Algorithmic Challenges: Suffix Array

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Algorithms on Strings
Data Structures and Algorithms

Outline

1. Suffix Array
2. General Construction Strategy
3. Initialization
4. Sort Doubled Cyclic Shifts
5. Updating Classes and Full Algorithm

Construct Suffix Array

Input: String S

Output: All suffixes of S in lexicographic order

Alphabet

We assume the alphabet is ordered, that is, for any two different characters in the alphabet one of them is considered smaller than another. For example, in English

$$\text{'a'} < \text{'b'} < \text{'c'} < \cdots < \text{'z'}$$

Definition

String S is lexicographically smaller than string T if $S \neq T$ and there exist such i that:

- $0 \leq i \leq |S|$
- $S[0..i-1] = T[0..i-1]$ (assume $S[0..-1]$ is an empty string)
- Either $i = |S|$ (then S is a prefix of T) or $S[i] < T[i]$

Examples

"ab" $<$ "bc" $(i = 0)$

"abc" $<$ "abd" $(i = 2)$

"abc" $<$ "abcd" $(i = 3)$

Suffix Array Example

$S = ababaa$

Suffixes in lexicographic order:

a

aa

$abaa$

$ababaa$

baa

$babaa$

Avoiding Prefix Rule

- Inconvenient rule: if S is a prefix of T, then $S < T$
- Append special character '\$' smaller than all other characters to the end of all strings
- If S is a prefix of T, then $S\$$ differs from $T\$$ in position $i = |S|$, and $\$ < T[|S|]$, so $S\$ < T\$$

Example

$S =$ "ababaa" $\Rightarrow S' =$ "ababaa\$"

Suffixes in lexicographic order:

\$

$a\$$

$aa\$$

$abaa\$$

$ababaa\$$

$baa\$$

$babaa\$$

Example

$S =$ "ababaa" $\Rightarrow S' =$ "ababaa\$"

Suffixes in lexicographic order:

a

aa

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Storing Suffix Array

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- Storing them all is too much memory
- Store the order of suffixes $O(|S|)$
- Suffix array is this order

Example

$S = ababaa\$$

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Suffixes are numbered by their starting positions: $ababaa\$$ is 0, $abaa\$$ is 2

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Suffix array: $order = [6]$

Example

$S = ababa\textcolor{blue}{a\$}$

Suffixes are numbered by their starting positions: $ababaa\$$ is 0, $abaa\$$ is 2

Suffix array: $order = [6, \textcolor{blue}{5}]$

Example

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Suffix array: $order = [6, 5, 4, 2, 0, 3]$

Example

$S = ababaa\$$

Suffixes are numbered by their starting positions: *ababaa*\$ is 0, *abaa*\$ is 2

Suffix array: $order = [6, 5, 4, 2, 0, 3, 1]$

Example

$S = ababaa\$$

Suffixes are numbered by their starting positions: $ababaa\$$ is 0, $abaa\$$ is 2

Suffix array: $order = [6, 5, 4, 2, 0, 3, 1]$

- OK, you know how to store suffix array

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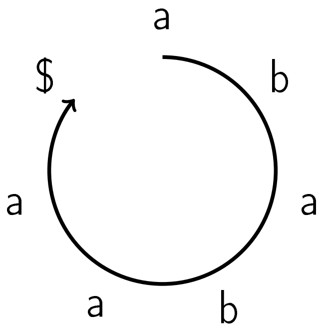
- But how to construct it?

