# Topics of Complex Social Networks: Domination, Influence and Assortativity

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## Python Code at github:

https://github.com/mboudour/GraphMultilayerity/tree/master/vartopics

## Slides at slideshare:

http://www.slideshare.net/MosesBoudourides/topics-of-complex-social
-networks-domination-influence-and-assortativity

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# A. Dominating Sets

# Dominating Sets in a Graph

## Definition

Let $G = (V, E)$ be a (simple undirected) graph. A set $S \subseteq V$ of vertices is called a **dominating set** (or **externally stable**) if every vertex $v \in V$ is either an element of $S$ or is adjacent to an element of $S$.

## Remark

A set $S \subseteq V$ is a dominating set if and only if:

- for every $v \in V \setminus S$, $|N(v) \cap S| \geq 1$, i.e., $V \setminus S$ is *enclaveless*;
- $N[S] = V$;
- for every $v \in V \setminus S$, $d(v, S) \leq 1$.

Above $N(v)$ is the open neighborhood of vertex $v$, i.e., $N(v) = \{w \in V : (u, w) \in E\}$, $N[v]$ is the closed neighborhood of vertex $v$, i.e., $N[v] = N(v) \cup \{v\}$, $d(v, x)$ is the geodesic distance between vertices $v$ and $x$ and $d(v, S) = \min\{d(v, s) : s \in S\}$ is the distance between vertex $v$ and the set of vertices $S$.

## Standard Reference

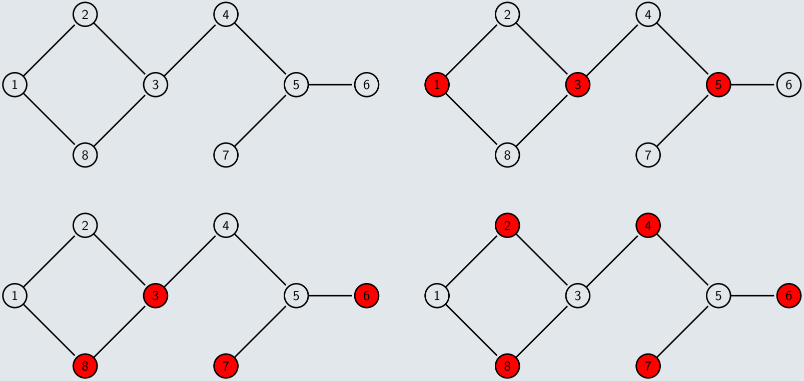
Haynes, Teresa W., Hedetniemi, Stephen T., & Slater, Peter J. (1998). *Fundamentals of Domination in Graphs*. New York: Marcel Dekker.

## Definition

Let $S$ be a set of vertices in graph $G$.

- $S$ is called **independent** (or **internally stable**) if no two vertices in $S$ are adjacent.
- If $S$ is a dominating set, then
  - $S$ is called **minimal dominating set** if no proper subset $S' \subset S$ is a dominating set;
  - $S$ is called a **minimum dominating set** if the cardinality of $S$ is minimum among the cardinalities of any other dominating set;
  - The cardinality of a minimum dominating set is called the **domination number** of graph $G$ and is denoted by $\gamma(G)$;
  - $S$ is called **independent dominating set** if $S$ is both an independent and a dominating set.

## Example




The sets $\{1, 3, 5\}$, $\{3, 6, 7, 8\}$ and $\{2, 4, 6, 7, 8\}$ (**red** circles) are all minimal dominating sets. However, only the first is a minimum dominating set. Apparently $\gamma = 3$ for this graph.

**Theorem (Ore, 1962)**

A dominating set $S$ is a minimal dominating set if and only if, for each vertex $u \in S$, one of the following two conditions holds:

1. $u$ is such that $N(u) \subseteq V \setminus S$, i.e., $u$ is an *isolate* of $S$,
2. there exists a vertex $v \in V \setminus S$ such that $N(v) \cap S = \{u\}$, i.e., $v$ is a *private neighbor* of $u$.

**Theorem (Ore, 1962)**

Every connected graph $G$ of order $n \geq 2$ has a dominating set $S$.

# Complexity of the Dominating Set Problem

## Theorem (Johnson, 1974)

The dominating set problem is $\mathcal{NP}$–complete.

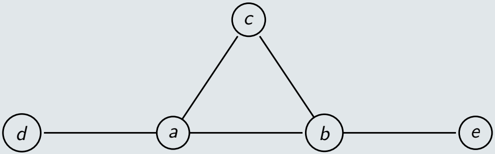
## Algorithmic Computation of Dominating Sets

- As the dominating set problem is $\mathcal{NP}$–complete, there are certain efficient algorithms (typically based on linear and integer programming) for its approximate solution.
- Here, we are algorithms that have been already implemented in Python's Sage. However, these implementations return only one of the minimum or independing (minimum) dominating sets.

## A Brute Force Heuristics (Berge)

- Given two elements $x$ and $y$ of a set, let
  - "$x + y$" denote *logical summation* of $x$ and $y$ (i.e., "$x$ or $y$")
  - and "$x \cdot y$" denote their *logical multiplication* (i.e., "$x$ and $y$").
- Thus, for each vertex $v$, $N[v] = v + u_1 + \ldots + u_k$, where $u_1, \ldots, u_k$ are all vertices adjacent to $v$.
- In this way, $\text{Poly}(G) = \prod_{v \in V} N[v]$ becomes a polynomial in graph vertices.
- Notice that each monomial of $\text{Poly}(G)$ can be simplified by removing numerical coefficients and repeated (more than once) vertices in it.
- Therefore, each simplified monomial of $\text{Poly}(G)$ including $r$ vertices is a minimal dominating set of cardinality $r$ and a minimum dominating set corresponds to the minimal $r$.

