Global Sensitivity Analysis

How to perform comprehensive, efficient, and robust Global Sensitivity Analysis?

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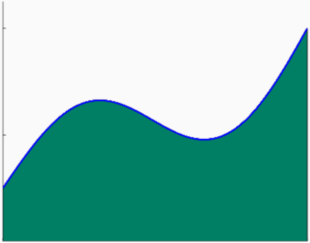
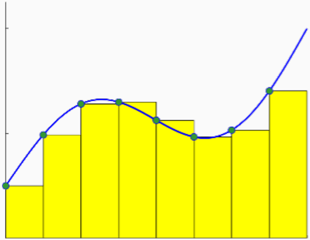
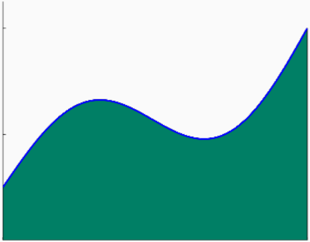
What is the area under the green curve?

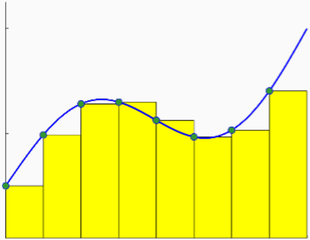




Integration in one dimension

What is the area under the green curve?



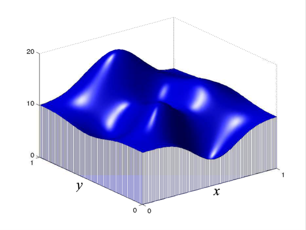


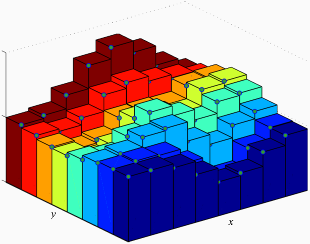
Approximation: Area $\approx$ sum of the areas of the 8 rectangles

We get a better approximation with more rectangles

Integration in two dimension

What is the volume under the green surface?



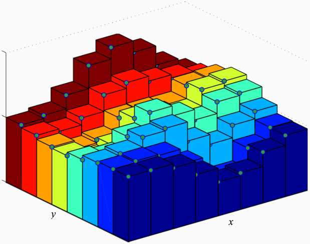


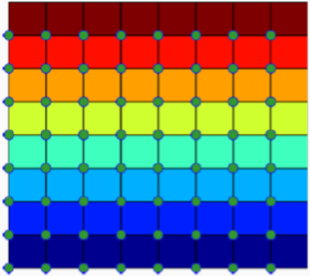
Approximation: Volume $\approx$ sum of the volumes of the $88 = 64$ rectangular prisms

We get a better approximation with more prisms

Integration in two dimension

What is the volume under the green surface?





Approximation: Volume $\approx$ sum of the volumes of the $88 = 64$ rectangular prisms

We get a better approximation with finer grids

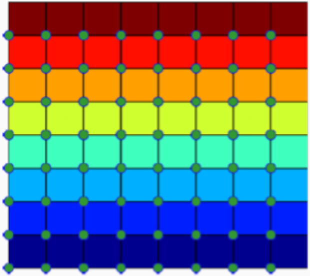
8^2 dots $\rightarrow$ k^2 dots

Integration in s dimension

Integral in high dimensions $\rightarrow$ "hyper volume"

Approximation by product grids:

- k^2
- k^3
- ...
- ...
- k^s points in hyper cube

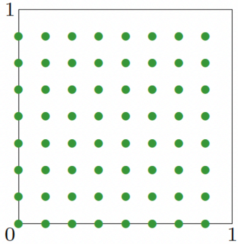


Approximation: $s = 360, k = 2; 2^{360}$ is astronomical (2^{20} one million)

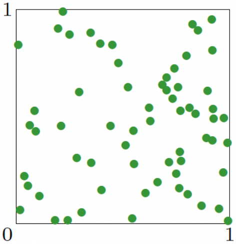
We need to stay away from product grids in high dimensions

Monte carlo methods

Integral ≈ average of function values at random points in the hyper cube.
drawbacks: big gaps, clusters, slow convergence