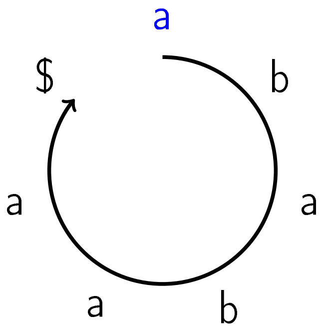
Outline

Sorting Cyclic Shifts

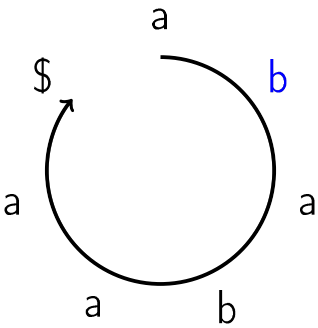


Sorting Cyclic Shifts

ababaa$



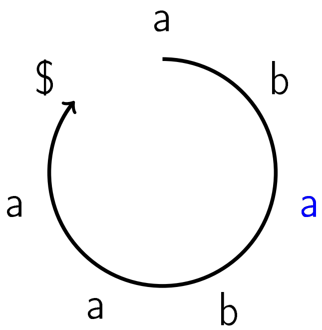
Sorting Cyclic Shifts



ababaa$

babaa$a

Sorting Cyclic Shifts

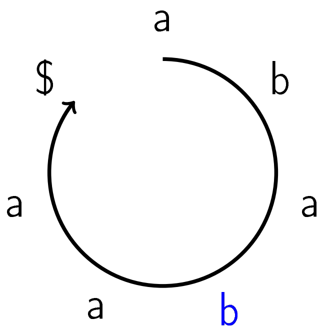


ababaa$

babaa$a

abaa$ab

Sorting Cyclic Shifts



ababaa$

babaa$a

abaa$ab

baa$aba

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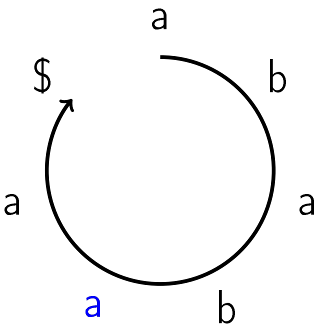

ababaa$

babaa$a

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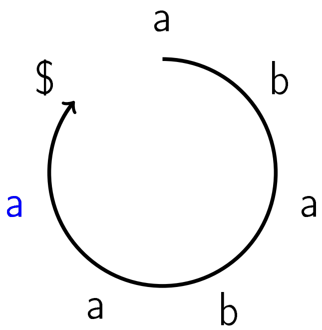
aa$abab

Sorting Cyclic Shifts



ababaa$

babaa$a

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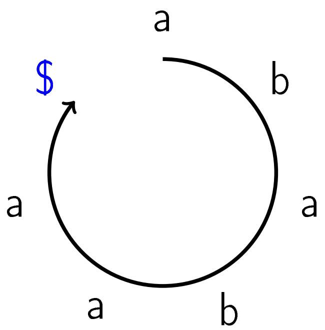
aa$abab

a$ababa

Sorting Cyclic Shifts



ababaa$

babaa$a

abaa$ab

baa$aba

aa$abab

a$ababa

$ababaa

Sorting Cyclic Shifts

ababaa\$

babaa\$a

abaa\$ab

baa\$aba

aa\$abab

a\$ababa

\$ababaa

Sorting Cyclic Shifts

ababaa\$	\$ababaa
babaa\$a	a\$ababa
abaa\$ab	aa\$abab
baa\$aba	abaa\$ab
aa\$abab	ababaa\$
a\$ababa	baa\$aba
\$ababaa	babaa\$a

Sorting Cyclic Shifts

ababaa$	$ababaa
babaa$a	a$ababa
abaa$ab	aa$abab
baa$aba	abaa$ab
aa$abab	ababaa$
a$ababa	baa$aba
$ababaa	babaa$a

Sorting Cyclic Shifts

ababaa$	$ababaa	$
babaa$a	a$ababa	a$
abaa$ab	aa$abab	aa$
baa$aba	abaa$ab	abaa$
aa$abab	ababaa$	ababaa$
a$ababa	baa$aba	baa$
$ababaa	babaa$a	babaa$

Lemma

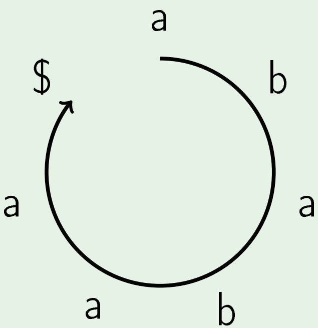
After adding to the end of string S character \$ which is smaller than all other characters, sorting cyclic shifts of S and suffixes of S is equivalent.