## Example




As $N[a] = a + b + c + d, N[b] = a + b + c + e, N[c] = a + b + c, N[d] = a + d, N[e] = b + e$, we find:

$$\begin{aligned}
\mathrm{Poly}(G) &= N[a]\,N[b]\,N[c]\,N[d]\,N[e] = \\
&= (a+b+c+d)(a+b+c+e)(a+b+c)(a+d)(b+e) = \\
&= (a+b+c)^3(a+d)(b+e) + \\
&\quad +(a+b+c)^2(d+e)(a+d)(b+e) = \\
&= ab + ae + bd + \\
&\quad +abc + abd + abe + ace + ade + bcd + bde + cde + \\
&\quad +abcd + abde + abce + acde + bcde + \\
&\quad +abcde.
\end{aligned}$$

Therefore,

- $\Big\{\{a,b\},\{a,e\},\{b,d\}\Big\}$ is the collection of minimum dominating sets (each one of cardinality 2 equal to the domination number of this graph),
- $\Big\{\{a,b,c\},\{a,b,d\},\{a,b,e\},\{a,c,e\},\{a,d,e\},\{b,c,d\},$ $\{b,d,e\},\{c,d,e\}\Big\}$ is the collection of minimal dominating sets of cardinality 3,
- $\Big\{\{a,b,c,d\},\{a,b,d,e\},\{a,b,c,e\},\{a,c,d,e\},\{a,d,e\},$ $\{b,c,d,e\}\Big\}$ is the collection of minimal dominating sets of cardinality 4 and
- $\{a,b,c,d,e\}$ is the minimal dominating set of cardinality 5.

**Definition**

Let $G = (V, E)$ be a (simple undirected) graph.

- Let $U \subset V$ be a set of vertices in a graph $G$ and let $u \in U$.
  - The subgraph induced by $U$ (in $G$) is called an **egocentric (sub)graph** (in $G$) if
    $$U = N[u],$$
    i.e., if all vertices of $U$ are dominated by $u$;
  - vertex $u$ is the **ego** of the egocentric (sub)graph $N[u]$;
  - vertices $w \in N(u)$ are called **alters** of ego $u$.
- Let $S \subset V$ be a dominating set in $G$ and let $v \in S$.
  - The subgraph induced by $N[v]$ (in $G$) is called a **dominating egocentric (sub)graph**;
  - vertex $v$ is the **dominating ego**;
  - vertices $w \in N(v)$ are the **dominated alters** by $v$.
  - Graph $G$ is called **multiple egocentric** or **$|S|$-egocentric graph corresponding to the dominating set** $S$.

### Remark

- Given a (simple undirected) graph, typically, there is a multiplicity of dominating sets for that graph.
- Therefore, any egocentric decomposition of a graph depends on the dominating set that was considered in the generation of the constituent egocentric subgraphs.

# Private and Public Alters

## Definition

Let $S$ be a dominating set in graph $G = (V, E)$ ($|S| \geq 2$) and let $v \in S$ an ego.

- An alter $w \in N(v)$ ($w \notin S$) is called **private alter** if $N(w) \subset N[v]$.
- An alter $w \in N(v)$ ($w \notin S$) is called **public alter** if $N(w) \setminus N[v] \neq \emptyset$.

## Remark

- Two adjacent egos $u, v$ are not considered alters to each other! Nevertheless, they define an ego–to–ego edge (bridge).
- A private alter is always adjacent to a single ego.
- A public alter is adjacent either to at least two egos or to a single ego and to another alter which is adjacent to a different ego.

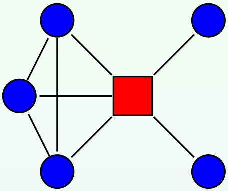


Figure : Five private alters (**blue** circles) adjacent to an ego (**red** square).

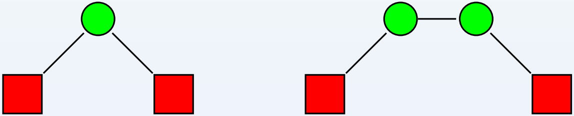


Figure : *Left*: one public alter (**green** circle) shared by two egos (**red** squares). *Right*: two public alters (**green** circles), each one adjacent to a different ego (**red** square).

# Dominating and Dominated Bridges

## Definition

Let $S$ be a dominating set in a graph $G = (V, E)$ ($|S| \geq 2$).

- If two egos $v_1, v_2 \in S$ are adjacent, then edge $(v_1, v_2) \in E$ is called **dominating edge** or **bridge of egos**.
- If two private alters $w_1 \in N(v_1), w_2 \in N(v_2)$ are adjacent (possibly $v_1 = v_2$), then edge $(w_1, w_2) \in E$ is called **dominated edge among private alters** or **bridge of private alters**.
- If two public alters $w_1 \in N(v_1), w_2 \in N(v_2)$ are adjacent (possibly $v_1 = v_2$), then edge $(w_1, w_2) \in E$ is called **dominated edge among public alters** or **bridge of public alters**.
- If a private alter $w_1 \in N(v_1)$ and a public alter $w_2 \in N(v_2)$ are adjacent (possibly $v_1 = v_2$), then edge $(w_1, w_2)$ is called **dominated edge among private–public alters** or **bridge of private–to–public alters**.

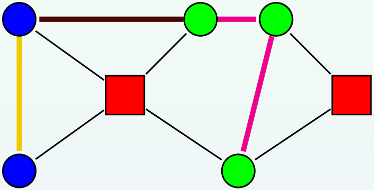


Figure : One edge (bridge) among private alters (**yellow** line), two edges (bridges) among public alters (**magenta** lines) and one edge (bridge) among private–to–public alters (**brown** lines).

