Quantum Computing and Cryptography

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Outline

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Introduction

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- Science of secret ($\kappa\rho\nu\pi\tau o\varsigma$)
- Two complementary parts: cryptography and cryptanalysis

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Historically

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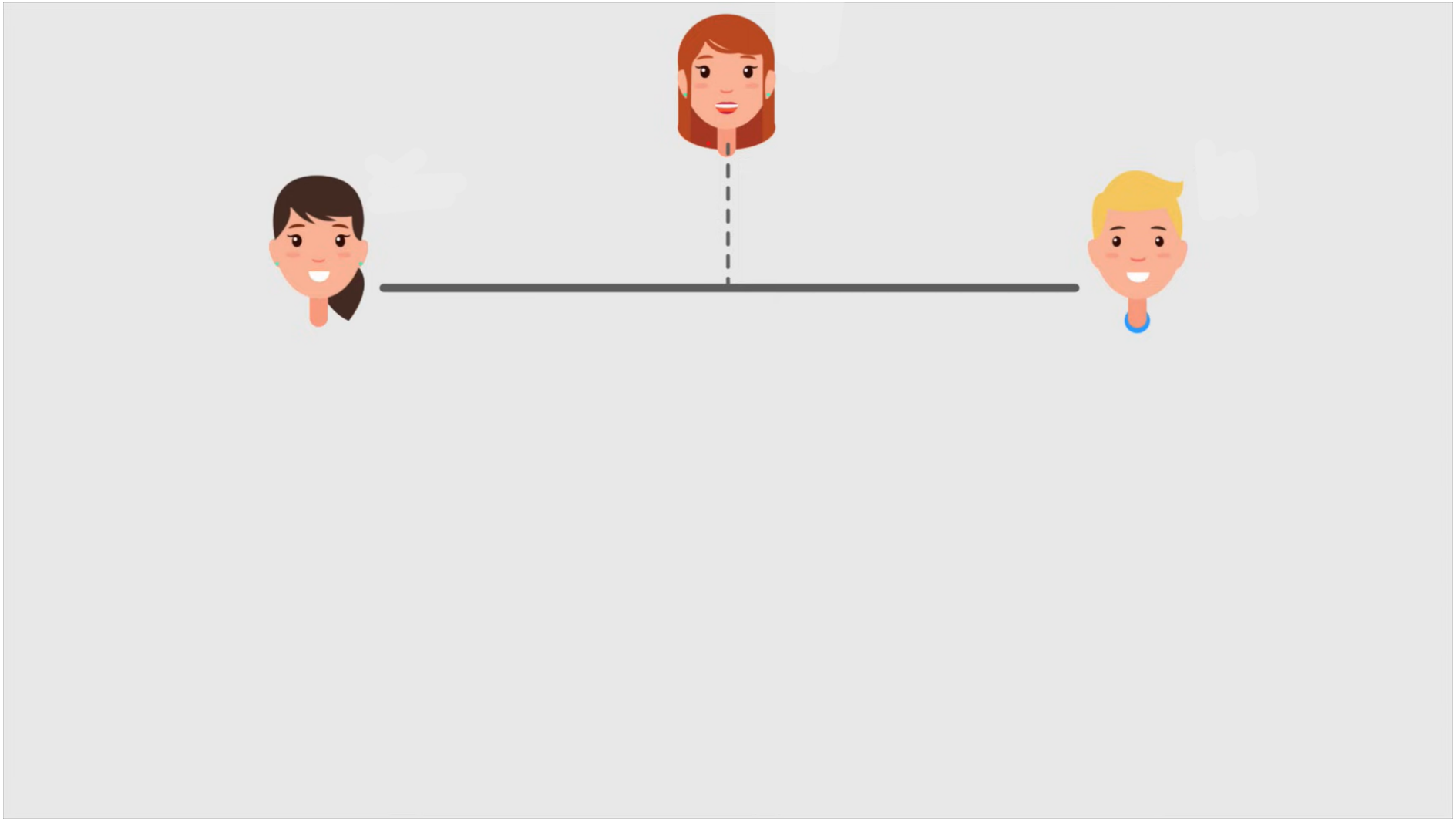
Nowadays

- Cryptography: Create protocols to protect a communication
- Cryptanalysis: Measure the security level of those protocols

Outline

RSA



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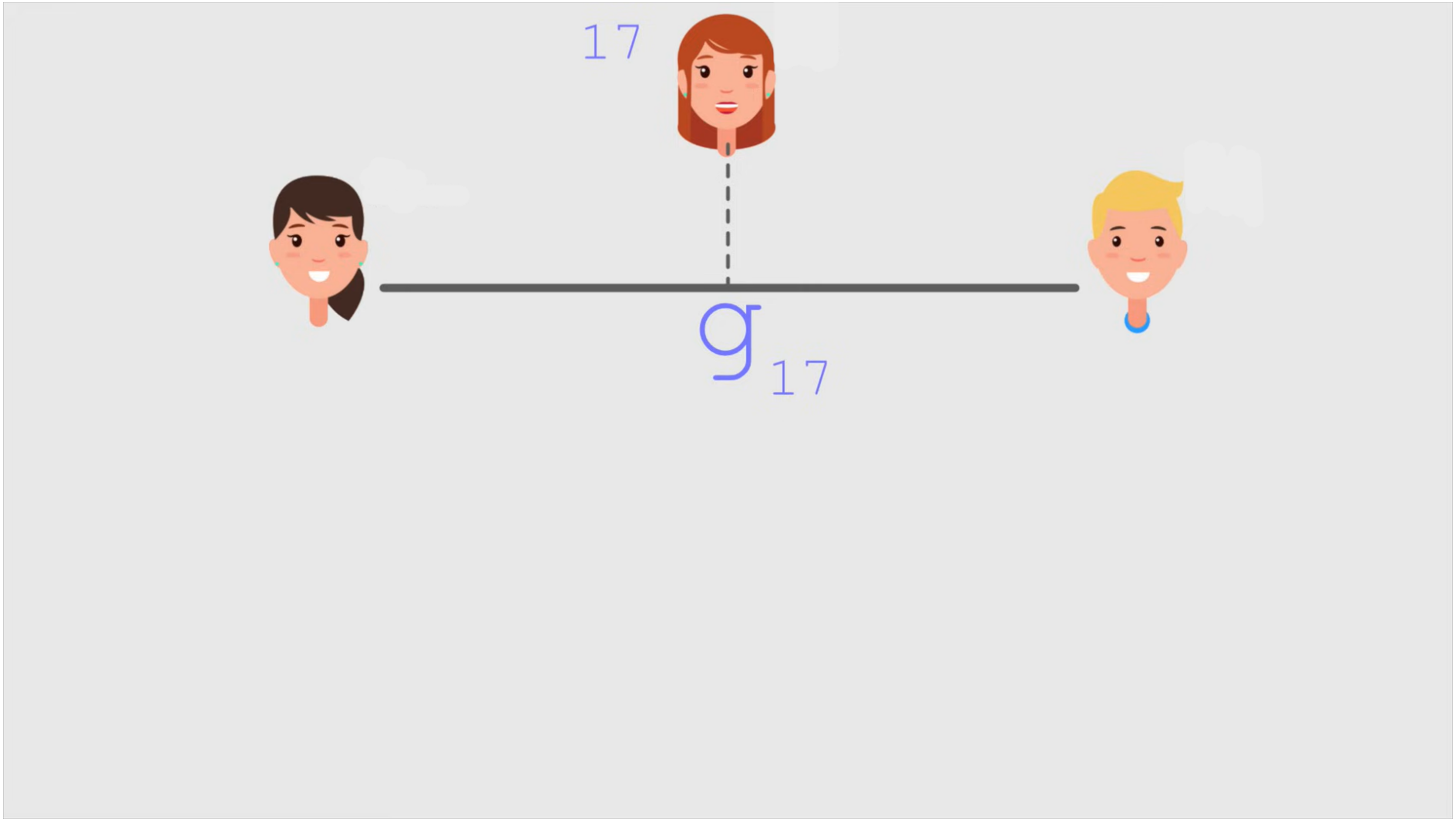

