An Introduction to Discrete Choice Theory (for ABM)

Discrete choice for ABM

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Introduction

Why Discrete Choice?

Discrete Choice Theory/Modelling tries to model the way individuals (or collectives) make a choice from a selection (two or more) of ***discrete*** alternatives.

Applied in:

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Applied in:

economics : What car to buy?

transportation : Which mode of transport to take?

business : What people buy?

Probabilities?

Who owns an electric vehicle, how will that change in the future?
Assumption that education level is important

Sample of 600 people randomly selected. Asked if they own an EV and how many years of education they have.

Table 1: Electric Vehicle Ownership

EV	**low**	**med**	**high**	
Y	10	100	120	230
N	140	200	30	370
	150	300	150	600

Probabilities?

Dependent variable:

$$y = \begin{cases} 1 & \text{owner} \\ 2 & \text{not owner} \end{cases}$$

Independent variable (education level):

$$x = \begin{cases} 1 & \text{low} \\ 2 & \text{med} \\ 3 & \text{high} \end{cases}$$

Probabilities?

We have a sample and use the sample population to estimate population.

$$\hat{p}(y = 1) = p(y = 1)$$

Joint probabilities:

$$\hat{p}(y = 1, x = 2) = \frac{100}{600} = 0.1667$$

Marginal probabilities:

$$\hat{p}(y = 1) = \sum_{k=1}^{3} \hat{p}(1, k) = \frac{10}{600} + \frac{100}{600} + \frac{120}{600} = 0.383$$

Probabilities

Conditional probabilities:

$$\begin{aligned}
\hat{p}(y = 1|x = 2) &= \hat{p}(y = 1, x = 2)/\hat{p}(x = 2) \\
&= 0.1667/0.5 = 0.333 \\
\hat{p}(y = 1|x = 1) &= 0.067 \\
\hat{p}(y = 1|x = 3) &= 0.8
\end{aligned}$$

This is a simple model to predict y given x. Note this assumes things don't change.

Probability Model

Conditional probabilities:

$$\hat{p}(y = 1|x = 1) = \pi_1 = 0.067$$
$$\hat{p}(y = 1|x = 2) = \pi_2 = 0.333$$
$$\hat{p}(y = 1|x = 3) = \pi_3 = 0.8$$

Future level of education changes (0.1, 0.6, 0.3)

$$\begin{aligned} p(y = 1) &= \sum_i p(y = 1|x = i)p(x = i) \\ &= 0.1\pi_1 + 0.6\pi_2 + 0.3\pi_3 \\ &= 0.4467 \end{aligned}$$

So if education level changes from (0.25,0.5,0.25) to (0.1,0.6,0.3) then ownership increases from 38.33% to 44.67%

Choice Modelling

Ben-Akiva and Lerman (Ben-Akiva and Lerman 1985) define a choice as a sequence of steps:

- Definition of the choice problem
- Generation of Alternatives
- Evaluation of attributes of alternatives
- Choice/Selection
- Implementation

Choice set is finite, selection is made based on *attributes* of choices. *Decision rule* defines how the choice is made from attributes.

Discrete Choice Modelling

There is a decision maker (agent) and a ***choice set*** of discrete options. There are other theories that deal with continuous choice options, but the constraints on the choice set discrete choice are:

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There is a decision maker (agent) and a ***choice set*** of discrete options. There are other theories that deal with continuous choice options, but the constraints on the choice set discrete choice are:

- Choices are mutually exclusive (you cannot choose two options together)
- The Choice Set must be exhaustive - for your model to work
- Number of alternatives must be finite

Note that there is freedom in designing the choice set and that will greatly impact the derived model. If two options A and B are similar they can be grouped.

Choice Modelling

Attributes

Each alternative in the choice set has a number of ***attributes*** that determine the decision. (e.g., price, travel time, etc.)

Decision Rule

A way to weigh up the alternatives and decide one option:

- Dominance
- Satisfaction
- Utility
- etc.

