[...]nkcja korelacji – ściśle

$$\langle\hat\Psi^+(x)\hat\Psi(x')\rangle = Tr\left(\hat\varrho\,\hat\Psi^+(x)\hat\Psi(x')\right) = \frac{1}{Z}\sum_{\{n_k\}} \langle\{n_k\}|\, e^{-\beta\sum_k \varepsilon_k \hat n_k}\,\hat\Psi^+(x)\hat\Psi(x')|\{n_k\}\rangle$$

Suma po wszystkich obsadzeniach poziomów takich, że $\sum_k n_k = N$

To już obliczyliśmy

$$\hat\Psi(x) = \sum_q \frac{1}{\sqrt{L}} e^{iqx}\,\hat a_q$$

$$\frac{1}{Z}\sum_{\{n_k\}} \langle\{n_k\}|\frac{1}{L}\, e^{-\beta\sum_k \varepsilon_k n_k} \sum_{q,q'} e^{i(q'x'-qx)}\,\hat a_q^+\hat a_{q'}|\{n_k\}\rangle =$$

$$= \frac{1}{Z}\sum_{\{n_k\}}\frac{1}{L}\sum_{q,q'} e^{i(q'x'-qx)}\, e^{-\beta\sum_k\varepsilon_k n_k}\underbrace{\langle\{n_k\}|\hat a_q^+\hat a_{q'}|\{n_k\}\rangle}_{=\,\delta_{q,q'}n_q} =$$

$$= \frac{1}{Z}\sum_{\{n_k\}}\sum_q \frac{1}{L} e^{iq(x'-x)}\, e^{-\beta\sum_k\varepsilon_k n_k}\, n_q = \sum_q \frac{1}{L}\langle n_q\rangle e^{iq(x'-x)} =$$

$$= \frac{1}{Z}\sum_{n_{-\infty}=0}^{\infty}\cdots\sum_{n_\infty=0}^{\infty}\frac{1}{L}\sum_q e^{iq(x'-x)}\, e^{-\beta\sum_k\varepsilon_k n_k}\, n_q\,\delta_{N,\sum_k n_k} =$$

Korzystamy z $\delta_{N,\sum_k n_k} = \frac{1}{2\pi}\int_0^{2\pi} d\xi\, e^{i\xi(N-\sum_k n_k)}$ i mamy

$$= \frac{1}{2\pi Z}\frac{1}{L}\sum_q e^{iq(x'-x)}\int_0^{2\pi} d\xi\, e^{i\xi N}\underbrace{\sum_{n_{-\infty}=0}^{\infty}\cdots\sum_{n_\infty=0}^{\infty} n_q\prod_k e^{-(i\xi+\beta\varepsilon_k)n_k}}_{} =$$

Wszystkie sumy, oprócz q-tej, to sumy szeregów geometrycznych.
Dla q-tej sumy mamy sumę typu $\sum_{n=0}^{\infty} n q^n = \frac{q}{(1-q)^2}$.

$$= \frac{1}{2\pi Z}\frac{1}{L}\sum_q e^{iq(x'-x)}\int_0^{2\pi} d\xi\, e^{i\xi N}\prod_{k\neq q}\frac{1}{\left(1-e^{-(i\xi+\beta\varepsilon_k)}\right)}\cdot\frac{e^{-(i\xi+\beta\varepsilon_q)}}{\left(1-e^{-(i\xi+\beta\varepsilon_q)}\right)^2} =$$

$$= \frac{1}{2\pi Z}\frac{1}{L}\sum_q e^{iq(x'-x)}\, I_q\,,\ \text{gdzie}$$

$$I_q = \int_0^{2\pi} d\xi\, e^{i\xi N}\prod_{k\neq q}\frac{1}{\left(1-e^{-(i\xi+\beta\varepsilon_k)}\right)}\,\frac{e^{-(i\xi+\beta\varepsilon_q)}}{\left(1-e^{-(i\xi+\beta\varepsilon_q)}\right)^2}$$

) Obliczanie całek I_q

Dokonujemy zamiany $z = e^{i\xi}$ i dostajemy

$$I_q = \frac{1}{i}\oint_{C(1)} dz\, z^N \prod_{k\neq q} \frac{z}{z - e^{-\beta\varepsilon_k}} \quad \frac{e^{-\beta\varepsilon_q}}{(z - e^{-\beta\varepsilon_q})^2}$$

Całkujemy funkcję $f(z) = z^N \frac{e^{-\beta\varepsilon_q}}{(z - e^{-\beta\varepsilon_q})^2} \prod_{k\neq q} \frac{z}{z - e^{-\beta\varepsilon_k}}$ po okręgu jednostkowym.

