Evolutionary prisoner's dilemma game on a square lattice

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A simplified prisoner's game is studied on a square lattice when the players interacting with their neighbors can follow two strategies: to cooperate (C) or to defect (D) unconditionally. The players updated in random sequence have a chance to adopt one of the neighboring strategies with a probability depending on the payoff difference. Using Monte Carlo simulations and dynamical cluster techniques we study the density c of cooperators in the stationary state. This system exhibits a continuous transition between the two absorbing state when varying the value of temptation to defect. In the limits $c \to 0$ and 1 we have observed critical transitions belonging to the universality class of directed percolation.

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I. INTRODUCTION

The evolutionary prisoner's dilemma games were introduced by Axelrod [1] to study the emergence of cooperation rather than exploitation among selfish individuals. Since the pioneering work of Axelrod this approach has become a fruitful tool in the area of political and behavior sciences, biology and economics [2–4].

In the prisoner's dilemma (PD) game each of two players have to decide simultaneously whether it wishes to cooperate with the other or to defect. The rewards dependent on their choices are expressed by 2×2 payoff matrices in agreement with the four possibilities. Assuming symmetric game the players get rewards R (P) if both choose to cooperate (defect). In the remaining two cases the defector's and cooperator's payoff are T (temptation to defect) and S (sucker's payoff) respectively. The elements of payoff matrix satisfy the following conditions: $T > R > P > S$ and $2R > T + S$. In this game the mutual cooperation leads to the highest total (average) payoff. The highest individual payoff (T) can only be reached against the other player decreasing the average payoff. These features makes the PD game so interesting in the mentioned areas.

In earlier studies N contestants played an iterated round-robin prisoner's dilemma game. The population of contestants, which apply different algorithms to choose between defection and cooperation in the knowledge of previous decisions, was modified according to a Darwinian selection rule round by round. For example, eliminating the worst player the best one will have an offspring inheriting the parent's strategy. In a different interpretation, the worst player adopts the best algorithm. Computer tournaments (simulations) were performed to study how the population of contestants varies with time [1]. Evidently, the final (stationary) state depends on the initial population. The simulations have clarified the emergence of mutual cooperation among all the players under some conditions. In these tournaments the winner the so-called 'Tit for Tat' (TFT) algorithm has a crucial

role. This very simple algorithm cooperates in the first round and later it reciprocates the partner's previous decision. It forces the players to cooperate mutually and maintains this state against defectors.

Beside the homogeneous system with players following TFT algorithm, the state where all the players choose to defect has proved to be stationary too, more precisely, spare cooperators will be suppressed due to the evolutionary rule in the large N limit. More precisely, only a sufficiently large portion of mutual cooperators can survive among defectors. The emergence of uniform cooperation becomes easier when – combining the evolutionary game with spatial effects – the players interact much more with their neighbors than with those who are far away as it is typical in real populations. The spatial effects promote the survival of cooperators even if we do not use any kind of elaborate strategies such as the TFT.

Recently Nowak and May [5] have introduced a spatial evolutionary PD game. In this model individuals located on a lattice play with their neighbors and with themself. The strategical complexities and memories of past encounters are neglected by considering only two simple kinds of individuals: those who cooperate (C) and those who defect (D) unconditionally. The evolutionary rule was also simplified by using discrete time steps. Between two rounds individuals adopt the strategy that has received the highest payoff among its neighbors including themself. This deterministic model is equivalent to a two-state cellular automaton where the next state at a given lattice point is determined by the states on the surrounding points. The outcome depends on the initial configuration and the rescaled payoff matrix described by a single parameter b characterizing the measure of temptation to defect (see the matrix in Sec. 2). This model with and without self-interaction was investigated on different lattice structures (square, triangle, cubic). The most exhaustive analysis is performed on a square lattice taking into account the interactions with the first and second neighbors and self-interaction. Nowak and May observed a rich variety of spatial and temporal dy-

namics dependent on the value of b. For example, the cooperators can invade the word of defectors along straight border lines while defectors gain along irregular boundaries for a given interval of b. Furthermore, the above rules conserve the symmetries of the initial state for adequate boundary conditions. Due to the discrete nature of total payoff there appear sharp steps when varying b.

Introducing stochastic evolutionary rules between two rounds Nowak *et al.* [6] have extended the above model. Although the stochasticity simplifies the dynamics it does not change the basic observations that cooperators and defectors can coexist. The randomness destroys the straight border lines as well as other symmetries that appear in the deterministic model.

