Funkcja korelacji – ściśle

$$\langle \psi^\dagger(x)\hat\psi(x')\rangle = \mathrm{Tr}\left(\hat\varrho\, \hat\psi^\dagger(x)\hat\psi(x')\right) = \frac{1}{Z}\sum_{\{n_k\}} \langle\{n_k\}|\, e^{-\beta\sum_k \varepsilon_k \hat n_k}\, \hat\psi^\dagger(x)\hat\psi(x')\,|\{n_k\}\rangle$$

To już obliczyliśmy

$$\hat\psi(x) = \sum_q \frac{1}{\sqrt{L}} e^{iqx}\,\hat a_q$$

Suma po wszystkich obsadzeniach poziomów takich, że $\sum_k n_k = N$

$$\frac{1}{Z}\sum_{\{n_k\}} \langle\{n_k\}|\frac{1}{L}\, e^{-\beta\sum_k \varepsilon_k n_k} \sum_{q,q'} e^{i(q'x'-qx)}\,\hat a_q^\dagger \hat a_{q'}|\{n_k\}\rangle =$$

$$= \frac{1}{Z}\sum_{\{n_k\}} \frac{1}{L}\sum_{q,q'} e^{i(q'x'-qx)}\, e^{-\beta\sum_k \varepsilon_k n_k} \underbrace{\langle\{n_k\}|\hat a_q^\dagger \hat a_{q'}|\{n_k\}\rangle}_{=\delta_{q,q'} n_q} =$$

$$= \frac{1}{Z}\sum_{\{n_k\}}\sum_q \frac{1}{L}\, e^{iq(x'-x)}\, e^{-\beta\sum_k \varepsilon_k n_k}\, n_q = \sum_q \frac{1}{L}\langle n_q\rangle e^{iq(x'-x)} =$$

$$= \frac{1}{Z}\sum_{n_{-\infty}=0}^{\infty}\cdots\sum_{n_\infty=0}^{\infty} \frac{1}{L}\sum_q e^{iq(x'-x)}\, e^{-\beta\sum_k \varepsilon_k n_k}\, n_q\, \delta_{N,\sum_k n_k} =$$

Korzystamy z $\delta_{N,\sum_k n_k} = \frac{1}{2\pi}\int_0^{2\pi} d\xi\, e^{i\xi(N-\sum_k n_k)}$ i mamy

$$= \frac{1}{2\pi Z}\frac{1}{L}\sum_q e^{iq(x'-x)}\int_0^{2\pi} d\xi\, e^{i\xi N} \underbrace{\sum_{n_{-\infty}=0}^{\infty}\cdots\sum_{n_\infty=0}^{\infty} n_q \prod_k e^{-(i\xi+\beta\varepsilon_k)n_k}} =$$

Wszystkie sumy, oprócz q-tej, to sumy szeregów geometrycznych.
Dla q-tej sumy mamy szereg typu $\sum_{n=0}^{\infty} n q^n = \frac{q}{(1-q)^2}$.

$$= \frac{1}{2\pi Z}\frac{1}{L}\sum_q e^{iq(x'-x)}\int_0^{2\pi} d\xi\, e^{i\xi N}\prod_{k\neq q}\frac{1}{\left(1-e^{-(i\xi+\beta\varepsilon_k)}\right)}\cdot\frac{e^{-(i\xi+\beta\varepsilon_q)}}{\left(1-e^{-(i\xi+\beta\varepsilon_q)}\right)^2} =$$

$$= \frac{1}{2\pi Z}\frac{1}{L}\sum_q e^{iq(x'-x)}\, I_q\,,\quad \text{gdzie}$$

$$I_q = \int_0^{2\pi} d\xi\, e^{i\xi N}\prod_{k\neq q}\frac{1}{\left(1-e^{-(i\xi+\beta\varepsilon_k)}\right)}\,\frac{e^{-(i\xi+\beta\varepsilon_q)}}{\left(1-e^{-(i\xi+\beta\varepsilon_q)}\right)^2}$$

② Obliczanie całek I_q

Zamieniamy zmienne $z = e^{i\xi}$ i dostajemy

$$I_q = \frac{1}{i}\oint_{C(1)} dz\, z^N \prod_{k\neq q} \frac{z}{z - e^{-\beta\varepsilon_k}} \quad \frac{e^{-\beta\varepsilon_q}}{(z - e^{-\beta\varepsilon_q})^2}$$

Całkujemy funkcję $f(z) = z^N \frac{e^{-\beta\varepsilon_q}}{(z - e^{-\beta\varepsilon_q})^2} \prod_{k\neq q} \frac{z}{z - e^{-\beta\varepsilon_k}}$ po okręgu jednostkowym.