Utility Maximisation

Assumption

A decision maker evaluates each alternative using a utility function, then chooses the option that has the maximal utility.

$$\text{if } U(\mathit{car}) > U(\mathit{bus}) \text{ then } \mathit{car}$$

Assume utility is a linear combination of the parameters

$$U(\mathit{car}) = \beta_1(\mathit{cost}) + \beta_2(\mathit{traveltime}) + \beta(\mathit{parkingcost}) + \dots$$

Parameters represent tastes - which might vary per person (agent), or by type of agent. These include socio-economic characteristics (e.g., age, income, gender)

Discrete Choice Theory

Table 2: Choices for travelling to work

Alternatives	time (t)	cost (p)	comfort (o)
car	t_c	c_c	o_c
bus	t_b	c_b	o_b
walk	t_w	c_w	o_w

$$U_i = U(t_i, c_i, o_i)$$

Only one alternative is possible, and we pick i such that $U_i > U_k \forall k$

Discrete Choice Theory - Utility Function

More generally Assume additive utility function of the form:

$$U_i = -\beta_t t_i - \beta_c c_i + \beta_o o_i$$

Generally, a decision maker $\underline{n}$ makes a choice among J alternatives. The decision maker picks alternative $i \in J$ where the utility is maximised:

$$U_{ni} > U_{nj} \quad \forall j \in J \quad j \neq i$$

Random Utility

Estimating the parameters

In discrete choice theory typically the ***researcher/modeller*** is trying to estimate which attributes are most important to the decision maker in the decision process.

Observations

The researcher observes some attributes of the alternatives x_{nj}, and some attributes of the decision maker labelled s_n. A function is then defined that relates these attributes to the utility of alternative j

$$V_{nj} = V(x_{nj}, s_n)$$

This is the ***representative utility***

Random Utility

Missing information

The researcher will use data to estimate things from a sample of the population (data set), there will be some attributes that the researcher cannot measure. The random utility model explicitly adds these unobserved (or error) terms as ϵ_{nj}. These are the terms that impact the utility but are not captured by V_{nj}

More generally the sources of uncertainty might be:

- Unobserved attributes
- Unobserved taste variations
- Measurement error

Random Utility Model

Assumes that instead of utility being a real value, it instead defined by a function and is subject to noise/uncertainty.

Dependent variable is latent and the expected utility of choice j for decision maker n is the defined as:

$$U_{nj} = V_{nj} + \epsilon_{nj}$$

Where ϵ_{nj} captures all the factors that impact utility but are not included in V_{nj} – the unknowns.

Random Utility Model

ϵ_{nj} is just defined as the difference between V_{nj} and U_{nj}. Therefore the characteristics (e.g., the distribution) of ϵ_{nj} depend on the researchers specification of V_{nj}.

$\epsilon_{nj} \forall j$ is not known and therefore treated as random. The joint density of the random vector is denoted as $f(\epsilon_n)$, where:

$$\epsilon_n = \langle \epsilon_{n1}, \ldots \epsilon_{nJ} \rangle$$

Probability of Choice

The the probability that decision maker n chooses alternative i

$$\begin{aligned} P_{ni} &= Prob(U_{ni} > U_{nj} \quad \forall j \neq i) \\ &= Prob(V_{ni} + \epsilon_{ni} > V_{nj} + \epsilon_{nj} \forall j \neq i) \\ &= Prob(\epsilon_{nj} - \epsilon_{ni} < V_{ni} - V_{nj} \forall j \neq i) \end{aligned}$$

Probability of choice

$$
\begin{aligned}
P_{ni} &= Prob(\epsilon_{nj} - \epsilon_{ni} > V_{ni} - V_{nj} \quad \forall j \neq i) \\
&= \int_{\epsilon} I(\epsilon_{nj} - \epsilon_{ni} < V_{ni} - V_{nj} \quad \forall j \neq i)\, f(\epsilon_n)\, \delta\epsilon_n
\end{aligned}
$$

$I(x)$ is the indicator function which is 1 if the predicate x is true, otherwise 0.

This is a multidimensional integral over the density of the unobserved portion of utility. Different discrete choice models are obtained from different assumptions about the distribution of the unobserved portion of utility

Probability of choice

$$P_{ni} = \int_{\epsilon} I(\epsilon_{nj} - \epsilon_{ni} < V_{ni} - V_{nj} \quad \forall j \neq i)\, f(\epsilon_n)\, \delta\epsilon_n$$

The integral takes a closed form only for certain $f(\epsilon_n)$.

