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Subject: Convex Optimization

Convex Optimization
Assignment 1

Question 1

$\sqrt{x}$ is concave for $x > 0$.
$\frac{1}{u}$ is convex for $u > 0$
So by DCP rules, $\frac{1}{\sqrt{x}}$ is concave.

The Function $g(x, y) = \frac{x^2}{y}$ is jointly convex for $x, y > 0$.(increasing for $x > 0$ and decreasing for $y > 0$.
So by DCP and general vector composition rules:

$$g(\frac{1}{\sqrt{x}}, y) = \frac{\frac{1}{\sqrt{x}^2}}{y} = \frac{1}{xy}$$

is convex.

Question 2

part a

Minimizing the sum of the disk areas:

$$\text{First objective Function } \sum_{i=1}^{n} \pi r_i^2$$

Minimizing sum of the disk perimeters:

$$\text{Second objective Function } \sum_{i=1}^{n} 2\pi r_i$$

These are convex functions because we have: $r_i > 0$
1st constraint: First k disks are fixed. $\rightarrow$ set of linear constraints $\Rightarrow$ convex
2nd constraint: $\|c_i - c_j\|_2 \leq r_i + r_j$ for each (i, j) $\in$ I $\Rightarrow \|c_i - c_j\|_2 - (r_i + r_j) \leq 0$
$\|c_i - c_j\|_2$ is constant and $r_i + r_j$ is linear. $\Rightarrow$ convex
So:

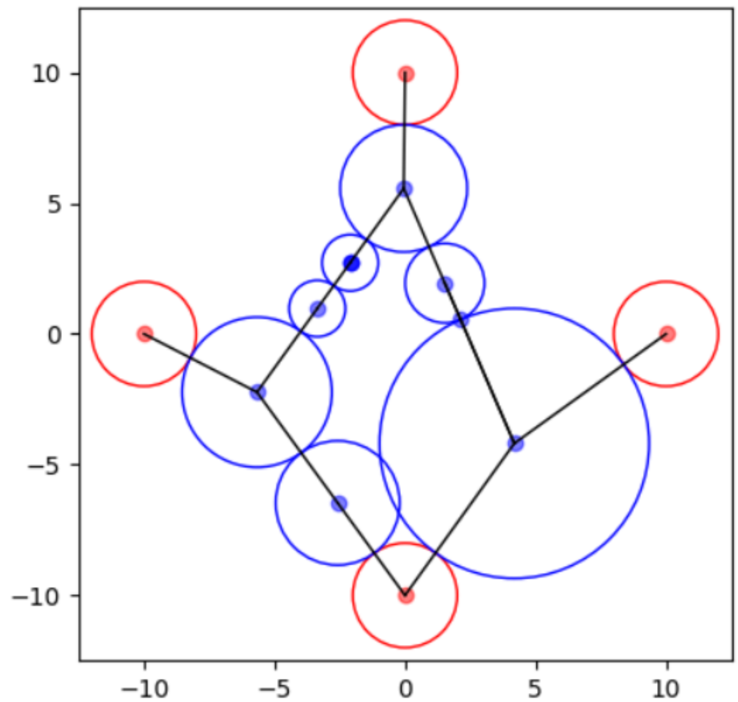
$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} \pi r_i^2 \\ \text{subject to} & c_i = c_i^{\text{fix}}, \quad r_i = r_i^{\text{fix}}, \quad i = 1, \ldots, k \\ & r_i \geq 0, \quad i = 1, \ldots, n \\ & \|c_i - c_j\|_2 \leq r_i + r_j, \quad (i, j) \in \mathcal{I}. \end{array}$$

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} 2\pi r_i \\ \text{subject to} & c_i = c_i^{\text{fix}}, \quad r_i = r_i^{\text{fix}}, \quad i = 1, \ldots, k \\ & r_i \geq 0, \quad i = 1, \ldots, n \\ & \|c_i - c_j\|_2 \leq r_i + r_j, \quad (i, j) \in \mathcal{I}. \end{array}$$

