Fixation in a generalized voter model

Prelims Research Abstract

1 Introduction

The voter model [1] is one of the simplest multi-particle dynamics that yields rich behavior. In its most basic form, it describes a situation where individuals take one of two states, $+1$ or -1. Following a given network of connections, a person will talk to one of his neighbors uniformly at random and adopt that neighbor's opinion. The probability of a person adopting a given opinion is thus the concentration of that opinion among his friends. There are exactly two absorbing states, corresponding to complete consensus for either opinion. Despite the crudeness of the model, it has been applied across many fields for a wide variety of purposes, often under different names. The voter model appears in physics for surface catalysis [2], in social science for opinion dynamics [3] and cultural or linguistic evolution [4], and in population genetics for neutral genetic drift [5].

A two state process in which individuals hold only one opinion is grossly unrealistic when attempting to model the spread of political opinions. People hold a spectrum of opinions on topics like abortion, the Iraq war, tax policy, foreign aid, etc. However, these opinions often sort themselves into coherent categories. A pair of democrats would likely agree about many topics by virtue of having the same political party, as would a pair of republicans. We consider this to indicate a drive for internal consistency. There is also a general desire by many people to adopt the behavior of his or her peers. This is considered a drive for conformity. It is conceivable that the interplay of these two forces can cause interesting behavior in the spread of opinions [6]. We create a network topology into which conformity and consistency are imbedded with differing strengths and study how this topology affects the voter model, with special emphasis on the time to consensus.

Let there be N individuals, each with an opinion, $+1$ or -1, on M issues. With probability p, an individual will select a topic and interact with another individual on the same topic. Conversely, with probability $q = 1 - p$ an individual selects two topics internally. Consistency is thus enforced with a weight q and conformity with a weight p. For a given topic, any individual is accessible from any other individual and, within a given person, any internal topic from any internal topic.

The $M = 1$ (or $N = 1$, mutatis mutandis) case is equivalent to a standard mean field. This is the simplest topology for the voter model, since a connection to random walks makes solving for many properties relatively simple [5]. See, for instance, [7]. There are N sites and only two possible states per site, so the system can be characterized at any given time by only the number of plus opinions, denoted by n. This analogy changes the problem to that of a random walker on a line. For N large enough that $1/N \approx 1/(N-1)$, the probability of hopping in either direction is $\frac{n}{N}(1-\frac{n}{N})$. The mean extinction time from a state, T_n, can be calculated directly with a self-consistency argument. In the continuum limit,

$$-N^2 = x(1-x)\frac{d^2}{dx^2}T(x). \quad (1)$$

This is easily integrable under the boundary condition that $T(0) = T(1) = 0$, giving

$$T_{mf}(x) = -N^2(x\log(x) + (1-x)\log(1-x)). \tag{2}$$

Using the results of the mean field, we can guess the scaling of the extinction time with respect to N, M, and p. For values of p near $1/2$, the $N \to 1$ or $M \to 1$ situations approach the mean field voter model with M or N voters, respectively. If we let $N = 1$, the average time to extinction, $T(N = 1, M)$, is that of the one dimensional mean field, so the extinction time of this process scales with system size as M^2. All edges are equally weighted, so the system is also symmetric under exchange of N and M. The most reasonable guess to allow both characteristics is

$$T(N, M, p = 0.5) = N^2M^2.$$

If $p << 1$, almost all of the dynamics occurs within individuals. Each person is almost always at consensus in himself, and so is like a single node in the one dimensional mean field. The system goes to extinction as N^2, but with a rescaled time to account for the average time to flip a whole person instead of just one site. To properly scale the time, first note that only in $1/p$ time steps is there an exchange. That exchange only confers one conflicting site, however, and often this dissent will not cause a switch in the whole person. The probability of a single introduced opinion causing internal convergence is the opinion's initial concentration, $1/M$. An opinion will need to be introduced M times before flipping an individual's state. Exchanging $M \leftrightarrow N$ and $p \leftrightarrow q$ simultaniously gives the complementary scaling in for $p \to 1$. This leads to the estimate

$$T(N, M, p) = \begin{cases} \frac{NM^2}{q} & p \to 1 \\ \frac{MN^2}{p} & p \to 0 \end{cases} \tag{3}$$

We can use network theory to make these estimates more rigorous. An adjacency matrix element A_{ij} denotes the probability that, having selected node i, node j will be picked to transmit its state.

$$A_{ij} = \begin{cases} \frac{q}{M-1} & \text{If } i \text{ and } j \text{ are in the same voter} \\ \frac{p}{N-1} & \text{If } i \text{ and } j \text{ are of the same topic} \\ 0 & \text{Otherwise} \end{cases} \tag{4}$$

A standard technique of solving the voter model on a lattice [8] can be adapted to the network approach [9]. Let the state of the voters be given by the set of all nodes, $\{\sigma_i\} = S$, and let S_k^i be the same state as S, but with $\sigma_k \to -\sigma_k$. The probability of being in state S at time t is $P(S, t)$. The transition rate for $\sigma_i \to -\sigma_i$ is

$$W_i(S) = \frac{1}{2}(1 - \sigma_i \sum_j A_{ij}\sigma_j) \tag{5}$$

The master equation for $P(S)$ is the Chapman-Komolgorov equation

$$\frac{dP(S)}{dt} = \sum_i [W_i(S_i^i)P(S_i^i) - W_i(S)P(S)]. \tag{6}$$

The fixation time for a network can be estimated by looking at the decay of the density ρ of edges that connect disagreeing nodes [2]. The density is related to the average over all correlations by

$$\rho = \frac{1}{2}\left(1 - \overline{\langle\sigma_i\sigma_j\rangle}\right). \tag{7}$$

The time constant with which the correlations approach one should thus be proportional to the mean extinction time.

