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The Price is Right – An Application of the GARCH Option Pricing Model

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1 Introduction

Options are contracts that give the owner the right to buy or sell an asset at a certain price at a certain time. While “European” options may only be exercised at the time of maturity, “American” options can be exercised throughout the whole period of time until expiration. Options are traded immensely all over the world and allow investors to build hedging portfolios which protect them from price risks. A “Call” option gives the right to buy and a “Put” option gives the right to sell. The theory of option pricing tries to find the right price for this contract, the premium. The right price is the price that prevents arbitrage and is realized in a complete market.

The standard option pricing model is the Black and Scholes option pricing model Black and Scholes (2019). The Black-Scholes model assumes homoskedastic asset returns, meaning that the returns have a constant variance. It also assumes normal distributed returns. Those two assumptions are not backed by empirical findings and observations of financial time series.

This paper uses the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) option pricing model based on the Esscher transform used in Zhu and Ling (2015), that incorporates more characteristics of asset return time series and require less restrictive assumptions. It allows for different return distributions and accounts for heteroskedasticity, time-varying variance, in the returns. Here, I will only incorporate the heteroskedasticity and keep the normality assumption. The option prices are then predicted using a Monte Carlo simulation.

Time series of asset returns commonly have clustered variance, periods with persistent high variation and periods with persistent low variation. The GARCH model captures this characteristic - great shocks lead to high variance in reality and in the model.

In this paper, I will compare the predictive power of the Black-Scholes option pricing model to that of the GARCH option pricing model by comparing their option price predictions with realized option trades. I use a subset of 1624 out of 26437 available call option trades between the 01.03.2016 and the 01.03.2018 on the S&P 500 index. I do this to find out whether the GARCH option pricing model can provide better price predictions than the Black-Scholes model, and if so whether the advantage (or disadvantage) is consistent throughout options with different characteristics in terms of moneyness and maturity.

Although there already is comparative literature on GARCH-type option pricing models that usually use the Black-Scholes model as a benchmark, it is still important to repeat the comparison with new data as the market dynamics can change anytime. To use options for risk management the knowledge of their value is important and the modeling should happen with as little error as possible.

The idea is that by incorporating the heteroskedasticity of the asset time series, the GARCH option prices will be closer to the market value than the Black-Scholes price predictions. The performances are thoroughly evaluated by providing an absolute, squared and relative error measure. To capture systematic mispricing also the bias is calculated. The comparison reveals that the GARCH model provides better predictions for options at-the-money and slightly in-the-money - with high prediction errors. The Black-Scholes model is better for deep in-the-money options where both models provide reliable estimates. The precision of both Black-Scholes and GARCH is heavily subject to the degree of moneyness while the maturity affects both models equally. Apart from that, I report predictions from a mixed model that uses a technically wrong forecast that delivers better option price predictions.

2 Literature

The fundamental model of option pricing was formulated by Fisher Black and Myron Scholes in 1973 Black and Scholes (2019). The Black-Scholes option pricing model has strict assumptions. One of those is that the stock price of the underlying asset is normal distributed with constant variance. This assumption inspired others to extend the BS model to allow for stochastic - random - volatility. Hull and White Hull and White (1987) found that the Black-Scholes model tends to overprice options at-the-money. The overpricing increases in the maturity. For deep in- or out-of-the money options it underprices options. Slight improvements to the Black-Scholes predictions where found by Wiggins (1987) using a stochastic volatility model for daily asset returns for eight stocks, the S&P 500 as well as the CRSP value-weighted index. Melino and Turnbull (1990) used stochastic volatility to price options on spot foreign currency using a volatility diffusion model. They find that allowing for volatility changes their model has a better fit to the empirical distribution of the Canadian Dollar - US Dollar exchange rate producing better option price predictions, although their model is still biased. Their model provides a “striking” improvement over the Black-Scholes model, especially for long-term options.

An alternative way of modeling the volatility of a financial time series was proposed in Engle (1982) with the Autoregressive Conditional Heteroskedasticity

Model (ARCH Model). The ARCH model allows for stochastic and time dependent conditional volatility, conditional on previous observations. The basic ARCH(1) model assumes the asset returns to be conditionally normal distributed with varying variance, depending on the last past return observation. A very high or very low (negative) return leads to a high return variance in the next period. This property is founded in empirical observations of financial time series. Shephard (1996) provides an overview and comparison of ARCH and stochastic volatility models.

The ARCH model provides the base of the Generalized Autoregressive Conditional Heteroskedasticity Model (GARCH Model) proposed by Bollerslev (1986). The Generalized ARCH model extends the ARCH model to allow for a dependency in conditional volatility. Since the proposal in 1986 many extensions have been introduced. Bollerslev (2008) provides an overview of the extensions. Some of those extensions found their way into option pricing models.

The Exponential GARCH, EGARCH, Nelson (1991) allows for asymmetry in the shock impact on the next periods volatility referred to as the leverage effect. It additionally allows for a negative effect of returns on future volatility which is assumed not to be the case in the basic GARCH model. According to Bollerslev (2008) the EGARCH "complicates the construction of unbiased forecasts for the level of future variances" due to its necessary "logarithmic transformation". This can be troublesome for option pricing because of its reliance on variance forecasts. Ding et al. (1993) introduce a different asymmetric GARCH model: the Asymmetric Power GARCH (APARCH), a generalization of the Nonlinear GARCH (NGARCH) proposed by Higgins and Bera (1992), the Glosten, Jagannathan and Runkle GARCH (GJR-GARCH) introduced by Glosten et al. (1993) and the Threshold GARCH (TGARCH) proposed by Zakoian (1994). The APARCH allows for different levels of asymmetry and the volatility power to be estimated.

In 1995 Duan (1995) derived an option pricing model based on the GARCH model and the locally risk-neutral valuation relationship (LRNVR). Using this LRNVR, Duan derives a risk-neutral pricing measure $\mathcal{Q}$ from the physical probability measure $\mathcal{P}$. This GARCH model is assuming normal distributed asset returns. An alternative to using the LRNVR is the application of the Esscher transform. Originally proposed by Esscher (1932) to transform probabilities in a risk assessment context. It was picked up by Gerber et al. (1993) and applied on option pricing using several different stochastic processes for the asset returns. Both Duan (1995) and Gerber et al. (1993) build their model on the concept of risk-neutral pricing through the derivation of a martingale pricing measure $\mathcal{Q}$ from Harrison and Pliska (1981).

There is a plentitude of GARCH option pricing models that differ on the base of the underlying return distributions. Siu et al. (2004) extend the Gerber et al. (1993) approach for option pricing to GARCH models and provide a general derivation of risk-neutral pricing measures. They give a parametric example of a GARCH model governed by a shifted negative gamma innovation. This gives the additional flexibility for the GARCH model to incorporate a higher kurtosis of returns. The return process under $\mathcal{P}$ and under $\mathcal{Q}$ is subject to a Gamma distribution with a fixed shape parameter and a time varying scale parameter. This allows for a time-varying first and second moment and a fixed third and fourth moment, subject to the shape parameter that is to be estimated using a method of moments estimator.

