8.022 Problem Set VI

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1 Problem I

A)

$$\sigma(x) = \sigma_0 \frac{L^4}{x^4} \;\rightarrow\; \rho(x) = \frac{x^4}{\sigma_0 L^4}$$
$$dR = \frac{\rho}{A} dl = \frac{4x^4}{\pi d^2 \sigma_0 L^4} dl$$
$$\Rightarrow R = \frac{8}{\pi d^2 \sigma_0 L^4} \int_0^{\frac{L}{2}} x^4 dl = \frac{L}{20\pi d^2 \sigma_0} \approx 1273\Omega$$

B)

$$I = \frac{V}{R} = \frac{20V\pi d^2 \sigma_0}{L}$$
$$J = \frac{I}{A} = \frac{80V\sigma_0}{L} \quad \text{(Independent of x)}$$
$$E(x) = \frac{J}{\sigma(x)} = \frac{80Vx^4}{L^5}$$

C)

$$\vec{\nabla} \cdot E(\vec{x}) = \frac{r(x)}{\varepsilon_0}$$
$$\Rightarrow r(x) = \frac{320Vx^3\varepsilon_0}{L^5}$$

2 Problem II - 4.45

$$I = \frac{3 \cdot 1.5V}{5 \cdot 100\Omega} = 9 \cdot 10^{-3}A$$

$$\text{Open-circuit Voltage: } 2 \cdot (1.5V) - 3 \cdot (9 \cdot 10^{-3}A)(100\Omega) = 0.3V$$

$$\text{Short-circuit Voltage: } 1.5V - 2 \cdot (9 \cdot 10^{-3}A)(100\Omega) = -0.3V$$

$$\varepsilon_{eq} = 0.3V$$

$$I_{eq} = |I_{open} - I_{short}| = |\frac{3V}{300\Omega} - \frac{1.5V}{200\Omega}| = 2.5 \cdot 10^{-3}$$

$$R_{eq} = \frac{\varepsilon_{eq}}{I_{eq}} = 120\Omega$$

3 Problem III

$$I = \int \vec{J_1} \cdot d\vec{A} = \int \vec{J_2} \cdot d\vec{A}$$

$$\vec{J_1} \cdot \vec{n_1} = \vec{J_2} \cdot \vec{n_2}$$

$$\rightarrow J_1 \cos\theta_1 = J_2 \cos\theta_2 \rightarrow J_2 = \frac{J_1 \cos\theta_1}{\cos\theta_2}$$

$$E_1^{||} - E_2^{||} = 0 \ \rightarrow \ E_1 \sin\theta_1 = E_2 \sin\theta_2$$

$$\frac{J_1}{\sigma_1} \sin\theta_1 = \frac{J_2}{\sigma_2} \sin\theta_2$$

$$\Rightarrow J_2 = \frac{J_1 \sigma_2}{\sigma_1 \sin\theta_2} \sin\theta_1$$

$$\frac{J_1 \cos\theta_1}{\cos\theta_2} = \frac{J_1 \sigma_2}{\sigma_2 \sin\theta_2} \sin\theta_1$$

$$\rightarrow \theta_2 = \arctan\left(\frac{\sigma_2}{\sigma_1} \tan\theta_1\right)$$

$$J_2 = \frac{J_1 \cos(\theta_1)}{\cos\arctan\left(\frac{\sigma_2}{\sigma_1} \tan\theta_1\right)}$$

Placing a cube with length s with the interface bisecting it,

we can use Gauss' law to get the following:

$$\oiint_S \vec{E} \cdot d\vec{s} = s^2 E_2 \cos\left(\frac{\pi}{2} - \theta_2\right) - s^2 E_1 \cos\left(\frac{\pi}{2} + \theta_1\right) = \frac{Q_{enc}}{\varepsilon_0}$$

$$\Rightarrow \sigma = \frac{Q_{enc}}{s^2} = \varepsilon_0\left(\frac{J_2}{\sigma_2} \sin\theta_2 + \frac{J_1}{\sigma_1} \sin\theta_1\right)$$

Wherein J_2 and θ_2 are the values found above

4 Problem IV

Expressing the circuit as a Thevenin equivalent circuit, we can simplify the problem to a loop with one voltage source V_T, one resistor R_T, and the capacitor.

$$V_T = \mathcal{E}(1 - \frac{R_3}{R_1 + R_3})$$

$$R_T = R_2 + \frac{R_1 R_3}{R_1 + R_3}$$

$$V_T = V_C + I_T R_T = \frac{Q}{C} = \frac{Q}{C} + \dot{Q} R_T$$

$$\Rightarrow \int \frac{1}{V_t - \frac{Q}{C}} dQ = \int \frac{1}{R_T} dt \rightarrow Q(t) = C V_T (1 - e^{-\frac{t}{R_T C}})$$

$$\Rightarrow V_C = V_T (1 - e^{-\frac{t}{R_T C}})$$

$$I_2(t) = \dot{Q}(t) = \frac{V_T}{R_T} e^{-\frac{t}{R_T C}}$$

$$\mathcal{E} - I_0 R_3 = V_C + I_2 R_2 \rightarrow I_0 = \frac{\mathcal{E} - (V_C + I_2(t)}{R_3}$$

$$\text{Voltage on Capacitor: } V_C(t) = V_T (1 - e^{-\frac{t}{R_T C}})$$

$$\text{Current through Battery: } I_0(t) = \frac{1}{R_3} (\mathcal{E} - V_T (1 - e^{-\frac{t}{R_T C}}) - \frac{V_T R_2}{R_T} e^{-\frac{t}{R_T C}})$$

$$\text{Where } V_T = \mathcal{E}(1 - \frac{R_3}{R_1 + R_3}) \text{ and } R_T = R_2 + \frac{R_1 R_3}{R_1 + R_3}$$

