The rank of a smooth map

Let $f: N \to M$ be a smooth map, $\dim N = n$, $\dim M = m$

Def . The rank of f at a point $p \in N$ is the rank of the linear transformation $f_{*,p}: T_p(N) \to T_{f(p)}(M)$

- f has constant rank in a nbh. of p if there exists a nbh $p \in U$ st. $\operatorname{rk} f_{*,q} = \operatorname{rk} f_{*,p}$ for all $q \in U$.

Constant rank theorem for manifolds

If $f: N \to M$ has constant rank k in a nbh of $p \in N$, then there exist charts $\phi: U \to \mathbb{R}^n$ on N st. $p \in U$ and $\phi(p) = 0$

$\psi: V \to \mathbb{R}^m$ on M st. $f(p) \in V$ and $\psi(f(p)) = 0$

such that $f(U) \subseteq V$ and $\psi \circ f \circ \phi^{-1}(r^1, \dots, r^k, r^{k+1}, \dots, r^n) = (r^1, \dots, r^k, 0, \dots, 0)$

Constant rank level set theorem

Suppose $f: N \to M$ is smooth, $\dim N = n$, $\dim M = m$. Let $c \in M$ be in the image of f and suppose f has constant rank k in an open set containing $f^{-1}(c)$. Then $f^{-1}(c)$ is a submanifold of dimension $n-k$.

Proof Given $p \in f^{-1}(c)$, choose charts ϕ, ψ as in the constant rank theorem

$$
\begin{tikzcd}
(N, p) \arrow[r, "f"] & (M, f(p)) \\
\cup & \cup \\
(U, p) \arrow[d, "\phi"'] \arrow[r, phantom, "f"] & (V, f(p) = c) \arrow[d, "\psi"] \\
(\phi(U), 0) \arrow[r] & (\psi(V), 0)
\end{tikzcd}
$$

$$(r^1, .., r^k, r^{k+1}, .., r^n) \longmapsto (r^1, .., r^k, 0, .., 0)$$

<u>Claim</u>: $\phi(U \cap f^{-1}(c)) = \phi(U) \cap (\{0\} \times \mathbb{R}^{n-k})$

Write $\phi = (x^1, ..., x^n)$

$$U \cap f^{-1}(c) = \{ q \in U : f(q) = c \} = \{ q \in U : (\psi \circ f \circ \phi^{-1})\, \phi(q) = 0 \}$$

$$= \{ q \in U : (\psi \circ f \circ \phi^{-1})\,(x^1(q), .., x^n(q)) = 0 \}$$

$$= \{ q \in U : x^1(q) = .. = x^k(q) = 0 \}$$ Hence ϕ is adapted to $f^{-1}(c)$

$\square$

Ex. We will show that the subspace $O(n)$ of orthogonal matrices in $GL_n(\mathbb{R})$ is a submanifold.

$\boxed{\text{One can show } \dim O(n) = \frac{n^2-n}{2}}$

Let $f: GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $f(A) = A^T A$.

Then $O(n) = f^{-1}(I_n)$. Will show that f has constant rank on all of $GL_n(\mathbb{R})$.

Given $C \in GL_n(\mathbb{R})$, let $\ell_C, r_C : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ be defined by $\ell_C(A) = CA$, $r_C(A) = AC$, diffeomorphisms.

Claim $f \circ r_C = \ell_{C^T} \circ r_C \circ f : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$,

Given $A \in GL_n(A)$, $f \circ r_C(A) = f(AC) = (AC)^T AC = C^T A^T AC$

$$= C^T f(A) C = \ell_{C^T} \circ r_C \circ f(A).$$

By the chain rule $(f \circ r)_{*,A} = f_{*,AC} \circ (r_C)_{*,A}$

$$(\ell_{C^T} \circ r_C \circ f)_{*,A} = (\ell_{C^T})_{*,A^TAC} \circ (r_C)_{*,A^TA} \circ f_{*,A}$$

Hence $\operatorname{rk} f_{*,AC} = \operatorname{rk} f_{*,A}$ for all $C \in GL_n(\mathbb{R})$, since all $r_{(-)*}, \ell_{(-)*}$ are isomorphisms. This shows f has constant rank.

<u>Def</u> • A smooth map $f: N \to M$ is a submersion at $p \in N$ if $f_{*,p}: T_p(N) \to T_{f(p)}(M)$ is surjective.

This means that $\dim N \geq \dim M$ and $\mathrm{rk}\, f_{*,p} = \dim M$.

• f is a submersion if it is a submersion at all $p \in N$.

