# The Halpin-Tsai Equations: A Review

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The Halpin-Tsai equations are based upon the "self-consistent micromechanics method" developed by Hill. Hermans employed this model to obtain a solution in terms of Hill's "reduced moduli". Halpin and Tsai have reduced Hermans' solution to a simpler analytical form and extended its use for a variety of filament geometries. The development of these micromechanics relationships, which form the operational bases for the composite analogy of Halpin and Kardos for semi-crystalline polymers, are reviewed herein.

## INTRODUCTION

Random or nearly random distributions of reinforcements, finite in dimension and arranged in a matrix, constitute many naturally occurring and synthetic materials. In fact, crystalline polymers are hypothesized to behave (1-6), with respect to their mechanical properties, as a multiphase composite solid. In most naturally occurring and synthetic materials, the spatial orientation of the discontinuous reinforcement is between a truly random three-dimensional arrangement and a truly random two-dimensional arrangement. Halpin and Pagano (7, 8) have shown that such materials can be modeled mathematically as laminated systems. Subsequent work has supported this hypothesis (1-6, 9). This composite laminate-analogy has been extended to semicrystalline polymers by Halpin and Kardos (1-3, 6).

The concept of the composite analogy for semicrystalline polymers has its basis in the fact that it is possible to construct a material having isotropic mechanical properties from layers or plies of a material which has anisotropic mechanical properties. The mechanical properties become isotropic as the number of distinct equal-angular ply orientations increases (7, 8). The mechanical properties of the quasi-isotropic state become a unique function of the anisotropic properties of the reference ply material. Anisotropic ply properties are, in turn, defined by the specifics of the heterogeneous phases which make up the oriented ply structure. This methodology is denoted by the term micromechanics.

The objective of this communication is to review the development of the micromechanics relationships known as the Halpin-Tsai equations. These relationships are central to the efforts to develop relationships sensitive to differences in crystalline morphology and the attendant changes in the mechanical characteristics of semicrystalline polymeric solids. The micromechanics employed in this development are based upon the "self-consistent method" developed by Hill (10). Hill rigorously modeled the composite as a single fiber, encased in a cylinder of matrix; with both embedded in an unbounded homogeneous medium which is macroscopically indistinguishable from the composite. Hermans (11) employed this model to obtain a solution in terms of Hill's "reduced moduli". Halpin and Tsai have reduced Hermans' results to a simpler analytical form and extended its use to a wide variety of reinforcement geometries. Crystallite morphology in semicrystalline polymers ranges from fibriller-like crystallite geometries in fibers and highly drawn bulk sheets to a ribbon or lamella–like geometry in spherilitic bulk polymers. This range of morphologies can be approached rigorously through use of the Halpin-Tsai equations.

## DETAILS OF THE DERIVATION OF THE HALPIN-TSAI EQUATIONS

### Relations Between Hill's Notation and Stiffness and Engineering Constants

The Halpin-Tsai equations are the handy forms of Hill's generalized self-consistent model results with engineering approximations to make them suitable for the designing of composite materials. We will therefore first discuss Hill's results (10). Hill assumed a composite cylinder model in which the embedded phase consisted of continuous and perfectly aligned cylindrical fibers. Both materials were assumed to be homogeneous and elastically transversely isotropic about the fiber direction. The cross-section and spatial arrangement of the fibers were subject merely to a requirement of statistical homogeneity and transverse isotropy. Alternatively, Kerner's (13) method may be used in which each fiber is assumed to behave as though it were surrounded by a concentric matrix cylinder; outside this cylinder lies a body with the properties of the composite. The main results following either assumption are the same (12).

For the above mentioned composite material which is transversely isotropic about the fiber direction, taken as the "1" direction, Hooke's law can be written from any standard book on elasticity (12) as

$$\begin{bmatrix} \bar\sigma_1 \\ \bar\sigma_2 \\ \bar\sigma_3 \\ \bar\sigma_4 \\ \bar\sigma_5 \\ \bar\sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \bar\epsilon_1 \\ \bar\epsilon_2 \\ \bar\epsilon_3 \\ \bar\epsilon_4 \\ \bar\epsilon_5 \\ \bar\epsilon_6 \end{bmatrix}$$

$C_{44} = \frac{1}{2}(C_{22} - C_{23}); C_{55} = C_{66}$ for transverse isotropy.

The above relations between the stress and strain components averaged throughout the composite may be rewritten as (overhead bar indicates average)

$$\bar\sigma_1 = C_{11}\bar\epsilon_1 + C_{12}(\bar\epsilon_2 + \bar\epsilon_3) \quad (1)$$

$$\bar\sigma_2 = C_{12}\bar\epsilon_1 + C_{22}\bar\epsilon_2 + C_{23}\bar\epsilon_3 \quad (2)$$

$$\bar\sigma_3 = C_{12}\bar\epsilon_1 + C_{23}\bar\epsilon_2 + C_{22}\bar\epsilon_3 \quad (3)$$

$$\bar\sigma_4 = C_{44}\bar\epsilon_4 = \tfrac{1}{2}(C_{22} - C_{23})\bar\epsilon_4 \quad (4)$$

$$\bar\sigma_5 = C_{55}\bar\epsilon_5 \quad (5)$$

$$\bar\epsilon_6 = C_{66}\bar\epsilon_6 \quad (6)$$

We will now show relations between Hill's notation and the standard notation used above. Hill used the following notation to write these relations as

$$\tfrac{1}{2}(\bar\sigma_2 + \bar\sigma_3) = k(\bar\epsilon_2 + \bar\epsilon_3) + l\bar\epsilon_1 \quad (7)$$

$$\bar\sigma_1 = l(\bar\epsilon_2 + \bar\epsilon_3) + n\bar\epsilon_1 \quad (8)$$

$$(\bar\sigma_2 - \bar\sigma_3) = 2m(\bar\epsilon_2 - \bar\epsilon_3) \quad (9)$$

$$\bar\sigma_{23} = 2m\bar\epsilon_{23} \text{ (i.e., } \bar\sigma_4 = m\bar\epsilon_4) \quad (10)$$

$$\bar\sigma_{13} = 2\mu\bar\epsilon_{13} \quad (11)$$

$$\bar\sigma_{12} = 2\mu\bar\epsilon_{12} \quad (12)$$

The $C_{ij}$ terms can be expressed in Hill's notation by the following manipulation,

$$\tfrac{1}{2}(Eq\ 2 + Eq\ 3) = \tfrac{1}{2}(\bar\sigma_2 + \bar\sigma_3) = C_{12}\bar\epsilon_1 + \tfrac{1}{2}(C_{22} + C_{23})(\bar\epsilon_2 + \bar\epsilon_3)$$