Hubermann and Glance [7] have studied a similar model using continuous time simulations where players are chosen randomly and immediately updated. Their results support that the above conclusions are not affected by whether we use continuous or discrete time in the stochastic case [6].

In the present work we study a PD game with a slightly different continuous time evolution on square lattice. In the modified model the players need less intelligence to decide whether they adopt one of the neighboring strategy or not. Using systematic Monte Carlo simulations and generalized mean-field techniques we calculate the density of cooperators as a function of b for different noise levels. It will be shown that the transitions from the active state (coexistence of defectors and cooperators) to the absorbing ones (all D or all C) exhibit universal behavior.

II. THE MODEL

The players located on a square lattice can follow only two simple strategies: C (always cooperate) and D (always defect). Due to this simplification this system can be handled with the Ising formalism and we can use the sophisticated techniques developed in non-equilibrium statistical physics. Each player plays a PD game with itself and with its neighbors. The total payoff of a certain player is the sum over all interactions. The elements of payoff matrix can be rescaled because the evolutionary rule depends on the payoff differences between the players. Accepting the idea suggested by Nowak and May [5] we choose $R = 1$, $P = S = 0$ and $T = b$. Thus, the payoff to player A against B is given by the matrix:

$_A{}^B$	C	D
C	1	0
D	b	0

where $b > 1$.

Two systems will be considered subsequently. In the first case only the first neighbors are taken into account. This means that the total payoff of a defector surrounded by cooperators is $4b$ while the cooperator's payoff is 5 in

the same surroundings. In the second case the neighborhood includes the first- and second-neighbors. Thus the payoffs of the defector and cooperator are $8b$ and 9 in the sea of cooperators.

The randomly chosen player X revises its strategy according to the following rules. This player selects one of its neighbours Y with equal probability. Given the total payoffs (E_X and E_Y) from the previous round, player X adopts the neighbor's strategy with the probability:

$$W = \frac{1}{1 + \exp\left[-(E_Y - E_X)/K\right]} \qquad (1)$$

where E_Y is the neighbor's payoff and K characterizes the noise introduced to permit irrational choices. For successful strategy adoption the new state as well as the new payoffs are updated. Notice that the decision is not affected by the variation of total payoff involving the change in the surroundings. Starting from a random initial state the above process is repeated many times.

For $K = 0$ the player X adopts Y's strategy if $E_Y > E_X$. In this case the randomness is represented by the selection of the players X and Y. The finite value of K characterizes the range of payoff difference within which the irrational decision can typically appear. At present, our analysis is constrained to noise levels $K < 1$.

Monte Carlo (MC) simulations are performed varying the value of b for fixed K values. We have determined the density c of cooperators using periodic boundary conditions. The system size was varied from $L = 200$ to 1000; the large sizes are required to suppress the statistical error in the critical regions ($c \to 0$ or 1).

The above models are also investigated by the generalized mean-field technique that proved to be very efficient for studying dynamical systems such as the one-dimensional stochastic cellular automata [8–10] and driven lattice gases [11–13]. In fact the introduction of the above evolutionary rule is motivated by the demand to make the model more convenient for this method. In the present case we have adapted the two-dimensional method to determine the probability of the configurations appearing on 2-, 4-, 5- and 6-point clusters [13]. It is expected that the larger the cluster we use the more accurate the prediction given by this technique. At the level of 6-point approximation – taking the consistency conditions and symmetries into account – we have to determine 20 parameters by solving a set of equation of motions for the configuration probabilities in the stationary state. Details of this calculations are given in previous papers [12,13].

III. RESULTS

For both models the $c = 0$ (all D) and 1 (all C) states are independent of time because the evolutionary rule cannot create a new strategy which can spread away under advantageous conditions. The uniform cooperation

($c = 1$) is a stable state if b does not exceed a threshold value b_{c1} which is larger than 1. This means that any constellation of defectors will be defeated if $b < b_{c1}$. In the same way the $c = 0$ state remains stable for $b > b_{c2}$. Henceforth we will concentrate on those states which the cooperators and defector can coexist in, that is, when $b_{c1} < b < b_{c2}$.
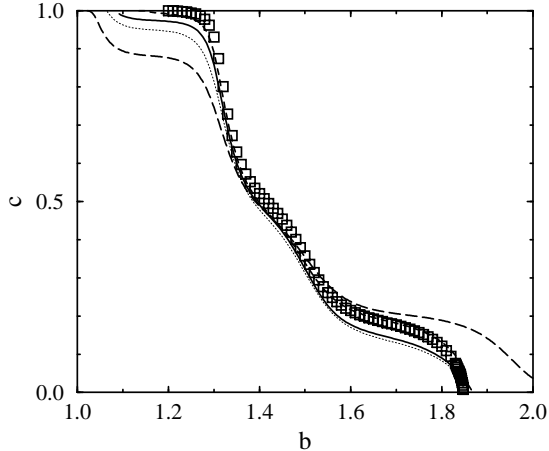

FIG. 1. Density of cooperators as a function of temptation to defect for $K = 0.1$. The MC data are plotted by squares, the results of generalized mean-field technique for different cluster sizes are indicated by long-dashed (2), dotted (4), dashed (5) and solid (6) lines.