$\boxed{q=0}$ Wszystkie bieguny są drugiego rzędu. Funkcję f zapisujemy jako

$$f(z) = z^N \frac{e^{-\beta\varepsilon_0}}{(z - e^{-\beta\varepsilon_0})^2} \prod_{k=1}^{\infty} \frac{z^2}{(z - e^{-\beta\varepsilon_k})^2}$$

Residua obliczamy ze wzoru

$$\operatorname*{Res}_{z=e^{-\beta\varepsilon_p}} f = \lim_{z\to e^{-\beta\varepsilon_p}} \frac{d}{dz} (z - e^{-\beta\varepsilon_p})^2 f(z)$$

Dla $\underline{\underline{p=0}}$ musimy różniczkować wyrażenie

$$z^N e^{-\beta\varepsilon_0} \prod_{k=1}^{\infty} \frac{z^2}{(z - e^{-\beta\varepsilon_k})^2}$$

Korzystamy z wzoru na pochodną iloczynu $\frac{d}{dx}\prod_{j=1}^{\infty} g_j(x) = \prod_{j=1}^{\infty} g_j(x) \sum_{l=1}^{\infty} \frac{g_l'(x)}{g_l(x)}$ i

po różniczkowaniu dostajemy

$$N z^{N-1} e^{-\beta\varepsilon_0} \prod_{k=1}^{\infty} \frac{z^2}{(z - e^{-\beta\varepsilon_k})^2} + z^N e^{-\beta\varepsilon_0} \prod_{k=1}^{\infty} \frac{z^2}{(z - e^{-\beta\varepsilon_k})^2} \left(-\frac{2}{z} \sum_{l=1}^{\infty} \frac{e^{-\beta\varepsilon_l}}{(z - e^{-\beta\varepsilon_l})} \right)$$

Wobec czego

$$\operatorname*{Res}_{z=e^{-\beta\varepsilon_0}} f = e^{-\beta N\varepsilon_0} \prod_{k=1}^{\infty} \frac{1}{(1 - e^{-\beta(\varepsilon_k - \varepsilon_0)})^2} \left[N + 2 \sum_{l=1}^{\infty} \frac{1}{1 - e^{\beta(\varepsilon_0 - \varepsilon_l)}} \right]$$

Natomiast dla $\underline{p\neq 0}$ różniczkujemy $\quad p > 0$

$$z^{N+2} \frac{e^{-\beta\varepsilon_0}}{(z - e^{-\beta\varepsilon_0})^2} \prod_{\substack{k=1\\k\neq p}}^{\infty} \frac{z^2}{(z - e^{-\beta\varepsilon_k})^2}$$

Po zróżniczkowaniu mamy

$$e^{-\beta\varepsilon_0} \frac{z^{N+1}(Nz - e^{-\beta\varepsilon_0}(N+2))}{(z - e^{-\beta\varepsilon_0})^3} \prod_{\substack{k=1\\k\neq p}}^{\infty} \frac{z^2}{(z - e^{-\beta\varepsilon_k})^2} + z^{N+2} \frac{e^{-\beta\varepsilon_0}}{(z - e^{-\beta\varepsilon_0})^2} \prod_{\substack{k=1\\k\neq p}}^{\infty} \frac{z^2}{(z - e^{-\beta\varepsilon_k})^2} \left(-\frac{2}{z} \sum_{\substack{l=1\\l\neq p}} \frac{e^{-\beta\varepsilon_l}}{z - e^{-\beta\varepsilon_l}} \right)$$

Wobec czego

$$\operatorname*{Res}_{z=e^{-\beta\varepsilon_p}} f = e^{-\beta N\varepsilon_p} \frac{e^{-\beta(\varepsilon_0 - \varepsilon_p)}}{(1 - e^{-\beta(\varepsilon_0 - \varepsilon_p)})^2} \prod_{\substack{k=1\\k\neq p}}^{\infty} \frac{1}{(1 - e^{-\beta(\varepsilon_k - \varepsilon_p)})^2} \left[N + 2 \sum_{\substack{l=0!\\l\neq p}} \frac{1}{1 - e^{\beta(\varepsilon_p - \varepsilon_l)}} \right]$$