Christoffersen et al. (2006) use an inverse Gaussian distribution to allow for a time-varying third moment. In an empirical test the authors find their model to be performing better than the Black-Scholes option pricing model, significantly reducing the negative bias that the Black-Scholes model has for in-the-money call prices. Their derivation of the risk-neutral pricing measure is also based on the Esscher transform.

The model of Chorro et al. (2012) uses generalized hyperbolic innovations for their GARCH models and compare the performance to GARCH models based on the negative inverse Gamma, hyperbolic and Normal distributions. They also incorporate GARCH extensions like the EGARCH, GJR-GARCH and APARCH. Zhu and Ling (2015) use the Esscher transform to derive option pricing models based on the Normal distribution, the shift negative Gamma distribution and shift negative inverse gaussian distribution. They incorporate a GARCH-in-mean (GIM) model and an Autoregressive Moving Average (ARMA) model for the conditional mean of the return and EGARCH, NGARCH and GJR-GARCH models for the conditional variance. They test their pricing models on options on Hang Seng Index options and S&P 500 options. Under assumption of a normal distribution they find theta the EGARCH, GJR-GARCH and EGARCH models, both with GIM and ARMA conditional mean specifications, perform superior compared to the Black-Scholes model and a GIM-GARCH model. They outperform the benchmark models even more when assuming a shift negative Gamma distribution or a shift negative inverse gaussian distribution. In their study the GIM-EGARCH with shift negative Gamma innovations performs best.

This paper is based on the application of the Esscher transform in Zhu and Ling (2015) for the basic GARCH model with normal innovations but with a constant mean.

3 Option Theory

European call (put) options give the owner the right to buy (sell) a stock to the strike price at the time of maturity. Another "vanilla" option is the American option, exercisable at every point in time until maturity. There are several other options, labeled "exotic". Some examples are the Bermuda option, a mix between the European and the American option or the Barrier option where a certain stock price level must be passed for the contract to be activated. The price of the option is called the premium.
Options are versatile in their use cases, Berk and DeMarzo (2019):

Options can be used for risk management. A call option is an insurance against an increasing stock price while a put option insures against a decreasing price. The strike price respectively defines the maximum or minimum price for the buying or selling.

Options can also serve for speculation. If one believes a stock price to fall they can make a gain by purchasing a put option at the market price. Another use are the payment of employees. Additional to a fixed salary some companies pay their employees with call stock options. Those reward the employees with a gain depending on how high the stock price is.

The payout profile, or the intrinsic value, of European options depends on the time of maturity T, the stock price at that time S_T and the strike price K.

$$\pi_C(S_T, K) = \max\{S_T - K, 0\} \tag{1}$$

$$\pi_P(S_T, K) = \max\{K - S_T, 0\} \tag{2}$$

It is important to distinguish between the intrinsic value and the extrinsic value. The intrinsic value is the value when exercised and the extrinsic value is the value for which the option contract is traded.

The relation between strike price K and spot price S is called moneyness ($\frac{S}{K}$). A call option is called in-the-money when $S - K > 0$. It is out-of-the money for $S - K < 0$. For put options it is the other way around. When the option is in-the-money at maturity there is a gain, when out-of-the-money there is none.

The value of a European call option can easily be translated into the value of a European put option by using the put-call parity Berk and DeMarzo (2019)

$$C_t = P_t + S_t - d_{t,T}K \tag{3}$$

C_t (P_t) denotes the market value of a call option at time t with strike price K and maturity at time T. $d_{t,T}$ is the discount factor for K, which is the value at t of a zero-coupon bond that pays out 1 at T.

There are different models aiming to calculate the market price based on the characteristics of the traded option. The most prominent model is the Black-Scholes model.

3.1 Black-Scholes Model

The Black-Scholes model assumes that S_T is log-normal distributed and $\ln S_T$ consequently normal distributed. A call option price can easily be computed using the "Black-Scholes formula", Berk and DeMarzo (2019):

$$C_0 = S_0 \mathrm{N}(d_1) - Ke^{-rT}\mathrm{N}(d_2) \tag{4}$$

with

$$d_1 = \frac{\ln\left(\frac{S_0}{Ke^{-rT}}\right) + \sigma^2 T/2}{\sigma\sqrt{T}} \qquad \text{and} \qquad d_2 = d_1 - \sigma\sqrt{T} \tag{5}$$

and N denoting the cumulative distribution function of the standard normal distribution. It is assumed that S_T is log-normal distributed consequently $\ln S_T$ normal distributed.

The Black-Scholes formula has five inputs. Three of those, S_t, K, T are part of the option contract and the risk-free interest rate r is common knowledge. According to Berk and DeMarzo (2019), there are two ways to determine the value of the parameter σ in practice. One way is to estimate the volatility based on historical stock data. This assumes implicitly that the distribution of the stock price remains the same over time. The other way is to calculated the implied volatility. It is called implied volatility because it is implied by option prices currently traded. This is supposed to incorporate market beliefs on the future volatility of the stock.

The implied volatility of an option is calculated by inserting all known information of an option trade (C_t, S_t, K, r, T) into the Black-Scholes formula and solving for σ. This σ is the volatility that the trading parties agreed on when making the trade. Consequently it is the volatility they believed to be accurate for the time until T.

4 GARCH Models

The Autoregressive Conditional Heteroskedasticity (ARCH) model was proposed by Engle (1982) to capture heteroskedasticity in economic time series. The presence of heteroskedasticity in a series of stock returns means that the variance of the returns varies over time and is therefore not constant. The model has been extended by Bollerslev (1986) to the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model.

The basic GARCH (1,1) process is defined as

$$y_t = \sigma_t \epsilon_t \tag{6}$$

$$\sigma_t^2 = \omega + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2 \tag{7}$$

with $\omega, \alpha, \beta > 0$, $\alpha + \beta < 1$ and $\epsilon_t | \mathcal{I}_{t-1} \sim \mathrm{N}(0,1)$. $\mathcal{I}_{t-1}$ is the information set available at $t-1$.

GARCH processes capture the heteroskedasticity of financial time series very well. An important aspect is that they capture the volatility clustering of those time series. Here also lies the critical difference between GARCH based option pricing and Black-Scholes option pricing. The volatility σ_t in the GARCH model is dynamic and the volatility σ in the Black-Scholes model is constant.

There are many extensions to the GARCH model that i.e. allow for asymmetry when $y_t < 0$ and $y_t \geq 0$. It is also common to choose a different conditional distribution for ϵ_t. A popular choice for the modeling of financial time series is the t-distribution because of its fatter tails in the density function which is closer to the observed distribution of financial returns.

4.1 Estimation

My model of choice is the GARCH(1,1) model with constant mean μ under $\mathcal{P}$:

$$y_t = \mu + \sigma_t \epsilon_t \tag{8}$$

$$\sigma_t^2 = \omega + \alpha(y_{t-1} - \mu)^2 + \beta\sigma_{t-1}^2 \tag{9}$$

with $\omega, \alpha, \beta > 0$, $\alpha + \beta < 1$ and $\epsilon_t | \mathcal{I}_{t-1} \sim \mathrm{N}(0, 1)$.

