18.02 Problem Set 1

Problem set 1 is due on Monday February 10 at 11:55 pm. Solutions will be posted on Stellar shortly afterwards. Turn in the homework using Gradescope. For instructions on using Gradescope, see Problem Set 0.

1. Level curves and partial derivatives

Reference: Pages 852-854 (graphs and level curves) and 868-871 (partial derivatives).

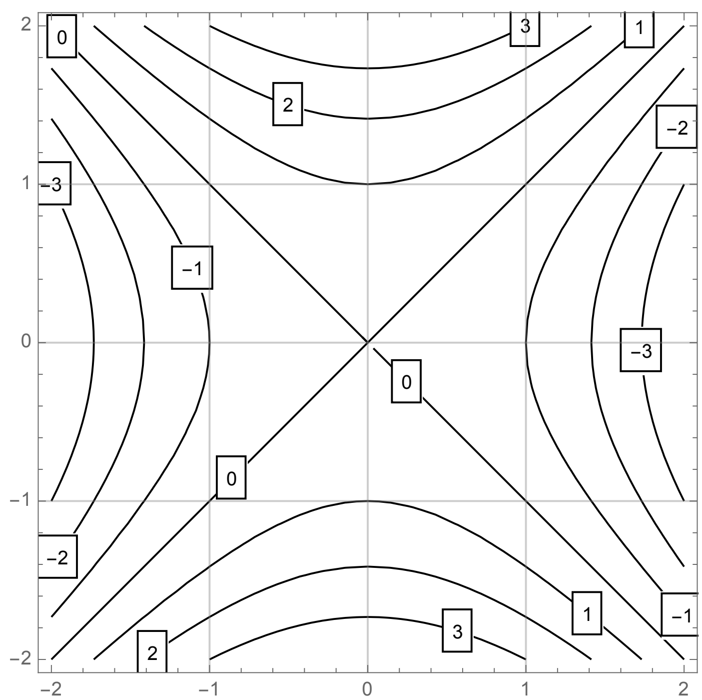
1. (Computing partial derivatives, 10 points) For the following functions, compute the partial derivatives with respect to x and with respect to y:

a.) $xy^2 + y$.

b.) $\frac{x}{x+y}$.

c.) $\sin(xy^2)$.

2. (10 points) Let $f(x, y) = y^2 - x^2$. Here is a picture of the level curves of f.



Now we will compute the partial derivatives of f and compare the information from the partial derivatives to the picture.

a.) Compute f_x and f_y.

b.) Is $f_y(1,-1)$ positive or negative? How can you see this in your picture?
c.) Which is bigger $f_y(.5,.5)$ or $f_y(1,1)$? How can you see this in your picture?

3. (10 points) Suppose we are interested in studying how fast the liver metabolizes protein P. The metabolism rate depends on the level of hormone X and hormone Y. Let x denote the level of hormone X and y the level of hormone Y. If $x < 1$, then increasing the amount of hormone Y increases the metabolic rate. But if $x > 1$, then increasing the amount of hormone Y decreases the metabolic rate. Which of the following formulas could be the formula for the metabolic rate in terms of x and y? Explain your reasoning.
a.) $10 - y^2 + x$.
b.) $10 - xy + y$.
c.) $10 - x^2 + xy$.

2. Linear approximation

Reference: Section 13.6.

4. (10 points) Let $f(x,y) = \sin(xy^2 + y) + 2e^x$.
a.) Compute the linear approximation of f around $(0,0)$, and use it to approximate $f(0.1, 0.2)$.
b.) We want to find a value of y near to zero so that $f(0,y) = 1.99$. Approximate this value of y.

There are two ways to write the linear approximation. One way is in terms of Δx and Δy.

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y.$$

The right-hand side is called the linear approximation to f around (x_0, y_0). Sometimes instead it is nice to write the linear approximation as a function of x and y. If we think of $x = x_0 + \Delta x$ and $y = y_0 + \Delta y$, then we can rewrite the last formula as

$$f(x,y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

The second form can be helpful when we want to think of the linear approximation as a function of x and y. The original function f is a function of x and y. The linear approximation of f around a point (x_0, y_0) is a new function of x and y, which is often much simpler than f, and which does a good job of approximating f near to (x_0, y_0). We will illustrate this in the next problem.

5. (10 points) Let us return to the function $f(x, y) = y^2 - x^2$ that we explored in problem 3 above.

a.) Compute the linear approximation to f around the point $(1, 1)$, and write your answer as a function of x and y. Let $L(x, y)$ denote this linear approximation.

b.) Consider the square in the plane where $.5 \leq x \leq 1.5$ and $.5 \leq y \leq 1.5$. Inside this square draw the level curves of L of heights $-1, 0, 1$. Compare this picture to the picture of the level curves of f from Problem 2. They should look fairly similar, especially close to the point $(1, 1)$.

6. (10 points) We experimentally measured two quantities called x and y. We determined that x is equal to 1 to within a measurement error of .01 and y is equal to 9 to within a measurement error of .01. The thing that we really care about is

$$z = \frac{x}{x+y}.$$

(Btw, this function appeared in Problem 1b above.)