Comparing this with *Eq 7*

$$C_{12} = l;\ \tfrac{1}{2}(C_{22} + C_{23}) = k$$

Comparison of *Eqs 1* and *8* gives,

$$C_{12} = l;\ C_{11} = n$$

Comparing *Eqs 2* and *3* with *Eq 9*

$$C_{22} - C_{23} = 2m$$

Therefore $C_{22} = k + m; C_{23} = k - m$. From *Eqs 5, 6, 11*, and *12*, $C_{55} = C_{66} = \mu$.

The $C_{ij}$ matrix is now rewritable in Hill's notation.

$$C_{ij} = \begin{bmatrix} n & l & l & 0 & 0 & 0 \\ l & k+m & k+m & 0 & 0 & 0 \\ l & k+m & k+m & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

We can correlate Hill's notation with the moduli of elasticity since we know the relations between $C_{ij}$ and $k$, $l$, $m$, $n$, $\mu$, and $C_{ij}$ are expressed in terms of the moduli of elasticity. *Table 1* summarizes these relations (they apply to both constituent and composite moduli) for transversely isotropic phases.

### Derivation of Hermans' Results

We will now discuss the generalized self-consistent model analysis of a composite with aligned continuous fibers by J. J. Hermans (11). The composite is treated by assuming each fiber to behave as though it were surrounded by a cylinder of pure matrix; outside this cylinder lies a body with the properties of the composite.

(Notation: The subscripts '$f$' and '$m$' denote the fiber (embedded) phase and matrix phase respectively. The volume fractions are $\phi$ and $(1 - \phi)$, respectively. Symbols with a bar and without the material subscript represent the composite properties. Cylindrical co-ordinates, $r$, $\theta$, and $z$ are used; $z$ is in the fiber direction.)

Each fiber of radius $a$ and elastic constants $k_f$, $l_f$, $n_f$, $m_f$, $\mu_f$ is imagined to be embedded in a cylinder of radius $R$ in which the elastic constants are $k_m$, $l_m$, $n_m$, $m_m$, $\mu_m$ (Hill's notation is used here also). This cylinder in turn is surrounded by an unbounded homogeneous medium which is macroscopically indistinguishable from the composite. Suppose that a uniform transverse radial stress $s$ is applied at infinity and that $\epsilon_z$ is kept equal to zero by a necessary longitudinal stress $\sigma_z$. Under these

**Table 1. Relations Between Hill's (10) Notation and Stiffness and Engineering Constants**

| Moduli in Hill's notation | $C_{ij}$ | Engineering constants |
|---|---|---|
| k (plane strain bulk modulus) | $1/2(C_{22} + C_{23})$ | $\frac{E_{22}}{2(1 - \nu_{23} - 2\nu_{12}\nu_{21})}$ |
| l | $C_{12} = C_{13}$ | $\frac{\nu_{21}E_{11}}{(1 - \nu_{23} - 2\nu_{12}\nu_{21})} = \frac{\nu_{12}E_{22}}{(1 - \nu_{23} - 2\nu_{12}\nu_{21})}$ |
| n | $C_{11}$ | $\frac{(1 - \nu_{23})E_{11}}{(1 - \nu_{23} - 2\nu_{12}\nu_{21})}$ |
| m (transverse shear modulus) | $C_{44}$ | $G_{23}$ |
| $\mu$ (longitudinal shear modulus) | $C_{55} = C_{66}$ | $G_{12}$ |
| l/(2k) | $C_{12}/(C_{22} + C_{23})$ | $\nu_{12}$ |
| $n - l^2/k$ | $C_{11}\left[\frac{2C_{12}^2}{C_{22} + C_{23}}\right]$ | $E_{11}$ |
| $\frac{4(kn - l^2)m}{(k + m)n - l^2}$ | $\frac{[C_{11}(C_{22} + C_{23}) - 2C_{12}^2](C_{22} - C_{23})}{C_{12}C_{22} - C_{12}^2}$ | $E_{22}$ |

conditions the displacement in the radial direction, $u_r$, would be a function of $r$ alone, (since there is symmetry about $\theta$) and may be written as

$$u_r = Ar + B/r,\ u_\theta = 0$$

The corresponding strains are given by

$$\epsilon_r = \frac{\partial u_r}{\partial r} = A - \frac{B}{r^2};\ \epsilon_\theta = \frac{1}{r}\frac{\partial u_\theta^{\,0}}{\partial \theta} + \frac{u_r}{r} = \frac{u_r}{r} = A + B/r^2$$

$$\epsilon_r + \epsilon_\theta = \left(A - \frac{B}{r^2} + A + \frac{B}{r^2}\right) = 2A$$

$$\epsilon_r - \epsilon_\theta = -2B/r^2$$

$$\epsilon_{r\theta} = ½\left(\frac{1}{r}\frac{\partial u_r^{\,0}}{\partial \theta} - \frac{u_\theta^{\,0}}{r} + \frac{\partial u_\theta^{\,0}}{\partial r}\right) = 0$$

Using constitutive *Eq 7*, $\sigma_r + \sigma_\theta = 4kA$. ("1"$\rightarrow z$, "2"$\rightarrow r$, "3"$\rightarrow \theta$ are the relations between the two co-ordinate systems.) From *Eq 9*, $\sigma_r - \sigma_\theta = \frac{-4mB}{r^2}$. From *Eq 10*, $\sigma_{r\theta} = 0$. In the core (fiber; $r < a$), because of finite strain $B = B_f = 0$, the strain in the core is uniform. In the shell ($a < r < R$); $A = A_m$ and $B = B_m$. In the remainder of the body ($r > R$), where properties match with the composite,