Different forms of choice model work by assuming that the unobserved portion of the utility follows different distributions:

Logit: Extreme value distribution (iid)

Nested Logit: Generalised extreme value

Mixed Logit: Extreme value distribution (iid) and something else

Probit: Multi-variate normal

Only nested logit and logit have closed form solutions for the integral - the others are estimated numerically.

Meaning of choice - Example

$$V_c = \alpha T_c + \beta M_c$$
$$V_b = \alpha T_b + \beta M_b$$

Where T and M represent the time and money for each mode of transport car (c) and bus (b). α and β are estimated by the modeller.

Suppose, $V_c = 3$ and $V_b = 4$, based on the observed factors bus is the better option. The final choice depends on the unobserved factors - if they are sufficiently better for bus i.e., $\epsilon_c - \epsilon_b < 1$ then $U_b > U_c$.

Linear probability model

Types of Choice Model

The models can be classified according to the number of alternatives:

- Binary (Binomial) choice models (2 alternatives)
- Multinomial choice models (>2 alternatives)
  - Assume no correlation in unobserved factors
  - Allow correlation in unobserved factors

We will only look at the binary choice in lectures - but you can look in detail at other options.

Linear probability model

$$y = \beta_0 + \beta_1 x + \epsilon; i.i.d(0, \sigma^2)$$

$$y = \begin{cases} 0 & \text{not} \\ 1 & \text{college} \end{cases}$$

$$E(y|x) = \sum_i P(y = y_i|x)y_i = P(y = 1|x)$$

$$E(y|x) = \beta_0 + \beta_1 x = P(y = 1|x)$$

Problems with LPM

Suppose: You want to predict if an individual will go to college based on their parental wage. You collect data and come up with a linear probability model

$$y = 0.3 + 0.2 lp_{wage} + \epsilon; i.i.d(0, \sigma^2)$$

$$P(y = 1|x) = 0.3 + 0.2 lp_{wage}$$

when $lp_{wage} = -5$, then $P(y = 1|x) = -0.7$
when $lp_{wage} = 10$, then $P(y = 1|x) = 2.3$

But probabilities $\in [0, 1]$

Problems with LPM

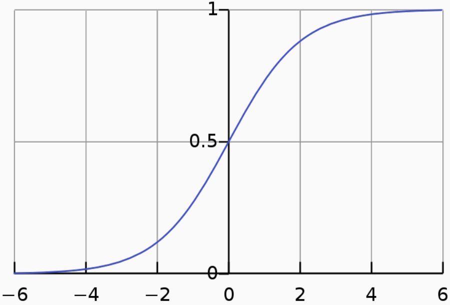
Solution:

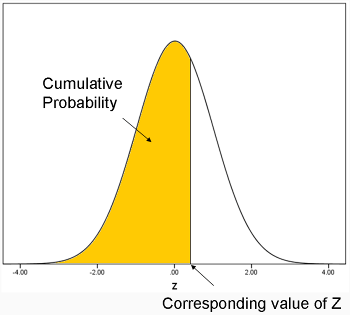
$$P(y = 1|x) = F(0.3 + 0.2lp_{wage}) \in [0, 1]$$

where $F(-\infty) = 0$, and $F(+\infty) = 1$
Can you think of reasonable $F()$?

Problems with LPM

Solution:





Taste variation (endogeneity)

Unobserved heterogeneity is typically about unobservable componenents of the effects that you are estimating. Unobserved heterogeneity might be that some people have higher returns (e.g., increases in wages) from going to college than others. Let the returns for person i be $\beta + b_i$ with $\mathbb{E}(b_i) = 0$. We have

$$y_i = x_i(\beta + b_i) + w_i'\gamma + \epsilon_i,$$

where y_i is (typically, log) income, x_i is years of education, and w_i is a set of other controls. An example of endogeneity is when x_i is correlated with ϵ_i (e.g., education is correlated with IQ, which is not among our other predictors).

If we estimate a single coefficient, we have

$$y_i = x_i\beta + w_i'\gamma + (\epsilon_i + bx_i) = x_i\beta + w_i'\gamma + \tilde{\epsilon}_i$$

See that the included variable x_i is correlated with the error term $\tilde{\epsilon}_i$,

Logit Model

Logit Model

The logit model is perhaps the most widely used discrete choice model.

It assumes the ϵ_{ni} is iid extreme value for all i.

Independent and Identically Distributed (iid)
For each choice i, the unobserved components ϵ_{ni} are assumed to follow the same distribution and are mutually independent.

