Interior-Point Algorithms

- barrier function
- primal barrier method
- perturbed optimality conditions
- Newton's method
- primal-dual method for LPs

Interior-point algorithms

The Simplex algorithm:

- "walks" the edges of the polyhedral feasible set
- worst-case complexity is exponential (may need to visit **every** vertex)
- experience (and some analysis) suggests average polynomial complexity

Interior-point (IP) are a radical departure from the simplex method:

- IP algorithms traverse the interior of the polyhedral set
- (impractical) polynomial algorithm for LP first proposed by Kachian (1979)
- Karmarkar (1984) offered first "practical" polynomial LP algorithm
  - AT&T wouldn't release details
  - patented the KORBYX, a computer that implemented the method
  - appeared in front-page of New York Times

Eliminate nonnegativity constraints

Apply to the primal LP problem in standard form:

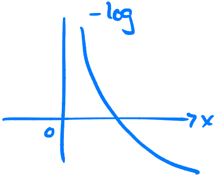
$$\underset{x}{\text{minimize}} \quad c^T x \quad \text{subject to} \quad Ax = b,\ x \geq 0$$

The core difficulty in LP is the prescence of the constraint $x \geq 0$

Eliminate nonnegativity constraint via **barrier function**:

$$B_\mu(x) = c^T x - \mu \sum_j \log x_j$$

- $-\log x_j \to \infty$ as $x_j \to 0^+$ (def'd as $+\infty$ for $x_j \leq 0$)
- $-\mu \sum_j \log x_j \to \infty$ as any $x_j \to 0^+$



Barrier function

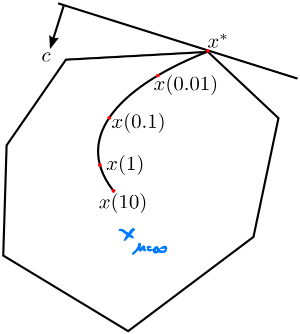
(P_μ) $\quad\quad \underset{x}{\text{minimize}} \quad B_\mu(x) \quad \text{subject to} \quad Ax = b$ $\quad \boxed{x \geq 0}$

- minimizer of the barrier problem depends on μ:

$B_\mu(x) = c^T x - \mu \sum_j \log x_j$

$$x_\mu \quad \text{solves} \quad P_\mu$$

- minimizer of P_μ is unique for each μ because of convexity of B_μ



"Analytic Center"

$x_\infty = \underset{Ax=b}{\operatorname{argmin}} - \sum_j \log x_j$

Example 1: $\underset{x}{\text{minimize}} \quad x \quad \text{subj to} \quad x \geq 0$ $\quad x \in \mathbb{R}$

$$B_\mu(x) = x - \mu \log x \quad \Longrightarrow \quad x(\mu) = \mu$$

$$\frac{dB_\mu(x)}{dx} = 1 - \frac{\mu}{x} = 0 \Rightarrow \boxed{x_\mu = \mu}$$

$$\lim_{\mu \searrow 0^+} x_\mu = 0.$$

$$\begin{bmatrix}0\\1\\0\end{bmatrix}^T\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_2$$

Example 2: $\underset{x_1,x_2,x_3}{\text{minimize}} \quad x_2 \quad \text{subj to} \quad x_1 + x_2 + x_3 = 1, \; x \geq 0$

$$\min \quad B_\mu(x) = x_2 - \mu\log(x_1) - \mu\log(x_2) - \mu\log(x_3)$$

$B^1_\mu(x_1, x_2, x_3)$

st $x_1 + x_2 + x_3 = 1$

$\Updownarrow$

Eliminate $x_3 = 1 - x_1 - x_2$:

$$\underset{x_1,x_2}{\text{minimize}} \quad x_2 - \mu\log(x_1) - \mu\log(x_2) - \mu\log(1 - x_1 - x_2) = B^2_\mu(x_1, x_2)$$

$$\frac{dB^2_\mu(x)}{dx_1} = -\frac{\mu}{x_1} + \frac{\mu}{1-x_1-x_2} = 0$$

$$\frac{dB^2_\mu(x)}{dx_2} = 1 - \frac{\mu}{x_2} + \frac{\mu}{1-x_1-x_2} = 0$$

solve for x_1 and x_2 in terms of μ

- $x_1(\mu) = \frac{1-x_2(\mu)}{2} \to \frac{1}{2}$
- $x_2(\mu) = \frac{1+2\mu-\sqrt{1+9\mu^2+2\mu}}{2} \to 0$
- $x_3(\mu) = \frac{1-x_2(\mu)}{2} \to \frac{1}{2}$