$2\bar{k}A = S$ ($\bar{k}$ is the modulus for in-plane strain)

The volume average of $\epsilon_r + \epsilon_\theta$ is

$$\overline{\epsilon_r} + \overline{\epsilon_\theta} = 2A_f\phi + 2A_m\pi N(R^2 - a^2) + 2Ac^* \quad (13)$$

where $N$ is the number of fibers per unit volume and $c^* = 1 - \pi NR^2$. Also if $\bar{k}$ is the modulus of the composite as a whole, the stress averaged through the unit volume is

$$\bar{k}(\overline{\epsilon_r} + \overline{\epsilon_\theta}) = 2k_fA_f\phi + 2k_mA_m\pi N(R^2 - a^2) + 2k_mAc^* \quad (14)$$

Substituting for $(\overline{\epsilon_r} + \overline{\epsilon_\theta})$ from *Eq 13*

$$\bar{k} = \frac{k_fA_f\phi + k_mA_m\pi N(R^2 - a^2) + k_mAc^*}{A_f\phi + A_m\pi N(R^2 - a^2) + Ac^*}$$

As there are $N$ fibers in a unit volume, each having $\pi R^2$ as the average volume available to it, we may assume $\pi R^2N = 1$ which simplifies the above equation to

$$\bar{k} = \frac{k_fA_f\phi + k_mA_m(1 - \phi)}{A_f\phi + A_m(1 - \phi)} \quad (15)$$

The ratio of $A_f/A_m$ is to be calculated from the boundary conditions; $u_r$ and $\sigma_r$ are continuous at $r = a$. The same conditions are valid at $r = R$, but these are not necessary to evaluate $\bar{k}$.

It should be again mentioned here that the theory assumes each fiber to behave as though it were surrounded by a concentric cylinder which has the elastic moduli of the matrix. This determines the stress and strain field inside and around the fiber, but one should not conclude that a volume fraction $c^*$ of the system consists of material with modulus $\bar{k}$ (11). From the first boundary condition that $u_r$ is continuous at $r = a$,

$$A_f = A_m + B_m/a^2$$

$\sigma_r = 2(kA - mB/r^2)$. Using the second boundary condition that $\sigma_r$ is continuous at $r = a$, gives $k_fA_f = k_mA_m - m_mB_m/a^2$. Substituting $B_m/a^2 = A_f - A_m$ from the first equation above into the second equation yields

$$(k_f + m_m)A_f = (k_m + m_m)A_m$$

Substituting for $A_f$ in terms of $A_m$ in *Eq 15*, we get

$$\bar{k} = \frac{k_m(k_f + m_m)(1 - \phi) + k_f(k_m + m_m)\phi}{(k_f + m_m)(1 - \phi) + (k_m + m_m)\phi} \quad (16)$$

The results for the remaining moduli based on an approach similar to that outlined for $\bar{k}$ are quoted below from Hermans (13)

$$\bar{m} = m_m\frac{2\phi m_f(k_m + m_m) + 2(1 - \phi)m_m m_m + (1 - \phi)k_m(m_f + m_m)}{2\phi m_m(k_m + m_m) + 2(1 - \phi)m_f m_m + (1 - \phi)k_m(m_f + m_m)} \quad (17)$$

and

$$\bar{\mu} = \frac{\mu_f + \mu_m[\mu_m(1-\phi) + 2\mu_f\mu_m\phi]}{\mu_f + \mu_m(1-\phi) + 2\mu_m\phi} \tag{18}$$

**Kerner's Results**

Since the Halpin-Tsai equations can also be shown to be the approximate form of Kerner's equations for particulate reinforced composites, we will summarize the derivation of Kerner's equations to gain more necessary background.

Kerner (13) deduced the shear modulus $\bar{G}$ and bulk modulus $\bar{k}$ of a macroscopically isotropic and homogeneous composite in terms of the moduli and concentration of its components. These components were assumed to be in the form of grains suspended in and bonded to some uniform suspending medium. It was assumed also that the grains were distributed spatially at random, and that in the mean they were spherical.

(Notation: Quantities referring to any grain species are labelled with subscript $i$; index $m$ refers to the suspending fluid and symbols with a bar and without subscript refer to the composite properties.)

$\bar{K}$ = bulk modulus ($k$ in Hill's notation is plane strain bulk modulus and different from $\bar{K}$.)

$\bar{G}$ = shear modulus

Consider a large mass of the composite and let it be subject to a simple hydrostatic compression. The mass behaves elastically in average like some uniform material having a shear modulus $\bar{G}$ and bulk modulus $\bar{K}$ which is to be expressed in terms of the moduli and volume fractions of the components.

The total change in volume in the composite will be the sum of the volume changes in its different parts. The desired results in terms of $K$ and $G$ are;

$$\bar{K} = \frac{\sum_i \dfrac{K_i\phi_i}{(3K_i + 4G_m)} + \dfrac{K_m(1-\Sigma\phi_i)}{3K_m + 4G_m}}{\sum_i \dfrac{\phi_i}{3K_i + 4G_m} + \dfrac{(1-\Sigma\phi_i)}{3K_m + 4G_m}}$$

For a binary (two components, including a matrix) composite we use the subscript $f$ to denote the property of a filler phase.

$$\bar{K} = \frac{\dfrac{K_f\phi}{3K_f + 4G_m} + \dfrac{K_m(1-\phi)}{3K_m + 4G_m}}{\dfrac{\phi}{3K_f + 4G_m} + \dfrac{(1-\phi)}{3K_m + 4G_m}} \tag{19}$$

For the shear modulus we have

$$\bar{G} = \frac{\dfrac{\phi G_f}{(7-5\nu_m)G_m + (8-10\nu_m)G_f} + \dfrac{(1-\phi)}{15(1-\nu_m)}}{\dfrac{\phi G_m}{(7-5\nu_m)G_m + (8-10\nu_m)G_f} + \dfrac{(1-\phi)}{15(1-\nu_m)}} \tag{20}$$

Hashin's formulas for a hollow fiber reinforced composite (14) can also be reduced to the above generalized formula.

