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A hybrid scheme for absorbing edge reflections in numerical modeling of wave propagation

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ABSTRACT

We propose an efficient scheme to absorb reflections from the model boundaries in numerical solutions of wave equations. This scheme divides the computational domain into boundary, transition, and inner areas. The wavefields within the inner and boundary areas are computed by the wave equation and the one-way wave equation, respectively. The wavefields within the transition area are determined by a weighted combination of the wavefields computed by the wave equation and the one-way wave equation to obtain a smooth variation from the inner area to the boundary via the transition zone. The results from our finite-difference numerical modeling tests of the 2D acoustic wave equation show that the absorption enforced by this scheme gradually increases with increasing width of the transition area. We obtain equally good performance using pseudospectral and finite-element modeling with the same scheme. Our numerical experiments demonstrate that use of 10 grid points for absorbing edge reflections attains nearly perfect absorption.

INTRODUCTION

Many scientific and engineering problems require numerical solutions of partial differential equations. The three main numerical methods are finite difference, pseudospectral, and finite element. Because of the finite computational domain, a persistent problem in the numerical solution of wave equations is the artificial reflections from boundaries or edges, introduced by a truncated computational domain. The simplest scheme to avoid boundary reflections is to enlarge the computation domain, which delays the boundary reflections and wraparound beyond the maximum time involved in the modeling (Cerjan et al., 1985). This scheme obviously increases computing costs; therefore, other schemes, generally termed *absorbing boundary conditions* (ABCs), have been developed.

Essentially three kinds of ABCs address this problem. The first uses a different form of wave equation to predict the wavefield at the boundaries. In the nonboundary region, the wavefield is calculated by wave equations. The boundary wavefield is usually estimated by some approximation, such as a one-way wave equation, field extrapolation, or impedance condition (e.g., Engquist and Majda, 1977; Liao et al., 1984; Hu et al., 2007). One-way wave equations have been widely used for absorbing boundary reflections in numerical modeling (e.g., Clayton and Engquist, 1977; Reynolds, 1978; Mur, 1981; Keys, 1985). However, the absorption effects, which are highly dependent on incident angle, are usually better for normally incident waves than nonnormally incident waves. Even though the edge-reflected direct waves' amplitudes are substantially reduced compared with the amplitudes of the direct waves, they are strong compared with the amplitudes of reflected waves from the reflectors in a model. Therefore, relatively few developments have been reported (e.g., Zhou and McMechan, 2000; Heidari and Guddati, 2006). High-order one-way wave equations can reduce edge reflections further (Heidari and Guddati, 2006).

The second kind of ABC attenuates the wavefield in the vicinity of the boundary. This method, introduced by Cerjan et al. (1985), uses an exponential function to attenuate the wavefield within a damping strip near the boundary. Some applications and improvements of this technique have been reported in the literature (e.g., Kosloff and Kosloff, 1986; Sochacki et al., 1987; Cao and Greenhalgh, 1998; Sarma et al., 1998; Tian et al., 2008). It is generally very difficult to find a proper attenuation function to absorb incident waves perfectly because they propagate with different velocities and in different directions.

The third kind of ABC is the perfectly matched layer (PML) proposed by Bérenger (1994), which is an absorbing material boundary condition. In this method, the computational region is wrapped by a material designed to absorb the outgoing waves. Decay factors are included in formulations used in PML modeling. Perfect absorption

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for any incident wave can be achieved only by setting appropriate decay factors. PML has been widely used, and many developments have been reported in applications to seismic wave-propagation problems (e.g., Zeng et al., 2001; Komatitsch and Tromp, 2003; Wang and Tang, 2003; Hu et al., 2007; Gao and Zhang, 2008).

Of these three methods, the prediction-based method has the lowest computing expense, and it moderately absorbs the boundary reflections. The damping or attenuating method has moderate computational expense but may perform the worst of the three. PML is the most expensive computationally but generally obtains the best absorption. We propose a hybrid scheme for absorbing edge reflections; it has the advantages of small memory, minimal computing time, and nearly perfect absorption.