## Notation

- The set of all private alters in $G$ is denoted as:

$$V_{\text{private}} = \{w \in V \setminus S\colon \exists v \in S \text{ s.t. } w \in N(v) \setminus S \text{ and } N(w) \subset N[v]\}.$$

- The set of all public alters in $G$ is denoted as:

$$V_{\text{public}} = \{w \in V \setminus S : N(w) \setminus N[v] \neq \varnothing, \forall v \in S\}.$$

- The set of all dominating bridges (among egos) in $G$ is denoted as:

$$E_{\text{ego}} = \{(v_1, v_2) \in E\colon v_1, v_2 \in S\}.$$

- The set of all dominated bridges among private alters in $G$ is denoted as:

$$E_{\text{private}} = \{(v_1, v_2) \in E\colon v_1, v_2 \in V_{\text{private}}\}.$$

- The set of all dominated bridges among public alters in $G$ is denoted as:

$$E_{\text{public}} = \{(v_1, v_2) \in E\colon v_1, v_2 \in V_{\text{public}}\}.$$

- The set of all dominated bridges among private–public alters in $G$ is denoted as:

$$E_{\text{private - public}} = \{(v_1, v_2) \in E\colon v_1 \in V_{\text{private}}, v_2 \in V_{\text{public}}\}.$$

## Definition

Let $G \setminus S$ be the graph remaining from $G = (V, E)$ after removing a dominating set $S$ (together with all edges which are incident to egos in $S$). Sometimes, $G \setminus S$ is called **alter (sub)graph** (or **alter–to–alter (sub)graph**) (of $G$). In other words, $G \setminus S$ is the subgraph induced by $V \setminus S$ and the set of edges of $G \setminus S$ is the set

$$E_{\text{private}} \cup E_{\text{public}} \cup E_{\text{private}\,-\,\text{public}}.$$

## Proposition

If $S$ is a dominating set of a graph $G = (V, E)$, then

$$|S| + |V_{\text{private}}| + |V_{\text{public}}| = |V|,$$

$$\sum_{v \in S} \text{degree}(v) + |E_{\text{private}}| + |E_{\text{public}}| + |E_{\text{private}\,-\,\text{public}}| - |E_{\text{ego}}| = |E|.$$

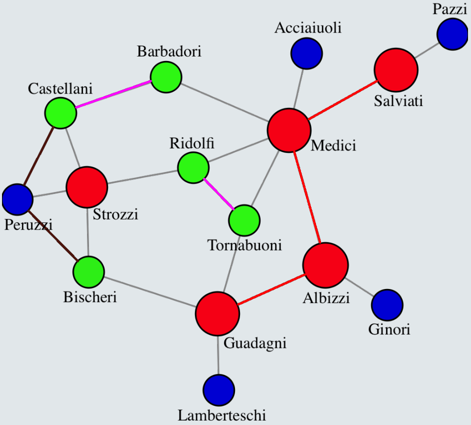
## **The Network of the 15 Florentine Families**: All Dominating Egocentric Subgraphs

| Minimum Dominating Set ($S$) | IDS[1] | $\lvert V_{\text{private}}\rvert$ | $\lvert V_{\text{public}}\rvert$ | $\sum_{v\in S}$ degree($v$) | $\lvert E_{\text{private}}\rvert$ | $\lvert E_{\text{public}}\rvert$ | $\lvert E_{\text{private - public}}\rvert$ | $\lvert E_{\text{ego}}\rvert$ |
|---|---|---|---|---|---|---|---|---|
| $\{A, C, K, J, M\}$ | | 3 | 7 | 14 | 0 | 5 | 2 | 1 |
| $\{C, E, F, I, J\}$ | ✓ | 2 | 8 | 15 | 0 | 5 | 0 | 0 |
| $\{C, F, I, J, M\}$ | ✓ | 1 | 9 | 12 | 0 | 8 | 0 | 0 |
| $\{C, I, K, J, M\}$ | | 2 | 8 | 13 | 0 | 8 | 0 | 1 |
| $\{C, D, F, I, M\}$ | | 2 | 8 | 14 | 0 | 7 | 0 | 1 |
| $\{C, E, D, F, L\}$ | | 3 | 7 | 17 | 0 | 5 | 0 | 2 |
| $\{A, C, E, K, J\}$ | | 4 | 6 | 17 | 0 | 2 | 2 | 1 |
| $\{A, C, F, J, M\}$ | ✓ | 2 | 8 | 13 | 0 | 5 | 2 | 0 |
| $\{C, E, I, K, J\}$ | | 3 | 7 | 16 | 0 | 5 | 0 | 1 |
| $\{A, C, D, F, M\}$ | | 3 | 7 | 15 | 0 | 4 | 2 | 1 |
| $\{A, C, E, D, K\}$ | | 5 | 5 | 19 | 0 | 2 | 2 | 3 |
| $\{A, C, D, K, M\}$ | | 4 | 6 | 16 | 0 | 4 | 2 | 2 |
| $\{C, E, K, J, L\}$ | | 3 | 7 | 16 | 0 | 5 | 0 | 1 |
| $\{C, E, F, J, L\}$ | ✓ | 2 | 8 | 15 | 0 | 5 | 0 | 0 |
| $\{C, E, D, F, I\}$ | | 3 | 7 | 17 | 0 | 5 | 0 | 2 |
| $\{A, C, E, D, F\}$ | | 4 | 6 | 18 | 0 | 2 | 2 | 2 |
| $\{C, D, I, K, M\}$ | | 3 | 7 | 15 | 0 | 7 | 0 | 2 |
| $\{C, E, D, I, K\}$ | | 4 | 6 | 18 | 0 | 5 | 0 | 3 |
| $\{A, C, E, F, J\}$ | ✓ | 3 | 7 | 16 | 0 | 2 | 2 | 0 |
| $\{C, E, D, K, L\}$ | | 4 | 6 | 18 | 0 | 5 | 0 | 3 |

**The 15 Florentine Families**: $A$ = Strozzi, $B$ = Tornabuoni, $C$ = Medici, $D$ = Albizzi, $E$ = Guadagni, $F$ = Pazzi, $G$ = Acciaiuoli, $H$ = Bischeri, $I$ = Peruzzi, $J$ = Ginori, $K$ = Salviati, $L$ = Castellani, $M$ = Lamberteschi, $N$ = Ridolfi, $O$ = Barbadori.