$\boxed{q=0}$ Wszystkie bieguny są drugiego rzędu. Funkcję f zapisujemy jako

$$f(z) = z^N \frac{e^{-\beta\varepsilon_0}}{(z - e^{-\beta\varepsilon_0})^2} \prod_{k=1}^{\infty} \frac{z^2}{(z - e^{-\beta\varepsilon_k})^2}$$

Residua obliczamy ze wzoru

$$\operatorname*{Res}_{z=e^{-\beta\varepsilon_p}} f = \lim_{z\to e^{-\beta\varepsilon_p}} \frac{d}{dz}\left(z - e^{-\beta\varepsilon_p}\right)^2 f(z)$$

Dla $\underline{\underline{p=0}}$ musimy różniczkować wyrażenie

$$z^N e^{-\beta\varepsilon_0} \prod_{k=1}^{\infty} \frac{z^2}{(z - e^{-\beta\varepsilon_k})^2}$$

Korzystamy z wzoru na pochodną iloczynu $\frac{d}{dx}\prod_{j=1}^{\infty} g_j(x) = \prod_{j=1}^{\infty} g_j(x) \sum_{l=1}^{\infty} \frac{g_j'(x)}{g_j(x)}$ i

po różniczkowaniu dostajemy

$$N z^{N-1} e^{-\beta\varepsilon_0} \prod_{k=1}^{\infty} \frac{z^2}{(z - e^{-\beta\varepsilon_k})^2} + z^N e^{-\beta\varepsilon_0} \prod_{k=1}^{\infty} \frac{z^2}{(z - e^{-\beta\varepsilon_k})^2}\left(-\frac{2}{z}\sum_{l=1}^{\infty} \frac{e^{-\beta\varepsilon_l}}{(z - e^{-\beta\varepsilon_l})}\right)$$

Wobec czego

$$\operatorname*{Res}_{z=e^{-\beta\varepsilon_0}} f = e^{-\beta N\varepsilon_0} \prod_{k=1}^{\infty} \frac{1}{(1 - e^{-\beta(\varepsilon_k-\varepsilon_0)})^2}\left[N + 2\sum_{l=1}^{\infty} \frac{1}{1 - e^{\beta(\varepsilon_0-\varepsilon_l)}}\right]$$

Natomiast dla $\underline{p\neq 0}$ różniczkujemy $\qquad p>0$

$$z^{N+2} \frac{e^{-\beta\varepsilon_0}}{(z - e^{-\beta\varepsilon_0})^2} \prod_{\substack{k=1\\k\neq p}}^{\infty} \frac{z^2}{(z - e^{-\beta\varepsilon_k})^2}$$

Po zróżniczkowaniu mamy

$$e^{-\beta\varepsilon_0}\frac{z^{N+1}(Nz - e^{-\beta\varepsilon_0}(N+2))}{(z - e^{-\beta\varepsilon_0})^3} \prod_{\substack{k=1\\k\neq p}}^{\infty} \frac{z^2}{(z - e^{-\beta\varepsilon_k})^2} + z^{N+2}\frac{e^{-\beta\varepsilon_0}}{(z - e^{-\beta\varepsilon_0})^2} \prod_{\substack{k=1\\k\neq p}}^{\infty} \frac{z^2}{(z - e^{-\beta\varepsilon_k})^2}\left(-\frac{2}{z}\sum_{\substack{l=1\\l\neq p}} \frac{e^{-\beta\varepsilon_l}}{z - e^{-\beta\varepsilon_l}}\right)$$

Wobec czego

$$\operatorname*{Res}_{z=e^{-\beta\varepsilon_p}} f = e^{-\beta N\varepsilon_p} \frac{e^{-\beta(\varepsilon_0-\varepsilon_p)}}{(1 - e^{-\beta(\varepsilon_0-\varepsilon_p)})^2} \prod_{\substack{k=1\\k\neq p}}^{\infty} \frac{1}{(1 - e^{-\beta(\varepsilon_k-\varepsilon_p)})^2}\left[N + 2\sum_{\substack{l=0!\\l\neq p}} \frac{1}{1 - e^{\beta(\varepsilon_p-\varepsilon_l)}}\right]$$

