Fluktuacje - ściśle

$\Delta^2 n_0 = \langle n_0^2 \rangle - \langle n_0 \rangle^2$

Musimy obliczyć $\langle n_0^2 \rangle$

$$\langle n_0^2 \rangle = \frac{1}{2\pi Z} \int_0^{2\pi} d\xi\, e^{i\xi N} \sum_{n_{-\infty}=0}^{\infty} \dots \sum_{n_\infty=0}^{\infty} n_0^2 \prod_k e^{-(i\xi+\beta\varepsilon_k)n_k} = \frac{1}{2\pi Z} \int_0^{2\pi} d\xi\, e^{i\xi N} \prod_{k=1}^{\infty}$$

$$\frac{1}{(1-e^{-(i\xi+\beta\varepsilon_k)})^2} \sum_{n_0=0}^{\infty} n_0^2\, e^{-(i\xi+\beta\varepsilon_0)n_0}$$

Korzystamy z wzoru na sumę ciągu $\sum_{n=0}^{\infty} n^2 q^n = \frac{q+q^2}{(1-q)^3}$

Musimy obliczyć całki:

$$\int_0^{2\pi} d\xi\, e^{i\xi N} \prod_{k=1}^{\infty} \frac{1}{(1-e^{-(i\xi+\beta\varepsilon_k)})^2} \frac{e^{-2i\xi}}{(1-e^{-(i\xi+\beta\varepsilon_0)})^3} = \int_0^{2\pi} d\xi\, e^{i\xi(N-2)} \prod_{k=1}^{\infty} \frac{1}{(1-e^{-(i\xi+\beta\varepsilon_k)})^2} \frac{1}{(1-e^{-(i\xi+\beta\varepsilon_0)})^3}$$

$$\int_0^{2\pi} d\xi\, e^{i\xi N} \prod_{k=1}^{\infty} \frac{1}{(1-e^{-(i\xi+\beta\varepsilon_k)})^2} \frac{e^{-i\xi}}{(1-e^{-(i\xi+\beta\varepsilon_0)})^3} = \int_0^{2\pi} d\xi\, e^{i\xi(N-1)} \prod_{k=1}^{\infty} \frac{1}{(1-e^{-(i\xi+\beta\varepsilon_k)})^2} \frac{1}{(1-e^{-(i\xi+\beta\varepsilon_0)})^3}$$

Obliczymy tylko pierwszą a następnie dokonamy zamiany $N \to N+1$

Robimy podstawienie $z = e^{i\xi}$ i dostajemy $\frac{1}{i}\oint_{C(1)} dz\, z^N \prod_{k=1}^{\infty} \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2} \frac{1}{(z-e^{-\beta\varepsilon_0})^3}$.

Całkujemy funkcję $f(z) = z^N \prod_{k=1}^{\infty} \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2} \frac{1}{(z-e^{-\beta\varepsilon_0})^3}$ po $C(1)$.

Bieguny 2 rzędu dla $p \neq 0$. Korzystamy ze wzoru $\operatorname{Res}_{z=e^{-\beta\varepsilon_p}} f = \lim_{z \to e^{-\beta\varepsilon_p}} \frac{d}{dz}(z-e^{-\beta\varepsilon_p})^2 f(z)$

Musimy różniczkować wyrażenie

$$\frac{z^{N+2}}{(z-e^{-\beta\varepsilon_0})^3} \prod_{\substack{k=1 \\ k\neq p}}^{\infty} \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2}$$

Dostajemy

$$\frac{z^{N+1}\left((N-1)z-(N+2)e^{-\beta\varepsilon_0}\right)}{(z-e^{-\beta\varepsilon_0})^4} \prod_{\substack{k=1 \\ k\neq p}}^{\infty} \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2} + \frac{z^{N+2}}{(z-e^{-\beta\varepsilon_0})^3} \prod_{\substack{k=1 \\ k\neq p}}^{\infty} \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2} \left(\frac{2}{z} \sum_{\substack{l=1 \\ l\neq p}}^{\infty} \frac{e^{-\beta\varepsilon_k}}{e^{-\beta\varepsilon_k}-z} \right)$$

Stąd

$$\operatorname{Res}_{z=e^{-\beta\varepsilon_p}} f = \frac{e^{-\beta\varepsilon_p(N-2)}}{(1-e^{-\beta(\varepsilon_0-\varepsilon_p)})^3} \prod_{\substack{k=1 \\ k\neq p}}^{\infty} \frac{1}{(1-e^{-\beta(\varepsilon_k-\varepsilon_p)})^2} \left(N - \frac{1}{1-e^{-\beta(\varepsilon_0-\varepsilon_p)}} + 2\sum_{\substack{l=0 \\ l\neq p}}^{\infty} \frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_l)}} \right)$$

② Dla $p=0$ mamy biegun 3 rzędu. Wykorzystujemy wzór $\operatorname{Res}_{z=e^{-\beta\varepsilon_0}} f = \frac{1}{2}\lim_{z\to e^{-\beta\varepsilon_0}} \frac{d^2}{dz^2}(z-e^{-\beta\varepsilon_0})$

Musimy dwukrotnie różniczkować wyrażenie

$$z^N \prod_{k=1}^{\infty} \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2}$$

