The exterior derivative

Let M be a smooth manifold

Recall that $\Omega^k(M)$ is the vector space of smooth k-forms on M. For $k=0$, $\Omega^0(M) = C^\infty(M)$.

The differential $d : \Omega^0(M) \to \Omega^1(M)$ is defined by

$df = \{ df_p \in T_p^*M : p \in M \}$, where $df_p(v) = v(f)$, $v \in T_pM$

Notice: $d(f \cdot g) = (df)g + f(dg)$:

For each $p \in M$ and $v \in T_pM$,

$$d(fg)_p(v) = v(fg) = v(f)\,g(p) + f(p)\,v(g)$$
$$= df_p(v)\,g(p) + f(p)\,dg_p(v)$$

Def An $\mathbb{R}$-linear map of deg 1 $D: \Omega^*(M) \to \Omega^*(M)$ is a sequence of $\mathbb{R}$-linear maps $D: \Omega^k(M) \to \Omega^{k+1}(M)$, $k \geq 0$.

D is an antiderivation of deg. 1 if

$$D(\omega \wedge \tau) = D(\omega) \wedge \tau + (-1)^k \omega \wedge D(\tau), \text{ for } \omega \in \Omega^k(M),\ \tau \in \Omega^\ell(M)$$

Def An exterior derivation on M is an $\mathbb{R}$-linear map of deg 1, $D: \Omega^*(M) \to \Omega^*(M)$, such that

(i) D is an antiderivation of deg. 1.

(ii) $D \circ D = 0$ (i.e., $\Omega^k(M) \xrightarrow{D} \Omega^{k+1}(M) \xrightarrow{D} \Omega^{k+2}(M)$ is 0)

(iii) $D = d: \Omega^0(M) \to \Omega^1(M)$ (the differential).

Let M be a manifold and $\phi = (x^1 .. x^n) : U \to \mathbb{R}^n$ a chart.

Every $\omega \in \Omega^k(M)$ can be written uniquely as

$$\omega = \sum_I a_I \, dx^I \text{, where } a_I : U \to \mathbb{R} \text{ are smooth}$$

Here $I = (1 \le i_1 < \dots < i_k \le n)$, $dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_k}$.

Let $d_U = d : \Omega^0(U) \to \Omega^1(U)$, $d_U(f) = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$.

For $k \geq 1$ define $d_U : \Omega^k(U) \to \Omega^{k+1}(U)$ by

$$d_U\left(\sum_I a_I \, dx^I\right) = \sum_I da_I \wedge dx^I = \sum_I \sum_{i=1}^n \frac{\partial f}{\partial x^i} \wedge dx^i \wedge dx^I.$$

Prop $d_U : \Omega^*(U) \to \Omega^*(U)$ is the unique exterior derivative on $\Omega^*(U)$.

Proof As in the case $U \subseteq \mathbb{R}^n$, see Prop 4.7 and 4.8 $\square$.

Let $\psi = (y^1, \dots, y^n) : V \to \mathbb{R}^n$ be another chart on M.

We similarly get $d_V : \Omega^*(V) \to \Omega^*(V)$ exterior derivative on V.

It follows by uniqueness that d_U and d_V "restrict" to the same exterior derivative on $\Omega^*(U \cap V)$.

Construction of the exterior derivative on M

Define $d : \Omega^k(M) \to \Omega^{k+1}(M)$ for $k \geq 1$.

Given $\omega \in \Omega^k(M)$ and $p \in M$, choose a chart $\phi = (x^1, \dots, x^n) : U \to \mathbb{R}^n$ such that $p \in U$.
Now use $d_U : \Omega^k(U) \to \Omega^{k+1}(U)$:
Let $(d\omega)_p = d_U(\omega|_U)_p \in A_{k+1}(T_pM)$
(This is independent of the choice of ϕ).

Letting p vary, this gives a smooth section $d\omega : M \to A_{k+1}(TM)$, hence an element $d\omega \in \Omega^{k+1}(M)$.

Prop $d : \Omega^*(M) \to \Omega^*(M)$ is an exterior derivative.