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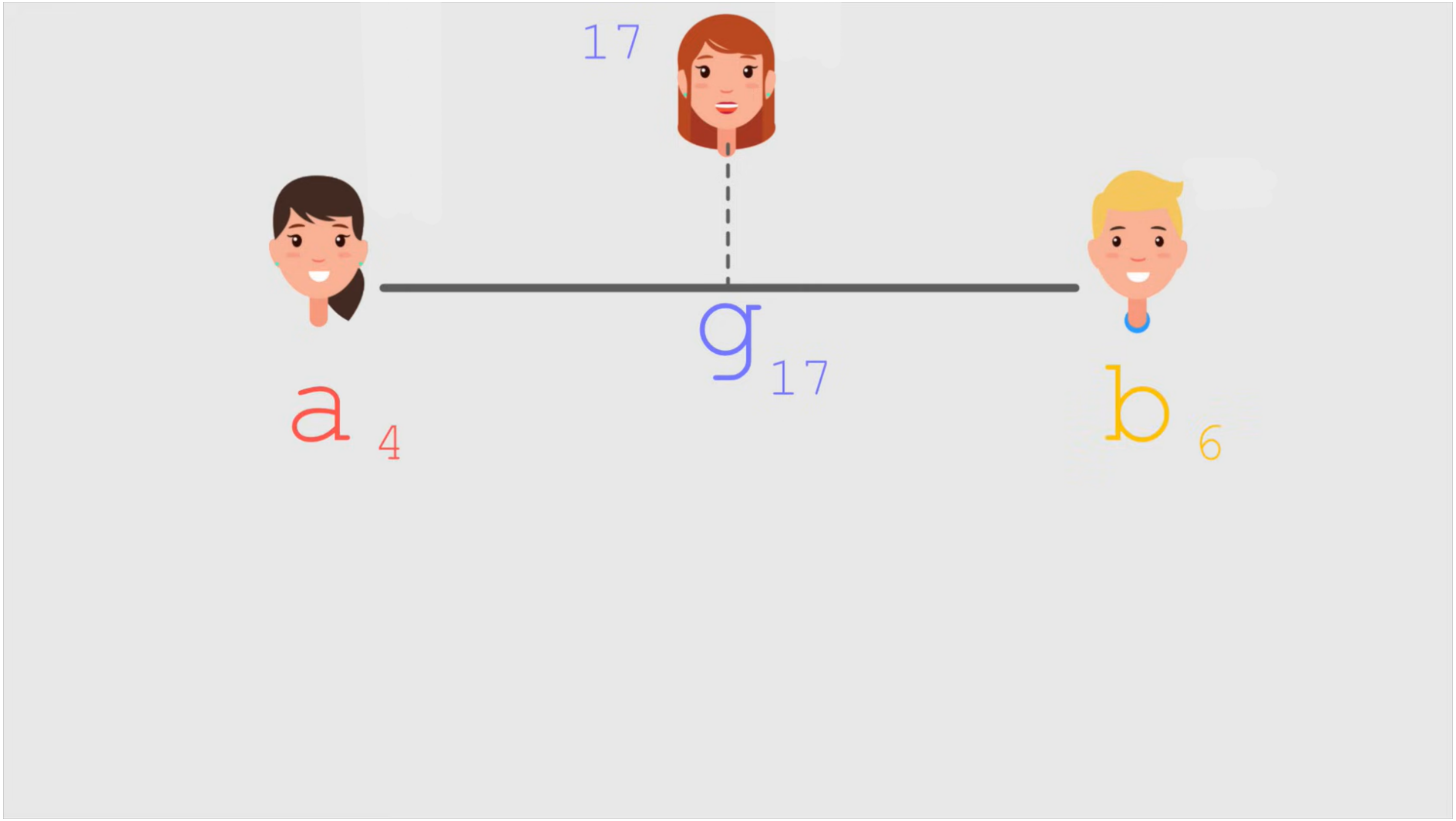

