Funkcja korelacji

$$\langle \hat\psi^+(x)\hat\psi(x')\rangle = \mathrm{Tr}\left(\hat\varrho\, \hat\psi^+(x)\hat\psi(x')\right) = \frac{1}{Z}\sum_{\{n_k\}} \langle \{n_k\}|\, e^{-\beta\sum_k \varepsilon_k \hat n_k}\, \hat\psi^+(x)\hat\psi(x')|\{n_k\}\rangle$$

To już obliczaliśmy

Suma po wszystkich obsadzeniach poziomów takich, że $\sum_k n_k = N$

$$\hat\psi(x) = \sum_q \frac{1}{\sqrt{L}} e^{iqx}\,\hat a_q$$

$$\frac{1}{Z}\sum_{\{n_k\}} \langle\{n_k\}|\frac{1}{L} e^{-\beta\sum_k \varepsilon_k n_k}\sum_{q,q'} e^{i(q'x'-qx)}\,\hat a_q^+\hat a_{q'}|\{n_k\}\rangle =$$

$$= \frac{1}{Z}\sum_{\{n_k\}}\frac{1}{L}\sum_{q,q'} e^{i(q'x'-qx)}\, e^{-\beta\sum_k\varepsilon_k n_k}\,\underbrace{\langle\{n_k\}|\hat a_q^+\hat a_{q'}|\{n_k\}\rangle}_{=\delta_{q,q'}n_q} =$$

$$= \frac{1}{Z}\sum_{\{n_k\}}\sum_q \frac{1}{L} e^{iq(x'-x)}\, e^{-\beta\sum_k\varepsilon_k n_k}\, n_q = \sum_q \frac{1}{L}\langle n_q\rangle e^{iq(x'-x)}$$

$$= \frac{1}{Z}\sum_{n_{-\infty}=0}^{\infty}\cdots\sum_{n_\infty=0}^{\infty}\frac{1}{L}\sum_q e^{iq(x'-x)}\, e^{-\beta\sum_k\varepsilon_k n_k}\, n_q\,\delta_{N,\sum_k n_k} =$$

Korzystamy z $\delta_{N,\sum_k n_k} = \frac{1}{2\pi}\int_0^{2\pi} d\xi\, e^{i\xi(N-\sum_k n_k)}$ i mamy

$$= \frac{1}{2\pi Z}\frac{1}{L}\sum_q e^{iq(x'-x)}\int_0^{2\pi} d\xi\, e^{i\xi N}\sum_{n_{-\infty}=0}^{\infty}\cdots\sum_{n_\infty=0}^{\infty} n_q \prod_k e^{-(i\xi+\beta\varepsilon_k)n_k} =$$

Wszystkie sumy, oprócz q-tej, to sumy szeregów geometrycznych.
Dla q-tej sumy mamy szereg typu $\sum_{n=0}^{\infty} n q^n = \frac{q}{(1-q)^2}$.

$$= \frac{1}{2\pi Z}\frac{1}{L}\sum_q e^{iq(x'-x)}\int_0^{2\pi} d\xi\, e^{i\xi N}\prod_{k\neq q}\frac{1}{\left(1-e^{-(i\xi+\beta\varepsilon_k)}\right)}\cdot\frac{e^{-(i\xi+\beta\varepsilon_q)}}{\left(1-e^{-(i\xi+\beta\varepsilon_q)}\right)^2} =$$

$$= \frac{1}{2\pi Z}\frac{1}{L}\sum_q e^{iq(x'-x)}\, I_q\,,\quad \text{gdzie}$$

$$I_q = \int_0^{2\pi} d\xi\, e^{i\xi N}\prod_{k\neq q}\frac{1}{\left(1-e^{-(i\xi+\beta\varepsilon_k)}\right)}\,\frac{e^{-(i\xi+\beta\varepsilon_q)}}{\left(1-e^{-(i\xi+\beta\varepsilon_q)}\right)^2}$$

2) Obliczanie całki I_q

Zamieniamy zmienne $z = e^{i\xi}$ i dostajemy

$$I_q = \frac{1}{i}\oint_{C(1)} dz\, z^N \prod_{k\neq q} \frac{z}{z-e^{-\beta\varepsilon_k}} \quad \frac{e^{-\beta\varepsilon_q}}{(z-e^{-\beta\varepsilon_q})^2}$$

Całkujemy funkcję $f(z) = z^N \frac{e^{-\beta\varepsilon_q}}{(z-e^{-\beta\varepsilon_q})^2}\prod_{k\neq q}\frac{z}{z-e^{-\beta\varepsilon_k}}$ po okręgu jednostkowym.