③ Całkę I_0 obliczamy z tw. o residuach $I_0 = 2\pi i \sum \mathrm{Res}$

$$I_0 = 2\pi \Big\{ e^{-\beta N\varepsilon_0} \prod_{k=1}^{\infty} \frac{1}{(1-e^{-\beta(\varepsilon_k-\varepsilon_0)})^2}\Big[N + 2\sum_{l=1}^{\infty}\frac{1}{1-e^{-\beta(\varepsilon_0-\varepsilon_l)}}\Big] +$$

$$+\sum_{p=1}^\infty e^{-\beta N\varepsilon_p}\frac{e^{-\beta(\varepsilon_0-\varepsilon_p)}}{(1-e^{-\beta(\varepsilon_0-\varepsilon_p)})^2}\prod_{\substack{k=1\\k\neq p}}^\infty \frac{1}{(1-e^{-\beta(\varepsilon_k-\varepsilon_p)})^2}\Big[N+2\sum_{\substack{l=0\\l\neq p}}^\infty \frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_l)}}\Big]\Big\} =$$

$$= 2\pi\sum_{p=0}^\infty e^{-\beta N\varepsilon_p}\, e^{-\beta(\varepsilon_0-\varepsilon_p)}\prod_{\substack{k=0\\k\neq p}}^\infty \frac{1}{(1-e^{-\beta(\varepsilon_k-\varepsilon_p)})^2}\Big[N+2\sum_{\substack{l=0\\l\neq p}}^\infty\frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_l)}}\Big]$$

$\boxed{q\neq 0}$ Mamy biegun 1 rzędu, 3 rzędu a reszta jest 2 rzędu

Zapiszmy funkcję f jako

$$f(z) = z^N \frac{e^{-\beta\varepsilon_q}}{(z-e^{-\beta\varepsilon_q})^2}\,\frac{z}{(z-e^{-\beta\varepsilon_q})}\prod_{\substack{k=1\\k\neq q}}^\infty \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2}\,\frac{z}{z-e^{-\beta\varepsilon_0}} =$$

$$= z^N e^{-\beta\varepsilon_q}\frac{1}{(z-e^{-\beta\varepsilon_q})(z-e^{-\beta\varepsilon_0})}\prod_{k=1}^\infty\frac{z^2}{(z-e^{-\beta\varepsilon_k})^2} = \qquad (*)$$

$$= z^{N+2} e^{-\beta\varepsilon_q}\frac{1}{(z-e^{-\beta\varepsilon_q})^3}\,\frac{1}{z-e^{-\beta\varepsilon_0}}\prod_{\substack{k=1\\k\neq q}}^\infty \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2} \qquad (**)$$

Biegun 1 rzędu $\underline{p=0}$ Wspomagamy się przedstawieniem $(**)$

$$\mathop{\mathrm{Res}}_{z=e^{-\beta\varepsilon_0}} f = e^{-\beta N\varepsilon_0}\cdot e^{-\beta(\varepsilon_q-\varepsilon_0)}\frac{1}{(1-e^{-\beta(\varepsilon_q-\varepsilon_0)})^3}\prod_{\substack{k=1\\k\neq q}}^\infty \frac{1}{(1-e^{-\beta(\varepsilon_k-\varepsilon_0)})^2}$$

Bieguny 2 rzędu $\underline{p\neq 0 \wedge p\neq q}$. Używamy przedstawienia $(*)$

Musimy różniczkować wyrażenie

$$z^{N+2}e^{-\beta\varepsilon_q}\frac{1}{(z-e^{-\beta\varepsilon_q})(z-e^{-\beta\varepsilon_0})}\prod_{\substack{k=1\\k\neq p}}^\infty\frac{z^2}{(z-e^{-\beta\varepsilon_k})^2}$$

Dostajemy

$$e^{-\beta\varepsilon_q}\cdot\frac{z^{N+1}\left(e^{-\beta(\varepsilon_0+\varepsilon_q)}(N+2) - z(N+1)(e^{-\beta\varepsilon_0}+e^{-\beta\varepsilon_q}) + Nz^2\right)}{(z-e^{-\beta\varepsilon_q})^2\,(z-e^{-\beta\varepsilon_0})^2}\prod_{\substack{k=1\\k\neq p}}^\infty \frac{z^2}{(z-e^{-\beta\varepsilon_k})^2} +$$

$$+ e^{-\beta\varepsilon_q}\frac{z^{N+2}}{(z-e^{-\beta\varepsilon_q})(z-e^{-\beta\varepsilon_0})}\prod_{\substack{k=1\\k\neq p}}^\infty\frac{z^2}{(z-e^{-\beta\varepsilon_k})^2}\left(-\frac{2}{z}\sum_{\substack{l=1\\l\neq p}}^\infty\frac{e^{-\beta\varepsilon_l}}{z-e^{-\beta\varepsilon_l}}\right)$$