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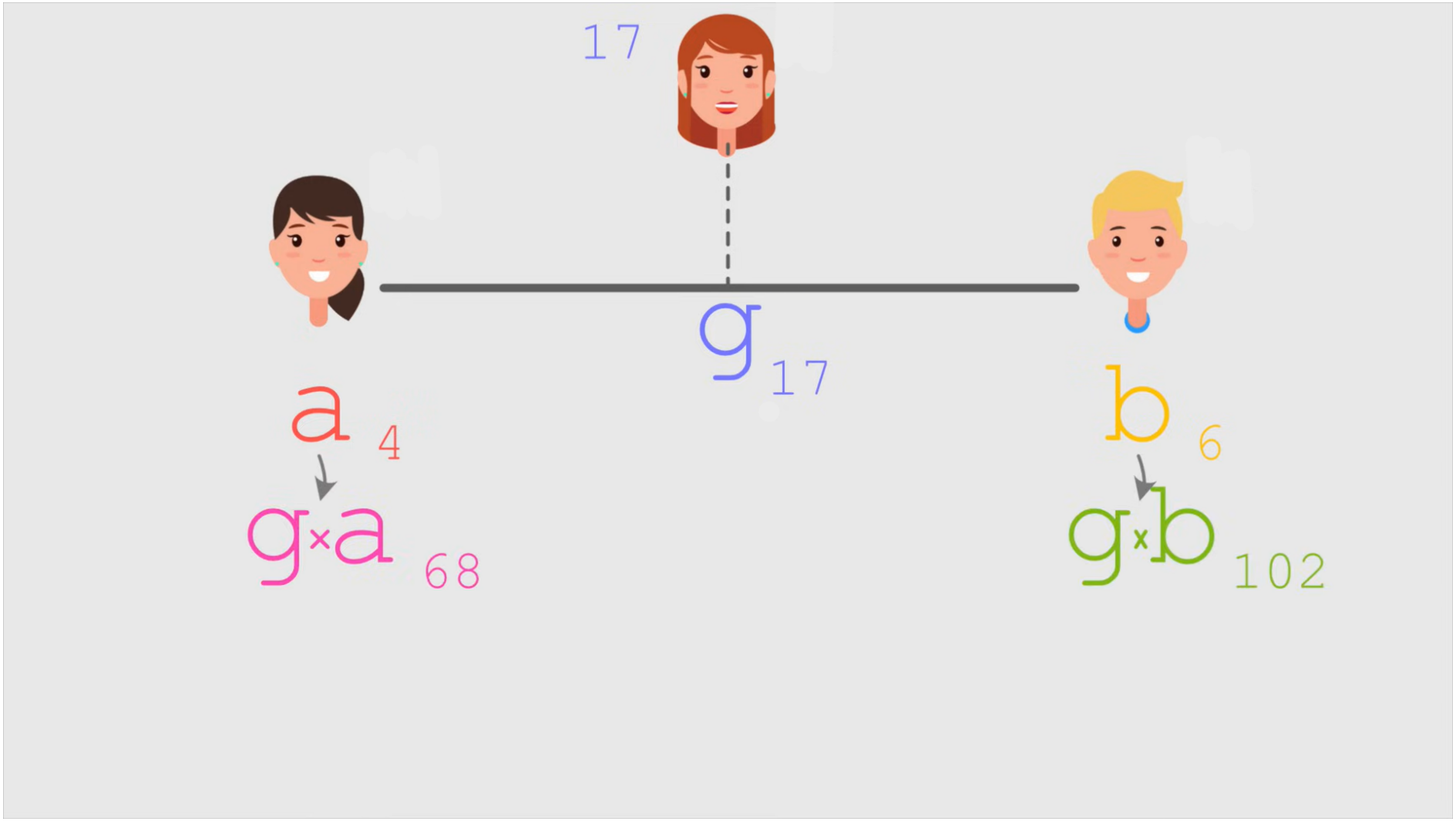

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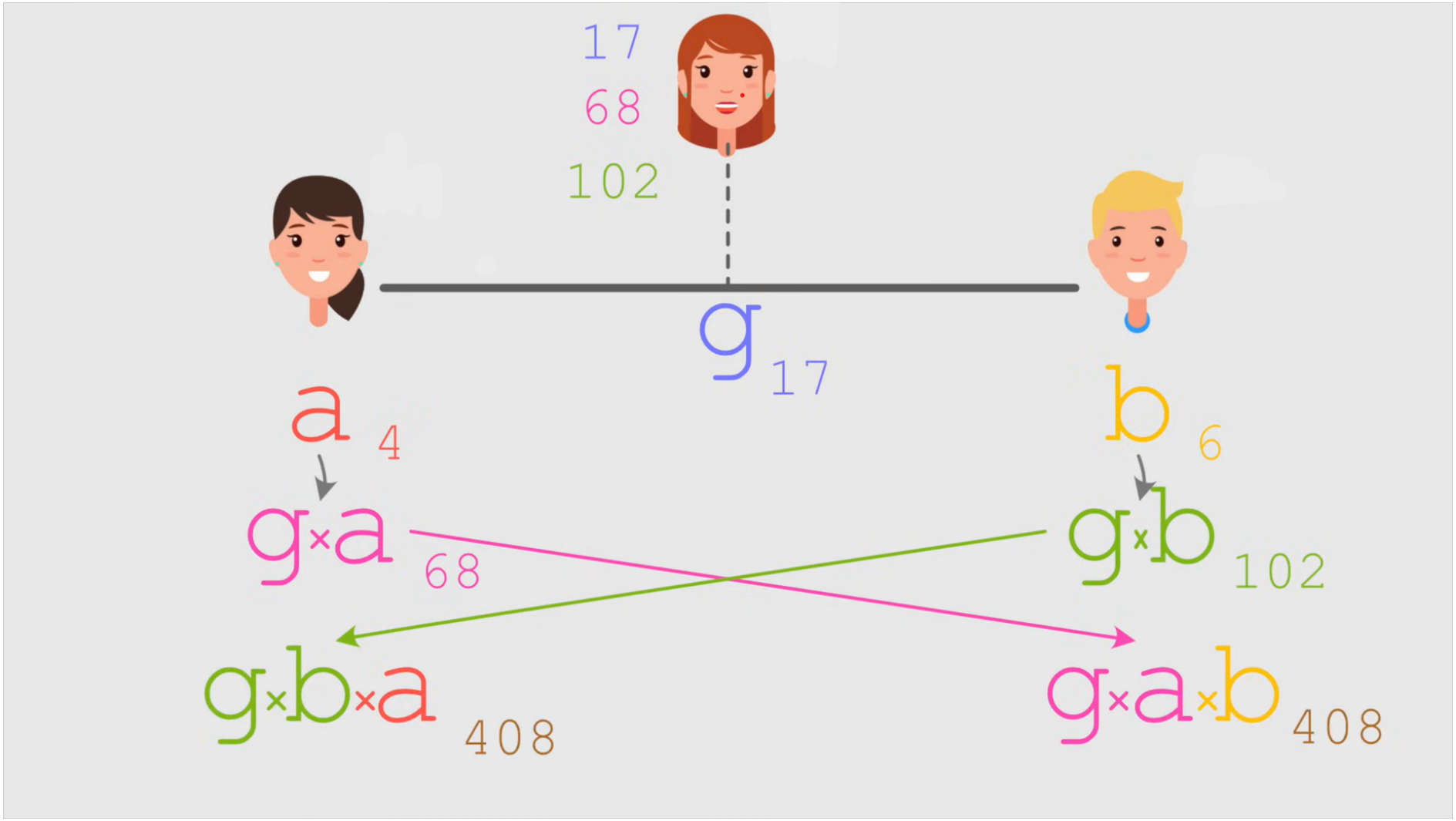