$\boxed{q=0}$ Wszystkie bieguny są drugiego rzędu. Funkcję f zapisujemy jako

$$f(z) = z^N \frac{e^{-\beta\varepsilon_0}}{(z-e^{-\beta\varepsilon_0})^2}\prod_{k=1}^{\infty} \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2}$$

Residua obliczamy ze wzoru

$$\operatorname*{Res}_{z=e^{-\beta\varepsilon_p}} f = \lim_{z\to e^{-\beta\varepsilon_p}} \frac{d}{dz}\left(z-e^{-\beta\varepsilon_p}\right)^2 f(z)$$

Dla $\underline{p=0}$ musimy różniczkować wyrażenie

$$z^N e^{-\beta\varepsilon_0}\prod_{k=1}^{\infty} \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2}$$

Korzystamy z wzoru na pochodną iloczynu $\frac{d}{dx}\prod_{j=1}^{\infty} g_j(x) = \prod_{j=1}^{\infty} g_j(x)\sum_{l=1}^{\infty} \frac{g_l'(x)}{g_l(x)}$ i po różniczkowaniu dostajemy

$$N z^{N-1} e^{-\beta\varepsilon_0}\prod_{k=1}^{\infty} \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2} + z^N e^{-\beta\varepsilon_0}\prod_{k=1}^{\infty}\frac{z^2}{(z-e^{-\beta\varepsilon_k})^2}\left(-\frac{2}{z}\sum_{l=1}^{\infty} \frac{e^{-\beta\varepsilon_l}}{(z-e^{-\beta\varepsilon_l})}\right)$$

Wobec czego

$$\operatorname*{Res}_{z=e^{-\beta\varepsilon_0}} f = e^{-\beta N\varepsilon_0}\prod_{k=1}^{\infty} \frac{1}{(1-e^{-\beta(\varepsilon_k-\varepsilon_0)})^2}\left[N + 2\sum_{l=1}^{\infty} \frac{1}{1-e^{-\beta(\varepsilon_0-\varepsilon_l)}}\right]$$

Natomiast dla $\underline{p\neq 0}$ różniczkujemy $\quad p>0$

$$z^{N+2}\frac{e^{-\beta\varepsilon_0}}{(z-e^{-\beta\varepsilon_0})^2}\prod_{\substack{k=1\\k\neq p}}^{\infty} \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2}$$

Po zróżniczkowaniu mamy

$$e^{-\beta\varepsilon_0}\frac{z^{N+1}\left(Nz - e^{-\beta\varepsilon_0}(N+2)\right)}{(z-e^{-\beta\varepsilon_0})^3}\prod_{\substack{k=1\\k\neq p}}^{\infty}\frac{z^2}{(z-e^{-\beta\varepsilon_k})^2} + z^{N+2}\frac{e^{-\beta\varepsilon_0}}{(z-e^{-\beta\varepsilon_0})^2}\prod_{\substack{k=1\\k\neq p}}^{\infty}\frac{z^2}{(z-e^{-\beta\varepsilon_k})^2}\left(-\frac{2}{z}\sum_{\substack{l=1\\l\neq p}}\frac{e^{-\beta\varepsilon_l}}{z-e^{-\beta\varepsilon_l}}\right.$$

Wobec czego

$$\operatorname*{Res}_{z=e^{-\beta\varepsilon_p}} f = e^{-\beta N\varepsilon_p}\frac{e^{-\beta(\varepsilon_0-\varepsilon_p)}}{(1-e^{-\beta(\varepsilon_0-\varepsilon_p)})^2}\prod_{\substack{k=1\\k\neq p}}^{\infty}\frac{1}{(1-e^{-\beta(\varepsilon_k-\varepsilon_p)})^2}\left[N+2\sum_{\substack{l=0!\\l\neq p}}\frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_l)}}\right]$$

