

## The tangent bundle

Let  $M$  be a smooth manifold,  $\dim M = n$ .

Def The tangent bundle of  $M$  is the (disjoint) union of the tangent space  $TM = \bigcup_{p \in M} T_p M$

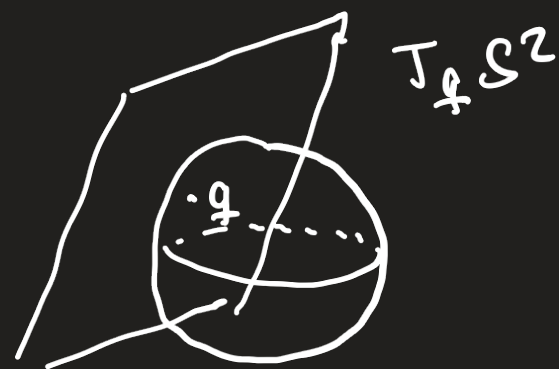
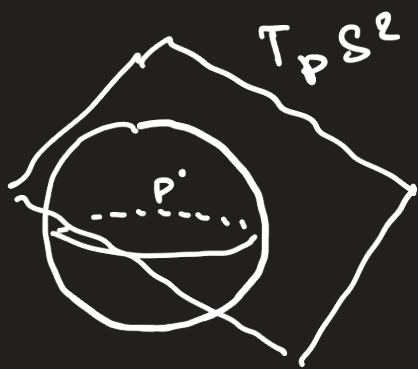
Notice : If  $p \neq q$ , then  $T_p M \cap T_q M = \emptyset$

Remark If  $M \subseteq \mathbb{R}^N$  is a submanifold, then

$$i_{*,p} : T_p M \rightarrow T_p \mathbb{R}^N \cong \mathbb{R}^N$$

is injective for all  $p \in M$ . The images of  $T_p M$  and  $T_q M$  in  $\mathbb{R}^N$  will intersect even though  $T_p M$  and  $T_q M$  do not.

Ex  $M = S^2 \subseteq \mathbb{R}^3$



We often write elements of  $TM = \bigcup_{p \in M} T_p M$  in the form  $(p, v)$  for  $v \in T_p M$ .

Goal: Show  $TM$  is a smooth manifold of dimension  $2n$ .

First we need to define a topology on  $TM$ .

Let  $\phi: U \rightarrow \phi(U) \subseteq \mathbb{R}^n$  be a chart on  $M$ .

$$\text{Write } TU = \bigcup_{p \in U} T_p U = \bigcup_{p \in U} T_p M$$

For each  $p \in U$  we have an isomorphism

$$\phi_{*,p}: T_p M \longrightarrow T_{\phi(p)} \phi(U) \cong \mathbb{R}^n$$

Hence we get a bijective map

$$\check{\phi}: TU \longrightarrow \phi(U) \times \mathbb{R}^n, (p, v) \mapsto (\phi(p), \phi_{*,p}(v))$$

Def A subset  $A \subseteq TU$  is open iff  $\check{\phi}(A)$  is open in  $\phi(U) \times \mathbb{R}^n$

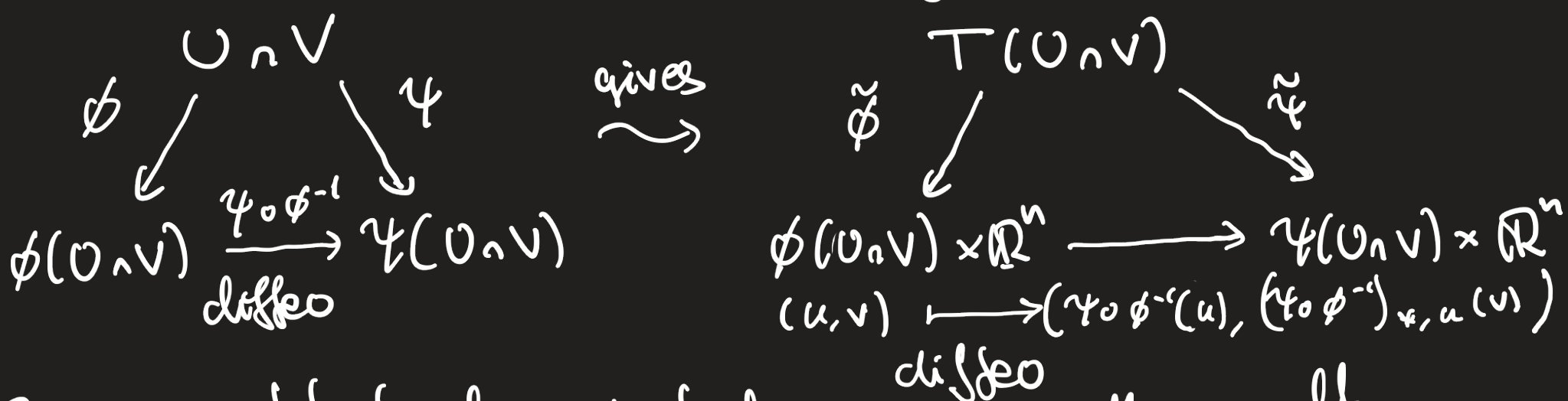
With this topology  $\check{\phi}$  is a homeomorphism

