$, respectively.

A more illuminating way to write equation (3) is

$$\widehat{\nabla \cdot}\ \hat{\mathbf{U}} = ik(\mathbf{l} \cdot \hat{\mathbf{U}}), \tag{4}$$

where $k = \sqrt{k_x^2 + k_y^2}$ is the wavenumber at (k_x, k_y) and $\mathbf{l}$ is the unit wave normal vector $(k_x/k, k_y/k)$ pointing in the direction of (k_x, k_y). Each point (k_x, k_y) in the Fourier domain represents plane waves traveling in the direction $\mathbf{l}$ with spatial frequency k. At any given (k_x, k_y) there are two wave types, distinguishable from each other by their distinct velocities and particle motion directions. In isotropic media the wave types are P and S and the particle motions are simple: P-waves shake along the direction of travel ($\|\mathbf{l}$), while S-waves shake perpendicular to the direction of travel ($\perp\mathbf{l}$). Thus, the $\mathbf{l} \cdot \hat{\mathbf{U}}$ in equation (4) admits P-waves (because the P particle motion is parallel to $\mathbf{l}$) but zeros S-waves (because the S particle motion is orthogonal to $\mathbf{l}$).

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The divergence operator is not purely a wave-type separation operator because of the additional presence of the factor k. This factor acts like a derivative operator in the space domain, amplifying higher spatial frequencies. An operator that rejects S-waves while passing P-waves of all frequencies equally can be constructed by inverse Fourier transforming the operator

$$i\mathbf{l}\cdot . \tag{5}$$

The left plot in Figure 1 shows a graphical Fourier-domain representation. [The extraneous "i" in equation (5) keeps the operator real in the space domain; $\mathbf{l}$ is *anti*symmetric about the origin.]

If the operator in equation (5) is an ideal isotropic wave-separation operator, why is $\nabla\cdot$ normally used instead? The problem is that $i\mathbf{l}\cdot$ is discontinuous at ($k_x = 0, k_y = 0$), where $\mathbf{l}$ becomes undefined. This discontinuity is not automatically fatal, because wave fields typically have no energy at zero frequency anyway. Unfortunately, the discontinuity at ($k_x = 0$, $k_y = 0$) in the Fourier domain also means the operator $i\mathbf{l}\cdot$ has tails extending off to infinity in the space domain. It is thus less compact than the divergence operator $\widehat{\nabla\cdot} = ik\mathbf{l}\cdot$, which is a point differential operator in the space domain.

The anisotropic, 2-D case

Our isotropic wave-type separation theory generalizes to the anisotropic case in an obvious way. The principal change is that the particle motion directions of the wave modes are no longer simply $\|\mathbf{l}$ or $\perp\mathbf{l}$; instead the particle motion directions may be complicated functions of k_x and k_y. In the Fourier domain the anisotropic wave-type separation operator has the form $i\mathbf{v}(k_x, k_y)\cdot$, where $\mathbf{v}(k_x, k_y)$ is a unit vector giving the particle motion direction of the desired mode on the plane wave (k_x, k_y).

How do we find the required particle motion directions? The Christoffel equation is just the elastic wave equation Fourier transformed over space and time. It has the form

$$\underline{\mathbf{M}}\underline{\mathbf{C}}\underline{\mathbf{M}}^T\mathbf{v}_n = -\rho\omega_n^2\mathbf{v}_n, \tag{6}$$

where $\underline{\mathbf{M}}$ is a matrix involving only k and $\mathbf{l}$, $\underline{\mathbf{C}}$ is the stiffness matrix that defines the elastic properties of the medium, ρ is the density of the medium, ω_n is the temporal frequency for solution n, and $\mathbf{v}_n$ is the associated particle motion direction (Auld, 1973). For a given $\underline{\mathbf{C}}$, k, and $\mathbf{l}$, equation (6) always has two orthonormal solutions $\mathbf{v}_1$ and $\mathbf{v}_2$, each of which corresponds to one wave type.

Algorithm description

Now we have everything we need to extend our isotropic wave-separation algorithm to handle anisotropy:

(1) Decide which mode ("quasi-P or quasi-S") this operator will pass.

(2) For all (k_x, k_y):

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(A) Substitute k_x and k_y into equation (6) and find the two solutions $\mathbf{v}_1$ and $\mathbf{v}_2$.