A product rule with 64 points



64 random points

Low discrepancy sequences

Discrepancy

Given a sequence $x_1, ..., x_N$ in $[0,1)$, its extreme discrepancy is:

$$D_N(P) = \sup_{B \in J} \left| \frac{A(B;P)}{N} - \lambda_s(B) \right|$$

$$\prod_{i=1}^{s} [a_i, b_i) = \{\mathbf{x} \in \mathbf{R}^s : a_i \leq x_i \leq b_i\}$$

where $0 \leq a_i \leq b_i \leq 1$

The star-discrepancy $D_N^*(P)$ is defined similarly, except that the supremum is taken over the set J^* of rectangular boxes of the form:

$$\prod_{i=1}^{s} [0, u_i)$$

where u_i is in the half-open interval $[0,1)$

Low Discrepancy Sequences

Why are these sequences important in applications?

Theorem

A sequence $x_1, ..., x_N$ in I^s is u.d. mod 1 iff

$$\frac{1}{N}\sum_{i=1}^{N} f(x_i) = \int_{\bar{I}^s} f(u)\, du$$

for all Riemann integrable functions f on I^s

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Low Discrepancy Sequences

Why are these sequences important in applications?

Theorem

$D_N(x_n) \to 0$ iff x_n is a u.d. mod 1 sequence.

Remark

Sequences with best known bounds for star-discrepancy satisfy $O((logN)^s/N)$. (The term low-discrepancy sequence is used for these sequences.). This is important because of the Koksma-Hlawka inequality.

Koksma-Hlawka inequality

$$\left| \frac{1}{N} \sum_{i=1}^{N} f(x_i) - \int_{\bar{I}^s} f(u)\, du \right| \leq V(f)\, D_N^*(x_1, \ldots, x_N).$$

It tells us about the asymptotic behaviour of the error $O((logN)^s/N)$.

Quasi Monte-carlo sequence

(t,m, s) nets

Definition

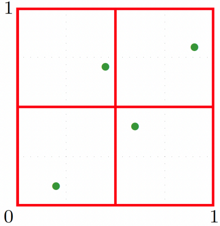
For $0 \leq t \leq m$, a finite sequence of b^m points in $[0, 1)^s$ is a $(t, m, s) - \textit{net}$ in base b if every elementary interval in base b of volume b^{t-m} contains exactly b^t points of the sequence.

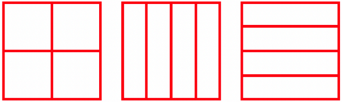
(t,m, s) nets

It is all about having the right number of points in various subdivisions.

2D Example:

we want to place 4 points in the unit square so that there is exactly one point in each of the 4 rectangles of the same shape and size,given by the three possible subdivisions:





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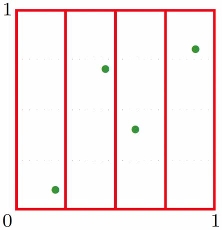

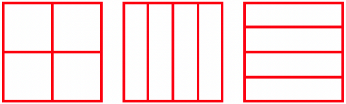

(t,m, s) nets

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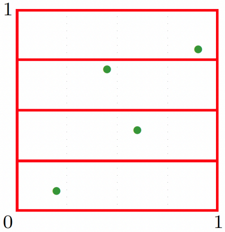
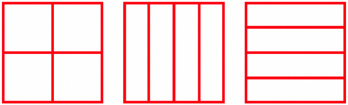
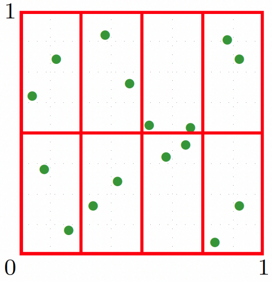


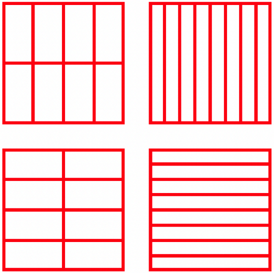


(t,m, s) nets

2D Exercise:

draw a "$(1, 4, 2) - net$ in base 2", that is, we want to place 16 points in the unit square so that there are exactly 2 points in each of the 8 rectangles of the same shape and size, given by the fourpossible subdivisions:





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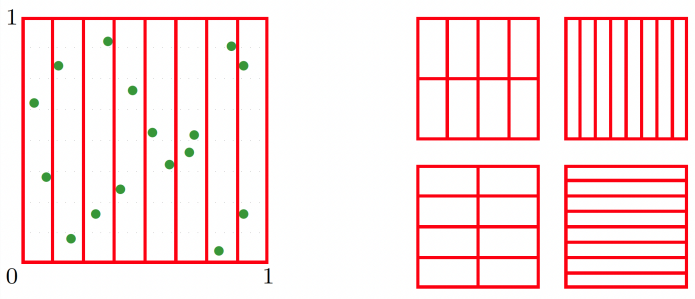

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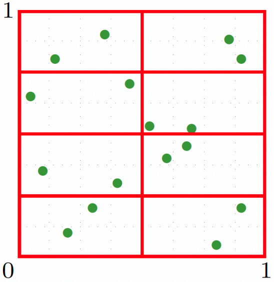

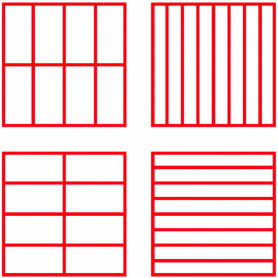

(t,m, s) nets

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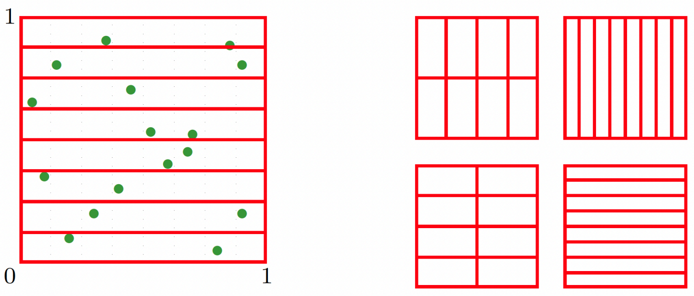


(t, s) nets

Definition: An infinite sequence of points $q_1, q_2, ...$ is called a (t, s)-sequence in base b if the finite sequence $q_{kb^m+1}, ..., q_{(k+1)b^m}$ is a $(t, m, s) - net$ in base b for all $k \geq 0$ and $m \geq t$.

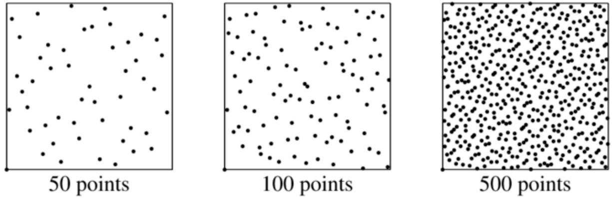
Sobol sequences

The most popular QMC approach uses Sobol sequencesx(x^i) which have the property that for small dimensions $d < 40$ the subsequence:

$$2^m \leq i < 2^{m+1}$$

of length 2^m has precisely 2^{md} points in each of the little cubes of volume 2^d formed by bisecting the unit hypercubein each dimension, and similar properties hold with otherpieces.

Sobol sequences



We can reuse points compared to Monte Carlo

Sobol sequences

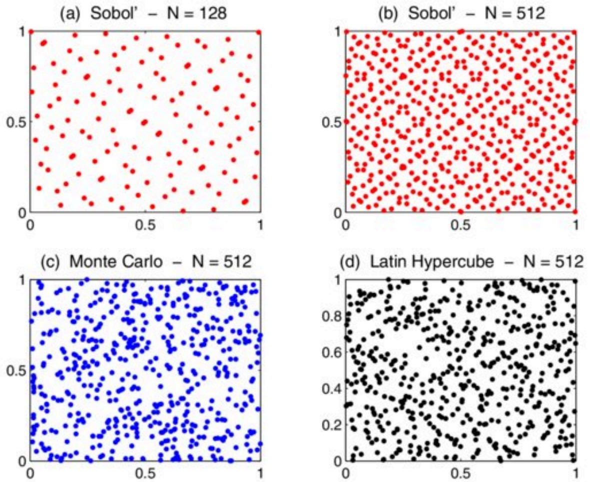


Fig. 2. Space-filling process of $[0, 1]^2$ using a two-dimensional Sobol' sequence, compared to MCS and LH

Questions?