Diskretna matematika 2

Gradiva za vaje iz diskretne matematike 2

Univerza v Ljubljani, Fakulteta za računalnišvo in informatiko

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1. Introduction

1.1 What is Sage?

Algorithms in this Notes are implemented in Python programming language using SageMath (https://www.sagemath.org).

> SageMath is a free open-source mathematics software system licensed under the GPL. It builds on top of many existing open-source packages: NumPy, SciPy, matplotlib, Sympy, Maxima, GAP, FLINT, R and many more.

You can download binaries at http://www.sagemath.org/download.html for Mac, and Windows.

Note: Binaries for Windows are avaliable up to version 9.3 (late 2021). For newer versions you will need to install it in WSL. Follow the instructions athttps://doc.sagemath.org/html/en/installation/index.html.

There is also a cloud version available at https://cocalc.com/
Documentation can be found at https://doc.sagemath.org/html/en/index.html.
We will moslty use *graph theory* package https://doc.sagemath.org/html/en/reference/graphs/index.html

1.2 Some examples of Sage Graph Theory objects and methods

For representing undirected graphs we use the `Graph` class, while for representing directed graphs we use the `DiGraph` class.

1.2.1 Undirected graphs

Undirected graph is represented using `Graph` class.

```
G = Graph({0:[1,2,3], 4:[0,2], 6:[1,2,3,4,5]})
```

There are many methods to access the graph properties. For example, to get a list of vertices use `vertices` method.

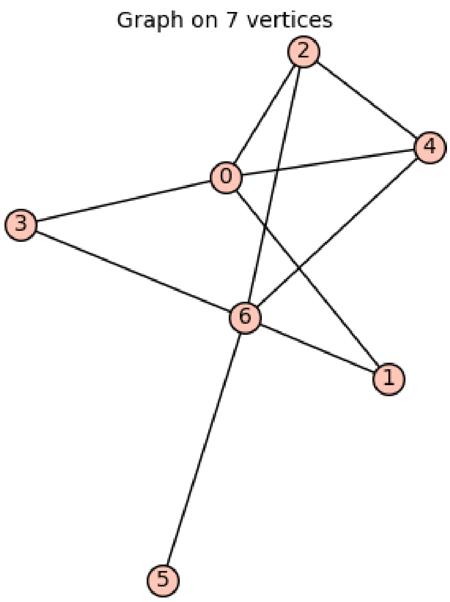
```
G.vertices()
```

```
[0,1,2,3,4,5,6]
```

To display the graph, simply execute a cell with the graph variable name.

```
G
```

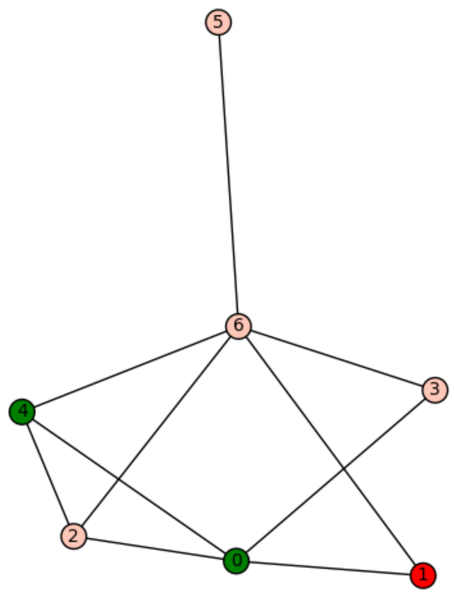
The output is a graphical representation of the graph. If we do not specify vertex coordinates (see below), Sage will use a spring embedder layout algorithm to compute the coordinates.



If a graph is too large, it will not be displayed. In this case, or if you need to specify other display options, you can use the `plot` method. There are many options for the `plot` method, see https://doc.sagemath.org/html/en/reference/plotting/sage/graphs/graph_plot.html for details.