For the estimation under pricing measure $\mathcal{P}$ I use a Quasi-Maximum-Likelihood estimator based on the conditional normality of ϵ_t. I maximize the log-likelihood function with respect to γ with $\gamma = (\omega, \alpha, \beta, \mu)'$:

$$\ell_T(\gamma) = -\frac{T}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\left(\log\sigma_t^2(\gamma) + \frac{(y_t - \mu)^2}{\sigma_t^2(\gamma)}\right) \tag{10}$$

It is a Quasi-Maximum-Likelihood estimator because the normality assumption is hurt. If the normality assumption is hurt and the innovations are generated by a different distribution the estimator still converges in probability but importantly the covariance matrix must be estimated with a robust estimator Rahbek and Søndergaard Pedersen (2020).

4.2 GARCH option pricing

For the GARCH option pricing I follow the approach of Zhu and Ling (2015) which is based on the application of Esscher transforms in option pricing by Gerber et al. (1993). The Esscher transform is used to transform the stochastic process of the returns into a martingale. Only then the process is discountable without creating arbitrage opportunities.

The general formula to determine the value of a European call option is given by

$$\mathrm{E}[m_{t,T}\max(S_T - K, 0)|\mathcal{F}_t], \tag{11}$$

the expectation of the discounted option payoff, with $m_{t,T}$ as the discount factor discounting the option payoff $\max(S_T - K, 0)$ from period T back to t. If I were to forecast the process $\{S_t\}$ and predict $\hat{S}_T$ based on my model assumptions "under the physical probability measure $\mathcal{P}$" the discounted price would not be arbitrage free.

Hence, I need to predict $\hat{S}_T$ using a "risk-neutralized measure $\mathcal{Q}$". The process under $\mathcal{Q}$ is directly derived from the assumptions on the asset price movements under $\mathcal{P}$.

Zhu and Ling (2015) provide a general derivation for those models: For the log-return of an asset with price S_t that "follows a discrete stochastic process under the physical probability measure $\mathcal{P}$, i.e.,

$$y_t = \log \frac{S_t}{S_{t-1}} \qquad \text{and} \qquad y_t = \mu_t + \eta_t \sigma_t \qquad \text{under } \mathcal{P} \tag{12}$$

where $\eta_t | F_{t-1} \sim D(0,1)$, $D(0,1)$ denotes some distribution $F(.)$ with zero mean and unit variance, and F_t is the information set up to time t; $\mu_t \in F_{t-1}$ and $h_t \in F_{t-1}$ are the conditional mean and the conditional variance of y_t, respectively." The moment generating function (MGF) of this process is generally given by

$$M_t^{\mathcal{P}}(z) = \mathrm{E}^{\mathcal{P}}[e^{zy_t}|\mathcal{F}_{t-1}] \tag{13}$$

and depicts the starting point of Zhu and Ling's derivation of the risk-neutralized Esscher measure $\mathcal{Q}$. They derive the MGF of the process under $\mathcal{Q}$ as a function of the MGF under $\mathbb{P}$:

$$M_t^{\mathcal{Q}}(z) = \frac{M_t^{\mathcal{P}}(z + \theta_t)}{M_t^{\mathcal{P}}(\theta_t)} \tag{14}$$

With a θ_t must be chosen so, that the martingale condition is fulfilled:

$$\frac{M_t^{\mathcal{P}}(1 + \theta_t)}{M_t^{\mathcal{P}}(\theta_t)} = \mathrm{e}^r. \tag{15}$$

They also provide the stochastic discount factor (SDF) used to discount the option payoff:

$$m_{t-1,t} = \mathrm{e}^{\theta_t y_t} \left[M_t^{\mathcal{P}}(1 + \theta_t) \right]^{-1} \tag{16}$$

which can be multiplied over periods:

$$m_{t,T} = \prod_{i=t+1}^{T} m_{i-1,i} \tag{17}$$

$m_{t,T}$ discounts the predicted asset value $\hat{S}_T$ from period T back to period t. With

$$\hat{S}_T = S_t \exp\Big(\sum_{i=t+1}^{T} y_i\Big) \tag{18}$$

and y_i forecast under $\mathcal{Q}$. $\mathrm{E}[m_{t,T}\max(S_T - K, 0)|\mathcal{F}_t]$ can be approximated via simulation. For this I simulate M paths for $\{y_t\}_t^T$ that lead to M different values $\hat{S}_T$. To determine the option price I discount the intrinsic option values back to time t. The premium is then calculated as:

$$P_C = \frac{1}{M}\sum_{j=1}^{M} m_{t,T}^{(j)} \max(\hat{S}_T^{(j)} - K) \tag{19}$$

and put option price respectively.

4.3 Constant Mean GARCH(1,1) under Q

For my specific case, I assume the stochastic process of (y_t, σ_t^2) under the physical probability measure $\mathcal{P}$ as

$$y_t = \mu + \sigma_t \epsilon_t \tag{20}$$

$$\sigma_t^2 = \omega + \alpha(y_{t-1} - \mu)^2 + \beta\sigma_{t-1}^2 \tag{21}$$

with $\epsilon_t|\mathcal{I}_{t-1} \sim \mathrm{N}(0,1)$ According to Zhu and Ling (2015) the moment generating function under $\mathcal{P}$ is given by

$$M_t^{\mathcal{P}}(z) = \exp\left(z\mu + \frac{z^2\sigma_t^2}{2}\right) \tag{22}$$

The θ_t that solves the martingale condition $\mathrm{e}^r = \frac{M_t^{\mathbb{P}}(1+\theta_t)}{M_t^{\mathbb{P}}(\theta_t)}$ is given by Zhu and Ling (2015) as $\theta_t = \frac{r-\mu}{\sigma_t^2} - \frac{1}{2}$. The moment generating function according to the authors is then

$$M_t^{\mathcal{Q}}(z) = \frac{M_t^{\mathcal{P}}(z+\theta_t)}{M_t^{\mathcal{P}}(\theta_t)} \tag{23}$$

$$M_t^{\mathcal{Q}}(z) = \exp\left[z\big(r - \frac{\sigma_t^2}{2}\big) + \frac{z^2\sigma_t^2}{2}\right] \tag{24}$$

which according to the authors translates into a process under $\mathcal{Q}$ of

$$y_t = r - \frac{\sigma_t^2}{2} + \epsilon_t \tag{25}$$

$$\sigma_t^2 = \omega + \alpha(y_{t-1} - \mu)^2 + \beta\sigma_{t-1}^2 \tag{26}$$

with $\epsilon_t|\mathcal{I}_{t-1} \sim \mathrm{N}(0, \sigma_t^2)$.

Rahbek and Søndergaard Pedersen (2020) express the SDF as:

$$m_{t-1,t} = \frac{\exp(\theta_t y_t)}{\exp\left[(1+\theta_t)\mu + \frac{(1+\theta_t)^2\sigma_t^2}{2}\right]}. \tag{27}$$

4.4 Pricing Algorithm

The process of y_t and σ_t^2 under the risk neutralized Esscher measure $\mathcal{Q}$ is used to make discountable asset price predictions S_T. The algorithm is described in Rahbek and Søndergaard Pedersen (2020).