<u>Rmk</u> If f is a submersion at p, then f has maximal rank at p. This implies that f is a submersion in a nbh of p: In local coordinates, if the $m \times n$ matrix $\left[\partial f^i / \partial x^j (p) \right]$ has rank m, then there exists an invertible $m \times m$ submatrix. This submatrix has non-zero determinant in a nbh. of p.

<u>Consequence</u>: f has constant rank in a nbh of p, so the constant rank theorem applies: In local coordinates:

$(x^1, \ldots x^m, x^{m+1}, \ldots, x^n) \mapsto (x^1, \ldots x^m)$.

Rmk The regular level set theorem is a consequence of the constant rank level set theorem: If $f: N \to M$ is smooth and $c \in M$ is a regular value in the image of f, then f is a submersion at all $p \in f^{-1}(c)$. Hence f is also a submersion in an open set containing $f^{-1}(c)$. Thus $f^{-1}(c)$ is a submanifold with dimension $n-m$.

Def • $f: N \to M$ is an immersion at $p \in N$ if
$f_{*,p}: T_p(N) \to T_{f(p)}(M)$ is injective.
This means $n \leq m$ and $\operatorname{rk} f_{*,p} = \dim N$.

• f is an immersion if it is an immersion at all $p \in N$.

Rem If f is an immersion at p, then f has maximal rank at p. Hence f has constant rank in a nbh. of p, so the constant rank theorem: In local coordinates

$$(x^1, \ldots, x^n) \longmapsto (x^1, \ldots, x^n, 0, \ldots, 0)$$

Ex $f: \mathbb{R} \to \mathbb{R}^2$, $f(x) = (x^2, x^3)$, $J(f)(x) = \begin{bmatrix} 2x \\ 3x^2 \end{bmatrix}$ not an immersion at $x=0$

Theorem Let $N \subseteq M$ be a submanifold. Then $i: N \to M$ is an immersion.

proof $p \in N$, $i_{*,p}: T_p(N) \to T_p(M)$, must show injective.

Choose an adapted chart at p: $\phi: U \to \mathbb{R}^m$, such that

$\phi(U \cap N) = \phi(U) \cap (\mathbb{R}^n \times \{0\})$

$$\begin{tikzcd} N \arrow[r, "i"] & M \\ p \in U \cap N \arrow[u, phantom, "\cup"] \arrow[r] \arrow[d, "\phi|_N"'] & U \arrow[u, phantom, "\cup"] \arrow[d, "\phi"] \\ \mathbb{R}^n \supseteq \phi(U \cap N) \arrow[r] & \phi(U) \subseteq \mathbb{R}^m \end{tikzcd}$$

$(r^1, \dots, r^n) \mapsto (r^1, \dots, r^n, 0, \dots, 0)$

since $\phi \circ i \circ (\phi|_N)^{-1}$ has rank n, also i has rank n.

$\square$

Rem. Let $M \subseteq \mathbb{R}^n$ be a submanifold. Then $i_{*,p} : T_p(M) \to T_p(\mathbb{R}^n) \cong \mathbb{R}^n$ is injective. So we can identify $T_p(M)$ with a linear subspace of $\mathbb{R}^n$:

$$T_p(M) \cong \{ \dot{\alpha}(0) \in \mathbb{R}^n : \alpha : (-\varepsilon, \varepsilon) \to M \subseteq \mathbb{R}^n \text{ smooth}, \alpha(0) = p \}.$$

Ex. Let $f : \mathbb{R}^3 \to \mathbb{R}$ smooth. If $c \in \mathbb{R}$ is a regular value, then $M := f^{-1}(c) \subseteq \mathbb{R}^3$ is a smooth surface.

If $\alpha : (-\varepsilon, \varepsilon) \to M$ smooth curve with $\alpha(0) = p$, then $f \circ \alpha = c$ constant, so