**Derivation of Hill's Relations for $E_{11}$ and $\nu_{12}$**

The universal relations between the main overall elastic moduli of a fiber composite with transversely isotropic phases as derived by Hill (10) are shown below. These results are useful in developing the approximate Halpin-Tsai formulae for the tensile modulus ($E_{11}$) and Poisson ratio ($\nu_{12}$) for continuous fiber composites

$$\bar{\nu}_{12} = \frac{\bar{l}}{2\bar{k}} = \nu_f\phi + \nu_m(1-\phi) + \left[\frac{\nu_f - \nu_m}{\left(\dfrac{1}{k_f} - \dfrac{1}{k_m}\right)}\right]\left(\frac{1}{\bar{k}} - \frac{\phi}{k_f} - \frac{(1-\phi)}{k_m}\right) \tag{21}$$

$$\bar{E}_{11} = n - \frac{\bar{l}}{2\bar{k}} = \phi(E_{11})_f + (1-\phi)(E_{11})_m - 4\left[\frac{(\nu_{12})_f - (\nu_{12})_m}{\left(\dfrac{1}{k_f} - \dfrac{1}{k_m}\right)}\right]\left(\frac{1}{\bar{k}} - \frac{\phi}{k_f} - \frac{(1-\phi)}{k_m}\right) \tag{22}$$

The Reuss bound is based on the assumption that uniform stress is present in the composite and it is of the following form for the plane-strain bulk modulus

$$\frac{1}{\bar{k}} \leq \frac{\phi}{k_f} + \frac{(1-\phi)}{k_m} = \frac{1}{k_r}$$

Hence from *Eq 22*, $E_{11} \geq \phi(E_{11})_f + (1-\phi)(E_{11})_m$ and from *Eq 21*, $\nu_{12} \gtrless \phi(\nu_{12})_f + (1-\phi)(\nu_{12})_m$ accordingly as

$$(\nu_1 - \nu_2)(k_1 - k_2) \gtrless 0$$

Halpin and Tsai (7) have approximated the above relations to give reasonable estimates of the continuous fiber-reinforced composite elastic coefficients as follows:

$$\bar{E}_{11} = (E_{11})_f\phi + (E_{11})_m(1-\phi) \tag{23}$$

$$\bar{\nu}_{12} = (\nu_{12})_f\phi - (\nu_{12})_m(1-\phi) \tag{24}$$

The results in *Eqs 21* and *22* or *23* and *24* are independent of either the geometrical distribution or shape of the parallel fibers (ribbons, etc.) and the constituents are either isotropic or transversely isotropic with respect to the longitudinal direction. The approximation in *Eqs 23* and *24* to achieve the "rule of mixtures" is the neglect of the interaction between the constituents due to the difference in their Poisson ratios, i.e., by the use of the parallel spring model. Extensive experience indicates that correlations of *Eqs 23* and *24* with both experiment and machine calculations are insensitive to this assumption.

**Rearrangement of Various Equations to Develop the Halpin-Tsai Equations**

Herein we show how Halpin and Tsai rearranged the formulae resulting from exact theory of elasticity to conform to a generalized formula as follows:

$$\frac{\bar{p}}{p_m} = \frac{1 + \zeta\eta\phi}{1 - \eta\phi}$$

where

$$\eta = \frac{\dfrac{p_f}{p_m} - 1}{\dfrac{p_f}{p_m} + \zeta} = \frac{M_R - 1}{M_R + \zeta} \tag{25}$$

$\bar{p}$ = a composite modulus, $k$, $m$ or $\bar{\mu}$, $p_f$ = corresponding fiber modulus, $k_f$, $m_f$ or $\mu_f$ respectively, $p_m$ = corresponding matrix modulus, $k_m$, $m_m$, or $\mu_m$, respectively, and $\zeta$ = a measure of reinforcement geometry which depends on loading conditions.

● We rewrite *Eq 16* for the plane-strain bulk modulus $\bar{k}$ as follows, after dividing numerator and denominator by $k_m$.

$$\frac{\bar{k}}{k_m} = \frac{\left(\frac{k_f}{k_m} + \frac{m_m}{k_m}\right)(1-\phi) + \frac{k_f}{k_m}\left(1 + \frac{m_m}{k_m}\right)\phi}{\left(\frac{k_f}{k_m} + \frac{m_m}{k_m}\right)(1-\phi) + \left(1 + \frac{m_m}{k_m}\right)\phi}$$

Put $\frac{k_f}{k_m} = M_R$ and $\frac{m_m}{k_m} = \zeta$

$$\frac{\bar{k}}{k_m} = \frac{(M_R+\zeta)(1-\phi) + M_R(1+\zeta)\phi}{(M_R+\zeta)(1-\phi) + (1+\zeta)\phi} = \frac{M_R + \zeta + \zeta\phi\,(M_R-1)}{M_R + \zeta - \phi\,(M_R+1)}$$

$$= \frac{1 + \zeta\phi\frac{(M_R-1)}{(M_R+\zeta)}}{1 - \phi\frac{(M_R-1)}{(M_R+\zeta)}}, \quad \text{Put } \eta = \frac{M_R-1}{M_R+\zeta}$$

$\frac{\bar{k}}{k_m} = \frac{1+\zeta\eta\phi}{1-\eta\phi}$. This is the Halpin-Tsai equation form with $\zeta = \frac{m_m}{k_m}$

● *Equation 17* for the transverse shear modulus $\bar{m}$ is rewritten by, (1) dividing numerator and denominator by $m_m^2$, (2) using $M_R = \frac{m_f}{m_m}$, (3) dividing the numerator and denominator by $\frac{k_m}{m_m} + 2$, and (4) denoting $(k_m/m_m)/(k_m/m_m + 2) = \zeta$ as

$$\frac{\bar{m}}{m_m} = \frac{\phi M_R(1+\zeta) + (1-\phi)(M_R+\zeta)}{\phi(1+\zeta) + (1-\phi)(M_R+\zeta)} = \frac{M_R + \zeta + \zeta\phi\,(M_R+1)}{M_R + \zeta - \phi\,(M_R-1)} = \frac{1+\zeta\eta\phi}{1-\eta\phi}$$

where $\eta = \frac{M_R-1}{M_R+\zeta}$.