Partial Cyclic Shifts

Definition

Substrings of cyclic string S are called partial cyclic shifts of S

Partial Cyclic Shifts Example



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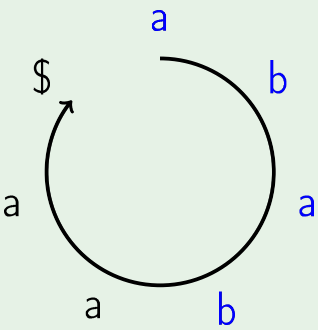
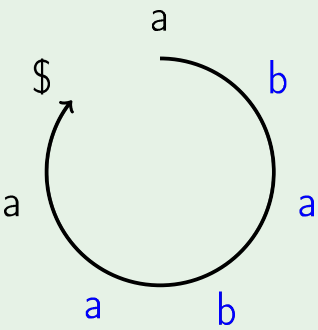


Cyclic shifts of length 4:

abab

Partial Cyclic Shifts Example



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abab

baba

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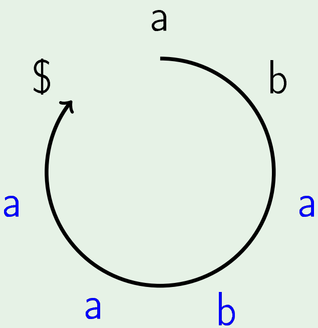


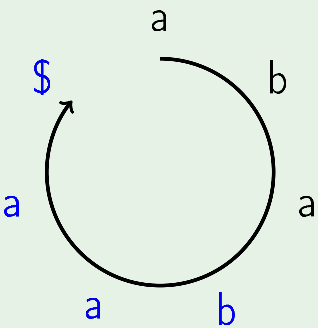
Cyclic shifts of length 4:

abab

baba

abaa

Partial Cyclic Shifts Example



Cyclic shifts of length 4:

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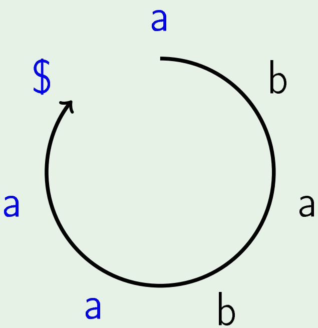
baba

abaa

baa$

Partial Cyclic Shifts Example



Cyclic shifts of length 4:

abab

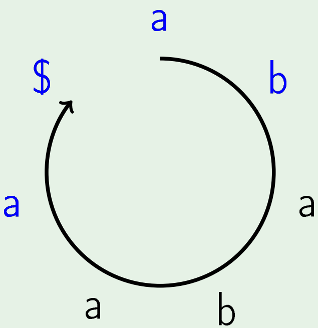
baba

abaa

baa$

aa$a

Partial Cyclic Shifts Example



Cyclic shifts of length 4:

abab

baba

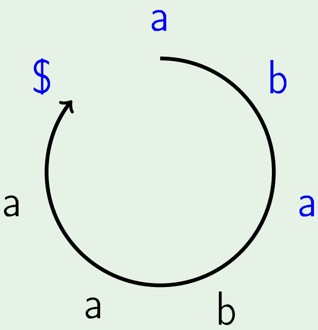
abaa

baa$

aa$a

a$ab

Partial Cyclic Shifts Example



Cyclic shifts of length 4:

abab

baba

abaa

baa$

aa$a

a$ab

$aba

General strategy

- Start with sorting single characters of S

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- Cyclic shifts of length $L = 1$ sorted

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- While $L < |S|$, sort shifts of length $2L$

General strategy

- Start with sorting single characters of S
- Cyclic shifts of length $L = 1$ sorted
- While $L < |S|$, sort shifts of length $2L$
- If $L \geq |S|$, cyclic shifts of length L sort the same way as cyclic shifts of length $|S|$

