Fast Quantum Gates with Trapped Ions

We explore the interaction of a trapped ion with a laser in the strong-field regime. In particular, coherent control techniques may be used to manipulate laser pulses and allow for the realization of a novel two-qubit gate scheme. The proposed gate operation has a number of advantages over other current gate schemes, including insensitivity to the temperature of the ions and the possibility of being orders of magnitude faster than the trap period. Here we investigate the theoretical and experimental prospects of this fast quantum gate.

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I. INTRODUCTION

A viable system for the implementation of a quantum computer is being actively persued by a number of research groups in varying shapes and forms. Trapped ions have proven to be one of the most promising of these systems, having already demonstrated many of the fundamental aspects necessary for a quantum information processor [1]. Advances in this area include the demonstration of two quantum bit (qubit) gates, and multiqubit entangled states [2,3,4]. These essential components have been accomplished by coupling the internal states of the ions via the mutual Coulomb-coupled motion ofi on crystal in the harmonic potential of the trap [5]. The basic quantum computation scheme envisioned here requires the ability to separate and shuttle pairs (or more) ions in some complex trap array, while keeping the motional state of the ions at the lowest possible levels. Recent results [6,7] indicate that arbitrary shuttling protocols may be a challenge to realize. Moreover, anomalous heating of the motional state of the ions continues to plague these traps and is a constant source of decoherence [8].

As though making a direct rebuttal to these limitations, a remarkable scheme for achieving two-qubit gate operations with trapped ions was put forward by Garcia-Ripoll, Zoller, and Cirac [9]. This new method proposes to use ultrafast laser pulses and the mechanical effects these can have on an ion to couple the motion and internal states, in place of the resolved sideband technique commonly employed. The distinguishing characteristics of this scheme are: (1) gate speeds orders of magnitude faster than the trap period; and (2) insensitivity to the temperature of the ion.

The scheme was further expanded by Duan [10], where it was shown that the gate explained above could be utilized in the construction of a scalable architecture that avoids the need to shuttle and separate ions. In doing so, Duan was able to remove what many view as the ultimate limiting factor on the total computation time that would be required.

In the rest of this paper, we seek to understand the theory that has led to this incredible proposal in quantum computation and touch on the experimental progress in its implementation.

II. SINGLE ION DYNAMICS

We begin with the discussion of the interaction of a single ion with a strong laser pulse. In doing so, we are able to disseminate the general mathematical formulation in a simple context. The concepts expressed here will serve as the foundation for our discussion of the full two qubit fast gate.

A. Hamiltonian

Let us consider a single ion trapped in a harmonic potential. For the sake of simplicity, we consider only the one dimensional case, and leave the higher order extrapolation as an exercise to the reader (besides, this allows the writer to ignore a bunch of subscripts, as everything is along the z-axis). The Hamiltonian describing this situation is then given by

$$H = H_o + H_I = \hbar\omega_z a^\dagger a - \mu \cdot E \tag{1}$$

where the first term simply describes the motion of the ion in the harmonic potential, and the second term accounts for the interaction between the atom and the light field. Assuming a two level system, a single-mode laser on resonance, and using the rotating wave approximation, we can rewrite the interaction portion of the Hamiltonian (in a frame that rotates at the frequency of the applied light field) as

$$\begin{aligned} H_I &= \frac{\mu E_o}{2}\left(\sigma_+ e^{ikz} + \sigma_- e^{-ikz}\right) \\ &= \frac{\hbar\Omega}{2}\left(\sigma_+ e^{ikz} + \sigma_- e^{-ikz}\right) \end{aligned} \tag{2}$$

where we have defined the Rabi frequency $\Omega = \frac{\mu E_o}{\hbar}$. The $\sigma_\pm$ are the usual Pauli spin-flip operators

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{3}$$

which simply illustrates that the applied laser field couples the two states in the atom (this is why the only nonzero terms are the off-diagonal elements).

B. Laser Pulse Interaction

We would now like to explore the effect of a laser pulse on a given initial state. Here on out, the ground state will be associated with the label $|0\rangle$, while the excited state will be $|1\rangle$. The evolution of a state is, of course, dictated by the interaction Hamiltonian (Eq. 2).

