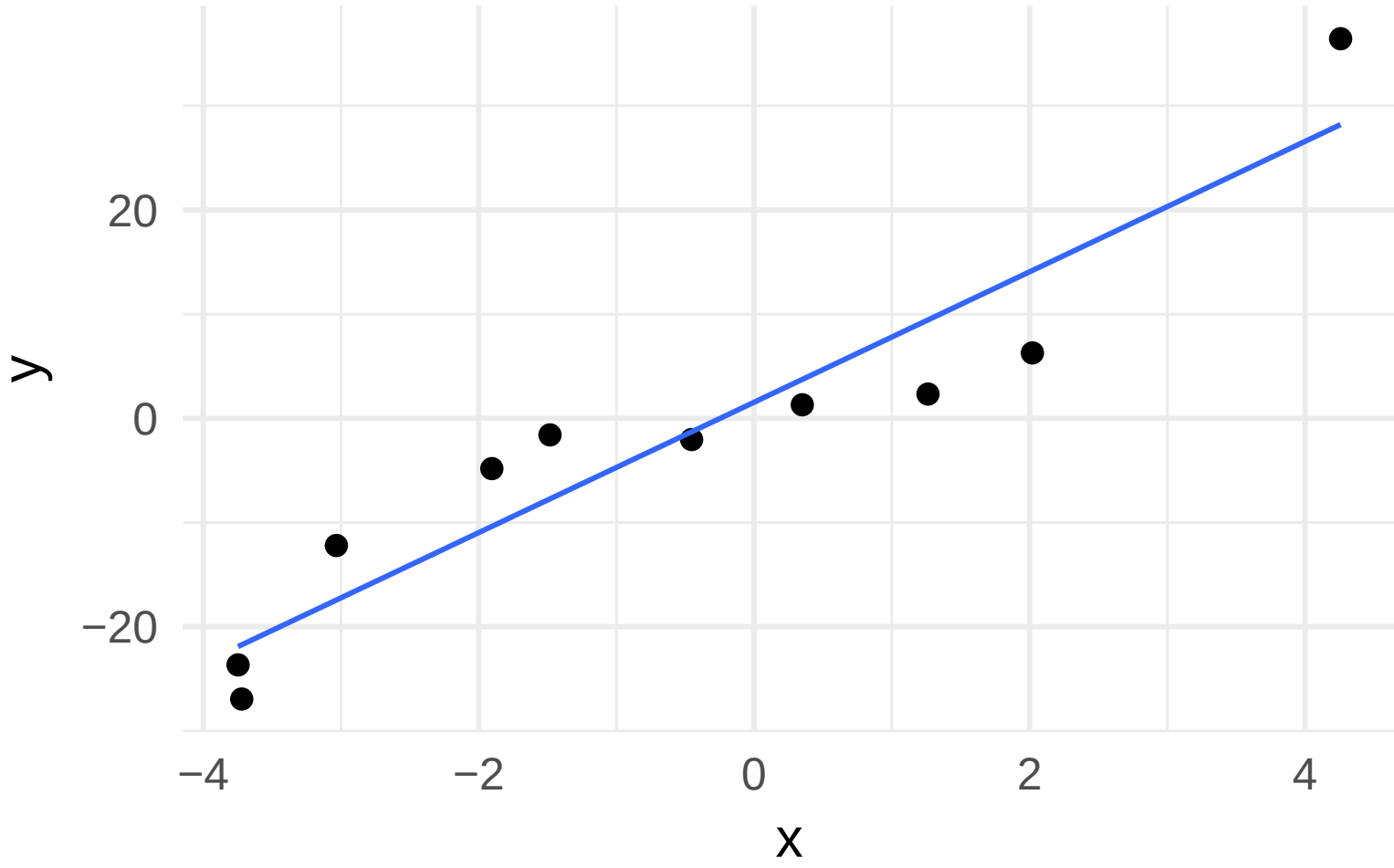
The problem with parameters: using information theory to evaluate models