● *Equation 18* for longitudinal shear modulus is also rewritten to conform with the Halpin-Tsai format.

$$\frac{\bar{\mu}}{\mu_m} = \frac{M_R + 1 + \phi(M_R-1)}{M_R + 1 - \phi(M_R-1)}$$

$$= \frac{1 + \phi\left(\frac{M_R-1}{M_R+1}\right)}{1 - \phi\left(\frac{M_R-1}{M_R+1}\right)} = \frac{1+\zeta\eta\phi}{1-\eta\phi}$$

Here $\zeta = 1$, $\eta = \frac{M_R-1}{M_R+\zeta}$, and $\frac{\mu_f}{\mu_m} = M_R$.

The above formulae are for a continuous fiber-reinforced composite with transversely isotropic phases and are derived by using a self-consistent model and put in the generalized form of the Halpin-Tsai equation.

Kerner's equations for an isotropic particulate composite can also be rearranged to conform to the Halpin-Tsai format.

● *Equation 19* for the bulk modulus of the composite is modified by setting $K_f/K_m = M_R$, dividing by $3K_m$ in numerator and denominator and calling $4G_m/3K_m = \zeta$. Thus,

$$\frac{\bar{K}}{K_m} = \frac{M_R + \zeta - \phi M_R - \phi\zeta + M_R\phi + M_R\phi\zeta}{M_R + \zeta - M_R\phi - \zeta\phi + \phi + \zeta\phi}$$

$$= \frac{1+\zeta\eta\phi}{1-\eta\phi} \text{ with } \zeta = \frac{4G_m}{3K_m},\ \eta = \frac{M_R-1}{M_R+\zeta}$$

● *Equation 20* for the shear modulus of the composite is rewritten as

$$\frac{\bar{G}}{G_M} = \frac{\phi M_R(1+\zeta) + (1-\phi)(M_R+\zeta)}{\phi(1+\zeta) + (1-\phi)(M_R+\zeta)}$$

$$= \frac{M_R + \zeta + \phi(M_R-1)}{M_R + \zeta - \phi(M_R-1)}$$

$$= \frac{1+\zeta\eta\phi}{1-\eta\phi} \quad \text{where } \zeta = \frac{7-5\nu_m}{8-10\nu m};$$

$$\eta = \frac{M_R-1}{M_R+\zeta};\ M_R = \frac{G_f}{G_m}$$

In the above development of the Halpin-Tsai equations $M_R = \frac{P_f}{P_m}$ in every case; however, $\zeta$ has different expressions for different composite moduli and is a unique function of the Poisson's ratio of the matrix in each specific case, as illustrated in *Table 2*. The range of values $\zeta$ can take is also indicated.

One should notice that $\zeta$ is also less than 2 and approximately unity when the transverse crossection of the reinforcement possesses an effective aspect ratio of unity. Note that a fiber and a spherical or cubical particle have comparable transverse moduli.

Hashin's formulae (14) for hollow-fiber-reinforced composites can also be reduced to a Halpin-Tsai format. The required modification in the symbol $M_R$ for particular cases is given below.

a) For the bulk modulus $K$,

$$M_R = \frac{1-a^2}{1 + \frac{a^2}{(1-2\phi)}}\left(\frac{K_f}{K_m}\right)$$

b) For the shear modulus $G_{23}$

$$M_R = \frac{1-a^2}{1+a^2}\left(\frac{G_f}{G_m}\right)$$

where $a$ is the ratio of the inside and outside radii of the hollow fiber. For a solid fiber, $a = 0$;

$$M_R = \frac{P_f}{P_m}.$$

**Table 2. $\zeta$, Filler Geometry Reinforcement Parameter in the Halpin-Tsai Equations as it Appears in Various Equations in Terms of Engineering Moduli**

| Composite modulus | $\zeta$ |
|---|---|
| $\bar{k}$ (Plane-strain bulk modulus for continuous fiber-reinforced composite) | $\zeta_{\bar{k}} = \frac{m_m}{k_m} = \frac{E_{22}/2(1+\nu_{23})}{E_{22}/2(1-\nu_{23}-2\nu_{12}\nu_{21})}$<br>$= \frac{(1-\nu_{23}-2\nu_{12}\nu_{21})_m}{(1+\nu_{23})_m}$<br>$\zeta_{\bar{k}}$ would be $<1$ ($\zeta \simeq 1$) |
| $\bar{m}$ (Transverse shear modulus $\bar{G}_{23}$ for continuous fiber-reinforced composite) | $\zeta_{\bar{m}} = \frac{k_m/m_m}{(k_m/m_m)+2} = \frac{1+\nu_{23}}{3-\nu_{23}-4\nu_{12}\nu_{21}}$<br>If matrix were isotropic and incompressible<br>$\zeta_{max} = \frac{1.5}{1.5} = 1;\ \zeta \le 1$ |
| $\bar{\mu}$ (Longitudinal shear modulus) | $\zeta_\mu = 1.0$ |
| $\bar{K}$ (Bulk modulus for particulate composite) | $\zeta_{\bar{K}} = \frac{4G_m}{3K_m} = \frac{4\,E_m/2(1+\nu_m)}{3\,E_m/3(1-2\nu_m)}$<br>$\zeta_{\bar{K}} = \frac{2(1-2\nu_m)}{(1+\nu_m)}$<br>$\zeta_{\bar{K}max} = 2$ as $\nu_m \to 0$; $\zeta_{\bar{K}min} = 0$ as $\nu_m \to 0.5$<br>$2 > \zeta_{\bar{K}} > 0$ |
| $\bar{G}$ (Shear modulus of particulate composite) | $\zeta_{\bar{G}} = \frac{7-5\nu_m}{8-10\nu_m}$ ; $\nu_m$ / $\zeta$: 0 / 7/8; 0.25 / 5.75/5.5; 0.5 / 1.5<br>$\zeta_{\bar{G}} = 1$ |

## DISCUSSION OF THE PARAMETERS IN THE HALPIN-TSAI EQUATIONS

Halpin and Tsai (7) have made further simplifications by directly employing the engineering constants $E_f$, $E_m$, $G_f$, $G_m$ for $P_f$ and $P_m$ in the equations summarized in *Table 2* for purposes of interpolation from existing micromechanics calculations. Thus $\bar{P}$ would yield directly an estimate of $E_{22}$, $G_{12}$, $G_{23}$ and also $\nu_{23}$ if analogous equations were constructed for it.