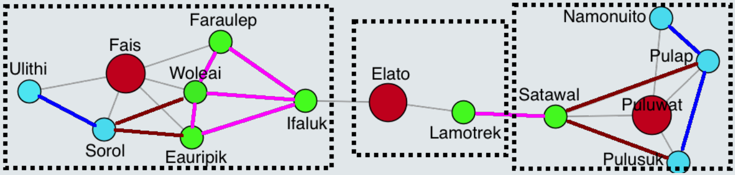
[1] Independent Dominating Set (IDS).

## Example (Florentine Families Network)




$|S| = 5$ (5 egos, **red** big circles);
$|V_{\mathrm{private}}| = 5$ (5 private alters, **blue** circles);
$|V_{\mathrm{public}}| = 5$ (5 public alters, **green** circles);
$|E_{\mathrm{private}}| = 0$ (no edges among private alters);
$|E_{\mathrm{public}}| = 2$ (2 edges among public alters, **magenta** lines);
$|E_{\mathrm{private-public}}| = 2$ (2 edges among private–to–public alters, **brown** lines).
$|E_{\mathrm{ego}}| = 3$ (3 among egos, **red** lines).

# Example (The Voyaging Network among 14 Western Carolines Islands, Hage & Harary, 1991)




$|S| = 3$ (3 egos, **red** big circles);
$|V_{\text{private}}| = 5$ (5 private alters, **blue** circles);
$|V_{\text{public}}| = 6$ (6 public alters, **green** circles);
$E_{\text{private}} = 3$ (3 edges among private alters, **blue** lines);
$E_{\text{public}} = 6$ (6 edges among public alters, **magenta** lines);
$E_{\text{private - public}} = 4$ (4 edges among private–to–public alters, **brown** lines).
(Three communities enclosed in dotted rectangles.)

| Minimum Dominating Set ($S$) | IDS | $\lvert V_{\text{private}}\rvert$ | $\lvert V_{\text{public}}\rvert$ | $\sum_{v\in S}$ degree($v$) | $\lvert E_{\text{private}}\rvert$ | $\lvert E_{\text{public}}\rvert$ | $\lvert E_{\text{private - public}}\rvert$ | $\lvert E_{\text{ego}}\rvert$ |
|---|---|---|---|---|---|---|---|---|
| $\{G, I, M\}$ | ✓ | 5 | 6 | 11 | 3 | 6 | 4 | 0 |
| $\{C, I, M\}$ | ✓ | 5 | 6 | 11 | 3 | 6 | 4 | 0 |

**The Dominant Western Carolines Islands**: $C$ = Puluwat, $G$ = Pulap, $I$ = Fais, $M$ = Elato.

# Community Partitions in a Graph

**Definition (Newman & Girvan, 2004)**

Let $G = (V, E)$ be a graph.

- A **clustering** of vertices of $G$ is a partition of the set of verticies $V$ into a (finite) family $\mathcal{C} \subset 2^V$ of subsets of vertices, often called **modules**, such that $\bigcup_{C \in \mathcal{C}} C = V$ and $C \cap C' = \varnothing$, for each $C, C' \in \mathcal{C}, C \neq C'$.
- A clustering $\mathcal{C}$ may be assessed by a quality function $Q = Q(\mathcal{C})$, called **modularity**, which is defined as:

$$
\begin{aligned}
Q &= \sum_{C \in \mathcal{C}} \left[ \frac{|E(C)|}{|E|} - \left( \frac{2|E(C)| + \sum_{C' \in \mathcal{C}, C \neq C'} |E(C, C')|}{2|E|} \right)^2 \right] \\
&= \sum_{C \in \mathcal{C}} \left[ \frac{|E(C)|}{|E|} - \left( \frac{\sum_{v \in C} \text{degree}(v)}{2|E|} \right)^2 \right],
\end{aligned}
$$

  where $E(C)$ is the number of edges inside module $C$ and $E(C, C')$ is the number of edges among modules $C$ and $C'$. Essentially, modularity compares the number of edges inside a given module with the expected value for a randomized graph of the same size and same degree sequence.
- A clustering that maximizes modularity is usually called **community partition** and the corresponding modules are called **communities**.

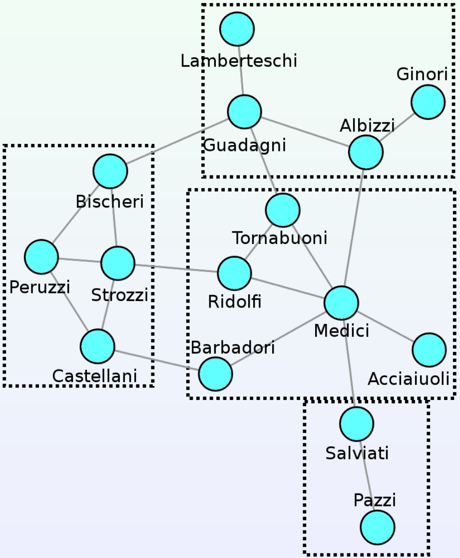
**Theorem (Brandes, Delling, Gaertler, Görke, Hoefer, Nikoloski & Wagner, 2008)**

Modularity is strongly $\mathcal{NP}$–complete.