For the constant mean GARCH(1,1) the algorithm follows:

1. For $i = 1, \dots, M$ I generate GARCH(1,1) forecasts for $j = t+1, \dots, T$ under $\mathcal{Q}$.

$$\sigma_j^{(i)2} = \hat{\omega} + \hat{\alpha}(y_{j-1}^{(i)} - \hat{\mu})^2 + \hat{\beta}\sigma_{j-1}^{(i)2} \tag{28}$$

$$\text{draw} \quad \epsilon_j^{(i)} \sim \mathrm{N}(0, \sigma_j^{(i)2}) \tag{29}$$

$$y_j^{(i)} = r - \frac{\sigma_j^{(i)}}{2} + \epsilon_j^{(i)} \tag{30}$$

   The starting value for the conditional variance $\sigma_{j-1}^{(i)2}$ must be cautiously chosen. A high value tends to a forecast with high variance as the next steps variance will depend on the previous draw. I use the historical variance as initial value. The log-return forecasts $y_{t+1}^{(i)}, \dots, y_{t+T}^{(i)}$ are used to calculate the prediction for the asset price at time T, where S_t is known:

$$\hat{S}_T^{(i)} = S_t \exp(\sum_{j=1}^{T} y_j^{(i)}) \tag{31}$$

2. With $(\sigma_j^{(i)2}, y_j^{(i)})$ I calculate $\theta_j^{(i)}$ and the stochastic discount factor $m_{j-1,j}^{(i)}$ for

$j = t+1, \dots, T$:

$$\theta_j^{(i)} = \frac{r - \hat{\mu}}{\sigma_j^{(i)2}} \tag{32}$$

$$m_{j-1,j}^{(i)} = \frac{\exp(\theta_j^{(i)} y_j^{(i)})}{\exp\left[(1+\theta_j^{(i)})\hat{\mu} + \frac{(1+\theta_j^{(i)})^2 \sigma_j^{(i)2}}{2}\right]} \tag{33}$$

The obtained one-period discount factors is multiplied into a T-period discount factor:

$$m_{t,T}^{(i)} = \prod_{j=t+1}^{T} m_{j-1,j}^{(i)} \tag{34}$$

3. The GARCH option price is then obtained by calculating the average of the M option prices $\psi^{(i)}$:

$$P_C = \frac{1}{M} \sum_{i=1}^{M} m_{t,T}^{(i)} \max(\hat{S}_T^{(i)} - K) \tag{35}$$

4.4.1 Pricing Algorithm for GARCH under P

The right way to use the GARCH model for option pricing is by deriving the risk-neutral pricing measure $\mathcal{Q}$. Only then, the obtained discounted asset prices are arbitrage free. I discovered that when the forecasting algorithm is incorrectly applied and the conditional mean under $\mathcal{P}$ is used instead of the conditional mean under $\mathcal{Q}$ the obtained predication biases are similar to those of the Black-Scholes model with the opposite sign, see section **Results**.

The pricing algorithm for GARCH $\mathcal{P}$ differs only in one aspect from GARCH $\mathcal{Q}$. The conditional mean under measure $\mathcal{P}$ is used:

$$y_j^{(i)} = \hat{\mu} + \sigma_j^{(i)} \epsilon_j^{(i)} \tag{36}$$

$\theta, m_{j-1,j}, \sigma$ stay the same.

The theory that the biases cancel each other out and lead to better option price predictions is developed based on table 1. I simply use the average of the GARCH $\mathcal{P}$ and Black-Scholes predictions. It is important to add that the biases continue canceling each other out to some degree when the Black-Scholes bias shifts from negative to positive for options with higher moneyness (table 6).

The mixed model has no theoretical backup but I decided not to keep it from the readers because of its performance. It is also important to note that the GARCH $\mathcal{P}$ still incorporates most elements from GARCH $\mathcal{Q}$. The only difference is the mean equation.

5 Data

I will be using two data sets. One will cover the index value of the S&P 500 index from 1st of March 2005 to the 29th of February 2016. The S&P 500 is an index provided by Standard and Poor's, an index and credit rating provider. The S&P 500 contains stocks of the 500 companies that have the largest market capitalization in the United States of America. On the 31st of December 2021 that corresponded to a capitalization of USD 5.4 trillion or 80% of US market capitalization[1]. Other important indices are the Dow Jones Industrial Average, (DJIA) tracking 30 hand-picked large blue-chip companies based in the US, or the NASDAQ 100, tracking the 100 largest companies listed at the technology focused National Association of Securities Dealers Automated Quotations stock exchange. With 80% of the US market capitalization the S&P 500 is a solid index for the state of the US economy. Though it does leave out small companies.

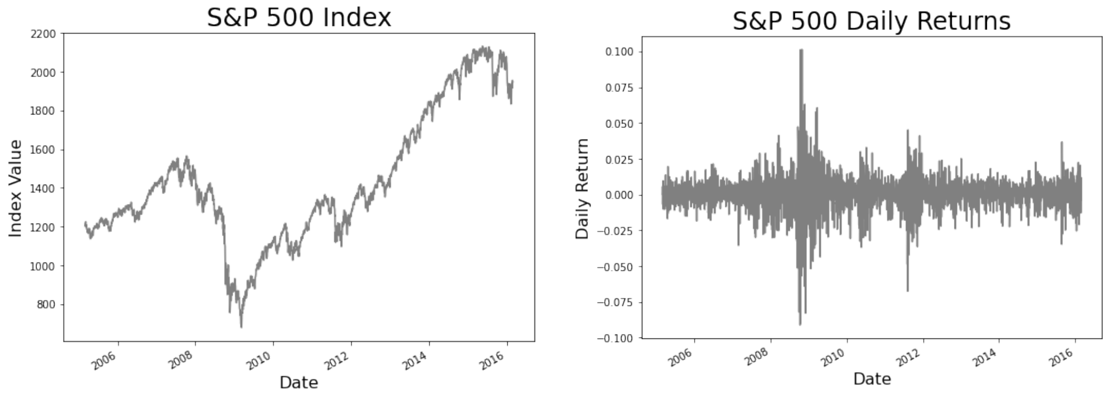
The selection of 2005 as the starting year is arbitrary but reflects the trade off between more observations for a more precise estimation (early starting year) and keeping the assumption that the underlying data generating process remains unchanged uncritical (late starting year). I selected the 29th of February as the end of the data because the 1st of March 2016 is the start of the data set on option trades. Traders trading on the 1st of March naturally only have information on the S&P 500 until the moment they trade. I make the simplifying assumption that for all the option trade data I use the participants use they information until the 29th of February 2016. This means assuming that the GARCH(1,1) estimates for 2005 to 2016 are the same as for 2005 to 2018 and that the historical volatility used in Black-Scholes option pricing is the same for 2005 to 2016 as from 2005 to 2018.

The returns are computed as $y_t = \log(\frac{S_t}{S_{t-1}})$.

