Submanifolds (= Regular submanifolds)

Let M be a manifold of dimension n.

<u>Def</u> A subset $S \subseteq M$ is a submanifold of dimension k if for each $p \in S$, there exists a chart $\phi : U \to \mathbb{R}^n$ on M st $p \in U$ and $\phi(U \cap S) = \phi(U) \cap (\mathbb{R}^k \times \{0\})$

Equivalently, writing $\phi = (x^1, .., x^n)$, then
$U \cap S = \{ q \in U : x^{k+1}(q) = .. = x^n(q) = 0 \}$
such ϕ is called a chart adapted to S.

<u>Rmk</u> Suppose $\phi = (x^1, .., x^n)$ is a chart on M such that
$U \cap S = \{ q \in U : x^{j_1}(q) = . = x^{j_{n-k}}(q) = 0 \}$ for $1 \leq j_1 < .. < j_{n-k} \leq n$.
Then we can permute the coordinates to get an adapted chart.

Have proved:

<u>Proposition</u> Let $S \subseteq M$ be a k dimensional submanifold. Then the adapted charts define a smooth structure on S such that S is a k-dimensional manifold.

<u>Ex</u> $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$ is a submanifold of $\mathbb{R}^2$.

Consider $F: \mathbb{R}^2 \to \mathbb{R}^2$, $F(x,y) = (x, x^2+y^2-1)$

Then $S^1 = \{(x,y) \in \mathbb{R}^2 : F^2(x,y) = 0\}$.

(Notice: F is not injective, $F(x,y) = F(x,-y)$.)

Jacobian matrix $J(F) = \begin{bmatrix} 1 & 0 \\ 2x & 2y \end{bmatrix}$. $\det J(F) = 2y$.

Inverse function theorem: If $y \neq 0$, then there exists a nbh U of (x,y) st. $F: U \to F(U)$ is a diffeomorphism.

Get adapted chart for all $(x,y): y \neq 0$, $F(U \cap S^1) = F(U) \cap (\mathbb{R} \times \{0\})$

Similarly, using $G(x,y) = (y, x^2+y^2-1)$, we get an adapted chart for (x,y) st $x \neq 0$. $\square$

Exercise Let $L = \{(x,y) \in \mathbb{R}^2 : x \cdot y = 0,\ x \geq 0,\ y \geq 0\}$

show that L is homeomorphic to $\mathbb{R}$, but is not a submanifold of $\mathbb{R}^2$.

Def. Let $F : N \to M$ be a smooth map.

Given $c \in M$, the set $F^{-1}(c) = \{p \in N : F(p) = c\}$ is called the level set at level c

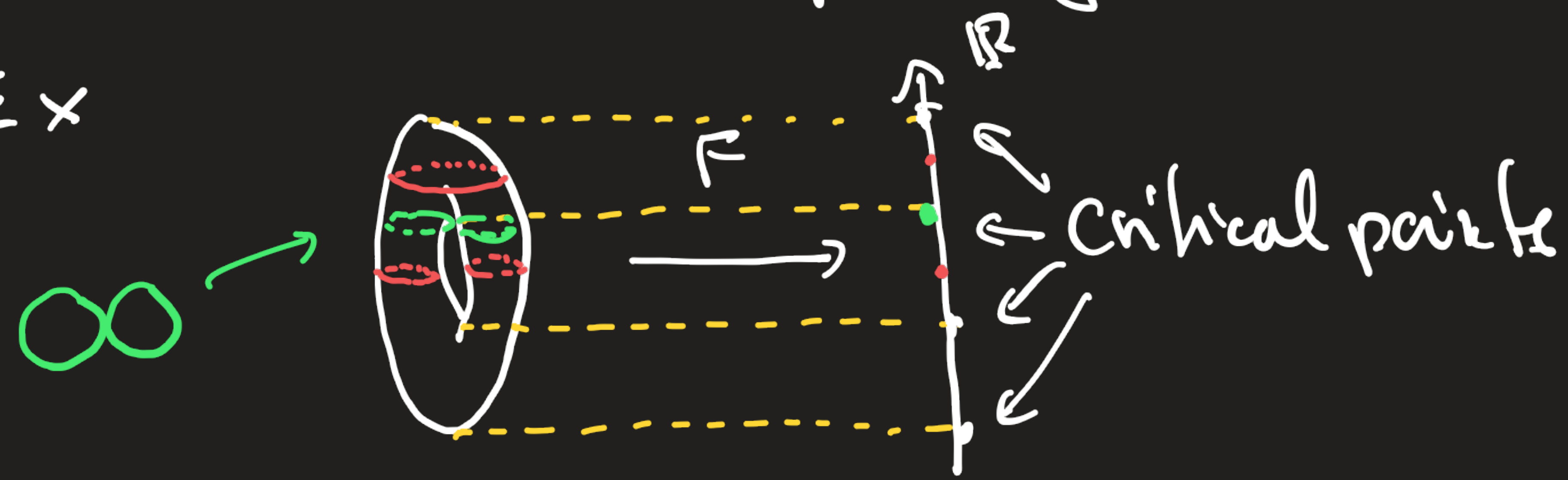
- $p \in N$ is a regular point if $F_{*,p} : T_p N \to T_{F(p)} M$ is surjective
- $p \in N$ is a critical point if $F_{*,p}$ is not surjective
- $c \in M$ is a regular value if all $p \in F^{-1}(c)$ are regular points.
- $c \in M$ is a critical value if there exists a critical point $p \in F^{-1}(c)$.