(B) Decide whether solution 1 or 2 corresponds to the desired mode chosen in step (1). Call this solution $\mathbf{v}$.

(C) Both $\mathbf{v}$ and $-\mathbf{v}$ are equally valid normalized solutions. Choose the one that is consistent with particle motion directions already determined at adjacent (k_x, k_y) points.

(D) Store the result for this (k_x, k_y) in $\mathbf{L}(k_x, k_y)$.

}

(3) Inverse Fourier transform the operator $i\mathbf{L}\cdot$, obtaining the desired (x, y)-domain wave-separation operator.

In practice compactness of the operator in the space domain is desirable. There are two Fourier-domain discontinuities to avoid, at the origin and the Nyquist frequency.

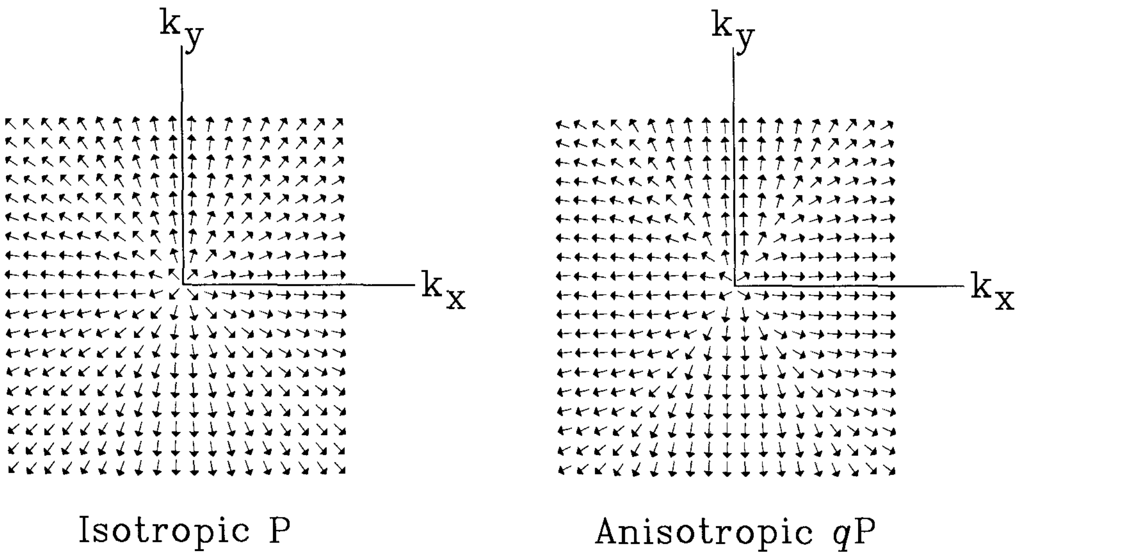

FIG. 1. Left: Fourier-domain plot of the operator for passing only P-waves in isotropic media. Right: Fourier-domain plot of the operator for passing only P-waves in the anisotropic medium used in Figure 2.

The first can be removed by multiplying the operator by k, following the example of the divergence operator. The second can be avoided by a suitable Fourier-domain window function.

A FINITE-DIFFERENCE EXAMPLE

To demonstrate the method, we generated a wave field using a finite-difference elastic modeling program. The model is split into three parts: an upper left part, an upper right part, and a bottom part. The upper left part is isotropic ($C_{11} = 227$, $C_{33} = 227$, $C_{55} = 54$, and $C_{13} = 119$); the upper right part is anisotropic ($C_{11} = 341$, $C_{33} = 227$, $C_{55} = 54$, and $C_{13} = 30$). These two media were chosen to have the same vertical velocities. The bottom part of the model grades uniformly from the isotropic elastic constants at the left edge to the anisotropic ones at the right edge. The density is uniform throughout.

The left plot in Figure 1 shows the P wave-field separation operator appropriate for the isotropic elastic constants. The right plot in Figure 1 shows the P wave-field separation operator appropriate for the anisotropic elastic constants. To avoid the Fourier-domain discontinuities at the origin and the Nyquist frequency, we also incorporated a derivative (k) and a raised cosine taper [$.5(1 + \cos(\pi k/k_{\mathrm{Nyquist}}))$] in our wave-type separation program.

