An Introduction to Game Theory

Building ABM with Game Theory

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Basic Game Theory

Introduction

Why game theory?

Game theory provides a way to reason about decision making. It can be used to explain why people/agents make particular decisions given particular circumstances.

This provides one way for us to imbue our agents with decision making - they consider decisions as games, and pick particular *strategies* in that game.

Traditional game theory considers one shot games and don't allow the players to adapt - but **Evolutionary Game Theory** does this.

A Classic Game

The Prisoners Dilemma (Poundstone 1992) is a classic game from game theory.

There are two players (prisoners), they have both been arrested and interviewed separately. They have two choices, either **C**onfess to the police, or stay **Q**uiet.

If both prisoners i or j confess they receive 4 years in prison, if only one confesses and the other stays quiet they receive 1 and 5 years respectively. If they both stay quiet they receive 2 years each.

Here utility should be negative.

A Classic Game

Players i and j have two possible strategies C and Q. The payoffs are as below:

		Player j	
		C	Q
Player i	C	$-4, -4$	$-1, -5$
	Q	$-5, -1$	$-2, -2$

Prisoner's Dilemma payoff matrix

Notation

n : number of players

s_i : A pure strategy of player i

$S_i = s_i^1, \ldots, s_i^m$: the straegy set of player i. Player i has m possible strategies.

$s = (s_1, \ldots, s_n)$: the strategy profile of all n players

$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$: the strategy profile of all other $n-1$ players, can write $s = (s_i, s_{-i})$

$u_i(s_i, s_{-i})$: The payoff to player i as determined by profile played by all n players

$\mathbf{S}$: the set of all possible strategy profiles, i.e., set of all s

What is a game?

Formally a game consists of:

Players: $1, 2, ..., i, ..., n$

Strategies: $S_i = s_i^1, s_i^2, ..., s_i^m$

Preferences: $S = \{(s_1, ..., s_n), ...\}, u : S \rightarrow R^n$

A Strategy Profile includes one strategy for each player. S is the set of strategy profiles. s is a particular strategy profile for a game e.g., $(s_1, ..., s_I)$

What is a game : Rock, paper, scissors

Players: $1, 2$

Strategies: $S_1 = \{R, P, S\} = S_2 = \{R, P, S\}$

Strategy profile : (R, P) player 1 is rock, player 2 is paper

Utility : $u_1(R, P) = -1, u_2(R, P) = 1 \ldots u_1(S, P) = ?$

All strategies $S = \{(R, R), (S, R), (P, R), \ldots, (P, P)\}$

Pure and Mixed Strategies

A ***pure*** strategy: $S_i = s_i^1, s_i^2, ..., s_i^m$, One of m possible strategies.

A mixed strategy σ_i for a player is a probability distribution over the m possible strategies.

$$\sigma_i : S \to [0,1] \quad s.t.$$

$$\sigma_i = (p_i^1, p_i^2, \ldots, p_i^m)$$

Note that $\sum_x p_i^x = 1$. A pure strategy is just a mixed strategy with one strategy i set to probability 1 and everything else zero.

Mixed Strategy utilities

If the other players play $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ then the expected utility of playing σ_i is

$$\sigma_i(s_i^1)u_i(s_i^1, s_{-i}) + \sigma_i(s_i^2)u_i(s_i^2, s_{-i}) + \ldots + \sigma_i(s_i^m)u_i(s_i^m, s_{-i})$$

Basic Game Theory

Prisoner's dilemma, Dominance & Nash Equilibria

Prisoner's dilemma

Players i and j have two possible strategies C and Q. The payoffs are as below:

		Player j C	Player j Q
Player i	C	$-4, -4$	$-1, -5$
	Q	$-5, -1$	$-2, -2$

Prisoner's Dilemma payoff matrix

Solution Concepts

A Solution concept is a rule to find a solution to the game - this is a formal rule that predicts how a game will be played.

There are a number of classic solution concepts for games, we will look at some:

- Nash Equilibrium (Nash 1950)
- Dominance (Fudenberg and Tirole 1991)
- Rationalizability (Fudenberg and Tirole 1991)

Nash Equilibrium

A (pure-strategy) ***Nash equilibrium*** is a strategy profile s with the property that no player i can do better by choosing a strategy different from s_i^*, given that every other player j adheres to s_j^*.

formally

A strategy profile $(s_1^*, \ldots, s_n^*)$ is a Nash equilibrium if, for every player i, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$ for every strategy possible strategy $s_i \in S_i$.

Nash Equilibrium - Prisoners Dilemma

		Player j C	Player j Q
Player i	C	$-4, -4$	$-1, -5$
	Q	$-5, -1$	$-2, -2$

Prisoner's Dilemma payoff matrix

Set of strategy profiles $S = (C, Q), (C, Q), (Q, C), (Q, Q)$

Is there a strategy profile that is a Nash Equilibrium?

Nash Equilibrium - Prisoners Dilemma

		Player j C	Player j Q
Player i	C	$-4, -4$	$-1, -5$
Player i	Q	$-5, -1$	$-2, -2$

Prisoner's Dilemma payoff matrix

- $(Q, Q), u(s_i) = -2, u(s_j) = -2$. Both i and j can do better by picking C
- $(C, C), u(s_i) = -4, u(s_j) = -4$. Neither can do better in picking another strategy
- $(C, Q), u(s_i) = -1, u(s_j) = -5$. j can do better by picking C
- $(Q, C), u(s_i) = -5, u(s_j) = -1$. i can do better by picking C
  (C,C) is the Nash Equilibrium

Mixed Strategy Nash Equilibrium

A **mixed strategy Nash equilibrium** involves at least one player playing a randomized strategy and no player being able to increase his or her expected payoff by playing an alternate strategy. A Nash equilibrium in which no player randomizes is called a **pure strategy Nash equilibrium**.