First we consider the model with first-neighbor interactions. Figure 1 shows the b-dependence of the density c of cooperators in the coexistence region for $K = 0.1$. As indicated c decreases monotonously with increasing b until the second threshold b_{c2} where the cooperators vanish.
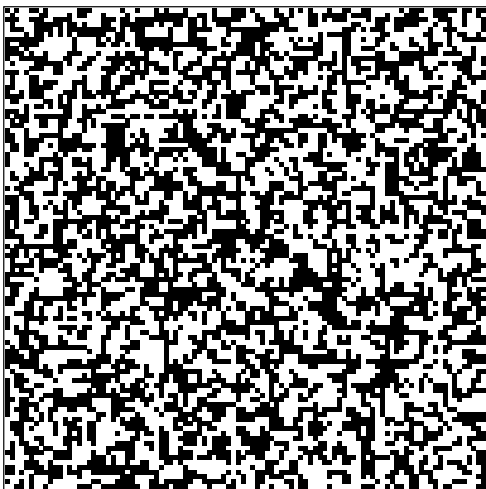

FIG. 2. Distribution of cooperators (white boxes) and defectors (black boxes) for $b = 1.4$ and $T = 0.1$ ($c = 0.515$).

The results of both the MC simulations and the generalized mean-field method refer to step-like behavior becoming more and more striking if we decrease the value of K. The sharp steps appear at the break points (e.g. $b = 4/3,\ 3/2$) described by Nowak and May [5]. Inside the coexistence region the mean-field results of 4-, 5- and 6-point approximations agree satisfactorily with the simulations while the pair approximation yields well-marked difference. The best agreement is found for the 5-point approximations (dashed line).

A typical snapshot on the steadt state distribution of cooperators and defectors is illustrated in Fig. 2 for $b = 0.4$ and $T = 0.1$. This snapshot as well as the subsequent ones are a 100×100 portion of the full 400×400 lattice. In this case the pair- (2-point) approximation gives a satisfactory description of the short-range correlations.

Notice furthermore that the mean-field predictions are not adequate when c tends either to 0 or 1. Namely, the 4- and 6-point approximations predict a continuous (linear) transition, the 5-point approximation indicates a first-order one while the simulations suggest power law behavior if $c \to 0$. Similar situation has already been observed for a one-dimensional stochastic cellular automaton [10]. The mentioned deviations are not surprising because the mean-field approximations are not capable to handle the critical transitions exhibiting enhanced fluctuations and long-range correlations.

In the limit $c \to 0$ the cooperators can survive if they form scattered colonies in the background of defectors as illustrated in Fig. 3. In general, any compact colony formation would be more preferable for cooperators, however, the defectors make them rare.

FIG. 3. The cooperators (white boxes) form colonies in the sea of defectors (black boxes) when their density goes to 0.

Visualizing the time-dependent configuration one can observe how the colonies try to spread away. Their center, size and shape change continuously and a separated colony can disappear without trace. Two colonies

can unite providing better opportunity for their survival, or oppositely, a colony can divide into two (or more) parts. Similar phenomena can be observed in a wide range of dynamical processes described by the directed percolation (DP) [14], the Reggeon field theory [15], the surface-reaction [16] and Schlögl models [17] as well as the branching and annihilating random walks [18]. Grassberger [19] and Janssen [20] conjectured that all one-component models with a single absorbing state belong to the universality class of directed percolation. Exceptions can appear if the dynamics conserves some symmetries (e.g. parity of offsprings).