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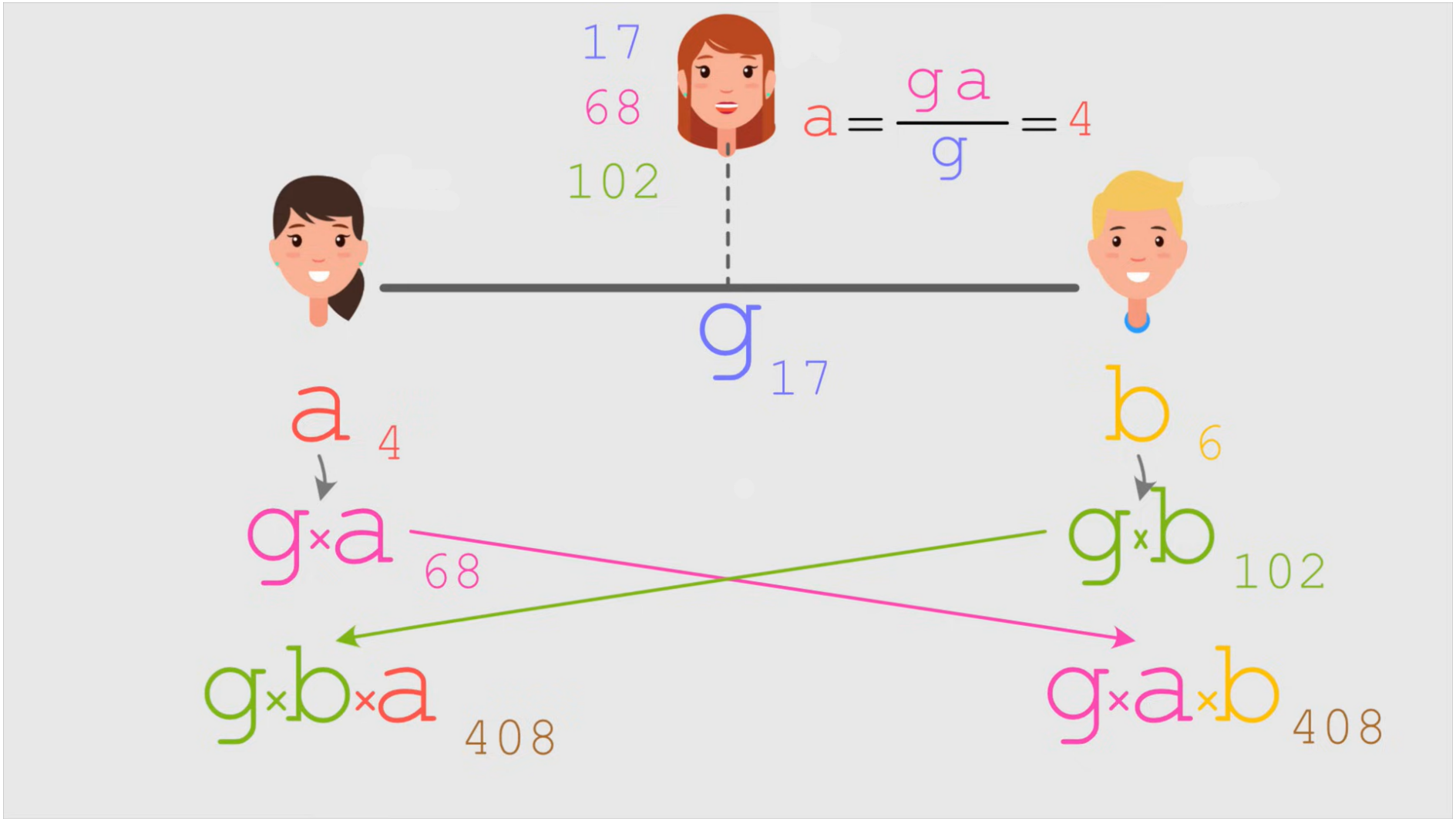

