Tangent spaces

M smooth manifold, $p \in M$.

Consider pairs (U, f), where $p \in U$ nhb and $f: U \to \mathbb{R}$ C^∞.

Equivalence relation: $(U, f) \sim (V, g)$ if there exists $p \in W \subseteq U \cap V$ such that $f|_W = g|_W$.

A function germ at p is an equivalence class $[U, f]$

$C_p^\infty(M)$ is the $\mathbb{R}$-algebra of functions germs at p

A derivation at p is an $\mathbb{R}$-linear function $D: C_p^\infty(M) \to \mathbb{R}$ such that $D(f \cdot g) = D(f)\, g(p) + f(p)\, D(g)$

<u>Def</u> The tangent space at p, $T_p(M)$, is the vector space of all point derivations $D: C_p^\infty(M) \to \mathbb{R}$.

Rmk Let $p \in M$ and let $U \subseteq M$ be open st. $p \in U \subseteq M$.

Then $C_p^\infty(U) \to C_p^\infty(M)$ is an isomorphism

(The inverse takes (V, f) to $(V \cap U, f|_{U \cap V})$)

Hence get an isomorphism $T_p(U) \xrightarrow{\simeq} T_p(M)$

Ex Let $U \subseteq M$ open $\phi : U \to \mathbb{R}^n$ a chart on M ($\dim M = n$)

Write $\phi(p) = (x^1(p), .., x^n(p))$. $x^1 .. x^n : U \to \mathbb{R}$.

Let $r^1, .., r^n$ be the standard coordinates on $\mathbb{R}^n$

Recall, given $f : U \to \mathbb{R}$ smooth,

$$\frac{\partial}{\partial x^i}\Big|_p (f) = \frac{\partial(f \circ \phi^{-1})}{\partial r^i} \phi(p) \in \mathbb{R}.$$

Claim : $\frac{\partial}{\partial x^i}\Big|_p : C_p^\infty(M) \to \mathbb{R}$ is a derivation at p.

$$\frac{\partial}{\partial x^i}\Big|_p (f \cdot g) = \frac{\partial((f \circ \phi^{-1})(g \circ \phi^{-1}))}{\partial r^i} \phi(p)$$

$$= \frac{\partial(f \circ \phi^{-1})}{\partial r^i} \phi(p) \cdot g(p) + f(p) \cdot \frac{\partial(g \circ \phi^{-1})}{\partial r^i} \phi(p) = \frac{\partial}{\partial x^i}\Big|_p (f) \, g(p) + f(p) \frac{\partial}{\partial x^i}\Big|_p (g).$$

Let $F: N \to M$ be a smooth map, $p \in N$.

<u>Def</u> The differential of F at p is the linear map

$F_{*,p} : T_p(N) \longrightarrow T_{F(p)}(M)$. defined by

$F_{*,p}(X_p)(f) = X_p(f \circ F)$, for $X_p \in T_p(N)$, $f \in C^\infty_{F(p)}(M)$

<u>Notice</u>: $f \circ F$ is defined in a nbh. of p

<u>Claim</u>: $F_{*,p}(X_p)$ is a point derivation

$$F_{*,p}(X_p)(f \cdot g) = X_p((f \circ F) \cdot (g \circ F))$$
$$= X_p(f \circ F) \cdot g(F(p)) + f(F(p)) \cdot X_p(g \circ F)$$
$$= F_{*,p}(X_p)(f) \cdot g\,F(p) + f\,F(p)\;F_{*,p}(X_p)(g)$$

Let $N \xrightarrow{F} M \xrightarrow{G} P$ be smooth maps, $p \in N$

$$\begin{tikzcd} T_p(N) \arrow[r, "F_{*,p}"] \arrow[dr, "(G\circ F)_{*,p}"'] & T_{F(p)}(M) \arrow[d, "G_{*,F(p)}"] \\ & T_{GF(p)}(P) \end{tikzcd}$$