METHOD

We start with the one-way wave-equation method for absorbing boundary reflections. In this scheme, wavefields at the boundary grid points are calculated by a one-way wave equation; those at the nonboundary grid points are determined by a wave equation (e.g., Clayton and Engquist, 1977). One-way wave equations generally are exact for waves that propagate in a certain direction, so the waves traveling in the other directions are still reflected from the boundary. In fact, the calculated wavefields are inexact even in this certain direction because of the errors resulting from finite-difference discretization of the one-way wave equation. Additionally, when the waves propagate from the nonboundary area to the boundary, a sharp variation occurs between the wavefields at the nonboundary and boundary grids because they are determined by different forms of the wave equation. For these two reasons, some residual edge reflections always exist.

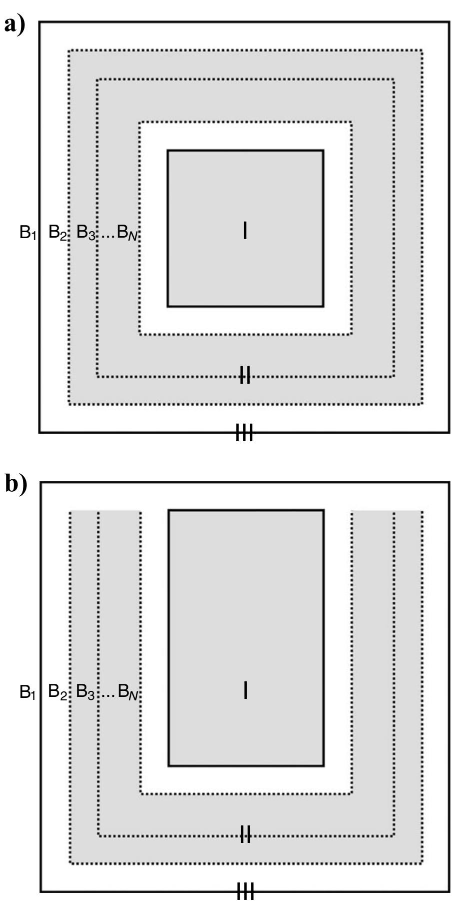

Figure 1. The new ABC scheme (new ABCs) (a) without and (b) with free surface. See the text for details.

To decrease the sharp variation in the wavefield between nonboundary and boundary grids, we introduce a transition area to solve the wave equation numerically. We divide the computational domain into (I) an inner area, (II) a transition area from boundary B_2 to B_N, and (III) boundary B_1, shown in Figure 1 for numerical problems without and with a free surface. The three areas in Figure 1b differ from those in Figure 1a because of the requirement of a free surface. The wavefield values within area I and boundary B_1 are computed by the wave equation and the one-way wave equation, respectively. However, the wavefield values at each grid within area II are first computed by the wave equation and one-way wave equation, and the final wavefield used for further propagation is a weighted average of the two at each grid. This ascertains a smooth variation from area I to B_1 via area II. Obviously, the degree of variation and thus the boundary reflections decrease as N increases. When $N = 1$, this scheme reduces to the conventional scheme.

At a given time step, the procedure is comprised of three steps. First, we calculate the wavefield values $P_i^{(1)}$ within areas I and II by wave equations. Second, we calculate the wavefield values $P_i^{(2)}$ at B_1 and area II by the one-way wave equations. Third, we weigh the values within area II from steps 1 and 2, $P_i = (1 - w_i)P_i^{(1)} + w_i P_i^{(2)}$, where $P_i^{(1)}$ are values updated using the wave equations and $P_i^{(2)}$ are values updated using the one-way wave equations for B_i grid points (Figure 1), P_i are the final wavefield values for B_i grid points, and w_i is a weight that varies from zero to one so that $P_i = P_i^{(1)}$ at the boundary of area I and $P_i = P_i^{(2)}$ at the boundary of area III. Here, w_i varies linearly.