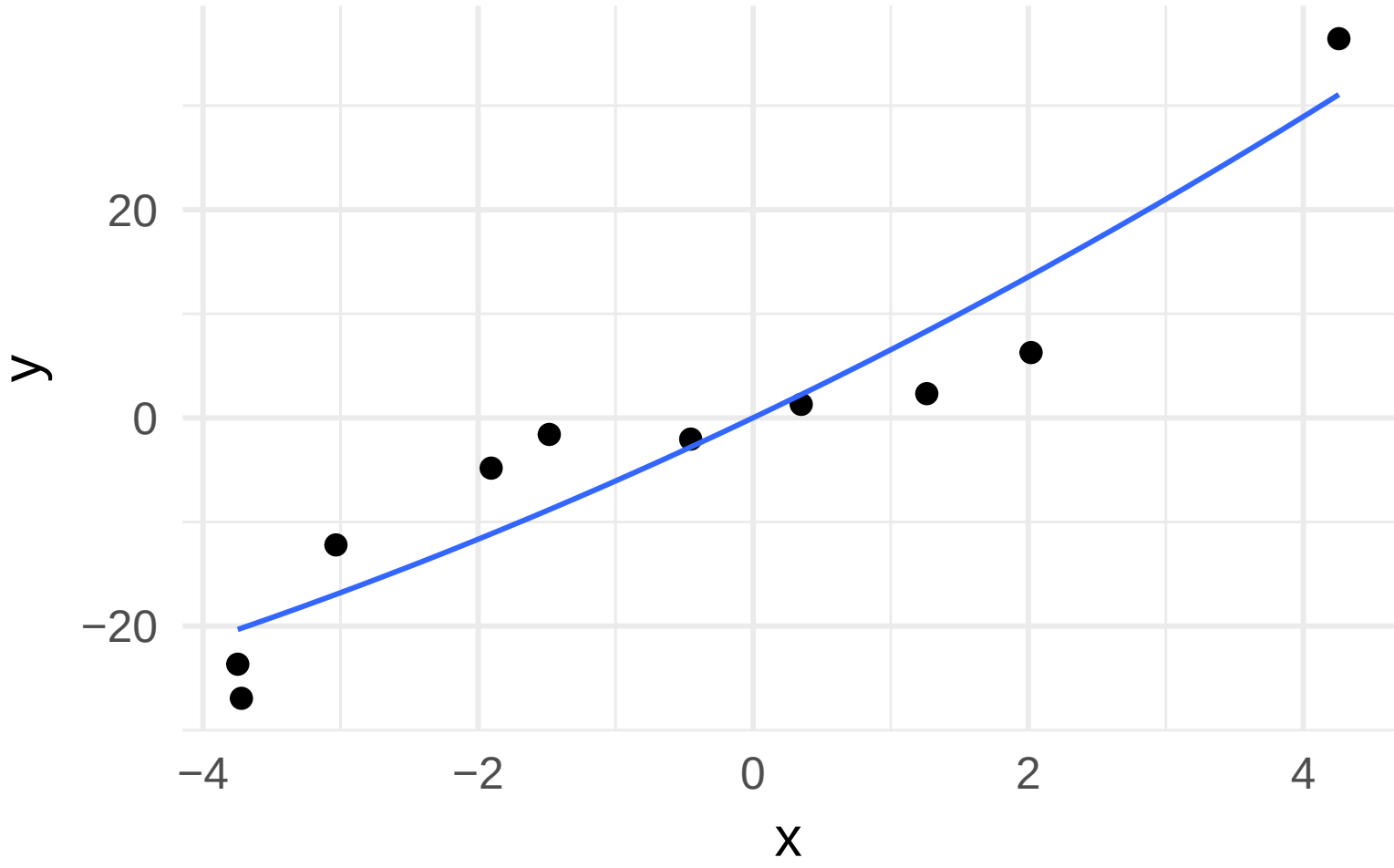
Max Joseph

March 16, 2017

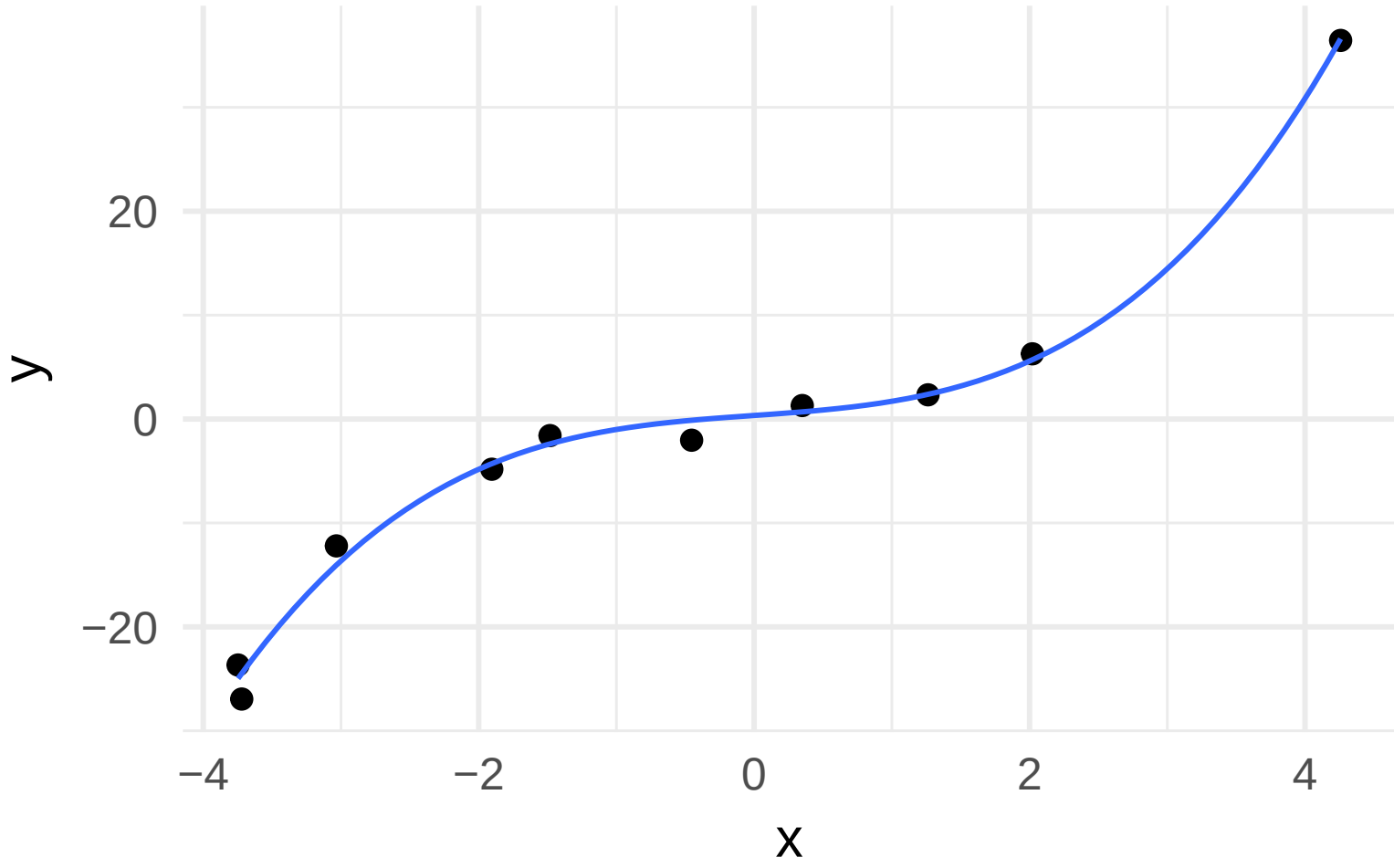
$$\hat{y} = \beta_0 + \beta_1 x$$



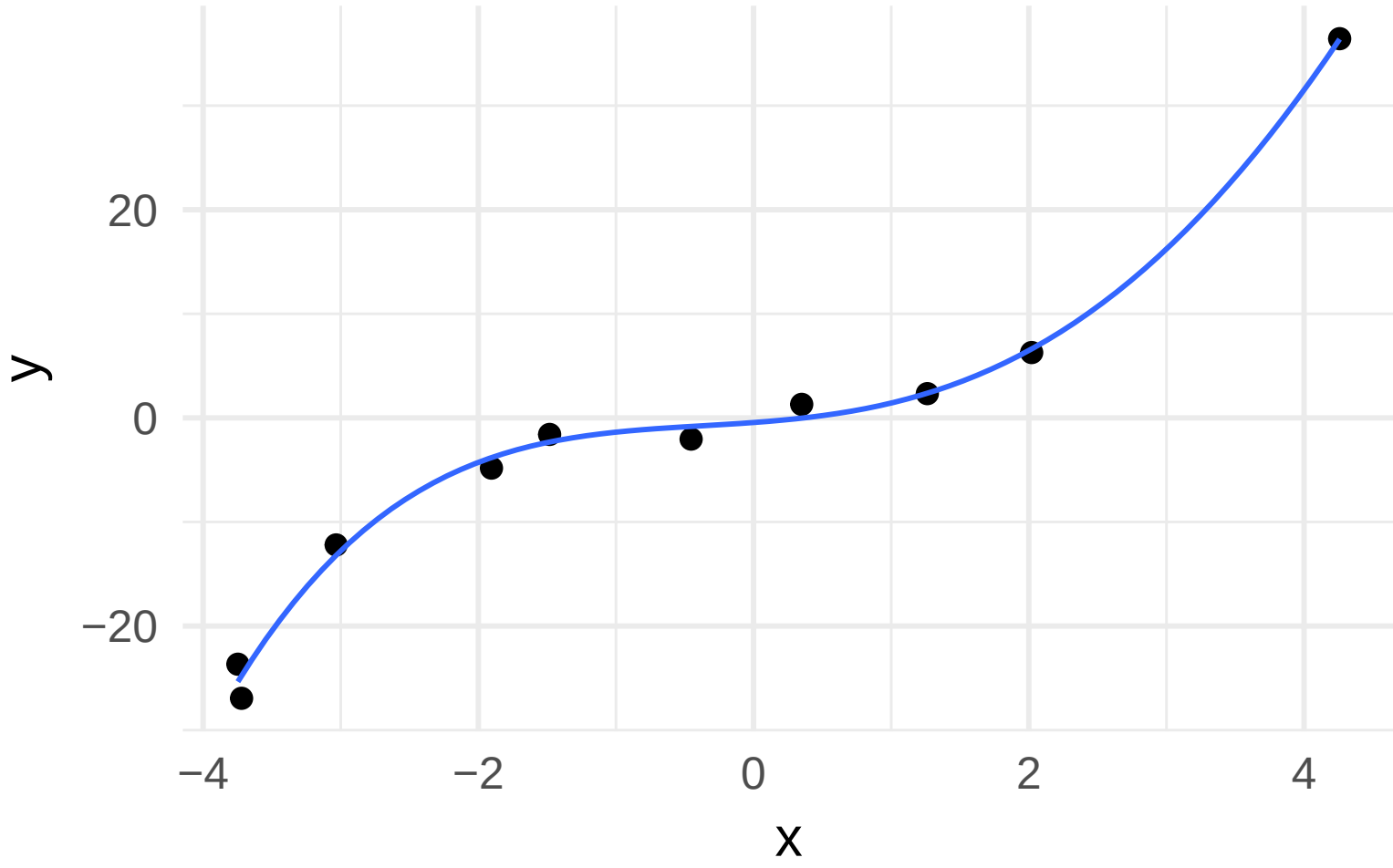
$$\hat{y} = \beta_0 + \beta_1 x + \beta_2 x^2$$



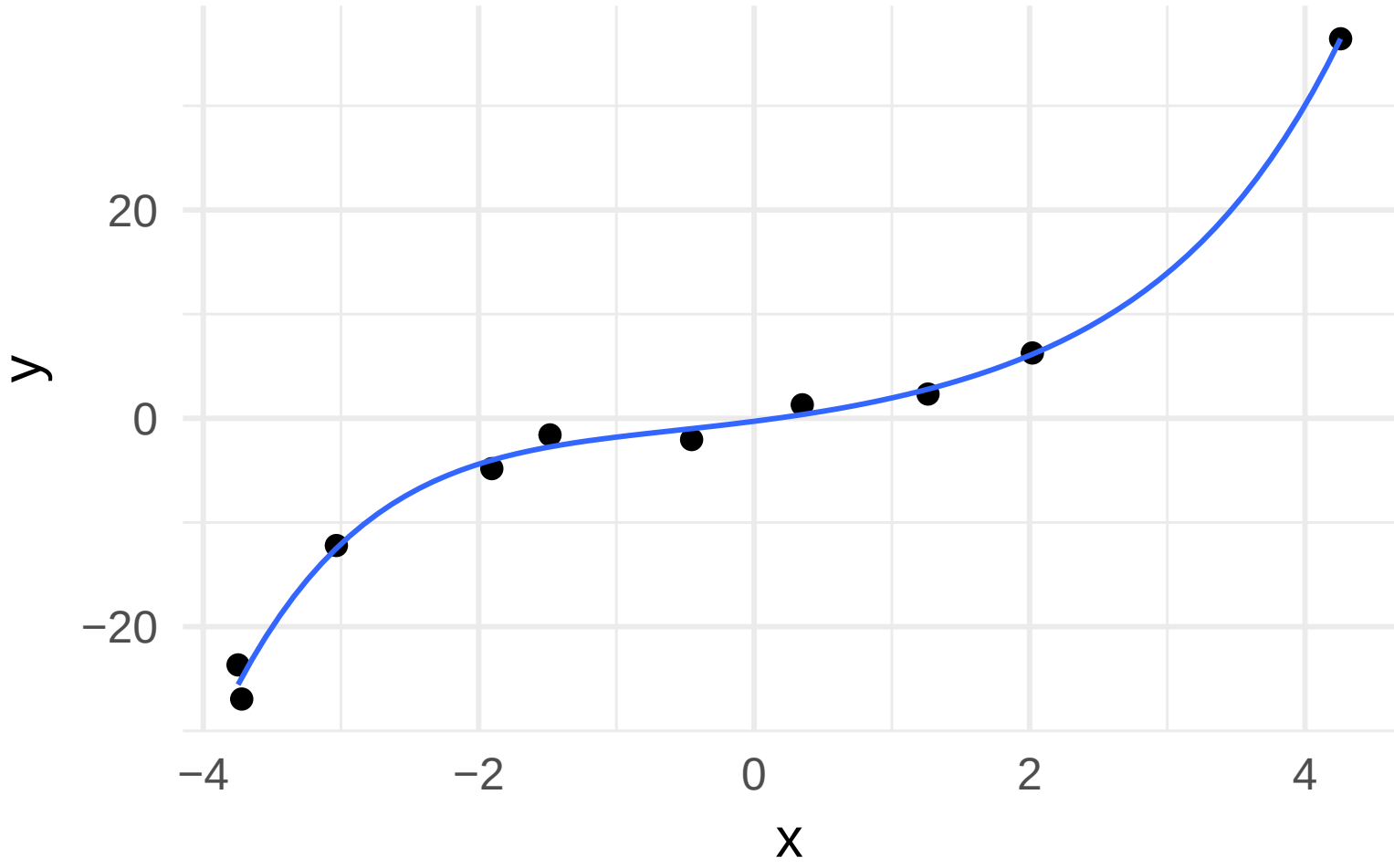
$$\hat{y} = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3$$