+) Otrzymujemy

$$\operatorname*{Res}_{z=e^{-\beta\varepsilon_p}} f = e^{-\beta N\varepsilon_p} e^{-\beta(\varepsilon_q-\varepsilon_p)} \frac{1}{\left(1-e^{-\beta(\varepsilon_q-\varepsilon_p)}\right)\left(1-e^{-\beta(\varepsilon_0-\varepsilon_p)}\right)} \prod_{\substack{k=1\\k\neq p}}^{\infty} \frac{1}{\left(1-e^{-\beta(\varepsilon_k-\varepsilon_p)}\right)^2} \Bigg[N +$$

$$+ \frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_q)}} + \frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_0)}} + 2\sum_{\substack{l=1\\l\neq p}}^{\infty} \frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_l)}} \Bigg]$$

Pozostał biegun 3 rzędu. Obliczamy go ze wzoru

$$\operatorname*{Res}_{z=e^{-\beta\varepsilon_q}} f = \frac{1}{2}\lim_{z\to e^{-\beta\varepsilon_q}} \frac{d^2}{dz^2}\left(z-e^{-\beta\varepsilon_q}\right)^3 f$$

Wykorzystujemy przedstawienie (**). Musimy dwukrotnie różniczkować

$$z^{N+2} e^{-\beta\varepsilon_q} \frac{1}{z-e^{-\beta\varepsilon_0}} \prod_{\substack{k=1\\k\neq q}}^{\infty} \frac{z^2}{\left(z-e^{-\beta\varepsilon_k}\right)^2}$$

Druga pochodna iloczynu

$$\left(\prod_j f_j\right)'' = \left(\prod_j f_j\right)\left[\sum_{j\neq j'} \frac{f_j' f_{j'}'}{f_j f_{j'}} + \sum_{l=1}^{\infty} \frac{f_l''}{f_l}\right]$$

Dostajemy

$$e^{-\beta\varepsilon_q} \frac{z^N\left(e^{-2\beta\varepsilon_0}(N+1)(N+2) - 2e^{-\beta\varepsilon_0}N(N+2)z + N(N+1)z^2\right)}{\left(z-e^{-\beta\varepsilon_0}\right)^3} \prod_{\substack{k=1\\k\neq q}}^{\infty} \frac{z^2}{\left(z-e^{-\beta\varepsilon_k}\right)^2} +$$

$$+ 2e^{-\beta\varepsilon_q} \frac{z^{N+1}\left(z(N+1) - e^{-\beta\varepsilon_0}(N+2)\right)}{\left(z-e^{-\beta\varepsilon_0}\right)^2} \prod_{\substack{k=1\\k\neq q}}^{\infty} \frac{z^2}{\left(z-e^{-\beta\varepsilon_k}\right)^2} \left(-\frac{2}{z}\sum_{\substack{l=1\\l\neq q}}^{\infty} \frac{e^{-\beta\varepsilon_l}}{z-e^{-\beta\varepsilon_l}}\right) +$$

$$+ e^{-\beta\varepsilon_q} \frac{z^{N+2}}{z-e^{-\beta\varepsilon_0}} \prod_{\substack{k=1\\k\neq q}}^{\infty} \frac{z^2}{\left(z-e^{-\beta\varepsilon_k}\right)^2} \left[\frac{4}{z^2} \sum_{\substack{l,l'=1\\l,l'\neq q\\l\neq l'}} \frac{e^{-\beta(\varepsilon_l+\varepsilon_{l'})}}{\left(z-e^{-\beta\varepsilon_l}\right)\left(z-e^{-\beta\varepsilon_{l'}}\right)} + \frac{2}{z^2}\sum_{\substack{l=1\\l\neq q}}^{\infty} \frac{e^{-\beta\varepsilon_l}\left(e^{-\beta\varepsilon_l}+z\right)}{\left(z-e^{-\beta\varepsilon_l}\right)^2}\right]$$