5.1 S&P 500 Index

The S&P 500 grew from 1203.6 on the 1st of March 2005 to 1947.1 on the 29th of February 2016. The annualized rate of return is 4.474%. The US GDP grew from

[1]https://www.spglobal.com/spdji/en/indices/equity/sp-500/#overview, 01.05.2022



(a) Index (b) Returns

Figure 1: S&P 500

USD 13.04 trillion in 2005 to USD 18.75 trillion in 2016[2] which translates into an annualized growth rate of 0.03357. The log-returns have an average of 0.0001714 and a sample volatility of 0.01206.

The graphic of the S&P 500 log-returns in figure 1b show the volatility clustering of the returns.

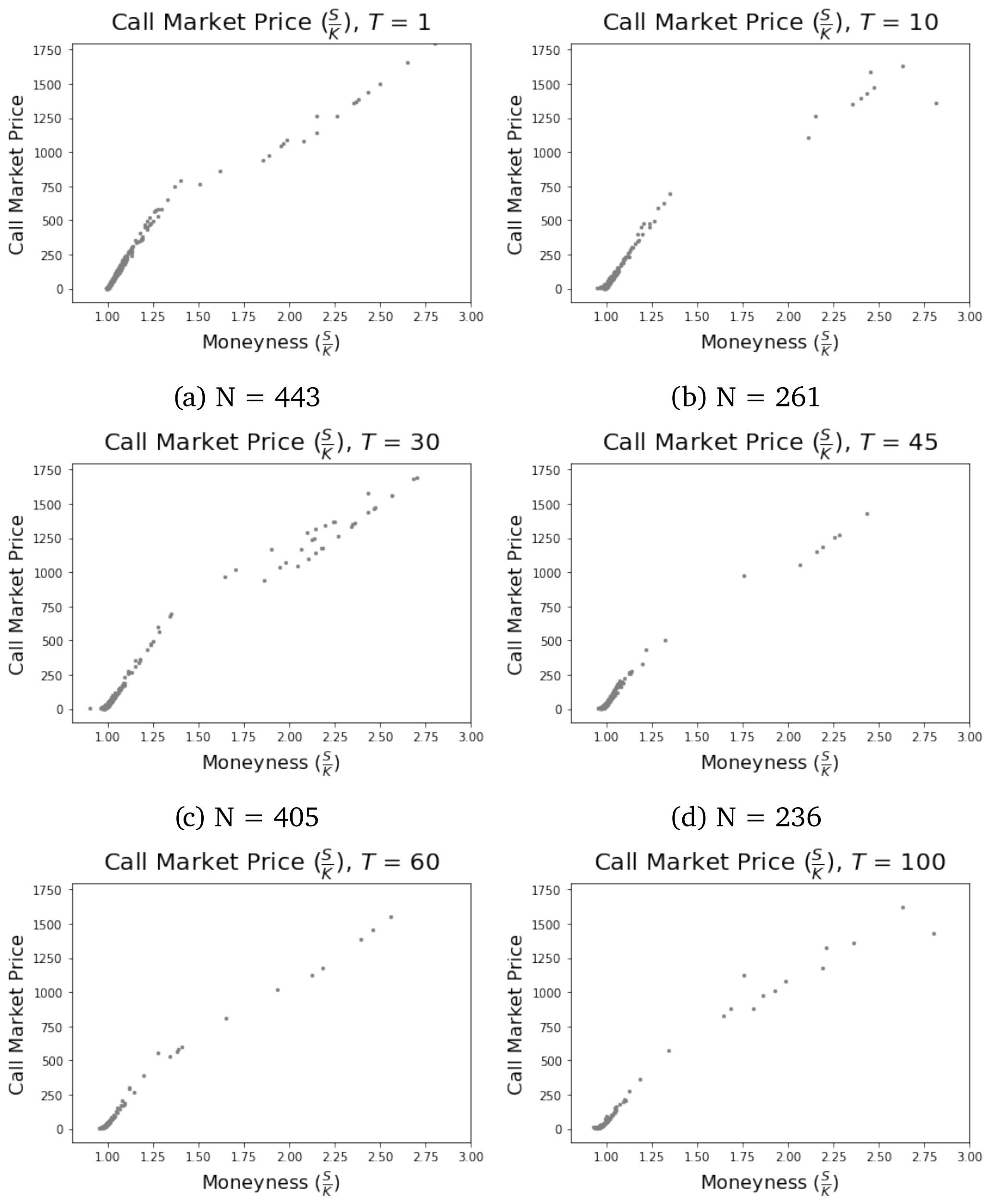
5.2 S&P 500 Options

As a second data set I use data on 26437 option trades provided by my supervisor Philipp Christian Kless. The data set also includes the safe interest rates at the time of the transactions. The set includes the option premium, the spot price, the strike price, the maturity and the safe interest rate.

From those 26437 I create 6 sub samples: 443 observed trades with maturity $T = 1$, 261 observed trades with maturity $T = 10$, 405 observed trades with maturity $T = 30$, 236 observed trades with maturity $T = 45$, 155 observed trades with maturity $T = 60$ and 124 observed trades with maturity $T = 100$. This is a total of 1624 observations.

Figure 2 shows the market prices of the call options within those sub samples subject to their moneyness. The option price is increasing in the moneyness. This is not surprising as the intrinsic value depends on S and K. Figure 3 illustrates how the value of the options is driven by the maturity for given moneyness. The maturity itself does not affect the intrinsic value, only the extrinsic market value.

[2]World Bank



(a) N = 443

(b) N = 261

(c) N = 405

(d) N = 236

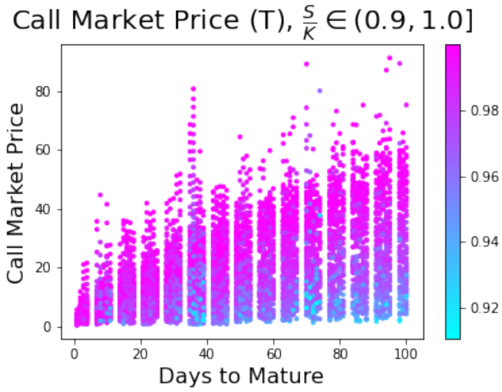
(e) N = 155

(f) N = 124

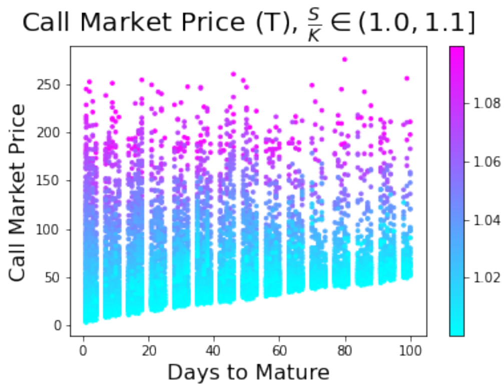
Figure 2: Call option market prices subject to their moneyness for given maturities

6 Estimation

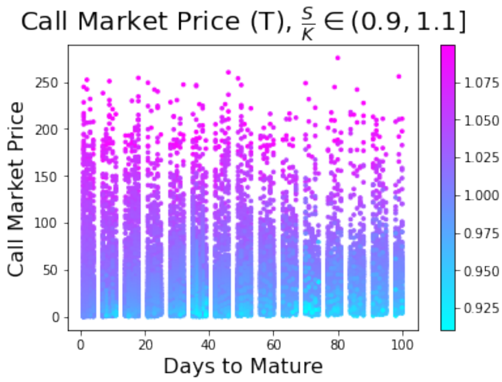
The GARCH option pricing method from Zhu and Ling (2015) is divided into two steps. 1. The estimation of the parameters under $\mathcal{P}$. 2. The Monte Carlo simulation to obtain option price estimates. The estimation is run with the *arch* package in python Sheppard (2021). The parameter estimates can be found in table **GARCH parameter estimates**. The standard errors are estimated using a robust standard error estimator.