Notice It may happen that $F^{-1}(c) = \emptyset$. Then c is a regular value by definition.

Regular level set Theorem. Let $F: N \to M$ be smooth and suppose $\dim N = n$, $\dim M = m$. If $c \in M$ is a regular value in the image of F, then $F^{-1}(c)$ is an $(n-m)$-dim. submanifold of N.

Corollary Let $F: N \to \mathbb{R}$ be a smooth map. If $c \in \mathbb{R}$ is a regular value (in the image of F) then $F^{-1}(c)$ is an $(n-1)$-dim. submanifold of N.

Ex



<u>Ex</u> $F : \mathbb{R}^3 \to \mathbb{R}$, $F(x,y,z) = x^2 + y^2 + z^2$

$F_{*,p} : T_p \mathbb{R}^3 \to T_p \mathbb{R}$, Jacobian matrix $[2x \quad 2y \quad 2z]$.

The only critical point is $(0,0,0)$

so $F(0,0,0) = 0$ is the only critical value

<u>Consequence</u>: If $c > 0$, then

$$S = \{ (x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = c \}$$

is a 2-dimensional submanifold of $\mathbb{R}^3$ (a sphere).

Proof of the regular level set theorem $\quad c \in M$ regular value

Chart a chart $\psi : V \to \mathbb{R}^m$ on M s.t. $\psi(c) = 0$

Then $F^{-1}(c) = (\psi \circ F)^{-1}(0)$

Write $\psi = (y^1, \dots, y^m)$, let $F^i = y^i \circ F : F^{-1}(V) \to \mathbb{R}$.

Then $F^{-1}(c) = \{ p \in F^{-1}(V) : F^i(p) = 0 \text{ for } i = 1, \dots, m \}$

Given $p \in F^{-1}(c)$, must find a chart on N adapted to $F^{-1}(c)$.

Choose a chart $\phi : U \to \mathbb{R}^n$ on N s.t. $p \in U \subseteq F^{-1}(V)$

$$\begin{tikzcd}
N \arrow[r, "F"] & M \\
F^{-1}(c) \subseteq F^{-1}(V) \arrow[u, phantom, "\cup" description] \arrow[r, dashed, "F"] & V \arrow[u, phantom, "\cup" description] \arrow[d, "\psi"] \\
U \arrow[u, phantom, "\cup" description] \arrow[d, "\phi"'] & \mathbb{R}^m \\
\mathbb{R}^n &
\end{tikzcd}$$

$$T_p(N) \xrightarrow{F_{*,p}} T_{F(p)}(M)$$

bases $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}$ $\quad \left\{ \frac{\partial}{\partial y^i}\Big|_{F(p)} \right\}$

The Jacobian is $\left[\partial F^i / \partial x^j (p) \right]$

$\left(= \text{Jacobian for } \psi \circ F \circ \phi^{-1} \text{ at } \phi(p) \right)$

Write $\phi = (x^1, \dots, x^n)$

$F_{*,p}$ surjective $\Rightarrow \left[\partial F^i / \partial x^j (p) \right]$ has rank $= m$.

Since $\left[\partial F^i / \partial x^j (p) \right]$ has rank m, there is an invertible $m \times m$ submatrix. By permuting the coordinates, we may assume that $\left\{ \partial F^i / \partial x^j (p) \right\}_{\substack{1 \le i \le m \\ 1 \le j \le m}}$ is invertible.