③ Całkę I_0 obliczamy z tw. o residuach $I_0 = 2\pi i \sum Res$

$$I_0 = 2\pi \Big\{ e^{-\beta N \varepsilon_0} \prod_{k=1}^{\infty} \frac{1}{(1-e^{-\beta(\varepsilon_k-\varepsilon_0)})^2}\Big[N + 2\sum_{l=1} \frac{1}{1-e^{-\beta(\varepsilon_0-\varepsilon_l)}}\Big] +$$

$$+\sum_{p=1}^{\infty} e^{-\beta N \varepsilon_p} \frac{e^{-\beta(\varepsilon_0-\varepsilon_p)}}{(1-e^{-\beta(\varepsilon_0-\varepsilon_p)})^2} \prod_{\substack{k=1\\k\neq p}}^{\infty} \frac{1}{(1-e^{-\beta(\varepsilon_k-\varepsilon_p)})^2}\Big[N + 2\sum_{\substack{l=0\\l\neq p}}^{\infty} \frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_l)}}\Big]\Big\} =$$

$$= 2\pi \sum_{p=0}^{\infty} e^{-\beta N \varepsilon_p} e^{-\beta(\varepsilon_0-\varepsilon_p)} \prod_{\substack{k=0\\k\neq p}}^{\infty} \frac{1}{(1-e^{-\beta(\varepsilon_k-\varepsilon_p)})^2}\Big[N + 2\sum_{\substack{l=0\\l\neq p}}^{\infty} \frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_l)}}\Big]$$

$\boxed{q \neq 0}$ Mamy biegun 1 rzędu, 3 rzędu a reszta jest 2 rzędu

Zapiszmy funkcję f jako

$$f(z) = z^N \frac{e^{-\beta\varepsilon_q}}{(z-e^{-\beta\varepsilon_q})^2} \frac{z}{(z-e^{-\beta\varepsilon_q})} \prod_{\substack{k=1\\k\neq q}}^{\infty} \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2} \frac{z}{z-e^{-\beta\varepsilon_0}} =$$

$$= z^N e^{-\beta\varepsilon_q} \frac{1}{(z-e^{-\beta\varepsilon_q})(z-e^{-\beta\varepsilon_0})} \prod_{k=1}^{\infty} \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2} = \qquad (*)$$

$$= z^{N+2} e^{-\beta\varepsilon_q} \frac{1}{(z-e^{-\beta\varepsilon_q})^3} \frac{1}{z-e^{-\beta\varepsilon_0}} \prod_{\substack{k=1\\k\neq q}}^{\infty} \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2} \qquad (**)$$

Biegun 1 rzędu $\underline{p=0}$ Wspomagamy się przedstawieniem $(**)$

$$\operatorname*{Res}_{z=e^{-\beta\varepsilon_0}} f = e^{-\beta N\varepsilon_0} \cdot e^{-\beta(\varepsilon_q-\varepsilon_0)} \frac{1}{(1-e^{-\beta(\varepsilon_q-\varepsilon_0)})^3} \prod_{\substack{k=1\\k\neq q}}^{\infty} \frac{1}{(1-e^{-\beta(\varepsilon_k-\varepsilon_0)})^2}$$

Bieguny 2 rzędu $\underline{p\neq 0 \wedge p\neq q}$. Używamy przedstawienia $(*)$

Musimy różniczkować wyrażenie

$$z^{N+2} e^{-\beta\varepsilon_q} \frac{1}{(z-e^{-\beta\varepsilon_q})(z-e^{-\beta\varepsilon_0})} \prod_{\substack{k=1\\k\neq p}}^{\infty} \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2}$$

Dostajemy

$$e^{-\beta\varepsilon_q} \cdot \frac{z^{N+1}\big(e^{-\beta(\varepsilon_0+\varepsilon_q)}(N+2) - z(N+1)(e^{-\beta\varepsilon_0}+e^{-\beta\varepsilon_q}) + Nz^2\big)}{(z-e^{-\beta\varepsilon_q})^2(z-e^{-\beta\varepsilon_0})^2} \prod_{\substack{k=1\\k\neq p}}^{\infty} \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2} +$$

$$+ e^{-\beta\varepsilon_q} \frac{z^{N+2}}{(z-e^{-\beta\varepsilon_q})(z-e^{-\beta\varepsilon_0})} \prod_{\substack{k=1\\k\neq p}}^{\infty} \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2} \Big(-\frac{2}{z} \sum_{\substack{l=1\\l\neq p}}^{\infty} \frac{e^{-\beta\varepsilon_l}}{z-e^{-\beta\varepsilon_l}}\Big)$$