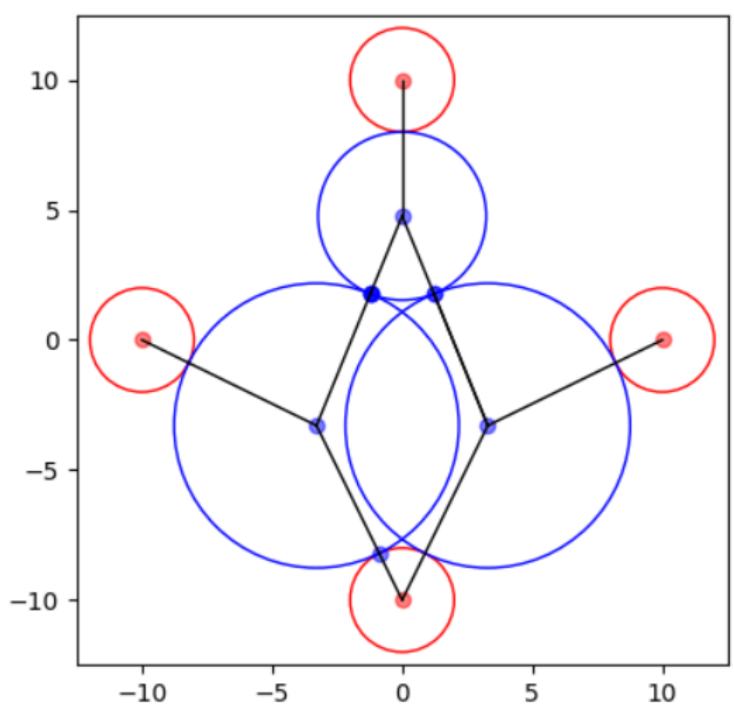
part b

Results are as below:

Optimal total area: 210.77



Optimal total perimeter: 139.35



Minimizing the Perimeter:

The perimeter here refers to the total boundary length of all the disks. The text notes that when minimizing the perimeter, which is essentially the sum of the radii of the disks (also referred to as the '1 norm' since it involves the sum of absolute values), the resulting solution tends to have a few disks with large radii and several disks with zero radii. This behavior is characterized as typical of 'L1 minimization,' where the optimization process tends to prioritize a sparse set of larger values (disks with large radii) while allowing others to be eliminated (disks with zero radii).

Minimizing the Area:

The area in this context is the sum of the squares of the radii of the disks (referred to as the 'L2 norm squared'). When minimizing the area, the solution tends to have fewer disks with large radii, and importantly, none of the disks have zero radii. This behavior is associated with 'L2 minimization,' where the optimization process distributes the values more evenly, avoiding complete elimination (zero radii) and favoring a balance among the radii.

In summary, the choice of the objective function (minimizing perimeter or area) influences the distribution of radii among the disks. Minimizing the perimeter tends to result in a solution with a few large disks and several with zero radii, while minimizing the area leads to a solution with fewer large disks and none with zero radii. The choice between L1 and L2 norms reflects different optimization preferences and trade-offs in the system.

Question 3

part a

$$\hat{y} = Ax$$

$x = (a, b) \in R^{2B}$ is a vector of cosine and sinusoid coefficients and $A = [C, S] \in R^{n \times 2B}$ with $C, S \in R^{n \times B}$ and:

$$C_{tj} = \cos(2\pi(f_{min} + j - 1)\frac{t}{n})$$

$$S_{tj} = \sin(2\pi(f_{min} + j - 1)\frac{t}{n})$$

Let $s_t a_t^T x \geq 0$ where a_i^T is i-th row of A, to ensure that the signs of $\hat{y}$ are consistent with s.

Linear equality constraint: $\|\hat{y}\|_1 = s^T A x = n$

(The signs are given so this constraint is convex in general)

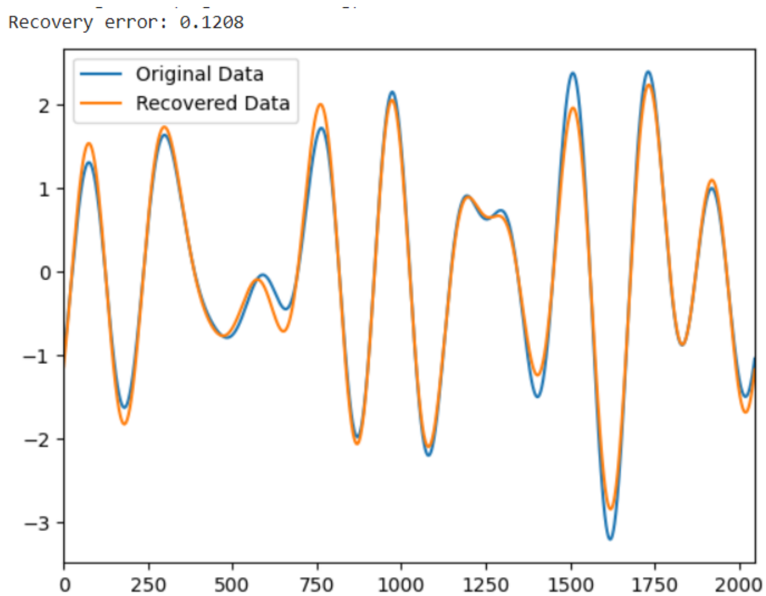
So:

$$\begin{array}{ll} \text{minimize} & \|Ax\|_2 \\ \text{subject to} & s_t a_t^T x \geq 0, \quad t = 1, \ldots, n \\ & s^T A x = n. \end{array}$$

Suppose that $x^\star$ is the solution of the problem. So the min value is $y^\star = Ax^\star$.

A prevalent error involved approaching the aforementioned problem without incorporating the normalization constraint. The flawed rationale behind this approach was the idea of solving the homogeneous problem first, followed by scaling the result to ensure its l_1 norm is one. However, this approach proves ineffective, as the sole solution to the homogeneous problem is x = 0 (given that x = 0 is feasible). Surprisingly, despite this, the method yielded numerical outcomes significantly superior to x = 0. This can be attributed to the fact that the solvers produced a minute x, for which Ax exhibited the correct sign. It's important to note that this does not imply the absence of a substantial error.

part b



The recovery error stands at 0.1208, and it's remarkably impressive, considering the scant information we had at our disposal.

