Suma statystyczna – ściśle

$$Z_N(\beta) = \sum_{n_{-\infty}}^{\infty} \dots \sum_{n_{\infty}=0}^{\infty} e^{-\beta \left(\sum_j \varepsilon_j n_j\right)} \delta_{N, \sum_j n_j}$$

Całkowa reprezentacja delty Kroneckera

$$\delta_{N,\sum_j n_j} = \frac{1}{2\pi}\int_0^{2\pi} d\xi \, e^{i\xi\left(N - \sum_j n_j\right)}$$

$$Z_N(\beta) = \frac{1}{2\pi}\int_0^{2\pi} d\xi \, e^{i\xi N} \prod_{j=-\infty}^{\infty} \frac{1}{1-e^{-(i\xi+\beta\varepsilon_j)}} = \frac{1}{2\pi}\int_0^{2\pi} d\xi \, \frac{e^{i\xi N}}{1-e^{-(i\xi+\beta\varepsilon_0)}} \prod_{j=1}^{\infty} \frac{1}{\left(1-e^{-(i\xi+\beta\varepsilon_j)}\right)^2}$$

Wprowadzamy zmienną $z = e^{i\xi}$ i dostajemy

$$Z_N(\beta) = \frac{1}{2\pi i} \oint_{C(1)} dz \, \frac{z^N}{z-e^{-\beta\varepsilon_0}} \prod_{j=1}^{\infty} \frac{z^2}{\left(z-e^{-\beta\varepsilon_j}\right)^2}$$

Całkujemy $f(z) = \frac{z^N}{z-e^{-\beta\varepsilon_0}} \prod_{j=1}^{\infty} \frac{z^2}{\left(z-e^{-\beta\varepsilon_j}\right)^2}$ po $C(1)$.

Biegun 1 rzędu

$$\operatorname*{Res}_{z=e^{-\beta\varepsilon_0}} f = e^{-\beta\varepsilon_0 N} \prod_{j=1}^{\infty} \frac{1}{\left(1-e^{-\beta(\varepsilon_j-\varepsilon_0)}\right)^2}$$

Reszta to bieguny drugiego rzędu. Pochodna iloczynu $\left(\prod_j g_j(x)\right)' = \prod_j g_j(x) \sum_j \frac{g_j'(x)}{g_j(x)}$ i $\operatorname*{Res}_{z=e^{-\beta\varepsilon_p}} f = \lim_{z\to e^{-\beta\varepsilon_p}} \frac{d}{dz}\left(z-e^{-\beta\varepsilon_p}\right)^2 f$. Musimy różniczkować wyrażenie

$$\frac{z^{N+2}}{z-e^{-\beta\varepsilon_0}} \prod_{\substack{j=1\\ j\neq p}}^{\infty} \frac{z^2}{\left(z-e^{-\beta\varepsilon_j}\right)^2}$$

Otrzymujemy

$$\frac{z^{N+1}\left((N+2)z - (N+2)e^{-\beta\varepsilon_0} - z\right)}{\left(z-e^{-\beta\varepsilon_0}\right)^2} \prod_{\substack{j=1\\ j\neq p}}^{\infty} \frac{z^2}{\left(z-e^{-\beta\varepsilon_j}\right)^2} + \frac{z^{N+2}}{z-e^{-\beta\varepsilon_0}} \prod_{\substack{j=1\\ j\neq p}}^{\infty} \frac{z^2}{\left(z-e^{-\beta\varepsilon_j}\right)^2} \left(\frac{2}{z} \sum_{\substack{l=1\\ l\neq p}}^{\infty} \frac{e^{-\beta\varepsilon_l}}{e^{-\beta\varepsilon_l} - z}\right)$$

Stąd

$$\operatorname*{Res}_{z=e^{-\beta\varepsilon_p}} f = \frac{e^{-\beta\varepsilon_p N}}{1-e^{-\beta(\varepsilon_0-\varepsilon_p)}} \prod_{\substack{j=1\\ j\neq p}}^{\infty} \frac{1}{\left(1-e^{-\beta(\varepsilon_j-\varepsilon_p)}\right)^2} \left(N + 1 + \frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_0)}} + 2\sum_{\substack{l=1\\ l\neq p}}^{\infty} \frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_l)}}\right)$$