Example

$S = ababaa\$$

Example

$S = ababaa\$$

6	$\$$
0	a
2	a
4	a
5	a
1	b
3	b

Example

$S = ababaa\$$

6	$\$$	$order = [6, 0, 2, 4, 5, 1, 3]$
0	a	
2	a	
4	a	
5	a	
1	b	
3	b	

Example

$S = ababaa\$$

6	$\$a$
5	$a\$$
4	aa
0	ab
2	ab
1	ba
3	ba

Example

$S = ababaa\$$

6	$\$a$	$order = [6, 5, 4, 0, 2, 1, 3]$
5	$a\$$	
4	aa	
0	ab	
2	ab	
1	ba	
3	ba	

Example

$S = ababaa\$$

6 $\$aba$

5 $a\$ab$

4 $aa\$a$

2 $abaa$

0 $abab$

3 $baa\$$

1 $baba$

Example

$S = ababaa\$$

$6\ \$aba$

$5\ a\$ab$

$4\ aa\$a$

$2\ abaa$

$0\ abab$

$3\ baa\$$

$1\ baba$

$order = [6, 5, 4, 2, 0, 3, 1]$

Example

$S = ababaa\$$

6 $\$ababaa\$$

5 $a\$ababaa$

4 $aa\$ababa$

2 $abaa\$aba$

0 $ababaa\$a$

3 $baa\$abab$

1 $babaa\$ab$

Example

$S = ababaa\$$

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5 $a\$ababaa$

4 $aa\$ababa$

2 $abaa\$aba$

0 $ababaa\$a$

3 $baa\$abab$

1 $babaa\$ab$

Example

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1 $babaa\$ab$

Example

$S = ababaa\$$

6 $\$$ $\qquad order = [6, 5, 4, 2, 0, 3, 1]$

5 $a\$$

4 $aa\$$

2 $abaa\$$

0 $ababaa\$$

3 $baa\$$

1 $babaa\$$

Outline

1. Suffix Array
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5. Updating Classes and Full Algorithm

Sorting single characters

- Alphabet Σ has $|\Sigma|$ different characters

Sorting single characters

- Alphabet Σ has $|\Sigma|$ different characters
- Use counting sort to compute *order* of characters

```
SortCharacters(S)
order ← array of size |S|
count ← zero array of size |Σ|
for i from 0 to |S| − 1:
    count[S[i]] ← count[S[i]] + 1
for j from 1 to |Σ| − 1:
    count[j] ← count[j] + count[j − 1]
for i from |S| − 1 down to 0:
    c ← S[i]
    count[c] ← count[c] − 1
    order[count[c]] ← i
return order
```

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    c ← S[i]
    count[c] ← count[c] − 1
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return order
```

Lemma

Running time of `SortCharacters` is $O(|S| + |\Sigma|)$.

Proof

We know this is the running time of the counting sort for $|S|$ items that can take $|\Sigma|$ different values. $\square$

Equivalence classes

- C_i — partial cyclic shift of length L starting in i
- C_i can be equal to C_j — then they are in one equivalence class
- Compute $class[i]$ — number of *different* cyclic shifts of length L that are strictly smaller than C_i
- $C_i == C_j \Leftrightarrow class[i] == class[j]$

Example

$S = ababaa\$$

6	$\$$	$order = [6, 0, 2, 4, 5, 1, 3]$
0	a	$class = [\ ,\ ,\ ,\ ,\ ,\ ,\ ]$
2	a	
4	a	
5	a	
1	b	
3	b	

Example

$S = ababaa\$$

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1	b	
3	b	

Example

$S = ababaa\$$

6	$\$$
0	a
2	a
4	a
5	a
1	b
3	b

$order = [6, 0, 2, 4, 5, 1, 3]$

$class = [1, \ , \ , \ , \ , \ , 0]$

Example

$S = ababaa\$$

6	\$	$order = [6, 0, 2, 4, 5, 1, 3]$
0	a	$class = [1, \ , \ , \ , \ , \ , 0]$
2	a	
4	a	
5	a	
1	b	
3	b	

Example

$S = ababaa\$$

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2	a	
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5	a	
1	b	
3	b	

Example

$S = ababaa\$$

$6 \quad \$$

$0 \quad a$

$2 \quad a$

$4 \quad a$

$5 \quad a$

$1 \quad b$

$3 \quad b$

$order = [6, 0, 2, 4, 5, 1, 3]$

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Example

$S = ababaa\$$

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Example

$S = ababaa\$$

6	$
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2	a
4	a
5	a
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3	b

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Example

$S = ababaa\$$

$6 \quad \$$

$0 \quad a$

$2 \quad a$

$4 \quad a$

$5 \quad a$

$1 \quad b$

$3 \quad b$

$order = [6, 0, 2, 4, 5, 1, 3]$

$class = [1, 2, 1, 2, 1, 1, 0]$

Example

$S = ababaa\$$

6	$\$$
0	a
2	a
4	a
5	a
1	b
3	b

$$order = [6, 0, 2, 4, 5, 1, 3]$$

$$class = [1, 2, 1, 2, 1, 1, 0]$$

```
ComputeCharClasses(S, order)

class ← array of size |S|
class[order[0]] ← 0
for i from 1 to |S| − 1:
    if S[order[i]] ≠ S[order[i − 1]]:
        class[order[i]] = class[order[i − 1]] + 1
    else:
        class[order[i]] = class[order[i − 1]]
return class
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for i from 1 to |S| − 1:
    if S[order[i]] ≠ S[order[i − 1]]:
        class[order[i]] = class[order[i − 1]] + 1
    else:
        class[order[i]] = class[order[i − 1]]
return class
```