Figure 2 shows the results of applying operators appropriate for each set of elastic constants to the whole. The top two plots in the figure show the input wave field, which contains

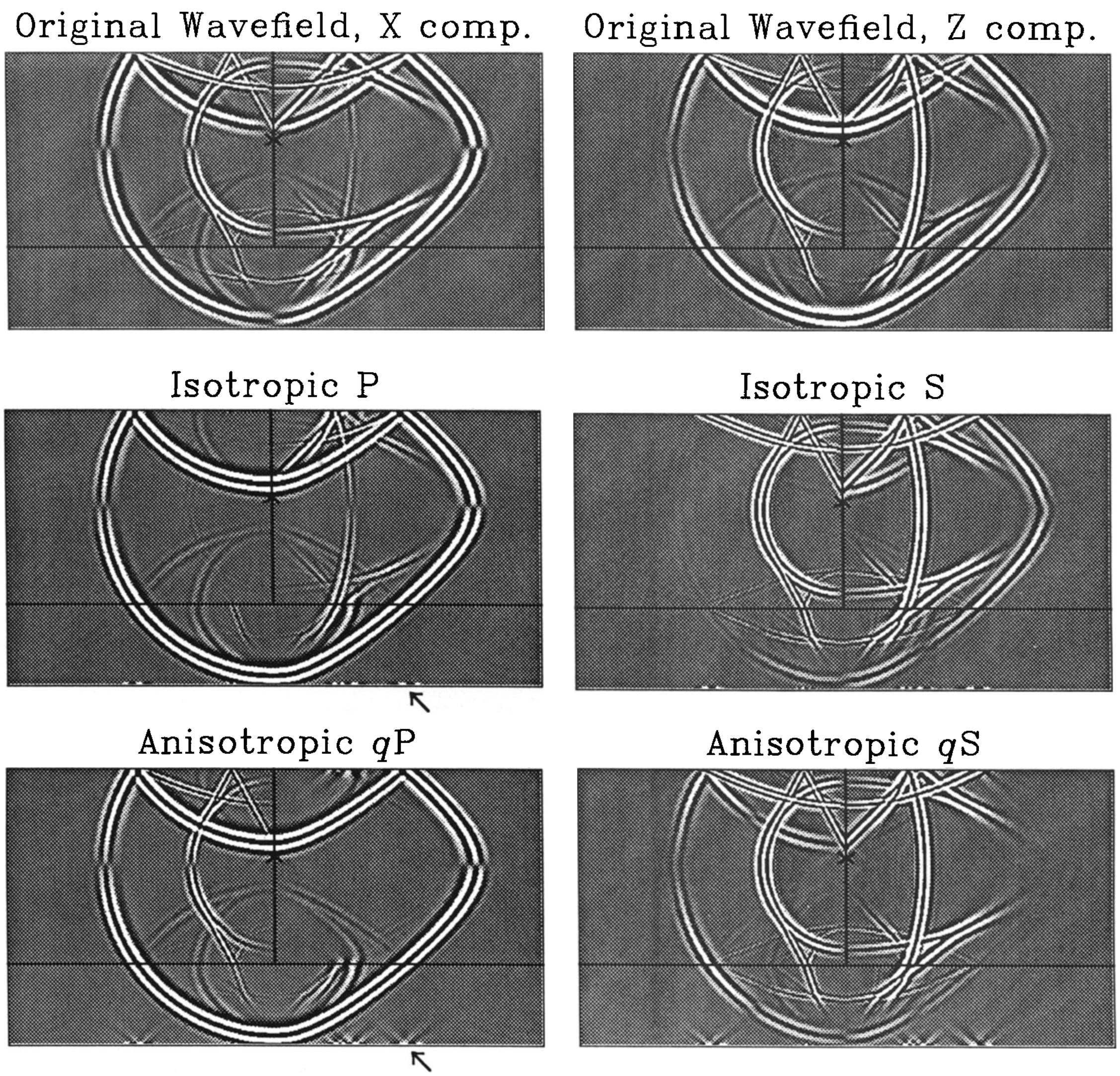

FIG. 2. Top: Snapshot of the X (left) and Z (right) components of a finite-difference wave field. The grid is 512 wide by 256 tall, with absorbing boundaries on the left, right, and bottom borders and a free-surface boundary at the top. The model consists of two layers. The upper layer is divided into left and right halves. The left half is isotropic and the right half is anisotropic. Underneath is a single layer that grades smoothly from isotropic on the left to anisotropic on the right. The source is a vertical impulsive point force at the "×". Middle: The result of applying isotropic wave-field separation operators (divergence and curl) to the entire medium; the separation works in the upper left block. Bottom: The result of applying anisotropic wave-field separation operators to the entire medium; the separation works in the upper right block. The lower four plots are all plotted with the same parameters: square root amplitude gain and heavy clipping have been applied to make weak events and artifacts visible.