Stąd

$$\operatorname*{Res}_{z=e^{-\beta\varepsilon_q}} f = e^{-\beta N\varepsilon_q} \frac{1}{1-e^{-\beta(\varepsilon_0-\varepsilon_q)}} \prod_{\substack{k=1\\k\neq q}}^{\infty} \frac{1}{\left(1-e^{-\beta(\varepsilon_k-\varepsilon_q)}\right)^2} \Bigg\{ \frac{N(N+1)}{2} + N\frac{1}{\left(1-e^{-\beta(\varepsilon_q-\varepsilon_0)}\right)^2}$$

$$+ 2\left(N+1+\frac{1}{1-e^{-\beta(\varepsilon_q-\varepsilon_0)}}\right)\sum_{\substack{l=1\\l\neq q}}^{\infty} \frac{1}{1-e^{-\beta(\varepsilon_q-\varepsilon_l)}} + \sum_{\substack{l=0\\l\neq q}}^{\infty} \frac{1}{\left(1-e^{-\beta(\varepsilon_q-\varepsilon_l)}\right)^2} +$$

$$2\sum_{\substack{l,l'=1\\l,l'\neq q\\l\neq l'}}^{\infty} \frac{1}{\left(1-e^{-\beta(\varepsilon_q-\varepsilon_l)}\right)\left(1-e^{-\beta(\varepsilon_q-\varepsilon_{l'})}\right)} + \frac{1}{2}\sum_{\substack{l=1\\l\neq q}}^{\infty} \frac{1}{\operatorname{sh}^2\left(\frac{1}{2}\beta(\varepsilon_q-\varepsilon_l)\right)} - \frac{N}{4}\frac{1}{\operatorname{sh}^2\left(\frac{1}{2}\beta(\varepsilon_q-\varepsilon_0)\right)} \Bigg\}$$

(5) Z tw. o residuach $I_q = 2\pi i \sum \mathrm{Res}$

$$I_q = 2\pi \Big\{ e^{-\beta N\varepsilon_0} e^{-\beta(\varepsilon_q-\varepsilon_0)} \frac{1}{(1-e^{-\beta(\varepsilon_q-\varepsilon_0)})^3} \prod_{\substack{k=1\\ k\neq q}}^{\infty} \frac{1}{(1-e^{-\beta(\varepsilon_k-\varepsilon_0)})^2} +$$

$$+ \sum_{\substack{p=1\\ p\neq q}}^\infty e^{-\beta N\varepsilon_p} e^{-\beta(\varepsilon_q-\varepsilon_p)} \frac{1}{(1-e^{-\beta(\varepsilon_q-\varepsilon_p)})(1-e^{-\beta(\varepsilon_q-\varepsilon_0)})} \prod_{\substack{k=1\\ k\neq p}}^\infty \frac{1}{(1-e^{-\beta(\varepsilon_k-\varepsilon_p)})^2}\Big[$$

$$N + \frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_q)}} + \frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_0)}} + 2\sum_{\substack{l=1\\ l\neq p}}^\infty \frac{1}{1-e^{-\beta(\varepsilon_p-\varepsilon_l)}}\Big] + e^{-\beta N\varepsilon_q}\frac{1}{1-e^{-\beta(\varepsilon_0-\varepsilon_q)}}$$

$$\prod_{\substack{k=1\\ k\neq q}}^\infty \frac{1}{(1-e^{-\beta(\varepsilon_k-\varepsilon_q)})^2}\Big[\frac{N(N+1)}{2} + N\frac{1}{(1-e^{-\beta(\varepsilon_q-\varepsilon_0)})^2} + 2\Big(N+1+\frac{1}{1-e^{-\beta(\varepsilon_q-\varepsilon_0)}}\Big)$$

$$\sum_{\substack{l=1\\ l\neq q}}^\infty \frac{1}{1-e^{-\beta(\varepsilon_q-\varepsilon_l)}} + \sum_{\substack{l=0\\ l\neq q}}^\infty \frac{1}{(1-e^{-\beta(\varepsilon_q-\varepsilon_l)})^2} + 2\sum_{\substack{l,l'=1\\ l,l'\neq q\\ l\neq l'}}^\infty \frac{1}{(1-e^{-\beta(\varepsilon_q-\varepsilon_l)})(1-e^{-\beta(\varepsilon_q-\varepsilon_{l'})})}$$

$$+\frac{1}{2}\sum_{\substack{l=1\\ l\neq q}}^\infty \frac{1}{\sinh^2(\frac{1}{2}\beta(\varepsilon_q-\varepsilon_l))} - \frac{N}{4} \frac{1}{\sinh^2(\frac{1}{2}\beta(\varepsilon_q-\varepsilon_0))}\Big]$$

Ostatecznie

$$\langle\hat\psi^\dagger(x)\hat\psi(x')\rangle = \frac{1}{Z}\frac{1}{2}\Big(\frac{I_0}{2\pi} + 2\sum_{q=1}^\infty \frac{I_q}{2\pi}\cos(q(x'-x))\Big)$$