The critical part of the assumption is that the unobserved factors are uncorrelated over alternatives, as well as having the same variance for all alternatives.

Logit Model

Is this assumption restrictive?

Recall that ϵ_{nj} is the difference between the real utility U_{nj} and the observed utility V_{nj} - as specified by the modeller.

The independence assumption, the error of alternative *i* provides no information about the error of *j*. OR...

The modeller specifies V_{nj} sufficiently well that the unobserved part ϵ_{nj} is essentially white noise.

IID - when does this not hold

One example of when this fails:

We have a bag containing two coins, with different probability of heads p_1 and p_2. Pick a coin and toss is N times, X_i in indicator function of heads on the i^{th} toss.

$$\begin{aligned} P\{X_i = 1\} =& \frac{1}{2} P\{X_i = 1 | coin_1\} + \frac{1}{2} P\{X_i = 1 | coin_2\} \\ =& \frac{p_1 + p_2}{2} \end{aligned}$$

$$\begin{aligned} P\{X_i = 1, X_j = 1\} =& \frac{P\{X_i = 1, X_j = 1 | coin_1\} + P\{X_i = 1, X_j = 1 | coin_2\}}{2} \\ =& \frac{p_1^2 + p_2^2}{2} \end{aligned}$$

$$P\{X_i = 1, X_j = 1\} \neq P\{X_i = 1\} P\{X_j = 1\}$$

They are not independent!

Homework: Under what condition do we get independence? Can you think of an intuitive explanation?

Logit Model

The reason for the Logit Model being so popular is simply one of convenience. This is at the cost of the assumption - unobserved factors could be related between choice.

E.g., People who don't chose bus because of overcrowding, might have the same reaction to rail travel.

Logit Model

$$U_{nj} = V_{nj} + \epsilon_{nj}$$

Logit model assumes each ϵ_{nj} identically distributed extreme value. (Gumbel, Type I and extreme value). The extreme value distribution gives slightly fatter tails than a normal, which means it can include outliers. Usually, however, the difference between extreme value and independent normal errors is indistinguishable empirically.

Standard (maximum) Gumbel distribution is $\mu = 0$ and $\beta = 1$

$$f(x) = e^{-x} e^{-e^{-x}}$$

See:
https://www.itl.nist.gov/div898/handbook/eda/section3/eda366g.htm

Logit Model

Then the density for each unobserved component of utility is:

$$f(\epsilon_{nj}) = e^{-\epsilon_{nj}} e^{-e^{-\epsilon_{nj}}}$$

Cumulative

$$F(\epsilon_{nj}) = e^{-e^{-\epsilon_{nj}}}$$

Logit Model

Recall that the difference between the two random terms is crucial:

In the case of gumbel ϵ_{nji}

$$\epsilon_{nji} = \epsilon_{nj} - \epsilon_{ni}$$

Actually follows a logistic distribution.

$$F(\epsilon_{nji}) = \frac{e^{\epsilon_{nji}}}{1 + e^{\epsilon_{nji}}}$$

Derive Logit Model

$$P_{ni} = Prob(V_{ni} + \epsilon_{ni} > \epsilon_{nj} + V_{nj} \quad \forall j \neq i)$$
$$= Prob(\epsilon_{nj} < \epsilon_{ni} + V_{ni} - V_{nj} \quad \forall j \neq i)$$

if ϵ_{ni} is considered given, the above expresion is the cumulative distribution for ϵ_{nj} evaluated at $\epsilon_{ni} + V_{ni} - V_{nj}$. This cumulative distrbution over all $j \neq i$ is the product of all individual cumulative distributions:

$$P_{ni}|\epsilon_{ni} = \prod_{j \neq i} e^{-e^{(\epsilon_{ni} + V_{ni} - V_{nj})}}$$

Derive Logit Model

ϵ_{ni} isn't given, so choice probability is integral of $P_{ni}|\epsilon_{ni}$ over all values of ϵ_{ni} weighted by density.

$$P_{ni} = \int \left(\prod_{j \neq i} e^{-e^{(\epsilon_{ni} + V_{ni} + V_{nj})}} \right) e^{-\epsilon_{ni}} e^{-e^{-\epsilon_{ni}}}$$

The integral leads to a closed form expression - which is why we assume the Gumbel.

