MoMath Workshop: Topological Crochet

Introduction

This handout is prepared for the MoMath workshop series *Topological Crochet*. The purpose of the workshop is to share the powerful method, by which we start a project by building a ribbon graph with foundation chains. This initial set up totally determines the topology of our final work. The key steps in these projects are:

- Build a foundation chain ribbon graph by chain *twists* and stitch *joins*, which we will cover later. Use stitch markers to hold the joined stitches together.
- Crochet a round on top of the chain graph to solidify the structure. Each location that the stitch markers hold in place should be crocheted into once. All stitch markers are safely removed at the end of this step.
- Use our personal taste and imagination to design rounds of stitches to achieve geometric and artistic effects.
- Crochet along the boundary of our work around wire to strengthen the sculpture.

The first step is the key to the whole project, since it decides the topology. For passionate participants who are newer to crochet, it is advisable to practice chain stitches at home before the workshop.

A *twist* in this handout always means **180-degree** unless otherwise noted.

Material needed

There is no restriction as to the type of **yarn** we can use. But it is much easier if we avoid extremely dark colors at the beginning, as it is important for us to see each stitch. For the same reason *at least DK weight is recommended*. Multi-ply worsted weight (non-merino) wool has been working best for me.

In addition to a good **crochet hook** for the hands, **split ring stitch markers** are essential for topological crochet. They are used to hold the joined stitches together before we can crochet into them. Each place where stitches join requires multiple markers, depending on the valence of that point (we will see examples later).

Blocking pins, though not essential, are very helpful when setting up the foundation chain graph. I make empty temari balls with different sizes and colors to accommodate different projects. **Scrap yarn balls** are good choices too.

The Möbius Band

Twist and *join* are two basic operations of topological crochet. To practice these, we will make a Möbius band, which is simple, elegant and topologically interesting. Step-by-step instructions for this project are shown in Figure 1. We need one **foundation chain** of about 27 stitches (Figure 1(a)). Twisting it and joining the end chain stitches produces a narrow Möbius band (Figure 1(b)). We then add a round of **single crochet**

(a) *Make a foundation chain of about 27 stitches.*

(b) *Twist it and join the end chains together by a blue stitch marker.*

(c) *Insert crochet hook into the joined stitch and chain one.*

(d) *Start single crocheting and mark the first single stitch.*

(e) *Encounter joined chain stitch with the blue stitch marker.*

(f) *Crochet into the joined chain stitch and remove the marker.*

(g) *Stop crocheting before the marked first single stitch.*

(h) *Cut the yarn.*

(i) *Pull the yarn end through and remove the hook.*

(j) *Thread a darning needle with the yarn end and insert the needle in the first sc marked by the white marker.*

(k) *Remove the white marker and pull the yarn end through the last stitch.*

(l) *The final work after weaving in the ends.*

Figure 1: *Making a Möbius band.*

stitches (Figures 1(c)-1(g)) to make it prettier and more stable. Figures 1(h)-1(k)) shows how to fasten off neatly. The final work shown in Figure 1(l) produces a satisfying result.

It is important to remember that the two end chain stitches that were joined should **always** be treated as if they were a single stitch whenever we crochet into them. This applies to all joined stitches in all projects for this workshop.

Some other tips that might help:

- Be very familiar with what a foundation chain looks like, when untwisted.
- To twist, it's easier to hold the end with the working yarn with your dominant hand, and the start of the chain the other hand, and twist the chain with the non-dominant hand before moving to the dominant hand. This way we don't need to turn our work back and forth unnecessarily many times to insert the crochet hook back. As shown in Figures 1(a)-1(d), the right end of the foundation chain hardly moves position during the twist and join operations.
- One might notice the constant need to turn the work when performing single stitches. This is a characteristic of this style of topological crochet: the *natural flow* of the chains guides our crochet hooks. Care has to be taken when crocheting so that such flow is respected and no unwanted local twists are created. This can be done by always holding the piece so that
  - you have an unobstructed front view of the segment of the row you are working on and
  - the working yarn is in the back of that row.
- Each crocheter has their preference of whether or not to include the back bumps when crochet into the foundation chain. To reduce unnecessary confusion, make a choice and don't change it in the middle of a project.
- It is easier to work on a table top instead of your lap.