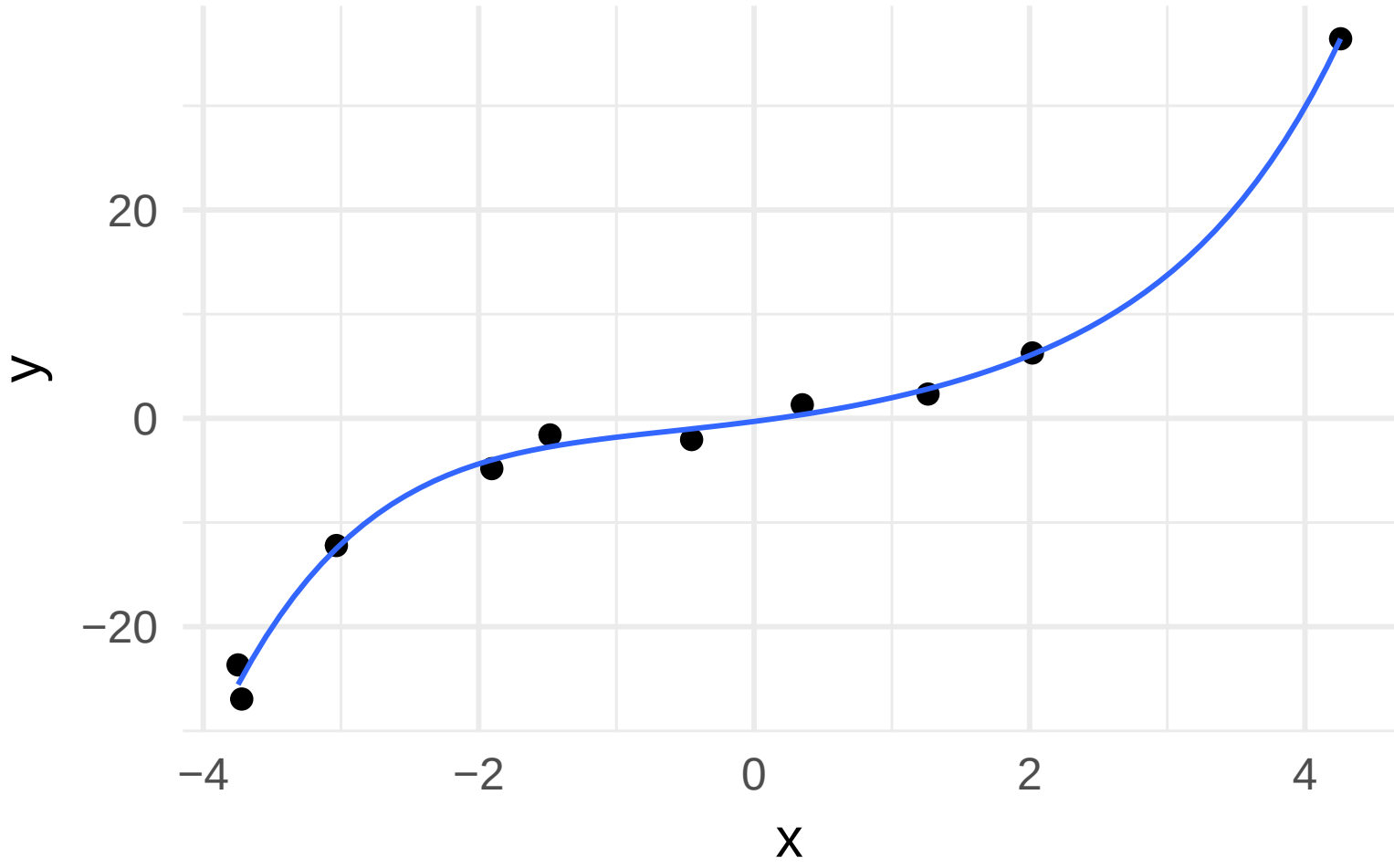
$$\hat{y} = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 x^4$$



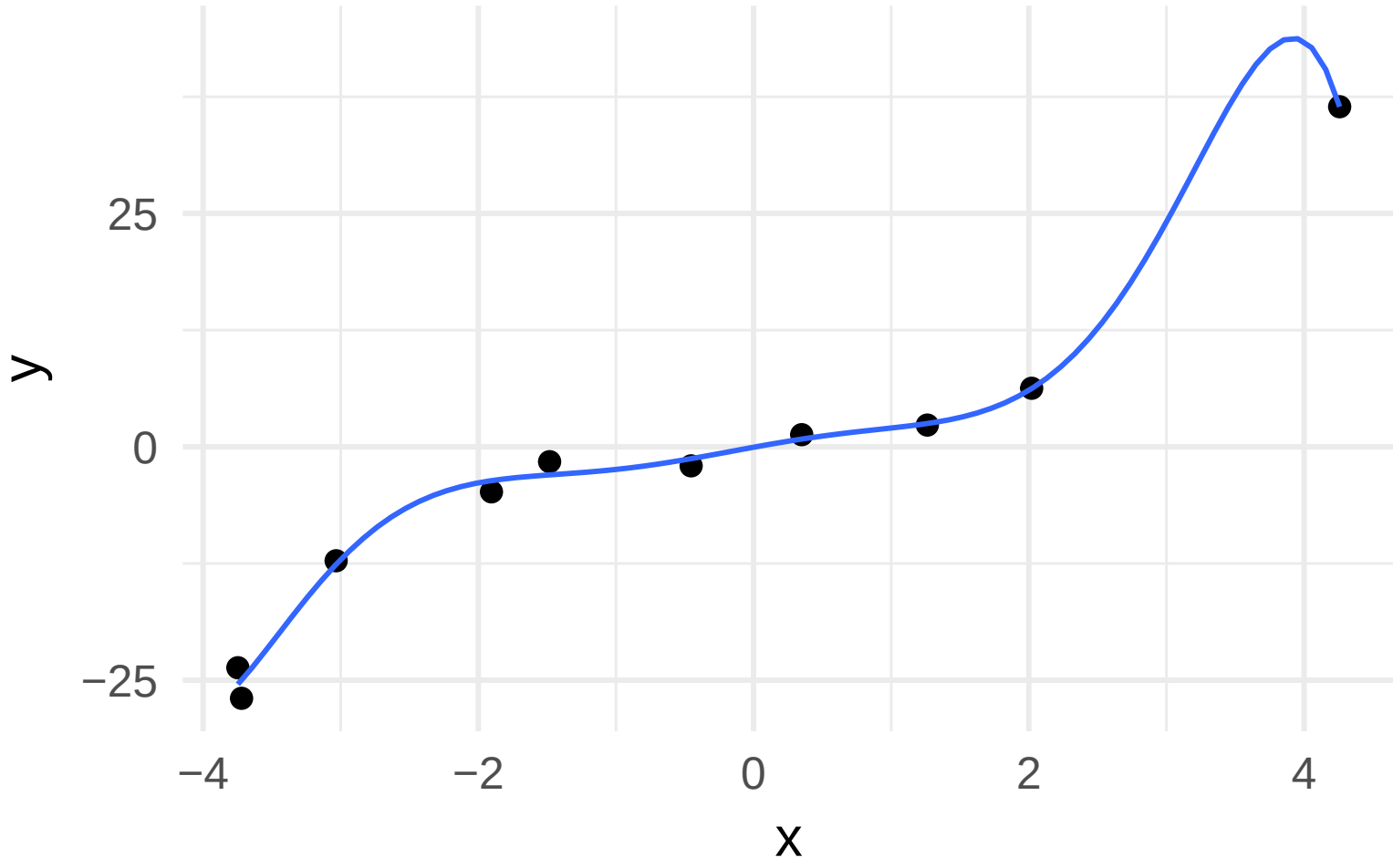
$$\hat{y} = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 x^4 + \beta_5 x^5$$