Question 4

part a

$$Pr(N_t \text{ events occur}) = e^{-\lambda_t}\frac{\lambda_t^{N_t}}{(N_t)!} \Rightarrow \log(Pr(N_t \text{ events occur})) = -\lambda_t + N_t\log(\lambda_t) - \log(N_t!)$$

So log-likelihood of collection of N_ts is :

$$LR = \prod_{t=1}^{24} Pr(N_t \text{ events occur}) = \sum_{t=1}^{24} -\lambda_t + N_t\log(\lambda_t) - \log(N_t!)$$

So we should minimize negative log-likelihood:

$$\frac{\partial(-LR)}{\partial\lambda_t} = 1 - \frac{N_t}{\lambda_t} = 0 \Rightarrow \lambda_t = N_t$$

This means if you observe N_t events, you will guess that this is the mean of the number of events that occur.

part b

We should solve the convex optimization problem

$$\text{minimize} \sum_{t=1}^{24} \lambda_t - N_t\log(\lambda_t) + \log(N_t!) + \rho(\sum_{t=1}^{23}(\lambda_t - \lambda_{t+1})^2 + (\lambda_{24} - \lambda_1)^2); \textbf{subject to } \lambda, \rho \geq 0 \textbf{ , } \lambda \in R^{24}$$

Let's call the function above f.

part c

As $\rho \to \infty$, since we should let $\frac{\partial f}{\partial \rho} = 0$ we get

$$(\sum_{t=1}^{23} \lambda_t - \lambda_{t+1})^2 + (\lambda_{24} - \lambda_1)^2 = 0 \Rightarrow \text{all } \lambda_t s \text{ are equal}$$

So we should minimize the following function:

$$\sum_{t=1}^{24} \hat{\lambda} - N_t \log(\hat{\lambda}) + \log(N_t!)$$

which is equal to minimizing the following function:

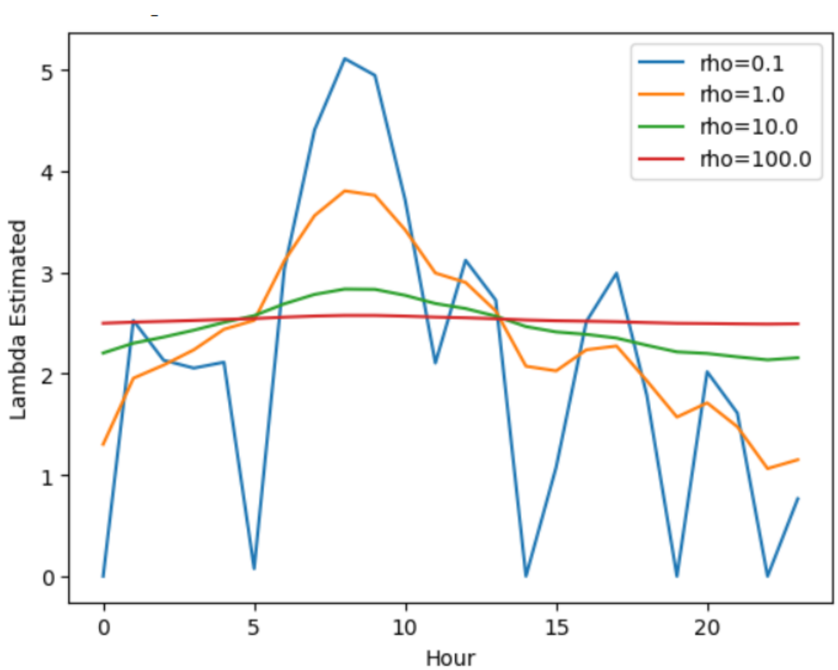
$$g(\lambda) = \sum_{t=1}^{24} \hat{\lambda} - N_t \log(\hat{\lambda}) = 24\hat{\lambda} - \log(\hat{\lambda}) \sum_{t=1}^{24} N_t$$

$$\frac{\partial g(\lambda)}{\partial \lambda} = 24 - \frac{\sum_{t=1}^{24} N_t}{\hat{\lambda}} = 0 \Rightarrow \hat{\lambda} = \frac{\sum_{t=1}^{24} N_t}{24}$$

So the solution is the constant Poisson model $\hat{\lambda} = \frac{\sum_{t=1}^{24} N_t}{24}$.

part d

The estimated rates are as below:



(You can observe that the larger ρ is, the smoother λ is.)

part e

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rho = 0.1, log likelihood = -83.29
rho = 1, log likelihood = -37.75
rho = 10, log likelihood = -41.71
rho = 100, log likelihood = -43.76
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By the log-likelihood for rhos shown above, we would choose $\rho = 1$, since it gets the highest log-likelihood on the test set.