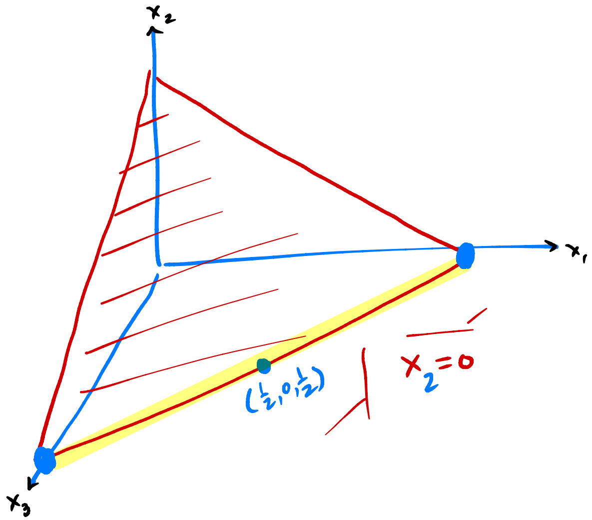
$$\lim_{\mu \searrow 0^+} x_2(\mu) = \frac{1-\sqrt{1}}{2} = 0$$

This problem has infinitely many solutions:

$$X^* = \{\, x \mid x = (x_1, 0, x_3), \; x_1 + x_3 = 1, x \geq 0 \,\}$$

Note that the solution that we converge to isn't basic.

$\min\ B_\mu(x) \quad st \quad Ax=b$



Primal barrier method

$\min c^T x$ st $Ax = b, x \geq 0 \Rightarrow \min c^T x - \mu \sum \log x_j$ st $Ax = b$

solve a sequence of linearly constrained **nonlinear** functions:

barrier parameter

> choose $x_0 > 0$, $\mu_0 > 0 (\approx 1)$, $\tau < 1$
> **repeat**
>
> $\quad x_{k+1}$ minimizes $B_{\mu_k}(x)$ subj to $Ax = b$
> $\quad \mu_{k+1} \leftarrow \tau \mu_k$
>
> **until** μ_k is "small"

$x_{k+1} = x_{\mu_k}$

under mild conditions, $x_k \to x^*$

(P) $\min_x c^T x$ st $Ax = b, x \geq 0$

(D) $\max_{y,z} b^T y$ st $A^T y + z = c, z \geq 0$

$$\min \quad g(y,z) = -b^T y - \mu \sum_i \log z_j$$
$$\text{st} \quad \begin{bmatrix} A^T & I \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = c$$

$$\begin{bmatrix} \nabla_y g(y,z) \\ \nabla_z g(y,z) \end{bmatrix} = \begin{bmatrix} A \\ I \end{bmatrix} w \quad \& \quad A^T y + z = c$$

$$\begin{bmatrix} -b \\ -\mu Z^{-1} e \end{bmatrix} = \begin{bmatrix} A \\ I \end{bmatrix} w \iff \begin{array}{l} Aw = -b \\ -\mu Z^{-1} e = w \end{array}$$

Perturbed optimality conditions

primal LP: $-\frac{\mu}{x_1}, -\frac{\mu}{x_2}, \ldots, -\frac{\mu}{x_n}$

minimize $\underbrace{c^T x - \mu \sum_j \log x_j}_{f(x)}$

subject to $Ax = b$

$\nabla f(x) = c - \mu \begin{bmatrix} 1/x_1 \\ \vdots \\ 1/x_n \end{bmatrix} = c - \mu X^{-1} e, \quad X = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}$

Optim cond's:

$$c + \mu X^{-1} e = A^T y$$
$$Ax = b \ (x > 0)$$

min $f(x)$ st $Ax = b$ | $\nabla f(x) = A^T y$ for some y, $Ax = b$

dual LP:

maximize $\overbrace{b^T y + \mu \sum_j \log z_j}^{-g(y,z)}$

subject to $A^T y + z = c$

Optim cond's:

$$Aw = -b$$
$$-\mu Z^{-1} e = w$$
$$A^T y + z = c \ (z > 0)$$

Tie these optimality cond's together by identifying $x \equiv -w$ and noting

$$\mu Z^{-1} e = x \quad \Longleftrightarrow \quad \mu \frac{1}{z_j} = x_j \quad \Longleftrightarrow \quad x_j z_j = \mu$$

write both optimality conditions simultaneously as $\overset{n}{(x}, \overset{m}{y}, \overset{n}{z}) = 2n+m$ vars

$A^T y + z = c \quad (z \geq 0)$

$Ax = b \quad (x \geq 0)$

$x_j \cdot z_j = 0 \quad j = 1 \ldots n$

Nec & Suff Opt Conds for Pr & Du LPs

$$\boxed{\begin{aligned} A^T y + z &= c, \\ Ax &= b, \\ x_j z_j &= \mu, \end{aligned}} \quad \begin{aligned} &z > 0 \quad n \\ &x > 0 \quad m \\ &j = 1, \ldots, n \end{aligned}$$

$2n+m$ eqn's

$X = \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{bmatrix}$ $Z = \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix}$

$x_j \cdot z_j = Xz$

Newton's method

$$F_\mu(x, y, z) = \begin{bmatrix} Ax - b \\ A^T y + z - c \\ Xz - \mu e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