4) Otrzymujemy

$$\operatorname*{Res}_{z=e^{-\beta\varepsilon_p}} f = e^{-\beta N\varepsilon_p} e^{-\beta(\varepsilon_q-\varepsilon_p)} \frac{1}{\left(1-e^{-\beta(\varepsilon_q-\varepsilon_p)}\right)\left(1-e^{-\beta(\varepsilon_0-\varepsilon_p)}\right)} \prod_{\substack{k=1\\k\neq p}}^{\infty} \frac{1}{\left(1-e^{-\beta(\varepsilon_k-\varepsilon_p)}\right)^2}\Bigg[N+$$

$$+\frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_q)}}+\frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_0)}}+2\sum_{\substack{l=1\\l\neq p}}^{\infty}\frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_l)}}\Bigg]$$

Pozostał biegun 3 rzędu. Obliczymy go ze wzoru

$$\operatorname*{Res}_{z=e^{-\beta\varepsilon_q}} f = \frac{1}{2}\lim_{z\to e^{-\beta\varepsilon_q}} \frac{d^2}{dz^2}\left(z-e^{-\beta\varepsilon_q}\right)^3 f$$

Wykorzystujemy przedstawienie (**). Musimy dwukrotnie różniczkować

$$z^{N+2} e^{-\beta\varepsilon_q} \frac{1}{z-e^{-\beta\varepsilon_0}} \prod_{\substack{k=1\\k\neq q}}^{\infty} \frac{z^2}{\left(z-e^{-\beta\varepsilon_k}\right)^2}$$

Druga pochodna iloczynu

$$\Big(\prod_j f_j\Big)'' = \Big(\prod_j f_j\Big)\Bigg[\sum_{j\neq j'}\frac{f_j' f_{j'}'}{f_j f_{j'}}+\sum_{l=1}\frac{f_l''}{f_l}\Bigg]$$

Dostajemy

$$e^{-\beta\varepsilon_q}\,\frac{z^N\left(e^{-2\beta\varepsilon_0}(N+1)(N+2)-2e^{-\beta\varepsilon_0}N(N+2)z+N(N+1)z^2\right)}{\left(z-e^{-\beta\varepsilon_0}\right)^3}\prod_{\substack{k=1\\k\neq q}}^{\infty}\frac{z^2}{\left(z-e^{-\beta\varepsilon_k}\right)^2}+$$

$$+2e^{-\beta\varepsilon_q}\,\frac{z^{N+1}\left(z(N+1)-e^{-\beta\varepsilon_0}(N+2)\right)}{\left(z-e^{-\beta\varepsilon_0}\right)^2}\prod_{\substack{k=1\\k\neq q}}^{\infty}\frac{z^2}{\left(z-e^{-\beta\varepsilon_k}\right)^2}\left(-\frac{2}{z}\sum_{\substack{l=1\\l\neq q}}^{\infty}\frac{e^{-\beta\varepsilon_l}}{z-e^{-\beta\varepsilon_l}}\right)+$$

$$+e^{-\beta\varepsilon_q}\,\frac{z^{N+2}}{z-e^{-\beta\varepsilon_0}}\prod_{\substack{k=1\\k\neq q}}^{\infty}\frac{z^2}{\left(z-e^{-\beta\varepsilon_k}\right)^2}\Bigg[\frac{4}{z^2}\sum_{\substack{l,l'=1\\l,l'\neq q\\l\neq l'}}\frac{e^{-\beta(\varepsilon_l+\varepsilon_{l'})}}{\left(z-e^{-\beta\varepsilon_l}\right)\left(z-e^{-\beta\varepsilon_{l'}}\right)}+\frac{2}{z^2}\sum_{\substack{l=1\\l\neq q}}^{\infty}\frac{e^{-\beta\varepsilon_l}\left(e^{-\beta\varepsilon_l}+z\right)}{\left(z-e^{-\beta\varepsilon_l}\right)^2}\Bigg]$$