**Algorithmic Computation of Dominating Sets**

Here, we are using the community detection algorithm (through modularity maximization) of the **Louvain method** (Blondel, Guillaume, Lambiotte & Lefebvre, 2008) as implemented in Python by **Thomas Aynaud**.

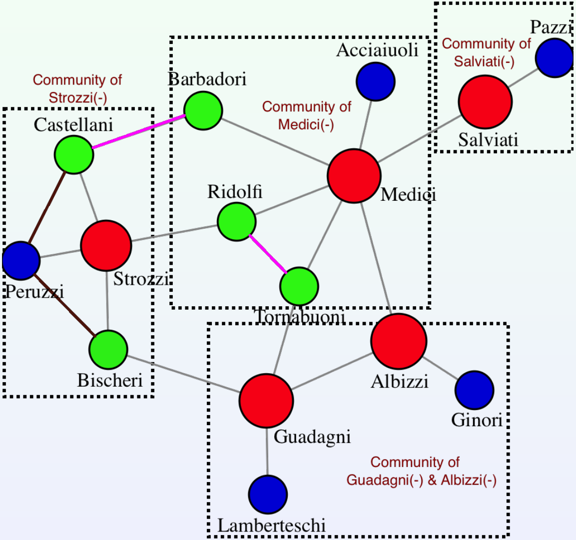
# Communities in the Network of Florentine Families



# **The Florentine Families Network**: All Dominating Egocentric Subgraphs and their Community Partitions

| Minimum Dominating Set ($S$) | $\gamma$ | $\lvert\mathcal{C}\rvert$ | Community Partition ($\mathcal{C}$) |
|---|---|---|---|
| $\{A, C, K, J, M\}$ | 5 | 4 | $\{A(-), C(-), J + M, K(-)\}$ |
| $\{C, E, F, I, J\}$ | 5 | 4 | $\{I, C(-), E(-) + J, F\}$ |
| $\{C, F, I, J, M\}$ | 5 | 4 | $\{I, C(-), J + M, F\}$ |
| $\{C, I, K, J, M\}$ | 5 | 4 | $\{I, C(-), J + M, K(-)\}$ |
| $\{C, D, F, I, M\}$ | 5 | 4 | $\{I, C(-), D(-) + M, F\}$ |
| $\{C, E, D, F, L\}$ | 5 | 4 | $\{L(-), C(-), E(-) + D(-), F\}$ |
| $\{A, C, E, K, J\}$ | 5 | 4 | $\{A(-), C(-), E(-) + J, K(-)\}$ |
| $\{A, C, F, J, M\}$ | 5 | 4 | $\{A(-), C(-), J + M, F\}$ |
| $\{C, E, I, K, J\}$ | 5 | 4 | $\{I, C(-), E(-) + J, K(-)\}$ |
| $\{A, C, D, F, M\}$ | 5 | 4 | $\{A(-), C(-), D(-) + M, F\}$ |
| $\{A, C, E, D, K\}$ | 5 | 4 | $\{A(-), C(-), E(-) + D(-), K(-)\}$ |
| $\{A, C, D, K, M\}$ | 5 | 4 | $\{A(-), C(-), D(-) + M, K(-)\}$ |
| $\{C, E, K, J, L\}$ | 5 | 4 | $\{L(-), C(-), E(-) + J, K(-)\}$ |
| $\{C, E, F, J, L\}$ | 5 | 4 | $\{L(-), C(-), E(-) + J, F\}$ |
| $\{C, E, D, F, I\}$ | 5 | 4 | $\{I, C(-), E(-) + D(-), F\}$ |
| $\{A, C, E, D, F\}$ | 5 | 4 | $\{A(-), C(-), E(-) + D(-), F\}$ |
| $\{C, D, I, K, M\}$ | 5 | 4 | $\{I, C(-), D(-) + M, K(-)\}$ |
| $\{C, E, D, I, K\}$ | 5 | 4 | $\{I, C(-), E(-) + D(-), K(-)\}$ |
| $\{A, C, E, F, J\}$ | 5 | 4 | $\{A(-), C(-), E(-) + J, F\}$ |
| $\{C, E, D, K, L\}$ | 5 | 4 | $\{L(-), C(-), E(-) + D(-), K(-)\}$ |

**The 15 Florentine Families**: $A$ = Strozzi, $B$ = Tornabuoni, $C$ = Medici, $D$ = Albizzi, $E$ = Guadagni, $F$ = Pazzi, $G$ = Acciaiuoli, $H$ = Bischeri, $I$ = Peruzzi, $J$ = Ginori, $K$ = Salviati, $L$ = Castellani, $M$ = Lamberteschi, $N$ = Ridolfi, $O$ = Barbadori.




**Five egos distributed inside four communities.**

Egos are colored **red**, private alters **blue** and public alters **green**.

Dotted rectangles embrace vertices lying inside the same communities.