An approximate LP solution (x, y, z), with $(x, z) > 0$ satisfies

$$F_\mu(x, y, z) = 0$$

Apply Newton's method for root finding to these equations, eg,

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} + \alpha \begin{bmatrix} p_x \\ p_y \\ p_y \end{bmatrix}$$

where p is a Newton step: $\nabla F_\mu(x,y,z)\, p = -F_\mu(x,y,z)$

$$J_k p = -F_k \quad \Longleftrightarrow \quad \begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ Z_k & 0 & X_k \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} b - Ax_k \\ c - A^T y_k - z_k \\ \mu e - X_k z_k \end{bmatrix}$$

Primal-dual method for LPs

choose $x_0 > 0$, y_0, $z_0 > 0$, $\tau < 1$
$\gamma_0 \leftarrow x_0^T z_0$, $k \leftarrow 0$
while $\gamma_k > \epsilon$ **do**

$\mu_k = \tau(x_k^T z_k)/n$

Solve $J_k p = -F_k$ for $p = (p^x, p^y, p^z)$ `[Newton step]`

$$\beta_k^x = \min\left\{1,\ .995 \min_{\{j|p_j^x<0\}} -\frac{x_j^k}{p_j^x}\right\}$$
$$\beta_k^z = \min\left\{1,\ .995 \min_{\{j|p_j^z<0\}} -\frac{z_j^k}{p_j^z}\right\}$$

$x_{k+1} \leftarrow x_k + \beta_k^x p^x$
$y_{k+1} \leftarrow y_k + \beta_k^z p^y$
$z_{k+1} \leftarrow z_k + \beta_k^z p^z$

$k \leftarrow k + 1$
end

Linear Algebra

The main work is in computing the step directions (p_x, p_y, p_z) via

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ Z_k & 0 & X_k \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} b - Ax_k \\ c - A^T y_k - z_k \\ \mu e - X_k z_k \end{bmatrix} =: \begin{bmatrix} r_p \\ r_d \\ r_\mu \end{bmatrix}$$

It's common to eliminate p_z and solve the block 2-by-2 system

$$\begin{bmatrix} -X_k^{-1} Z_k & A^T \\ A & \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \begin{bmatrix} r_p \\ r_d - X_k^{-1} r_\mu \end{bmatrix}$$

Quadratic Programming (QP)

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^TQx + c^Tx \\ \text{subject to} & Ax = b,\ x \geq 0 \end{array}$$

This is a much more general problem than LP. (Setting $Q = 0$ gives an LP.)

Example: The "distance" between two polyhedra

$$\mathcal{P}_1 = \{ x \mid A_1x \leq b_1 \} \qquad \text{and} \qquad \mathcal{P}_2 = \{ x \mid A_2x \leq b_2 \}$$

is defined by the solution of the quadratic program

$$\begin{array}{ll} \underset{x_1, x_2}{\text{minimize}} & \frac{1}{2}\|x_1 - x_2\|_2^2 \\ \text{subject to} & A_1x_1 \leq b_1 \quad \text{and} \quad A_2x_2 \leq b_2 \end{array}$$

Primal-dual approach for QPs

QP:

minimize $\quad \frac{1}{2}x^TQx + c^Tx - \mu\sum_j \log x_j$
subject to $\quad Ax = b$

LP:

minimize $\quad c^Tx - \mu\sum_j \log x_j$
subject to $\quad A^Ty + z = c$

Optim cond's:

$$\begin{aligned} Qx + c + \mu X^{-1}e &= A^Ty \\ Ax &= b \ (x > 0) \end{aligned}$$

Optim cond's:

$$\begin{aligned} c + \mu X^{-1}e &= A^Ty \\ Ax &= b \ (x > 0) \end{aligned}$$

Define z such that $x_jz_j = \mu$. The optimality conditions become

$$\begin{aligned} -Qx + A^Ty + z &= c, \quad z > 0 \\ Ax &= b, \quad x > 0 \\ x_jz_j &= \mu, \quad j = 1, \ldots, n \end{aligned}$$

$$F_\mu(x, y, z) = \begin{bmatrix} Ax - b \\ -Qx + A^Ty + z - c \\ Xz - \mu e \end{bmatrix}$$

$$\begin{aligned} A^Ty + z &= c, \quad z > 0 \\ Ax &= b, \quad x > 0 \\ x_jz_j &= \mu, \quad j = 1, \ldots, n \end{aligned}$$

$$F_\mu(x, y, z) = \begin{bmatrix} Ax - b \\ A^Ty + z - c \\ Xz - \mu e \end{bmatrix}$$

Newton's method for QPs

$$F_\mu(x,y,z) = \begin{bmatrix} Ax - b \\ A^T y + z + Qx - c \\ Xz - \mu e \end{bmatrix}$$

An approximate LP solution (x,y,z), with $(x,z) > 0$ satisfies

$$F_\mu(x,y,z) = 0$$

Apply Newton's method for root finding to these equations, eg,

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} + \alpha \begin{bmatrix} p_x \\ p_y \\ p_y \end{bmatrix}$$

where p is a Newton step:

$$J_k p = -F_k \quad \iff \quad \begin{bmatrix} A & 0 & 0 \\ -Q & A^T & I \\ Z_k & 0 & X_k \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} b - Ax_k \\ c + Qx_k - A^T y_k - z_k \\ \mu e - X_k z_k \end{bmatrix}$$