# 2. A Picture and an Equation
## (Introduction to Causal Set Theory)

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In the previous blog post, we aimed at the questions which compel us to start a pursuit for a quantum theory of gravity. From now on, we will begin paving the path for the discrete versions of spacetime. A famous adage, "A picture is worth a thousand words," and this is profoundly true in science. The relation between physics and figures is as old as the history of physics itself. A typical science student started getting used to figures (particularly *free-body diagrams*) while solving everyday mechanics' problems via Newton's laws. In the second half of the last century, Feynman promoted diagrams to another level and used these to study complex quantum interactions and associate equations to different segments of these diagrams. In the next section, we will briefly review different types of diagrams in quantum field theory and Einstein's relativity.

## 1 An Equation and a Picture

The idea to model processes (or equations) with pictures dates back to the iconoclastic figure of the twentieth century, Richard P. Feynman, who introduced his famous diagrams in 1949 [1]. Feynman diagrams provide simple visualization to abstract mathematical ideas. These are quantum processes that happen in space and time. Imagine electrons with solid. The arrows point in the positive time axis represent particles and those in the backward time direction indicate antiparticles. They come close, interact with each other and exchange a photon, represented by a wiggly line. The first published Feynman diagram is shown in Figure 1. Employing Feynman rules, one can assign mathematical formulas to each part of the diagram. It turns out one will get the same result as one would obtain by doing laborious and cumbersome mathematics of quantum fields.

Feynman diagrams have proved so useful, they have become an integral part of quantum field theory. Penguin diagrams are another class of diagrams inspired by Feynman diagrams. A penguin diagram superimposed on an image of a Gentoo penguin[1] is shown in Figure 2.

Not only in field theory, but diagrams are also extensively used in relativity. These include simple Minkowski spacetime diagrams of special relativity to the complex Penrose–Carter (or *sim-*

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[1] https://commons.wikimedia.org/wiki/File:Penguin_diagram.JPG#/media/File:Penguin_diagram.JPG

*ply* Penrose) diagrams representing the causal structure of spacetimes containing black holes. First Penrose diagram, shown in Figure 3, appeared in [2].

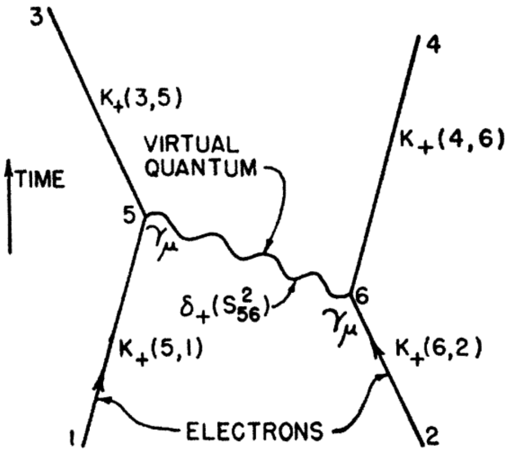


Figure 1: Feynman diagram [1]

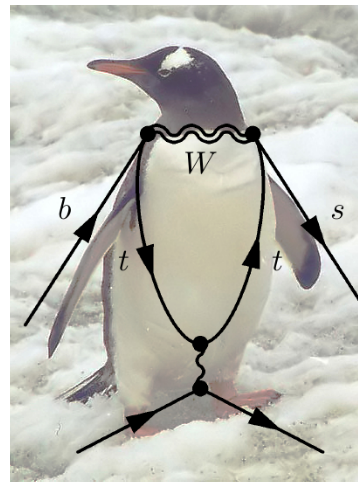


Figure 2: Penguin diagram (Wikipedia)