$$P_{ni} = \frac{e^{V_{ni}}}{\sum_j e^{V_{nj}}}$$

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$$P_{ni} = \frac{e^{V_{ni}}}{\sum_j e^{V_{nj}}}$$

Derive Logit Model

From the ij-response set $\mathcal{R}^{J_i}$ the probability that player $i \in N$ will choose action j is given by:

$$\sigma_{ij}(\bar{u}_i) = P\big[\hat{u}_{ij} \geq \hat{u}_{ik} \ \ \forall k = 1, ..., J_i \ k \neq j\big]$$

$$= P\big[\bar{u}_{ij} + \mathcal{E}_{ij} \geq \bar{u}_{ik} + \mathcal{E}_{ik} \ \ \forall k = 1, ..., J_i \ k \neq j\big]$$

$$= P[\mathcal{E}_{ik} \leq \bar{u}_{ij} - \bar{u}_{ik} + \mathcal{E}_{ij} \ \ \forall k = 1, ..., J_i \ k \neq j]$$

$$= P\big[\mathcal{E}_{i1} \leq \bar{u}_{ij} - \bar{u}_{i1} + \mathcal{E}_{ij}, ..., \mathcal{E}_{i(j-1)} \leq \bar{u}_{ij} - \bar{u}_{i(j-1)} + \mathcal{E}_{ij},$$

$$\mathcal{E}_{i(j+1)} \leq \bar{u}_{ij} - \bar{u}_{i(j+1)} + \mathcal{E}_{ij}, ..., \mathcal{E}_{iJ_i} \leq \bar{u}_{ij} - \bar{u}_{iJ_i} + \mathcal{E}_{ij}\big]$$

$$= \int_{-\infty}^{+\infty} \Bigg[\prod_{\substack{k=1 \\ k\neq j}}^{J_i} P\big[\mathcal{E}_{ik} \leq \bar{u}_{ij} - \bar{u}_{ik} + \varepsilon_{ij}\big]\Bigg] f_i(\varepsilon_{ij}) \ d\varepsilon_{ij}$$

$$\textit{with pdf } f_i(\varepsilon_{ij}) = \frac{dF_i(\varepsilon_{ij})}{d\varepsilon_{ij}} = \lambda \ \exp(-\exp(-\lambda\varepsilon_{ij} - \gamma)) \ \exp(-\lambda\varepsilon_{ij} - \gamma), \ \textit{so we have}$$

Derive Logit Model

$$\sigma_{ij}(\bar{u}_i) = \int_{-\infty}^{+\infty} \left[\prod_{\substack{k=1\\k\neq j}}^{J_i} \exp(-\exp(-\lambda(\bar{u}_{ij}-\bar{u}_{ik}+\varepsilon_{ij})-\gamma)) \right] \lambda \exp(-\exp(-\lambda\varepsilon_{ij}-\gamma)) \exp(-\lambda\varepsilon_{ij}-\gamma)\, d\varepsilon_{ij}$$

Let $z = \exp(-\lambda\varepsilon_{ij} - \gamma)$ *, such that the above equation becomes*

$$\sigma_{ij}(\bar{u}_i) = \int_{0}^{+\infty} \left[\prod_{\substack{k=1\\k\neq j}}^{J_i} \exp(-\exp(-\lambda(\bar{u}_{ij}-\bar{u}_{ik}))z) \right] \exp(-z)\, dz$$

$$= \int_{0}^{+\infty} \exp\left(-z\left\{ \sum_{\substack{k=1\\k\neq j}}^{J_i} \exp(-\lambda(\bar{u}_{ij}-\bar{u}_{ik})) \right\}\right) \exp(-z)\, dz$$

$$= \int_{0}^{+\infty} \exp\left(-z\left\{ 1+\sum_{\substack{k=1\\k\neq j}}^{J_i} \exp(-\lambda(\bar{u}_{ij}-\bar{u}_{ik})) \right\}\right) dz$$

$$= -\frac{1}{1+\sum_{\substack{k=1\\k\neq j}}^{J_i} \exp(-\lambda(\bar{u}_{ij}-\bar{u}_{ik}))} \exp\left(-z\left\{ 1+\sum_{\substack{k=1\\k\neq j}}^{J_i} \exp(-\lambda(\bar{u}_{ij}-\bar{u}_{ik})) \right\}\right)\Bigg|_0^{+\infty}$$