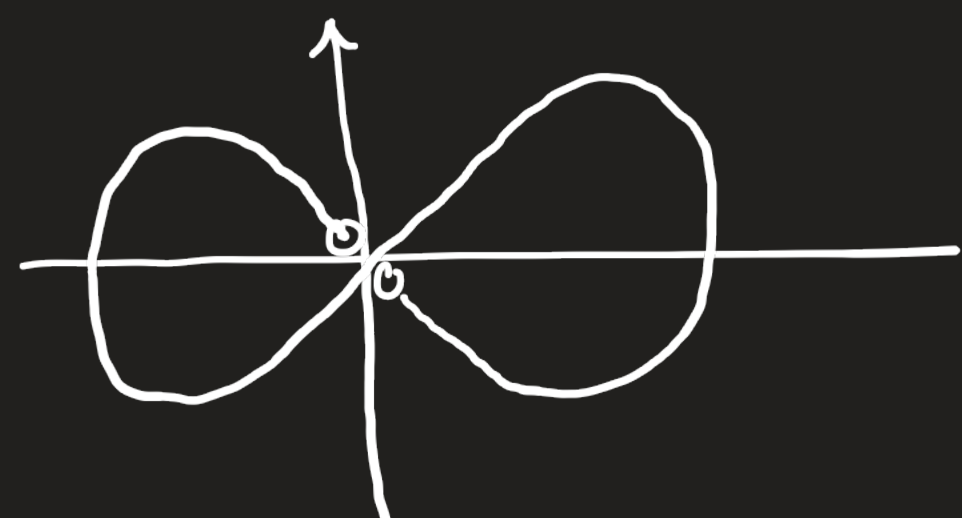
$$0 = \frac{d}{dt}\Big|_{t=0} (f \circ \alpha) = \frac{\partial f}{\partial x}(p) \cdot \dot{\alpha}^1(0) + \frac{\partial f}{\partial y}(p) \cdot \dot{\alpha}^2(0) + \frac{\partial f}{\partial z}(p) \cdot \dot{\alpha}^3(0).$$

Hence $T_p(M) \cong \{ (v^1, v^2, v^3) \in \mathbb{R}^3 : \frac{\partial f}{\partial x}(p) \cdot v^1 + \frac{\partial f}{\partial y}(p) \cdot v^2 + \frac{\partial f}{\partial z}(p) \cdot v^3 = 0 \}$

Question: If $f: N \to M$ is an injective immersion is $f(N) \subseteq M$ then necessarily a submanifold?

- No, not in general!

Ex $f: \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right) \to \mathbb{R}^2$, $f(t) = (\cos t, \sin 2t)$.



"$f\left(-\frac{\pi}{2}\right)$" $= (0,0)$

$f\left(\frac{\pi}{2}\right) = (0,0)$

"$f\left(\frac{3\pi}{2}\right)$" $= (0,0)$

f is injective, $f_{*,t}: T_t\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right) \to T_{f(t)}(\mathbb{R}^2)$ has

Jacobian $\begin{bmatrix} -\sin t \\ 2\cos 2t \end{bmatrix}$, $\sin t = 0 \Rightarrow t = p \cdot \pi$, but $\cos(2p\pi) = 1$.

Hence f is an injective immersion

But the figure space is not a manifold!

Def A smooth map $f: N \to M$ is an embedding if

(i) f is an immersion

(ii) $f: N \to f(N)$ is a homeomorphism ($f(N) \subseteq M$ has the subspace top.)

Theorem If $f: N \to M$ is an embedding, then $f(N)$ is a submanifold of M.

Proof f immersion $\Rightarrow$ f has constant rank n. Let $p \in N$.

By the rank theorem there are charts:

$$\begin{array}{ccc} (N, p) & \xrightarrow{f} & (M, f(p)) \\ \cup\!\!\mathsf{I} & & \cup\!\!\mathsf{I} \\ (U, p) & \xrightarrow{f} & (V, f(p)) \\ \phi \downarrow & & \downarrow \psi \\ \mathbb{R}^n \supseteq (\phi(U), 0) & \longrightarrow & (\psi(V), 0) \subseteq \mathbb{R}^m \\ (r^1, \dots, r^n) & \longmapsto & (r^1, \dots, r^n, 0, \dots, 0) \end{array}$$

we may assume that $\psi(V) \cap (\mathbb{R}^n \times \{0\}) = \phi(U) \times \{0\}$ by shrinking V if necessary

$f(U) \subseteq f(N)$ is open in the subspace topology.

Hence there exists an open set $V' \subseteq M$ s.t. $f(U) = f(N) \cap V'$.

Notice $V \cap V' \cap f(N) = V \cap f(U) = f(U)$

Claim $\psi : V \cap V' \to \mathbb{R}^m$ is an adapted chart at $f(p)$:

$$\psi(V \cap V' \cap f(N)) = \psi(V \cap V') \cap (\mathbb{R}^n \times \{0\})$$

Let $q \in V \cap V' \cap f(N)$, the $q = f(q')$, $q' \in U$.

Write $\phi = (x^1, .., x^n)$. Then

$\psi(q) = \psi f(q') = (\psi f \phi^{-1})(\phi(q'))$

$= (x^1(q'), \ldots, x^n(q'), 0, \ldots, 0)$

This gives the inclusion $\subseteq$.

The other inclusion $\supseteq$ follows from the last comment on the previous page. $\square$