EXAMPLES

To examine the effectiveness of this scheme, we model the 2D acoustic wave equation by all three numerical schemes (finite difference, pseudospectral, and finite element) for a homogeneous model and a horizontally layered model. The 2D acoustic wave equation is

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{V^2}\frac{\partial^2 p}{\partial t^2}, \qquad (1)$$

where $p = p(x,z,t)$ is the wavefield and V is the spatially varying velocity. This wave equation is used by the finite-difference and pseudospectral methods. A conventional finite-element algorithm with linear bases and square grids is used in our finite-element modeling. We use the same finite-difference formulations and discretizations of the one-way wave-equation boundary conditions, i.e., combining first order (A1) in corners and second order (A2) on regular boundaries, as presented by Clayton and Engquist (1977; their appendix). Nonabsorbing (i.e., reflecting) boundary conditions and new ABCs (our method) with different values of N are included in the modeling. When $N = 1$, our ABC is the same as that of Clayton and Engquist (1977).

Finite-difference modeling of 2D acoustic wave equation

Figure 2 shows the snapshots computed by the finite-difference method for a homogeneous acoustic model without ABC and with the Clayton-Engquist and new ABCs. Although the conventional

Clayton-Engquist ABC absorbs much of the incident wave energy, boundary reflections can be seen in Figure 2b. Figure 2b-d demonstrates that the boundary reflections decrease with increasing width N for the new ABCs. Nearly perfect absorption is obtained when $N = 10$. These conclusions can also be drawn from Figure 3, which shows the direct and residual edge-reflected waves resulting from the Clayton-Engquist and new ABCs.

Figure 4 displays the seismograms computed by the finite-difference method for a horizontally layered acoustic model without ABC, with the Clayton-Engquist ABC, and with the new ABC. Figure 4b suggests that the boundary reflections are still strong compared with the reflections from the true reflectors for the Clayton-Engquist ABC. The new ABC achieves nearly perfect absorption, as shown in Figure 4c.