```
ComputeCharClasses(S, order)

class ← array of size |S|
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for i from 1 to |S| − 1:
    if S[order[i]] ≠ S[order[i − 1]]:
        class[order[i]] = class[order[i − 1]] + 1
    else:
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return class
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return class
```

Lemma

The running time of `ComputeCharClasses` is $O(|S|)$.

Proof

One for loop with $O(|S|)$ iterations. $\square$

Outline

1. Suffix Array
2. General Construction Strategy
3. Initialization
4. **Sort Doubled Cyclic Shifts**
5. Updating Classes and Full Algorithm

Idea

- C_i — cyclic shift of length L starting in i

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- C_i — cyclic shift of length L starting in i
- C'_i — doubled cyclic shift starting in i
- $C'_i = C_i C_{i+L}$ — concatenation of strings
- To compare C'_i with C'_j, it's sufficient to compare C_i with C_j and C_{i+L} with C_{j+L}

Example

$S = ababaa\$$

$L = 2$

$i = 2$

$C_i = C_2 = ab$

$C_{i+L} = C_{2+2} = C_4 = aa$

$C'_i = C'_2 = abaa = C_2 C_4$

Sorting pairs

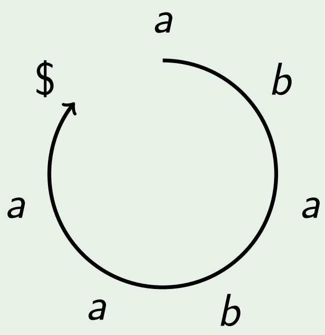
- First sort by second element of pair

Sorting pairs

- First sort by second element of pair
- Then **stable** sort by first element of pair

Example

$L = 2$



$C_6 = \$a$

$C_5 = a\$$

$C_4 = aa$

$C_0 = ab$

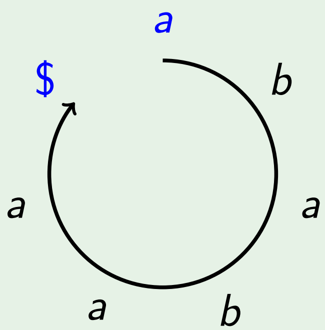
$C_2 = ab$

$C_3 = ba$

$C_1 = ba$

Example

$L = 2$



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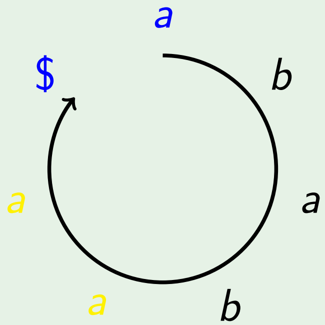


$C_4' = aa\$a$

$C_5 = a\$$

$C_4 = aa$

$C_0 = ab$

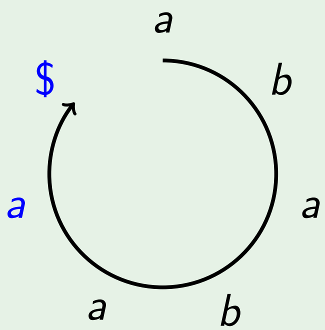
$C_2 = ab$

$C_3 = ba$

$C_1 = ba$

Example

$L = 2$



$C_4' = aa\$a$

$C_5 = a\$$

$C_4 = aa$

$C_0 = ab$

$C_2 = ab$

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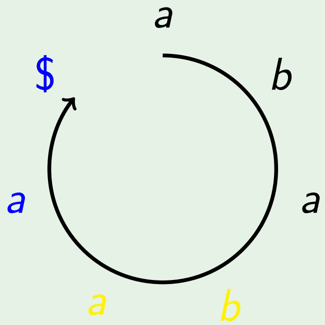
$C_1 = ba$

Example

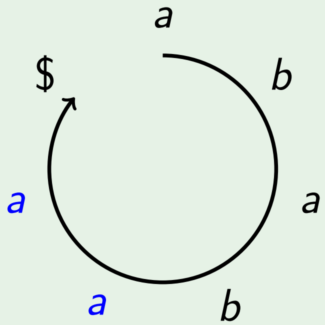
$L = 2$



$$C'_4 = aa\$a$$
$$C'_3 = baa\$$$
$$C_4 = aa$$
$$C_0 = ab$$
$$C_2 = ab$$
$$C_3 = ba$$
$$C_1 = ba$$

Example

$L = 2$



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$C_4 = aa$