complicated interacting wavefronts of both wave types. If the separation operators work correctly, they should admit one wave type while exactly zeroing the other. From the figure, we can see that each set of operators works well in the appropriate part of the medium: the isotropic operators work in the upper left block, and the anisotropic operators work in the upper right block. The separations are not quite perfect because of slight numerical anisotropy in the finite-difference modeling program; a small amount of the undesired wave type does get through (about 1 percent). As expected, none of the separations works very well in the central portion of the lower block, where neither set of operators is appropriate.

DISCUSSION

Can wave separation work in heterogeneous media?

Another kind of error is visible along the bottom edges of the lower four plots in Figure 2 (the arrows point to one example). The wave-type separation operators were applied in the Fourier domain with no padding. To the operators the model is thus cyclic, and the free surface at the top of the model abuts directly against the absorbing boundary at the bottom. This creates both a heterogeneity in the elastic constants and a discontinuity in the wave field itself.

Since the operators were derived assuming a uniform homogeneous medium, it is not clear how well they should work across free surfaces and other such elastic-constant heterogeneities. The most important factor is how compact the operators are in the space domain; a compact operator is more tolerant of heterogeneities of any kind. Figure 3 shows a closeup of the impulse response of the discretized isotropic and anisotropic wave separation operators. The discretized isotropic operator shown is relatively compact, while the anisotropic operator possesses "tails" extending out from the center of the operator. The arrows below two of the plots in Figure 2 point to the same free-surface wraparound artifact for the isotropic and anisotropic cases. As we expected from Figure 3, in the isotropic case the artifact is much smaller.

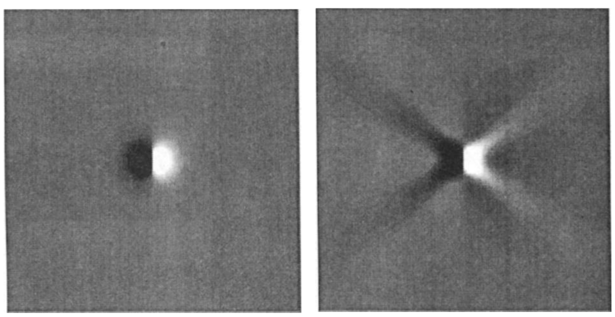

FIG. 3. P wave-type separation operator impulse responses for the isotropic and anisotropic media in Figure 2. The operators were calculated on a 2048^2 grid using a smoothed vertical impulse source. A 201^2 subgrid centered on the impulse is shown. Both plots are plotted with the same parameters; as in Figure 2, square root amplitude gain and heavy clipping have been applied to make low-amplitude features visible. The isotropic operator is clearly more compact than the anisotropic one, although the "tails" of the anisotropic operator are at most a few percent of the amplitude of the central spike.

We know that in the continuous limit, the isotropic operators collapse to the point differential operators divergence and curl. We cannot expect such a collapse for arbitrary operators defined in the Fourier domain, any more than we can expect the one-dimensional Fourier transform of some general function to collapse to a point. The tails of the discrete anisotropic wave-separation operator in Figure 3 warn that in the continuous limit anisotropic wave-separation operators are probably not normally point differential operators like divergence and curl.

While disturbing in theory, such tails should not be much of a problem in practice. Even in the extremely anisotropic "worst case" example in Figure 2, the tails are only noticeable along the large discontinuity at the top and bottom of the model. The amplitude of the artifacts there is only a few percent of the amplitude of the waves reflecting at the free surface.

Do we really need to know the elastic constants?

We know that for isotropic media $\nabla \cdot$ separates the two wave types regardless of the density, P-wave velocity, or S-wave velocity. Equation (6), on the other hand, seems to require complete knowledge of both the elastic constants and density. Are the anisotropic and isotropic cases somehow fundamentally different? No. Equation (6) solves for both the particle motion directions $\mathbf{v}_n$ and the phase velocities ω_n/k for each mode. However, for separating wave types only the particle motion direction matters, which gives us some latitude in specifying the medium. For example, the value of ρ is clearly irrelevant, since it scales the phase velocities but has no effect on the particle motion directions. Likewise, the stiffness matrix $\underset{\sim}{C}$ can be freely scaled up and down.