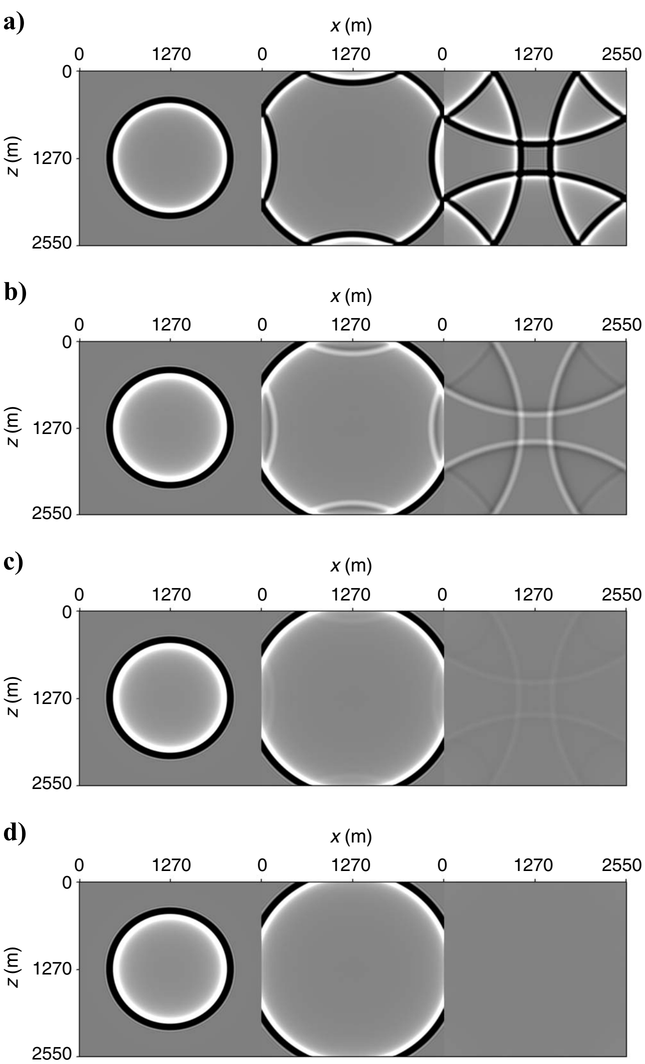

Figure 2. Snapshots computed by finite-difference modeling for a 2D homogeneous acoustic model with (a) non-ABCs, (b) Clayton-Engquist ABCs, and (c, d) new ABCs. Each figure includes three panels with three snapshots at (left to right) 300, 500, and 800 ms. The model velocity is 3000 m/s, grid size is 10×10 m, grid measures 256×256, and time step is 1 ms. Twentieth- and second-order finite differences are used for spatial and temporal discretization, respectively. A source pulse of 20-Hz sine function with one period length is located at the center of the model. The free-surface condition is not included.

We adopt the mirror-image symmetry boundary condition when calculating spatial derivatives to maintain $2M$-order accurate finite differences near the boundaries. For the free surface, the mirror-image inverse symmetry boundary condition is used instead. However, it is unnecessary that $M = N$. For instance, with the decrease of wavefield frequency, a smaller M can be used.

Pseudospectral modeling of 2D acoustic wave equation

Figure 5 shows the snapshots computed by the pseudospectral method for the homogeneous acoustic model without and with the new ABC. The results of modeling with the Clayton-Engquist ABC ($N = 1$) are not shown because that scheme is unstable for the pseudospectral method. From this figure, we observe that the new ABC with $N = 10$ reaches nearly perfect absorption. Figure 6 shows the seismograms computed by the pseudospectral method for the horizontally layered acoustic model without ABC, with a model with extended edges, and with the new ABC. The seismograms computed

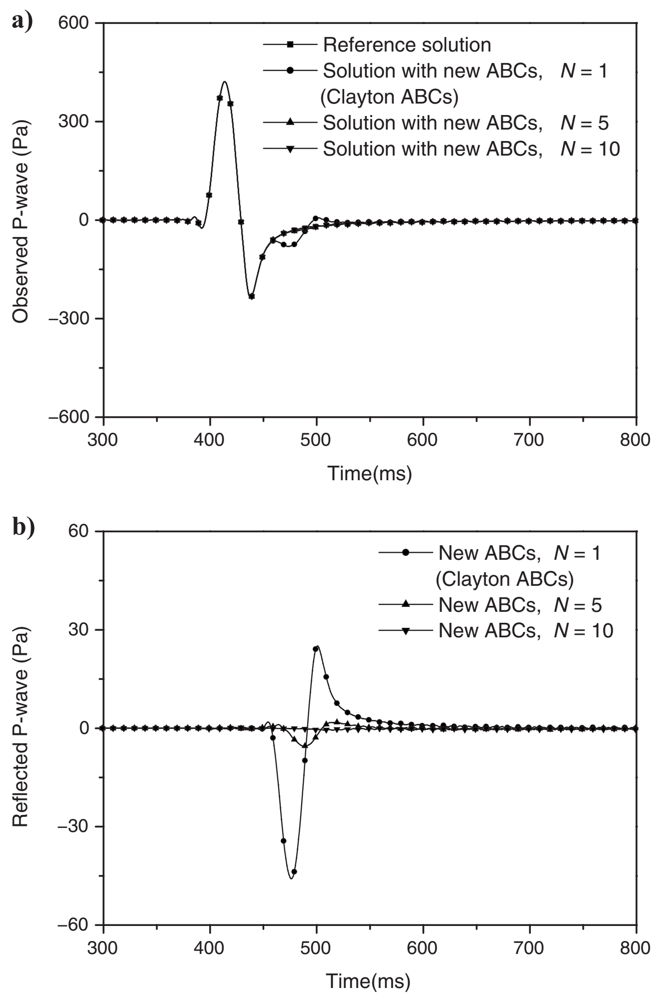

Figure 3. The received P-wave at (100,1070) m and the artificially reflected wave caused by Clayton-Engquist ABCs and new ABCs. All parameters are the same as those in Figure 2. The reference solution is calculated by extending the model boundaries. (a) Reference solution, the solution with Clayton-Engquist ABCs, and the solution with new ABCs. (b) The reflected P-wave, i.e., the difference between the reference solution and the solution with Clayton-Engquist ABCs or new ABCs.