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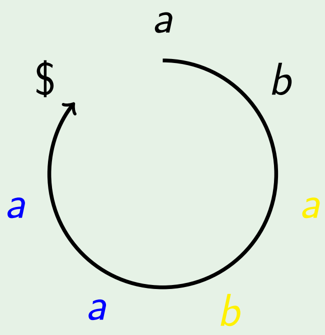

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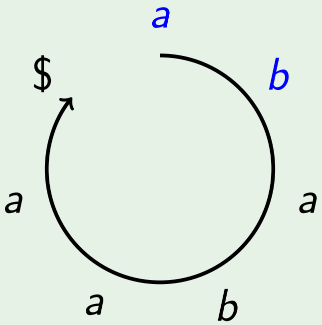

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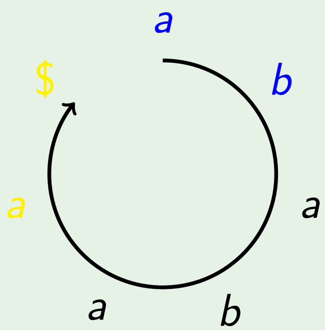


$C'_4 = aa\$a$

$C'_3 = baa\$$

$C'_2 = abaa$

$C'_5 = a\$ab$

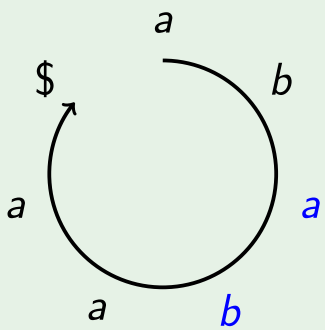
$C_2 = ab$

$C_3 = ba$

$C_1 = ba$

Example

$L = 2$



$C'_4 = aa\$a$

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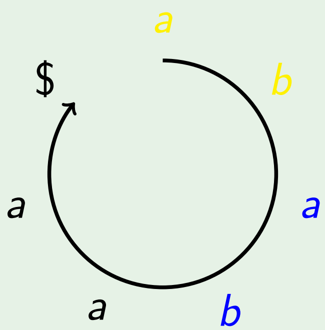

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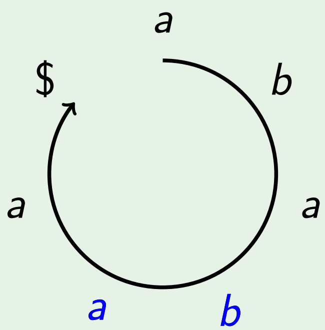

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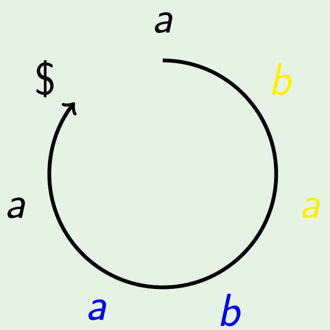

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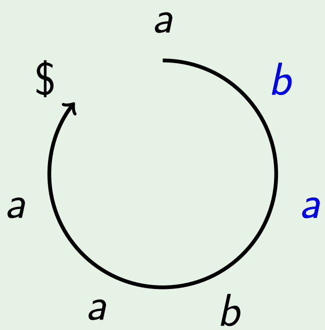

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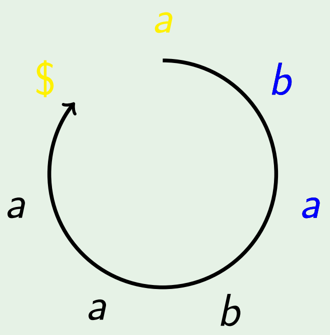

$C'_4 = aa\$a$

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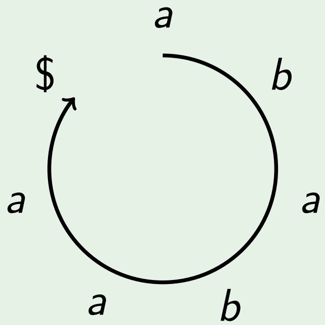

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$C'_1 = baba$

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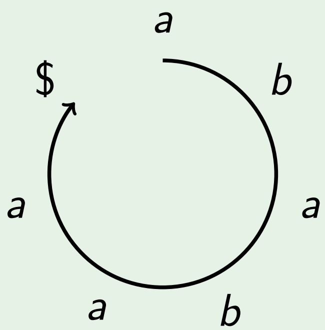

$C'_4 = aa\$a$

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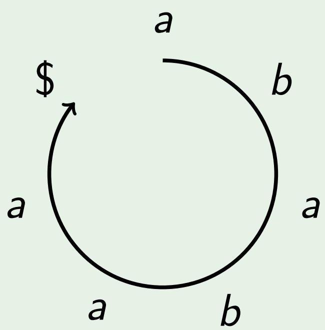

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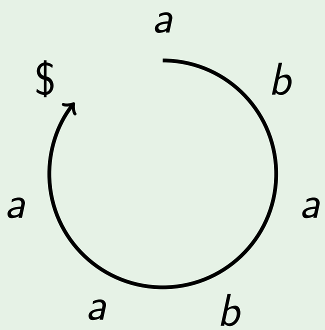

$C'_6 = \$aba$

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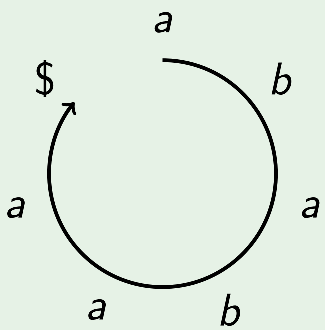

$C_6' = \$aba$

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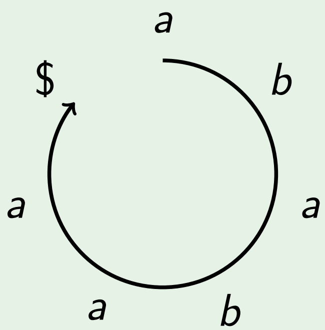

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Sorting doubled cyclic shifts

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Sorting doubled cyclic shifts

- $C'_{order[0]-L}, C'_{order[1]-L}, \cdots, C_{order[|S|-1]-L}$ are sorted by second element of pair
- Need a stable sort by first elements of pairs
- Counting sort is stable!
- We know equivalence classes of single shifts for counting sort

SortDoubled(S, L, $order$, $class$)