As is shown in Poyatos, et. al. [11], the evolution can be most easily described if we first perform the unitary operation

$$U = e^{-\imath kz|1\rangle\langle 1|} \tag{4}$$

Using this operator, the Hamiltonian becomes

$$\begin{aligned} \tilde{H}_I &= U H_I U^\dagger \\ &= e^{-\imath kz|1\rangle\langle 1|} \frac{\hbar\Omega}{2} \left(\sigma_+ e^{\imath kz} + \sigma_- e^{-\imath kz}\right) e^{\imath kz|1\rangle\langle 1|} \end{aligned} \tag{5}$$

To simplify this expression further, we will need to use the operator identity

$$e^B X e^{-B} = X + [B, X] + \frac{1}{2!}[B, [B, X]] + \ldots \tag{6}$$

a consequence of the Baker-Hausdorff lemma (see, for instance, page 80 of Meystre). We then observe that

$$\begin{aligned} [\sigma_+ e^{\imath kz}, \imath kz\,|1\rangle\langle 1|] &= ([\sigma_+, \imath kz]\,|1\rangle\langle 1| \\ &\quad + \imath kz[\sigma_+, |1\rangle\langle 1|])\, e^{\imath kz} \\ &\quad + \sigma_+ \left([e^{\imath kz}, \imath kz]\,|1\rangle\langle 1| \right.\\ &\quad \left. + \imath kz[e^{\imath kz}, |1\rangle\langle 1|]\right) \\ &= (0 + \imath kz\,(-\,|1\rangle\langle 0|))\, e^{\imath kz} \\ &\quad + \sigma_+ (0 + 0) \\ &= -\imath kz e^{\imath kz}\,|1\rangle\langle 0| \\ &= -\imath kz e^{\imath kz} \sigma_+ \end{aligned}$$

and

$$\begin{aligned} [\imath kz\,|1\rangle\langle 1| \;&,\; [\sigma_+ e^{\imath kz}, \imath kz\,|1\rangle\langle 1|]] \\ &= [\imath kz\,|1\rangle\langle 1|\,, -\imath kz e^{\imath kz}\sigma_+] \\ &= -k^2 z^2 e^{\imath kz}\sigma_+ \end{aligned}$$

Thus, as we plug in numbers for the series given by Eq. 6, we find

$$\begin{aligned} e^{-\imath kz|1\rangle\langle 1|} \;\; \tfrac{\hbar\Omega}{2}\sigma_+ e^{\imath kz} \;\; e^{\imath kz|1\rangle\langle 1|} & \\ &= \frac{\hbar\Omega}{2}(\sigma_+ e^{\imath kz} - \imath kz\sigma_+ e^{\imath kz} \\ &\quad - \frac{1}{2}k^2 z^2 e^{\imath kz}\sigma_+ + \ldots) \\ &= \frac{\hbar\Omega}{2}\sigma_+ e^{\imath kz}(1 \\ &\quad - \imath kz - \frac{1}{2}k^2 z^2 + \ldots) \\ &= \frac{\hbar\Omega}{2}\sigma_+ e^{\imath kz} e^{-\imath kz} \\ &= \frac{\hbar\Omega}{2}\sigma_+ \end{aligned}$$

Following a similar construction for the σ_- component of Eq. 5, it is found that

$$\tilde{H}_I = \frac{\hbar\Omega}{2}\left(\sigma_+ + \sigma_-\right) = \frac{\hbar\Omega}{2}\sigma_x \tag{7}$$

Given this remarkably simple expression for the Hamiltonian, the evolution of any initial state is easily derived, obtaining

$$\begin{aligned} |\Psi(\tau)\rangle &= U^\dagger \exp\left[-\imath \int \tilde{H}_I d\tau\right] U\,|\Psi(0)\rangle \\ &= U^\dagger \exp\left[-\imath \frac{\hbar}{2}\sigma_x \int \Omega d\tau\right] U\,|\Psi(0)\rangle \\ &= U^\dagger \exp\left[-\imath \frac{\hbar\Omega}{2}\sigma_x \tau\right] U\,|\Psi(0)\rangle \end{aligned} \tag{8}$$

where in the last line, for simplicity, we have made the assumption that the Rabi frequency Ω is time independent (this would correspond to a square-wave laser pulse).

