Exam Scientific Computing
18 May 2016, 13.00-16.00; G3.10

Question 1 (25 points)
Reaction-diffusion equation

Consider the general form of a reaction-diffusion system

$$\frac{\partial U_1}{\partial t} = f_1(U_1, U_2, ..) + D_{U_1}\nabla^2 U_1$$
$$\frac{\partial U_2}{\partial t} = f_2(U_1, U_2, ..) + D_{U_2}\nabla^2 U_2$$
...

a) About the general form: what is the meaning of functions $f_1, f_2, ..$ what are these functions describing? What is the role of the diffusion coefficients D in these systems?

b) how could you derive an equation for $f_1, f_2, ..$?
c) There are many studies in complex system theory about reaction-diffusion systems, why are these equations interesting from a complex system point of view?
d) An example of a reaction-diffusion system is the Fitzhugh Nagumo equation:

$$f_u = \lambda u - u^3 - k - \delta v$$
$$f_v = \frac{u - v}{\tau}$$

Where k, δ, τ, λ are constants. Derive an explicit (forward Euler, fully-discrete) numerical scheme for approximating solutions of this equation.

Question 2 (25 points)
Heat equation

Consider the one-dimensional heat equation $u_t = cu_{xx}$ with initial condition u(0,x) = f(x), and $0 \le x \le 1$ and $t \ge 0$.

a) Desribe (in a diagram) the problem domain (use the boundary values, initial values)
b) Derive a semi-discrete scheme for this equation
c) Describe the semi-discrete scheme in matrix form
d) Consider a fully discrete numerical solution of the heat equation. Both the time derivative and the spatial derivative are discretized with centred finite differencing. One can now construct two types of numerical schemes, explicit or implicit. What is the main difference between explicit and implicit schemes?
e) Draw the stencils for both the explicit and implicit scheme for the fully discrete numerical solution of the heat equation.
f) Derive both the explicit and implicit numerical solution of the heat equation.
g) Use Von Neumann stability analysis to derive stability conditions for both the explicit and implicit numerical scheme. What do you conclude?

Question 3 (25 points)

Numerical Solution to the wave equation

Consider the one-dimensional wave equation $u_{tt} = c\, u_{xx}$, with initial condition $u(0,x) = f(x)$, and $0 \leq x \leq 1$ and $t \geq 0$.

a) The wave equation is a hyperbolic equation. What is the main feature of a hyperbolic equation, and how does it distinguish itself from a parabolic equation?

b) Draw the stencils for the explicit scheme for the fully discrete numerical solution of the wave equation.

d) Derive the explicit numerical solution of the wave equation.

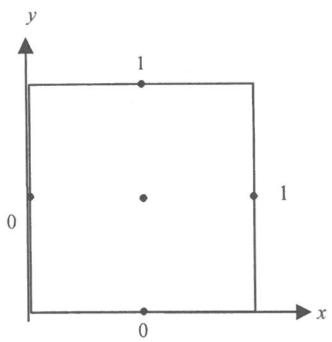
e) Proof that the central finite difference scheme for u_{xx} has an accuracy $O(\Delta x^2)$.

f) What is the accuracy of the fully discrete numerical solution of the wave equation? Besides accuracy, two other important properties of a numerical scheme are consistency and stability. What does it mean that a numerical scheme is consistent and stable?

g) Proof that the scheme is consistent.

Question 4 (25 points)

Poisson equation



Consider a finite difference solution of the Poisson equation $u_{xx} + u_{yy} = 1 + y$ on the unit square using boundary conditions and mesh points as shown in the picture. Use a second order accurate, centred finite difference scheme to compute the approximate value of the solution at the centre of the square (so, at $x = y = ½$).

good luck!