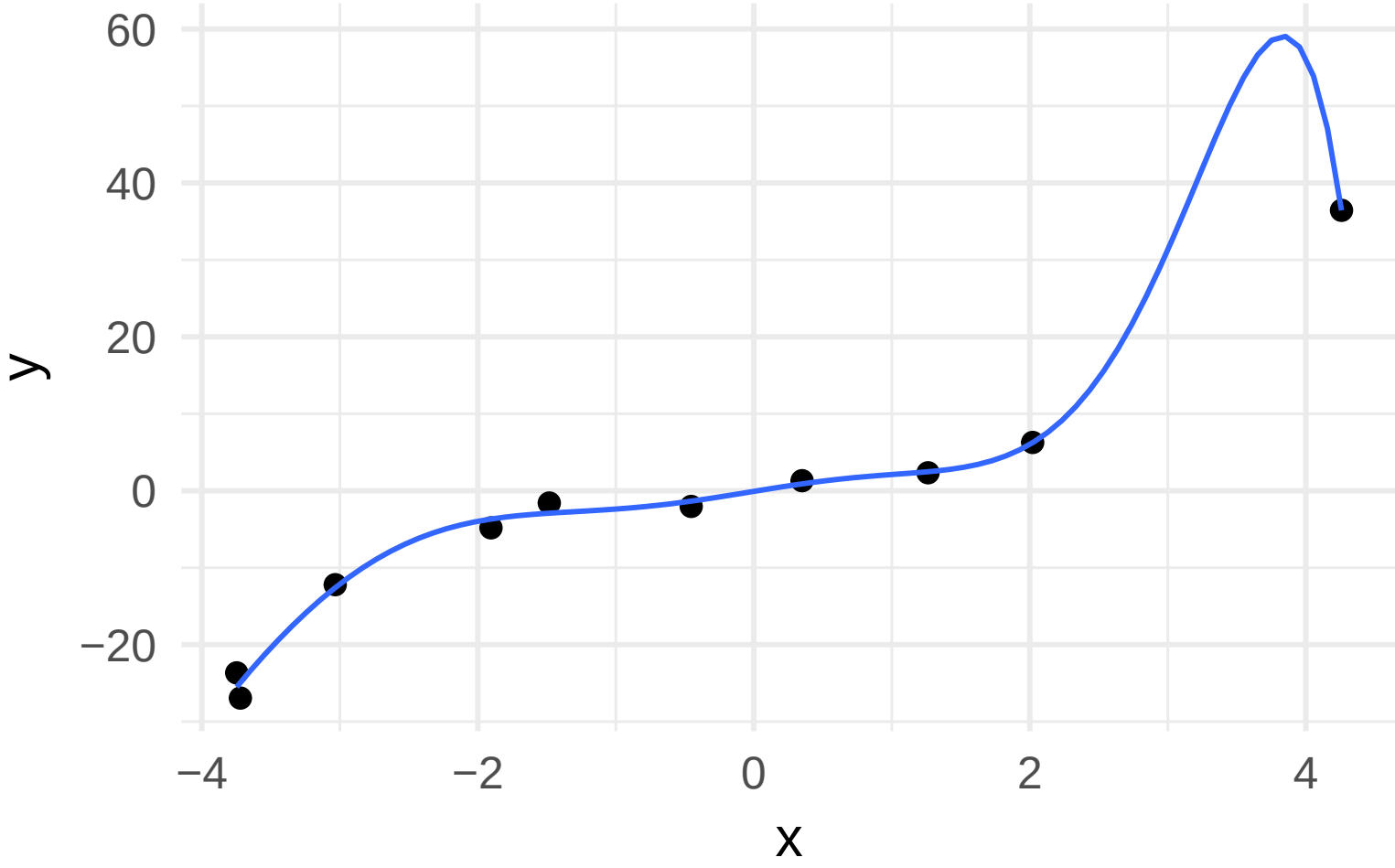
$$\hat{y} = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 x^4 + \beta_5 x^5 + \beta_6 x^6$$



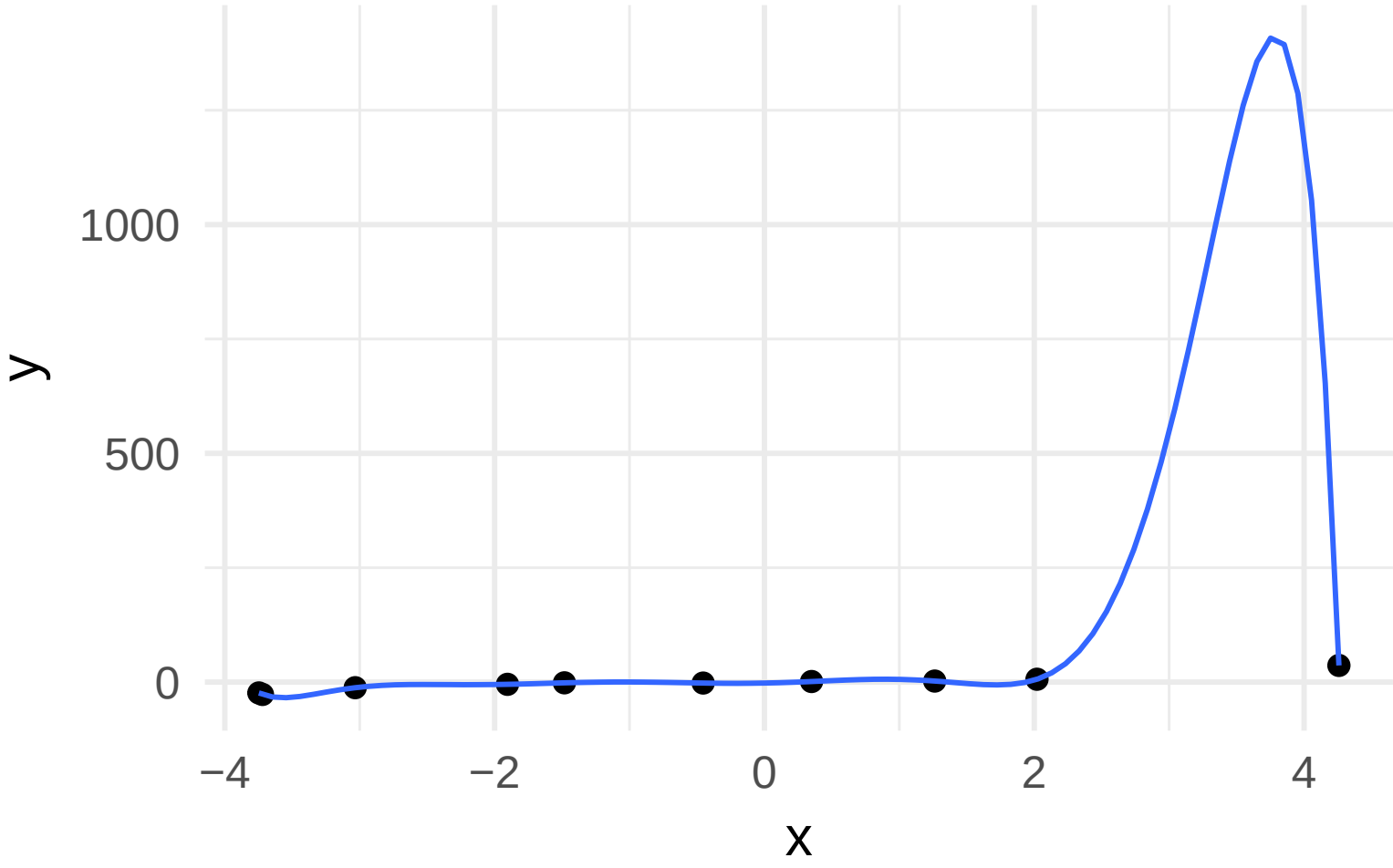
$$\hat{y} = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 x^4 + \beta_5 x^5 + \beta_6 x^6 + \beta_7 x^7$$



$$\hat{y} = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 x^4 + \beta_5 x^5 + \beta_6 x^6 + \beta_7 x^7 + \beta_8 x^8$$



$$\hat{y} = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 x^4 + \beta_5 x^5 + \beta_6 x^6 + \beta_7 x^7 + \beta_8 x^8 + \beta_9 x^9$$