Directly employing the engineering constants involves approximation only in the formula for $\bar{k}$, the plane strain bulk modulus and $\bar{K}$, the bulk modulus for particulate composites, by calling them $E_{22}$ for the respective composites. This is a good approximation as the Halpin-Tsai equations involve ratios like $\frac{P_f}{P_m}$ and $\frac{\bar{P}}{P_m}$ which do not change much, relatively.

$$\frac{k_f}{k_m} = \frac{\{E_{22}/2(1-\nu_{23}-2\nu_{12}\nu_{21})\}_f}{\{E_{22}/2(1-\nu_{23}-2\nu_{12}\nu_{21})\}_m} \doteq \frac{(E_{22})_f}{(E_{22})_m}$$

$$\frac{K_f}{K_m} = \frac{\{E/3(1-2\nu)\}_f}{\{E/3(1-2\nu)\}_m} \doteq \frac{E_f}{E_m}$$

The essential approximation of Halpin and Tsai is that the computations for $\bar{E}_{22}$ or $\bar{E}$ and the $\zeta$ terms of *Table 2* are insensitive to the differences in the constituent Poisson ratios. This approximation is comparable to the simplification of $\bar{E}_{11}$ and $\nu_{12}$ in the section on the derivation of Hill's relations for $\bar{E}_{11}$ and $\nu_{12}$.

Reliable estimates for the $\zeta$ factor can be obtained by comparison of the Halpin-Tsai equations with the numerical micromechanics solutions employing the formal elasticity theory. *Figures 1 (a)* and *(b)* show the predictions for the transverse stiffness $E_{22}$ and the longitudinal shear stiffness $G_{12}$, using $\zeta_{E22} = 2$ and $\zeta_{G12} = 1$ in the Halpin-Tsai equations.

The predicted dependence of moduli on the filler/matrix stiffness ratios for several volume fractions of filler are compared with the results of Adams and Doner (15) shown by solid curves. The agreement is very good up to a volume fraction of about 0.70. The exact elasticity calculations predict the moduli to rise faster with increasing volume fraction of reinforcement above 0.70 as compared to the Halpin-Tsai equation, *Figs. 1* and *2*. Hewitt and de Malherbe (16) have suggested the following modifications to compensate for this effect:

$$\zeta_{G12} = 1 + 40\,\phi^{10}$$

$$\zeta_{E22} = 2 + 40\,\phi^{10}$$

We will now show how the limiting values of the parameters $\zeta$ and $\eta$ in *Table 2* affect the Halpin-Tsai Equations. The equations once more are as follows,

$$\eta = \frac{M_R - 1}{M_R + \zeta}$$

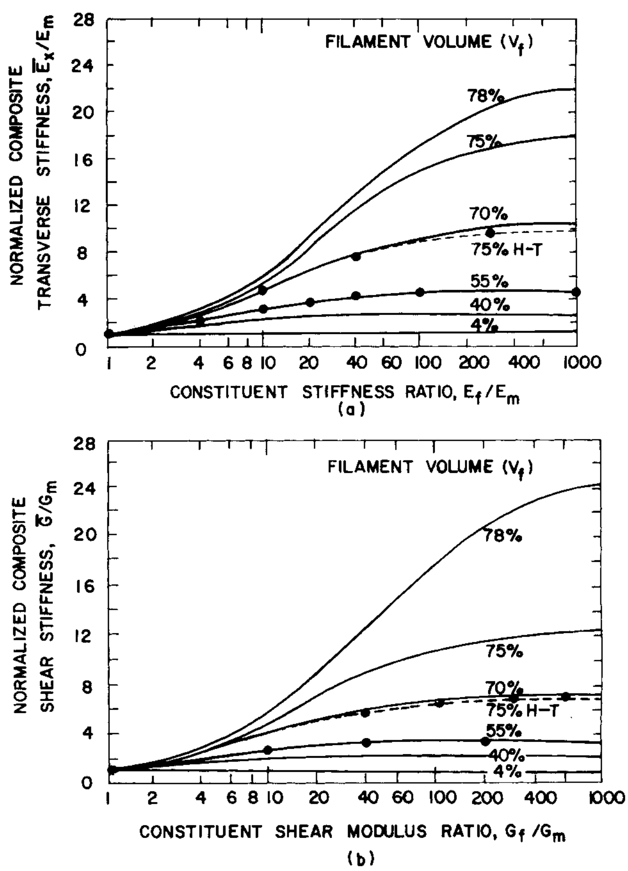


*Fig. 1. Comparison of Halpin-Tsai calculation (solid circles) with Adams and Doner's (15) square array calculations for (a) transverse stiffness, and (b) longitudinal shear stiffness. The Halpin-Tsai predictions for $\phi = 0.75$ are shown by the dotted line. (a) $\zeta_{E22} = 2$; (b) $\zeta_{G12} = 1$.*

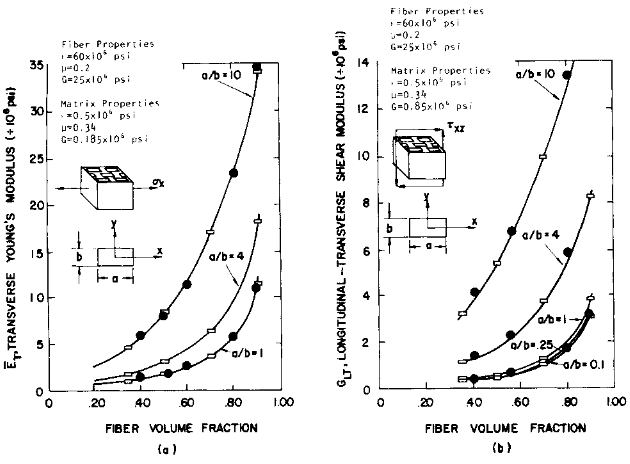


*Fig. 2. Comparison of Halpin-Tsai calculation (circles) with Foye's calculation (17) for (a) the transverse stiffness, and (b) the longitudinal shear stiffness of a composite containing rectangularly shaped fibers. (a) $(\zeta_{E22})_{cont.} = 2(a/b)$; (b) $\log \zeta_{G12} = \sqrt{3}$ $\log (a/b)$.*

$$\frac{\bar{P}}{P_m} = \frac{1 + \zeta\eta\phi}{1 - \eta\phi}$$

where $\zeta$ is a measure of the reinforcement geometry as we have developed above.