# B. Network Influence and Scalar Attribute Assortativity

# The Friedkin–Johnsen Model of Social Influence

- Let $G = (V, E)$ be a (simple undirected) graph.
- Vertices represent **persons**.
- For $i \in V$ and at each time $t$, person $i$ holds an opinion (or attitude) $x_i^t$, where $x_i^t$ is a considered to be a **scalar attribute** of graph vertices that takes values in the interval $[0, 1]$.
- Each person interacts with all her graph neighbors and updates her opinion at the subsequent time $t + 1$ as follows:

$$x_i^{t+1} = \sigma_i \sum_{j \in N(i) \neq \varnothing} \frac{1}{k_i} A_{ij} x_i^t + (1 - \sigma_i) x_i^t,$$

where $\mathbf{A} = A_{ij}$ is the adjacency matrix of the graph, $k_i$ is the degree of $i$, $N(i)$ is the set of neighbors of $i$ and $\sigma_i \in [0, 1]$ is the **susceptibility** of $i$ to the influence of her neighbors. Let $\mathbf{S}$ be the diagonal matrix such that $S_{ii} = \frac{\sigma_i}{k_i}$, for all $i \in V$. Note that when $k_i = 0$, then $A_{ij} = 0$, for all $j \in V$ and, thus, $\mathbf{S}$ is well defined, if $G$ is assumed to be free of isolated vertices.

## Proposition

If $G$ is a connected graph, then, for any person $i \in V$ and any initial opinion $x_i^0 \in [0, 1]$, there exists a **consensus** opinion $x^\infty \in [0, 1]$ such that

$$\lim_{t \to \infty} x_i^t = x^\infty.$$

In fact, $x^\infty$ is given by

$$\mathbf{x}^\infty = (\mathbf{I} - \mathbf{SA})^{-1}(\mathbf{I} - \mathbf{S})\mathbf{x}^0,$$

where $\mathbf{I}$ is the unit matrix (1's on the diagonal and 0's elsewhere).

# Contrary Influence

- The previous mechanism of social influence converges to a consesus, because at each time step each person's opinion is compromized with the opinions of her neighbors.
- However, one can also think of a mechanism of **negative influence**, when at each time step each person's opinion tends to diverge from the opinions of her neighbors in the following way:

$$x_i^{t+1} = \sigma_i D(x_i^t) + (1 - \sigma_i) x_i^t,$$

where, denoting $y_i^t = \sum_{j \in N(i) \neq \varnothing} \frac{1}{k_i} A_{ij} x_i^t$,

$$D(x_i^t) = \begin{cases} \max\{2x_i^t - y_i^t, 0\}, & \text{when } x_i^t \leq y_i^t \\ \min\{2x_i^t - y_i^t, 1\}, & \text{when } x_i^t > y_i^t. \end{cases}$$

**Proposition**

For the model of negative influence, if $G$ is a connected graph, then, for any person $i \in V$ and any initial opinion $x_i^0 \in [0, 1]$,

$$\lim_{t \to \infty} x_i^t = 0 \text{ or } 1.$$

# Scalar Attribute Assortativity Coefficient

- Let $G = (V, E)$ be a graph with adjacency matrix $A_{ij}$, $k_i$ be the degree of $i \in V$ and $|E| = m$.
- Let each vertex $i$ possess a scalar attribute $x_i$ ($\in \mathbb{R}$).
- Then one can define (cf. **Mark Newman**, 2003), the (normalized) **scalar attribute assortativity** as follows:

$$r = \frac{\sum_{i,j \in V} (A_{ij} - \frac{k_i k_j}{2m}) x_i x_j}{\sum_{i,j \in V} (k_i \delta_{ij} - \frac{k_i k_j}{2m}) x_i x_j},$$

  where $\delta_{ij}$ is the Kronecker delta. This is an example of a (Pearson) correlation coefficient with a covariance in its numerator and a variance in the denominator.
- Clearly, $r \in [-1, 1]$ and, as we will discuss later in the case of enumerative attributes, $r$ is a measure of assortative (or disassortative) mixing according to the scalar attribute $x_i$, with vertices having similar values of this attribute being more (or less) likely to be connected by an edge.

# C. Multilayer Networks and Enumerative Attribute Assortativity

- Let
$$G = (V, E)$$
be a (simple undirected) graph with (finite) set of vectors $V$ and (finite) set of edges $E$.
- The terms "graph" and "(social) network" are used here interchangeably.
- If $S_1, S_2, \ldots, S_s \subset V$ (for a positive integer $s$) are $s$ nonempty (sub)sets of vertices of $G$, the (finite) family of subsets
$$\mathcal{S} = \mathcal{S}(G) = \{S_1, S_2, \ldots, S_s\},$$
forms a **partition** (or $s$–**partition**) in $G$, when, for all $i, j = 1, 2, \cdots, \ s, i \neq j$,
$$S_i \cap S_j = \varnothing \text{ and}$$
$$V = \bigcup_{i=1}^{s} S_i.$$
- $S_1, S_2, \ldots, S_s$ are called **groups** (of vertices) of partition $\mathcal{S}$.

- For every graph, there exists (at least one) partition of its vertices.
- Two trivial graph partitions are:
  - the $|V|$–partition into singletons of vertices and

$$\mathcal{S}_{\text{point}} = \mathcal{S}_{\text{point}}(G) = \{\{v\} \colon v \in V\},$$

  - the 1–partition into the whole vertex set

$$\mathcal{S}_{\text{total}} = \mathcal{S}_{\text{total}}(G) = \{V\}.$$

- If $G$ is a bipartite graph with (vertex) parts $U$ and $V$, the bipartition of $G$ is denoted as:

$$\mathcal{S}_{\text{bipartition}} = \mathcal{S}_{\text{bipartition}}(G) = \{U, V\}.$$

# Examples of Nontrivial Graph Partitions

1. **Structural (endogenous) partitions**:
   - Connected components
   - Communities
   - Chromatic partitions (bipartitions, multipartitions)
   - Degree partitions

   - Time slicing of temporal networks
   - Domination bipartitions (ego– and alter–nets)
   - Core–Periphery bipartitions

2. **Ad hoc (exogenous) partitions**:
   - Vertex attributes (or labels or colors)
   - Layers (or levels)

# Multilayer Partitions and Multilayer Graphs

- A **multilayer partition** of a graph $G$ is a partition

$$\mathcal{L} = \mathcal{L}(G) = \{L_1, L_2, \ldots, L_\ell\}$$

  into $\ell \geq 2$ groups of vertices $L_1, L_2, \ldots, L_\ell$, which are called **layers**.

