Laws of Total Expectation and Total Variance

Definition of conditional density. Assume and arbitrary random variable X with density f_X. Take an event A with $P(A) > 0$. Then the conditional density $f_{X|A}$ is defined as follows:

$$f_{X|A}(x) = \begin{cases} \frac{f(x)}{P(A)} & x \in A \\ 0 & x \notin A \end{cases}$$

Note that the support of $f_{X|A}$ is supported only in A.

Definition of conditional expectation conditioned on an event.

$$E(h(X)|A) = \int_A h(x)\, f_{X|A}(x)\, dx = \frac{1}{P(A)} \int_A h(x)\, f_X(x)\, dx$$

Example. For the random variable X with density function

$$f(x) = \begin{cases} \frac{1}{64}\, x^3 & 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

calculate $E(X^2 \mid X \geq 1)$.

Solution.

Step 1.

$$P(X \geq 1) = \int_1^4 \frac{1}{64}\, x^3\, dx = \frac{1}{256} \left[\frac{1}{4} x^4\right]_{x=1}^{x=4} = \frac{255}{256}$$

Step 2.

$$E(X^2 \mid X \geq 1) = \frac{1}{P(X \geq 1)} \int_{\{x \geq 1\}} x^2\, f(x)\, dx = \frac{256}{255} \int_1^4 x^2 \left(\frac{1}{64}\, x^3\right) dx$$

$$= \frac{256}{255} \int_1^4 \left(\frac{1}{64}\, x^5\right) dx = \left(\frac{256}{255}\right) \left(\frac{1}{64}\right) \left[\frac{1}{6} x^6\right]_{x=1}^{x=4} = \frac{8192}{765}$$

Definition of conditional variance conditioned on an event.

$$\mathrm{Var}(X|A) = E(X^2|A) - E(X|A)^2$$

Example. For the previous example , calculate the conditional variance $\mathrm{Var}(X|X \geq 1)$

Solution.

We already calculated $E(X^2 \mid X \geq 1)$. We only need to calculate $E(X \mid X \geq 1)$.

$$E(X \mid X \geq 1) = \frac{1}{P(X \geq 1)} \int_{\{x \geq 1\}} \int x\, f(x)\, dx = \frac{256}{255} \int_1^4 x \left(\frac{1}{64}\, x^3\right) dx$$

$$= \frac{256}{255} \int_1^4 \left(\frac{1}{64}\, x^4\right) dx = \left(\frac{256}{255}\right)\left(\frac{1}{64}\right)\left[\frac{1}{5} x^5\right]_{x=1}^{x=4} = \frac{4096}{1275}$$

Finally:

$$\mathrm{Var}(X|X \geq 1) = E(X^2|X \geq 1) - E(X|X \geq 1)^2 = \frac{8192}{765} - \left(\frac{4096}{1275}\right)^2 = \frac{630784}{1625625}$$

Definition of conditional expectation conditioned on a random variable. Suppose two random variables X and Y. To define $E(X|Y)$ we need the conditional density function. For this , let $f_{X,Y}(x, y)$ be the joint density of the pair $\{X, Y\}$. Thsn the conditional density $f_{X|Y}$ is defined

$$f_{X|Y}(x|y) = \begin{cases} \frac{f_{X,Y}(x,y)}{f_Y(y)} & f_Y(y) > 0 \\ \\ 0 & f_Y(y) = 0 \end{cases}$$

Then $E(h(X) \mid Y)$ is defined to be the random variable that assigns the value $\int_{-\infty}^{\infty} h(x)\, f_{X|Y}(x|y)\, dx$ to y in the continuous case, and assigns the value $\sum_x h(x)\, f_{X|Y}(x|y)$ in the discrete case.

Example. Take the following joint density:

	X=1	X=2	X=3	$f_Y(y)$
$Y = 1$	0.07	0.1	0.23	0.4
$Y = 2$	0.25	0.08	0.07	0.4
$Y = 3$	0.05	0.04	0.11	0.2
$f_X(x)$	0.37	0.22	0.41	1

Describe the random variable $E(Y^2|X)$

Solution. $E(Y^2|X=1) = \sum\limits_y y^2 f_{Y|X}(y|1) = \sum\limits_y y^2 \frac{f(X=1\,,\,Y=y)}{f_X(1)}$

$$= \frac{1}{f_X(1)} \sum_y y^2 \, f(X=1\,,\,Y=y) = \frac{1}{0.37}\Big\{(1)^2(0.07) + (2)^2(0.25) + (3)^2(0.05)\Big\} = \frac{1.52}{0.37}$$

Similarly:

$E(Y^2|X=2) = \dfrac{0.78}{0.32}$

$E(Y^2|X=3) = \dfrac{1.5}{0.41}$

So then:

$$E(Y^2|X) = \begin{cases} \frac{1.52}{0.37} & \text{with probability } P(X=1) = 0.37 \\ \frac{0.78}{0.32} & \text{with probability } P(X=2) = 0.22 \\ \frac{1.5}{0.41} & \text{with probability } P(X=3) = 0.41 \end{cases}$$

Law of Total Expectation.

$$E(X) = E(E[X|Y])$$

Law of Total Variance.

$$\text{Var}(X) = E\Big(\text{Var}[X\,|\,Y]\Big) + \text{Var}\Big(E[X\,|\,Y]\Big)$$

Proof. By definition we have

$$\text{Var}(X|Y) = E(X^2|Y) - \{E(X|Y)\}^2$$

BX taking E of both sides we get:

$$\begin{aligned}
E[\mathrm{Var}(X|Y)] &= E[E[X^2|Y]] - E[\{E(X|Y)\}^2] \\
&= E(X^2) - E[\{E(X|Y)\}^2] \qquad \text{law of iterated expectations} \\
&= \Big\{E(X^2) - \{E(X)\}^2\Big\} - \Big\{E[\{E(X|Y)\}^2] - \{E(X)\}^2\Big\} \\
&= \mathrm{Var}(X) - \Big\{E[\{E(X|Y)\}^2] - \{E[E(X|Y)]\}^2\Big\} \\
&= \mathrm{Var}(X) - \mathrm{Var}[E(X|Y)]
\end{aligned}$$

By moving terms around , the claim follows.

Note: Using similar arguments , one can prove the following:

Example (from the Dean's note): Two urns contain a large number of balls with each ball marked with one number from the set $\{0, 2, 4\}$. The proportion of each type of ball in each urn is displayed in the table below:

	Number on Ball		
	0	2	4
A	0.6	0.3	0.1
B	0.1	0.3	0.6

An urn is randomly selected and then a ball is drawn at random from the urn. The number on the ball is represented by the random variable X.

(a) Calculate the hypothetical means (or conditional means)

$$E[X|\theta = A] \quad \text{and} \quad E[X|\theta = B]$$

(b) Calculate the variance of the hypothetical means: $\mathrm{Var}[E[X|\theta]]$.

(c) Calculate the process variances (or conditional variances)

$$\mathrm{Var}[X|\theta = A] \quad \text{and} \quad \mathrm{Var}[X|\theta = B]$$

(d) Calculate the expected value of the process variance: $E[\mathrm{Var}[X|\theta]]$.

(e) Calculate the total variance (or unconditional variance) $\mathrm{Var}[X]$ and show that it equals the sum of the quantities calculated in (b) and (d) .

Solution: **Part (a)**

$$E[X|\theta = A] = (0.6)(0) + (0.3)(2) + (0.1)(4) = 1.0$$

$$E[X|\theta = B] = (0.1)(0) + (0.3)(2) + (0.6)(4) = 3.0$$

Part (b)

$$E[X] = E[E[X|\theta]] = (\frac{1}{2})(1.0) + (\frac{1}{2})(3.0) = 2.0$$

$$\mathrm{Var}[E[X|\theta]] = (\frac{1}{2})(1.0 - 2.0)^2 + (\frac{1}{2})(3.0 - 2.0)^2 = 1.0$$

Part (c)

$$\mathrm{Var}[X|\theta = A] = (0.6)(0 - 1.0)^2 + (0.3)(2 - 1.0)^2 + (0.1)(4 - 1.0)^2 = 1.8$$

$$\mathrm{Var}[X|\theta = B] = (0.1)(0 - 3.0)^2 + (0.3)(2 - 3.0)^2 + (0.6)(4 - 3.0)^2 = 1.8$$

Part (d)

$$E[\mathrm{Var}[X|\theta]] = (\frac{1}{2})(1.8) + (\frac{1}{2})(1.8) = 1.8$$

Part (e)

$$\begin{aligned}
\mathrm{Var}[X] &= \tfrac{1}{2}\Big[(0.6)(0 - 2.0)^2 + (0.3)(2 - 2.0)^2 + (0.1)(4 - 2.0)^2\Big] \\
&+ \tfrac{1}{2}\Big[(0.1)(0 - 2.0)^2 + (0.3)(2 - 2.0)^2 + (0.6)(4 - 2.0)^2\Big] \\
&= 2.8 \\
&\Rightarrow \qquad \mathrm{Var}(X) = \mathrm{Var}[E[X|\theta]] + E[\mathrm{Var}[X|\theta]]
\end{aligned}$$