Mixed Strategy Nash Equilibrium

Note that randomization requires equality of expected payoffs. If a player is supposed to randomize over strategy A or strategy B, then both of these strategies must produce the same expected payoff. Otherwise, the player would prefer one of them and wouldn't play the other.

Mixed Strategy Nash Equilibrium - Matching Pennies

		Player j H	Player j T
Player i	H	$+1, -1$	$-1, +1$
	T	$-1, +1$	$+1, -1$

Matching Penny payoff matrix

What is the mixed strategy Nash equilibrium?

Mixed Strategy Nash Equilibrium - Battle of Sexes

W

M		Baseball(p)	Ballet (1-p)
	Baseball(q)	$3, 2$	$1, 1$
	Ballet(1-q)	$0, 0$	$2, 3$

Battle of Sexes payoff matrix

What is the mixed strategy Nash equilibrium?

Mixed Strategy Nash Equilibrium - Battle of Sexes

		W		
		Baseball(p)	Ballet (1-p)	M's Payoff
	Baseball(q)	$3, 2$	$1, 1$	$3p + 1(1 - p)$
M	Ballet(1-q)	$0, 0$	$2, 3$	0P + 2(1-p)
	W's Payoff	2q + 0(1-q)	1q + 3(1-q)	

Battle of Sexes payoff matrix

What is the mixed strategy Nash equilibrium? $p = 1/4$, $q = 3/4$

Mixed Strategy Nash Equilibrium - Battle of Sexes

W

		Baseball(p)	Ballet (1-p)
M	Baseball(q)	3/16	9/16
	Ballet(1-q)	1/16	3/16

Battle of Sexes payoff matrix

More than half of the time (Baseball, Ballet) is the outcome and the two people are not together. This lack of coordination is generally a feature of mixed strategy equilibria. The M's payoff is 1½. Thus, both do worse than coordinating on their less preferred outcome. **But this mixed strategy Nash equilibrium, undesirable as it may seem, is a Nash equilibrium in the sense that neither party can improve his or her own payoff, given the behavior of the other party.**

Mixed Strategy Nash Equilibrium - Chicken

		Column: Swerve	Column: Don't
Row	Swerve	$0, 0$	$-1, 1$
	Don't	$1, -1$	$-4, -4$

Chicken payoff matrix

What is the mixed strategy Nash equilibrium? **Try to solve this**

Risk dominance and payoff dominance

Risk dominance and payoff dominance are two related refinements of the Nash equilibrium (NE) solution concept in game theory,

A Nash equilibrium is considered payoff dominant if it is Pareto superior to all other Nash equilibria in the game.When faced with a choice among equilibria, all players would agree on the payoff dominant equilibrium since it offers to each player at least as much payoff as the other Nash equilibria. Conversely, a Nash equilibrium is considered risk dominant if it is less risky. This implies that the more uncertainty players have about the actions of the other player(s), the more likely they will choose the strategy corresponding to it.

Risk dominance and payoff dominance

		Player 2	
		Hunt	Gather
Player 1	Hunt	$5, 5$	$0, 4$
	Gather	$4, 0$	$2, 2$

		Player 2	
		H	G
Player 1	H	A, a	C, b
	G	B, c	D, d

is a coordination game if : A $>$ B, D $>$ C, : a $>$ b, d $>$ c. The strategy pairs (H, H) and (G, G) pure Nash equilibria. A mixed Nash equilibrium where player 1 plays H with probability p $=$ (d-c)/(a-b-c$+$d) and G with probability 1–p; player 2 plays H with probability q $=$ (D-C)/(A-B-C$+$D) and G with probability 1–q.

Risk dominance and payoff dominance

		Player 2	
		H	G
Player 1	H	pq	$p(1-q)$
	G	$(1-p)q$	$(1-p)(1-q)$

Therefore, P(H,H) = pq = (d-c)(D-C)/K and P(G,G) = (1-p)(1-q) = (a-b)(A-B)/K where K = (a-b-c+d)(A-B-C+D)

Risk dominance and payoff dominance

Strategy pair (H, H) payoff dominates (G, G) if $A \geq D$, $a \geq d$, and at least one of the two is a strict inequality: $A > D$ or $a > d$.

Strategy pair (G, G) risk dominates (H, H) if the product of the deviation losses is highest for (G, G) (Harsanyi and Selten, 1988, Lemma 5.4.4). In other words, if the following inequality holds: $(C - D)(c - d) \geq (B - A)(b - a)$. If the inequality is strict then (G, G) strictly risk dominates (H, H).

Dominance

Compare two strategies A and B to see which one is better:

- B is **equivalent** to A: choosing B always gives the same outcome as choosing A, no matter what the other players do.
- B **strictly dominates** A: choosing B always gives a better outcome than choosing A, no matter what the other players do.
- B **weakly dominates** A: choosing B always gives at least as good an outcome as choosing A, no matter what the other players do, and there is at least one set of opponents' action for which B gives a better outcome than A.
- B is **weakly dominated** by A: there is at least one set of opponents' actions for which B gives a worse outcome than A, while all other sets of opponents' actions give A the same payoff as B. (Strategy A **weakly dominates** B).
- B is **strictly dominated** by A: choosing B always gives a worse outcome than choosing A, no matter what the other player(s) do. (Strategy A **strictly dominates** B).

Dominance

This notion can be generalized beyond the comparison of two strategies.

