Quantum Computing and Cryptography

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Outline

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1. Intro
2. RSA
3. Quantum computing
   - Introduction to the quantum world
   - Quantum algorithms
4. Post-Quantum cryptography
   - Intro to PQ cryptography
   - Lattice cryptography
     - What is a lattice ?
   - Limits of PQ cryptography
5. Conclusion

Introduction

What is Cryptography?

- Science of secret $\kappa\rho\upsilon\pi\tau o\varsigma$
- Two complementary parts: cryptography and cryptanalysis

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Nowadays

- Cryptography: creating protocols to protect a communication
- Cryptanalysis: Measuring the security level of those protocols

Outline

RSA

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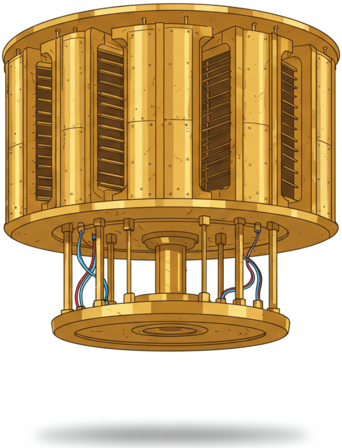
Quantum computing



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Qubits

Classical bit

$b \in \{0, 1\}$

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- $|-\rangle \rightarrow 0$ (50%), 1 (50%)

NOT gate

X gate

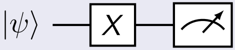
- $X\left|0\right\rangle \rightarrow \left|1\right\rangle$
- $X\left|1\right\rangle \rightarrow \left|0\right\rangle$

NOT gate

X gate

- $X\,|0\rangle \rightarrow |1\rangle$
- $X\,|1\rangle \rightarrow |0\rangle$

Circuit representation



Hadamard gate

H gate

- $H |0\rangle \rightarrow |+\rangle$
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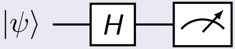

Outline

Bernstein-Vazirani Problem

Problem Definition

Given an oracle for a function f:

$$f : \{0,1\}^n \to \{0,1\}$$

$$f(\mathrm{x}) = \mathrm{x} \cdot \mathrm{s}$$

where s is a secret bit string. Find s with the fewest oracle calls. ($\cdot$ is the bitwise dot product, XOR sum).

Classical Algorithm - Example

Example (n=2)

To find s $= s_0 s_1$:

Requires 2 queries.

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- Query $f(10) = 1 \cdot s_0 + 0 \cdot s_1 = s_0$

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Example (n=2)

To find s $= s_0 s_1$:

- Query $f(10) = 1 \cdot s_0 + 0 \cdot s_1 = s_0$
- Query $f(01) = 0 \cdot s_0 + 1 \cdot s_1 = s_1$

Requires 2 queries.

Classical Algorithm - General Case

Classical complexity: $\mathcal{O}(n)$

We need to isolate each bit of s by querying with inputs that have a single '1'. This requires n queries for an n-bit string.

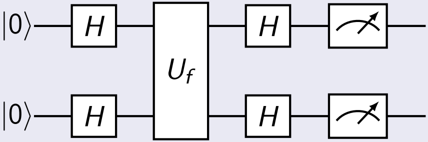
Quantum Algorithm - Overview

Quantum complexity: $\mathcal{O}(1)$

The quantum algorithm can find s with just **one** query. It uses superposition to query all possible inputs simultaneously.

Quantum Algorithm - Circuit

Circuit Diagram



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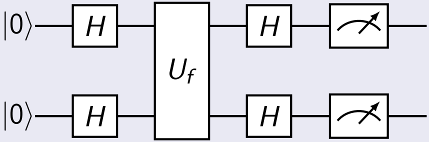

Explanation

Quantum Algorithm - Circuit

Circuit Diagram

$$
\begin{array}{l}
|0\rangle - \boxed{H} - \boxed{} - \boxed{H} - \boxed{\nearrow} - \\
\qquad\qquad\qquad\;\; U_f \\
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- H : Hadamard gates on all n input qubits (creates superposition).

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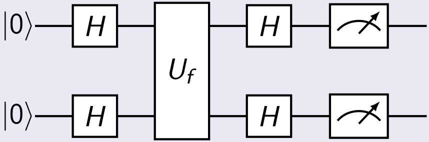

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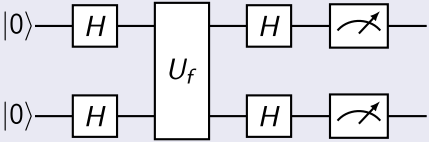

Explanation

- H : Hadamard gates on all n input qubits (creates superposition).
- U_f : The quantum oracle.
- Final Hadamards and measurement reveal s.

Shor's Algorithm

Complexity Gain

Classical factoring is very slow (roughly $\mathcal{O}(e^{\sqrt[3]{n}})$). Shor's algorithm is much faster (polynomial, $\mathcal{O}(n^3)$).

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- Requires a large number of high-quality (low-error) qubits (roughly $2n$ for an n-bit number).
- We currently don't have quantum computers large and stable enough to break practical RSA encryption.

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What it is not:

Cryptography **using** quantum technologies

- Many cases where it is unusable
- Considered unreliable

The problems

- Codes
- Hash functions
- Multivariates polynomials systems
- Isogenies
- Lattices

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Why lattices ?

- Well spread
- Good results

Encryption/Key encapsulation	
Crystals-Kyber	Lattices
Signatures	
Crystals-Dilithium	Lattices
Falcon	Lattices
Sphincs+	Hash

Table: Results from the NIST

Outline

Some definitions

The unformal definition

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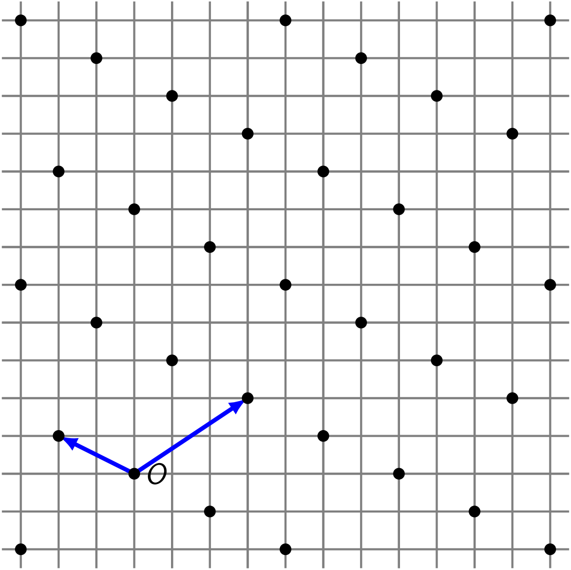
$\rightarrow$ We have vectors, dot/scalar product and matrices

Example



Learning with error problem

TODO

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- Used to manipulate private data (e.g. Medical data, data science)

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Sizes of the keys and data

TODO	

Not necessarly robust to classical computer

- Example : Supersingular isogenies Diffie-Hellman key exchange

Outline

Conclusion