If we plug in $x = 1$ and $y = 9$, then we get $z = .1$ But there are measurement errors: x is not exactly 1 and y is not exactly 9, and so we can't say that z is exactly .1 There is some measurement error in z.

a.) What is the order of magnitude of the measurement error in z? Is it best described as 10^{-2}, 10^{-3}, or 10^{-4}? Explain your reasoning.

b.) Our experiment has suffered budget cuts, and we don't have the money to do it as accurately as in the original plan. We can either measure x to within an error of .01 and y to within an error of .03, or we can measure x to within an error of .03 and y to within an error of .01. The thing we really care about is $z = \frac{x}{x+y}$, and we want to approximate z as accurately as we can. We know from experience that x will be fairly close to 1 and y will be fairly close to 9. Which plan is better? Explain your reasoning.

3. Vectors

7. (10 points, 2 for each part)

a. Find the length of the vector $(3, 4)$, which we write as $|(3, 4)|$.

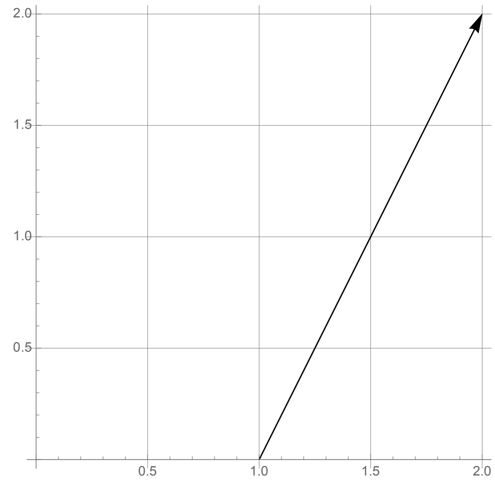
b. Find a vector which is in the same direction as $(3, 4)$ and has length 10.

c. Find a vector which is in the same direction as $(3, 4)$ and has length 1.

d. If the vector $(a, 6)$ is in the same direction as $(3, 4)$, then what is a?

e. Find a vector which is perpendicular to $(3, 4)$.

8. a. (3 points) What is the vector that starts at $(1, 0)$ and ends at $(2, 2)$? The vector is drawn in the picture below:



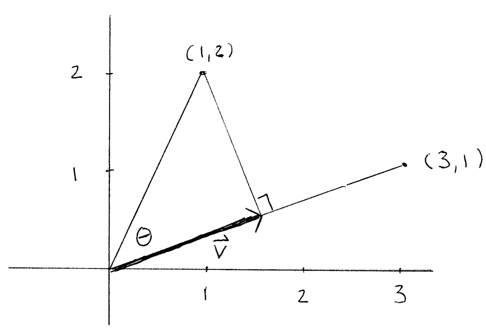
b. (3 points) Imagine the straight line from $(1,0)$ to $(2,2)$. Suppose that we start at $(1,0)$ and then follow this line one third of the way to $(2,2)$. What point do we end up at?

c. (4 points) Imagine that we start at the point $(1,0)$ and go a distance .5 along the straight line to $(2,2)$. What point do we end up at? (Remark. The numbers in the answer are a bit messy and they involve a square root.)

9a. (5 points) Find a unit normal vector to the line defined by $x + 2y = 2$.

b. (5 points) As a sanity check, sketch the line $x + 2y = 2$ and sketch the vector that you found in part a.

10. (15 points) This problem is based on the following picture:

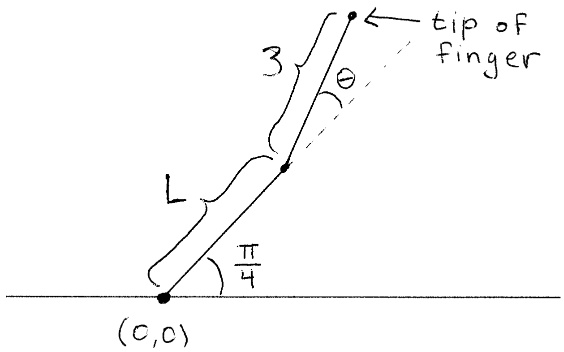


a.) Find the cosine of the angle θ in the picture.

b.) Find the length of the vector $\vec{v}$ in the picture (hint: knowing $\cos\theta$ helps).

c.) Find the vector $\vec{v}$ in the picture.

11. (10 points). In class we discussed a robot arm. In this problem we consider a slightly different robot arm, as in the following picture.



There is a joint at $(0, 0)$, and a bar comes out of this joint. The bar is at a fixed angle $\pi/4$, but its length is adjustable. We write L for the length of the bar. At the end of the first bar is a second joint, and a bar of length 3 comes out of the second joint. Let θ be the angle between the first bar and the second bar as in the picture. We will refer to the end of the second bar as the tip of the robot's finger.

In terms of L and θ, find the position of the tip of the robot's finger.

(In the coming problem sets, we will use calculus to study how to control this type of robot – for instance, how to get it to move its finger to a desired location.)