Ostatecznie mamy

$$Z_N(\beta) = e^{-\beta\varepsilon_0 N} \prod_{j=1}^{\infty} \frac{1}{\left(1-e^{-\beta(\varepsilon_j-\varepsilon_0)}\right)^2} + \sum_{p=1}^{\infty} \frac{e^{-\beta\varepsilon_p N}}{1-e^{-\beta(\varepsilon_0-\varepsilon_p)}} \prod_{\substack{j=1\\ j\neq p}}^{\infty} \frac{1}{\left(1-e^{-\beta(\varepsilon_j-\varepsilon_p)}\right)^2} \Bigg(N + 1 +$$

$$+ \frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_0)}} + 2\sum_{\substack{l=1\\ l\neq p}}^{\infty} \frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_l)}}\Bigg)$$

Suma statystyczna - pola klasyczne

$$Z_N(\beta) = \int \frac{d^2\alpha_{-m}}{\pi} \dots \int \frac{d^2\alpha_n}{\pi} \, e^{-\beta \sum_j \varepsilon_j |\alpha_j|^2} \, \delta\Big(N - \sum_j |\alpha_j|^2\Big)$$

Całkowa reprezentacja delty Diraca.

$$\delta\Big(N - \sum_j |\alpha_j|^2\Big) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \, e^{i\xi(N - \sum |\alpha_j|^2)}$$

Stąd $Z_N(\beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \, e^{i\xi N} I_{-m} \dots I_n$ $\qquad I_n = \int \frac{d^2\alpha_n}{\pi} e^{-|\alpha_n|^2(\beta\varepsilon_n + i\xi)} = \frac{1}{\beta\varepsilon_n + i\xi}$

$$Z_N(\beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \, e^{i\xi N} \prod_{j=-m}^{n} \frac{1}{\beta\varepsilon_j + i\xi} = (-1)^n \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\xi \, e^{i\xi N} \frac{1}{\xi - i\beta\varepsilon_0} \prod_{j=1}^{n} \frac{1}{(\xi - i\beta\varepsilon_j)^2}$$

Całkujemy funkcję $f(z) = \frac{e^{izN}}{z - i\beta\varepsilon_0} \prod_{j=1}^{n} \frac{1}{(z - i\beta\varepsilon_j)^2}$ po

Biegun 1 rzędu

$$\operatorname*{Res}_{z=i\beta\varepsilon_0} f = e^{-\beta\varepsilon_0 N} (-1)^n \prod_{j=1}^{n} \frac{1}{(\beta\varepsilon_0 - \beta\varepsilon_j)^2}$$

Bieguny 2 rzędu. Musimy różniczkować

$$\frac{e^{izN}}{z - i\beta\varepsilon_0} \prod_{\substack{j=1\\ j\neq p}}^{n} \frac{1}{(z - i\beta\varepsilon_j)^2}$$

Dostajemy

$$\frac{i e^{izN}(Nz - i\beta\varepsilon_0 N + i)}{(z - i\beta\varepsilon_0)^2} \prod_{\substack{j=1\\ j\neq p}}^{n} \frac{1}{(z - i\beta\varepsilon_j)^2} + \frac{e^{izN}}{z - i\beta\varepsilon_0} \prod_{\substack{j=1\\ j\neq p}}^{n} \frac{1}{(z - i\beta\varepsilon_j)^2} \Big(2 \sum_{\substack{l=1\\ l\neq p}}^{n} \frac{1}{i\beta\varepsilon_l - z}\Big)$$

stąd

$$\operatorname*{Res}_{z=i\beta\varepsilon_p} f = \frac{e^{-\beta\varepsilon_p N}}{\beta\varepsilon_0 - \beta\varepsilon_p} (-1)^n \prod_{\substack{j=1\\ j\neq p}}^{n} \frac{1}{(\beta\varepsilon_p - \beta\varepsilon_j)^2} \Big(N + \frac{1}{\beta\varepsilon_p - \beta\varepsilon_0} + 2\sum_{\substack{l=1\\ l\neq p}}^{n} \frac{1}{\beta\varepsilon_p - \beta\varepsilon_l}\Big)$$

Stąd ostatecznie

$$Z_N(\beta) = e^{-\beta\varepsilon_0 N} \prod_{j=1}^{n} \frac{1}{(\beta\varepsilon_j - \beta\varepsilon_0)^2} + \sum_{p=1}^{\infty} \frac{e^{-\beta\varepsilon_p N}}{\beta\varepsilon_0 - \beta\varepsilon_p} \prod_{\substack{j=1\\ j\neq p}}^{n} \frac{1}{(\beta\varepsilon_j - \beta\varepsilon_p)^2} \Big(N + \frac{1}{\beta\varepsilon_p - \beta\varepsilon_0} + 2\sum_{\substack{l=1\\ l\neq p}}^{n} \frac{1}{\beta\varepsilon_p - \beta\varepsilon_l}\Big)$$