Integration and Stoke's theorem

Stoke's theorem Let M be an oriented n-dim. manifold with boundary and let $\imath : \partial M \to M$ be the inclusion. Then

$$\int_M d\omega = \int_{\partial M} \imath^* \omega \quad \text{for all } \omega \in \Omega^{n-1}_c(M).$$

Rem. It follows that if $\partial M = \emptyset$, then $\int_M d\omega = 0$

We have proved the following lemma:

Lemma Stoke's theorem hold for $M = \mathbb{H}^n$ and $M = \mathbb{R}^n$.

Proof of Stoke's theorem

We first choose an oriented atlas $\{(U_\alpha, \phi_\alpha : U_\alpha \to \phi(U_\alpha))\}$ such that $\phi_\alpha(U_\alpha) = \mathbb{R}^n$ or $\phi_\alpha(U_\alpha) = \mathbb{H}^n$ for each α.

Claim Stoke's theorem holds if $\operatorname{supp} \omega \subseteq U_\alpha$ for some α.

$$\int_{U_\alpha} d\omega = \int_{\phi(U_\alpha)} (\phi_\alpha^{-1})^* d\omega = \int_{\phi(U_\alpha)} d\,(\phi_\alpha^{-1})^*\omega = \int_{\partial\phi(U_\alpha)} \imath^*(\phi_\alpha^{-1})^*\omega$$

$$= \int_{\partial\phi(U_\alpha)} (\phi_\alpha^{-1}\circ\imath)^*\omega$$, on the other side

$$\int_{\partial U_\alpha} \imath^*\omega = \int_{\phi_\alpha(\partial U_\alpha)} (\phi_\alpha|_{\partial U_\alpha}^{-1})^* \imath^*\omega = \int_{\partial\phi_\alpha(U_\alpha)} (\imath\circ\phi_\alpha|_{\partial U_\alpha}^{-1})^*\omega$$

These are equal since the diagram

$$\begin{tikzcd}
\partial U_\alpha \arrow[r, "\imath"] \arrow[d, "\phi_\alpha|_{\partial U_\alpha}"'] & U_\alpha \arrow[d, "\phi_\alpha"] \\
\partial\phi_\alpha(U_\alpha) \arrow[r, "\imath"] & \phi_\alpha(U_\alpha)
\end{tikzcd}$$

is commutative and the vertical diffeomorphisms are orientation preserving.

Now chose a partion of unity $\{\rho_\alpha : M \to [0,1]\}$ subordinate $\{U_\alpha\}$

Given $\omega \in \Omega_c^{n-1}(M)$, we write $\omega = \sum_\alpha \rho_\alpha \omega$ (finite sum).

$$\int_M d\omega = \int_M d\Big(\sum_\alpha \rho_\alpha \omega\Big) = \sum_\alpha \int_M d(\rho_\alpha \omega) = \sum_\alpha \int_{U_\alpha} d(\rho_\alpha \omega)$$

$$= \sum_\alpha \int_{\partial U_\alpha} \imath^*(\rho_\alpha \omega) = \sum_\alpha \int_{\partial M} \imath^*(\rho_\alpha \omega) = \int_{\partial M} \imath^*\Big(\sum_\alpha \rho_\alpha \omega\Big)$$

$$= \int_{\partial M} \imath^* \omega \qquad \square$$

<u>Rmk</u>. The fundamental theorem for line integrals and Green's theorem from multivariable calculus are special cases of Stoke's theorem (see textbook Section 23.6).

Relation to algebraic topology

If M is an n-dim manifold, we have defined a sequence of real vector spaces

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \to \dots \to \Omega^{n-1}(M) \xrightarrow{d} \Omega^n(M) \to 0$$

such that $d \circ d = 0$. This is a cochain complex.

The (co)homology groups of this cochain complex are the de Rham cohomology groups $H^k_{dR}(M)$, $k = 0, \dots, n$.

Fact: $H^k_{dR}(M)$ is isomorphic to the k-th singular homology group of M with coefficients in $\mathbb{R}$.