- Strategy B is strictly dominant if strategy B strictly dominates every other possible strategy.
- Strategy B is weakly dominant if strategy B weakly dominates every other possible strategy.
- Strategy B is strictly dominated if some other strategy exists that strictly dominates B.
- Strategy B is weakly dominated if some other strategy exists that weakly dominates B.

Dominance

A player's strategy strictly dominates another one if it gives a higher payoff, no matter what other players play.

The strategy of player i, s_i **strictly** ***dominates*** strategy s_i^* if $u_i(s_i, s_{-i}) > u_i(s_i^*, s_{-i})$ for every profile s_{-i} of opponents' strategies.

Weak Dominance

A player's strategy weakly dominates another one if it is at least as good as other strategies.

The strategy of player i, s_i **weakly** ***dominates*** strategy s_i^* if

$u_i(s_i, s_{-i}) \geq u_i(s_i^*, s_{-i})$ for every profile s_{-i} of opponents' strategies.

AND

$u_i(s_i, s_{-i}) > u_i(s_i^*, s_{-i})$ for some profile s_{-i} of opponents' strategies.

Another Game

		Player j	
		C	D
Player i	A	$1, 1$	$0, 0$
	B	$0, 0$	$0, 0$

Set of strategy profiles $S = (A, C), (A, D), (B, C), (B, D)$

Is there a strictly dominant strategy for any player? Are there Nash Equilibria?

Another Game

		Player j C	Player j D
Player i	A	$1, 1$	$0, 0$
Player i	B	$0, 0$	$0, 0$

Set of strategy profiles $S = (A, C), (A, D), (B, C), (B, D)$

Is there a strictly dominant strategy for any player? A and C Are there Nash Equilibria?

- $(A, C), u(s_i) = 1, u(s_j) = 1$. If i or j change (B,D) they do worse.
- $(A, D), u(s_i) = 0, u(s_j) = 0$. j can do better by picking C
- $(B, C), u(s_i) = 0, u(s_j) = 0$. i can do better by picking A
- $(B, D), u(s_i) = 0, u(s_j) = 0$. Neither can do better in picking another strategy

  (A,C) and (B,D) are Nash Equilibrium, (B,D) is weakly dominated

Basic Game Theory

Rationality & Predicting Behaviour

Rationalizibility

Rationalizability: basic problem: which actions should we expect from a player?

Believe something about the way other players play! Based on this assumption determine a best response for me to play.

I assume everyone is as smart (rational) as me, so their strategy will be a best reponse.

We play a best response to a best response.

Rationalizibility - Justified

Common knowledge of rationality: The assumption is that every player is rational and players (know) assume that every other player is rational.

Rational means that a player never plays a strictly dominated strategy.

A player can remove strictly dominated strategies from their model of the game.

This is iterated until it stops (no more dominated strategies available)

Iterated Elimination of dominated strategies (IEDS)

Player j

		L	C	R
	U	$3, 0$	$2, 1$	$0, 0$
Player i	M	$1, 1$	$1, 1$	$5, 0$
	D	$0, 1$	$4, 2$	$0, 1$

IEDS works in the case of eliminating strictly dominated, or weakly dominated strategies. See sections 1.1 and 2.1 of (Fudenberg and Tirole 1991)

Iterated Elimination of dominated strategies (IEDS)

Player j

		L	C	R
	U	$3, 0$	$2, \mathbf{1}$	$0, \mathbf{0}$
Player i	M	$1, 1$	$1, \mathbf{1}$	$5, \mathbf{0}$
	D	$0, 1$	$4, \mathbf{2}$	$0, \mathbf{1}$

R is strictly dominated by C

Iterated Elimination of dominated strategies (IEDS)

Player j

		L	C
	U	**3**, 0	**2**, 1
Player i	M	**1**, 1	**1**, 1
	D	0, 1	4, 2

M is strictly dominated by U

Iterated Elimination of dominated strategies (IEDS)

		Player j	
		L	C
Player i	U	$3, \mathbf{0}$	$2, \mathbf{1}$
	D	$0, \mathbf{1}$	$4, \mathbf{2}$

L is strictly dominated by C

Iterated Elimination of dominated strategies (IEDS)

		Player j
		C
Player i	U	**2**, 1
	D	**4**, 2

U is strictly dominated by D

Iterated Elimination of dominated strategies (IEDS)

		Player j
		C
Player i	D	$4, 2$

(D, C) is the nash equilibrium of the game.

Iterated Elimination of dominated strategies (IEDS)

Player j

		L	C	R
	U	$3, 1$	$0, \mathbf{1}$	$0, \mathbf{0}$
Player i	M	$1, 1$	$1, \mathbf{1}$	$5, \mathbf{0}$
	D	$0, 1$	$4, \mathbf{1}$	$0, \mathbf{0}$

Again, R is strictly dominated by both L and C

Iterated Elimination of dominated strategies (IEDS)

Player j

		L	C
	U	**3**, 1	**0**, 1
Player i	M	**1**, 1	**1**, 1
	D	**0**, 1	**4**, 1

No pure strategy that dominates, but the mixed strategy of U and D with equal probability ($\frac{1}{2}$) dominates M

Iterated Elimination of dominated strategies (IEDS)

		Player j L	Player j C
Player i	U	$3, 1$	$0, 1$
Player i	D	$0, 1$	$4, 1$

Game cannot be reduced further.

General Prisoners Dilemma

Let's again look at look at a 2-player game. Our prisoners dilemma (pay-offs positive now and in terms of cooperation and defection).