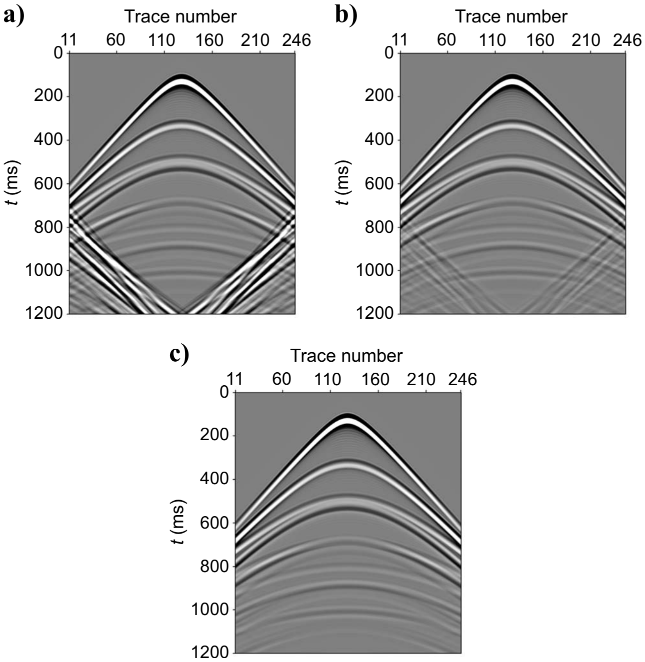

Figure 4. Seismograms computed by finite-difference modeling for a 2D horizontally layered acoustic model with (a) nonabsorbing, (b) Clayton-Engquist, and (c) new ABCs, with $N = 10$. Velocities of the six layers are 2000, 2500, 3000, 3400, 3700, and 4000 m/s, respectively, from shallow to deep. Depths of the interfaces are 400, 600, 900, 1100, and 1500 m. Grid size is 10×10 m, grid dimensions are 256×256, and the time step is 1 ms. Twentieth- and second-order finite differences are used for spatial and temporal discretization, respectively. A source pulse of 20-Hz sine function with one period length is located at the grid point (1280,200) m. Receivers are located on the surface. The free-surface condition is included. Trace numbers of seismograms range from 1 to 256. Only traces from 11 to 246 are shown because 10 grids points are used for new ABCs.

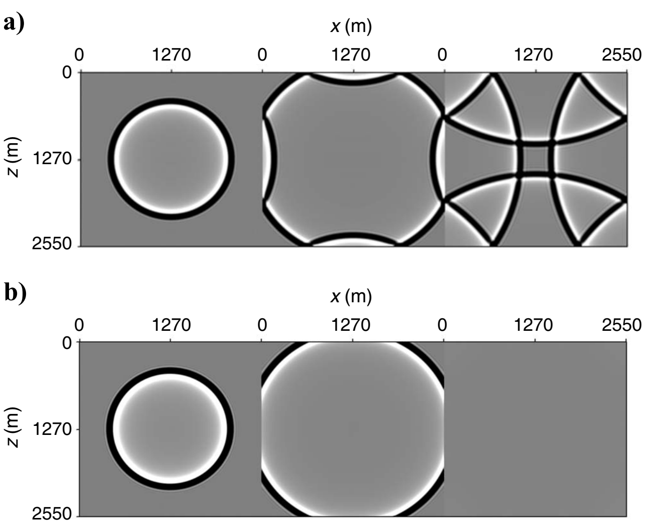

Figure 5. Snapshots computed by pseudospectral modeling for the 2D homogeneous acoustic model with (a) nonabsorbing and (b) new ABCs. The modeling with Clayton-Engquist ABCs is unstable. All parameters are the same as those in Figure 2.

by the new ABC with $N = 10$ are nearly the same as those obtained by the model with extended edges. These modeling tests demonstrate that the new ABC works well for the pseudospectral method but the Clayton-Engquist ABC does not.