Our MC data (shown in Fig. 1) refer to a power law behavior, that is

$$c \propto (b_{c2} - b)^\beta \qquad (2)$$

if $b \rightarrow b_{c2}$. The best fit is obtained for $b_{c2} = 1.8472(1)$ and $\beta = 0.56(3)$ which is consistent with the critical exponent ($\beta \approx 0.58$) of the two-dimensional directed percolation [21].
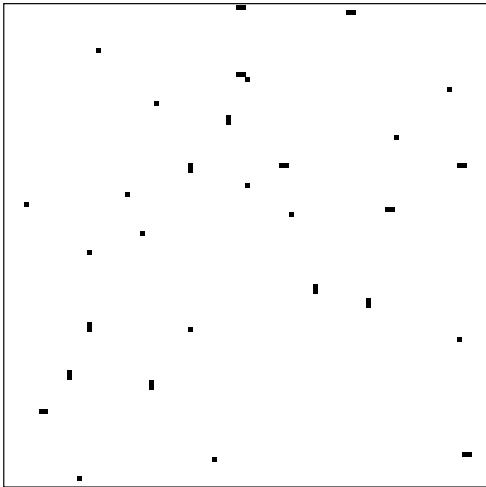

FIG. 4. Typical snapshot for high concentrat ion of cooperators (white boxes) for $b = 1.222$ and $T = 0.1$. The small 'gangs' of defectors (black boxes) walk randomly.

Contrary to the above pattern defectors form small isolated 'gangs' as demonstrated in Fig. 4 for a typical stationary state if $1 - c \ll 1$. A single defector surrounded by cooperators has the highest payoff (fitness) in this system. Sooner or later this defector will have a neighboring offspring which reduces its payoff immediately. (This process can be considered as a retaliation executed by the TFT algorithm if more elaborate strategies are permitted.) If $b < 4/3$ then one of the defectors will be defeated within a short time. The iteration of this process yields randomly walking gangs. Two colliding gangs can unite into one. Due to the possibility of irrational choices a single gang can divide into two or can disappear. Shortly, the gangs can be considered as

branching and annihilating random walkers whose critical behavior belongs to the DP universality class too.

In the deterministic model introduced by Nowak and May [5] isolated gangs with fixed positions can occur if $1 < b < 4/3$. The density of gangs (whose size alternates cyclically if $5/4 < b < 4/3$) depends on the initial state. In contrary to this feature the homogeneous cooperation can emerge in the stochastic models even for $b > 1$ as a consequence of the random walk and annihilation. Besides, the random walk causes the steady state density to be independent of the initial state.

Despite the mentioned expectation the MC data in Fig. 1 do not show any power law behavior in the limit $c \rightarrow 1$. This discrepancy can be resolved by reminding the reader that the critical behavior is controlled by a simple function of the diffusion constant and the rates of branching and annihilation. In the present case these parameters are strongly non-linear functions of b at low K. For higher value of K, however, the non-linear contributions are negligible in the close vicinity of b_{c1} and we expect the power law behavior to appear clearly. In order to check this statement we have repeated the same analyses at higher noise level.
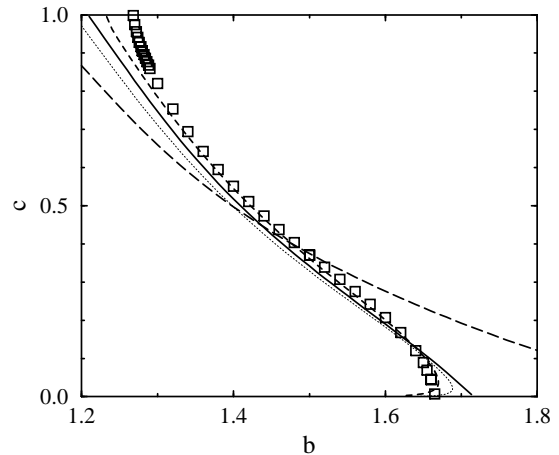

FIG. 5. Density of cooperators *vs.* b for $K = 0.5$ as suggested by MC simulations (squares) and generalized mean-field approximations whose level is indicated as in Fig. 1.

The results obtained for $K = 0.5$ are summarized in Fig. 5. As expected the MC data show power law behavior for both the limits $c \rightarrow 0$ and 1. Detailed numerical analysis results in $b_{c1} = 1.2687$, $\beta = 0.62(5)$ if $c \rightarrow 0$ and $b_{c2} = 1.6644(2)$, $\beta = 0.59(3)$ if $c \rightarrow 0$. These β values agree satisfactorily with the corresponding exponent of DP universality class. Notice, furthermore, that b_{c1} and b_{c2} depend on K. The determination of a $K - b$ phase diagram indicating the active and absorbing states goes beyond the purpose of the present work. Instead of it we have studied the model involving second-neighbor interactions.