- A graph with a multilayer partition is called a **multilayer graph**.

- Let $\mathcal{P} = \{P_1, P_2, \ldots, P_p\}$ and $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_q\}$ be two (different) partitions of vertices of graph $G = (V, E)$.
- Partition $\mathcal{P}$ is called **thicker** than partition $\mathcal{Q}$ (and $\mathcal{Q}$ is called **thinner** than $\mathcal{P}$), whenever, for every $j = 1, 2, \cdots, q$, there exists a $i = 1, 2, \cdots, p$ such that

$$Q_j \subset P_i.$$

- Partitions $\mathcal{P}$ and $\mathcal{Q}$ are called (**enumeratively**) **equivalent**, whenever, for all $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$,

$$P_i \cap Q_j \neq \varnothing.$$

- Obviously, if partitions $\mathcal{P}$ and $\mathcal{Q}$ are equivalent, then

$$pq \leq |V|.$$

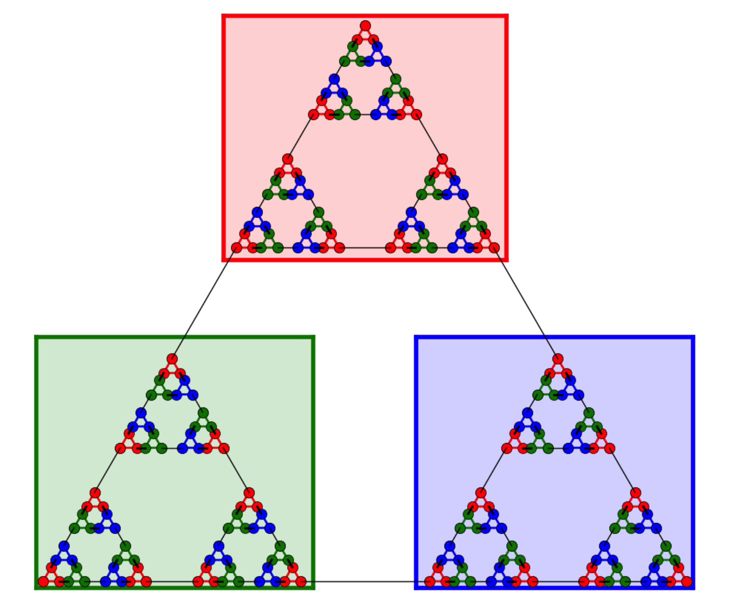
# Self–Similarity of Partitions

- A partition $\mathcal{P}$ of vertices of graph $G$ (such that $|V| \geq p^2$) is called (**enumeratively**) **self–similar**, whenever, for any $i = 1, 2, \cdots, p$, there is a $p$–(sub)partition

$$\mathcal{P}^i = \{P_1^i, P_2^i, \ldots, P_p^i\}$$

  of the (induced) subgraph $P_i$.
- In this case, graph $G$ is called (**enumeratively**) **self–similar** with respect to partition $\mathcal{P}$.

# A Sierpinski (Cantor–type) Graph with $k = 2.5$ and depth 4



# Graph Partitions as Enumerative Attribute Assignments

- An **assignment of enumerative attributes** (or **discrete attribute assignment**) to vertices of graph $G$ is a mapping

$$\mathcal{A} \colon V \to \{1, 2, \cdots, \alpha\}, \text{for some } \alpha \leq |V|,$$

  through which the vertices of $G$ are classified according to the values they take in the set $\{1, 2, \cdots, \alpha\}$.
- Every $p$–partition $\mathcal{P} = \{P_1, P_2, \ldots, P_p\}$ of vertices of $G$ corresponds to a $p$–assignment $\mathcal{A}_{\mathcal{P}}$ of enumerative attributes to vertices of $G$ distributing them in the groups of partition $\mathcal{P}$.
- Conversely, every assignment of enumerative attributes to vertices of $G$ taking values in the (finite) set $\{1, 2, \cdots, \alpha\}$ corresponds to a partition $\mathcal{P}^\alpha = \{P_1^\alpha, P_2^\alpha, \ldots, P_p^\alpha\}$ of the vertices of $G$ such that, for any $k = 1, 2, \cdots, \alpha$,

$$P_k^\alpha = \{v \in V \colon \mathcal{A}(v) = k\} = \mathcal{A}^{-1}(k).$$

- Thus, vertex partitions and enumerative vertex attribute assignments are coincident.

- Let $\mathcal{P}$ be a partition of $G$ into $p$ groups of vertices and $\mathcal{A}$ be an assignment of $\alpha$ discrete attributes to vertices of $G$.
- Partition $\mathcal{P}$ is called **compatible** with attribute assignment $\mathcal{A}$, whenever partition $\mathcal{P}$ is thinner than partition $\mathcal{P}^\alpha$, i.e., whenever, for every $k = 1, 2, \cdots, \alpha$, there exists (at least one) $i = 1, 2, \cdots, p$ such that

$$P_i = \mathcal{A}^{-1}(k).$$

- Apparently, if partition $\mathcal{P}$ is compatible with attribute assignment $\mathcal{A}$, then

$$p \geq \alpha.$$

- Trivially, every discrete attributes assignment (or every partition) is compatible to itself.

