AP Physics C: Mechanics - Notes

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Contents

1 Vectors and Motion in 2 Dimensions

Unit vector: length of 1, no units (used for direction)

- Denoted with "hat"
- $\hat{x}$, $\hat{y}$, $\hat{z}$ - usually denoted as $\hat{i}$, $\hat{j}$, $\hat{k}$

For a 2D velocity vector:

$$\vec{v} = v_x\hat{i} + v_y\hat{j}$$

1.1 Polar Notation

$$\vec{v} = \sqrt{{v_x}^2 + {v_y}^2} \text{ at } \theta^\circ$$

θ is angle relative to *positive x-axis*

1.2 Vector Addition

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$
$$\vec{B} = B_x\hat{i} + B_y\hat{j}$$
$$\vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j}$$

1.3 Scalar Multiplication

scalar · vector = vector

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$
$$2\vec{A} = 2A_x\hat{i} + 2A_y\hat{j}$$

1.4 Vector Multiplication

vector × vector

scalar/dot product: $\vec{A} \cdot \vec{B}$

- results in scalar

vector/cross product: $\vec{A} \times \vec{B}$

- results in vector

1.5 Dot Product

$$\vec{A} \cdot \vec{B} = |A||B|\cos\theta$$

- θ is the angle between the two vectors

Or,

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yBy$$

1.5.1 Dot Product Properties

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$
$$\vec{A} \cdot \vec{B} = 0 \text{ if } \vec{A} \text{ and } \vec{B} \text{ are orthogonal}$$

1.6 Cross Products

$$\text{If } \vec{C} = \vec{A} \times \vec{B},$$
$$|C| = |A||B| \sin\theta$$

the cross product is **orthogonal** to both original vectors (uses 3rd dimension)

1.6.1 Cross Product Properties

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$
$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

right hand rule - cross product goes "into" or "out of" page

2 Drag Force

- resistive force by **fluids** on **moving objects**
- **opposes** direction of motion

2.1 v^2 model

$$F_{drag} = -\frac{1}{2} C \rho A v^2$$

- C - drag coefficient
- ρ - density of fluid - greek letter "rho"
- A - cross-sectional area of object
- v - velocity of object
- negative sign just indicates direction opposite to motion

Sometimes, the coefficients $\frac{1}{2} C \rho A v^2$ are grouped into one coefficient (b or k)

- So, $F_{drag} = -bv^2$
- also called the drag coefficient
- specific to object and location
- calculated experimentally

2.2 v model

for **small objects** moving at **slow speeds**,

$$F_{drag} = -bv$$

2.3 Terminal Velocity

For an object in free fall, terminal velocity occurs when $\left|\vec{F}_{drag}\right| = \left|\vec{F}_{gravity}\right|$

$$\begin{aligned} F_{net} &= F_g - F_{drag} \\ 0 &= mg - bv_t \\ v_t &= \frac{mg}{b} \text{ (v model)} \\ v_t &= \sqrt{\frac{mg}{b}} \text{ (v^2 model)} \end{aligned}$$

3 Differential Equations

Differential equations define relationship between a function and one or more *derivatives* of that function

Drag force is modeled with a differential equation:

$$\begin{aligned} F_{drag} &= bv \\ ma &= bv \\ m\frac{dv}{dt} &= bv \end{aligned}$$

3.1 Integrating Differential Equations

$$\begin{aligned} m\frac{dv}{dt} &= bv && \text{rewrite equation to see dependent and independent variables} \\ \frac{1}{v}dv &= \frac{b}{m}dt && \text{separate each variable to one side} \\ \int \frac{1}{v}dv &= \int \frac{b}{m} && \text{integrate} \end{aligned}$$

4 Circular Motion

Centripetal force (F_c or $\sum F_c$) isn't really a force - it is the **net** force along the centripetal axis (the axis going through a point on the circular path and the center of said circle).

