<u>Ex 9.6</u> Let $F(x_0,\dots,x_n) \in \mathbb{R}[x_0\dots x_n]$ be a homogeneous polynomial of degree k. Then for $t \in \mathbb{R}$

$$F(tx_0,\dots,tx_n) = t^k F(x_0,\dots,x_n)$$

show $$\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i} = kF \quad \circledast$$

Use the chain rule to get

$$\left.\frac{d}{dt}\right|_t F(tx_0,\dots,tx_n) = \sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i}(tx_0\dots tx_n)$$

$$\frac{d}{dt} t^k F(x_0\dots x_n) = k\,t^{k-1} F(x_0,\dots x_n)$$

Set $t=1$, we get $\circledast$

Ex 9.7 $F(x_0, x_1, x_2) \in \mathbb{R}[x_0, x_1, x_2]$ homogeneous of deg. k.

$Z(F) = \{ [x_0, x_1, x_2] \in \mathbb{RP}^2 : F(x_0, x_1, x_2) = 0 \}$

Claim If $F : \mathbb{R}^3 \to \mathbb{R}$ is a submersion, then $Z(F)$ is a submanifold of $\mathbb{RP}^2$ of dimension 1.

Given $P = [x_0, x_1, x_2] \in Z(F)$. Suppose $x_0 \neq 0$.

Let $U_0 = \{ [x_0, x_1, x_2] : x_0 \neq 0 \}$ $\phi_0 : U_0 \to \mathbb{R}^2$,

$\phi_0([x_0, x_1, x_2]) = (x_1/x_0, x_2/x_0)$.

Let $f(x, y) = F(1, x, y)$. $: \mathbb{R}^2 \to \mathbb{R}$.

$$\begin{array}{ccc} U_0 & \xrightarrow{\phi_0} & \mathbb{R}^2 \\ \cup & & \cup \\ U_0 \cap Z(F) & \longleftrightarrow & Z(f) \\ [1, x, y] & \longleftarrow\!\shortmid & (x, y) \end{array}$$

suffices to show $Z(f) \subseteq \mathbb{R}^2$ is a submanifold.

$\frac{\partial f}{\partial x}(x, y) = \frac{\partial F}{\partial x}(1, x, y)$, $\frac{\partial f}{\partial y}(x, y) = \frac{\partial F}{\partial y}(1, x, y)$

We must show that one of

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial F}{\partial x_1}(1,x,y) \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = \frac{\partial F}{\partial x_2}(1,x,y)$$

is non-zero for (x,y) in $Z(f)$.

Suppose $\frac{\partial F}{\partial x_1}(1,x,y) = \frac{\partial F}{\partial x_2}(1,x,y) = 0$.

Know $\quad 0 = kF(1,x,y) = \frac{\partial F}{\partial x_0}(1,x,y) + x\frac{\partial F}{\partial x_1}(1,x,y) + y\frac{\partial F}{\partial x_2}(1,x,y)$

$$= \frac{\partial F}{\partial x_0}(1,x,y) \quad \lightning$$

Hence the result follows from the regular level set theorem.

Ex. E $\quad L = \{ (x,y) \in \mathbb{R}^2 : x \cdot y = 0 ,\ x \geq 0 ,\ y \geq 0 \}$

(1) show L is homeomorphic to $\mathbb{R}$.

$(x,y) \mapsto x - y$

(2) show L is not a submanifold of $\mathbb{R}^2$.

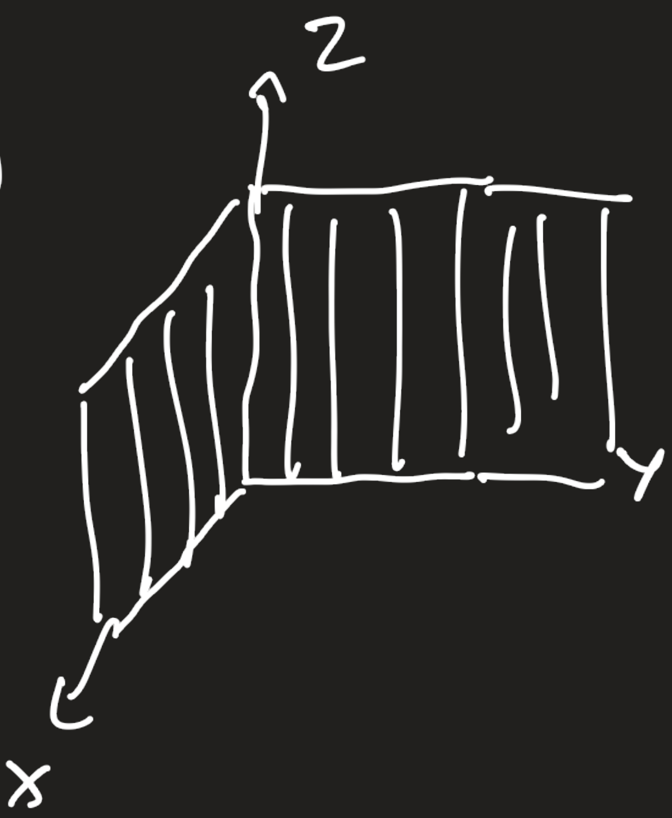
Suppose $F : U \to U' \subseteq \mathbb{R}^2$ is a chart on $\mathbb{R}^2$ that is adapted to L. Look at $\frac{\partial F^2}{\partial x}, \frac{\partial F^2}{\partial y}$

$$\frac{\partial F^2}{\partial x}(0,0) = \lim_{x \to 0} \frac{F^2(x,0) - F^2(0,0)}{x} = \lim_{x \to 0+} \frac{F^2(x,0) - F^2(0,0)}{x}$$

$= 0$ since by assumption $F(U \cap L) \subseteq U' \cap (\mathbb{R} \times \{0\})$

Similarly $\frac{\partial F^2}{\partial y}(0,0) = 0$, so $J(F)$ is singular at $(0,0)$ ↯

(3)



is not a submanifold of $\mathbb{R}^3$.