$$= 0 - \left(-\frac{1}{1+\sum_{\substack{k=1\\k\neq j}}^{J_i} \exp(-\lambda(\bar{u}_{ij}-\bar{u}_{ik}))} \right) = \frac{1}{1+\frac{1}{\exp(\lambda\bar{u}_{ij})}\sum_{\substack{k=1\\k\neq j}}^{J_i} \exp(\lambda\bar{u}_{ik})}$$

$$= \frac{\exp(\lambda\bar{u}_{ij})}{\sum_{k=1}^{J_i} \exp(\lambda\bar{u}_{ik})}$$

Limitations of Logit Model

1. Taste Variation (possible but limited)
2. Independence of Irrelevant Alternatives (IIA)
3. Assumes repeated choice has no correlation in observed factors

Taste Variation

The value or importance that decision makers place on each attribute of the alternatives varies, in general, over decision makers. Logit can do taste variation that varies systematically with respect to observed variables - but not random variation.

Independence of Irrelevant Alternatives

If a choice problem is changed to include a new alternative, we need to know how that alternative will impact existing ratios.

If A is preferred to B out of the choice set $\{A, B\}$, introducing a third option X, expanding the choice set to $\{A, B, X\}$, must not make B preferable to A.

For Logit:

$$\begin{aligned}\frac{P_{ni}}{P_{nk}} &= \frac{e^{V_{ni}} / \sum_j e^{V_{nj}}}{e^{V_{nk}} / \sum_j e^{V_{nj}}} \\ &= \frac{e^{V_{ni}}}{e^{V_{nk}}} = e^{V_{ni} - V_{nk}}\end{aligned}$$

The ratio only depends on i and k nothing else. The relative odds of choosing i over k is the same no matter what other alternatives are.

Independence of Irrelevant Alternatives - example

The red-bus, blue-bus example.

Assume you have two choices, ***blue bus*** (bb) or ***car*** (c) which are equally likely.

$$P_{nc} = P_{nbb} = \frac{1}{2}$$

In this case the ratio of the probabilities is:

$$\frac{P_c}{P_{bb}} = 1$$

Independence of Irrelevant Alternatives - example

Now assume that a red bus, which in terms of attributes and preference is exactly the same for the decision maker. So

$$\frac{P_{rb}}{P_{bb}} = 1$$

And due to the ***IIA assumption***

$$\frac{P_c}{P_{rb}} = 1$$

Therefore,

$$P_{nc} = P_{nbb} = P_{nrb} = \frac{1}{3}$$

A more reasonable assumption would be the probability of taking any bus is equal to taking the car, then which bus is irrelevant.

$$P_{nbb} = P_{nrb} = \frac{1}{4} \quad P_{nc} = \frac{1}{2}$$

Proportional Substitution

If we change the attribute of a particular alternative i so that i becomes more attractive. The probability increase in i results in a reduction in other choices $j \neq i$.

The logit model assumes that this reduction happens equally across all options. Changing i will impact the probability of k and l equally:

$$\frac{P_{nk}^1}{P_{nl}^1} = \frac{P_{nk}^0}{P_{nl}^0}$$

The ratio of the probabilities for k and l is the same after the change in j (1) as it was before the change

Generalised Extreme Value

Generalised Extreme Value

In generalized extreme value (GEV) models the unobserved portion of the utility for all alternatives are jointly distributed as a generalised extreme value (not Type I).

This cumulative joint distributions allows for correlations over alternatives within nests.

$$\exp\left(- \sum_{k=1}^{K} \left(\sum_{j \in B_k} e^{-\varepsilon_{nj}/\lambda_k} \right)^{\lambda_k} \right)$$

This distribution for the unobserved ccomponents of utility gives rise to the following choice probability for alternative $i \in B_k$

$$P_{ni} = \frac{e^{V_{ni}/\lambda_k} \left(\sum_{j \in B_k} e^{V_{nj}/\lambda_k} \right)^{\lambda_k - 1}}{\sum_{\ell=1}^{K} \left(\sum_{j \in B_\ell} e^{V_{nj}/\lambda_\ell} \right)^{\lambda_\ell}}$$

Nested Logit Model

Nested Logit is most common GEV.