		Player j	
		C	D
Player i	C	3	0
	D	5	1

Prisoner's Dilemma payoff matrix

General Prisoners Dilemma

		Player j	
		C	D
Player i	C	R	S
	D	T	P

Prisoner's Dilemma payoff matrix

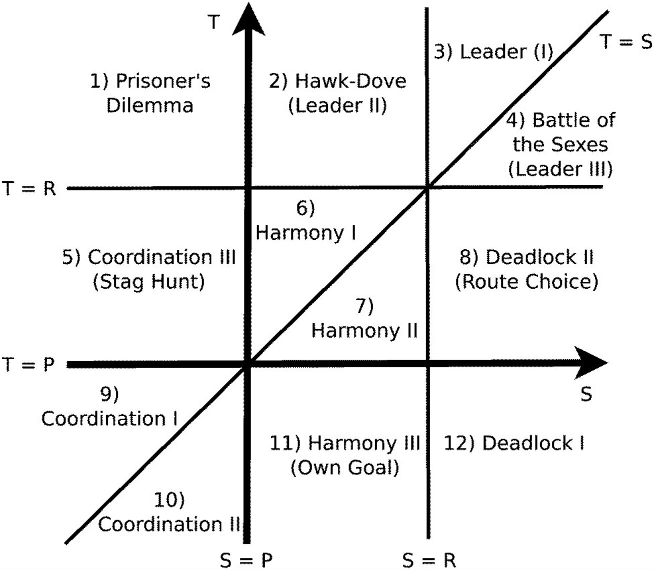
and to be a prisoner's dilemma game in the strong sense, the following condition must hold for the payoffs: T>R>P>S The payoff relationship R>P implies that mutual cooperation is superior to mutual defection, while the payoff relationships T>R and P>S imply that defection is the dominant strategy for both agents.

Prisoners Dilemma?

		Player j C	D
Player i	C	10	1
	D	8	5

Prisoner's Dilemma payoff matrix

Map of Games



T
T = S
3) Leader (I)
1) Prisoner's Dilemma
2) Hawk-Dove (Leader II)
4) Battle of the Sexes (Leader III)
T = R
6) Harmony I
5) Coordination III (Stag Hunt)
8) Deadlock II (Route Choice)
7) Harmony II
T = P
S
9) Coordination I
11) Harmony III (Own Goal)
12) Deadlock I
10) Coordination II
S = P
S = R

Evolutionary & Iterated Game Theory

Evolutionary & Iterated Game Theory

Introduction

Evolutionary Game Theory

Evolutionary game theory originated as an application of the mathematical theory of games to biological contexts, arising from the realization that frequency dependent fitness introduces a strategic aspect to evolution. Now applied by economists, sociologists, anthropologists and social scientists in general
Source: `https://plato.stanford.edu/entries/game-evolutionary/`

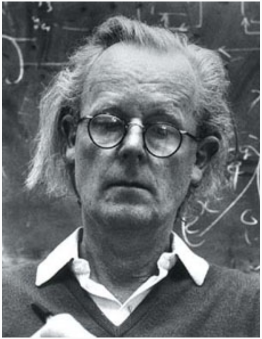


Figure 1: John Maynard Smith (Smith 1974)

Evolutionary Games

We've considered one games between 2 players, and found ways to reason about the strategies of each player.

We can consider games with:

- Populations of players (agents) and strategies.
- Multiple rounds with memory (Repeated Games)
- Adaptive strategies - players learn or change strategy

And also cases where all of these things happen.

Evolutionary Game Theory

Two approaches:

1. ***Evolutionary stable strategies***: a static conceptual analysis of evolutionary stability - are the stable populations?
2. ***Strategy dynamics***: analyse how the frequency of strategies change in the population and the evolutionary dynamics.

Evolutionary Games

From time to time players meet and play using their strategy - call these the agents behaviour. This behaviour can be learned (copied) or inherited.

Utilities now measured as Darwinian fitness, better strategies have a higher chance of reproducing and are therefore more likely to spread. This is Evolutionary Biology - evolutionary game theory has its routes there.

Evolutionary game theory (EGT) focuses on evolutionary dynamics that are frequency dependent. The fitness pay-off for a particular strategy (phenotype) depends on the population composition.

Evolutionary & Iterated Game Theory

Evolutionary Stable Strategies

Hawk and Dove Game

Two agents (animals) are fighting over some resource (payoff) V. Could be two animals fighting over food or a mate.

They can:

- Fight, (s_1) to the death (be a Hawk),
- Flee (s_2) and the fight ends peacefully (be a Dove)

Hawk and Dove Game part II

Winning the conflict leads to pay-off V, losing an escalated fight costs $C > 0$. Possible outcomes:

- Both Hawks - then they fight, ***50/50 win chance***. Both pay cost C
- Both Doves - then they don't fight and split the resource $V/2$
- One Hawk, One Dove - Hawk gets V and Dove gets 0. Hawk beats Dove.

Hawk and Dove Game part III

If player i plays dove? Then j always plays Hawk If player i plays Hawk? Then j?

$\frac{V-C}{2} > 0$: Hawk strictly dominates dove - Hawk, Hawk played

$$
\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{j} \\
 & & H & D \\
\cline{3-4}
\multirow{2}{*}{i} & H & (\frac{V-C}{2}, \frac{V-C}{2}) & (V, 0) \\
\cline{3-4}
 & D & (0, V) & (\frac{V}{2}, \frac{V}{2}) \\
\cline{3-4}
\end{array}
$$

Hawk and Dove Game part III

If player i plays dove? Then j always plays Hawk If player i plays Hawk? Then j?