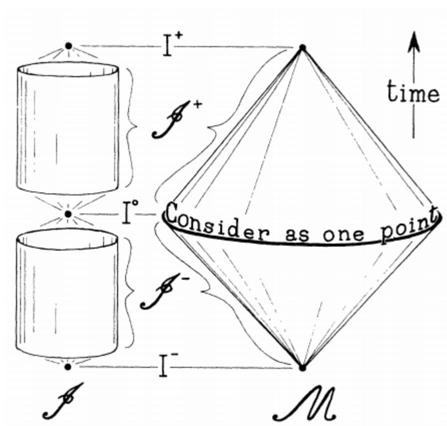


Figure 3: Penrose diagram [2]

Loosely speaking, the same idea is also being widely used in discrete causal approaches but in the reverse order. A spacetime diagram representing causal relations is being created using some established kinematical techniques. Then axioms of the theory are used to derive mathematical equations. In the next section, we will see how we can study the nature of spacetime atoms using pictures. In the end, we will also briefly visit the causal analog of the Schrodinger equation.

## 2 A Picture and an Equation

Causal theories provide a figurative approach to understand and model the universe around us. A glimpse of such a quantum universe is shown in the figure 4 [3]. Two different levels of structural hierarchy, employed in such causal universes, are:

- higher-level structures,

- lower-level structures.

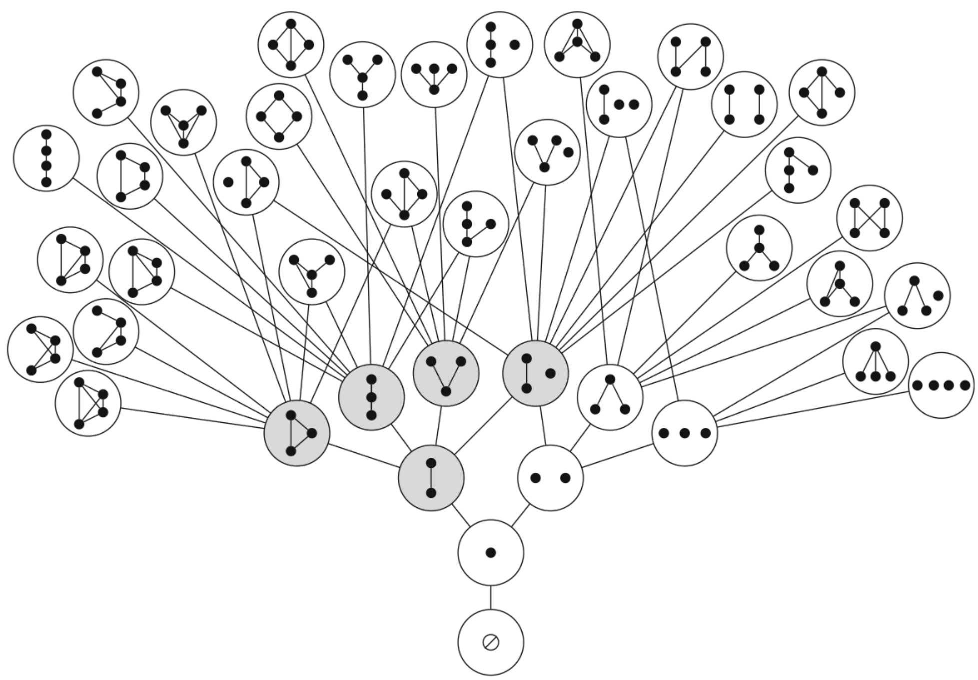


Figure 4: Quantum universe in discrete causal theory (taken from [3])

In figure 4, the big open nodes and long edges are the higher-level structures and represent only a small portion of the quantum multiverse. Lower-level structures are drawn inside these open nodes with filled balls (or dots). Some of these nodes are connected through a line with other filled nodes while some are not. This reminds about the "ball-and-stick" molecular model. However, in physics, they seem not to be as complicated as they are in chemistry[2]. These simple structures are extremely powerful information systems and represent the even simpler *classical universes* or *histories*. Both the terms, *universes* and *histories*, are almost the same and widely used in causal theories. However, when we talk about 'universes', it evokes the idea of a 'self-contained' systems (independent of each other) while the term 'histories' depicts an 'extended' process and does not only indicate these nodes as the snapshots of underlying structure at a particular instance (or spacetime). Classical histories (or simply *histories*) are more descriptive in the context of discrete causal theories, so we will try to stick with this all the time.