- could be weight, tension, friction, etc.
- is always orthogonal (perpendicular) to velocity
- depends on mass and tangential velocity of object and radius of circular path

4.1 Uniform Circular Motion

Uniform circular motion occurs when an object moving in a circle with a fixed radius has a constant tangential (and therefore angular) speed

frequency (f) is measured in Hertz Hz
period (T) is measured in seconds (s)

4.2 Velocity

$$v_t = \frac{\Delta x}{T} = \frac{2\pi r}{T} = 2\pi r f$$

4.3 Acceleration

$$a_c = \frac{v^2}{r} = \frac{4\pi^2 r}{T^2}$$

Centripetal acceleration always points toward the center of the circular path

- does not change *speed* of object; only changes *direction*

4.4 Force

Centripetal force is the sum of all forces on the centripetal axis

$$\sum F = ma$$
$$\sum F_c = ma_c$$
$$\sum F_c = \frac{mv^2}{r}$$

points toward the center of the circular path

5 Rotational Motion

rigid bodies: objects that do not change shape (deform)
translational motion: motion of the *center of mass* of the object
rotational motion: rotation around a fixed point (not always the center of mass)

Circular Motion	Rotational Motion
circular path	spins about axis of rotation
all parts of object move with same velocity	different points on the object travel at different speeds

5.1 Variables

- R - distance from point on object to axis of rotation
- θ - angular displacement - *measured in radians*
- S - Arc Length

$$S = R\theta$$

5.2 Angular Velocity

rate of change of θ

$$\omega = \frac{\Delta\theta}{t} = 2\pi f$$

- ω - angular velocity - *measured in radians/second*

all points on a rigid body have the same angular velocity

ω does **not** point in the direction of rotation

- use the right hand rule to determine the direction of $omega$ - always orthogonal to the rotational plane
  - usually described as "into page" or "out of page"
  - sometimes described as "clockwise" or "counterclockwise"

5.3 Angular Acceleration

rate of change of $omega$

$$\alpha = \frac{\Delta\omega}{t}$$

- α - angular acceleration - *measured in radians/second²*

α does **not** point in the direction of rotation - same as angular velocity (ω)

- use right hand rule just like angular velocity to determine direction of α

5.4 Kinematic Equations

Translational Kinematics	**Rotational Kinematics**
$\bar{v} = \frac{\Delta x}{t}$	$\bar{\omega} = \frac{\Delta\theta}{t}$
$\bar{v} = \frac{v_1 + v_2}{2}$	$\bar{\omega} = \frac{\omega_1 + \omega_2}{2}$
$\bar{a} = \frac{v - v_0}{t}$	$\bar{\alpha} = \frac{\omega - \omega_0}{t}$
$\Delta x = v_0 t + \frac{1}{2}at^2$	$\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
${v_f}^2 = {v_0}^2 + 2a\Delta x$	${\omega_f}^2 = {\omega_0}^2 + 2\alpha\Delta\theta$

6 Center of Mass (Discrete Objects)

- The point at which all **mass** of an object is though to be concentrated
- also thought of as "average location of mass"
- can be determined experimentally or mathematically
- the center of mass of *all objects* moves like a point particle *even if the object is rotating*
- could be inside or outside the object
- the geometric center of an object is *not* always its center of mass - only when the object has a uniform density
- unless specified, objects have a uniform density

6.1 Position of the Center of Mass

$$x_{cm} = \frac{1}{M}\sum m_i x_i$$

- $x_c m =$ position of the center of mass of the systeme
- M - total mass of the systeme
- m_i - mass of the i^{th} particle
- v_i - psoition of the i^{th} particle

A continuous object can be broken down into symmetric discrete pieces to find its center of mass

6.2 Velocity of the Center of Mass

$$v_c m = \frac{1}{M}\sum m_i v_i$$

- v_i - velocity of the i^{th} particle

6.3 Acceleration of the Center of Mass

Newton's third law ($F = ma$) actually refers to the center of mass of a system or object:

$$Ma_{cm} = \sum F_{external}$$

7 Center of Mass (Continuous Objects)

Not all continuous objects have a uniform mass density

- Mass density usually varies withlocation

$$x_{cm} = \frac{1}{M}\int x\ dm$$

- x - position of particle
- dm - infinitely small mass

position is given as a function of mass
integrate along the length of the object

7.1 Position of Center of Mass

1. Define mass density function:
$$\lambda = \frac{dm}{dx}$$
2. Separate variables:
$$dm = \lambda(x)dx$$
3. Integrate along length of object to find total mass of object:
4. Use integral form of center of mass formula (see above) to solve for the position of the center of mass
    - use total mass calculated in step 4