$\frac{V-C}{2} > 0$: Hawk strictly dominates dove - Hawk, Hawk played
$\frac{V-C}{2} < 0$: ?

$$
\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{j} \\
 & & H & D \\
\cline{3-4}
\multirow{2}{*}{i} & H & \left(\frac{V-C}{2}, \frac{V-C}{2}\right) & (V, 0) \\
\cline{3-4}
 & D & (0, V) & \left(\frac{V}{2}, \frac{V}{2}\right) \\
\cline{3-4}
\end{array}
$$

Hawk and Dove Game part IV

When $\frac{V-C}{2} < 0$

If player i plays D?

Then player j will play H and vice versa.

Pure strategy equilibria. $< D.H >$ and $< H, D >$

Check for ***mixed*** strategy equilibrium.

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{j} \\ & & H & D \\ \hline i & H & (\frac{V-C}{2}, \frac{V-C}{2}) & (V, 0) \\ \hline & D & (0, V) & (\frac{V}{2}, \frac{V}{2}) \\ \hline \end{array}$$

Hawk and Dove Game part V

When $\frac{V-C}{2} < 0$

$E(U_H)$ Expected utility for Hawk for player i?

Depends on mixed strategy of j. Lets say j plays H with frequency p_H.

Player i then plays against:

- H a fraction of p_H times
- D a fraction of $1 - p_H$ times

Then playing H, player i gets and expected utility

$$E(U_H) = \frac{V-C}{2} \times p_H + V \times (1 - p_H)$$

		j: H	j: D
i	H	$(\frac{V-C}{2}, \frac{V-C}{2})$	$(V, 0)$
	D	$(0, V)$	$(\frac{V}{2}, \frac{V}{2})$

Hawk and Dove Game part VI

When $\frac{V-C}{2} < 0$

$E(U_H)$ Expected utility for Dove for player i?

Lets say j plays H with frequency p_H.

Then playing D, player i gets and expected utility

$$E(U_D) = 0 \times p_H + \frac{V}{2} \times (1 - p_H)$$

		j: H	j: D
i	H	$(\frac{V-C}{2}, \frac{V-C}{2})$	$(V, 0)$
	D	$(0, V)$	$(\frac{V}{2}, \frac{V}{2})$

Hawk and Dove Game part VI

When $\frac{V-C}{2} < 0$

When will there be a chance for both strategies to survive? i.e., when there is no advantage between playing D and H

$$E(U_D) = E(U_H)$$

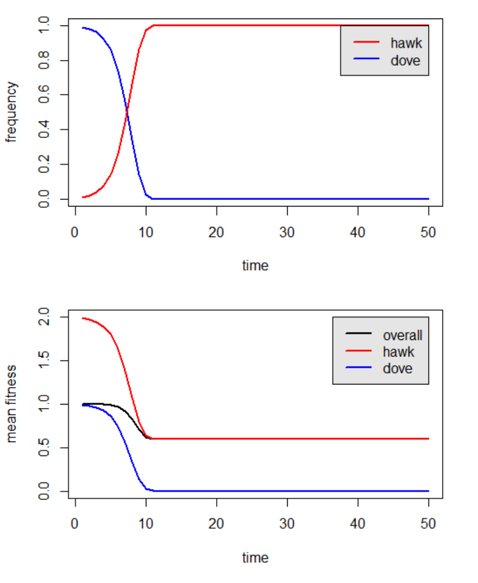
$$\frac{V-C}{2} \times p_H + V \times (1 - p_H) = 0 \times p_H + \frac{V}{2} \times (1 - p_H)$$

$$\frac{V}{2} p_H - \frac{C}{2} p_H + V - V p_H = \frac{V}{2} - \frac{V}{2} p_H$$

$$\frac{V}{2} = \frac{C}{2} p_H$$

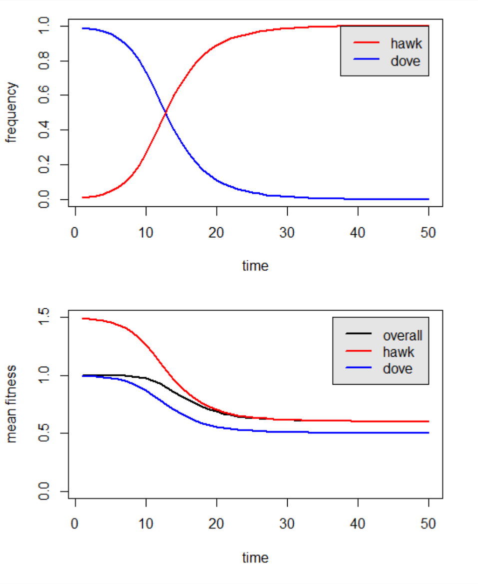
i \ j	H	D
H	$(\frac{V-C}{2}, \frac{V-C}{2})$	$(V, 0)$
D	$(0, V)$	$(\frac{V}{2}, \frac{V}{2})$

Hawk and Dove Game - Evolutionary Dynamics



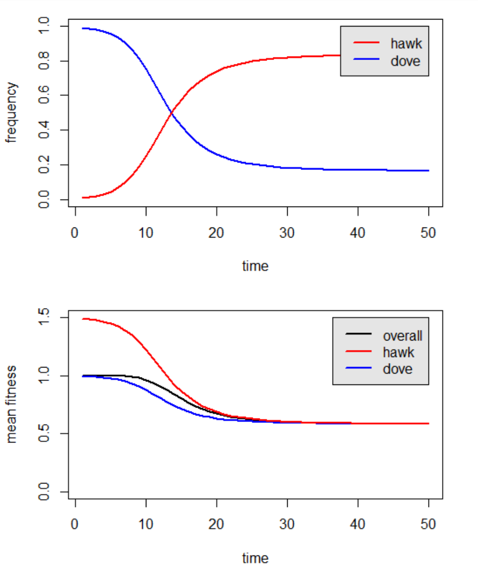
$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{j} \\ & & H & D \\ \hline \multirow{2}{*}{i} & H & 0.6 & 2 \\ \cline{3-4} & D & 0 & 1 \\ \hline \end{array}$$

Hawk and Dove Game - Evolutionary Dynamics



		j	
		H	D
i	H	0.6	1.5
	D	0.5	1

Hawk and Dove Game - Evolutionary Dynamics



		j	
		H	D
i	H	0.4	1.5
	D	0.5	1

Evolutionary & Iterated Game Theory

Replicator Dynamics

We have a population of players they have N possible strategies, or they are N different types.