By multiplying equation (6) by $\sigma_i\sigma_j$ and summing over all configurations S, we can derive the evolution equations for the ensemble average correlations

$$\frac{d}{dt}\langle s_i s_j\rangle = -\frac{1}{NM}\left(2\langle s_i s_j\rangle - \langle [As]_i s_j\rangle - \langle s_i [As]_j\rangle\right) \tag{8}$$

The conformity-consistency network defined by equation (4) is symmetric under exchange of any two topics and exchange of any two individuals, so we only have four distinct correlations. There is a trivial autocorrelation which is 1 for all time, a correlation r_p between individuals on one topic, a correlation r_q between topics within an individual, and the correlation between unconnected nodes, r_2. The three remaining equations are

$$\frac{d}{dt} r_q = \frac{2}{NM}\left(\frac{q}{M-1} - \left(p + \frac{q}{M-1}\right) r_q + p r_2\right) \tag{9}$$

$$\frac{d}{dt} r_p = \frac{2}{NM}\left(\frac{p}{N-1} - \left(q + \frac{p}{N-1}\right) r_p + q r_2\right) \tag{10}$$

$$\frac{d}{dt} r_2 = \frac{2}{NM}\left(\frac{p}{N-1} r_p + \frac{q}{M-1} r_p - \left(\frac{q}{M-1} + \frac{q}{N-1}\right) r_2\right) \tag{11}$$

This can be represented by a linear equation by making the substitutions

$$\mathbf{r} = \begin{pmatrix} r_q \\ r_p \\ r_2 \end{pmatrix} \tag{12}$$

$$v = \begin{pmatrix} \frac{q}{M-1} \\ \frac{p}{N-1} \\ 0 \end{pmatrix} \tag{13}$$

$$B = \begin{pmatrix} p + \frac{q}{M-1} & 0 & -p \\ 0 & q + \frac{p}{N-1} & -q \\ -\frac{p}{N-1} & -\frac{q}{M-1} & \frac{q}{M-1} + \frac{p}{N-1} \end{pmatrix} \tag{14}$$

giving the final form

$$\frac{d}{dt}\mathbf{r} = \frac{2}{NM}(v - B\mathbf{r})$$

The slowest time scale in the evolution of the correlations is proportional to the inverse of the smallest eigenvalue of B. In the limit of small p, matrix perturbation theory tells us that the smallest eigenvalue is $p/(N-1)$. To first order, the time constant is proportional to MN^2/p, in agreement with our earlier guess. Second order and higher terms can also be calculated, though in principle the eigenvalues are solutions to a cubic equation and therefore have a closed form.

Computer simulations have been done confirming the scaling properties we have shown and further showing that the smallest eigenvalues give the extinction time for any p. See Figure 1. In fact, the time constant τ appears to relate to the fixation time T by $T \approx 1.4\tau$ for all explored system parameters. The reason for this constant is not known. We are attempting to understand the exact connection between the correlations and the fixation time. Other future work is to ask how adding noise affects dynamics on a weighted network and to explore the use of population genetics tools such as coalescent theory [10] in these models.

This work has been done in collaboration with Len Sander and Scott Page.

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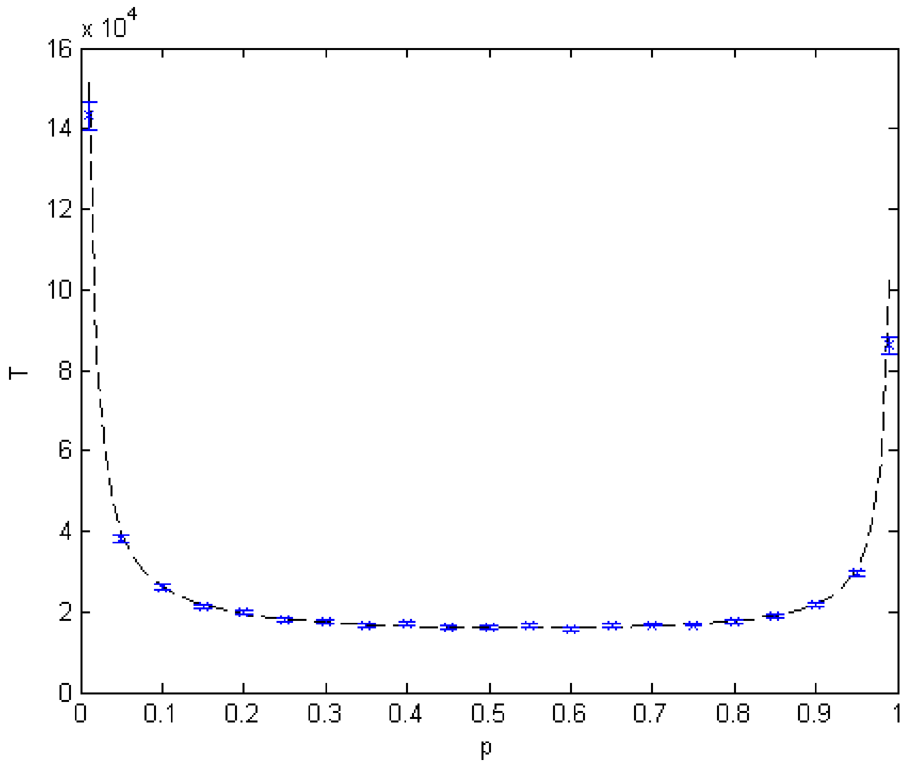

Figure 1: Simulation of mean fixation time for a network with N = 15, M = 10 averaged over 1000 runs. The data is shown in blue and the inverse of the smallest eigenvalue is shown in black. An order one constant factor was introduced by hand to the eigenvalues.