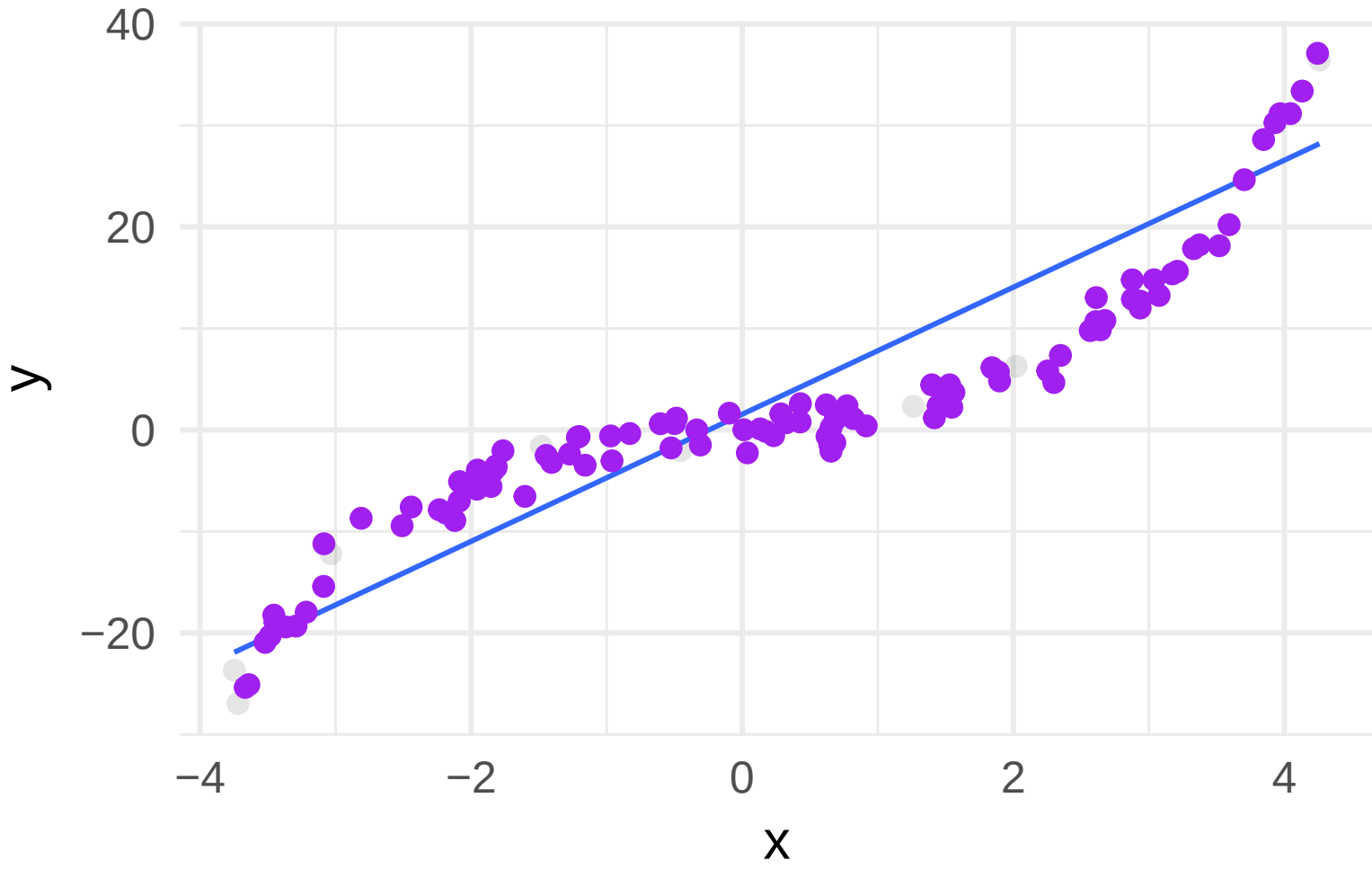


Overfitting

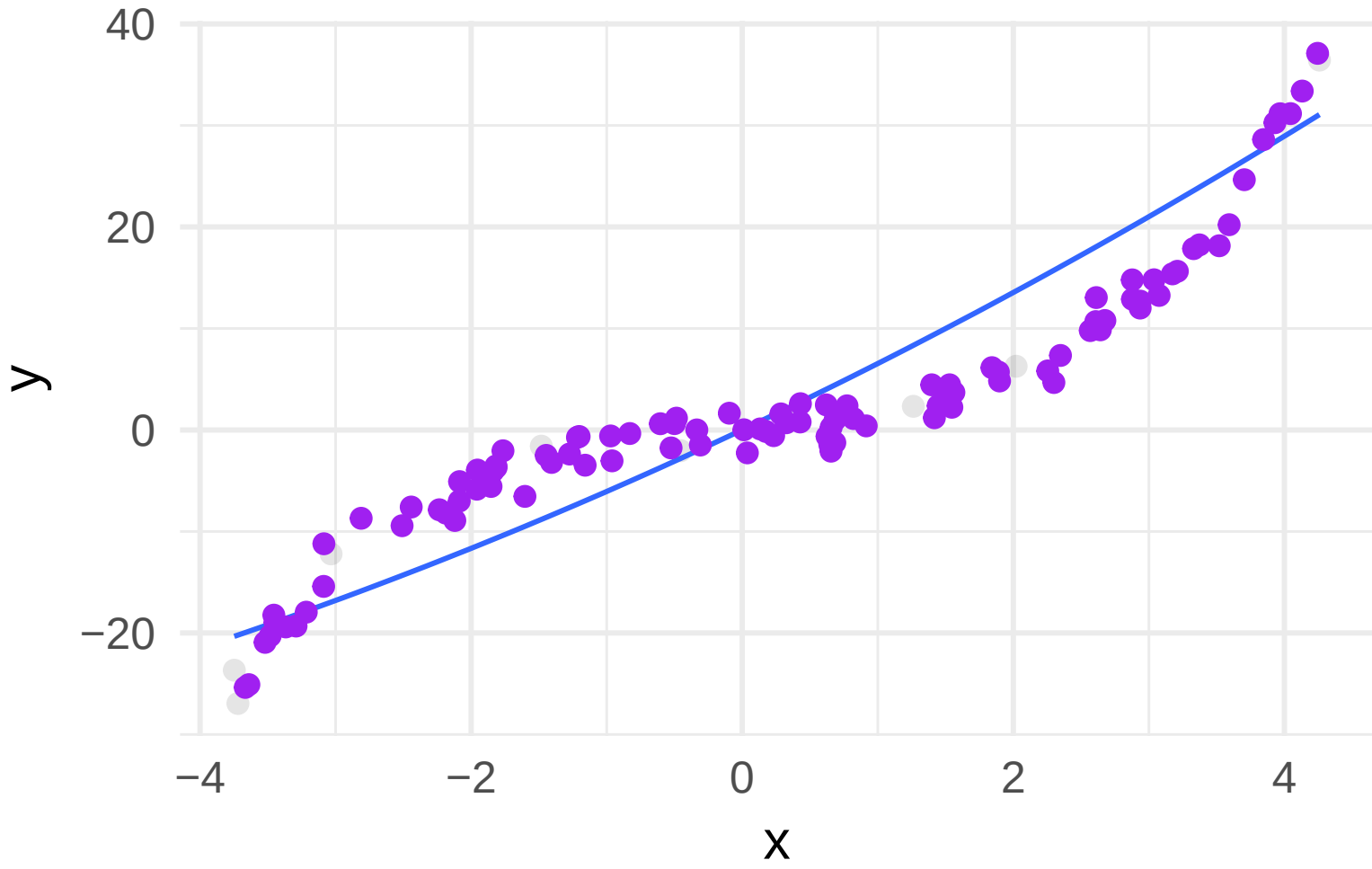
Parameters begin to fit to **noise**

- good fit to **training data**
- bad predictions for out of sample data

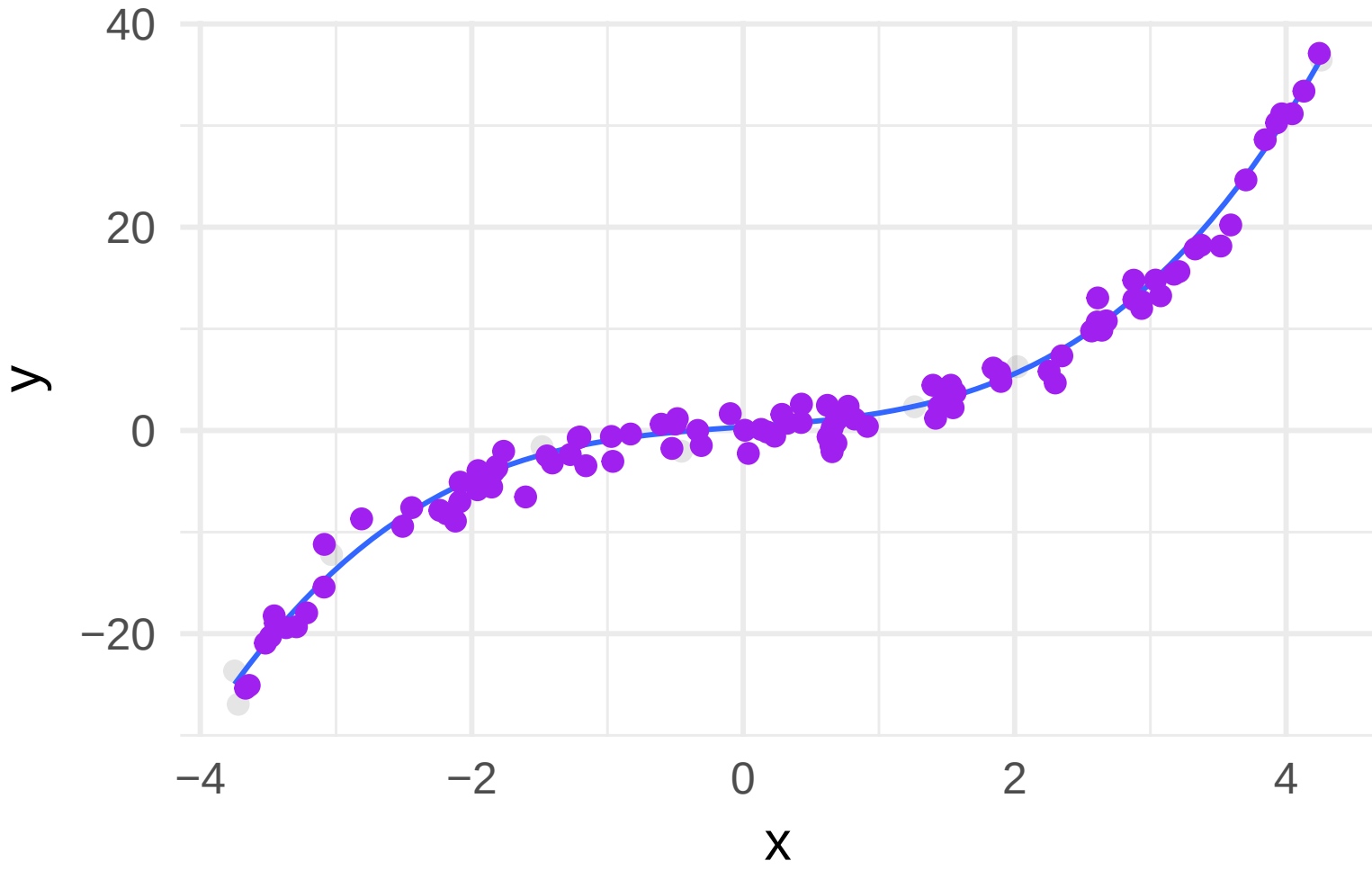
$$\hat{y} = \beta_0 + \beta_1 x$$