Nested Logit works by dividing the choice set into subsets called ***nests***. The nests are chosen such that:

- Choices within a nest are assumed to follow IIA.
- For choices in two different nests IIA does not hold

$$\frac{P_{ni}}{P_{nm}} = \frac{e^{V_{ni}/\lambda_k}\left(\sum_{j\in B_k} e^{V_{nj}/\lambda_k}\right)^{\lambda_k-1}}{e^{V_{nm}/\lambda_\ell}\left(\sum_{j\in B_\ell} e^{V_{nj}/\lambda_\ell}\right)^{\lambda_\ell-1}}$$

Nested Logit Model - an example

Consider a choice between car alone, **car pool, bus or rail.**

Assume that the original choice is

$$P_{nc} = 0.4 \quad P_{np} = 0.1 \quad P_{nb} = 0.3 \quad P_{nr} = 0.2$$

Then we examine the change in choice as we remove one option.

Table 3: Change in probabilities when one choice removed

Choice	original	car	pool	Bus	Rail
Car	.40	–	.45	.52	.48
Pool	.10	.20	–	.13	.12
Bus	.30	.48	.33	–	.40
Rail	.20	.32	.22	.35	–

Nested Logit Model - an example

Table 4: Change in probabilities when one choice removed

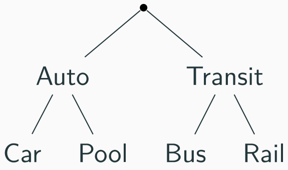
Choice	original	car	pool	Bus	Rail
Car	.40	–	.45 (+12.5%)	.52 (+30%)	.48 (+20%)
Pool	.10	.20 (+100%)	–	.13 (+30%)	.12(20%)
Bus	.30	.48 (+60%)	.33 (+10%)	–	.40 (+33%)
Rail	.20	.32 (+60%)	.22(+10%)	.35(+70%)	–

Notice that the probabilities for rail and bus always increase by the same rate when one of the other alternatives is removed - IIA holds between these alternatives, so Train and Bus can be mapped to one *nest* called ***transit***. The same thing for car and pool - put these into a nest called ***auto***

Nested Logit Model - an example

You can visualise these nested choices as a tree. From this it's clear that the decision maker actually makes a two-phase choice.

First the choice of whether by car or pubic transport, then the choice of own car or pool and bus or rail.



Probit Model

Probit Model

The probit models overcome the 3 problems of logit models: (taste variation, IIA, temporal correlation).

Probit models overcome this.

Probit Assumption

A probit model assumes that the unobserved components of utility ($\epsilon_{ni} \forall i$) are normally distributed.

Probit Model

Again utility is assumed to be observed and unobserved:

$$U_{nj} = V_{nj} + \epsilon_{nj} \quad \forall j$$

ϵ_n is then distributed normal with density:

$$\phi(\epsilon_n) = \frac{1}{(2\pi)^{J/2}|\Omega_n|^{1/2}} e^{-\frac{1}{2}\epsilon_n' \Omega_n^{-1} \epsilon_n}$$

where

$$\epsilon_n' = \langle \epsilon_{n1}, \ldots, \epsilon_{nJ} \rangle \quad \Omega_n \text{ is the covariance matrix}$$

Probit Model

The choice probability is then

$$
\begin{aligned}
P_{ni} &= Prob(V_{ni} + \epsilon_{ni} > V_{nj} + \epsilon_{nj} \quad \forall j \neq i) \\
&= \int I(V_{ni} + \epsilon_{ni} > V_{nj} + \epsilon_{nj} \quad \forall j \neq i) \quad \phi(\epsilon_n) \quad \delta\epsilon_n
\end{aligned}
$$

Here there is no closed form solution, it's evaluated numerically in this case.

Probit Model

- Taste variation: Probit can do this.
- Substitution: estimating a full covariance matrix allows the model to represent any substitution pattern.
- IIA: Probit probabilities do not exhibit IIA.

Taste Variation

$$U_{nj} = \beta_n x_{nj} + \epsilon_{nj}$$

β_n is the vector of coefficients for decision maker n representing that person's tastes. Suppose the β_n is normally distributed in the population with mean b and covariance $W : \beta_n belongsto N(b, W)$. The goal is to estimate the parameters b and W.