(a) N = 6847



(b) N = 7578



(c) N = 14425

Figure 3: Call option market prices subject to their maturities for given moneyness

GARCH parameter estimates

parameter	coefficient	t-statistic	p > |t|
μ	0.000573***	3.850	0.0001182
	(0.0001489)		
ω	0.000209***	3.681	0.000232
	(0.00005686)		
α	0.1050***	6.877	6.111×10^{-12}
	(0.01526)		
β	0.8761***	53.608	0.0000
	(0.01634)		

T = 2769

Robust standard errors in parentheses

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

To determine whether or not the data y_t are likely to be generated by a Normal distribution I use the Jarque-Bera test, proposed by Jarque and Bera (1980).

$$\mathcal{H}_0 : \text{the data is Normal distributed} \qquad \text{vs} \qquad \mathcal{H}_1 : \text{the data is not Normal distributed}$$

The test uses that the skewness of a Normal distribution is 0 and therefore the sample skewness should be close to 0 and analogously the kurtosis of a Normal distribution is 3. With a sample skewness of -0.38 and a sample kurtosis of 10.78 the hypothesis is rejected (p-value = 0.000). That means that the assumption of normality of both option pricing models is hurt. For the GARCH(1,1) model a different innovation with a higher kurtosis should be preferred.

7 Results

The model performances for maturities $T = 1, 10, 30, 45, 60, 100$ can be found in 7. The table summarizes the model performances with regards to the mean absolute error (MAE), mean squared error (MSE) and bias. The MAE shows by how much the models miss the option market value on average. It is measured in USD. The MSE squares those errors and by that weighs large errors more than small ones. Therefore, it punishes models with mostly high accuracy that produce singular large outliers. The bias is calculated by averaging the errors. If a model produces positive errors (over-pricing) and negative errors (under-pricing) of the same magnitude the bias will be close to zero. If the bias is positive in a large enough sample the model systematically over-prices options and if it is negative it systematically under-prices options. The mean percentage errors are calculated as the average of $100 \times \frac{|\text{True Price - Predicted Price}|}{\text{True Price}}$. The factor 100

Table 1 summarizes the performance over all considered maturities and over all moneyness. For $T = 1$ it shows that the Black-Scholes model and the GARCH option pricing model, labeled as GARCH $\mathbb{Q}$ perform almost identical. This is because for the GARCH model the historical variance was used as starting value and for the Black-Scholes model the historical volatility was used. They each have small a small MAE and MSE due to the low price of the options with a maturity of $T = 1$. They misprice those options on average by 16%. The mispricing increases in the maturity over around 50% for $T = 10$ to 77% (Black-Scholes) and 73% (GARCH) for $T = 100$. The overall bias is negative for both models because the data set is not balanced in regards to moneyness.

Those results, and the observation, that when - incorrectly - forecasting the return y_t under $\mathcal{P}$ (as $y_t = \mu + \sigma_t \epsilon$) instead of under $\mathcal{Q}$ (as $y_t = r - \frac{\sigma_t^2}{2} + \epsilon_t$), the GARCH option pricing model produces over-priced predictions, motivated the investigation if two wrongs can become one right by averaging the under-priced predictions of the Black-Scholes model and the over-priced predictions of the false GARCH model (labeled GARCH $\mathcal{P}$). The performance of this mixed model is far better than the performance of the Black-Scholes or the GARCH model. The MAEs and MSEs are much lower and the MPEs slightly lower with the exception of $T = 30$. Because I formulated the model based on the results of the original subsample, I tested the performance on a new subsample. The results for the new sample performance are summarized in Table 2 with similar results. The new subsample consists of options with a maturity of one day more or one day less maturity. The average could be replaced by a weighted average based on the magnitude of the biases. The fact that the MPEs are not as superior as the MAEs and MSEs means that the mixed model performs better for options with a high premium. Higher option premiums occur for options with higher maturity and/or higher moneyness.

For the options with moneyness below 1 (Table 3) the models are very far off. Those options have a low premium but the models struggle with given them a value at all. Black-Scholes and GARCH perform similarly bad. It is astounding that even for low maturities, where there is little room for uncertainty, the models cannot price the options right.

The models improve for options at-the-money or slightly in-the-money, with a moneyness between 1 and 1.1 (Table 4). They still perform weakly, worsening in maturity. The mixed model performs very well for high and low maturity options with a MPE of about 6% for $T = 1$ and $T = 100$. For every maturity except for $T = 1$ the GARCH model performs slightly better than Black-Scholes.

The overall performance massively improves for options with moneyness between 1.1 and 1.5 (Table 5). For $T = 1, 10, 30$ the MPEs for both Black-Scholes and GARCH are even below 1%, for Black-Scholes even until $T = 45$. Unfortunately the sample sizes within this group are very small so the sample errors cannot converge to true model errors but the results still indicate that from starting at slight in-the-moneyness the models begin to perform better and more reliable. Both models still under-price the options.

For the options with moneyness above 1.5 (Table 6) the models perform best but Black-Scholes outperforms GARCH for maturities of 30, 45, 60 and 100 days. For maturities of 1 day and days they perform similarly. Black-Scholes performs great with MPEs of below 1% for all maturities. For those options Black-Scholes no longer under-prices the options but starts to over-price them in almost every case. That can be seen because the bias is very close to the MAE. The GARCH model still under-prices options systematically. For maturities of 30, 45, 60 and 100 days it under-prices every single option ($-\text{MAE} = \text{Bias}$). The relationship between MPE and MSE shows that the GARCH model has problems pricing options with very high premiums. The relative error is low but absolute and especially squared error are high. The mixed model still outperforms both models but the difference between Black-Scholes and the mixed model is only slight.