3) Całkę I_0 obliczamy z tw. o residuach $I_0 = 2\pi i \sum \mathrm{Res}$

$$I_0 = 2\pi \Big\{ e^{-\beta N \varepsilon_0} \prod_{k=1}^{\infty} \frac{1}{(1-e^{-\beta(\varepsilon_k-\varepsilon_0)})^2} \Big[N + 2\sum_{l=1}^{\infty} \frac{1}{1-e^{-\beta(\varepsilon_0-\varepsilon_l)}} \Big] +$$

$$+\sum_{p=1}^{\infty} e^{-\beta N \varepsilon_p} \frac{e^{-\beta(\varepsilon_0-\varepsilon_p)}}{(1-e^{-\beta(\varepsilon_0-\varepsilon_p)})^2} \prod_{\substack{k=1\\k\neq p}}^{\infty} \frac{1}{(1-e^{-\beta(\varepsilon_k-\varepsilon_p)})^2} \Big[N + 2\sum_{\substack{l=0\\l\neq p}}^{\infty} \frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_l)}} \Big] \Big\} =$$

$$= 2\pi \sum_{p=0}^{\infty} e^{-\beta N \varepsilon_p} e^{-\beta(\varepsilon_0-\varepsilon_p)} \prod_{\substack{k=0\\k\neq p}}^{\infty} \frac{1}{(1-e^{-\beta(\varepsilon_k-\varepsilon_p)})^2} \Big[N + 2\sum_{\substack{l=0\\l\neq p}}^{\infty} \frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_l)}} \Big]$$

$\boxed{q\neq 0}$ Mamy biegun 1 rzędu, 3 rzędu a reszta jest 2 rzędu

Zapiszmy funkcję f jako

$$f(z) = z^N \frac{e^{-\beta\varepsilon_q}}{(z-e^{-\beta\varepsilon_q})^2} \frac{z}{(z-e^{-\beta\varepsilon_q})} \prod_{\substack{k=1\\k\neq q}}^{\infty} \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2} \frac{z}{z-e^{-\beta\varepsilon_0}} =$$

$$= z^N e^{-\beta\varepsilon_q} \frac{1}{(z-e^{-\beta\varepsilon_q})(z-e^{-\beta\varepsilon_0})} \prod_{k=1}^{\infty} \frac{z^2}{(z-e^{-\beta\varepsilon_k})} = \qquad (*)$$

$$= z^{N+2} e^{-\beta\varepsilon_q} \frac{1}{(z-e^{-\beta\varepsilon_q})^3} \frac{1}{z-e^{-\beta\varepsilon_0}} \prod_{\substack{k=1\\k\neq q}}^{\infty} \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2} \qquad (**)$$

Biegun 1 rzędu $\underline{\underline{p=0}}$ Wspomagamy się przedstawieniem $(**)$

$$\mathrm{Res}_{z=e^{-\beta\varepsilon_0}} f = e^{-\beta N\varepsilon_0} \cdot e^{-\beta(\varepsilon_q-\varepsilon_0)} \frac{1}{(1-e^{-\beta(\varepsilon_q-\varepsilon_0)})^3} \prod_{\substack{k=1\\k\neq q}}^{\infty} \frac{1}{(1-e^{-\beta(\varepsilon_k-\varepsilon_0)})^2}$$

Bieguny 2 rzędu $\underline{\underline{p\neq 0 \wedge p\neq q}}$. Używamy przedstawienia $(*)$

Musimy różniczkować wyrażenie

$$z^{N+2} e^{-\beta\varepsilon_q} \frac{1}{(z-e^{-\beta\varepsilon_q})(z-e^{-\beta\varepsilon_0})} \prod_{\substack{k=1\\k\neq p}}^{\infty} \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2}$$

Dostajemy

$$e^{-\beta\varepsilon_q} \cdot \frac{z^{N+1}\left(e^{-\beta(\varepsilon_0+\varepsilon_q)}(N+2) - z(N+1)(e^{-\beta\varepsilon_0}+e^{-\beta\varepsilon_q}) + Nz^2\right)}{(z-e^{-\beta\varepsilon_q})^2 (z-e^{-\beta\varepsilon_0})^2} \prod_{\substack{k=1\\k\neq p}}^{\infty} \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2} +$$

$$+ e^{-\beta\varepsilon_q} \frac{z^{N+2}}{(z-e^{-\beta\varepsilon_q})(z-e^{-\beta\varepsilon_0})} \prod_{\substack{k=1\\k\neq p}}^{\infty} \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2} \left(-\frac{2}{z} \sum_{\substack{l=1\\l\neq p}}^{\infty} \frac{e^{-\beta\varepsilon_l}}{z-e^{-\beta\varepsilon_l}} \right)$$