The Trefoil Knot

The main project in this workshop is to make a surface bounded by the trefoil knot, which is the simplest non-trivial knot. One can find many fascinating mathematical features and artistic implementations of it online. The following are the step-by-step instructions of our crochet sculpture. The third step uses treble crochet stitches (tr) as well as **adding** and **combining** of stitches. Those who are newer to crochet are free to skip this step during their first attempt. In [3] a video instruction of this project is available for participants' reference after the workshop.

- **Build the foundation.** Chain 49 stitches plus a few extra that we will remove later. Remove the crochet hook. Starting from the first stitch, mark every 16th chain stitch (Figure 2(a)), four in total (at numbers 1, 17, 33 and 49). If necessary, secure the first chain stitch in place on a temari ball or a yarn ball. Twist once to secure the second marked chain stitch. Twist again to join the first chain stitch with the third marked chain stitch. Without a temari ball this can be done by joining the **first** and **third** marked stitches with a **360-degree twist**. Place **three** stitch markers at this join (Figure 2(b)). Then twist once more to join the **second** marked chain stitch with the **fourth** marked chain stitch. Place **two** stitch markers at this join (Figure 2(c)). We have built a graph with two vertices, or *branch points* as we will call them, with three edges connecting them. Make sure all the edges twist in the **same** direction.
- **Round 1: remove the stitch markers.** Insert crochet hook into the second join (Figure 2(d)), undo all the extra chain stitches (Figure 2(e)), and chain 2 (ch2) to get ready for round 1 (Figure 2(f)). Crochet

(a) *Chain at least 49 and mark four stitches.*

(b) *Join the two stitches marked blue with a 360-degree twist.*

(c) *With another twist, join the stitches marked green.*

(d) *Insert the crochet hook back into the joined stitch marked green.*

(e) *Remove extra chain stitches.*

(f) *Start the first round with ch 2.*

(g) *Crochet along the first edge with a hdc followed by many sc.*

(h) *End this edge by another hdc, before the blue marker.*

(i) *Remove the blue marker, dc, and crochet along another edge.*

(j) *Continue till all edges are crocheted into on both sides and end round 1 by a slip stitch.*

(k) *After the first outer edge is finished of round 2.*

(l) *After round 2 is finished and fastened off.*

(m) *Start the boundary round with wire.*

(n) *Join the wire ends before finish.*

(o) *Ready to fasten off.*

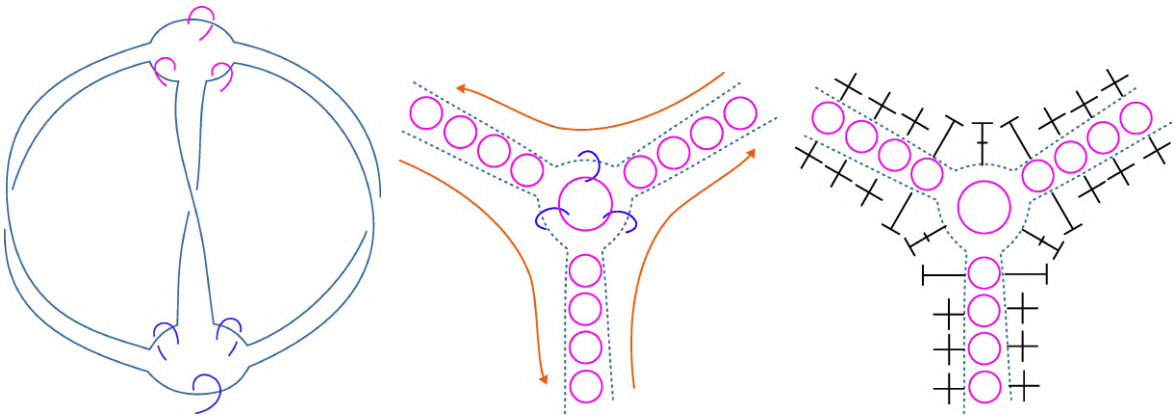
(p) *The final work after fasten off and weaving in the ends.*

Figure 2: *Making a surface bounded by the trefoil knot.*

round 1 with the following guideline: **double crochet (dc)** at either branch point, **half-double crochet (hdc)** at the two neighbors of either branch point and **single crochet (sc)** the rest (Figures 2(g)-2(j)). In total we will go through six sides of the edges, each with a half-double crochet stitches at each end, and 13 single crochet stitches in between. There will also be three double crochet stitches around each branch point. Whenever we crochet past a stitch marker, remove that one but not the rest in the same branch point. Use a slip stitch to end the round. For those who are newer to crochet, dc and hdc may be replaced by sc all around.