For example. we can specify vertex colors using a dictionary, where keys are colors and values are lists of vertices.

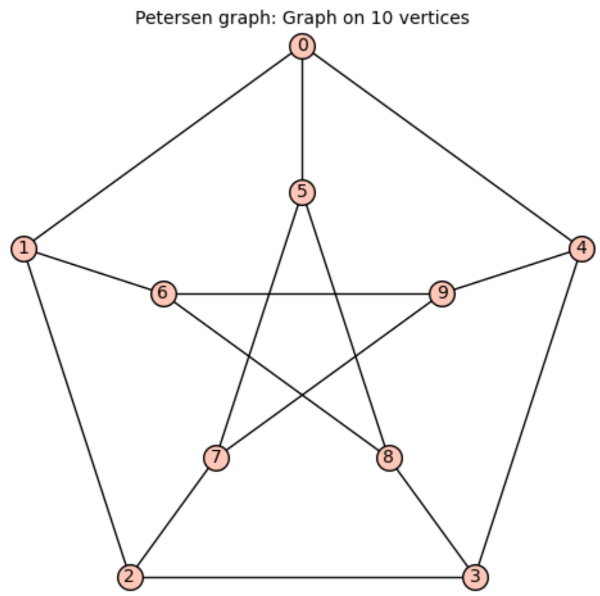
```
G.plot(vertex_colors={'red':[1],'green':[0,4]})
```



1.2.1.1 Some well-known graphs and graph families

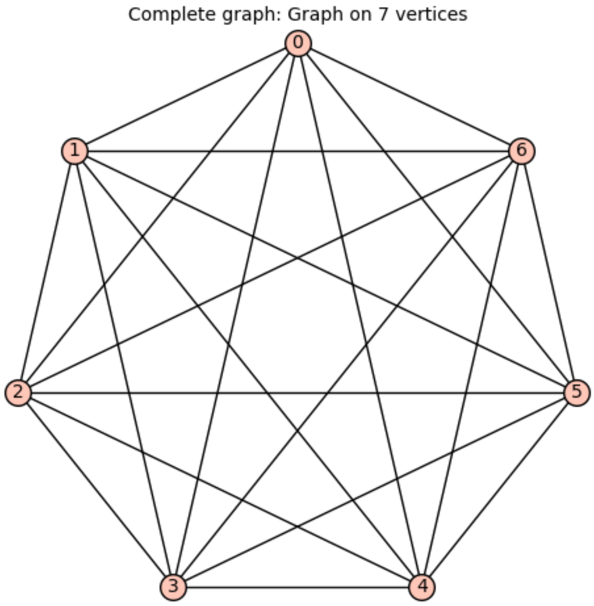
The famous Petersen graph.

```
graphs.PetersenGraph()
```



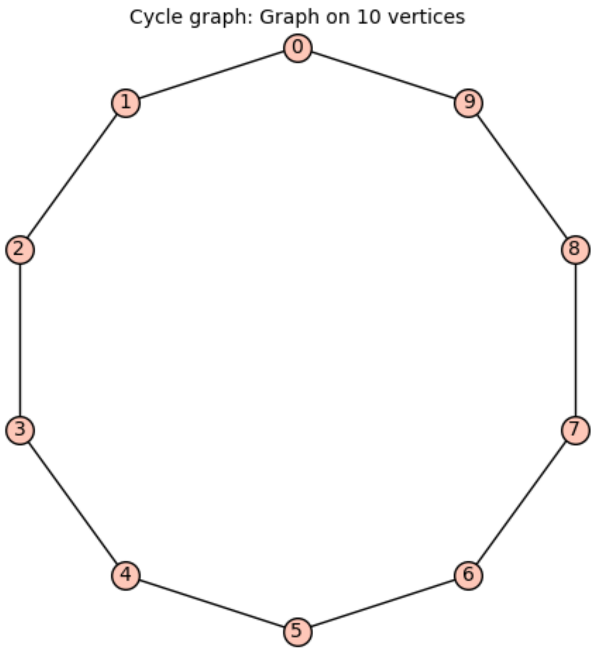
Complete graphs K_n.

```
graphs.CompleteGraph(7)
```