At the moment, we are not talking about what these histories are required to model in conventional physics or what could be the size of these nodes? However, being the most fundamental structural unit of spacetime, these are required to describe and represent local processes such as particle interactions but it is expected, the size of these histories would be tens of orders of magnitude smaller than the interacting particles or any other objects involved in these processes.

[2]We will figure out in future what the word *complication* means in this context.

# 3 Representation of Structures

The "ball-and-stick" or "node-and-edge" representation exists at both structural levels. However, these edges are not random but *directed* with initial and terminal nodes. These nodes are the *events* and the directed edges represent the *cause-and-effect* relation between these events. This is where causality comes in[3]. Such sets of events (or objects) are called *directed sets*[4]. Before proceeding, we will label the two-level structure in the following way:

- Higher-level structures as the quantum structures,
- Lower-level structures as the classical structures.

Many readers would be dubious about this labelling as it seems disturbing and unusual for many obvious reasons. Most trivial objection comes in if we think about the scales of natural phenomena. In physics, we use classical theory (such as Newtonian mechanics and Einstein's general relativity) to study large objects which we are referring here as lower-level structures. On the other hand, quantum theory is an approach to study small-scale phenomena and we are labelling it as a higher-level structure. However, this labelling becomes obvious if we think about the Feynman's path integral or path summation approach to quantum mechanics.

Richard Feynman, in 1948, presented his *path integral formulation* or *sum-over-histories* approach [4] which generalizes the action of classical mechanics. In classical theory, a particle can only have a unique trajectory from one point to another while in Feynman's approach, it can take any path it wants out of infinitely possible trajectories. Each possible path is assigned a phase (or amplitude) in some space and then all the phases (or amplitudes) are summed up to yield the quantum amplitude and the probabilities of these amplitudes can then be measured. In Feynman's own words [5]:

> "The electron does anything it likes ... It goes in any direction at any speed, forward and backward in time, however it likes, and then you add up the amplitudes and it gives you the wave-function."

Another reason to think about this unusual labeling of structures is the supremacy of quantum theory. It is accepted that quantum theory is more fundamental than the classical, that, the classical world is built up from the quantum principles. In our designation, higher-level structures are responsible for making up the lower-level structures, nonetheless, this disagreement is not serious. To understand this, let's try to comprehend what the word *classical* means.

- Technically, it refers to a *specific* or *unique* permissible physical scenario[5], for instance, a unique trajectory to reach from one point to another. This is the lower-level structure. But in quantum world, we talk about *all* the possible permissible scenarios, henceforth, it evokes a higher-level structure.

[3] At the moment, We cannot say whether this representation truly depicts a Lorentz invariant causal structure or not. We will answer this once we develop a scheme to see how these nodes come into existence.

[4] I will explain its technical details in the upcoming blogs.

[5] In discrete causal theories, these permissible scenarios are determined from the axioms of the theory and not from the classical equations of motion as we usually do in the electromagnetic theory or in general relativity. Histories satisfying these axioms are said to be *permissible* scenarios and treated on an equal footing. The histories, that do not satisfy axioms, are not even included in the theory. In contrast to continuum-based approaches, discrete causal theories do not distinguish between 'on-shell' and 'off-shell' scenarios.

- Conventionally, anything, measured at sufficient large scales and low energies, is said to be *classical*. This was evident even before the development of quantum theory, when there was no concept of indivisible atom. However, in quantum theory, Bohr's correspondence principle also provides a nontrivial justification to this definition. It states that [6] "the predictions of classical and quantum mechanics agree in the limit of large quantum numbers [the large scales]."