Let's look at the evolution of a particular state. The initial state is given as the product state $|0\rangle\,|\alpha\rangle$, where $|0\rangle$ indicates the ground state of the internal levels and $|\alpha\rangle$ represents a coherent motional state. Recall that coherent states are eigenvectors of the annihilation operator a. Moreover, the position operator that appears in the unitary operator we have defined can be written as

$$z = z_o\left(a + a^\dagger\right) = \left(\frac{\hbar}{2m\omega_z}\right)^{1/2}\left(a + a^\dagger\right)$$

Thus, using Eq. 8, after experiencing a laser pulse of length τ the system will be in the state

$$\begin{aligned}
|\Psi\rangle &= U^\dagger e^{\left[-\imath\frac{\hbar\Omega}{2}\sigma_x\tau\right]} U\,|0\rangle\,|\alpha\rangle \\
&= e^{\imath kz|1\rangle\langle 1|} e^{\left[-\imath\frac{\hbar\Omega}{2}\sigma_x\tau\right]} e^{-\imath kz|1\rangle\langle 1|}\,|0\rangle\,|\alpha\rangle \\
&= e^{\imath kz|1\rangle\langle 1|} e^{\left[-\imath\frac{\hbar\Omega}{2}\sigma_x\tau\right]}\,|0\rangle\,|\alpha\rangle \\
&= e^{\imath kz|1\rangle\langle 1|}\left(1 - \frac{\imath\hbar\Omega}{2}\sigma_x\tau - \frac{\hbar^2\Omega^2}{8}\sigma_x^2\tau^2\right. \\
&\quad \left. + \frac{\imath\hbar^3\Omega^3}{48}\sigma_x^3\tau^3 + \ldots\right)|0\rangle\,|\alpha\rangle \\
&= e^{\imath kz|1\rangle\langle 1|}\left(1 - \frac{\imath\hbar\Omega}{2}\sigma_x\tau - \frac{\hbar^2\Omega^2}{8}\tau^2\right. \\
&\quad \left. + \frac{\imath\hbar^3\Omega^3}{48}\sigma_x\tau^3 + \ldots\right)|0\rangle\,|\alpha\rangle \\
&= e^{\imath kz|1\rangle\langle 1|}\left(\cos\left(\frac{\hbar\Omega\tau}{2}\right) - \imath\sigma_x\sin\left(\frac{\hbar\Omega\tau}{2}\right)\right)|0\rangle\,|\alpha\rangle \\
&= \cos\left(\frac{\hbar\Omega\tau}{2}\right) e^{\imath kz|1\rangle\langle 1|}\,|0\rangle\,|\alpha\rangle \\
&\quad -\imath\sin\left(\frac{\hbar\Omega\tau}{2}\right) e^{\imath kz|1\rangle\langle 1|}\,|1\rangle\,|\alpha\rangle \\
&= \cos\left(\frac{\hbar\Omega\tau}{2}\right)|0\rangle\,|\alpha\rangle - \imath\sin\left(\frac{\hbar\Omega\tau}{2}\right)|1\rangle\,|\alpha \pm \imath kz_o\rangle
\end{aligned}$$

The last line of the above equation is the result we have sought! For any pulse length $\tau \neq \frac{m\pi}{\hbar\Omega}$ where m is an integer, $|\Psi\rangle$ is an entangled state between the spin and motion of the ion. Hence, in this section we have been able to fully demonstrate the mathematics that enable coupling of the internal and motional states of a trapped ion through use of ultrafast laser pulses.

III. FAST GATE SCHEME

Since we have already reviewed some of the more technical (i.e. mathematical) details of ion-laser-pulse interaction, in this section we will not attempt to repeat the complex mathematics. Instead, making use of the previous section as our mathematical foundation, we seek a qualitative understanding of the two qubit gate proposed by Garcia-Ripoll, Zoller, and Cirac [9].