- **Round 2: beautify.** This step is **optional**. Place the work so that you can trace the trefoil knot. For any of the three twisted edges, there is one side facing outward and one inward. For the outward side we will crochet taller and more stitches and for the inward side the opposite. Double check there are 16 total stitches on each side of the edges. We are going to crochet in a symmetric way. On the outer side, for the eight stitches from the branch point to the center, crochet as follows: **dc for the first four, 2dc for the next two, 2tr for the next two** – I'll call this an *outer half sequence (ohs)*. On the inner side, for the eight stitches from the branch point to the center, crochet: **dc, hdc, sc, sc, sc2tog, sc2tog** – I'll call this an *inner half sequence (ihs)*. Starting from any branch point, we'll crochet the **combined sequence of ohs, reversed ohs (Figure 2**(k)**), ihs and reversed ihs three times** (Figure 2(l)). Fasten off. This round is where the whole project resembles ordinary crochet the most – we choose the stitch arrangement to vary the size or to add other decorative aspects. The participants are encouraged to experiment with different arrangements to achieve different visual effects.

- **Round 3: color and wire the boundary.** This is the most exciting step as our sculpture will gradually reveal its three dimensional look as we work along its boundary. Cut a 30 inch long wire. Here I'm using gauge 20 copper. Use the second color to single crochet around the wire and along the boundary, starting from a point that is close to the beginning of an ohs (Figure 2(m)). This point is chosen so the wire ends will hide better. Stitches can be added when the boundary curvature is high, as long as the choice is symmetric. I choose to **add three on each outer side**. Before finishing the last inner side, join the wire ends and trim the extra (Figure 2(n)). Fasten off carefully (Figure 2(o)). After finishing, adjust the boundary so that it looks smooth and symmetric (Figure 2(p)).



(a) *Stitched marked chain graph.* (b) *Around each branch point.* (c) *Stitch diagram of the round one around each branch point.*

Figure 3: *Illustration of how to setup the chain graph and crochet round one.*

To summarize, in the first two steps we build the foundation chain graph and reinforce it. In the third step (Round 2) we add artistic elements, and in the fourth step (Round 3) we finish the work by decorating

it with desired colors and using metal to support the whole surface. The first, second, and fourth steps are essential, while the third step can be omitted completely.