Limiting values of $\eta$ are: a) For very rigid inclusions (i.e., $M_R \to \infty$), $\eta = 1$; b) For homogeneous material ($M_R = 1$), $\eta = 0$; and c) For voids ($M_R = 0$), $\eta = \frac{-1}{\zeta}$. Thus, the limiting values of $\zeta$ are $\zeta = 0$ and $\zeta = \infty$.

$$\text{a) } \zeta = 0;\ \frac{\bar{P}}{P_m} = \frac{1}{1 - \eta\phi} = \frac{1}{P_m\left(\dfrac{\phi}{P_f} + \dfrac{(1-\phi)}{P_m}\right)}$$

$$\therefore \frac{1}{\bar{P}} = \frac{\phi}{P_f} + \frac{(1-\phi)}{P_m}$$

$$\text{b) } \zeta = \infty,\ \therefore \eta = 0$$

$$\frac{\bar{P}}{P_m} = \frac{M_R + \zeta + \zeta(M_R - 1)\phi}{M_R + \zeta - \zeta(M_R - 1)\phi}$$

$$\lim_{\zeta\to\infty} \frac{\bar{P}}{P_m} = \frac{\lim\limits_{\zeta\to\infty} \dfrac{\partial \text{ Numerator}}{\partial \zeta}}{\lim\limits_{\zeta\to\infty} \dfrac{\partial \text{ Denominator}}{\partial \zeta}} \quad \text{(L' Hospital rule)}$$

$$\frac{\bar{P}}{P_m} = 1 + (M_R - 1)\phi$$

$$\bar{P}_{\zeta\to\infty} = P_f\phi + P_m(1 - \phi)$$

The above result is the same as the equations for $E_{11}$ and $\nu_{12}$ (*Eqs 23* and *24*) for the continuous fiber-reinforced composite. It is important to observe that $\bar{P}$ continually progressed from the lower bound when $\zeta = 0$ to the upper bound as $\zeta \to \infty$ at fixed volumetric concentration of reinforcement and packing geometry. It is generally recognized that the lower bound is associated with an effective reinforcement aspect ratio of unity and that the upper bound corresponds to an aspect ratio approaching infinity. Numerical solutions consistent with the governing equations of elasticity have been developed by Foye (17) for the effect of filament shape on transverse moduli. In general, the transverse moduli increases as the filaments become more elongated in the direction of the applied stress. Thus $\zeta$ must be a measure of the geometry (aspect ratio) of the reinforcement phase as both have identical effects.

*Figures 2(a)* and *(b)* show the Halpin-Tsai calculation (7) (circles in the figures) for the transverse stiffness and the longitudinal shear stiffness of a composite containing rectangularly shaped fibers. These are compared with Foye's (17) calculations for fibers in a diamond array for various filler geometries and as a function of the volume fraction of filler phase. *Figure 2(a)* equally well describes the dependence of longitudinal stiffness, $E_{11}$, of an oriented discontinuous fiber-reinforced composite on the aspect ratio $(a/b)$, where '$a$' is now the fiber length and '$b$' is the fiber width. The curve with $(a/b) = 1$ in *Fig. 2(b)* is also applicable for the shear modulus, $G_{12}$, of the discontinuous reinforced composite and describes its dependence on the filler volume fraction.

Further examination of the dependence of the transverse stiffness moduli and the shear moduli upon increasing aspect ratio of rectangular filaments gives confidence in the interpolation procedure. The factors $\zeta_E$ and $\zeta_G$ are functions of the width/thickness ratios and were found by fitting Foye's results as illustrated in *Figs. 2* and *3*. The resulting forms for the two terms are:

$$\zeta_{E22} = 2(a/b)$$

$$\log \zeta_{G12} = \sqrt{3} \log (a/b)$$

The last formula for $\zeta_{G12}$ is applicable to composites with filaments or platelets as the filler phase. In the limiting case for circular or square fibers with the fiber axis as direction 1, $\zeta_{G12} = 1$, since $a = b$ in this case. Also $\zeta_{G13} = 1$ and therefore $G_{12} = G_{13}$, satisfying the transverse isotropy. Also, for the transverse modulus $E_{22}$, $\zeta_{E22} = 2$ since $a = b$.

The Halpin-Tsai equations retain the same form for discontinuous layer or lameller-shaped reinforcements

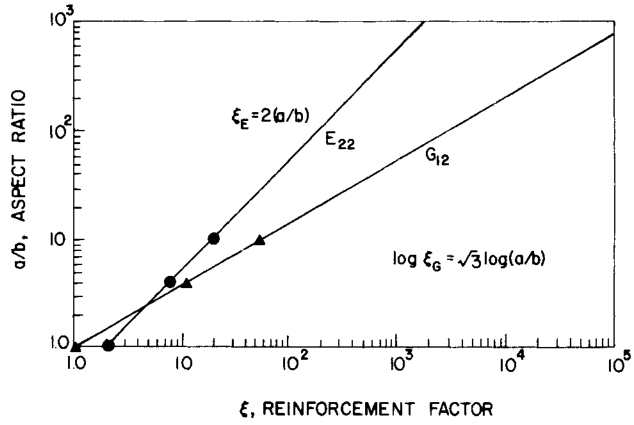


*Fig. 3. Correlation of coefficient, $\zeta$, in Eq 25 with the transverse aspect ratio of rectangularly shaped fibers.*

(2, 6, 18). For these composites $\bar{P}$ represents $E_{11}$, $E_{22}$, or $G_{12}$ and $\zeta$ is given by the following expressions.