We now consider two ions in a one-dimensional harmonic potential. In order to justify our claim of a derivation analogous to the single ion situation, we note that the Hamiltonian describing the dynamics of the ions in the trap with a laser on resonance is given by [9, Eq. 1]:

$$\begin{aligned}
H_{o_2} + H_{I_2} &= \hbar\omega_c a^\dagger a + \hbar\omega_r b^\dagger b \\
&\quad + \frac{\hbar\Omega}{2}\sigma_{1+} e^{\imath\eta_c\left(a^\dagger + a\right) + \imath/(2\eta_r)\left(b^\dagger + b\right)} \\
&\quad + \frac{\hbar\Omega}{2}\sigma_{2+} e^{\imath\eta_c\left(a^\dagger + a\right) - \imath/(2\eta_r)\left(b^\dagger + b\right)} + h.c.
\end{aligned} \tag{9}$$

where here ω_c and ω_r are the frequencies of the center of mass and stretch mode, respectively; a and b are the corresponding annihilation operators; and η_c and η_r are proportional to the Lamb-Dicke parameter $\eta = kz_o$.

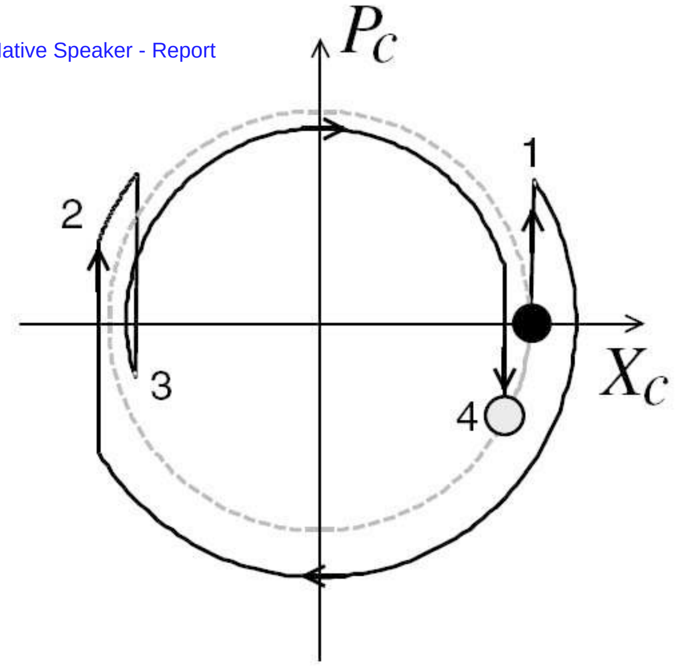

FIG. 1: Trajectory in phase space of the center-of-mass state of the two ions. Free harmonic oscillator evolution is depicted by motion along the arcs of circle, while the momentum kicks imparted by the laser pulses are represented by the vertical displacements. As is seen in the figure, a correctly chosen sequence of pulses ensures that the final state falls again on the dashed circle representing the path of free evolution. Hence, the motional state is unchanged, but a global phase is attained; a phase gate is implemented.

Let's compare Eq. 9 to Eqs. 1 and 2. In both the single and dual ion scenarios, we see the unperturbed Hamiltonian (H_o) is normal harmonic oscillator energy operator (though we have suppressed the zero-point energy in both cases). Likewise, in both cases the interaction Hamiltonian is simply that which is expected from the semiclassical interpretation of $H_I = -\mu\cdot\mathbf{E}$. The Hamiltonians in the cases of both one and two ions are completely analogous. Given that the complexity of the two ion case tends to cloud physical intuition, we thereby feel justified in claiming that the mathematics can be transferred from Section II without bypassing any physics (just a lot more math). In the rest of the section, then, we simply state and discuss the results from [9].