In the isotropic case there is one more degree of freedom, so we expect there should be one more in the anisotropic case as well. For qP and qSV waves in transverse isotropic media, the transformation

$$C_{33} + \delta \Rightarrow C_{33},$$

$$C_{11} + \delta \Rightarrow C_{11},$$

$$C_{44} + \delta \Rightarrow C_{44},$$

and

$$C_{13} - \delta \Rightarrow C_{13}$$

has no effect on the particle motion directions for any legal value of δ, and thus no effect on the corresponding wave-type separation operator. This last degree of freedom in $\underset{\sim}{C}$ can radically change the shapes of the wave surfaces, however.

Figure 4 shows an example of wave-type separation in an anisotropic, strongly heterogeneous finite-difference model. The same wave-type separation operator is appropriate in all three parts of the model. The lower left part is normalized Greenhorn shale (Jones and Wang, 1981) with density $\rho = 1.$ and elastic constants $c_{11} = 3.41$, $c_{13} = 1.07$, $c_{33} = 2.27$, and $c_{55} = .54$. The medium in the lower right part has the same particle motion direction and vertical qP velocity as Greenhorn Shale, but radically different qS behavior. This medium has density $\rho = 1.$ and elastic constants $c_{11} = 2.88$, $c_{13} = -.49$, $c_{33} = 2.27$, and $c_{55} = 1.35$. Finally, the upper part of the model grades linearly between the two sets of elastic constants from one edge to the other and has a constant density of $\rho = 1.2$. Note that the same operator works perfectly in both of the lower parts of the model. It does have some trouble in the upper continuously-varying medium, where the underlying homogeneous assumption breaks down.

Is it useful?

We have shown it is possible to separate anisotropic wave types in time snapshots if the elastic constants are known. This type of wave-type separation should be applicable in elastic migration or inversion algorithms, where it allows one to examine contributions to the image due to different reflected and converted waves separately. For example, by correlating upcoming qSV waves with downgoing qP waves, we obtain the qP-qSV reflection amplitude any place the waves overlap in space and time (Claerbout, 1985). By treating the various possible reflected waves separately, we should be able to better estimate changes in material parameters across interfaces.

Wave-type separation requiring time snapshots and known elastic constants is not directly useful for separating P and S arrivals in surface seismic data or VSPs. We do not record time snapshots (x, z, t = constant) in the field, and we do not normally know elastic constants.

The first objection need not be fatal; techniques such as reverse-time migration can be used to reconstruct time snapshots from field data. Alternatively, we can adapt the algorithm to the recording geometry. For VSP data, for example, if we know the elastic constants we can solve for k_x as a function of k_z and ω, and perform the separation in the (k_z, ω) domain. This allows us to use VSP data of the form (x = constant, z, t). Devaney and Oristaglio (1986) show how to do this in the continuous isotropic case. For surface data we can similarly solve for k_z as a function of k_x and ω, and use surface data of the form (x, z = constant, t). There is the additional complication in this case of allowing for the free surface, which should be possible if the elastic constants at the surface are known.

What if the elastic constants are not known? In theory it should be possible to set up an inverse problem and use the particle motion directions of distinct wavefronts to determine the necessary elastic constants. The trick is we have to identify distinct single-wave-type wavefronts first. Otherwise, for example, we could reverse-time migrate surface data using any elastic constants we wanted and exactly account for the data by treating each recorded arrival as a