Here, we provide further explanations. For the pseudospectral method, the edge reflections are caused by wraparound. For example, when the waves arrive at the left boundary, they wrap to the right boundary and then propagate from the right boundary to the inner area of the model through the transition area. In our scheme, the wavefield values of the one-way wave equation and the wave equation within the transition area are calculated; then the two wavefield values are weighted to obtain the final wavefield values. From B_1 to B_{N+1}, the weight for the one-way wave equation varies from one to zero and the weight for the wave equation varies from zero to one. This means the contribution of the wave equation to the final wavefield values varies from zero to nonzero and then gradually increases. The wrapped waves from the wave equation propagate from the boundary to the inner area; thus, they can be reduced significantly by our scheme. Our modeling experiments show the modeling is unstable when $N = 1$; the stability of the modeling improves with the increase of N; and the modeling is stable for a large number of time updates, yielding good results for $N = 10$.

Finite-element modeling of 2D acoustic wave equation

Figure 7 shows the snapshots computed by the finite-element method for the homogeneous acoustic model with the non-ABCs and the new ABCs. Figure 8 shows the seismograms computed by the finite-element method for the horizontally layered acoustic model without an ABC, with the Clayton-Engquist ABC, and with the new ABC. These figures clearly demonstrate that the new ABC works well for the finite-element method.

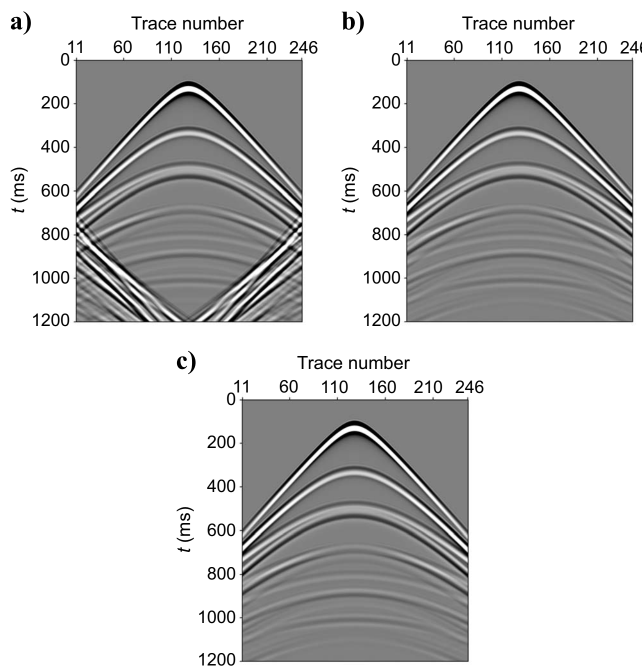

Figure 6. Seismograms computed by pseudospectral modeling for the 2D horizontally layered acoustic model (a) without ABCs, (b) using a model with extended edges, and (c) with new ABCs. The modeling with Clayton-Engquist ABCs is unstable. All parameters are the same as those in Figure 4.

DISCUSSION

Our numerical examples demonstrate that the hybrid ABC, which combines the one-way wave equation with the wave equation through a weight (equivalent to damping), performs very well. Although we show examples for acoustic waves only, it is possible to extend our scheme to any for which a one-way wave equation is available. For example, Clayton and Engquist (1977) present first- and second-order one-way wave equations for elastic wave equations, Hedstrom (1979) and Dong (1999) derive isotropic and transversely isotropic first-order one-way wave equations, and Guddati (2006) derives an arbitrary wide-angle one-way wave equation. Thus, our simple hybrid scheme may be used for more complex media such as anisotropic, porous, and viscoelastic media.