The crucial point in this scheme is illustrated in Fig. 1 (copied from Fig. 1 in [9]), which shows the trajectory of the center-of-mass state during the sequence of four laser pulses. Basically, each laser pulse gives the ion a momentum kick (vertical lines in the figure). If the timing of the pulses is chosen appropriately, at the end of the sequence, the ion will be returned to a point in phase space corresponding to free harmonic oscillator evolution;

in other words, the motional state will be the same as if only free evolution had occurred. However, during the pulse sequence a global phase factor appears, which is independent of the motional state. Thus, this series of pulses represents a controlled-phase gate, which, combined with single qubit rotations, represents a universal gate scheme [12].

Another incredible result that falls out of the derivation of the gate, is that it can be shown that for any given value of the total gate time T, there exists a sequence of laser pulses that implements the gate. In principle, then, this gate can be arbitrarily fast! Conventional gate schemes depend on the ability to resolve the motional sidebands of the ions, which inherently limits your gate speed to being less than the trap frequency (typically a few MHz). For the four pulse protocal illustrated in Fig. 1, was calculated for a (not optimized) fixed gate time of $T \cong 1.08\,(2\pi/\omega_c)$. However, they also derive a protocal that performs the gate in an arbitrarily short time, where the time T scales with the number of pulses N_p according to $N_p \propto T^{-3/2}$.

The gate scheme depicted here has advanced the prospects for a quantum information processor based on trapped ions. It allows one to avoid the issues associated with cooling the motion of the ions, as the protocal is independent of the motion state of the ions. Moreover, allows for arbitrarily fast gates, overcoming what was thought to be a fundamental limitation on a trap frequency. Combined with the technique of put forth by Duan [10] that bypasses the need for shuttling, we have an incredible new scheme for scalable quantum computation.

IV. EXPERIMENTAL IMPLEMENTATION

While the theoretical work outlined in the previous sections has been impressive, to say the least, the question of experimental implementation remains. As can be directly inferred from the publication dates of the theory work we have thus far reviewed, this quantum computation scheme is still very new. That being the case, it is perhaps not so surprising that the experimental work to date is somewhat scarce.

Nevertheless, in a recent article on arXiv, Madsen, et. al. [13] announced that they have used ultrafast optical pulses to drive picosecond Rabi oscillations in a single trapped ion. In addition, two counter-propagating laser pulses were utilized to first excite, then coherently deexcite the atom. As a result, the pulse pair imparts a $2\hbar k$ momentum kick to the ion. Clearly, this is the first step towards realizing the fast quantum gate scheme detailed above.

V. CONCLUSIONS

We have completed a detailed study of the interaction of a single trapped ion with laser pulses. By analogy, we were then able to put into context the fast gate scheme proposed by Garcia-Ripoll, Zoller, and Cirac. The remarkable results of this scheme were presented, as well as a short look at the experimental progress in realizing these fast gates. Overall, we have been achieved a greater understanding of the promises of theory, and an appreciation for the experimental work still ahead.

References

1. C. Monroe, Nature (London) **416**, 238 (2002)
2. C. A. Sackett, et. al. *Nature* **404**, 256 (2000)
3. D. Leibfried, et. al. *Nature* **438**, 639 (2005)
4. P. C. Haljan, et. al. *Phys. Rev. A* **72**, 062316 (2005)
5. D. J. Wineland, et. al. *NIST J. Res.* **103**, 259 (1998)
6. W. K. Hensinger, et. al. *Appl. Phys. Lett.* **88**, 034101 (2006)
7. M. A. Rowe, et. al. *Quantum Inf. Comp.* **2**, 257 (2002)
8. L. Deslauriers, et. al. quant-ph/0602003 (2006)
9. J. J. Garcia-Ripoll, et. al. *Phys. Rev. Lett.* **91**, 157901 (2003)
10. L. Duan *Phys. Rev. Lett.* **93**, 100502 (2004)
11. J. F. Poyatos, et. al. *Phys. Rev. A* **54**, 1532 (1996)
12. Nielsen and Chuang *Quantum Computation and Quantum Information* (2000)
13. M. J. Madsen, et. al. quant-ph/0603258 (2006)