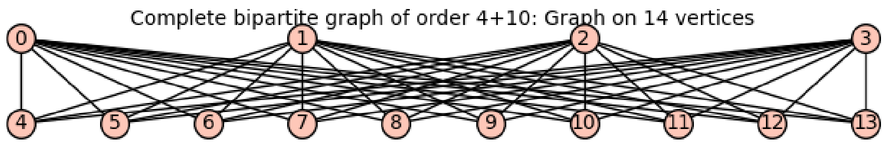
Cycle graphs C_n.

```
graphs.CycleGraph(10)
```



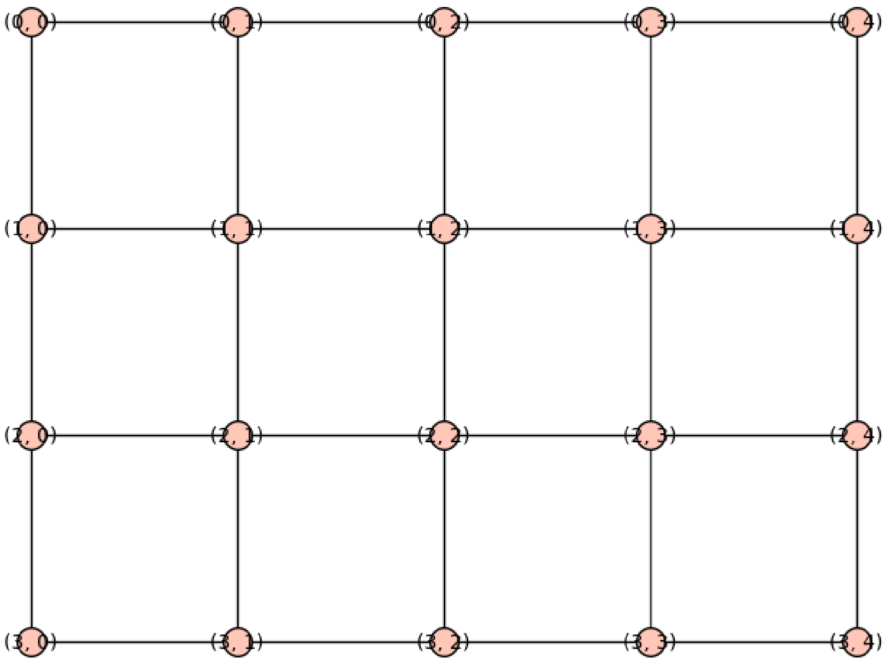
Complete bipartite graphs $K_{n,m}$.

```
graphs.CompleteBipartiteGraph(4, 10)
```



Grid graphs $G_{n,m}$.

