Funkcja korelacji - pola klasyczne

$M \equiv M_{cut}$

Pole atomowe $\psi(x) = \sum_{q=-M}^{M} \frac{1}{\sqrt{L}} e^{iqx} \alpha_q$

$$\langle \psi^\dagger(x)\psi(x')\rangle = \frac{1}{Z}\int \frac{d^2\alpha_{-M}}{\pi} \dots \int \frac{d^2\alpha_M}{\pi}\, \psi^\dagger(x)\psi(x')\, e^{-\beta\sum_k \varepsilon_k |\alpha_k|^2}\, \delta\Big(N-\sum_k |\alpha_k|^2\Big) =$$

$$= \frac{1}{Z}\frac{1}{L}\sum_{q,q'} e^{i(q'x'-qx)} \int \frac{d^2\alpha_{-M}}{\pi}\dots\int\frac{d^2\alpha_M}{\pi}\, \alpha_q^*\alpha_{q'}\, e^{-\beta\sum_k \varepsilon_k|\alpha_k|^2}\,\delta\Big(N-\sum_k|\alpha_k|^2\Big) =$$

Korzystamy z reprezentacji

$$\delta\Big(N-\sum_k|\alpha_k|^2\Big) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{i\xi(N-\sum_k|\alpha_k|^2)}$$

$$= \frac{1}{Z}\frac{1}{L}\frac{1}{2\pi}\sum_{q,q'} e^{i(q'x'-qx)}\int_{-\infty}^{\infty} d\xi\, e^{i\xi N}\underbrace{\int\frac{d^2\alpha_{-M}}{\pi}\dots\int\frac{d^2\alpha_M}{\pi}\,\alpha_q^*\alpha_{q'}\, e^{-\sum_k(\beta\varepsilon_k+i\xi)|\alpha_k|^2}}_{\text{Wyrazy } q\neq q' \text{ znikają}} =$$

Oznaczamy $$I_q = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{i\xi N}\int\frac{d^2\alpha_{-M}}{\pi}\dots\int\frac{d^2\alpha_M}{\pi}\,|\alpha_q|^2 e^{-\sum_k(\beta\varepsilon_k+i\xi)|\alpha_k|^2} =$$

$$= \int_{-\infty}^{\infty} d\xi\, e^{i\xi N}\, I_{-M}\dots\tilde{I}_q\dots I_M \quad , \text{ gdzie}$$

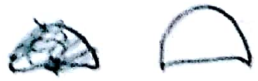
$$I_k = \int\frac{d^2\alpha_k}{\pi}\, e^{-(\beta\varepsilon_k+i\xi)|\alpha_k|^2} = \frac{1}{\beta\varepsilon_k+i\xi}$$

$$\tilde{I}_q = \int\frac{d^2\alpha_q}{\pi}\,|\alpha_q|^2 e^{-(\beta\varepsilon_q+i\xi)|\alpha_q|^2} = \frac{1}{(\beta\varepsilon_q+i\xi)^2}$$

$$= \frac{1}{Z}\frac{1}{L}\sum_q e^{iq(x'-x)}\, I_q$$

$\boxed{q=0}$ $$2\pi I_0 = \int_{-\infty}^{\infty} d\xi\, e^{i\xi N}\prod_{k=0}^{M}\frac{1}{(\beta\varepsilon_k+i\xi)^2} = \frac{1}{i^{2(M+1)}}\int_{-\infty}^{\infty} d\xi\, e^{i\xi N}\prod_{k=0}^{M}\frac{1}{(\xi-i\beta\varepsilon_k)^2} =$$

$$= (-1)^{M+1}\int_{-\infty}^{\infty} d\xi\, e^{i\xi N}\prod_{k=0}^{M}\frac{1}{(\xi-i\beta\varepsilon_k)^2}$$

Całkujemy $f(z) = e^{izN}\prod_{k=0}^{M}\frac{1}{(z-i\beta\varepsilon_k)^2}$ po 

② Wszystkie bieguny 2 rzędu

$$\operatorname*{Res}_{z=i\beta\varepsilon_p} f = \lim_{z\to i\beta\varepsilon_p} \frac{d}{dz}(z-i\beta\varepsilon_p)^2 f$$

Musimy różniczkować $\quad e^{izN}\prod_{\substack{k=0\\k\neq p}}^{n}\frac{1}{(z-i\beta\varepsilon_k)^2}$

Dostajemy $\quad iNe^{izN}\prod_{\substack{k=0\\k\neq p}}^{n}\frac{1}{(z-i\beta\varepsilon_k)^2} + e^{izN}\prod_{\substack{k=0\\k\neq p}}^{n}\frac{1}{(z-i\beta\varepsilon_k)^2}\cdot\left(-2\sum_{\substack{l=0\\l\neq p}}^{n}\frac{1}{z-i\beta\varepsilon_l}\right)$