5 Problem V

A)
Using right-hand rule the current should flow to the East, its magnitude can be found by:

$$F = BIL = mg = r_m LAg$$
$$\rightarrow J = \frac{I}{A} = \frac{r_m g}{B} = \frac{(8900\frac{kg}{m^3})(9.8\frac{m}{s^2})}{(5 \cdot 10^{-5}T)} \approx 1.74 \cdot 10^9 \frac{A}{m^2}$$

B)

$$P = IV = I^2 R = (JA)(\frac{rL}{A}) = J^2 ArL$$
$$\rightarrow \frac{P}{AL} = J^2 r = (1.74 \cdot 10^9 \frac{A}{m^2})^2 (1.7 \cdot 10^{-8}\Omega \cdot m) \approx 5.147 \cdot 10^{10} \frac{W}{m^3}$$

6 Problem VI - 6.44

$$B_z = \frac{\mu_0 I}{4\pi}\frac{2\pi b^2}{r^3} = \frac{\mu_0 I b^2}{2(b^2+z^2)^{\frac{3}{2}}} \tag{1}$$

$$\int \vec{B}\cdot\vec{s} = \frac{\mu_0 I b^2}{2}\int_0^\infty \frac{dz}{(b^2+z^2)^{3/2}}$$
$$\Rightarrow \frac{\mu_0 I b^2}{2}\left(\frac{z}{b^2\sqrt{z^2+b^2}}\right)\Big|_0^\infty = \frac{\mu_0 I b^2}{2}\cdot\frac{2}{b^2} = \mu_0 I$$

Given that the field is symmetric and the line integral is taken to infinity (at which point $\vec{B} \to 0$ we can neglect the return path, for if we were to imagine a circular return path we can observe that the line integral would go to zero as the circular radius goes to infinity, and it can therefore be neglected.

7 Problem VII

R is used as the radius of the arc in this problem

$$\text{Angle subtended by arc: } \phi = \frac{3\pi}{4}$$
$$B_{arc} = \frac{\mu_0 I \phi}{4\pi R} = \frac{3\mu_0 I}{16R}$$

For the infinite wires, we will create a triangle with the arc center, the end of the arc (R away from the center), and a point x away on the wire, a distance of $\sqrt{x^2 + R^2}$ from the arc center. We can then use the Bio-Savart Law to find the total contribution of one infinite wire, and multiply it by a factor of 2.

$$B_{wire} = \frac{\mu_0 I}{4\pi} \int_0^\infty \frac{d\vec{x} \times \vec{\hat{r}}}{r^2} = \frac{\mu_0 I}{4\pi} \int_0^\infty \frac{dx \sin\theta}{r^2} \rightarrow \frac{\mu_0 I R}{4\pi} \left(\frac{x}{R^2\sqrt{x^2 + R^2}}\right)\Big|_0^\infty = \frac{\mu_0 I}{4\pi R}$$
$$B_{center} = B_{arc} + 2 \cdot B_{wire} = \frac{3\mu_0 I}{16R} + \frac{\mu_0 I}{2\pi R} = \frac{\mu_0 I}{2R}\left(\frac{3\pi + 8}{8\pi}\right)$$

8 Problem VIII

A)

If orienting yourself to view the cross-sections of the wires with the current coming toward you on the right, the field is directed downwards.

$$B = 2\frac{\mu_0 I}{\pi\sqrt{\frac{d^2}{4} + B^2}}\sin(\theta) = \frac{\mu_0 I d}{2\pi(\frac{d^2}{4} + R^2)}$$

B)

$$P = IV \rightarrow I = \frac{7.5 \cdot 10^6 W}{750 \cdot 10^3 V} = 10A$$
$$B \approx 7.92 \cdot 10^{-8} T$$

C)

$$I = \frac{1000W}{100V} = 10A$$
$$B \approx 2.5 \cdot 10^{-9} T$$

D)

$$\rho = \frac{J}{c} \rightarrow \lambda = \frac{JA}{c} = \frac{I}{c}$$
$$E = \frac{\lambda}{\pi\varepsilon_0 r}\cos\theta = \frac{IR}{c\pi\varepsilon_0(R^2 + \frac{d^2}{4})}$$
$$\text{For two power lines: } E_{power} \approx 237\frac{V}{m}$$
$$\text{For two fridge wires: } E_{fridge} \approx 600\frac{V}{m}$$