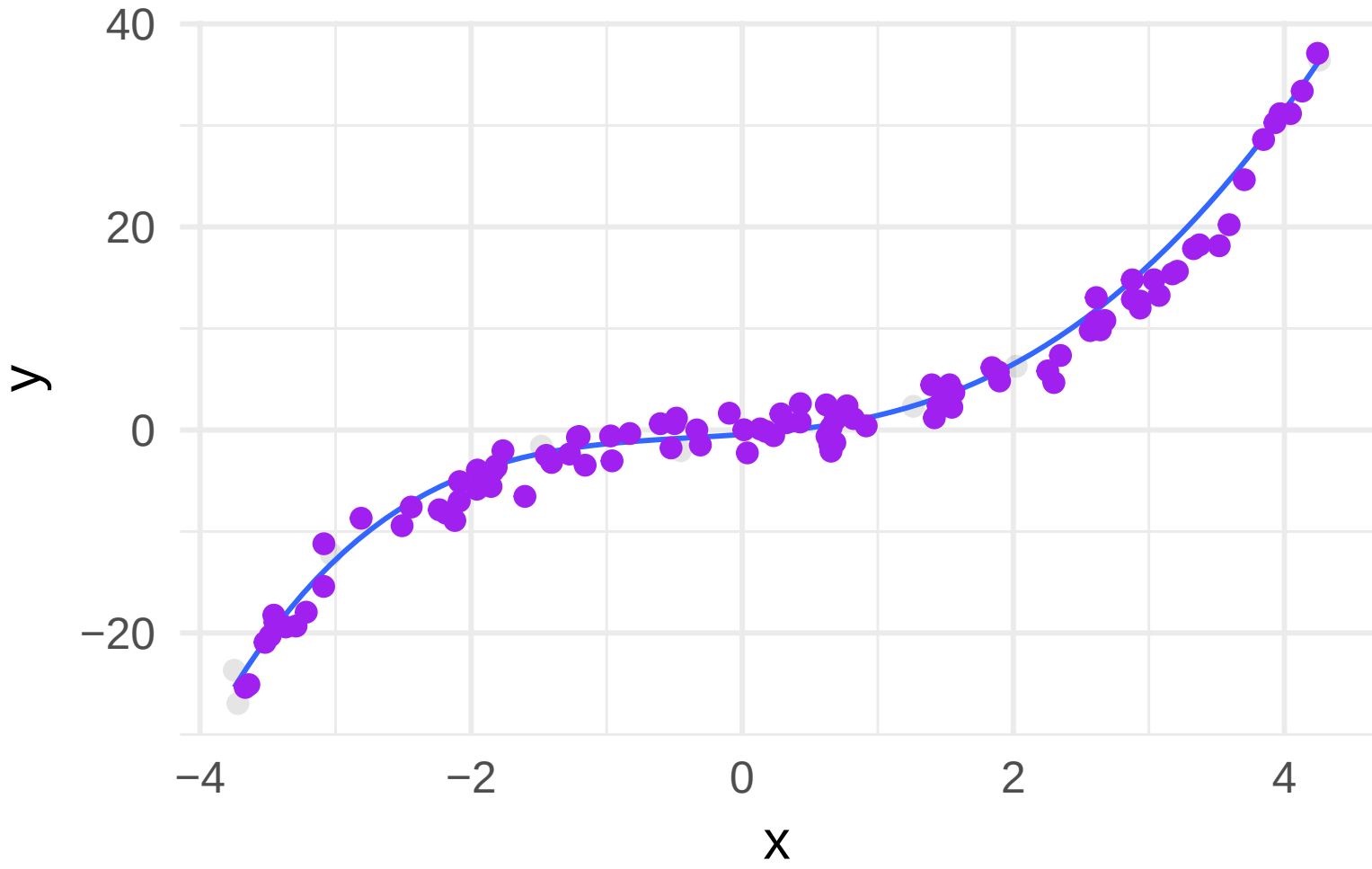
$$\hat{y} = \beta_0 + \beta_1 x + \beta_2 x^2$$



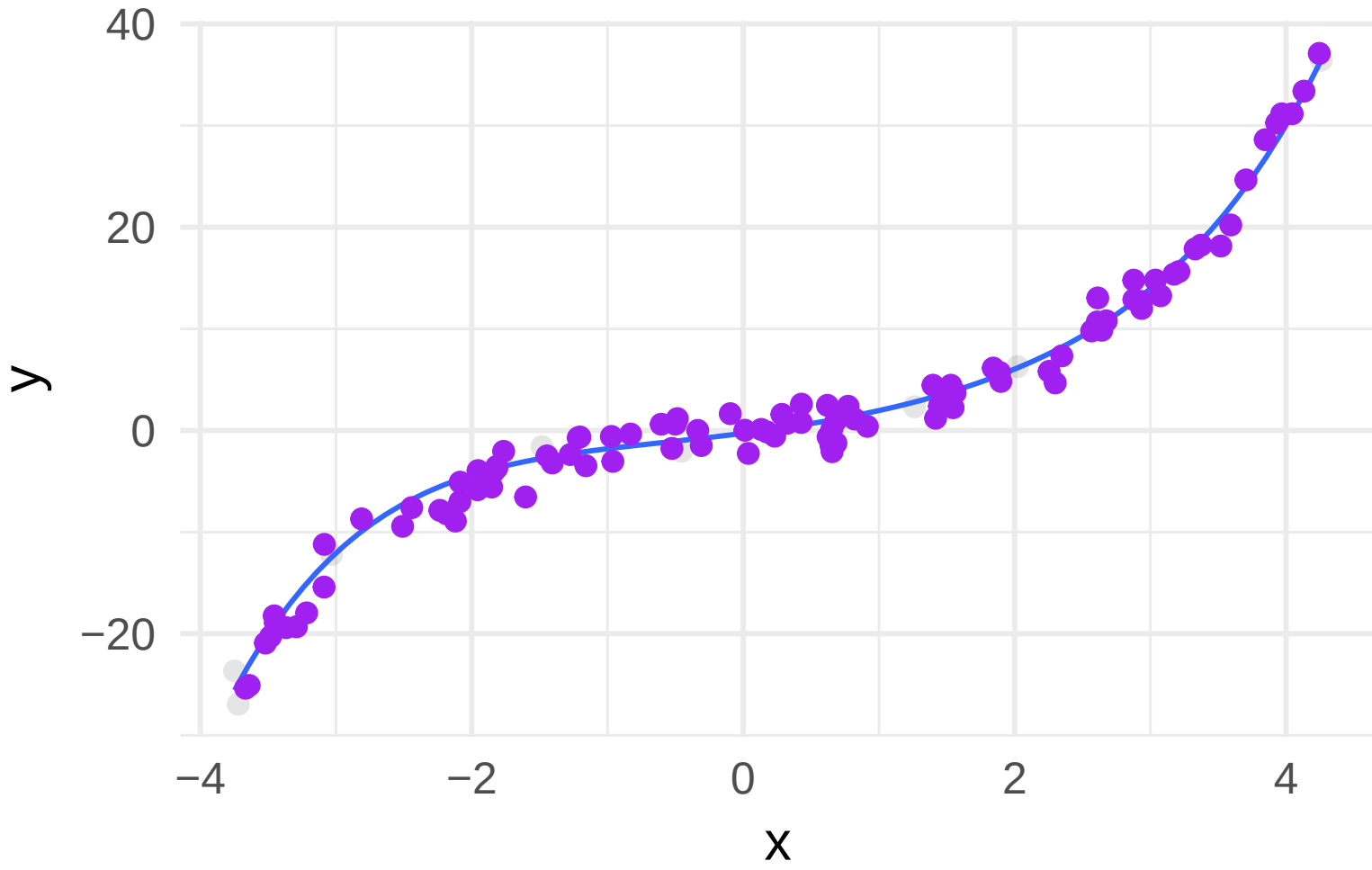
$$\hat{y} = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3$$



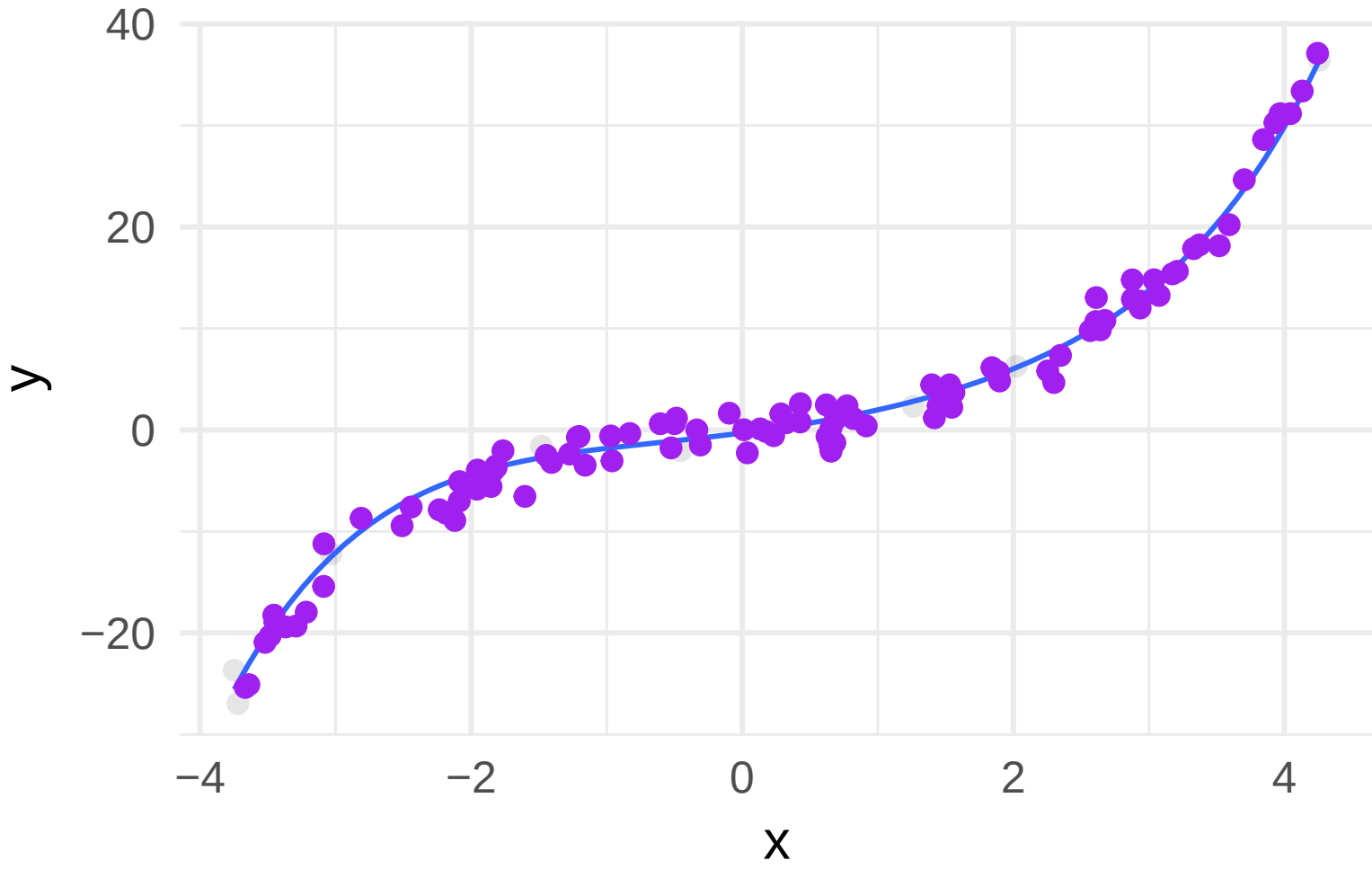
$$\hat{y} = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 x^4$$