Let  $(U, \phi)$  and  $(V, \psi)$  be charts on  $M$ .

Then  $TU \cap TV = T(U \cap V)$  by definition

Lemma The subspace topology on  $T(U \cap V)$  inherited from  $TU$  is the same as the subspace topology inherited from  $TV$ .

Proof The commutative diagram



The commutativity of the last diagram gives the result.  
 $(\mathbb{R}^n \xrightarrow{\phi_*^{-1}} T_p(U \cap V) \xrightarrow{\tilde{\psi}_*, p} \mathbb{R}^n \text{ equals } (\psi \circ \phi^{-1})_* \text{ by the chain rule}) \square$

Let  $\mathcal{B}$  be the following collection of subsets of  $TM$ :

$$\mathcal{B} = \bigcup_{\{(U, \phi) \text{ charts on } M\}} \{A \subseteq TU : \phi(A) \subseteq \phi(U) \times \mathbb{R}^n \text{ is open}\}$$

Proposition The collection  $\mathcal{B}$  is a basis for a topology on  $TM$ . This topology is second countable and Hausdorff.

Proof It follows from the previous lemma that  $\mathcal{B}$  is a basis for a topology.

Next check that the topology is second countable:

Let  $\mathcal{C}$  be a countable basis for the topology on  $M$ .

For each chart  $(U_\alpha, \phi_\alpha)$  and each  $p \in U_\alpha$  choose  $B_{\alpha,p} \in \mathcal{C}$  such that  $p \in B_{\alpha,p} \subseteq U_\alpha$ . Then  $\{B_{\alpha,p}\} \subseteq \mathcal{C}$  is countable.

Write  $\{B_{\alpha, \rho}\} = \{U_i : i \geq 1\}$ . This is a basis for the topology on  $M$  by construction. For each  $i$  there is a chart  $\phi_i : U_i \rightarrow \phi_i(U_i) \subseteq \mathbb{R}^n$ . Hence there is a countable atlas  $\{(U_i, \phi_i)\}$  on  $M$ .

For each  $i$  have  $\tilde{\phi}_i : TU_i \xrightarrow{\cong} \phi_i(U_i) \times \mathbb{R}^n$

Choose for each  $i$  a countable basis  $\{B_{i,j} : j \geq 1\}$  for  $TU_i$ .

Then  $\{B_{i,j} : i, j \geq 1\}$  is a countable basis for the topology on  $TM$ .

Exercise for next week:  $TM$  is Hausdorff.

□

Theorem Let  $M$  be a smooth  $n$ -dimensional manifold.

Then  $TM$  is a smooth manifold of dimension  $2n$ .

Proof We already know that  $TM$  is a topological manifold with charts  $\tilde{\phi}: TU \rightarrow \phi(U) \times \mathbb{R}^n$ .

In order to see the charts are  $C^\infty$  compatible we again consider the commutative diagrams

$$\begin{array}{ccc}
 & TU \cap TV & \\
 \tilde{\phi} \swarrow & & \searrow \tilde{\psi} \\
 \phi(U \cap V) \times \mathbb{R}^n & \xrightarrow{\quad} & \psi(U \cap V) \times \mathbb{R}^n \\
 (u, v) & \xrightarrow{\quad} & (\psi \circ \phi^{-1}(u), (\psi \circ \phi^{-1})_{*,u}(v)) \\
 & \text{diffeomorphism} &
 \end{array}$$

□