Stąd

$$N(N-1)z^{N-2}\prod_{k=1}^{\infty}\frac{z^2}{(z-e^{-\beta\varepsilon_k})^2} + 2N z^{N-1}\prod_{k=1}^{\infty}\frac{z^2}{(z-e^{-\beta\varepsilon_k})^2}\left(\frac{2}{z}\sum_{l=1}^{\infty}\frac{e^{-\beta\varepsilon_l}}{e^{-\beta\varepsilon_l}-z}\right) + z^N\prod_{k=1}^{\infty}\frac{z^2}{(z-e^{-\beta\varepsilon_k})^2}$$

$$\left(\frac{4}{z^2}\sum_{\substack{l,l'=1\\ l\neq l'}}^{\infty}\frac{e^{-\beta\varepsilon_l}\cdot e^{-\beta\varepsilon_{l'}}}{(z-e^{-\beta\varepsilon_l})(z-e^{-\beta\varepsilon_{l'}})} + \frac{2}{z^2}\sum_{l=1}^{\infty}\frac{e^{-\beta\varepsilon_l}(e^{-\beta\varepsilon_l}+2z)}{(z-e^{-\beta\varepsilon_l})^2}\right)$$

Stąd

$$\operatorname{Res}_{z=e^{-\beta\varepsilon_0}} f = e^{-\beta\varepsilon_0(N-2)}\prod_{k=1}^{\infty}\frac{1}{(1-e^{-\beta(\varepsilon_k-\varepsilon_0)})^2}\left(\frac{N(N-1)}{2} + 2N\sum_{l=1}^{\infty}\frac{1}{1-e^{-\beta(\varepsilon_0-\varepsilon_l)}} + \right.$$

$$\left. + 2\sum_{\substack{l,l'=1\\ l\neq l'}}^{\infty}\frac{1}{(1-e^{-\beta(\varepsilon_0-\varepsilon_l)})(1-e^{-\beta(\varepsilon_0-\varepsilon_{l'})})} + \sum_{l=1}^{\infty}\frac{1}{(1-e^{-\beta(\varepsilon_0-\varepsilon_l)})^2} + \frac{1}{2}\sum_{l=1}^{\infty}\frac{1}{\operatorname{sh}^2\left(\frac{\beta}{2}(\varepsilon_0-\varepsilon_l)\right)}\right)$$

Po uwzgl drugiej całki mamy

$$\langle n_0\rangle = \frac{1}{Z_N(\beta)}\left\{\sum_{p=1}^{\infty}\frac{e^{-\beta\varepsilon_p N}}{(1-e^{-\beta(\varepsilon_0-\varepsilon_p)})^3}\prod_{\substack{k=1\\ k\neq p}}\frac{1}{(1-e^{-\beta(\varepsilon_k-\varepsilon_p)})^2}\left[e^{-\beta(\varepsilon_0-\varepsilon_p)}\left(N+1-\frac{1}{1-e^{-\beta(\varepsilon_0-\varepsilon_p)}}\right.\right.\right.$$

$$\left.\left. + 2\sum_{\substack{l=0\\ l\neq p}}^{\infty}\frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_l)}}\right) + e^{-2\beta(\varepsilon_0-\varepsilon_p)}\left(N-\frac{1}{1-e^{-\beta(\varepsilon_0-\varepsilon_p)}} + 2\sum_{\substack{l=0\\ l\neq p}}^{\infty}\frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_l)}}\right)\right] +$$

$$e^{-\beta\varepsilon_0 N}\prod_{k=1}^{\infty}\frac{1}{(1-e^{-\beta(\varepsilon_0-\varepsilon_k)})^2}\left(N^2 + 2(2N+1)\sum_{l=1}^{\infty}\frac{1}{1-e^{-\beta(\varepsilon_0-\varepsilon_l)}} + 4\sum_{\substack{l,l'=1\\ l\neq l'}}^{\infty}\frac{1}{(1-e^{-\beta(\varepsilon_0-\varepsilon_l)})(1-e^{-\beta(\varepsilon_0-\varepsilon_{l'})})}\right.$$

$$\left.\left. + 2\sum_{l=1}^{\infty}\frac{1}{(1-e^{-\beta(\varepsilon_0-\varepsilon_l)})^2} + \sum_{l=1}^{\infty}\frac{1}{\operatorname{sh}^2\left(\frac{\beta}{2}(\varepsilon_0-\varepsilon_l)\right)}\right)\right\} =$$

$$= \frac{1}{Z_N(\beta)}\left\{\sum_{p=1}^{\infty}\frac{e^{-\beta\varepsilon_p(N-1)}\,e^{-\beta\varepsilon_0}}{(1-e^{-\beta(\varepsilon_0-\varepsilon_p)})^3}\prod_{\substack{k=1\\ k\neq p}}^{\infty}\frac{1}{(1-e^{-\beta(\varepsilon_k-\varepsilon_p)})^2}\left[1+(1+e^{-\beta(\varepsilon_0-\varepsilon_p)})\left(N-\frac{1}{1-e^{-\beta(\varepsilon_0-\varepsilon_p)}}\right.\right.\right.$$

$$\left.\left. + 2\sum_{\substack{l=0\\ l\neq p}}^{\infty}\frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_l)}}\right)\right] + e^{-\beta\varepsilon_0 N}\prod_{k=1}^{\infty}\frac{1}{(1-e^{-\beta(\varepsilon_k-\varepsilon_0)})^2}\left(N^2 + 2(2N+1)\sum_{l=1}^{\infty}\frac{1}{1-e^{-\beta(\varepsilon_0-\varepsilon_l)}} + \right.$$

$$\left.\left. + 4\sum_{\substack{l,l'=1\\ l\neq l'}}^{\infty}\frac{1}{(1-e^{-\beta(\varepsilon_0-\varepsilon_l)})(1-e^{-\beta(\varepsilon_0-\varepsilon_{l'})})} + 2\sum_{l=1}^{\infty}\frac{1}{(1-e^{-\beta(\varepsilon_0-\varepsilon_l)})^2} + \sum_{l=1}^{\infty}\frac{1}{\operatorname{sh}^2(\beta(\varepsilon_0-\varepsilon_l))}\right)\right\}$$