- $\zeta_{E_{11}} = 2\,(l/t)$, where $l$ is the length of the tape in the 1-direction and $t$ is the thickness of the tape in the 3-direction. As $\zeta \rightarrow \infty$, we know from the above discussion of limiting values of the Halpin-Tsai equations that they yield the modulus for the continuous fiber reinforced composite.
- $\zeta_{E_{22}} = 2(w/t)$, where $w$ is the width of the tape in the 2-direction. For circular or square fibers, $w = t$; therefore $\zeta_{E_{22}} = 2$.
- $\zeta_G = 1$, this gives a conservative estimate.

*Figures 2(a)* and *(b)* indicate good agreement between the Halpin-Tsai formulae and elasticity calculations for different aspect ratios and volume fractions. More importantly, as there are no exact theoretical solutions available for short fiber or discontinuous lamella reinforced composites, these formulae can be extended to predict the stiffnesses of unidirectionally oriented short-fiber-reinforced composites, which are the building blocks for composites with any type of orientation of the filler phases.

*Figure 4(a)* presents the experimentally observed (2, 4) stiffness variation in the fiber direction for an oriented short-fiber sheet as a function of the aspect ratio. At an aspect ratio of unity the system is a particulate reinforced composite and possesses a nearly isotropic modulus approaching in value the lower bound. As the reinforcement phase is extended in one dimension, and as $l/d$ increases, the composite modulus increases in that direction. Furthermore, the material becomes highly

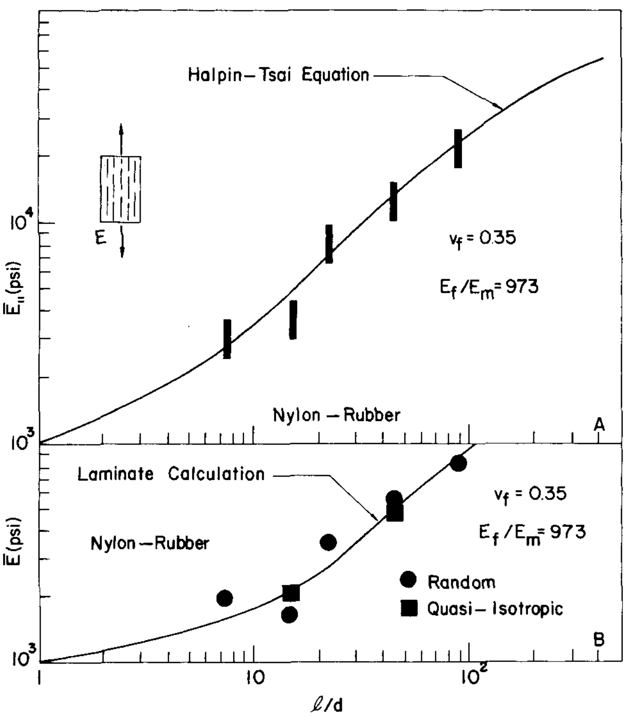


*Fig. 4. (a) Effect of fiber l/d on the effective longitudinal tensile stiffness of an unidirectionally oriented discontinuous nylon fiber/rubber composite. Bars represent experimental results. (b) Effect of fiber l/d on the effective tensile stiffness of a rubber filled with randomly oriented discontinuous nylon fibers. Solid circles and squares represent experimental results.*

anisotropic; that is, the elastic stiffness is strongly dependent on direction. *Figure 4(b)* presents the dependence of the effective extensional moduli, the solid circles, for randomly oriented short fiber composites as a function of aspect ratio at a constant volumetric composition of reinforcement. As the aspect ratio $l/d$ becomes large, the stiffness of the random array of short fibers in a matrix attains a plateau representing the equivalent continuous filament result. Note that at an aspect ratio of $10^2$, the random short-filament reinforcement yields a stiffness property which is 10 times greater than the equivalent spherically shaped particulate reinforcement at that volumetric concentration. These results provide independent experimental verification of the Halpin-Tsai equations.

Thus the Halpin-Tsai formulae enable one to calculate the effect of the volume fraction of the filler phase, the relative moduli of the constituents, and the reinforcement geometry on the composite moduli in a simple straight-forward way. The specific forms of the formulae are equally applicable to isotropic and transversely isotropic constituent phases.

## CONCLUSIONS

- The Halpin-Tsai equations are developed from rigorous elasticity calculations. The geometry factor $\zeta$ may be expressed in terms of combinations of engineering elastic constants and differences in Poissons ratios. The only assumption made in assigning various values for the geometrical factor is that the engineering stiffness expressions for the composite ply are insensitive to the differences in Poisson ratios of the constituent phases making up the ply. That this is indeed the case is borne out both by comparison with machine calculations based on the exact equations and by experimental data on carefully controlled short fiber systems.
- The Halpin-Tsai equations demonstrate the significant effect of reinforcement geometry alone on the stiffness properties of a unidirectionally oriented ply at both constant volume fraction and packing geometry; changing from a sphere (aspect ratio of one in the principal material direction) to a long fiber (aspect ratio approaching infinity) gives an order of magnitude or more increase in stiffness for both a unidirectional and a randomly distributed reinforcement depending on the constituent stiffness properties. Furthermore, the equations predict, in agreement with experimental demonstration, that progressing from a fiber to a tape or platelet by expanding the transverse reinforcement dimension also results in an additional order of magnitude or more improvement in the stiffness properties.
- In considering the prediction of crystalline polymer properties utilizing a micromacromechanics approach, the Halpin-Tsai equations can account very nicely for the principal material direction (the polymer backbone chain direction) being either colinear (extended chain crystal) or perpendicular to (folded-chain lamella) the principal geometric direction of the reinforcement. This is so because the equations recognize anisotropy as well as geometry through Hermans' equa-

tions which account for both isotropic and transversely isotropic phases.

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