Stąd mamy

$$\operatorname*{Res}_{z=i\beta\varepsilon_p} f = \underbrace{\pm i}_{(-1)^{n+1}} e^{-\beta\varepsilon_p N}\prod_{\substack{k=0\\k\neq p}}^{n}\frac{1}{(\beta\varepsilon_p-\beta\varepsilon_k)^2}\left[\cancel{2} N + 2\sum_{\substack{l=0\\l\neq p}}^{n}\frac{1}{\beta\varepsilon_p-\beta\varepsilon_l}\right]$$

Stąd $\quad I_0 = \sum_{p=0}^{n} e^{-\beta\varepsilon_p N}\prod_{\substack{k=0\\k\neq p}}^{n}\frac{1}{(\beta\varepsilon_p-\beta\varepsilon_k)^2}\left[N + 2\sum_{\substack{l=0\\l\neq p}}^{n}\frac{1}{\beta\varepsilon_p-\varepsilon_l}\right]$

$\boxed{q\neq 0}$ Mamy biegun 1 rzędu, biegun 3 rzędu a reszta to ~~p~~ bieguny 2 rzędu.

~~Biegun 1 rzędu~~ Zapisujemy funkcję f jako

~~Res f z=iβε₀~~ $\quad f(z) = e^{izN}\frac{1}{z-i\beta\varepsilon_0}\frac{1}{(z-i\beta\varepsilon_q)^3}\prod_{\substack{k=1\\k\neq q}}^{n}\frac{1}{(z-i\beta\varepsilon_k)^2} = \qquad (*)$

$$= e^{izN}\frac{1}{z-i\beta\varepsilon_0}\frac{1}{z-i\beta\varepsilon_q}\prod_{k=1}^{n}\frac{1}{(z-i\beta\varepsilon_k)^2} \qquad (**)$$

Wykorzystujemy $(**)$ i dostajemy dla bieguna 1 rzędu

$$\operatorname*{Res}_{z=i\beta\varepsilon_0} f = e^{-\beta\varepsilon_0 N}\frac{1}{i^{2n+1}}\frac{1}{\beta\varepsilon_0-\beta\varepsilon_q}\prod_{k=1}^{n}\frac{1}{(\beta\varepsilon_0-\beta\varepsilon_k)^2}$$

Bieguny 2 rzędu $\quad p\neq 0 \wedge p\neq q$. Wykorzystujemy $(**)$. Musimy różniczkować wyrażenie

$$e^{izN}\frac{1}{z-i\beta\varepsilon_0}\frac{1}{z-i\beta\varepsilon_q}\prod_{\substack{k=1\\k\neq p}}^{n}\frac{1}{(z-i\beta\varepsilon_k)^2}$$

Dostajemy $\quad e^{izN}\dfrac{\left[i(i\beta\varepsilon_0)(i\beta\varepsilon_q)N - i(i\beta\varepsilon_0)Nz + i\beta\varepsilon_0 - i(i\beta\varepsilon_q)Nz + i\beta\varepsilon_q + iNz^2 - 2z\right]}{(z-i\beta\varepsilon_0)^2(z-i\beta\varepsilon_q)^2}$

$$\times\prod_{\substack{k=1\\k\neq p}}^{n}\frac{1}{(z-i\beta\varepsilon_k)^2} + e^{izN}\frac{1}{z-i\beta\varepsilon_0}\frac{1}{z-i\beta\varepsilon_q}\prod_{\substack{k=1\\k\neq p}}^{n}\frac{1}{(z-i\beta\varepsilon_k)^2}\left(-2\sum_{\substack{l=1\\l\neq p}}^{n}\frac{1}{z-i\beta\varepsilon_l}\right)$$

3) Dostajemy

$$\operatorname*{Res}_{z=i\beta\varepsilon_p} f = \frac{1}{i^{2m-1}} e^{-\beta\varepsilon_p N} \frac{1}{\beta\varepsilon_p-\beta\varepsilon_0} \frac{1}{\beta\varepsilon_p-\varepsilon_q} \prod_{\substack{k=1\\k\neq p}}^{m} \frac{1}{(\beta\varepsilon_p-\beta\varepsilon_k)^2}\left[N + \frac{1}{\beta\varepsilon_p-\beta\varepsilon_q} + \frac{1}{\beta\varepsilon_p-\beta\varepsilon_0} + \right.$$

$$\left. + 2\sum_{\substack{l=1\\l\neq p}}^{m} \frac{1}{\beta\varepsilon_p-\beta\varepsilon_l}\right]$$

Został biegun 3 rzędu

$$\operatorname*{Res}_{z=i\beta\varepsilon_q} f = \frac{1}{2}\lim_{z\to i\beta\varepsilon_q} \frac{d^2}{dz^2}(z-i\beta\varepsilon_q)^3 f$$

Korzystamy z (*). Musimy dwukrotnie różniczkować wyrażenie

$$\frac{e^{izN}}{z-i\beta\varepsilon_0}\prod_{\substack{k=1\\k\neq q}}^{m}\frac{1}{(z-i\beta\varepsilon_k)^2}$$

