Exam Scientific Computing
26 March 2015, 9.00-12.00; G4.15

Question 1 (25 points)
Laplace equation

Consider the two-dimensional Laplace equation $\frac{\partial^2 c}{\partial x^2}+\frac{\partial^2 c}{\partial y^2}=0$ where c is a function $c(x,y)$

a) Explain how the Jacobi update procedure can be derived from this equation (explain how the equation is discretized, how are the derivatives discretized?
b) How can you improve convergence of this scheme (describe update scheme, stencil, memory requirements) ?
c) Give (in pseudo-code) an algorithm for approximating solutions of the two-dimensional Laplace equation (consider boundary conditions, convergence).
d) Give the matrix form of the one-dimensional Laplace equation.
e) How can you (in general) solve this type of matrix form equations (give a brief description of the method)?

Question 2 (25 points)
Reaction-diffusion equation

Consider the general form of a reaction-diffusion system

$$\frac{\partial U_1}{\partial t}=f_1(U_1,U_2,..)+D_{U_1}\nabla^2 U_1$$
$$\frac{\partial U_2}{\partial t}=f_2(U_1,U_2,..)+D_{U_2}\nabla^2 U_2$$
...

a) About the general form: what is the meaning of functions $f_1, f_{2,..}$ what are these functions describing? What is the role of the diffusion coefficients D in these systems?

b) how could you derive an equation for $f_1, f_{2,..}$?
c) There are many studies in complex system theory about reaction-diffusion systems, why are these equations interesting from a complex system point of view?
d) An example of a reaction-diffusion system is the Brusselator:

$$\frac{\partial u}{\partial t}=-(\beta+1)u+u^2v+\alpha+D_u\nabla^2 u$$
$$\frac{\partial v}{\partial t}=-u^2v+\beta u+D_v\nabla^2 v$$

Where α,β are constants. Derive an explicit (forward Euler, fully-discrete) numerical scheme for approximating solutions of this equation.

Please check the other side!

Question 3 (25 points)
Heat equation

Consider the one-dimensional heat equation $u_t = cu_{xx}$ with initial condition $u(0,x) = f(x)$, and $0 \leq x \leq 1$ and $t \geq 0$.

a) Desribe (in a diagram) the problem domain (use the boundary values, initial values)
b) Derive a semi-discrete scheme for this equation
c) Describe the semi-discrete scheme in matrix form
d) Consider a fully discrete numerical solution of the heat equation. Both the time derivative and the spatial derivative are discretized with centred finite differencing. One can now construct two types of numerical schemes, explicit or implicit. What is the main difference between explicit and implicit schemes?
e) Draw the stencils for both the explicit and implicit scheme for the fully discrete numerical solution of the heat equation.
f) Derive both the explicit and implicit numerical solution of the heat equation.
g) Use Von Neumann stability analysis to derive stability conditions for both the explicit and implicit numerical scheme. What do you conclude?

Question 4 (25 points)

Numerical Solution to the wave equation

Consider the one-dimensional wave equation $u_{tt} = c\, u_{xx}$, with initial condition $u(0,x) = f(x)$, and $0 \leq x \leq 1$ and $t \geq 0$.

a) The wave equation is a hyperbolic equation. What is the main feature of a hyperbolic equation, and how does it distinguish itself from a parabolic equation?

b) Draw the stencils for the explicit scheme for the fully discrete numerical solution of the wave equation.

d) Derive the explicit numerical solution of the wave equation.

e) Proof that the central finite difference scheme for u_{xx} has an accuracy $O(\Delta x^2)$.
f) What is the accuracy of the fully discrete numerical solution of the wave equation? Besides accuracy, two other important properties of a numerical scheme are consistency and stability. What does it mean that a numerical scheme is consistent and stable?

good luck!