- Let $\mathcal{P} = \{P_1, P_2, \ldots, P_p\}$ be a vertex partition of graph $G$.
- Identifying $\mathcal{P}$ to a $p$–assignment $\mathcal{A}_{\mathcal{P}}$ of enumerative attributes to the vertices of $G$ , one can define (cf. **Mark Newman**, 2003), the (normalized) **enumerative attribute assortativity** (or **discrete assortativity**) **coefficient of partition** $\mathcal{P}$ as follows:

$$r_{\mathcal{P}} = r_{\mathcal{P}}(\mathcal{A}_{\mathcal{P}}) = \frac{\operatorname{tr} \mathbf{M}_{\mathcal{P}} - ||\mathbf{M}_{\mathcal{P}}^2||}{1 - ||\mathbf{M}_{\mathcal{P}}^2||},$$

where $\mathbf{M}_{\mathcal{P}}$ is the $p \times p$ (normalized) **mixing matrix of partition** $\mathcal{P}$. Equivalently:

$$r_{\mathcal{P}} = \frac{\sum_{i,j \in V} (A_{ij} - \frac{k_i k_j}{2m}) \delta(\mathcal{A}_{\mathcal{P}}(i), \mathcal{A}_{\mathcal{P}}(j))}{2m - \sum_{i,j \in V} (\frac{k_i k_j}{2m}) \delta(\mathcal{A}_{\mathcal{P}}(i), \mathcal{A}_{\mathcal{P}}(j))},$$

where $\{A_{ij}\}$ is the adjacency matrix of graph $G$, $m$ is the total number of edges of $G$, $k_i$ is the degree of vertex $i$ and $\delta(x, y)$ is the Kronecker delta.

- In general,
$$-1 \leq r_{\mathcal{P}} \leq 1,$$
where
  - $r_{\mathcal{P}} = 0$ signifies that there is no assortative mixing of graph vertices with respect to their assignment to the $p$ groups of partition $\mathcal{P}$, i.e., graph $G$ is configured as a perfectly mixed network (the random null model).
  - $r_{\mathcal{P}} = 1$ signifies that there is a perfect assortative mixing of graph vertices with respect to their assignment to the $p$ groups of partition $\mathcal{P}$, i.e., the connectivity pattern of these groups is perfectly homophilic.
  - When $r_{\mathcal{P}}$ atains a minimum value, which lies in general in the range $[-1, 0)$, this signifies that there is a perfect disassortative mixing of graph vertices with respect to their assignment to the $p$ groups of partition $\mathcal{P}$, i.e., the connectivity pattern of these groups is perfectly heterophilic.

# Assortative Mixing Among Partitions

- Let $\mathcal{P} = \{P_1, P_2, \ldots, P_p\}$ and $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_q\}$ be two (different) partitions of vertices of graph $G$.
- Then, for any $i = 1, 2, \cdots, p, j = 1, 2, \cdots, q, P_i \cap Q_j \neq \varnothing$ and the intersection of $\mathcal{P}$ and $\mathcal{Q}$

$$\mathcal{P} \cap \mathcal{Q} = \{P_i \cap Q_j \colon i = 1, 2, \cdots, p, j = 1, 2, \cdots, q\}$$

  is also a vertex partition of $G$.
- Notice that partition $\mathcal{P} \cap \mathcal{Q}$ is compatible with any one of the following three discrete attribute assignments on $G$:
  - $\mathcal{A}_{\mathcal{P}}$, in which, for any group $P_i \cap Q_j$ of $\mathcal{P} \cap \mathcal{Q}$, all vertices of $P_i \cap Q_j$ are assigned a value in the set $\{1, 2, \cdots, p\}$,
  - $\mathcal{A}_{\mathcal{Q}}$, in which, for any group $P_i \cap Q_j$ of $\mathcal{Q} \cap \mathcal{Q}$, all vertices of $P_i \cap Q_j$ are assigned a value in the set $\{1, 2, \cdots, q\}$ and
  - $\mathcal{A}_{\mathcal{P} \cap \mathcal{Q}}$, in which, for any group $P_i \cap Q_j$ of $\mathcal{P} \cap \mathcal{Q}$, all vertices of $P_i \cap Q_j$ are assigned a value in the set $\{1, 2, \cdots, r\}$, for some $r \leq pq$.

- Thus, we may define a discrete assortativity coefficient of partition $\mathcal{P} \cap \mathcal{Q}$ according to each one of the three compatible attribute assignments.
- Here we will focus on the third case and we define the **discrete assortativity coefficient of the joint partition for** $\mathcal{P}$ **and** $\mathcal{Q}$ as follows:

$$r_{\mathcal{P}\mathcal{Q}} = r_{\mathcal{P}\cap\mathcal{Q}}(\mathcal{A}_{\mathcal{P}\cap\mathcal{Q}}).$$

- Apparently now,

$$r_{\mathcal{P}\mathcal{Q}} = r_{\mathcal{Q}\mathcal{P}}.$$

# Special Cases

- If $\mathcal{Q} = \mathcal{S}_{\text{point}}$, then $r_{\mathcal{P}\mathcal{S}_{\text{point}}}$ is the attribute assortativity coefficient $r_{\mathcal{P}}$ of graph $G$ equipped with the vertex attributes corresponding to partition $\mathcal{P}$, i.e.,

$$r_{\mathcal{P}S_{\text{point}}} = r_{\mathcal{P}}.$$

- If $\mathcal{P} = \mathcal{S}_{\text{total}}$, then, for any partition $\mathcal{P}$,

$$r_{\mathcal{P}\mathcal{S}_{\text{total}}} = 1.$$

- If $G$ is bipartite and $\mathcal{P} = \mathcal{S}_{\text{bipartition}}$, then, for any partition $\mathcal{P}$,

$$r_{\mathcal{P}\mathcal{S}_{\text{bipartition}}} \in [-1, 0).$$