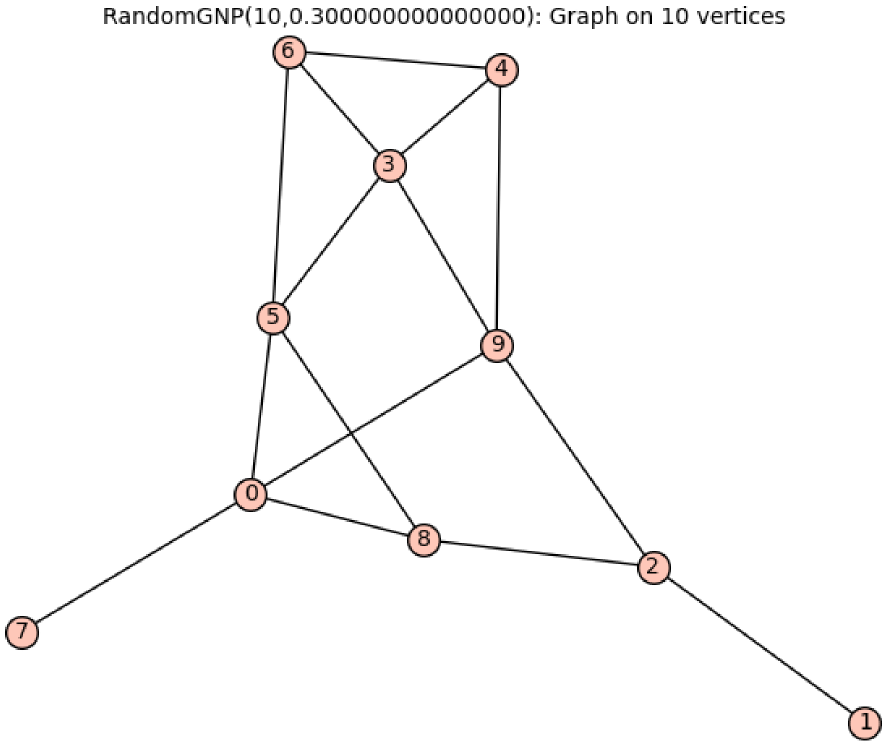
```
GG = graphs.GridGraph([4, 5])
GG.plot()
```



1.2.1.2 Randomly generated graphs

Random graph on 10 nodes. Each edge is inserted independently with probability 0.3.

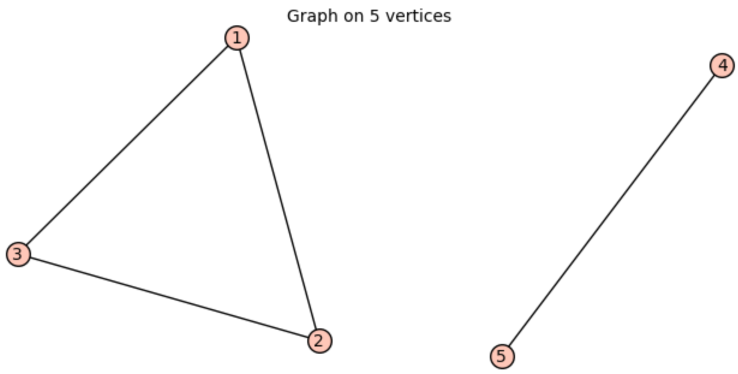
```
graphs.RandomGNP(10, 0.3)
```



1.2.1.3 Graph constructors

From a list of edges.

```
Graph([(1,2),(2,3),(3,1),(4,5)])
```

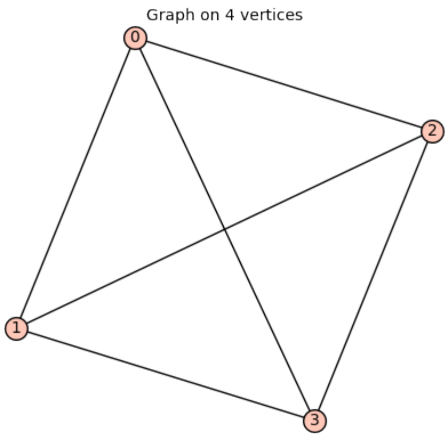


From an adjacency matrix.

```
m = matrix([[int(i != j) for i in range(4)] for j in range(4)])
m
```

```
[0 1 1 1]
[1 0 1 1]
[1 1 0 1]
[1 1 1 0]
```

```
Graph(m)
```