The game is temporal with discrete time, at each time step (t) we can measure the proportion of each type x_i^t, that is the fraction of type i at time t. A type i is just the agent that plays strategy i.

We say the payoff for a particular type (strategy) i is π_i

Replicator Dynamics

Think about a population of species (types), at any point during the evolutionary process you have a particular mix. This mix will depend on two things, firstly the *frequency* of each type and secondly the *fitness* of each type.

This can be in biology, but also social, in terms of behaviour selection. A player will copy their neighbours strategy if it is doing better.

Replicator Dynamics - Example

Consider three species

Table 1: Species

s_i	S_1	S_2	S_3
x_i^t	0.5	0.4	0.1
π_i	4	5	6

In this situation what happens intuitively with the population in the next time, how does the fraction of each type change?

The Replicator Equation

$$\begin{aligned} max(\pi_i) &: S_3 \text{ - strategy with highest payoff (rational)} \\ max(x_i^t) &: S_1 \text{ - most common strategy, copying (social)} \\ w_i &: \text{? - } w_i = x_i^t \times \pi_i \text{ - a combination of pay-off and frequency} \end{aligned}$$

Question? Why not weight $w_i = x_i^t + \pi_i$

NOTE: payoff should not be fixed, it depends on the rest of population - the games we play.

Replicator Equation

The replicator equation (Taylor and Jonker 1978) defines the rate of change of a particular species within the population.

In general terms the payoff π_i for strategy (type) s_i during a particular round of the game, will depend on how the strategy performs against each other strategy in the game $u(s_i, s_j)$ **AND** the likelihood of running into that strategy - x_j.

$$\pi_i = x_1 u(s_i, s_1) + x_2 u(s_i, s_2) + \ldots + x_n u(s_i, s_n)$$

Replicator Equation

Mean fitness

$$\pi_\mu = x_1\pi_1 + x_2\pi_2 + \ldots + x_n\pi_n$$

$$\pi_\mu = \sum_{i=1}^{n} x_i\pi_i$$

Then we ask the question, how does the frequency of each strategy in the population change over time? Does x_i increase or decrease?

$$\dot{x}_i = x_i \times [\pi_i - \pi_\mu]$$

Replicator Equation

$$\dot{x}_i = x_i \times [\pi_i - \pi_\mu]$$

Replicator Equation

$$\dot{x}_i = x_i \times [\pi_i - \pi_\mu]$$

Rate of change x_i

Replicator Equation

Fraction of strategy i in population

$$\dot{x}_i = x_i \times [\pi_i - \pi_\mu]$$

Rate of change x_i

Replicator Equation

Fraction of strategy i in population

$$\dot{x}_i = x_i \times [\pi_i - \pi_\mu]$$

payoff for strategy i in the population

Rate of change x_i

Replicator Equation

Fraction of strategy i in population

Mean payoff for all strategies in the population

$$\dot{x}_i = x_i \times [\pi_i - \pi_\mu]$$

payoff for strategy i in the population

Rate of change x_i

Evolutionary & Iterated Game Theory

Examples Replicator Equation

The Shakers and Bowers

Replicator dynamics gives us a way to model human decision making/behaviour.

If we assume that strategies or behaviours are more likely to increase if the behaviour is successful (by some measure) and they are popular.

The Shakers and Bowers - part I

Some people like to shake and some like to bow. Shaking has a higher payoff

		Player j S	Player j B
Player i	S	$(2, 2)$	$(0, 0)$
	B	$(0, 0)$	$(1, 1)$

A totally rational player would opt for S. Assume that initially: $x_S = x_B = \frac{1}{2}$

$$\pi_S = \frac{1}{2} \times 2 + \frac{1}{2} \times 0 = 1$$

$$\pi_B = \frac{1}{2} \times 0 + \frac{1}{2} \times 1 = \frac{1}{2}$$

$$\pi_\mu = \frac{1}{2} \times 1 + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$$

The Shakers and Bowers - part II

		Player j	
		S	B
Player i	S	$(2,2)$	$(0,0)$
	B	$(0,0)$	$(1,1)$

$$\dot{x_S} = \frac{1}{2} \times \left[1 - \frac{3}{4}\right] = \frac{1}{8}$$

$$\dot{x_B} = \frac{1}{2} \times \left[\frac{1}{2} - \frac{3}{4}\right] = -\frac{1}{8}$$

So bowers will decrease, and the population will converge to all shakers. The evolutionary stable strategy (ESS) would coincide with the best rational strategy.

The SUV game - part I

In some cases, the dynamics of a population game lead to a situation where we reach a non-optimal equilibrium.

The payoff of driving an SUV vs. a compact car.