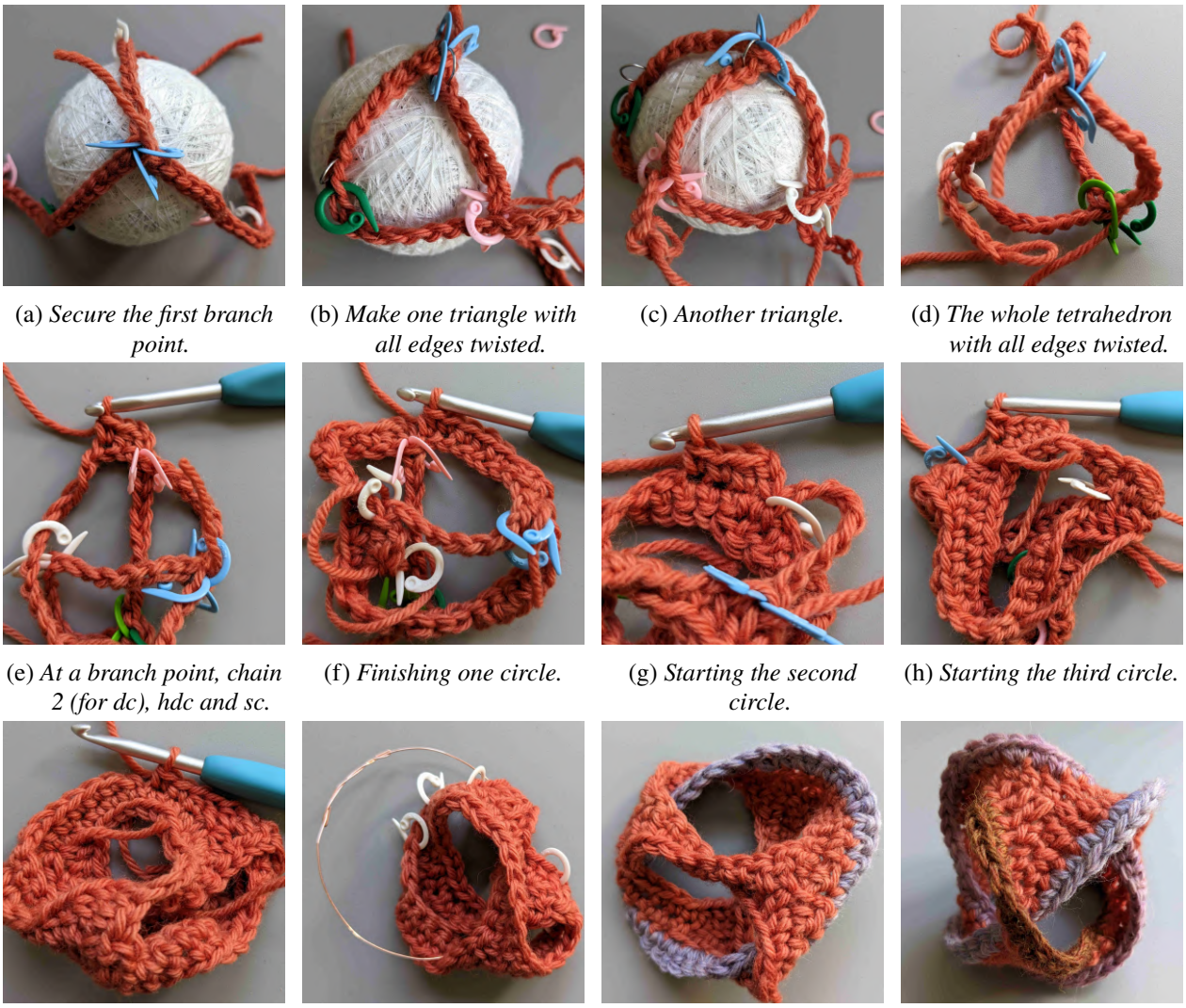
More tips:

- Once again, **extensive stitch marking** is always necessary, even when we think we fully understand. Since there is no natural global coordinate system it is very easy to forget how the edges around each branch point should line up. It is worthwhile to use a stitch marker **between every pair of neighboring edges**.
- To reduce the confusion of stitch counting, it's better to crochet in rounds instead of a spiral. Use turn chain stitches to provide needed height when starting a new round, if using the same yarn from the previous round.
- Let the wire bend along the boundary without too much intrinsic twist, because when the wire has high twist resistance it'll change the bend and introduce unnecessary tension on our stitches.
- Since the boundary is usually a non-planar curve, when we crochet along it we need to reposition the yarn ball back and forth; a smaller yarn ball is easier to maneuver. Sometimes the stitches are hard to reach; a long Tunisian crochet hook can help. Be extra careful not to miss any stitches.

The Borromean Rings

Another simple yet profound example is the Borromean rings. Here we are going to make an unorientable surface bounded by this link, and it has *chiral tetrahedral symmetry*, which is apparent in each step of the construction. A table of many other polyhedra based projects can be found in [1].

- **Build the foundation.** In this step we try to set up the foundation chain graph to be the *1-skeleton* of the tetrahedron, while all edges are twisted in the same direction. Let's choose the edge stitch number to be 9. Chain 37 with a few extra. Start a new row and chain 19 with a few extra. For each row starting from the first chain mark every ninth, 5 in total for the longer row and 3 for the shorter row. Join the **first** chain of the shorter row with the **center** marking of the longer row and put three stitch markers all around. This branch point will be one of the tetrahedron vertices. Make the rest of the tetrahedron by joining the marked chains appropriately, while twisting each edge of the tetrahedron in the same way. Place three stitch markers around each branch point except the one with the working yarn, where only two are needed. With this setup there are three separate circles in each round, each circle covering four edge sides.
- **Round 1: remove the stitch markers.** With crochet hook in one of the branch points, start crocheting with the exact same guideline as in the previous example. Each round consists of **three** unknotted components, and each consists of **four** edges. We can double check the latter while we crochet. Once we finish one component, fasten off and start another one till we finish all three.
- **Round 2: color and wire the boundary.** Cut 3 pieces of 22-gauge copper wire with equal length, about 10 to 11 inches. Since now the boundary has three simple components, we can attach the wire before we crochet, and that'll make the crocheting part easier. Make sure the overlapping is the same for all three components. We can also choose different colors of yarn for different components.