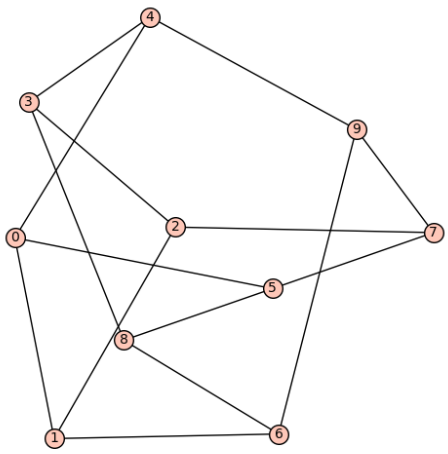
Graph to adjacency matrix.

```
M = G.adjacency_matrix()
m
```

```
[0 1 1 1 1 0 0]
[1 0 0 0 0 0 1]
[1 0 0 0 1 0 1]
[1 0 0 0 0 0 1]
[1 0 1 0 0 0 1]
[0 0 0 0 0 0 1]
[0 1 1 1 1 1 0]
```

From/to graph6 format (compressed string representation of a graph).

```
G = Graph('IheA@GUAo')
G.plot()
```



```
G.graph6_string()
```

```
'IheA@GUAo'
```

Query a graph from local database http://doc.sagemath.org/html/en/reference/graphs/sage/graphs/graph_database.html. For example to get a list of all graphs on 7 vertices with diameter 5.

```
Q = GraphQuery(display_cols=['graph6'], num_vertices=7, diameter=5)
Q.show()
```

```
Graph6
--------------------
F?`po
F?gqg
F@?]O
F@OKg
F@R@o
FA_pW
FEOhW
FGC{o
FIAHo
```

1.2.2 Basic graph manipulation

```
G = Graph({0:[1,2,3], 4:[0,2], 6:[1,2,3,4,5]});
```

1.2.2.1 Access edges, verices, neighbors, etc.

Access edges.

```
G.edges(labels=False)
```

```
[(0,1),(0,2),(0,3),(0,4),(1,6),(2,4),(2,6),(3,6),(4,6),(5,6)]
```

Note: Edges can have labels. To get a list of edges without labels, use `labels=False` option. Without this option we get

```
[(0,1,None),(0,2,None),(0,3,None),(0,4,None),(1,6,None),(2,4,None),
(2,6,None),(3,6,None),(4,6,None),(5,6,None)]
```

To check if there is an edge between two vertices use

```
G.has_edge(1,2)
```

```
False
```

Access vertices.

```
G.vertices()
```

```
[0,1,2,3,4,5,6]
```

Access neighbors of a vertex.

```
G.neighbors(0)
```

```
[1,2,3,4]
```

Degree of a vertex is a number of its neighbors

```
G.degree(0)
```

```
4
```

To list degrees of all vertices use

```
G.degree()
```

```
[4,3,3,2,3,2,5]
```

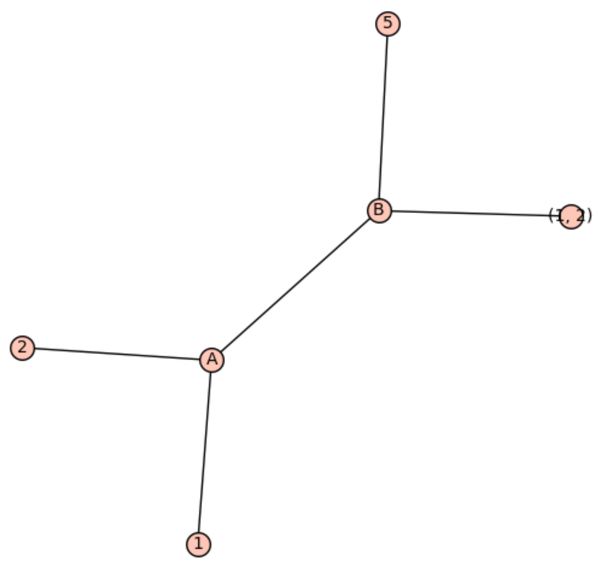
Access number of vertices, edges

```
[G.num_verts(),G.num_edges()]
```

```
[7,10]
```

Vertices of a graph can be any hashable objects, not just integers. For example:

```
X=Graph({'A':[1,2],'B':[(1,2),5,'A']})
X.plot()
```



1.2.2.2 Add/remove vertices, edges

Add a vertex.

```
G.add_vertex('a')
```

Method `add_vertex` without arguments adds a single vertex with the smallest available label.

```
newv = G.add_vertex()
newv
```

```
7
```

```
G.vertices(sort=False)
```

```
['a',7,0,1,2,3,4,5,6]
```

Unnumbered Section

Unnumbered Subsection

Unnumbered Subsubsection

2. In-text Element Examples

2.1 Referencing Publications

This statement requires citation [1]; this one is more specific [2, page 162].