		Player j	
		S	C
Player i	S	$(2,2)$	$(2,0)$
	C	$(0,2)$	$(3,3)$

A totally rational all knowing player would opt for C. It's a Nash equilibrium.

$$\pi_S = \frac{1}{2}\times 2 + \frac{1}{2}\times 2 = 2 \qquad \pi_C = \frac{1}{2}\times 0 + \frac{1}{2}\times 3 = \frac{3}{2} \qquad \pi_\mu = \frac{1}{2}\times 2 + \frac{1}{2}\times\frac{3}{2} = 1\frac{3}{4}$$

The SUV game - part II

		Player j: S	Player j: C
Player i	S	$(2,2)$	$(2,0)$
	C	$(0,2)$	$(3,3)$

$$\dot{x_S} = \frac{1}{2} \times \left[2 - 1\frac{3}{4}\right] = \frac{1}{8}$$

$$\dot{x_C} = \frac{1}{2} \times \left[\frac{3}{2} - 1\frac{3}{4}\right] = -\frac{1}{8}$$

So compacts will decrease, and the population will converge to all SUVs. The evolutionary stable strategy (ESS) doesn't coincide with the best rational strategy.

Evolutionary & Iterated Game Theory

Iterated Games

Adaptive Dynamics

So far we assumed players don't play (or remember playing) more than once. Players don't look for payback, they have just changed strategy for a fixed set e.g., Hawk or Dove.

Adaptive strategies

- The set of probabilities for moves is a function of the state of the player
- The state of a player depends on previous games, either of the player himself, or those of others.

In effect the player remembers previous rounds of the game.

Adaptive Dynamics

The core question is always: Why, when faced with an easy quick win at the expense of another, do many people or animals take a lower profit which does not harm the other.

Another way of looking at the problem might be: why do we have such a strong feeling of fairness? Why do we get angry seeing someone cheat another when he should have shared?

What we saw of the prisoners dilemma so far wouldn't explain this.

Iterated Prisoners Dilemma

Let's again look at look at a 2-player game. Our prisoners dilemma (modified pay-offs slightly).

		Player j C	Player j D
Player i	C	3	0
	D	5	1

Prisoner's Dilemma payoff matrix

Now we assume that players play multiple rounds and can 'respond' to strategies of the other players.

Iterated Prisoners Dilemma - Axelrod Competition

in 1980 Robert Axelrod (R. Axelrod and Hamilton 1981) asked for game theorists to propose strategies to play an iterated prisoners dilemma.

Each strategy would play against every other strategy in 200 iterations of the prisoners dilemma, the strategy could look at past history of the game when deciding it's strategy.

Maximum total pay-off would be always defect, with other player always cooperating - $200 \times 5 = 1000$

Iterated Prisoners Dilemma - Axelrod Competition

Let's try...

Some possible strategies?

- Random (50-50) C or D
- Always defect
- Always cooperate
- Cooperate until you defect then always defect (grudger)
- I will cooperate unless you defect 3 times in a row at which point I will defect forever.
- ?

Iterated Prisoners Dilemma - Axelrod Competition

For the original competition there was one winner - a strategy called Tit for Tat. The player simply copied the last option of the previous player.

Axelrod repeated the competition again in 1984, this time people knew the results of the previous game and everything was the same. Only this time the number of iterations was randomly determined.

Again Tit-for-Tat won.

Figure 2: Robert Axelrod

Iterated Prisoners Dilemma - Axelrod Competition

Let's try... Netlogo

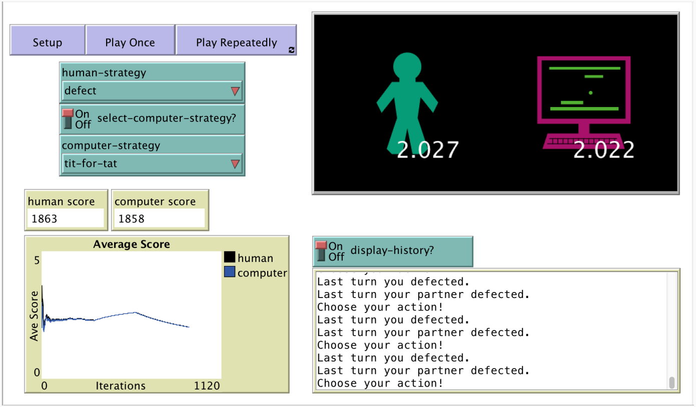


Figure 3: Iterated PD - NetLogo

Iterated Prisoners Dilemma - Axelrod Competition

Netlogo - population of different strategies - let's try.

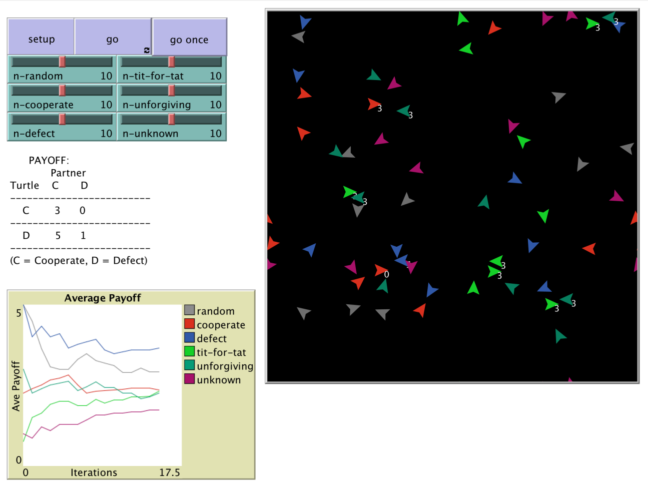


Figure 4: Iterated PD - NetLogo

Evolutionary & Iterated Game Theory

Evolving Strategies in Games

Evolutionary Prisoners Dilemma

Axelrod went on to think (R. Axelrod et al. 1987; R. M. Axelrod 1997) about what would happen if there were generations of agents. Where poorly adapted agents would die off - and better adapted agents succeed. How would the iterated strategies evolve over generations?