```
count ← zero array of size |S|
newOrder ← array of size |S|
for i from 0 to |S| − 1:
   count[class[i]] ← count[class[i]] + 1
for j from 1 to |S| − 1:
   count[j] ← count[j] + count[j − 1]
for i from |S| − 1 down to 0:
   start ← (order[i] − L + |S|) mod |S|
   cl ← class[start]
   count[cl] ← count[cl] − 1
   newOrder[count[cl]] ← start
return newOrder
```

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SortDoubled($S, L, order, class$)

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    count[cl] ← count[cl] − 1
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return newOrder
```

SortDoubled($S, L, order, class$)

$count \leftarrow$ `zero array of size` $|S|$
$newOrder \leftarrow$ `array of size` $|S|$
`for` i `from 0 to` $|S| - 1$:
  $count[class[i]] \leftarrow count[class[i]] + 1$
`for` j `from 1 to` $|S| - 1$:
  $count[j] \leftarrow count[j] + count[j - 1]$
`for` i `from` $|S| - 1$ `down to 0`:
  $start \leftarrow (order[i] - L + |S|) \mod |S|$
  $cl \leftarrow class[start]$
  $count[cl] \leftarrow count[cl] - 1$
  $newOrder[count[cl]] \leftarrow start$
`return` $newOrder$

SortDoubled(S, L, $order$, $class$)