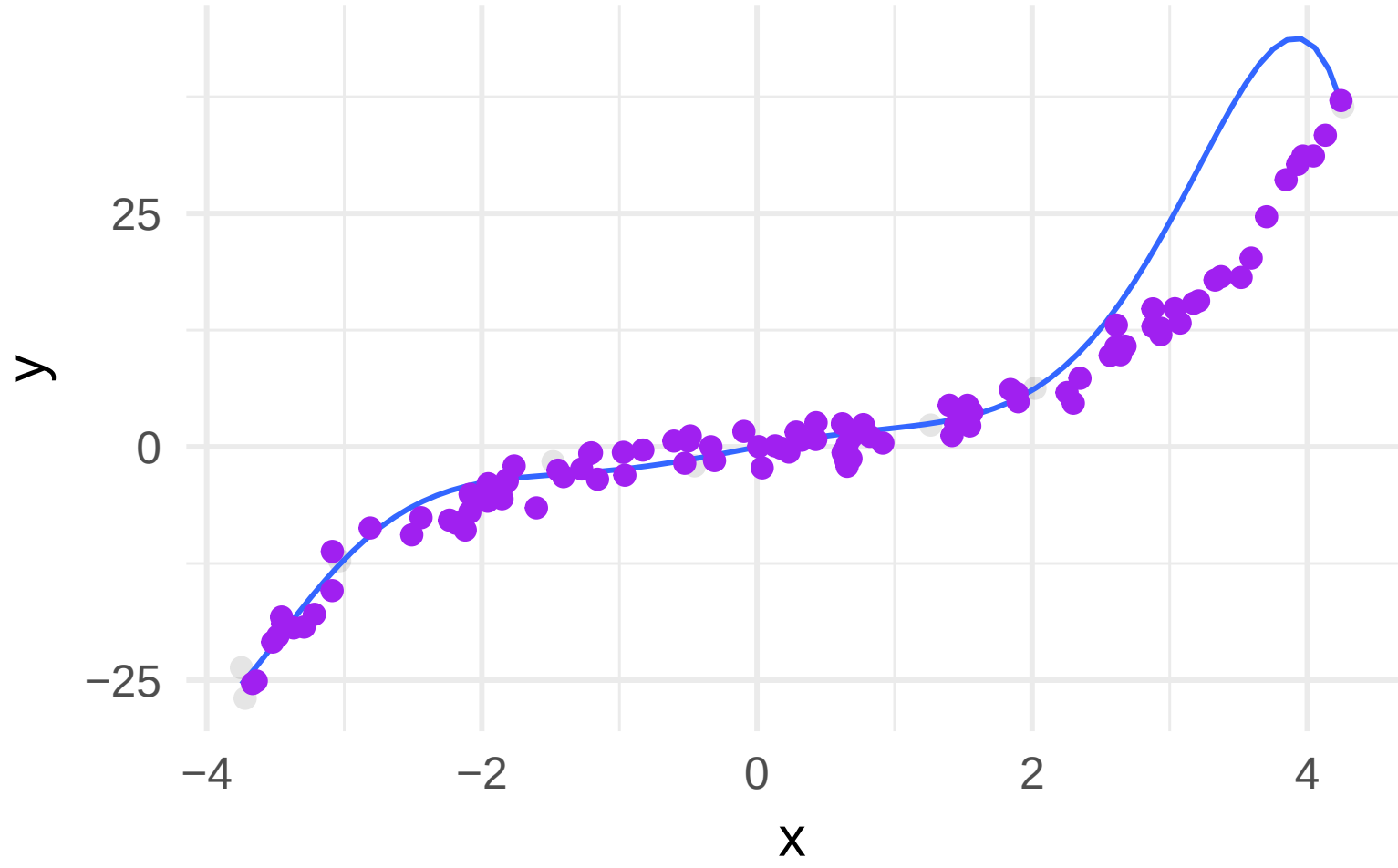
$$\hat{y} = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 x^4 + \beta_5 x^5$$



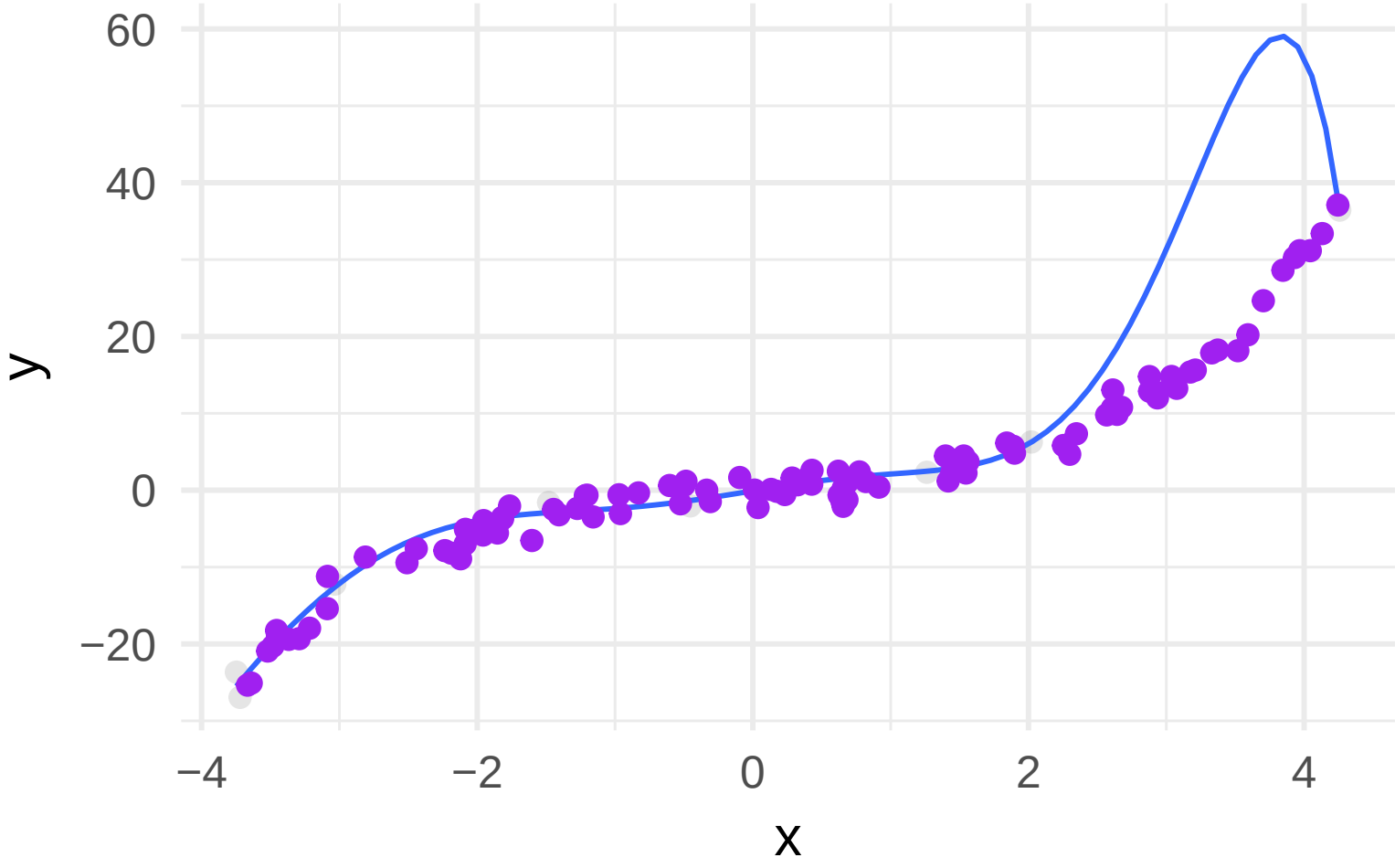
$$\hat{y} = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 x^4 + \beta_5 x^5 + \beta_6 x^6$$