4) Otrzymujemy

$$\operatorname*{Res}_{z=e^{-\beta\varepsilon_p}} f = e^{-\beta N\varepsilon_p}\, e^{-\beta(\varepsilon_q-\varepsilon_p)} \frac{1}{\left(1-e^{-\beta(\varepsilon_q-\varepsilon_p)}\right)\left(1-e^{-\beta(\varepsilon_0-\varepsilon_p)}\right)} \prod_{\substack{l=1\\ l\neq p}}^{\infty} \frac{1}{\left(1-e^{-\beta(\varepsilon_l-\varepsilon_p)}\right)^2}\Bigg[N+\frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_q)}}$$

$$+\frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_0)}}+2\sum_{\substack{l=1\\ l\neq p}}^{\infty}\frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_l)}}\Bigg]$$

Pozostał biegun 3 rzędu. Obliczamy go ze wzoru:

$$\operatorname*{Res}_{z=e^{-\beta\varepsilon_q}} f = \frac{1}{2}\lim_{z\to e^{-\beta\varepsilon_q}} \frac{d^2}{dz^2}\left(z-e^{-\beta\varepsilon_q}\right)^3 f$$

Wykorzystujemy przedstawienie (* *). Musimy dwukrotnie różniczkować

$$z^{N+2}e^{-\beta\varepsilon_q}\frac{1}{z-e^{-\beta\varepsilon_0}}\prod_{\substack{l=1\\ l\neq q}}^{\infty}\frac{z^2}{\left(z-e^{-\beta\varepsilon_l}\right)^2}$$

Druga pochodna iloczynu

$$\left(\prod_{j=1}^{\infty} f_j(x)\right)'' = \left(\prod_{j=1}^{\infty} f_j(x)\right)\left[\sum_{\substack{l,l'=1\\ l\neq l'}}^{\infty}\frac{f_l'(x)f_{l'}'(x)}{f_l(x)f_{l'}(x)}+\sum_{l=1}^{\infty}\frac{f_l''(x)}{f_l(x)}\right]$$

Wykorzystujemy to i dostajemy

$$e^{-\beta\varepsilon_q}\frac{z^N\left(e^{-2\beta\varepsilon_0}(N+1)(N+2)-2e^{-\beta\varepsilon_0}N(N+2)z+N(N+1)z^2\right)}{\left(z-e^{-\beta\varepsilon_0}\right)^3}\prod_{\substack{l=1\\ l\neq q}}^{\infty}\frac{z^2}{\left(z-e^{-\beta\varepsilon_l}\right)^2}+$$

$$2e^{-\beta\varepsilon_q}\frac{z^{N+1}\left(z(N+1)-e^{-\beta\varepsilon_0}(N+2)\right)}{\left(z-e^{-\beta\varepsilon_0}\right)^2}\prod_{\substack{l=1\\ l\neq q}}^{\infty}\frac{z^2}{\left(z-e^{-\beta\varepsilon_l}\right)^2}\cdot\left(-\frac{2}{z}\sum_{\substack{l=1\\ l\neq q}}^{\infty}\frac{e^{-\beta\varepsilon_l}}{z-e^{-\beta\varepsilon_l}}\right)+$$

$$+e^{-\beta\varepsilon_q}\frac{z^{N+2}}{z-e^{-\beta\varepsilon_0}}\prod_{\substack{l=1\\ l\neq q}}^{\infty}\frac{z^2}{\left(z-e^{-\beta\varepsilon_l}\right)^2}\left[\frac{4}{z^2}\sum_{\substack{l,l'=1\\ l\neq l'\\ l\neq q,\, l'\neq q}}^{\infty}\frac{e^{-\beta(\varepsilon_l+\varepsilon_{l'})}}{\left(z-e^{-\beta\varepsilon_l}\right)\left(z-e^{-\beta\varepsilon_{l'}}\right)}+\frac{2}{z^2}\sum_{\substack{l=1\\ l\neq q}}^{\infty}\frac{e^{-\beta\varepsilon_l}\left(e^{-\beta\varepsilon_l}+2z\right)}{\left(z-e^{-\beta\varepsilon_l}\right)^2}\right]$$

