Bayesian Inference

Session 10

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Summary

Bayesian Hypothesis Testing

Bayesian Hypothesis Testing

- In the Bayesian context, the problem of deciding about which hypothesis to accept is conceptually simple. Typically, one would compare the hypotheses $H_1, \ldots, H_k$ through their respective posterior probabilities, obtained via Bayes theorem as

$$p(H_i|\mathbf{x}) \propto p(\mathbf{x}|H_i)p(H_i).$$

- Once again, this setup can be framed as a decision problem. In addition to the posterior probabilities attached to the hypotheses (or states of nature), a loss structure with the possible actions can be incorporated.

Bayesian Hypothesis Testing

- In the context of hypothesis testing procedure framed as a decision problem, one would have $\mathcal{A} = \{d_0, d_1\}$, where d_i is to accept $H_i : \theta \in \Theta_i$, (for $i = 0, 1$) as the most plausible hypothesis. Under this setting $\Theta = \Theta_0 \cup \Theta_1$. In this situation a hypothesis testing procedure can be seen as a decision function $\delta : \Omega \to \{d_0, d_1\}$.
  Example 1:

Θ	δ	
	d_0	d_1
H_0	0	a
H_1	b	0

Table:

Bayesian Hypothesis Testing

In this sense, the Bayesian testing procedure can be presented as

$$\delta^* = \begin{cases} \mathsf{d}_1 & p(H_0|\mathbf{x}) \leq \frac{b}{a+b} \\ \mathsf{d}_0 & \text{otherwise.} \end{cases}$$

Assuming equal error loss, if $p(H_0|\mathbf{x}) > p(H_1|\mathbf{x})$ then H_0 should be accepted as the most plausible hypothesis for θ. In this case, it can be said that H_0 is preferable to H_0. Otherwise, H_1 is preferred to H_0.

Bayesian Hypothesis Testing

An alternative way to test hypotheses is by using **Bayes factors**, which are defined as follows:

Type of test

$$B_{01} = \frac{m_0(\mathbf{y})}{m_1(\mathbf{y})} = \frac{f(\mathbf{y}|\theta_0)}{f(\mathbf{y}|\theta_1)}, \qquad H_0 : \theta = \theta_0 \text{ vs. } H_1 : \theta = \theta_1$$

$$B_{01} = \frac{m_0(\mathbf{y})}{m_1(\mathbf{y})} = \frac{f(\mathbf{y}|\theta_0)}{\int_{\Theta_1} f(\mathbf{y}|\theta) f(\theta) d\theta}, \qquad H_0 : \theta = \theta_0 \text{ vs. } H_1 : \theta \in \Theta_1$$

$$B_{01} = \frac{m_0(\mathbf{y})}{m_1(\mathbf{y})} = \frac{\int_{\Theta_0} f(\mathbf{y}|\theta) f(\theta) d\theta}{\int_{\Theta_1} f(\mathbf{y}|\theta) f(\theta) d\theta}, \qquad H_0 : \theta \in \Theta_0 \text{ vs. } H_1 : \theta \in \Theta_1$$

Bayesian Hypothesis Testing

The concept of Bayes factors was introduced by Jeffreys (1961). The following table presents some cutoff points that can help interpret the values one would obtain from the Bayes factor.

Cutoff	Interpretation
$B_{01} < 1$	Evidence supporting H_1
$B_{01} \in (1, 3)$	Weak evidence in favor of H_0 (not enough to make a decision)
$B_{01} \in (3, 20)$	Strong evidence in favor of H_0
$B_{01} \in (20, 150)$	Solid evidence in favor of H_0
$B_{01} > 150$	Extreme evidence in favor of H_0

Table: Caption

Bayesian Hypothesis Testing

Example 1: Let us suppose that one is interested to test

$$\begin{aligned} H_0 : &\quad \theta = 1/2 \\ H_1 : &\quad \theta > 1/2, \end{aligned}$$

where θ represents the probability of success in a Bernoulli process. Assuming that $\theta \sim U(0,1)$ and after observing the first six trials (1,1,1,1,1,0) compute B_{01}.

Bayesian Hypothesis Testing

Example 2: One wants to perform inference about λ: the number of goals scored by the visitor team in a local tournament. The hypotheses of interest are

$$H_0 : \lambda = 1 \;\; vs \;\; H_1 : \lambda = 2$$

After observing seven matches, the number of goals scored by the visitor is 3, 1, 0,1,0,0,1.

Bayesian Hypothesis Testing

Example 3: Now the interest is about the number of goals scored by the local team. One wishes to perform the following test:

$$H_0 : \lambda \leq 1 \quad vs \quad H_1 : \lambda > 1$$

After observing the games of the first four dates of the season:

Scored goals	0	1	2	3
Frequency	19	13	3	1

Table:

Assuming that $\lambda \sim Gamma(0.1, 0.1)$ compute B_{01}.

Bayesian Hypothesis Testing

The Bayes factor can also be used to compare candidate models for **y**. Suppose that one has K candidate models, where the $i$ht model has likelihood $f_i(\mathbf{y}|\theta_i)$ and θ_i has density $f(\theta_i)$, for $i = 1, \ldots K$. Thus, the Bayes factor is given by:

$$B_{ij} = \frac{m_i(\mathbf{y})}{m_j(\mathbf{y})} = \frac{\int f_i(\mathbf{y}|\theta_i) f_i(\theta_i)}{\int f_j(\mathbf{y}|\theta_j) f_j(\theta_j)}$$

Bayesian Hypothesis Testing

Example 4: More hypothesis testing!
Let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be a random sample from the $N(\theta, 10)$ and assume one wishes to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. Supposing $\theta | H_1 \sim N(\mu, w)$ compute B_{01}.

The End