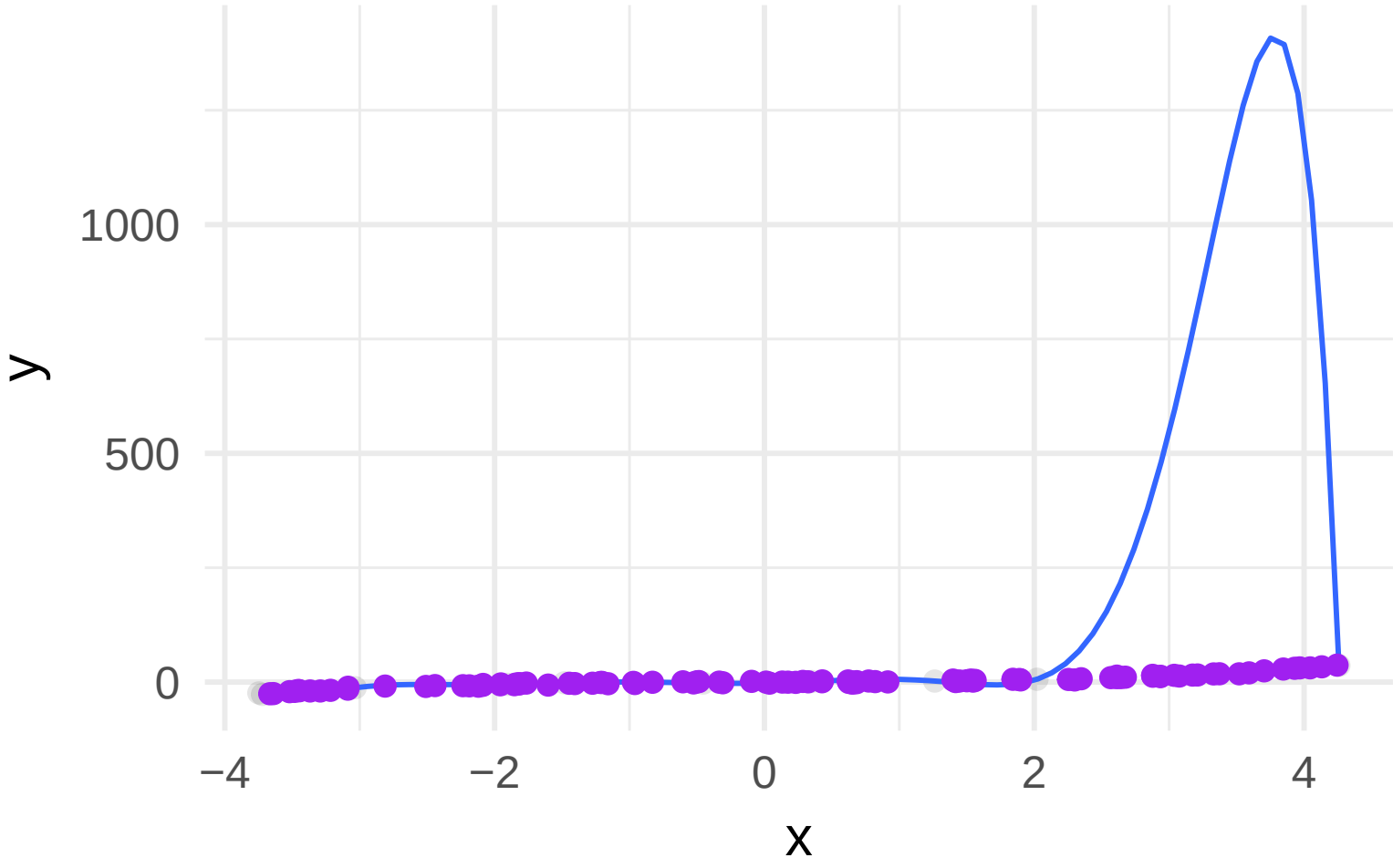
$$\hat{y} = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 x^4 + \beta_5 x^5 + \beta_6 x^6 + \beta_7 x^7$$



$$\hat{y} = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 x^4 + \beta_5 x^5 + \beta_6 x^6 + \beta_7 x^7 + \beta_8 x^8$$

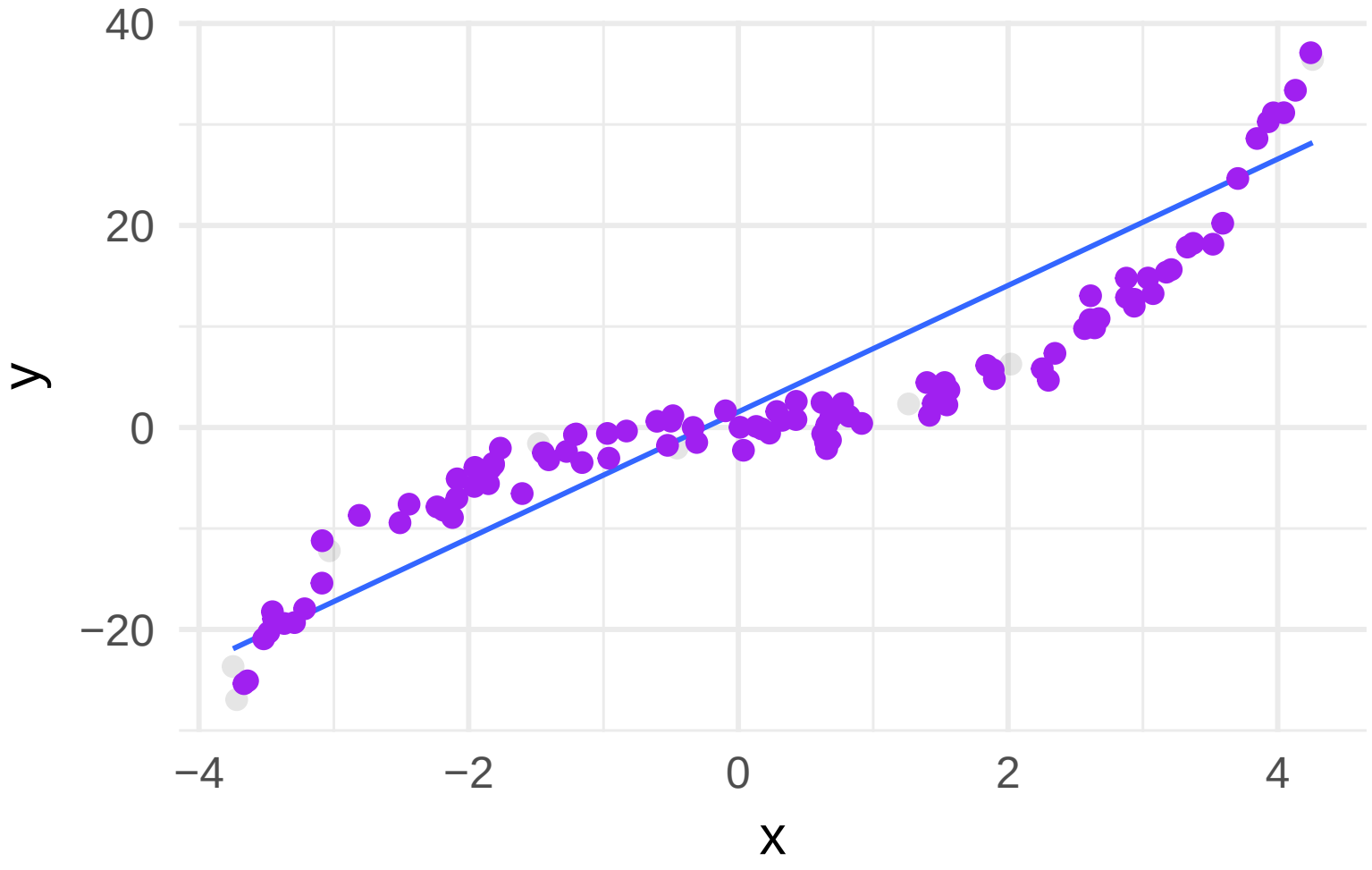


$$\hat{y} = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 x^4 + \beta_5 x^5 + \beta_6 x^6 + \beta_7 x^7 + \beta_8 x^8 + \beta_9 x^9$$



Underfitting

Model is too simplistic to capture signal



Today

Working toward **information criteria** to balance:

- model complexity
- out of sample predictive power

Roadmap

1. Information entropy
2. Kullback-Leiber divergence
3. Deviance
4. Akaike's information criterion

Information entropy

Uncertainty contained in a probability distribution

$$H(p) = -\sum_{i=1}^{n} p_i \log(p_i)$$

- $H(p)$: information entropy of a distribution p
- n: the number of possible outcomes
- p_i: the probability of outcome i

Activity

Compute the information entropy for your die!

$$H(p) = -\sum_{i=1}^{n} p_i \log(p_i)$$

What if we didn't know anything about dice?

Find an estimate of information entropy:

1. Estimate $p_1, p_2, p_3, ...$
2. Compute information entropy of your estimated distribution:

$$H(\hat{p}) = -\sum_{i=1}^{n} \hat{p}_i \log(\hat{p}_i)$$

Divergence

How far off is our model from the true distribution?

Example

We used $\hat{p}$ to estimate p

- What's the "divergence" between $\hat{p}$ and p?

Kullback-Leibler divergence

How far off is our model q from the true distribution p?

$$D_{\mathrm{KL}} = \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right)$$

Activity

Calculate KL divergence for the following sample sizes:

- 5
- 10
- 20
- 1000

$$D_{\mathrm{KL}} = \sum_{i=1}^{n} p_i(\log(p_i) - \log(q_i))$$

- Average difference in log probability between p and q

Bonus

What happens when our approximation q is exactly the same as p?

The problem with reality

We almost never know the true probability of events!

What *do* we have

Typically we have data $y_1, y_2, ..., y_n$

and some models (let's say two)

$$q, r$$

So we can ask

Which model seems closer to the true distribution p?

$$D_{\mathrm{KL}}(p, q) - D_{\mathrm{KL}}(p, r) = -(E \log(q_i) - E \log(r_i))$$

Deviance

We want to compare divergence of two models q and r:

$$D_{\mathrm{KL}}(p, q) - D_{\mathrm{KL}}(p, r) = -(E \log(q_i) - E \log(r_i))$$

In practice we use the Deviance

$$E \log(q_i) \propto$$

$$D(q) = -2 \sum_{i=1}^{n} log(q_i)$$

where $D(q)$ is the **Deviance**