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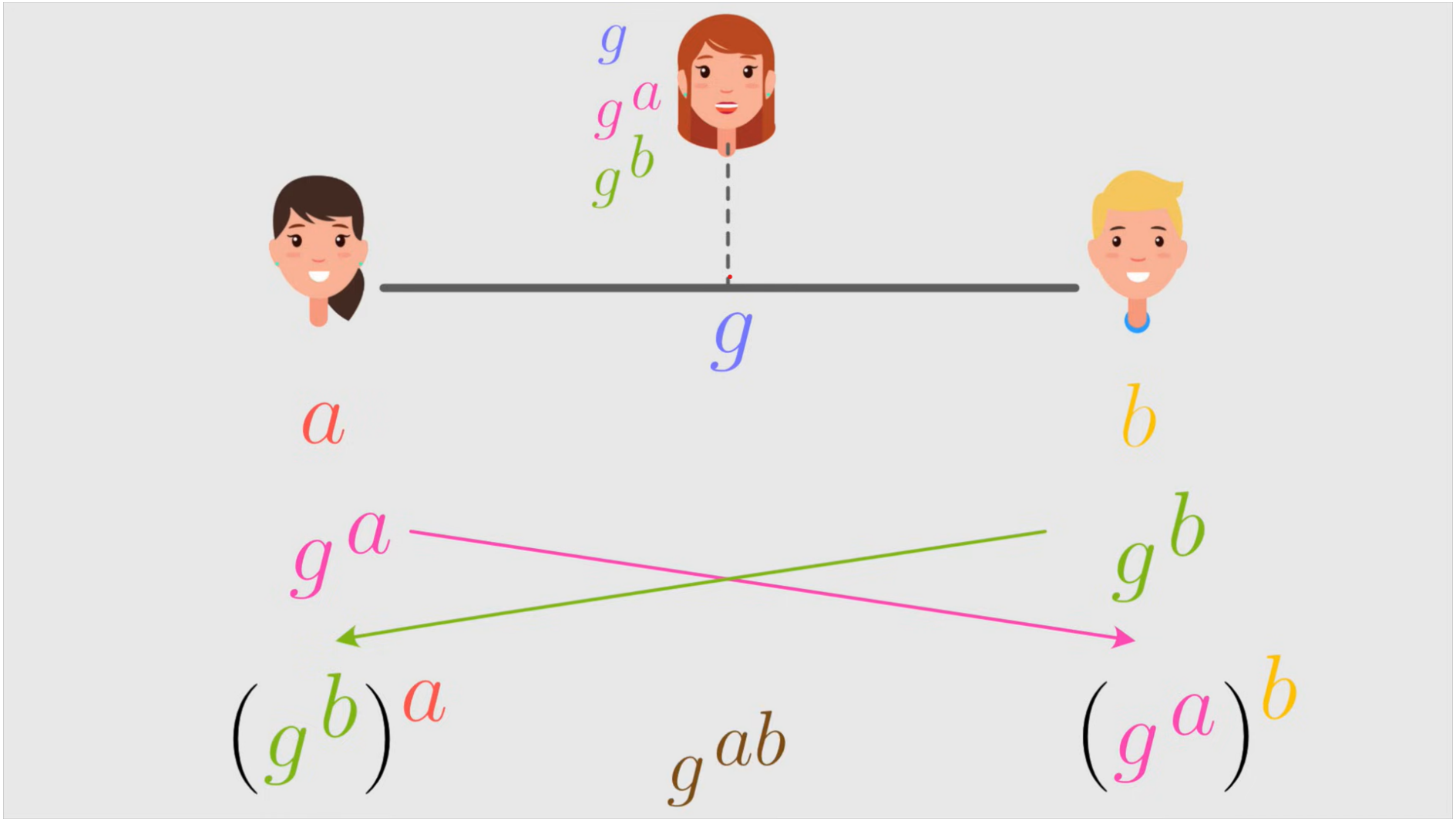

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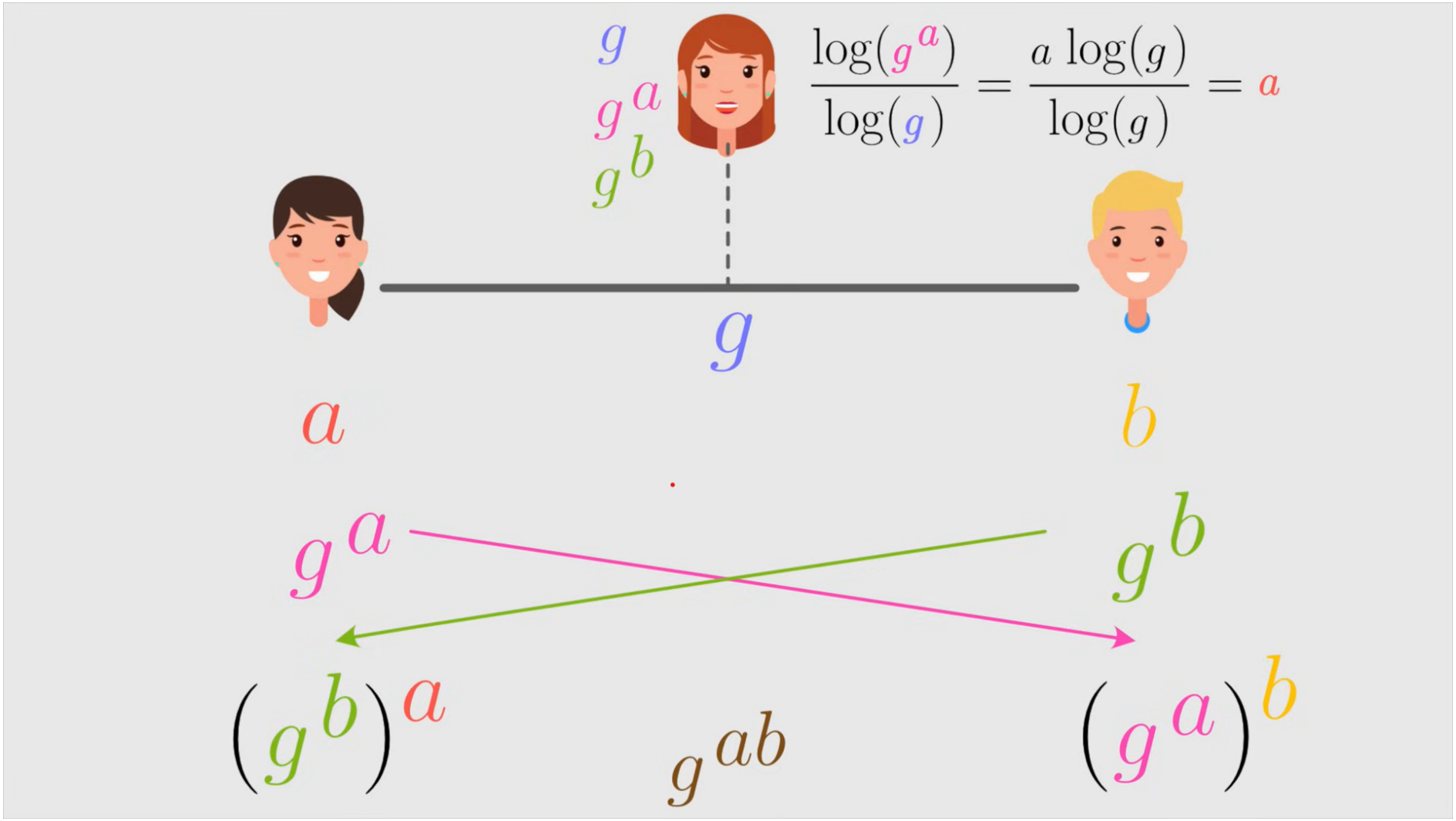