Define $\Phi : U \to \mathbb{R}^n$. $\Phi = (F^1, \dots, F^m, x^{m+1}, \dots, x^n)$

<u>Claim</u>: Φ is a chart on N in a nbh U_p of p.

$$\left[\partial \Phi^i / \partial x^j \right] = \begin{bmatrix} \left\{ \partial F^i / \partial x^j \right\}_{\substack{1 \le i \le m \\ 1 \le j \le m}} & \left\{ \partial F^i / \partial x^j \right\}_{\substack{1 \le i \le m \\ m+1 \le j \le n}} \\ \left\{ \partial x^i / \partial x^j \right\}_{\substack{m+1 \le i \le n \\ 1 \le j \le m}} & \left\{ \partial x^i / \partial x^j \right\}_{\substack{m+1 \le i \le n \\ m+1 \le j \le n}} \end{bmatrix}$$

$$= \begin{bmatrix} \left\{ \partial F^i / \partial x^j \right\}_{\substack{1 \le i \le m \\ 1 \le j \le m}} & ? \\ 0 & I \end{bmatrix}$$

Hence

$$\det \left[\partial \Phi^i / \partial x^j (p) \right] = \det \left\{ \frac{\partial F^i}{\partial x^j} \right\}_{\substack{1 \le i \le m \\ 1 \le j \le m}} \neq 0$$

By the inverse function theorem Φ is a chart in a possibly smaller nbh U_p of p.

This is in fact a chart adapted to $F^{-1}(c)$:

$$\Phi(U_p \cap F^{-1}(c)) = \Phi(U_p) \cap (\{0\} \times \mathbb{R}^{n-m}), \text{ since}$$

$F^1, \dots, F^m$ vanish on $F^{-1}(c)$. $\square$

Ex. Let $S \subseteq \mathbb{R}^3$ be defined by $\begin{cases} x^3+y^3+z^3=1 \\ x+y+z=0 \end{cases}$

<u>Claim</u> S is a 1-dim. submanifold of $\mathbb{R}^3$.

Let $F: \mathbb{R}^3 \to \mathbb{R}^2$, $F(x,y,z) = \begin{bmatrix} x^3+y^3+z^3 \\ x+y+z \end{bmatrix}$

Jacobian matrix $\begin{bmatrix} 3x^2 & 3y^2 & 3z^2 \\ 1 & 1 & 1 \end{bmatrix}$

(x,y,z) is a regular point iff rank $J(F) = 2$.

$$\begin{vmatrix} 3x^2 & 3y^2 \\ 1 & 1 \end{vmatrix} = 3(x^2-y^2) = 0 \Rightarrow x = \pm y$$

$$\begin{vmatrix} 3x^2 & 3z^2 \\ 1 & 1 \end{vmatrix} = 3(x^2-z^2) = 0 \Rightarrow x = \pm z.$$

Consequence: (x,y,z) is a critical point iff $|x|=|y|=|z|$.

If also $x+y+z=0$, then $x=y=z=0$

Consequence: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a regular value for F.

Ex $\mathrm{Mat}_{2\times 2}(\mathbb{R}) \cong \mathbb{R}^{2\times 2} \cong \mathbb{R}^4$.

Let $SL_2(\mathbb{R}) = \{ A \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) : \det A = 1 \}$.

Claim $SL_2(\mathbb{R})$ is a 3-dim submanifold of $\mathrm{Mat}_{2\times 2}(\mathbb{R})$.

$\det : \mathrm{Mat}_{2\times 2}(\mathbb{R}) \to \mathbb{R}$, $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$.

Jacobian matrix of det: $[d \ -c \ -b \ d]$

The only critical point $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Hence 1 is a regular value for det, so

$SL_2(\mathbb{R}) = \det^{-1}(1)$ is a 3-dim. submanifold of $\mathrm{Mat}_{2\times 2}(\mathbb{R}) \cong \mathbb{R}^4$.

<u>Claim</u> In general $SL_n(\mathbb{R})$ is an n^2-1-dim submanifold of $Mat_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$.

Consider $\det : Mat_{n\times n}(\mathbb{R}) \to \mathbb{R}$, $\det^{-1}(1) = SL_n(\mathbb{R})$.

Given $A = [a_{ij}]$, we know $\det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{1i} \det(A_{1i})$,

where A_{1i} is a submatrix of A.

Hence $\frac{\partial \det}{\partial x_{11}}(A) = \det(A_{11})$, $\frac{\partial \det}{\partial x_{12}}(A) = -\det(A_{12})$,

$\ldots, \frac{\partial \det}{\partial x_{1n}}(A) = (-1)^{n+1}\det(A_{1n})$

Hence if all $\frac{\partial \det}{\partial x_{1i}}(A) = 0$, we have $\det(A) = 0$

Hence 0 is the only critical value.

In particular, 1 is a regular value.