8 Rotational Intertia

Rotational inertia (a.k.a **moment of inertia**): ability of an object to resist changes in its rotational motion

symbol: I
scalar quantity
units: kg · m^2

8.1 Discrete Objects

$$I = \sum m_i R_i^2$$

- m_i - mass of object
- R - distance between axis of rotation and the object

The same object can have different rotational inertias depending on the location of the axis of rotation

8.2 Continuous Objects

$$I = \int R^2 \, dm$$

Object and Axis of Rotation	Rotational Inertia
Thin rod, about center	$\frac{1}{12}ML^2$
Thin rod, about end	$\frac{1}{3}ML^2$
Cylinder or disk, about center	$\frac{1}{2}MR^2$
Cylindrical hoop, about edge	MR^2

8.3 Parallel Axis Theorem

If the moment of inertia about the center of mass (I_{CoM}) is known, the moment of inertia about any parallel axis of rotation can be found using the formula:

$$I_{parallel} = I_{CoM} + Md^2$$

- M - mass of the objects
- d distance between the axes

8.4 Systems of Objects

For a system of discrete and/or continous objects:

$$I_{system} = I_1 + I_2 + I_3 + \cdots + I_n$$

9 Torque

Torque is the ability of a force to make an object rotate - a "twisting force"

symbol: τ ("tau")
units: $N \cdot m$
vector quantity

Magnitude depends on

- size of the force
- direction
- location at which the force is applied

torque is given as the cross product $R \times F$:

$$\tau = R \times F = RF \sin\theta$$

- R - distance from the axis of rotation to the applied force
- F - applied force
- θ - angle between R and F (when drawn tip-to-tail)

9.1 Direction of Torque

Use the right hand rule to find the direction of torque:

- point index finger in direction from axis of rotation to applied force
- point middle finger in direction of applied forces
- point thumb out - this is the direction of the torque

9.2 Torque in Newton's Second Law

$$\begin{aligned}
\tau_{net} &= R \times F \\
&= RF_{net} \\
&= Rma && \text{Newton's 2nd Law, } F = ma \\
&= Rm\alpha R && a = \alpha R \\
&= I\alpha && \text{rotational inertia, } I = mR^2
\end{aligned}$$

10 Rolling

Friction causes a torque that makes objects roll. Rolling objects experience both **rotational** and **translational** motion.

Objects can roll with or without **slipping**

10.1 Rolling Without Slipping

When an object rolls without slipping,

$$v_{cm} = \omega R$$
$$a_c m = \alpha R$$

- v_{cm} and a_{cm} are the velocities and accelerations of the center of mass, respectively
- ω and α are the angular velocity and acceleration of the object, respectively

- R is the distance from the axis of rotation to the point of contact with the ground

Rolling without slipping problems are solved by relating the rotational and translational motions of an object using the equation $a = R\alpha$

10.2 Rolling With Slipping

When an object rolls without slipping,

$$v_{cm} \neq \omega R$$
$$a_{cm} \neq \alpha R$$

Rolling without slipping problems are solved by figuring out **when** $v = R\omega$ (the object enters into rolling without slipping)

10.2.1 Two Types of Rolling With Slipping

1. $v_{cm} > \omega R$ - object is sliding around ground at some points (e.g. bowling ball sliding and rolling slightly before it 'catches grip' and rolls at full speed)
2. $v_{cm} < \omega R$ - object is spinning more than one revolution in the time it takes to advance by one circumference (e.g. car doing burnouts or losing traction)

11 Work and Energy

- energy is a scalar quantity (has no direction), but can be positive or negative'
- energy is always conserved in a closed system

11.1 Work

Work is the result of a force being applied to move an object across a displacement - measured in $N{\cdot}m$

$$W = F \cdot \Delta x = F\Delta x \cos\theta$$

- work is the **cross product** of force and displacement (vector quantity)
- θ is the angle between F and Δx when aligned tail to tail

11.1.1 Work-Energy Theorem

Work changes the energy of an object

$$\Sigma W = \Delta KE$$

- net work is equal to the change in kinetic energy of an object

11.2 Energy

Energy is measured in Joules (J) - equal to ($N \cdot m$)