Reminder: Modulo and Diffie-Hellman

Understanding Modulo

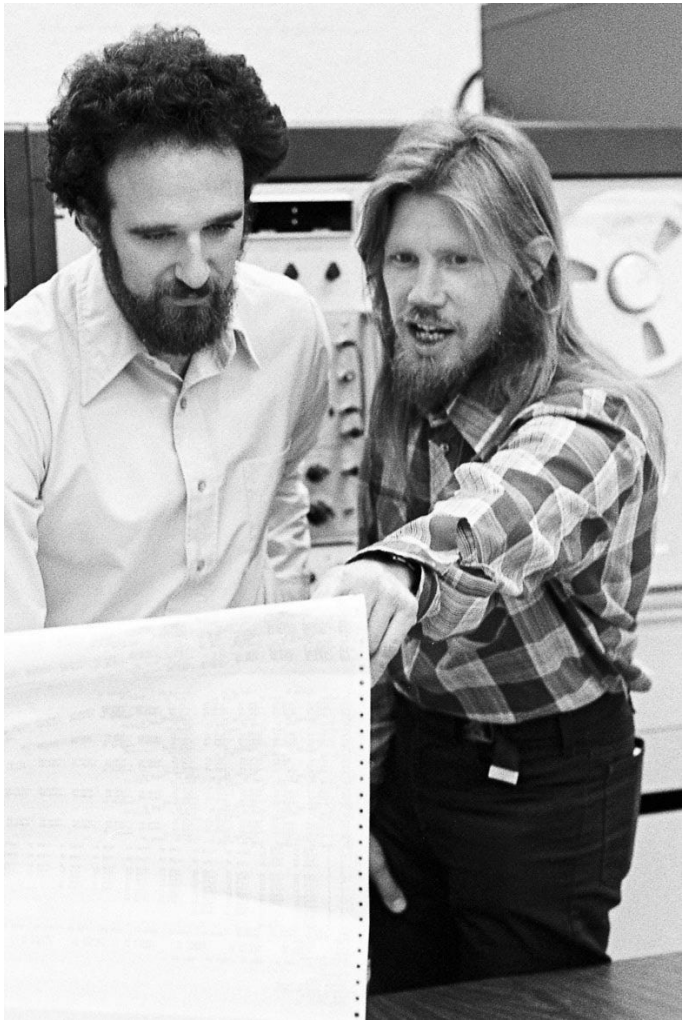
The modulo operation finds the remainder when dividing one number by another.

Example:

$$17 \mod 3 = 2$$

Because:

$$17 = 5 \times 3 + 2$$



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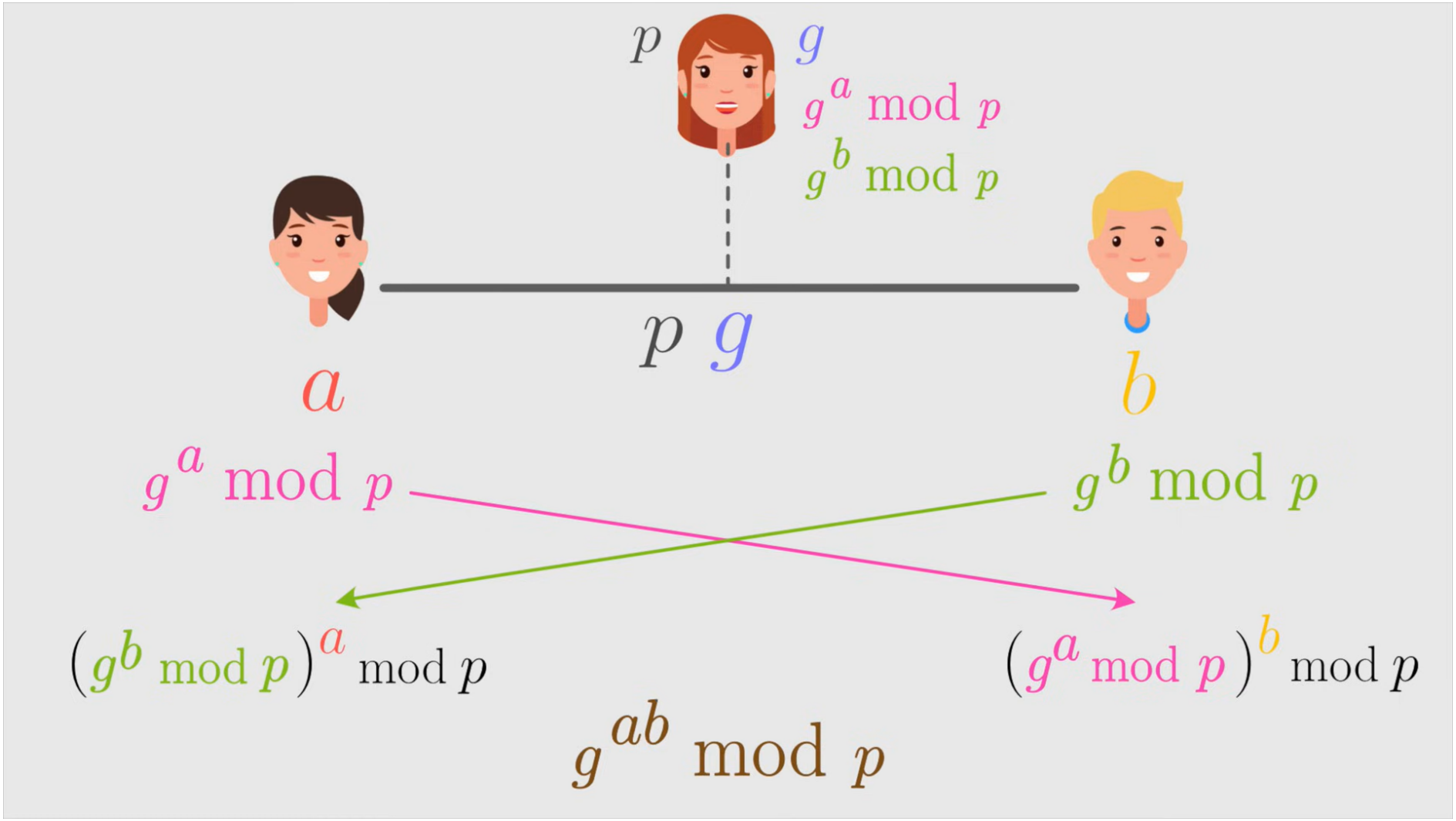
p
g
g^a mod p
g^b mod p
p g
a
b
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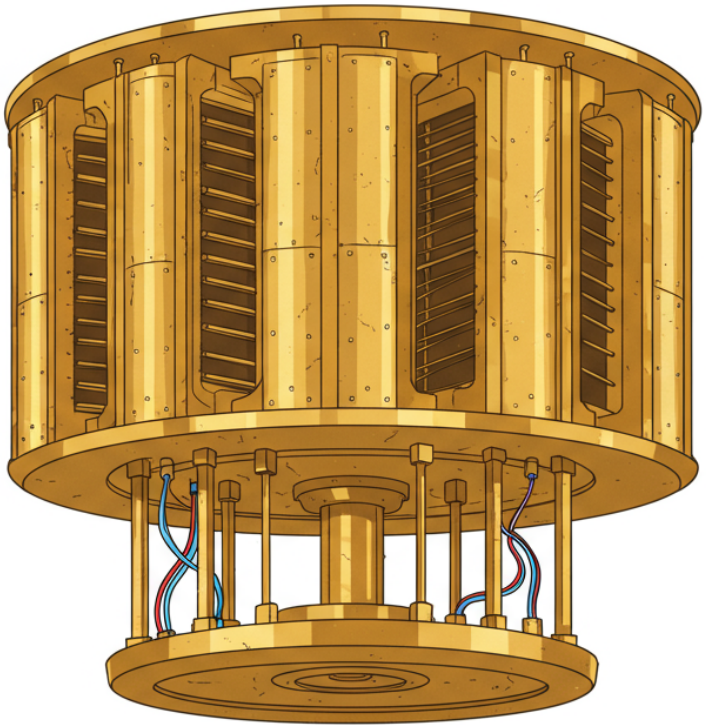
(g^b mod p)^a mod p
(g^a mod p)^b mod p
g^ab mod p