Taste Variation

The utility can be rewritten with n decomposed into its mean and deviations from its mean:

$$U_{nj} = b_n x_{nj} + \beta_n \tilde{x}_{nj} + \epsilon_{nj}$$

Taste Variation

The last two terms in the utility are random; denote their sum as η_{nj} to obtain $U_{nj} = b_n x_{nj} + \eta_{nj}$. The covariance of the η_{nj} 's depends on W as well as the x_{nj} 's, so that the covariance differs over decision makers.

Taste Variation

$$U_{n1} = \beta_n x_{n1} + \varepsilon_{n1},$$
$$U_{n2} = \beta_n x_{n2} + \varepsilon_{n2}.$$

Assume that β_n is normally distributed with mean b and variance σ_β. Assume that ε_{n1} and ε_{n2} are identically normally distributed with variance σ_ε. The assumption of independence is for this example and is not needed in general. The utility is then rewritten as

$$U_{n1} = bx_{n1} + \eta_{n1},$$
$$U_{n2} = bx_{n2} + \eta_{n2},$$

Taste Variation

where η_{n1} and η_{n2} are jointly normally distributed. Each has zero mean: $E(\eta_{nj}) = E(\tilde{\beta}_n x_{nj} + \varepsilon_{nj}) = 0$. The covariance is determined as follows. The variance of each is $V(\eta_{nj}) = V(\tilde{\beta}_n x_{nj} + \varepsilon_{nj}) = x_{nj}^2 \sigma_\beta + \sigma_\varepsilon$. Their covariance is

Taste Variation

$$\begin{aligned}\mathrm{Cov}(\eta_{n1}, \eta_{n2}) &= E[(\tilde{\beta}_n x_{n1} + \varepsilon_{n1})(\tilde{\beta}_n x_{n2} + \varepsilon_{n2})] \\ &= E\left(\tilde{\beta}_n^2 x_{n1} x_{n2} + \varepsilon_{n1}\varepsilon_{n2} + \varepsilon_{n1}\tilde{\beta}_b x_{n2} + \varepsilon_{n2}\tilde{\beta}_n x_{n1}\right) \\ &= x_{n1} x_{n2} \sigma_\beta .\end{aligned}$$

Taste Variation

The covariance matrix is

$$\Omega = \begin{pmatrix} x_{n1}^2\sigma_\beta + \sigma_\varepsilon & x_{n1}x_{n2}\sigma_\beta \\ x_{n1}x_{n2}\sigma_\beta & x_{n2}^2\sigma_\beta + \sigma_\varepsilon \end{pmatrix}$$

$$= \sigma_\beta \begin{pmatrix} x_{n1}^2 & x_{n1}x_{n2} \\ x_{n1}x_{n2} & x_{n2}^2 \end{pmatrix} + \sigma_\varepsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Summary & Applying Discrete Choice

Other Models

Mixed Logit

Allows taste variation, any substitution pattern and doesn't require IIA. It also does not require the unknowns are normally distributed.

The mixed logit requires simulation for estimating parameters and is still seen as 'the modern approach' despite being developed in the eighties.

Estimation

The work comes about in estimating the values of β for the choice models. We find the values of β that best match the sample data we have collected.

There are different ways this can be done (provided in software normally):

- Maximum likelihood methods (numerical optimization)
- Bayesian Methods (Markov chain Monte Carlo)
- Simulation-based estimators (draw random variables to estimate integral)

Summary

We've discussed (briefly) most of the classic discrete choice models:

- Logit - simple form, disadvantages
- Nested Logit - IIA within a nest, but not between nests
- Probit - Normally distributed unknowns
- Mixed Logit - a very general form of choice model.

You can also consider cases of ***binary*** choice (2 alternatives) or ***multi-nomial*** choice.

Applications to ABM

Discrete Choice Theory provides a mature methodology for developing choice algorithms. If you are developing an ABM where to try to capture choice e.g.,

- Transport/Mobility modelling (Modal choice)
- Migration/Urban modelling (house choice)
- Economic Modelling (Purchase choice)
- Transportation (EV Charging station choice)

These models give you a way to develop a model of choice within an agent. You can consider to do this at a population level, or at an aggregate level, or at an individual level (depending on data)

References

Ben-Akiva, Moshe E and Steven R Lerman (1985). *Discrete choice analysis: theory and application to travel demand*. Vol. 9. MIT press.