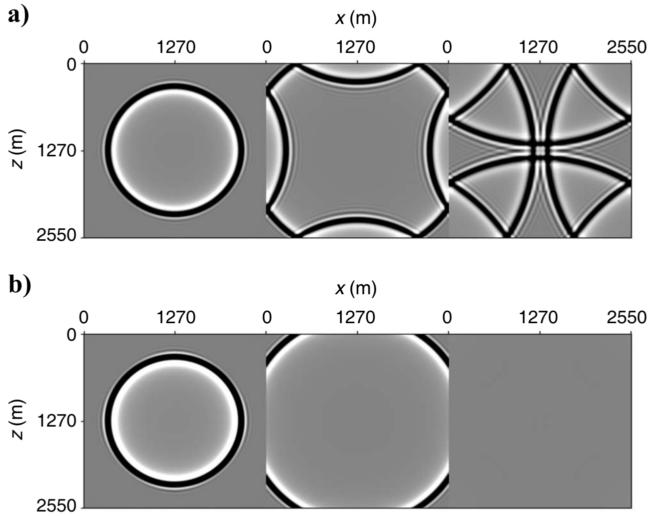

Figure 7. Snapshots computed by finite-element modeling for the 2D homogeneous acoustic model with (a) nonabsorbing and (b) new ABCs. Conventional finite-element equation, linear bases, and square grids are used in the modeling. All parameters are the same as those in Figure 2 except the source, which is defined by $p(x,z,t=0) = \exp(-0.0025((x-1280)^2+(z-200)^2))$ and $\partial p/\partial t|_{t=0}=0$.

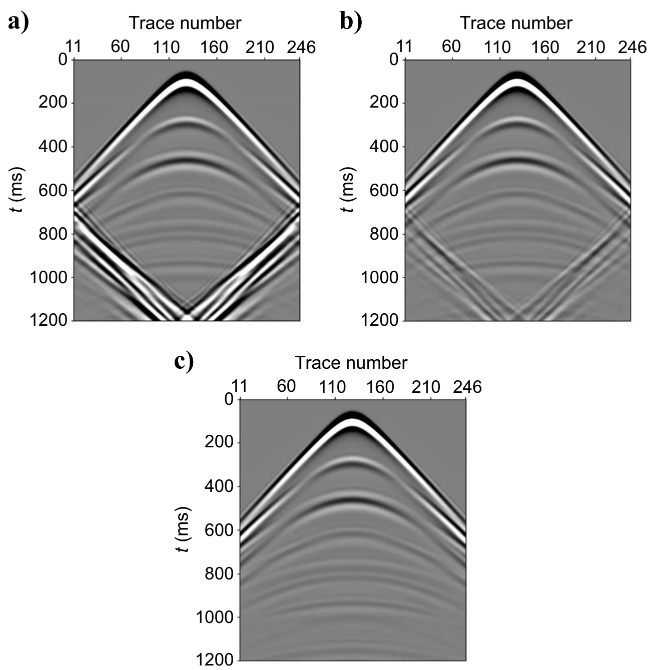

Figure 8. Seismograms computed by finite-element modeling for the 2D horizontally layered acoustic model with (a) nonabsorbing, (b) Clayton-Engquist, and (c) new ABCs. Conventional finite-element equation, linear bases, and square grids are used in the modeling. All parameters are the same as those in Figure 4 except the source, which is the same as in Figure 7.

CONCLUSIONS

We have presented a hybrid scheme for absorbing edge reflections encountered in the numerical solution of wave equations. This scheme enforces the wavefields from the inner area to the model boundaries to change smoothly via a transition area by using a weighted combination of wave and one-way wave equations; thus, it decreases edge reflections. We demonstrated the performance of this scheme with 2D acoustic-wave propagation modeling using finite-difference, pseudospectral, and finite-element methods. Unlike the pure one-way wave-equation-based scheme, we demonstrated numerically that our approach is stable in pseudospectral and finite-element modeling of the wave equation. The hybrid scheme also performs better than schemes based on the one-way wave equation and damping alone. Modeling results and our experience with the method indicate this scheme can be applied in numerically solving a variety of wave-propagation problems.

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