2.2 Link Examples

This is a URL link: LaTeX Templates. This is an email link: example@example.com. This is a monospaced URL link: `https://www.LaTeXTemplates.com`.

2.3 Lists

Lists are useful to present information in a concise and/or ordered way.

2.3.1 Numbered List

1. First numbered item
    a. First indented numbered item
    b. Second indented numbered item
        i. First second-level indented numbered item
2. Second numbered item
3. Third numbered item

2.3.2 Bullet Point List

- First bullet point item
    - First indented bullet point item
    - Second indented bullet point item
        - First second-level indented bullet point item
- Second bullet point item
- Third bullet point item

2.3.3 Descriptions and Definitions

Name Description
Word Definition
Comment Elaboration

2.4 International Support

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2.5 Ligatures

fi fj fl ffl ffi Ty Ty

Part Two Title

3. Mathematics

3.1 Theorems

3.1.1 Several equations

This is a theorem consisting of several equations.

Theorem 3.1 — Name of the theorem. In $E = \mathbb{R}^n$ all norms are equivalent. It has the properties:

$$\big| ||\mathbf{x}|| - ||\mathbf{y}|| \big| \leq ||\mathbf{x} - \mathbf{y}|| \tag{3.1}$$

$$|| \sum_{i=1}^{n} \mathbf{x}_i || \leq \sum_{i=1}^{n} ||\mathbf{x}_i|| \quad \text{where } n \text{ is a finite integer} \tag{3.2}$$

3.1.2 Single Line

This is a theorem consisting of just one line.

Theorem 3.2 A set $\mathscr{D}(G)$ in dense in $L^2(G)$, $|\cdot|_0$.

3.2 Definitions

A definition can be mathematical or it could define a concept.

Definition 3.1 — Definition name. Given a vector space E, a norm on E is an application, denoted $||\cdot||$, E in $\mathbb{R}^+ = [0, +\infty[$ such that:

$$||\mathbf{x}|| = 0 \;\Rightarrow\; \mathbf{x} = \mathbf{0} \tag{3.3}$$

$$||\lambda \mathbf{x}|| = |\lambda| \cdot ||\mathbf{x}|| \tag{3.4}$$

$$||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}|| \tag{3.5}$$

3.3 Notations

■ **Notation 3.1** Given an open subset G of $\mathbb{R}^n$, the set of functions φ are:

1. Bounded support G;
2. Infinitely differentiable;

a vector space is denoted by $\mathscr{D}(G)$.

3.4 Remarks

This is an example of a remark.

R The concepts presented here are now in conventional employment in mathematics. Vector spaces are taken over the field $\mathbb{K}=\mathbb{R}$, however, established properties are easily extended to $\mathbb{K}=\mathbb{C}$.

3.5 Corollaries

Corollary 3.1 — Corollary name. The concepts presented here are now in conventional employment in mathematics. Vector spaces are taken over the field $\mathbb{K}=\mathbb{R}$, however, established properties are easily extended to $\mathbb{K}=\mathbb{C}$.