Evolutionary Prisoners Dilemma

Now let us consider a game with N by N players on a grid. Each player has a particular strategy for each round of the game. Each player then plays a game with its neighbours. It's total score is the sum of all the games. A cell playing

C, who has 5 neighbours playing D and 3 neighbours playing C will get a score of $5 \times 0 + 3 \times 3$.

In a simple version of the model a player will change its strategy in the next round by adopting the same strategy as the most successful neighbour.

		Player j C	Player j Q
Player i	C	$3, 3$	$0, 5$
	Q	$5, 0$	$1, 1$

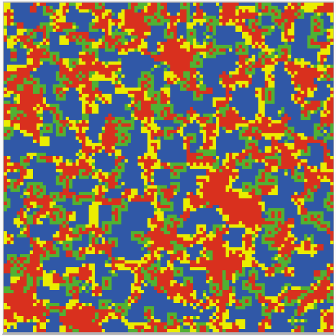


Figure 5: Evolutionary grid - Blue: played C twice, Red: played D twice, Green: new C, Yellow: new D

Evolutionary Game Theory Diagram

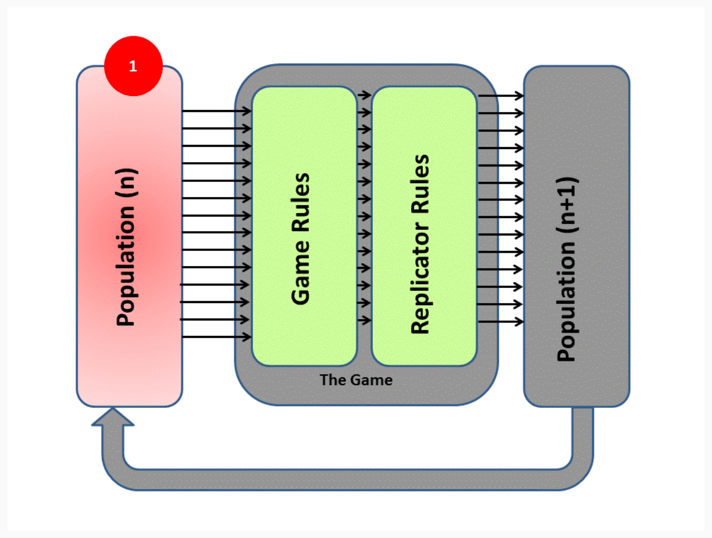


Figure 6: General Flow of Evolutionary Game (Source:Wikipedia)

Evolutionary Prisoners Dilemma

In principle now we have:

- Population based games: Replicator equation, Evolutionary PD
- Adaptive Strategies: Ways players adapt their behaviour.

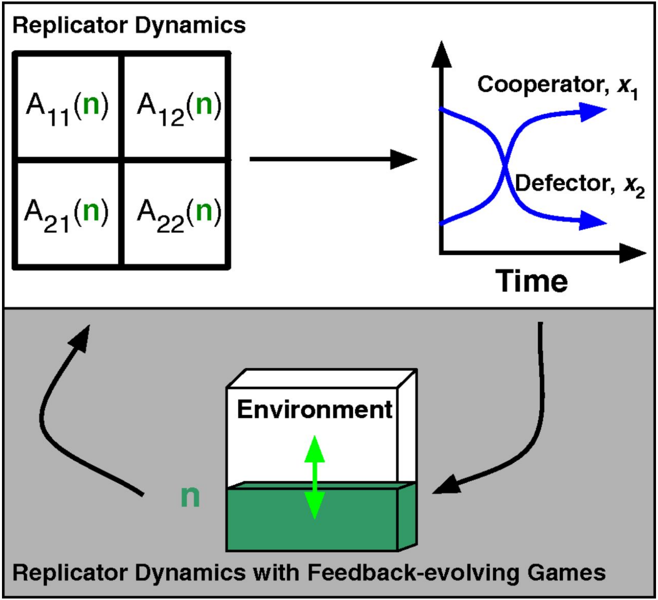
You can combine these things into a model where there is a population and instead of copying the best of my neighbours, an individual can adapt

Ritual fighting fish

John Maynard Smith and George Prince - explain the evolution of ritual fighting fish in animal contents.

Fighting within a species didn't lead to physical attack - just behaviour such as posturing, pushing, etc. Signals of surrender stopped the fight. This is not Darwian.

Tragedy of Commons (Weitz et al. 2016)



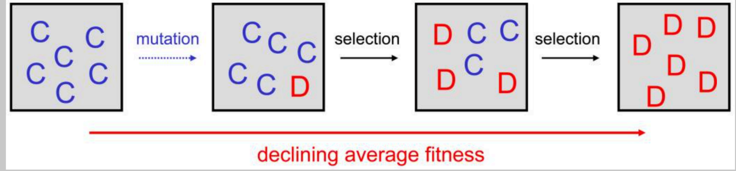
Tragedy of Commons (Weitz et al. 2016)

$$A(n) = (1-n)\begin{bmatrix} T & P \\ R & S \end{bmatrix} + n\begin{bmatrix} R & S \\ T & P \end{bmatrix}.$$

R>S and T >P

$$\epsilon\dot{x} = x(1-x)[r_1(x, A(n)) - r_2(x, A(n))],$$
$$\dot{n} = n(1-n)f(x),$$

Evolution of Cooperation



A cooperator is someone who pays a cost, c, for another individual to receive a benefit, b. A defector has no cost and does not deal out benefits. Cost and benefit are measured in terms of fitness. The total population size is given by N. There are i cooperators and $N - i$. **What is the fitness of cooperators and defectors?**

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