This comparison also clarifies an important issues in the understanding of classical and quantum theories. Conventionally, quantum theory is fundamental than its classical counterpart as the classical equations of motion are the approximations of quantum dynamical laws. However, technically, no theory is *more* fundamental than the other. Both are equally fundamental in their premises and represents different structural levels to explore the natural phenomena. But, on the structural basis, we can safely say that quantum (higher-level) structure is more *complete* than the classical (lower-level) structure because the quantum structure engirdles all the permissible physical scenarios while the classical only has unique or specific scenario.

# 4 How to Model a Picture with an Equation?

The way we have defined the hierarchical structure in the universe, it gives rise to a "structured configuration space". Such a space represents a *kinematic scheme* and in case of discrete causal theories, these schemes are encoded by "sequential evolutionary processes". A configuration space is a representation space of some underlying principle or phenomena and provides us all the possible states of a physical system. To understand the idea, let's visit a very simple example from classical mechanics. The position of a particle in a system can be represented by three coordinates $(x, y, z)$ or to describe its motion, we need different sets of the coordinates say $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ and it forms a function, $\psi(x_1, y_1, z_1, x_2, y_2, z_2)$, which lives in a six-dimensional configuration space. So, we can say, the configuration space represents the configuration or structure of the physical system[6].

It means, by adopting a generalized path summation approach, we can represent any quantum object with such a structured configuration space. And here comes the way to model a picture with an equation. To do is, it is necessary to find the contributions of all the paths in the configuration space and sum them up. In actual, to sum these contributions, it is needed to transform these structural points into *some sort of* mathematical objects and apply the suitable and pertinent mathematical operations. In the same fashion, the quantum amplitudes and other observables of the system can be found out. Such a scheme provides us the needed quantum-theoretic framework equipped with the predictive powers. In discrete causal theories, such an algebraic framework naturally arises from the underlying causal structure between the events[7] and this framework results in the dynamical laws for the theory. I will now just show a particular example of such a dynamical law, the causal Schrodinger-type equation.

[6]It is important to reiterate, configuration space is closely related to phase space or state space but not exactly the same.

[7]We will review this in detail in the future blog posts.

## 4.1 Causal Schrodinger-Type Equations

In non-relativistic quantum mechanics, time-dependent Schrodinger wave equation is

$$i\hbar\frac{d\psi}{dt} = \hat{H}\psi. \tag{2.1}$$

Here, $\psi$ is an element of the Hilbert space and $\hat{H}$ is the Hamiltonian operator. Since, it closely resembles with the classical heat equation, so its solutions are often dubbed as the wave functions or *preferably* state functions, since $\psi$ represents the state of the system. Now I will only write the *causal Schrodinger-type equation* [3] and state what the symbols mean without going into the conceptual detail and derivation of the equation. We will do this later. Moreover, I will not show how this equation corresponds to the conventional Schrodinger equation. Anyhow, the discrete causal version of Schrodinger equation is:

$$\psi^{-}_{R;\theta} = \theta(r)\sum_{r^{-}\prec r}\psi^{-}_{R;\theta}(r^{-}). \tag{2.2}$$

Here, $\psi^{-}_{R;\theta}$ represents a generalized state function, and negative sign as a superscript indicates it is a past state function. The $\theta(R)$ is the relation function and is analogous to the Lagrangian, rather than the Hamiltonian. Here, the relation function is defined for $R$ which is a subobject in the configuration space and possesses a special kind of histories, the co-relative histories, for elements $r$ and $r^{-}$. The symbol $\prec$ in $r^{-} \prec r$ reads as $r^{-}$ precedes $r$ or $r^{-}$ is to the past of $r$. This relation encodes causality in the theory. The causal Schrodinger-type equation can also be treated as the discrete heat equation, for obvious reasons stated for the Schrodinger equation in non-relativistic quantum theory.

However, if this equation is true, it must have the necessary information to recover the continuum structure from the discrete spacetime, or in this case, quantum field theory and the Einstein's field equations. But it is a long path to pave, so in the next few blog posts, we will try to understand what are metric recovery theorems and how we can recover the metric from the underlying discrete spacetime.

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