(a) *Secure the first branch point.* (b) *Make one triangle with all edges twisted.* (c) *Another triangle.* (d) *The whole tetrahedron with all edges twisted.*

(e) *At a branch point, chain 2 (for dc), hdc and sc.* (f) *Finishing one circle.* (g) *Starting the second circle.* (h) *Starting the third circle.*

(i) *Finishing the third circle.* (j) *Place a wire long along a circle.* (k) *The first circle is done.* (l) *The final work.*

Figure 4: *Making a surface bounded by Borromean rings with chiral tetrahedral symmetry.*

The Math We Promised

A quick review of topological surfaces, and why twist and join are important in making them can be found in [4]. In [5] ribbon graph structures of works of Perry were analyzed. One can use examples in these papers to practice working on different surfaces.

To contemplate what we are really making, the key observation is that crocheting rounds is simply performing the opposite of *deformation retraction* [2]. Such an action constructs a classical *mapping cylinder* M_f from $f : O \to Y$,where in our case O, a one dimensional closed space (hence the choice of letter O), is the knot or link on the boundary of our final piece, Y, almost the same as O except a finite number of branch points (hence the choice of letter Y), is the *skeletal graph* that lies in the center of the foundation chain graph and f is a *degree-2 branched covering*.

To see the latter, very vaguely speaking, any stitch s on the boundary O can be traced back vertically down the crocheted rounds to Y. Such a tracing is very clear in a plain knitted piece. The precise construction here is slightly more complicated, since in crochet a new stitch actually grows in between two old stitches in the lower round and also there will be adding and combining stitches. Nevertheless the mathematical picture remains the same. By construction M_f is a surface (result of degree-2 covering) that is *homotopy equivalent* to Y. The pre-images (or fibers) in M_f of the branch points can be considered as "seams" one could have sewn along had one decided to make M_f in a traditional patch-and-sew way with a differential manifold mindset.

In our current topological method, once we set up the foundation chain graph, we've made one "end"(which is not a boundary component) of the mapping cylinder where Y is located and its one narrow neighborhood. All the following rounds of stitches are merely rest of the cylinder part of M_f that is homeomorphic to $O \times I$. We finish crocheting on the other end (the true boundary) of M_f which is O. The beauty of crochet is that the foundation chain rows just have 2 sides to support degree 2 covering, and a single chain can fit as many sides as needed (as in magic ring) to serve as a branch point. Such a perfect connection between crochet and algebraic topology is quite satisfying and mysterious. Hopefully this workshop will be a seed for many new inspiring works to come.

References

[1] S. Dong. "Sculpting Mapping Cylinders: Seamless Crochet of Topological Surfaces." *Bridges Conference Proceedings*, Halifax, Canada, Jul. 26–31, 2023, pp. 559–566. https://archive.bridgesmathart.org/2023/bridges2023-559.html

[2] A. Hatcher. *Algebraic Topology*. Cambridge University Press. https://pi.math.cornell.edu/~hatcher/AT/AT.pdf.

[3] F. Herr. "Crocheting a Seifert surface | a math-y crochet project." https://www.youtube.com/watch?v=UgoGGRhlhPU

[4] C. H. Séquin. "2-Manifold Sculptures." *Bridges Conference Proceedings*, Baltimore, Maryland, USA, Jul. 29–Aug. 1, 2015, pp. 17–26. http://archive.bridgesmathart.org/2015/bridges2015-17.html.

[5] C. H. Séquin. "Homage to Charles O. Perry." *Bridges Conference Proceedings*, Stockholm, Sweden, Jul. 25–29, 2018, pp. 123–130. http://archive.bridgesmathart.org/2018/bridges2018-123.html.