<u>Theorem</u> (The chain rule)

This diagram is commutative:

$$G_{*,F(p)} \circ F_{*,p} = (G\circ F)_{*,p}$$

<u>Proof</u> Let $X_p \in T_p(N)$, let $f \in C^\infty_{GF(p)}(P)$

$$(G_{*,F(p)} \circ F_{*,p})(X_p)(f) = G_{*,F(p)}(F_{*,p}(X_p))(f)$$
$$= F_{*,p}(X_p)(f\circ G) = X_p(f\circ G\circ F) = (G\circ F)_{*,p}(X_p)(f).$$

$\square$

Corollary If $F: N \to M$ is a diffeomorphism, then $F_{*,p}: T_p(N) \to T_{F(p)}(M)$ is a linear isomorphism, $p \in N$.

Proof Know F has a smooth inverse $G: M \to N$ s.t.

$F \circ G = \mathbb{1}_M$, $G \circ F = \mathbb{1}_N$.

$$\begin{tikzcd}
T_p(N) \arrow[r, "F_{*,p}"] \arrow[rr, bend right, "(G \circ F)_{*,p} = (\mathbb{1}_N)_{*,p} = \mathbb{1}_{T_p(N)}"'] & T_{F(p)}(M) \arrow[r, "G_{*,F(p)}"] & T_{GF(p)}(N) = T_p(N)
\end{tikzcd}$$

Hence $G_{*,F(p)} \circ F_{*,p} = \mathbb{1}_{T_p(N)}$

Similarly, $F_{*,p} \circ G_{*,F(p)} = \mathbb{1}_{T_{F(p)}(M)}$

Conclusion: $G_{*,F(p)}$ is inverse to $F_{*,p}$ $\square$

Corollary If an open subset $U \subseteq \mathbb{R}^n$ is diffeomorphic to an open subset $V \subseteq \mathbb{R}^m$, then $m = n$.

Proof Let $F: U \to V$ be a diffeomorphism, let $p \in U$.

Then $F_{*,p}: T_p(U) \to T_{F(p)}(V)$ is an isomorphism.

We know $T_p(U) \cong \mathbb{R}^n$, $T_{F(p)}(V) \cong \mathbb{R}^m$, so $m = n$ $\square$

Let (U, ϕ) be a chart on M, $\phi : U \to \mathbb{R}^n$, $p \in U$.

Write $\phi = (x^1, \dots, x^n)$, $\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p \in T_p(M)$.

<u>Proposition</u> $\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p$ form a basis for $T_p(M)$.

<u>Proof</u> $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$ diffeomorphism, so

$$\phi_{*,p} : T_p(M) \cong T_p(U) \xrightarrow{\cong} T_{\phi(p)}(\phi(U)) \cong T_{\phi(p)}(\mathbb{R}^n)$$

is an isomorphism.

Know $\frac{\partial}{\partial r^1}\Big|_{\phi(p)}, \dots, \frac{\partial}{\partial r^n}\Big|_{\phi(p)}$ is a basis for $T_{\phi(p)}(\mathbb{R}^n)$

Suffices to show $\phi_{*,p}\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial}{\partial r^i}\Big|_{\phi(p)}$.

Let $f \in C^\infty_p(\mathbb{R}^n)$, then

$$\phi_{*,p}\left(\frac{\partial}{\partial x^i}\Big|_p\right)(f) = \frac{\partial}{\partial x^i}\Big|_p(f \circ \phi) = \frac{\partial(f \circ \phi \circ \phi^{-1})}{\partial r^i}\Big|_{\phi(p)} = \frac{\partial f}{\partial r^i}\Big|_{\phi(p)}$$

$$= \frac{\partial}{\partial r^i}\Big|_{\phi(p)}(f)$$

$\square$

Local expression for the differential

Let $F: N \to M$ be smooth. Given $p \in N$

Choose chart $\phi: U \to \mathbb{R}^n$ on N st $p \in U$.