The generalization of our techniques to investigate the density of cooperators in the second model is straightforward. The results of these calculations (see Fig. 6) refer to a behavior similar to those of the previous version. There are some minor differences. For example, the threshold values (b_{c1} and b_{c2}) are definitely smaller than those of the previous model. Furthermore, the convergence of the results of generalized mean-field approximation is slow. This fact indicates that the short-range correlations become more relevant if we take the second-neighbor interactions into account.
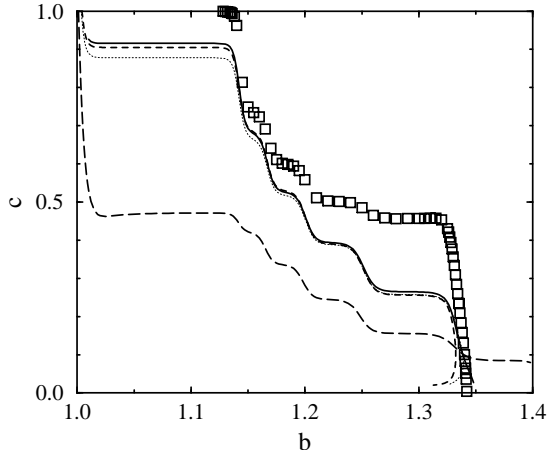

FIG. 6. Density of cooperators as a function of b for the model taking the second neighbor interactions into account at $K = 0.02$. Results are indicated as in Fig. 1.

The steps of the continuous $c(b)$ function (for $b = 8/7$, $7/6$, $6/5$ and $5/4$) becomes sharper when decreasing the value of K. For high noise levels the function becomes smooth and exhibits power law behavior with exponents close to the DP value at both ends of the active region. Inside the active phase the difference between the mean-field results and MC simulations decreases with increasing K.

IV. CONCLUSIONS

We have studied the evolution of cooperation among players who can follow only two simple strategies (C and D) and are placed on a square lattice. The individual receives payoffs from interactions with each of its neighbors and itself in a PD confrontation. An evolutionary rule is introduced by slightly modifying the model suggested by Nowak et al [6]. Namely, a randomly chosen player is to adopt one of its neighboring strategy with a probability dependent on the payoff difference. Two versions of the model have been investigated. In the first case the neighborhood is limited to the first-neighbors. In the second case we have increased the number of neighbors by taking into consideration the second-neighbors too.

The MC simulations have given direct evidence of the existence of two absorbing states ($c = 1$ if $b < b_{c1}$ and $c = 0$ if $b > b_{c2}$). It is remarkable that the homogeneous cooperation proved to be stable against the temptation to defect for $1 < b < b_{c1}$ due to the randomness and possibility of irrational choice. We have found significantly different (K-dependent) threshold values in the models we are interested in. It is expected that b_{c1} tends to 1 if we increase the number of neighbors.

For high density of defectors the cooperators forming compact blocks can spread if $b < b_{c2}$. Comparing the present models with the corresponding deterministic versions [5] we can state that the active region is reduced by the stochasticity. For example, in the deterministic version of our second model a competition between the C and D invasion processes can be observed if $9/5 < b < 2$ because the cooperators invade along straight lines meanwhile the defectors win along irregular boundaries. In this parameter range Mukherji et $al.$ [22] have observed that the cooperation is eliminated when introducing stochastic elements. This is not surprising because the C invasion along straight lines is not permitted in the stochastic models. At lower value of b, however, the spatial effects can facilitate the survival of cooperators [6,23]. In the present stochastic model the second threshold value of b is decreased by the randomness, namely, we have found $b_{c2} < 1.4$ for $K = 0.02$, 0.1 and 0.5 .

The generalized mean-field approximations have clarified the importance of short range correlations for both versions of the stochastic evolutionary PD game inside the coexistence region. Unfortunately, this technique is not applicable in the critical regions ($c \to 0$ and 1) where long-range correlations and fluctuations play a dominant role.

In these critical regions the MC simulations indicated clearly power law behavior, namely $c \propto (b_{c2} - b)^{\beta}$ and $1 - c \propto (b - b_{c1})^{\beta})$ at sufficiently high noise levels. The values of β deduced from the MC data agree well with the DP exponent for both versions. These findings corroborate the conjecture according to which the transitions in all one-component models to an absorbing state belong to the DP universality class in the absence of conserved symmetries. The curiosity of the present model is that here we have two different (non-symmetric) absorbing states whose stability regions are separated by the active phase. For low values of K the appearance of power law behavior against b is distorted by the strongly nonlinear b-dependence of the diffusion and annihilation. Due to the robustness of DP universality class similar critical behavior is expected for many other versions of stochastic evolutionary rules.

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