11.2.1 Types of Mechanical Energy

Kinetic Energy

$$KE = \frac{1}{2}mv^2$$

Rotational Kinetic Energy

$$RKE = \frac{1}{2}I\omega^2$$

Potential Energy
Gravitational

$$GPE = mgh$$

Elastic

$$EPE = \frac{1}{2}k(\Delta s)^2$$

- k - spring constant
- Δs - displacement of the spring (compression or extension)

11.3 Power

power is energy delivered over a period of time

$$P = \frac{\Delta E}{\Delta t} = \frac{W}{\Delta t} = \frac{F\Delta x}{\Delta t} = Fv$$

11.4 Potential Energy Curves

graph of potential energy vs position

For **conservative forces**,

$$F(x) = -\frac{d}{dx}U(x)$$

11.4.1 Equilibrium Points

Occur where the slope (equal to F_{net}) is zero

12 Impulse and Momentum

impulse is a force applied for a period of time

symbol: J
units: $N \cdot s$
vector quantity - same direction as force

$$J = F\Delta t$$
$$J = \int_{t_0}^{t_1} F(t)dt$$

momentum is the amount of motion of an object

symbol: p
units: $kg \cdot m/s$
vector quantity

$$p = mv$$

12.1 Conservation of Momentum

Impulse is a change in momentum

Where A and B are two objects in a system upon which no outside forces act,

$$J = \Delta p$$
$$\Delta p_B = -\Delta p_A$$
$$\Delta p_A + \Delta p_B = 0$$

When no outside forces act on a system, momentum is conserved (the net change in momentum is 0)

12.2 Collision Types

12.2.1 Elastic Collisions

- energy is consevered
- objects bounce off each other

12.2.2 Inelastic Collisions

- energy is NOT conserved
- objects bounce off each other

12.2.3 Perfectly Inelastic Collisions

- energy is NOT conserved
- objects stick to each other

12.3 Useful Equations

12.3.1 Head-on Elastic Collision

$$v_{1_0} + v_{1_f} = v_{2_0} + v_{2_f}$$

12.3.2 Perfectly Inelastic Ballistic Collission

When a projectile strikes a block on a pendulum,

$$v_0 = \frac{m + M}{m}\sqrt{2gh}$$

where

- v_0 - initial velocity of the bullet
- m - mass of the projectile
- M - mass of the block
- h - max height of the pendulum

12.4 Calculus with Momentum and Impulse

$$\int F(t)dt = J$$

13 Angular Impulse and Momentum

13.1 Angular Momentum

angular momentum is the amount of angular motion an obect has

symbol: L
units: $kg \cdot m^2/s$
"pseudo-vector" quantity (like torque)

$$L = I\omega = \vec{R} \times \vec{p} = Rp\sin\theta$$

- I - moment of inertia
- ω - angular velocity
- $\vec{R}$ - vector from aribtrary center of rotation to object
- $\vec{p}$ - linear momentum vector
- θ - angle between R and p (tip to tail)

NOTE: because any point can be chosen as the center of rotation, point particles can have an angular momentum

13.2 Angular Impulse

angular impulse is a torque applied for a period of time

symbol: ΔL
units: $N \cdot m \cdot s$
"pseudo-vector" quantity

$$\Delta L = \tau_{net}\Delta t$$

13.2.1 Newton's Third Law from Angular Impulse

$$\tau_{net} = \frac{dL}{dt} = I\alpha$$

14 Simple Harmonic Motion

- **periodic motion** is cyclic motion that repeats the same path in the same amount of time
- the **equilibrium position** is the position where the net force on the object is 0
- the **restoring force** is the force that acts opposite to displacement to bring the object back to the equilibrium position
- **displacement** is the position relative to the equilibrium position (measured in meters)
- **amplitude** is the maximum displacement of an object in periodic motion
- a **cycle** is a complete back and forth swing of the pendulum
- the **period** is the time it takes to complete one cycle
- the **frequency** is the number of cycles completed in a second (inverse of period)

simple harmonic motion occurs when the the restoring force is in the *opposite direction* of, and has a magnitude *proportional to* the displacement