Security and Attacks

Outline

Quantum computing



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Qubits

Classical bit

$b \in \{0, 1\}$

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NOT gate

X gate

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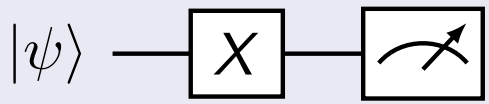
Circuit representation



Hadamard gate

H gate

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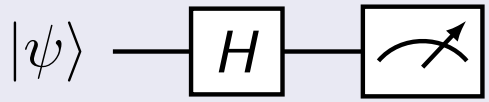

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Bernstein-Vazirani Problem

Problem Definition

Given an oracle for a function f:

$$f : \{0,1\}^n \to \{0,1\}$$

$$f(\mathrm{x}) = \mathrm{x} \cdot \mathrm{s}$$

where s is a secret bit string. Find s with the fewest oracle calls. ($\cdot$ is the bitwise dot product, XOR sum).

Classical Algorithm - Example

Example (n=2)

To find $s = s_0 s_1$:

Requires 2 queries.

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To find $s = s_0 s_1$:

- Query $f(10) = 1 \cdot s_0 + 0 \cdot s_1 = s_0$
- Query $f(01) = 0 \cdot s_0 + 1 \cdot s_1 = s_1$

Requires 2 queries.

Classical Algorithm - General Case

Classical complexity: $\mathcal{O}(n)$

We need to isolate each bit of s by querying with inputs that have a single '1'. This requires n queries for an n-bit string.

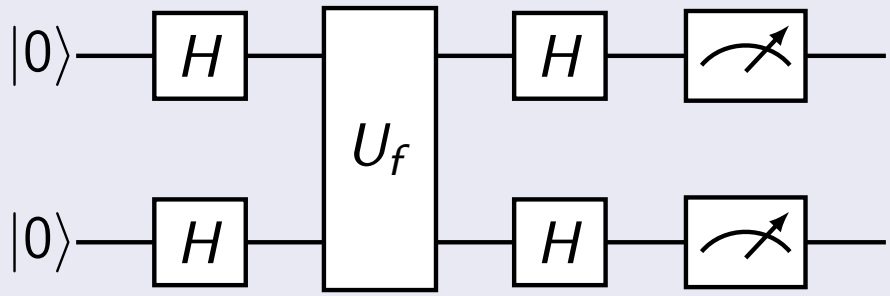
Quantum Algorithm - Overview

Quantum complexity: $\mathcal{O}(1)$

The quantum algorithm can find s with just **one** query. It uses superposition to query all possible inputs simultaneously.

Quantum Algorithm - Circuit

Circuit Diagram



Quantum Algorithm - Circuit

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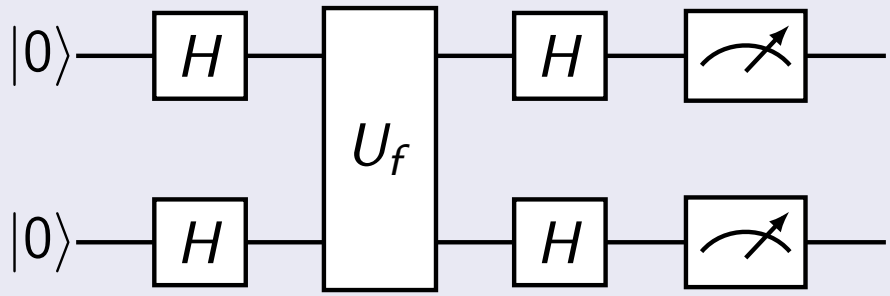

Explanation

- H : Hadamard gates on all n input qubits (creates superposition).
- U_f : The quantum oracle.
- Final Hadamards and measurement reveal s.

Shor's Algorithm

Complexity Gain

Classical factoring is very slow (roughly $\mathcal{O}(e^{\sqrt[3]{n}})$). Shor's algorithm is much faster (polynomial, $\mathcal{O}(n^3)$).

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Requirements

- Requires a large number of high-quality (low-error) qubits (roughly $2n$ for an n-bit number).
- We currently don't have quantum computers large and stable enough to break practical RSA encryption.