Choose chart $\psi: V \to \mathbb{R}^m$ on M st $F(U) \subseteq V$.

$$\begin{array}{ccc} T_p(N) & \xrightarrow{F_{*,p}} & T_{F(p)}(M) \\ \phi_{*,p} \downarrow & & \downarrow \psi_{*,F(p)} \\ T_{\phi(p)}(\mathbb{R}^n) & \longrightarrow & T_{\psi F(p)}(\mathbb{R}^m) \end{array}$$

Write $\phi = (x^1, \dots, x^n)$

$\psi = (y^1, \dots, y^m)$.

Basis $\left\{ \frac{\partial}{\partial x^1}\Big|_p \dots \frac{\partial}{\partial x^n}\Big|_p \right\}$ for $T_p(N)$

$\left\{ \frac{\partial}{\partial y^1}\Big|_{F(p)} \dots \frac{\partial}{\partial y^m}\Big|_{F(p)} \right\}$ for $T_{F(p)}(M)$.

__Proposition__ With respect to these bases, $F_{*,p}$ is given by the Jacobian matrix for $\psi \circ F \circ \phi^{-1}$ at $\phi(p)$.

(In the textbook this is written $\left[\frac{\partial F^i}{\partial x^j}(p) \right]$)

<u>Proof</u> We know $F_{*,p}\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \sum_{k=1}^{m} a_{k,j} \frac{\partial}{\partial y^k}\Big|_{F(p)}$,

for some real numbers a_{kj}.

Evaluate on $y^i \in C^\infty_{F(p)}(M)$:

$$a_{ij} = \left(\sum_{k=1}^{m} a_{kj} \frac{\partial}{\partial y^k}\Big|_{F(p)}\right)(y^i)$$

$$= F_{*,p}\left(\frac{\partial}{\partial x^j}\Big|_p\right)(y^i)$$

$$= \frac{\partial}{\partial x^j}\Big|_p (y^i \circ F)$$

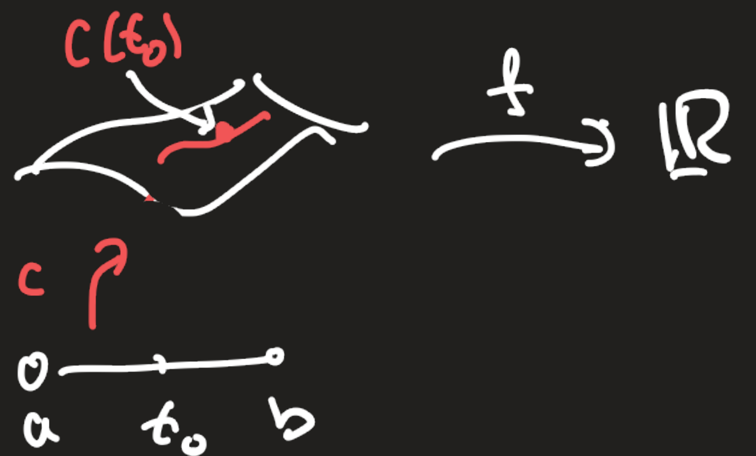
$$= \frac{\partial (y^i \circ F \circ \phi^{-1})}{\partial r^j}\phi(p) = \left\{\begin{array}{l}(i,j)\text{th entry of the} \\ \text{Jacobian for } \psi \circ F \circ \phi^{-1} \\ \text{at } \phi(p)\end{array}\right\}$$

(Alternative argument:
$\psi_{*,F(p)} \circ F_{*,p} \circ \phi^{-1}_{*\,\phi(p)} = (\psi \circ F \circ \phi^{-1})_{*,p}$ by the
chain rule.) $\square$

Curves on a smooth manifold

In general, a smooth curve $c : (a,b) \longrightarrow M$ defines a tangent vector $c'(t_0) \in T_{c(t_0)}(M)$ for all $t_0 \in (a,b)$:

$$c'(t_0) : C^\infty_{c(t_0)}(M) \longrightarrow \mathbb{R}, \quad c'(t_0)(f) = \frac{d(f \circ c)}{dt}(t_0).$$



$c'(t_0)$ is the "directional derivative" of $f \in C^\infty_{c(t_0)}(M)$ with respect to c.