Results Table

Method	N	T	BS	GARCH $\mathcal{Q}$	GARCH $\mathcal{P}$	$\frac{1}{2}(\text{BS} + \text{GARCH}\mathcal{P})$
MAE	443	1	1.608686	1.616234	2.424216	1.458417
MSE			4.050321	4.084724	9.236462	2.965710
Bias			-0.944312	-0.961156	0.614338	-0.164987
MPE			16.039491	16.106000	42.480919	15.339998
MAE	261	10	7.901219	7.668651	13.088910	5.748157
MSE			123.414933	117.473490	237.982160	49.272445
Bias			-7.545352	-7.282493	12.470218	2.462433
MPE			49.698316	49.071779	125.831959	48.777498
MAE	405	30	15.316095	14.739482	30.545333	8.268531
MSE			333.189101	287.956766	1103.547731	97.497214
Bias			-15.014114	-14.739482	30.202576	7.594231
MPE			67.148325	64.516836	205.042985	70.126587
MAE	236	45	21.692522	20.397272	33.566937	7.849534
MSE			618.393025	527.085461	1284.022272	93.970258
Bias			-21.303856	-20.397272	33.053400	5.874772
MPE			71.877981	68.138934	183.828391	60.107909
MAE	155	60	25.061033	24.536463	37.034568	8.050395
MSE			811.712160	730.274188	1613.092756	107.978224
Bias			-24.435747	-24.536463	35.876823	5.720538
MPE			71.383617	66.936813	159.520701	47.879696
MAE	124	100	33.148572	35.589751	45.974273	9.927457
MSE			1543.340134	1561.623233	2435.560356	176.846446
Bias			-31.543896	-35.589751	41.782008	5.119056
MPE			77.643475	73.433230	185.718710	58.930512

Performance of option pricing models within original sample

MAE = Mean Absolute Error; MSE = Mean Squared Error; Bias = Average Error

MPE = Mean Percentage Error, calculated as the average of $100 \times \frac{|\text{True Price - Predicted Price}|}{\text{True Price}}$

Table 1

Test Table

Method	N	T	BS	GARCH $\mathcal{Q}$	GARCH $\mathcal{P}$	$\frac{1}{2}(\text{BS} + \text{GARCH}\mathcal{P})$
MAE			2.533303	2.542905	3.668779	1.818706
MSE	337	2	11.139788	11.068570	21.441921	5.389166
Bias			-1.984780	-2.000268	1.666099	-0.159341
MPE			16.303411	16.138489	44.499785	15.992123
MAE			6.555933	6.235798	13.800504	5.501877
MSE	197	11	82.013151	79.182298	264.135486	40.958222
Bias			-6.166740	-5.769880	13.574074	3.703667
MPE			49.784154	49.002541	125.441524	48.96723
MAE			16.117069	15.087492	31.130731	9.134000
MSE	330	31	383.591240	319.316712	1141.792306	113.951169
Bias			-15.933297	-15.085697	30.918462	7.492582
MPE			67.816524	64.15482	205.183492	70.125321
MAE			20.878753	18.653868	36.450997	8.538461
MSE	184	46	602.523041	460.926653	1537.808137	112.170261
Bias			-20.580077	-18.653868	36.188209	7.804066
MPE			72.002103	67.100218	183.784126	60.980051
MAE			24.800601	23.473269	37.029433	8.168288
MSE	236	59	833.847999	685.075605	1590.675694	107.593863
Bias			-24.464219	-23.473269	36.422638	5.979209
MPE			71.192363	66.391254	159.815657	47.125684
MAE			35.245846	36.320391	44.398917	10.380368
MSE	112	101	1798.734424	1643.658567	2296.429174	186.074501
Bias			-34.194000	-36.320391	41.114009	3.460004
MPE			77.782967	73.150786	185.730951	58.245202

Performance of option pricing models within test sample to test hybrid model performance

MAE = Mean Absolute Error; MSE = Mean Squared Error; Bias = Average Error;

MPE = Mean Percentage Error, calculated as the average of $100 \times \frac{|\text{True Price - Predicted Price}|}{\text{True Price}}$

Table 2

Moneyness < 1

T	N	Method	BS	GARCH $\mathcal{Q}$	GARCH $\mathcal{P}$	$\frac{1}{2}$(BS + GARCH$\mathcal{P}$)
1	42	MAE	2.223810	2.223810	5.377955	1.615942
		MSE	6.296548	6.296548	31.179938	3.442918
		Bias	-2.223810	-2.223810	5.377955	1.577073
		MPE	72.139264	71.986843	325.048861	113.389572
10	92	MAE	10.416419	10.362100	15.521851	6.762646
		MSE	161.668257	160.171087	314.427093	59.300459
		Bias	-10.416419	-10.362100	15.105391	2.344486
		MPE	78.379370	78.956448	277.992042	110.853134
30	191	MAE	15.272263	15.006345	36.151086	10.838308
		MSE	307.000761	291.448191	1380.652659	148.140994
		Bias	-15.272263	-15.006345	36.151086	10.439411
		MPE	83.933018	83.878405	366.988221	134.205030
45	113	MAE	19.609673	19.324385	37.962592	10.707940
		MSE	510.280479	488.259915	1539.896923	151.120811
		Bias	-19.609673	-19.324385	37.962592	9.176460
		MPE	85.817709	85.929637	322.023606	114.677018
60	65	MAE	22.375150	21.836363	43.635072	11.884225
		MSE	649.455859	607.248647	2028.635806	195.273870
		Bias	-22.375150	-21.836363	43.635072	10.629961
		MPE	87.264294	87.789452	299.247866	102.056881
100	70	MAE	31.183464	29.334369	54.579276	13.805782
		MSE	1361.870461	1147.907696	3135.680031	274.058995
		Bias	-31.183464	-29.334369	54.579276	11.697906
		MPE	89.077486	89.494430	297.743022	101.086049

Performance of option pricing models for models with $\frac{S}{K} < 1$

MAE = Mean Absolute Error; MSE = Mean Squared Error; Bias = Average Error

MPE = Mean Percentage Error, calculated as the average of $100 \times \frac{|\text{True Price - Predicted Price}|}{\text{True Price}}$

Table 3

1 < Moneyness < 1.1

T	N	Method	BS	GARCH $\mathcal{Q}$	GARCH $\mathcal{P}$	$\frac{1}{2}$(BS + GARCH$\mathcal{P}$)
1	339	MAE	1.554967	1.565658	2.174981	1.406753
		MSE	3.957118	3.995803	7.516615	2.843399
		Bias	-0.920882	-0.945616	0.279066	-0.320908
		MPE	8.510647	8.598070	15.164533	5.929150
10	135	MAE	7.716035	7.346444	13.852963	6.089807
		MSE	127.377707	117.154494	241.976411	53.593043
		Bias	-7.494120	-7.088978	13.167573	2.836727
		MPE	27.568743	26.655401	53.595231	18.657882
30	164	MAE	19.295405	17.005655	31.786778	7.092070
		MSE	462.424069	355.204904	1103.644831	65.889511
		Bias	-19.295405	-17.005655	31.786778	6.245687
		MPE	48.149272	42.813589	78.632779	16.756697
45	106	MAE	26.874558	22.464808	33.010296	5.505851
		MSE	830.301632	596.638835	1203.407379	45.050196
		Bias	-26.874558	-22.464808	33.010296	3.067869
		MPE	52.492884	44.970148	65.724112	11.450020
60	73	MAE	32.079151	27.819277	37.830356	5.803704
		MSE	1134.923787	871.214520	1592.368332	52.085915
		Bias	-32.079151	-27.819277	37.830356	2.875602
		MPE	61.146167	52.705024	71.943345	10.689511
100	37	MAE	48.030570	41.600113	43.227894	4.577270
		MSE	2549.172971	1855.773186	2075.845312	55.051573
		Bias	-48.030570	-41.600113	43.227894	-2.401338
		MPE	64.439925	55.324734	58.301278	5.922591