Stąd

$$\operatorname*{Res}_{z=e^{-\beta\varepsilon_q}} f = e^{-\beta N\varepsilon_q}\frac{1}{1-e^{-\beta(\varepsilon_0-\varepsilon_q)}}\prod_{\substack{k=1\\k\neq q}}^{\infty}\frac{1}{\left(1-e^{-\beta(\varepsilon_k-\varepsilon_q)}\right)^2}\Bigg\{\frac{N(N+1)}{2}+N\frac{1}{\left(1-e^{-\beta(\varepsilon_q-\varepsilon_0)}\right)^2}$$

$$+2\left(N+1+\frac{1}{1-e^{-\beta(\varepsilon_q-\varepsilon_0)}}\right)\sum_{\substack{l=1\\l\neq q}}^{\infty}\frac{1}{1-e^{-\beta(\varepsilon_q-\varepsilon_l)}}+\sum_{\substack{l=0\\l\neq q}}^{\infty}\frac{1}{\left(1-e^{-\beta(\varepsilon_q-\varepsilon_l)}\right)^2}+$$

$$2\sum_{\substack{l,l'=1\\l,l'\neq q\\l\neq l'}}^{\infty}\frac{1}{\left(1-e^{-\beta(\varepsilon_q-\varepsilon_l)}\right)\left(1-e^{-\beta(\varepsilon_q-\varepsilon_{l'})}\right)}+\frac{1}{2}\sum_{\substack{l=1\\l\neq q}}^{\infty}\frac{1}{\operatorname{sh}^2\left(\frac{1}{2}\beta(\varepsilon_q-\varepsilon_l)\right)}-\frac{N}{4}\frac{1}{\operatorname{sh}^2\left(\frac{1}{2}\beta(\varepsilon_q-\varepsilon_0)\right)}\Bigg\}$$

⑤ Z tw. o residuach $I_q = 2\pi i \sum \mathrm{Res}$

$$I_q = 2\pi \Big\{ e^{-\beta N \varepsilon_0} e^{-\beta(\varepsilon_q - \varepsilon_0)} \frac{1}{\left(1-e^{-\beta(\varepsilon_q-\varepsilon_0)}\right)^3} \prod_{\substack{k=1\\k\neq q}}^{\infty} \frac{1}{\left(1-e^{-\beta(\varepsilon_k-\varepsilon_0)}\right)^2} +$$

$$+\sum_{\substack{p=1\\p\neq q}}^{\infty} e^{-\beta N\varepsilon_p} e^{-\beta(\varepsilon_q-\varepsilon_p)} \frac{1}{\left(1-e^{-\beta(\varepsilon_q-\varepsilon_p)}\right)\left(1-e^{-\beta(\varepsilon_0-\varepsilon_p)}\right)} \prod_{\substack{k=1\\k\neq p}}^{\infty} \frac{1}{\left(1-e^{-\beta(\varepsilon_k-\varepsilon_p)}\right)^2} \Big[$$

$$N + \frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_q)}} + \frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_0)}} + 2\sum_{\substack{l=1\\l\neq p}}^{\infty} \frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_l)}} \Big] + e^{-\beta N\varepsilon_q} \frac{1}{1-e^{-\beta(\varepsilon_0-\varepsilon_q)}}$$

$$\prod_{\substack{k=1\\k\neq q}}^{\infty} \frac{1}{\left(1-e^{-\beta(\varepsilon_k-\varepsilon_q)}\right)^2} \Big[\frac{N(N+1)}{2} + N \frac{1}{\left(1-e^{-\beta(\varepsilon_q-\varepsilon_0)}\right)^2} + 2\left(N+1+\frac{1}{1-e^{-\beta(\varepsilon_q-\varepsilon_0)}}\right)$$

$$\sum_{\substack{l=1\\l\neq q}}^{\infty} \frac{1}{1-e^{-\beta(\varepsilon_q-\varepsilon_l)}} + \sum_{\substack{l=0\\l\neq q}}^{\infty} \frac{1}{\left(1-e^{-\beta(\varepsilon_q-\varepsilon_l)}\right)^2} + 2\sum_{\substack{l,l'=1\\l,l'\neq q\\l\neq l'}}^{\infty} \frac{1}{\left(1-e^{-\beta(\varepsilon_q-\varepsilon_l)}\right)\left(1-e^{-\beta(\varepsilon_q-\varepsilon_{l'})}\right)}$$

$$+\frac{1}{2}\sum_{\substack{l=1\\l\neq q}}^{\infty} \frac{1}{\mathrm{sh}^2\left(\frac{1}{2}\beta(\varepsilon_q-\varepsilon_l)\right)} - \frac{N}{4}\frac{1}{\mathrm{sh}^2\left(\frac{1}{2}\beta(\varepsilon_q-\varepsilon_0)\right)} \Big]$$

Ostatecznie

$$\langle \hat{\psi}^{+}(x)\hat{\psi}(x')\rangle = \frac{1}{Z}\frac{1}{2}\left(\frac{I_0}{2\pi} + 2\sum_{q=1}^{\infty}\frac{I_q}{2\pi}\cos\left(q(x'-x)\right)\right)$$