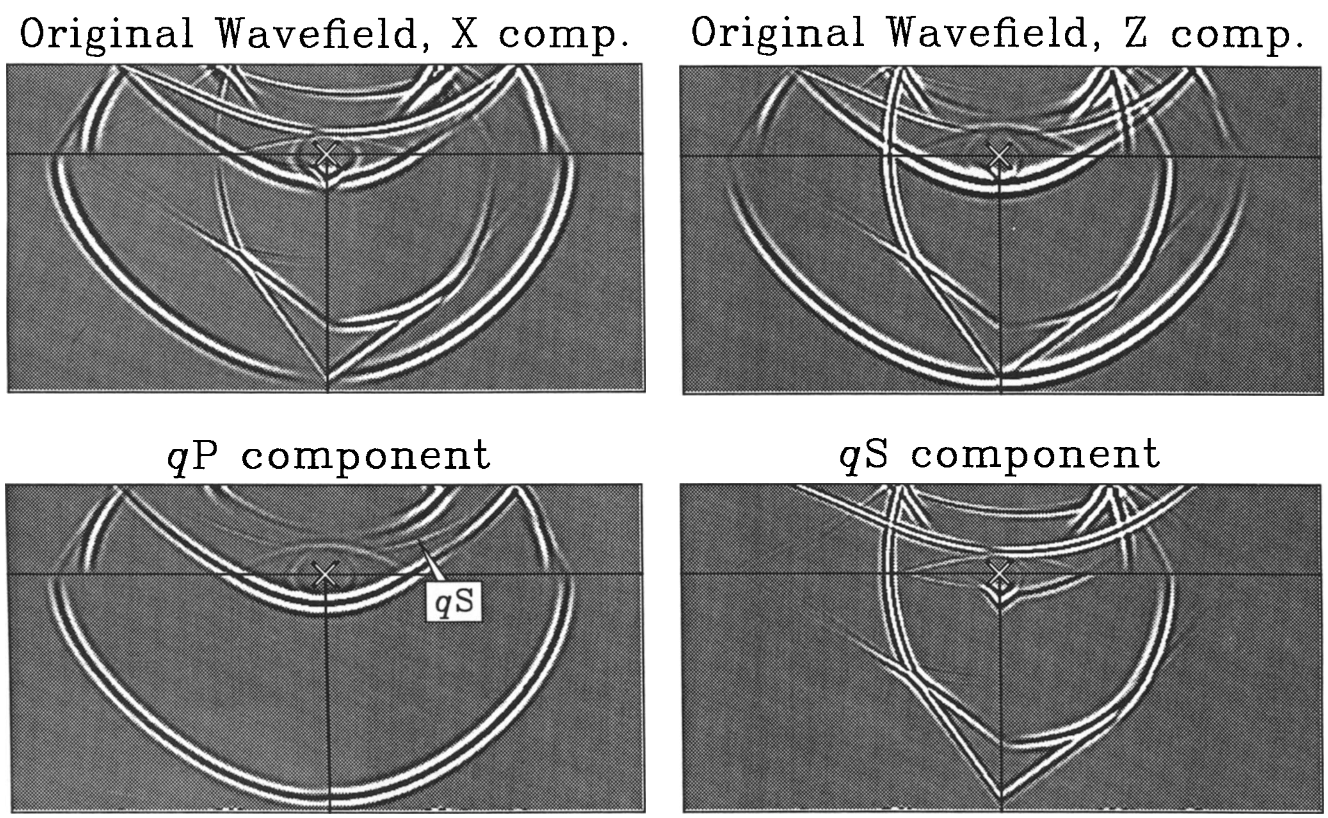

FIG. 4. Top: Snapshot of the X (left) and Z (right) components of a finite-difference wave field. The differencing grid and boundary conditions are the same as those in Figure 2. The source is a vertical impulse point force at the "×". Bottom: The result of applying anisotropic wave-field separation operators to the entire medium. The separation works well everywhere, except for a small amount of qS which appears in the separated qP component. Hard clipping has been applied to make weak events and artifacts more visible.

coincidental instantaneous overlap of a *P*-wave and an *S*-wave.

Extending to three dimensions

We might expect our 2-D theory to generalize to three dimensions in a straightforward way, but this is not the case. Anisotropic propagation in three dimensions introduces several novel complications that have no counterparts in two dimensions. Mode separation in three-dimensional anisotropic media will be the subject of a separate paper.

CONCLUSION

In the discrete wavenumber domain, anisotropic wave-type separation in two dimensions is only slightly more complicated than in the standard isotropic case. To perform the separation, we need to know the particle motion direction for each plane-wave propagation direction. This does not require complete knowledge of the elastic constants; there is an unconstrained 2-D null space that has no effect on the particle motion directions. In the isotropic case, this null space corresponds to the Lamé parameters λ and μ.

Although anisotropic wave-type separation operators do not reduce to simple differential operators such as divergence and curl in the continuous limit, their discrete representations are compact enough to be just as applicable to modeling and inversion algorithms as the standard isotropic operators.

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