Dostajemy

$$\frac{e^{izN}\left(2(i\beta\varepsilon_0)N^2z + 2i(i\beta\varepsilon_0)N+2 - (i\beta\varepsilon_0)^2N^2 - N^2z^2 - 2iNz\right)}{(z-i\beta\varepsilon_0)^3}\prod_{\substack{k=1\\k\neq q}}^{m}\frac{1}{(z-i\beta\varepsilon_k)^2} +$$

$$+2\,\frac{ie^{izN}\left(Nz+i-(i\beta\varepsilon_0)N\right)}{(z-i\beta\varepsilon_0)^2}\prod_{\substack{k=1\\k\neq q}}^{m}\frac{1}{(z-i\beta\varepsilon_k)^2}\left(-2\sum_{\substack{l=1\\l\neq q}}^{m}\frac{1}{(z-i\beta\varepsilon_l)}\right) +$$

$$+\frac{e^{izN}}{z-i\beta\varepsilon_0}\cdot\prod_{\substack{k=1\\k\neq q}}^{m}\frac{1}{(z-i\beta\varepsilon_k)^2}\left[4\sum_{\substack{l,l'=1\\l,l'\neq q\\l\neq l'}}^{m}\frac{1}{(z-i\beta\varepsilon_l)(z-i\beta\varepsilon_{l'})} + 6\sum_{\substack{l=1\\l\neq q}}^{m}\frac{1}{(z-i\beta\varepsilon_l)^2}\right]$$

Stąd

$$\operatorname*{Res}_{z=i\beta\varepsilon_q} f = -\frac{1}{i^{2m-1}}e^{-\beta\varepsilon_q N}\frac{1}{\beta\varepsilon_q-\beta\varepsilon_0}\prod_{\substack{k=1\\k\neq q}}^{m}\frac{1}{(\beta\varepsilon_q-\beta\varepsilon_k)^2}\left\{\frac{N^2}{2} + \frac{N}{\beta\varepsilon_q-\beta\varepsilon_0} + \frac{1}{(\beta\varepsilon_q-\beta\varepsilon_0)^2}\right.$$

$$+2\left[N+\frac{1}{\beta\varepsilon_q-\beta\varepsilon_0}\right]\cdot\left(\sum_{\substack{l=1\\l\neq q}}^{m}\frac{1}{\beta\varepsilon_q-\beta\varepsilon_l}\right) + 2\sum_{\substack{l,l'=1\\l,l'\neq q\\l\neq l'}}^{m}\frac{1}{(\beta\varepsilon_q-\beta\varepsilon_l)(\beta\varepsilon_q-\beta\varepsilon_{l'})} +$$

$$\left. + 3\sum_{\substack{l=1\\l\neq q}}^{m}\frac{1}{(\beta\varepsilon_q-\beta\varepsilon_l)^2}\right\}$$

4) Ostatecznie

$$I_q = \sum_{\substack{p=1\\p\neq q}}^{m} e^{-\beta\varepsilon_p N} \frac{1}{\beta\varepsilon_p-\beta\varepsilon_0}\,\frac{1}{\beta\varepsilon_p-\beta\varepsilon_q}\prod_{\substack{k=1\\k\neq p}}^{m} \frac{1}{(\beta\varepsilon_p-\beta\varepsilon_k)^2}\Big[N+\frac{1}{\beta\varepsilon_p-\beta\varepsilon_q}+\frac{1}{\beta\varepsilon_p-\beta\varepsilon_0}+$$

$$+\,2\sum_{\substack{l=1\\l\neq p}}^{m} \frac{1}{\beta\varepsilon_p-\beta\varepsilon_l}\Big] - e^{-\beta\varepsilon_0 N}\frac{1}{\beta\varepsilon_0-\beta\varepsilon_q}\prod_{k=1}^{m}\frac{1}{(\beta\varepsilon_0-\beta\varepsilon_k)^2} - e^{-\beta\varepsilon_q N}\frac{1}{\beta\varepsilon_q-\beta\varepsilon_0}\prod_{\substack{k=1\\k\neq q}}^{m} \frac{1}{(\beta\varepsilon_q-\beta\varepsilon_k)^2}$$

$$\Big\{ \frac{N^2}{2} + \frac{N}{\beta\varepsilon_q-\beta\varepsilon_0}+\frac{1}{(\beta\varepsilon_q-\beta\varepsilon_0)^2} + 2\Big[N+\frac{1}{\beta\varepsilon_q-\beta\varepsilon_0}\Big]\Big(\sum_{\substack{l=1\\l\neq q}}^{m} \frac{1}{\beta\varepsilon_q-\beta\varepsilon_l}\Big) + 2\sum_{\substack{l,l'=1\\l,l'\neq q\\l\neq l'}}^{m} \frac{1}{(\beta\varepsilon_q-\beta\varepsilon_l)}$$

$$\times\frac{1}{(\beta\varepsilon_q-\beta\varepsilon_{l'})} + 3\sum_{\substack{l=1\\l\neq q}}^{m} \frac{1}{(\beta\varepsilon_q-\beta\varepsilon_l)^2}\Big\}$$