Stąd

$$\operatorname*{Res}_{z=e^{-\beta\varepsilon_q}} f = e^{-\beta N\varepsilon_q}\frac{1}{1-e^{-\beta(\varepsilon_0-\varepsilon_q)}}\prod_{\substack{l=1\\ l\neq q}}^{\infty}\frac{1}{\left(1-e^{-\beta(\varepsilon_l-\varepsilon_q)}\right)^2}\Bigg\{\frac{N(N+1)}{2}+N\frac{1}{\left(1-e^{-\beta(\varepsilon_q-\varepsilon_0)}\right)^2}$$

$$+\sum_{\substack{l=0\\ l\neq q}}^{\infty}\frac{1}{\left(1-e^{-\beta(\varepsilon_q-\varepsilon_l)}\right)^2}+2\sum_{\substack{l,l'=1\\ l,l'\neq q\\ l\neq l'}}^{\infty}\frac{1}{\left(1-e^{-\beta(\varepsilon_q-\varepsilon_l)}\right)\left(1-e^{-\beta(\varepsilon_q-\varepsilon_{l'})}\right)}+\frac{1}{2}\sum_{\substack{l=1\\ l\neq q}}^{\infty}\frac{1}{\operatorname{sh}^2\left(\frac{1}{2}\beta(\varepsilon_q-\varepsilon_l)\right)}-\frac{N}{4}\frac{1}{\operatorname{sh}^2\left(\frac{1}{2}\beta(\varepsilon_q-\varepsilon_0)\right)}\Bigg\}$$

5) Obliczamy w końcu $I_q = 2\pi i \sum \mathrm{Res}$

$$I_q = 2\pi \Bigg\{ e^{-\beta N \varepsilon_0} \cdot e^{-\beta(\varepsilon_q - \varepsilon_0)} \frac{1}{\left(1 - e^{-\beta(\varepsilon_q-\varepsilon_0)}\right)^3} \prod_{\substack{k=1 \\ k\neq q}}^{\infty} \frac{1}{\left(1 - e^{-\beta(\varepsilon_k - \varepsilon_0)}\right)^2} +$$

$$+ e^{-\beta N \varepsilon_q} \frac{1}{1 - e^{-\beta(\varepsilon_0 - \varepsilon_q)}} \prod_{\substack{k=1 \\ k\neq q}}^{\infty} \frac{1}{\left(1 - e^{-\beta(\varepsilon_k-\varepsilon_q)}\right)^2} \Bigg[\frac{N(N+1)}{2} + N \frac{1}{\left(1 - e^{-\beta(\varepsilon_q - \varepsilon_0)}\right)^2} +$$

$$+ \sum_{\substack{k=0 \\ k\neq q}}^{\infty} \frac{1}{\left(1 - e^{-\beta(\varepsilon_q - \varepsilon_k)}\right)^2} + 2 \sum_{\substack{k,k'=1 \\ k,k'\neq q \\ k\neq k'}}^{\infty} \frac{1}{\left(1 - e^{-\beta(\varepsilon_q-\varepsilon_k)}\right)\left(1 - e^{-\beta(\varepsilon_q - \varepsilon_{k'})}\right)} + \frac{1}{2} \sum_{\substack{k=1 \\ k\neq q}}^{\infty} \frac{1}{\mathrm{sh}^2\left(\frac{1}{2}\beta(\varepsilon_q - \varepsilon_k)\right)} +$$

$$- \frac{N}{4} \frac{1}{\mathrm{sh}^2\left(\frac{1}{2}\beta(\varepsilon_q-\varepsilon_0)\right)} \Bigg] + \sum_{\substack{p=1 \\ p\neq |q|}}^{\infty} e^{-\beta N \varepsilon_p} \cdot e^{-\beta(\varepsilon_q - \varepsilon_p)} \frac{1}{\left(1 - e^{-\beta(\varepsilon_q - \varepsilon_p)}\right)\left(1 - e^{-\beta(\varepsilon_0 - \varepsilon_p)}\right)} \times$$

$$\times \prod_{\substack{k=1 \\ k\neq p}}^{\infty} \frac{1}{\left(1 - e^{-\beta(\varepsilon_k - \varepsilon_p)}\right)^2} \Bigg[N + \frac{1}{1 - e^{-\beta(\varepsilon_p - \varepsilon_q)}} + \frac{1}{1 - e^{-\beta(\varepsilon_p - \varepsilon_0)}} + 2 \sum_{\substack{k=1 \\ k\neq p}}^{\infty} \frac{1}{1 - e^{-\beta(\varepsilon_p - \varepsilon_k)}} \Bigg] \Bigg\}$$

stąd

$$\langle \hat{\psi}^+(x) \hat{\psi}(x') \rangle = \frac{1}{Z} \frac{1}{L} \left(\frac{I_0}{2\pi} + 2 \sum_{q=1}^{\infty} \frac{I_q}{2\pi} \cos\left(q(x'-x)\right) \right)$$