<u>Notice</u> By definition

$$c'(t_0) = c_{*,t_0}\left(\frac{d}{dt}\Big|_{t_0} \right) \in T_{c(t_0)}(M), \text{ where}$$

$c_{*,t_0} : T_{t_0}(\mathbb{R}) \longrightarrow T_{c(t_0)}(M)$, $\quad \frac{d}{dt}\Big|_{t_0}$ basis for $T_{t_0}(\mathbb{R})$.

Remark Suppose $M = U \subseteq \mathbb{R}^n$ open subset.

Let $c : (a,b) \to U$ be a smooth curve.

Write $c(t) = (c^1(t), \dots, c^n(t))$, $c^1 \dots c^n : (a,b) \to \mathbb{R}$.

Recall isomorphisms $T_p(U) \longleftrightarrow \mathbb{R}^n$

$$\left\{ \frac{\partial}{\partial r^1}\Big|_p \dots \frac{\partial}{\partial r^n}\Big|_p \right\} \longleftrightarrow \{e_1 \dots e_n\} \text{ standard basis vectors.}$$

Write $c'(t_0) = a^1(t_0) \frac{\partial}{\partial r^1}\Big|_{c(t_0)} + \dots + a^n(t_0) \frac{\partial}{\partial r^n}\Big|_{c(t_0)}$

Then $a^i(t) = c'(t_0)(r^i) = \dfrac{d(r^i \circ c)}{dt}(t_0) = \dfrac{dc^i}{dt}(t_0) =: \dot{c}^i(t_0)$,

- this is the usual derivative of $c^i : (a,b) \to \mathbb{R}$ at t_0.

Conclusion $c'(t_0) \in T_p(U)$ can be identified with the column vector $\begin{bmatrix} \dot{c}^1(t_0) \\ \vdots \\ \dot{c}^n(t_0) \end{bmatrix} \in \mathbb{R}^n$.

prop Given a tangent vector $X_p \in T_p(M)$, there exists a smooth curve $c : (-\varepsilon, \varepsilon) \to M$ s.t. $c(0) = p$ and $c'(0) = X_p$.

Proof Choose a chart $\phi : U \to \mathbb{R}^n$ s.t. $p \in U$.

Isomorphism $T_p(M) \xrightarrow{\phi_{*,p}} T_{\phi(p)}(\mathbb{R}^n) \cong \mathbb{R}^n$.

$\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}$ $\quad$ $\left\{ \frac{\partial}{\partial r^i}\Big|_{\phi(p)} \right\}$

Write $X_p = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i}\Big|_p$, then $\phi_{*,p}(X_p) = \sum_{i=1}^n a_i \frac{\partial}{\partial r^i}\Big|_{\phi(p)}$.

Let $\alpha : (-\varepsilon, \varepsilon) \to \phi(U)$, $\alpha(t) = \phi(p) + t \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

Then $\alpha'(0) = \sum_{i=1}^n a_i \frac{\partial}{\partial r^i}\Big|_{\phi(p)}$

Let $c = \phi^{-1} \circ \alpha : (-\varepsilon, \varepsilon) \to M$, claim $c'(0) = X_p$

$$c'(0) = c_{*,0}\left(\frac{d}{dt}\Big|_0\right) = \phi^{-1}_{*,\phi(p)} \circ \alpha_{*,0}\left(\frac{d}{dt}\Big|_0\right) = \phi^{-1}_{*,\phi(p)}\left(\alpha'(0)\right)$$
$$= X_p \qquad \square$$