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What it is not:

Cryptography **using** quantum technologies

- Many cases where it is unusable
- Deprecated from governmental institutions (NSA, ENISA[a], ANSSI)

[a] doi.org/10.2824/92307

The PQ problems

Families

- Codes
- Hash functions
- Isogenies
- Multivariates polynomials systems
- Lattices

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Why lattices ?

- Well spread
- Good results

Encryption/Key encapsulation	
Crystals-Kyber	Lattices
Signatures	
Crystals-Dilithium	Lattices
Falcon	Lattices
Sphincs+	Hash

Table: Results from the NIST

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Some definitions

The unformal definition

A arrangement of points in space, following a regular pattern

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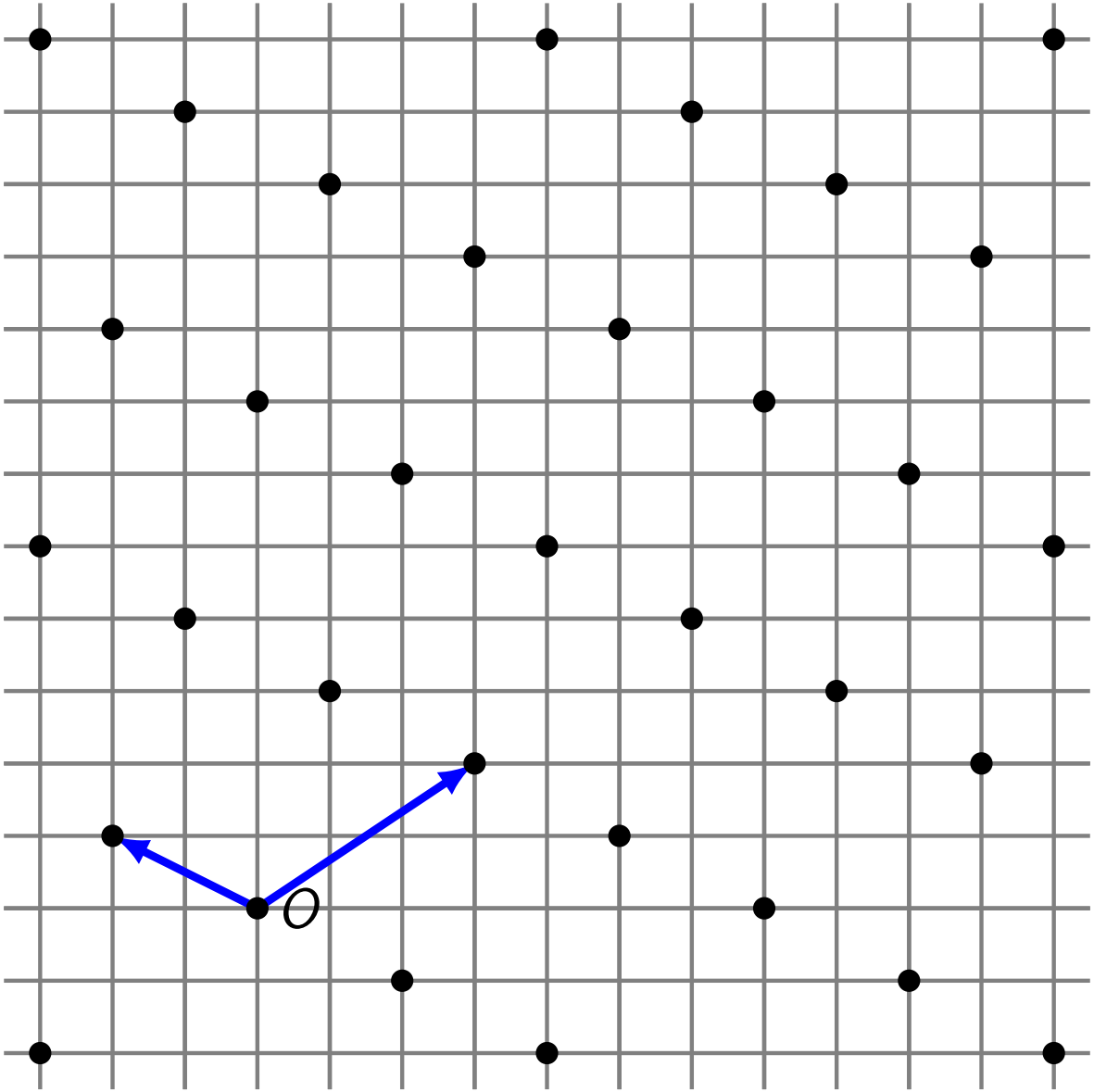
$\Rightarrow$ We have vectors, dot/scalar product and matrices

Example



Learning with error problem

One equation

For

- a secret vector s,
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solve $a \cdot s + e = b$

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Example in $\mathbb{Z}/13\mathbb{Z}$

$$\begin{bmatrix} 9 & 5 & 7 & 10 \\ 7 & 10 & 5 & 3 \\ 4 & 7 & 6 & 7 \\ 7 & 8 & 3 & 11 \\ 2 & 2 & 11 & 1 \\ 2 & 9 & 9 & 1 \end{bmatrix} \times \begin{bmatrix} 6 \\ 5 \\ 6 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 6 \\ 5 \\ 0 \\ 9 \end{bmatrix}$$

(Fully) homomorphic encryption

- We can evaluate (or compile[1]) a circuit on encrypted data

[1] https://heir.dev/

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For a function $f : P \times P \to P$ and encrypted data C_1 and C_2 :

$$eval(f, C_1, C_2) = Enc(f(Dec(C_1), Dec(C_2)))$$

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- Based on lattices (variant of LWE problem)

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Sizes of the keys and data

TODO	

Not necessarly robust to classical computer

- Example : Supersingular isogenies Diffie-Hellman key exchange

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Conclusion

Summary of Key Points

- Cryptography has evolved from simple secret writing to complex mathematical protocols.
- Quantum computing poses a significant threat to classical cryptographic schemes, particularly RSA.
- Post-Quantum cryptography offers potential solutions through lattice-based and other hard mathematical problems.

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Challenges and Future Perspectives

- Quantum computers are not yet powerful enough to break RSA in practice, but research is advancing rapidly.
- Post-Quantum cryptography must balance security, efficiency, and scalability.
- The transition to quantum-safe cryptography is crucial for securing future communications.