Proof We can check the conditions locally at each p. Hence the result follows since each d_U is an exterior derivative. $\square$

Prop (Uniqueness) Suppose $D : \Omega^*(M) \to \Omega^*(M)$ is an exterior derivative on M. Then $D = d$.

Proof Given $\omega \in \Omega^k(M)$ and $p \in M$, must show $D\omega_p = d\omega_p$.

Choose a chart $\phi = (x^1, \dots, x^n) : U \to \mathbb{R}^n$ such that $p \in U$.

Write $\omega|_U = \sum_I a_I \, dx^I$.

Use bump functions to extend $a_I, x^1 \dots x^n$ to functions $\tilde{a}_I, \tilde{x}^1 \dots \tilde{x}^n : M \to \mathbb{R}$ s.t. $\tilde{a}_I = a_I$, $\tilde{x}^i = x^i$ in a nbh. of p.

Let $\tilde{\omega} = \sum_I \tilde{a}_I \, d\tilde{x}^I$, then $\tilde{\omega} \in \Omega^k(M)$ and $\tilde{\omega} = \omega$ in a nbh. of p. Since D is a local operator,

$$(D\omega)_p = (D\tilde{\omega})_p = \sum_I \left(D\tilde{a}_I \wedge d\tilde{x}^I + \tilde{a}_I D(d\tilde{x}^I) \right)_p$$

$$= \sum_I (d\tilde{a}_I \wedge d\tilde{x}^I)_p = \sum_I (da_I \wedge dx^I)_p = (d\omega)_p \qquad \square$$

Let $F: N \to M$ be a smooth map. Recall the the pullback construction gives a linear map $F^*: \Omega^k(M) \to \Omega^k(N)$.

<u>Prop</u> $\underline{F^*(d\omega)} = \underline{d(F^*\omega)}$ for $\omega \in \Omega^k(M)$.

<u>Proof</u> We have checked that for $k=0$. Let $\omega \in \Omega^k(M), k \geq 1$.

Given $p \in M$, must show $F^*(d\omega)_p = d\,F^*(\omega)_p$.

choose charts $\left.\begin{array}{l}\phi = (x^1, \dots x^n) : U \to \mathbb{R}^n \text{ on } N \\ \psi = (y^1, \dots y^m) : V \to \mathbb{R}^m \text{ on } M\end{array}\right\}$ such that $p \in U$ and $F(U) \subseteq V$.

write $\omega_{|V} = \sum_I a_I \, dy^I$.

$$\underline{F^*(d\omega)}_{|U} = (F|_U)^*(d\omega_{|V}) = (F|_U)^*\left(\sum_I da_I \wedge dy^I\right)$$

$$= \sum_I F^*(da_I) \wedge F^*(dy^{i_1}) \wedge \dots \wedge F^*(dy^{i_k}) = \sum_I \underline{d(F^* a_I) \wedge d(F^* y^{i_1}) \wedge \dots \wedge d(F^* y^{i_k})}$$

$$\underline{d\,F^*(\omega)}_{|U} = d\,(F|_U)^*(\omega_{|U}) = d\left(\sum_I F^*(a_I)\, F^*(dy^{i_1}) \wedge \dots \wedge F^*(dy^{i_k})\right)$$

$$= d\left(\sum_I F^*(a_I)\, d(F^* y^{i_1}) \wedge \dots \wedge d(F^* y^{i_k})\right) = \sum_I \underline{d\,F^*(a_I) \wedge d(F^* y^{i_1}) \wedge \dots \wedge d(F^* y^{i_k})}$$

Rem The previous proposition shows that the diagram

$$\begin{tikzcd} \Omega^k(M) \arrow[r, "d"] \arrow[d, "F^*"'] & \Omega^{k+1}(M) \arrow[d, "F^*"] \\ \Omega^k(N) \arrow[r, "d"] & \Omega^{k+1}(N) \end{tikzcd}$$

is commutative.

In the textbook (Prop 19.5), this is proved without knowing that $F^*(\omega)$ is smooth.

This does not make sense!