3.6 Propositions

3.6.1 Several equations

Proposition 3.1 — Proposition name. It has the properties:

$$\big|\,||\mathbf{x}|| - ||\mathbf{y}||\,\big| \leq ||\mathbf{x}-\mathbf{y}|| \tag{3.6}$$

$$||\sum_{i=1}^{n}\mathbf{x}_i|| \leq \sum_{i=1}^{n}||\mathbf{x}_i|| \quad \text{where } n \text{ is a finite integer} \tag{3.7}$$

3.6.2 Single Line

Proposition 3.2 Let $f,g\in L^2(G)$; if $\forall\varphi\in\mathscr{D}(G)$, $(f,\varphi)_0=(g,\varphi)_0$ then $f=g$.

3.7 Examples

3.7.1 Equation Example

■ **Example 3.1** Let $G=\{x\in\mathbb{R}^2 : |x|<3\}$ and denoted by: $x^0=(1,1)$; consider the function:

$$f(x)=\begin{cases} \mathrm{e}^{|x|} & \text{si } |x-x^0|\leq 1/2 \\ 0 & \text{si } |x-x^0|>1/2 \end{cases} \tag{3.8}$$

The function f has bounded support, we can take $A=\{x\in\mathbb{R}^2 : |x-x^0|\leq 1/2+\varepsilon\}$ for all $\varepsilon\in\,]0;5/2-\sqrt{2}[$. ■

3.7.2 Text Example

■ **Example 3.2 — Example name.** Aliquam arcu turpis, ultrices sed luctus ac, vehicula id metus. Morbi eu feugiat velit, et tempus augue. Proin ac mattis tortor. Donec tincidunt, ante rhoncus luctus semper, arcu lorem lobortis justo, nec convallis ante quam quis lectus. Aenean tincidunt sodales massa, et hendrerit tellus mattis ac. Sed non pretium nibh. Donec cursus maximus luctus. Vivamus lobortis eros et massa porta porttitor. ■

3.8 Exercises

Exercise 3.1 This is a good place to ask a question to test learning progress or further cement ideas into students' minds. ■

3.9 Problems

Problem 3.1 What is the average airspeed velocity of an unladen swallow?

3.10 Vocabulary

Define a word to improve a students' vocabulary.

- **Vocabulary 3.1 — Word.** Definition of word.

4. Presenting Information and Results with a Long Chapter Title

4.1 Table

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Treatments	**Response 1**	**Response 2**
Treatment 1	0.0003262	0.562
Treatment 2	0.0015681	0.910
Treatment 3	0.0009271	0.296

Table 4.1: Table caption.

Referencing Table 4.1 in-text using its label.

4.2 Figure

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Figure 4.1: Figure caption.

Referencing Figure 4.1 in-text using its label.

Treatments	**Response 1**	**Response 2**
Treatment 1	0.0003262	0.562
Treatment 2	0.0015681	0.910
Treatment 3	0.0009271	0.296

Table 4.2: Floating table.

Figure 4.2: Floating figure.

Bibliography

Articles

[1] A. B. Jones and J. M. Smith. “Article Title”. In: *Journal title* 13.52 (Mar. 2022), pages 123–456. DOI: 10.1038/s41586-021-03616-x (cited on page 19).

Books

[2] J. M. Smith and A. B. Jones. *Book Title*. 7th. Publisher, 2021 (cited on page 19).

Index

A. Appendix Chapter Title

A.1 Appendix Section Title

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B. Appendix Chapter Title

B.1 Appendix Section Title

Lorem ipsum dolor sit amet, consectetur adipiscing elit. Aliquam auctor mi risus, quis tempor libero hendrerit at. Duis hendrerit placerat quam et semper. Nam ultricies metus vehicula arcu viverra, vel ullamcorper justo elementum. Pellentesque vel mi ac lectus cursus posuere et nec ex. Fusce quis mauris egestas lacus commodo venenatis. Ut at arcu lectus. Donec et urna nunc. Morbi eu nisl cursus sapien eleifend tincidunt quis quis est. Donec ut orci ex. Praesent ligula enim, ullamcorper non lorem a, ultrices volutpat dolor. Nullam at imperdiet urna. Pellentesque nec velit eget est euismod pretium.