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   newOrder[count[cl]] ← start
return newOrder
```

Lemma

The running time of `SortDoubled` is $O(|S|)$.

Proof

Three for loops with $O(|S|)$ iterations each. $\square$

Outline

Updating classes

- Pairs are sorted — go through them in order, if a pair is different from previous, put it into a new class, otherwise put it into previous class
- $$(P_1, P_2) == (Q_1, Q_2) \Leftrightarrow$$
  $$(P_1 == Q_1) \text{ and } (P_2 == Q_2)$$
- We know equivalence classes of elements of pairs

Example

$S = ababaa\$$

C'_6 $\$a$ $(0, 1)$ $\leftarrow$ $class = [1, 2, 1, 2, 1, 1, 0]$

C'_5 $a\$$ $(1, 0)$ $newOrder = [6, 5, 4, 0, 2, 1, 3]$

C'_4 aa $(1, 1)$ $newClass = [\ ,\ ,\ ,\ ,\ ,\ ,\ ]$

C'_0 ab $(1, 2)$

C'_2 ab $(1, 2)$

C'_1 ba $(2, 1)$

C'_3 ba $(2, 1)$

Example

$$S = ababaa\$$$

C'_6 $\$a$ $(0,1) \leftarrow class = [1,2,1,2,1,1,0]$

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C'_0 ab $(1,2)$

C'_2 ab $(1,2)$

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C_4' aa $(1, 1)$ $newClass = [\ ,\ ,\ ,\ ,\ , 1, 0]$

C_0' ab $(1, 2)$

C_2' ab $(1, 2)$

C_1' ba $(2, 1)$

C_3' ba $(2, 1)$

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C'_5 $a\$$ $(1, 0)$ $newOrder = [6, 5, 4, 0, 2, 1, 3]$

C'_4 aa $(1, 1)$ $newClass = [\ ,\ ,\ ,\ ,\ , 1, 0]$

C'_0 ab $(1, 2)$

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C'_4 aa $(1,1)$ $newClass = [\ ,\ ,\ ,\ ,2,1,0]$

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C'_2 ab $(1,2)$

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$S = ababaa\$$

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UpdateClasses(*newOrder*, *class*, *L*)

```
n ← |newOrder|
newClass ← array of size n
newClass[newOrder[0]] ← 0
for i from 1 to n − 1:
   cur ← newOrder[i], prev ← newOrder[i − 1]
   mid ← (cur + L), midPrev ← (prev + L) (mod n)
   if class[cur] ≠ class[prev] or
        class[mid] ≠ class[midPrev]:
      newClass[cur] ← newClass[prev] + 1
   else:
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```
UpdateClasses(newOrder, class, L)
```

$n \leftarrow |newOrder|$
$newClass \leftarrow$ array of size n
$newClass[newOrder[0]] \leftarrow 0$
for i from 1 to $n-1$:
  $cur \leftarrow newOrder[i], prev \leftarrow newOrder[i-1]$
  $mid \leftarrow (cur + L), midPrev \leftarrow (prev + L) \pmod{n}$
  if $class[cur] \neq class[prev]$ or
    $class[mid] \neq class[midPrev]$:
   $newClass[cur] \leftarrow newClass[prev] + 1$
  else:
   $newClass[cur] \leftarrow newClass[prev]$
return $newClass$

UpdateClasses($newOrder$, $class$, L)

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Lemma

The running time of `UpdateClasses` is $O(|S|)$.

Proof

One for loop with $O(|S|)$ iterations. $\square$

```
BuildSuffixArray(S)
order ← SortCharacters(S)
class ← ComputeCharClasses(S, order)
L ← 1
while L < |S|:
    order ← SortDoubled(S, L, order, class)
    class ← UpdateClasses(order, class, L)
    L ← 2L
return order
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Lemma

The running time of `BuildSuffixArray` is $O(|S| \log |S| + |\Sigma|)$.

Proof

- Initialization: `SortCharacters` in $O(|S| + |\Sigma|)$ and `ComputeCharClasses` in $O(|S|)$

Lemma

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- Initialization: `SortCharacters` in $O(|S| + |\Sigma|)$ and `ComputeCharClasses` in $O(|S|)$
- While loop iteration: `SortDoubled` and `UpdateClasses` run in $O(|S|)$

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Proof

- Initialization: `SortCharacters` in $O(|S| + |\Sigma|)$ and `ComputeCharClasses` in $O(|S|)$
- While loop iteration: `SortDoubled` and `UpdateClasses` run in $O(|S|)$
- $O(\log |S|)$ iterations while $L < |S|$ $\square$

Conclusion

- Can build suffix array of a string S in $O(|S| \log |S|)$ using $O(|S|)$ memory
- Can also sort all cyclic shifts of a string S in $O(|S| \log |S|)$
- Suffix array enables many fast operations with the string
- Next lesson you will learn to construct suffix tree from suffix array in $O(|S|)$ time, so you will be able to build suffix tree in total $O(|S| \log |S|)$ time!