In his paper *A simple proof for optimality of (S, s) policies*, Yu-Sheng Zheng calls upon the results of stationary optimal solutions to infinite-horizon inventory management problems to simplify previously complicated proofs of optimality. Referring mainly to the seminal work of Inglehart, Zheng reports that earlier proofs of the optimality of a stationary (S, s) policy were unnecessarily tedious because their authors depended upon the results of the finite-horizon case. The problem arose because in the finite horizon case, one must account for end-of-horizon effects, and therefore the proofs were "capable of characterizing only the *form* of an optimal policy, rather than the optimal policy *itself*." Zheng uses Markov decision process theory to show that the optimal policy consists of just two parameters: S^* and s^*, and that it holds for the case with fixed-plus linear ordering costs and quasiconvex holding and shortage costs. He is able to derive the proof for both the discounted and the average cost models.

An (S, s) policy stipulates that a decision maker need not order any additional units if the inventory level lies between two integers s and S, S > s; if the inventory level falls below s, then he should reorder a quantity such that the new inventory level becomes S. Hence, the range of possible inventory positions is bounded above by S and unbounded below. The goal of such a policy is to simultaneously minimize the costs of carrying too much or too little inventory and the cost of reordering.

Because the problem is over an infinite horizon, it is not necessary to formulate a dynamic programming functional equation. Rather, Zheng gives the total expected α-discounted cost with initial inventory level *i* as a function of the policy (s, S):

$$f_\alpha(i \mid s,S) = \begin{cases} c_\alpha(s,S)/(1-\alpha), & i \le s \\ c_\alpha(s,S)/(1-\alpha) + I_\alpha(s,i) - c_\alpha(s,S)M_\alpha(i-s), & i > s \end{cases},$$

where $I_\alpha(s,i)$ represents the expected α-discounted inventory and holding costs until the inventory level drops to or below s given initial inventory *i*, and $M_\alpha(j)$ represents the expected discounted time to deplete *j* units of inventory.

The challenge then becomes how to classify the values s and S that minimize the above function. Zheng first shows that there do exist parameters S^* and s^* which bound a region of inventory levels producing minimal expected costs. Then, using such an S^* and s^*, he shows case-wise (where the cases are for initial inventories at or below s^*, between S^* and s^*, and greater than S^*) that $f_\alpha(\cdot)$ satisfies the optimality equation

$$f_\alpha(i) = \inf_{j \ge i} \left[K\delta(j-i) + G_\alpha(j) + \alpha \sum_{l=o}^{\infty} p_l f_\alpha(j-l) \right].$$

After having then shown that $f_\alpha(\cdot)$ satisfies the optimality equation for all possible initial inventory positions, he concludes that there exists an optimal stationary (s, S) policy.

He lastly turns his attention to the average cost model. To make the costs bounded, he relaxes the problem to one where inventory can in fact take on values greater than S and thus allowing for negative orders. However, he is quick to point out that intuitively, this would never happen optimally.

Personally, I didn't find Zheng's proofs to be "simple" as he called them. Perhaps that is because we skipped (S, s) policies and spent little time on average cost models in my IOE 512 course. However, I do appreciate the strength of the arguments that come by

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being able to use a stationary policy in the infinite horizon problem. The paper assumes that the reader is familiar with the cost models used in a paper by Wagner and Veinott, which I was not, and also does a poor job of explaining the relationships between the cost models, which makes the proofs mildly inaccessible in my opinion. Nevertheless, if Zheng's claim that all previous attempts at proofs of the infinite horizon (S, s) model rely on finite horizon results, then this paper is indeed significant.

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