Performance of option pricing models for models with $1 < \frac{S}{K} < 1.1$

MAE = Mean Absolute Error; MSE = Mean Squared Error; Bias = Average Error

MPE = Mean Percentage Error, calculated as the average of $100 \times \frac{|\text{True Price - Predicted Price}|}{\text{True Price}}$

Table 4

1.1 < Moneyness < 1.5

T	N	Method	BS	GARCH $\mathcal{Q}$	GARCH $\mathcal{P}$	$\frac{1}{2}$(BS + GARCH$\mathcal{P}$)
1	44	MAE	1.576880	1.569036	1.982689	1.777652
		MSE	3.361745	3.409306	4.357201	3.730984
		Bias	-0.299365	-0.270459	-1.014091	-0.656728
		MPE	0.420090	0.415976	0.550514	0.484718
10	24	MAE	1.734911	1.642557	4.082190	1.727437
		MSE	3.701449	2.846226	18.972448	5.574853
		Bias	-0.643100	-0.031095	4.082190	1.719545
		MPE	0.546957	0.503042	1.255629	0.525749
30	17	MAE	3.272778	2.711407	9.922934	3.325078
		MSE	14.249880	9.611838	106.252925	13.553380
		Bias	-3.272778	-2.711407	9.922934	3.325078
		MPE	0.879007	0.694058	2.706583	0.913788
45	7	MAE	2.054480	3.573329	10.219001	4.967625
		MSE	10.179406	19.772527	122.795454	33.118810
		Bias	-0.283750	-3.573329	10.219001	4.967625
		MPE	0.494820	1.046906	3.502039	1.681197
60	9	MAE	6.460030	10.441842	5.859686	2.217508
		MSE	68.976757	112.737830	63.530196	6.372326
		Bias	-2.392781	-10.441842	5.859686	1.733452
		MPE	1.789596	2.434973	1.804255	0.565430
100	4	MAE	12.143207	24.734260	5.815786	3.236907
		MSE	194.628817	621.654105	50.577443	20.846417
		Bias	-11.112054	-24.734260	4.638240	-3.236907
		MPE	4.701533	8.146142	1.991053	1.368086

Performance of option pricing models for models with $1.1 < \frac{S}{K} < 1.5$

MAE = Mean Absolute Error; MSE = Mean Squared Error; Bias = Average Error

MPE = Mean Percentage Error, calculated as the average of $100 \times \frac{|\text{True Price - Predicted Price}|}{\text{True Price}}$

Table 5

1.5 < Moneyness

T	N	Method	BS	GARCH $\mathcal{Q}$	GARCH $\mathcal{P}$	$\frac{1}{2}$(BS + GARCH$\mathcal{P}$)
1	18	MAE	1.262853	1.266443	1.305362	1.283509
		MSE	2.247638	2.249506	2.352539	2.285065
		Bias	0.023384	0.003986	-0.205889	-0.091253
		MPE	0.105629	0.106073	0.110849	0.108195
10	10	MAE	2.060484	1.701327	2.007252	1.452318
		MSE	5.299264	4.067481	6.389707	3.560861
		Bias	1.611424	1.034078	-1.056392	0.277516
		MPE	0.145119	0.118885	0.138219	0.100646
30	33	MAE	1.997958	8.129003	2.553969	1.788258
		MSE	6.807434	76.935764	12.973377	4.701652
		Bias	1.708178	-8.129003	-1.652594	0.027792
		MPE	0.164893	0.585321	0.177915	0.135587
45	10	MAE	4.045769	22.381773	6.139991	2.409918
		MSE	19.583092	583.667414	60.015358	9.317677
		Bias	3.887250	-22.381773	-5.979481	-1.046116
		MPE	0.309285	1.457542	0.356273	0.149310
60	8	MAE	3.769639	32.375554	11.215650	3.963585
		MSE	16.315839	1138.504577	169.173730	23.025051
		Bias	3.769639	-32.375554	-11.215650	-3.723006
		MPE	0.313117	2.438778	0.811861	0.279061
100	13	MAE	7.836658	55.506312	19.812719	6.330257
		MSE	72.717622	3241.347527	423.330429	48.041209
		Bias	7.152206	-55.506312	-19.812719	-6.330257
		MPE	0.714117	4.731596	1.692077	0.519372

Performance of option pricing models for models with $\frac{S}{K} > 1.5$

MAE = Mean Absolute Error; MSE = Mean Squared Error; Bias = Average Error

MPE = Mean Percentage Error, calculated as the average of $100 \times \frac{|\text{True Price - Predicted Price}|}{\text{True Price}}$

Table 6

8 Conclusion

The performance of the Black-Scholes and the GARCH option pricing model depends highly on the moneyness and the maturity of the underlying option. The maturity affects both models equally making predictions more imprecise with longer maturity. I find that the Black-Scholes model under-prices options with low moneyness but overprices options with a moneyness above 1.5. This result is backed by Black (1975) who also finds that the Black-Scholes model underprices out-of-the-money and at-the-money options and overprices deep in-the-money options. The GARCH option pricing model under-prices all types of options with exception of low maturity high moneyness options.

I find that both Black-Scholes and GARCH have a very bad performance for options that are out-of-the-money or at-the-money. Lehar et al. (2002) find similarly bad performances for the Black-Scholes model for the British FTSE 100 index with percentage errors of up to 67% for out-of-the-money options, though in their application the Normal GARCH option pricing model performs better over all moneyness and maturity. This overall GARCH superiority was also found by Christoffersen et al. (2006). The overall performance of the Black-Scholes model is far better in Chorro et al. (2012) than in this paper. Both for CAC 40 options as well as for S&P 500 options. This discrepancy is especially large for options with an underlying moneyness below 1. Zhu and Ling (2015) find MPEs between 2.5% and 10% for both the Black-Scholes model and the GARCH model with Normal innovations (with a GARCH-in-mean-specification instead of constant mean) taking mostly options at-the-money or in-the money into consideration. In comparison to their findings the model performances that I find are more extreme with MPEs below 1% and up to almost 90%.

The fact that the GARCH option pricing model underprices options systematically is surprising. The consideration of heteroskedasticity allows the model to account for the insurance against volatility shocks. This shock insurance is valuable to investors and is therefore reflected in the market premium.

The combination of the Black-Scholes model and the GARCH option pricing model using pricing measure $\mathcal{P}$ into a mixed model provides by far the most stable results. Although this paper does not provide a theoretical foundation and the 50-50 weighting is arbitrary it gives empirical evidence for superior performance for S&P 500 index options compared to the Black-Scholes model and the GARCH option pricing model under risk-neutral pricing measure $\mathcal{Q}$. The driver of the good performance in this application are the mitigative balancing biases. When the Black-Scholes predication is overpricing the premium the GARCH under $\mathcal{P}$ is underpricing the premium and vice versa.

The choice to provide of multiple error measures is important to assess the systematic mispricing of the models. This is rarely done in the literature. To have more observations the maturities should have been grouped in the form of $T \in (95, 105)$.

The results should be interpreted with caution as the literature suggests less extreme prediction errors. Especially the high errors for options out-of-the-money and at-the-money are surprising and hurt the credibility of the results in general.

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