All simple harmonic oscillators have an **angular velocity**. The period of the oscillator is given by

$$T = 2\pi\sqrt{\frac{1}{\omega^2}}$$

For all simple harmonic oscillators,

$$a = \frac{d^2x}{dt^2} = -\omega^2 x$$

14.1 Simple Pendulums

For a pendulum, the restoring force is gravity and $x = \theta L$

The restoring force of a pendulum is given by

$$F_r = -mg\sin\theta$$

where θ is the angular displacement of the Pendulum

For small values of θ (less than 20°), $\sin\theta \approx \theta$, so

$$F_r = -mg\theta$$

For a pendulum,

$$\begin{aligned} F &= -mg\theta \\ ma &= -\frac{mg}{L}x \\ \frac{d^2x}{dt^2} &= -\frac{g}{L}x \end{aligned}$$

There are three possible solutions to the differential equation above, depending on the starting position of the pendulum

initial position	$x(t)$
$x_0 = 0$	$x(t) = A(\sin\omega t)$
$x = A$	$x(t) = A(\cos\omega t)$
$x = \theta_i L$	$x(t) = A(\cos\omega t + \phi)$

For a pendulum, the angular velocity is given by

$$\omega^2 = -\frac{g}{L}$$

14.2 Spring Mass Systems

The period of a spring mass system is given by

$$T = 2\pi\sqrt{\frac{m}{k}}$$

The angular velocity is given by

$$\omega^2 = \frac{k}{m}$$

14.2.1 Vertical Springs

If a spring stretches a distance of ΔL from its unstretched length when a mass is added to it,

$$\Delta L = \frac{mg}{k}$$

14.2.2 Useful Equations for Spring Mass Systems

$$v_{max} = A\sqrt{\frac{k}{m}}$$

$$v(x) = \pm v_{max}\sqrt{1 - \frac{x^2}{A^2}}$$

14.2.3 Multi Spring Systems

When multiple springs are connected, they can be thought of as a single **equivalent spring** with a unique spring constant.

14.2.3.1 Springs in Parallel Springs are considered to be connected **in parallel** if they are *both connected to an object* and *not connected to each other*. Springs in parallel act as a single **more effective** spring. For springs in parallel,

$$k_{equiv} = k_1 + k_2 + \cdots$$

14.2.3.2 Springs in Series Springs are considered to be connected **in series** if they are *connected to each other* and *only one of the springs is connected to an object*. Springs in series act as a single **less effective** spring. For springs in series,

$$\frac{1}{k_{equiv}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots$$

14.3 Physical Pendulums

A physical pendulum is a *rigid* extended body that rotates cyclically about a point

For a physical pendulum,

$$\frac{d^2\theta}{dt^2} = \frac{mgR_{cm}}{I}$$

- R_{cm} is the distance from the center of mass of the pendulum to the axis of rotation
- I is the rotational inertia of the pendulum

The angular velocity of a physical pendulum is given by

$$\omega^2 = \frac{mgR_{cm}}{I}$$

The period of a physical pendulum is given by

$$T = 2\pi\sqrt{\frac{I}{mgR_{cm}}}$$

14.4 Energy Transfer in Simple Harmonic Motion

The total mechanical energy of a simple harmonic oscillator is always constant

At *equilibrium*, the kinetic energy (and therefore velocity) is at a *maximum* and the potential energy is at a *minimum* (therefore net force and acceleration are 0)

At the *amplitude*, the kinetic energy (and therefore velocity) is 0 and the potential energy (and therefore net force and acceleration) is at a *maximum*

14.5 Resonance

When a force is applied to an object with a frequency close to the natural frequency (aka resonant frequency) of the object, the amplitude of the oscillation (and therefore the total energy in the system) increases.

14.6 Summary of Simple Harmonic Oscillator Angular Velocities and Periods

oscillator type	ω^2	T
simple pendulum	$\omega^2 = \frac{g}{L}$	$T = 2\pi\sqrt{\frac{L}{g}}$
spring-mass system	$\omega^2 = \frac{k}{m}$	$T = 2\pi\sqrt{\frac{m}{k}}$
physical pendulum	$\omega^2 = \frac{mgR_{cm}}{I}$	$T = 2\pi\sqrt{\frac{I}{mgR_{cm}}}$