Differential forms on open subsets of $\mathbb{R}^n$

Let $U \subseteq \mathbb{R}^n$ be an open subset.

Def. A k-form on U is a collection of alternating k-linear functions $\omega = \{\omega_p \in A_k(T_p\mathbb{R}^n) : p \in U\}$.

Rem A k-form on U is also called a differential form of degree k on U, or a k-covector field on U.

Ex A C^∞ function $f: U \to \mathbb{R}$ gives rise to a 1-form on U: $df = \{df_p \in A_1(T_p\mathbb{R}^n) : p \in U\}$, $df_p(X_p) = X_p(f)$

In particular, the coordinate functions $x^i : U \to \mathbb{R}$ define 1-forms dx^i, $i = 1, .. n$.

Have checked: $dx^1_p, .., dx^n_p$ is the dual basis for $(T_p\mathbb{R}^n)^\vee = A_1(T_p\mathbb{R}^n)$ wrt. the basis $\left\{ \frac{\partial}{\partial x^1}\Big|_p, .., \frac{\partial}{\partial x^n}\Big|_p \right\}$ for $T_p\mathbb{R}^n$.

A k-form can be written uniquely in the form

$$\omega = \sum_I a_I \, dx^I, \text{ where } dx^I_p = dx^{i_1}_p \wedge \dots \wedge dx^{i_k}_p, \ 1 \le i_1 < \dots < i_k \le n,$$

and $a_I : U \to \mathbb{R}$ is a function.

<u>Def</u> We say that ω is a C^∞ k-form if each a_I is a C^∞ function.

<u>Def.</u> $\Omega^k(U)$ is the real vector space of C^∞ k-forms on U.

By definition, $\Omega^0(U) = C^\infty(U) = \{C^\infty \text{ function } U \to \mathbb{R}\}$

Notice: If $U \subseteq \mathbb{R}^n$, then $\Omega^k(U) = 0$ if $k > n$.

This is because $A_k(T_p\mathbb{R}^n) = 0$ for $k > n$.

Wedge product $\wedge : \Omega^k(U) \times \Omega^l(U) \to \Omega^{k+l}(U)$.

Given $\omega \in \Omega^k(U)$, $\tau \in \Omega^l(U)$, $(\omega \wedge \tau)_p = \omega_p \wedge \tau_p \in A_{k+l}(T_p\mathbb{R}^n)$

<u>Notice</u> $\omega = \sum_I a_I dx^I$, $\tau = \sum_J b_J dx^J$, then

$$\omega \wedge \tau = \sum_{I,J} a_I b_J \, dx^I \wedge dx^J = \sum_{I,J \text{ disjoint}} a_I b_J \, dx^I \wedge dx^J$$

This shows that $\omega \wedge \tau$ is C^∞.

<u>Multiplicative unit</u> $1 \in \Omega^0(U) = C^\infty(U)$

<u>Proposition</u> The wedge product makes

$$\Omega^*(U) = \{ \Omega^k(U) : k \geq 0 \}$$

an anticomm. graded $\mathbb{R}$-algebra.

- Follows since $A_*(T_p\mathbb{R}^n) = \{ A_k(T_p\mathbb{R}^n) : k \geq 0 \}$ is an anticomm. graded $\mathbb{R}$-algebra for all $p \in U$.

Exterior derivative

The differential defines an $\mathbb{R}$-linear function

$$d : C^\infty(U) = \Omega^0(U) \to \Omega^1(U).$$

<u>Prop</u>: If $f : U \to \mathbb{R}$ is C^∞, then $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i$.

<u>Proof</u> We can write $df = \sum_{i=1}^{n} a_i dx^i$ for some functions $a^i : U \to \mathbb{R}$.

$$\left.\begin{aligned} df_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) &= \frac{\partial}{\partial x^i}\Big|_p (f) = \frac{\partial f}{\partial x^i}(p) \\ \left(\sum_{j=1}^{n} a_j(p)\, dx^j_p\right)\left(\frac{\partial}{\partial x^i}\Big|_p\right) &= a_i(p) \end{aligned}\right\} \Rightarrow a_i = \frac{\partial f}{\partial x^i}$$

$\square$

This shows that df is a C^∞ 1-form and that d is $\mathbb{R}$-linear.

Def For $k \geq 1$, the exterior derivative $d : \Omega^k(U) \to \Omega^{k+1}(U)$ is the $\mathbb{R}$-linear function defined as follows:

If $\omega = \sum_I a_I \, dx^I$, then $d\omega = \sum_I (da_I) \wedge dx^I$.

(Notice $d\omega = \sum_I \sum_{i=1}^n \frac{\partial a_I}{\partial x^i} dx^i \wedge dx^I$).

Ex. $U \subseteq \mathbb{R}^2$ open $d : \Omega^1(U) \to \Omega^2(U)$.

$$\omega = f\,dx + g\,dy$$

$$\begin{aligned} d\omega &= (df) \wedge dx + (dg) \wedge dy \\ &= \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right) \wedge dx + \left(\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy\right) \wedge dy \\ &= \frac{\partial f}{\partial y} dy \wedge dx + \frac{\partial g}{\partial x} dx \wedge dy \\ &= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy. \end{aligned}$$

Prop Properties of $d : \Omega^k(U) \to \Omega^{k+1}(U)$.

(i) $d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge (d\tau)$

(ii) $d \circ d = 0$: $\Omega^k(U) \xrightarrow{d} \Omega^{k+1}(U) \xrightarrow{d} \Omega^{k+2}(U)$

(iii) If X is a C^∞ vector field on U and $f \in \Omega^0(U) = C^\infty(U)$, then $df_p(X_p) = X_p(f)$ for all $p \in U$.

Proof (i) By linearity it suffices to consider $\omega = f\, dx^I \in \Omega^k(U)$, $\tau = g\, dx^J \in \Omega^l(U)$

$$d(\omega \wedge \tau) = d(f \cdot g\, dx^I \overset{(*)}{\wedge} dx^J) = \sum_{i=1}^n \frac{\partial (fg)}{\partial x^i} dx^i \wedge dx^I dx^J$$

$$= \sum_{i=1}^n \frac{\partial f}{\partial x^i} \cdot g\, dx^i \wedge dx^I \wedge dx^J + \sum_{i=1}^n f \frac{\partial g}{\partial x^i} dx^i \wedge dx^I \wedge dx^J$$

$$= \left(\sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i \wedge dx^I \right) \wedge g\, dx^J + (-1)^k f\, dx^I \wedge \left(\sum_{i=1}^n \frac{\partial g}{\partial x^i} dx^i \wedge dx^J \right)$$

$$= (d\omega) \wedge \tau + (-1)^k \omega \wedge (d\tau). \qquad \square$$

$\boxed{\text{Rmk. The formula for } d \text{ is ok also for } dx^I \wedge dx^J \text{ in } (*)}$

(ii) By linearity suffices to prove $d \circ d(\omega) = 0$ for $\omega = f\,dx^I$.

$$d \circ d(\omega) = d\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i \wedge dx^I\right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i \wedge dx^I$$

(Terms with $i=j$ vanish)

$$= \sum_{i<j} \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i \wedge dx^I + \sum_{i>j} \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i \wedge dx^I = 0$$

since $\dfrac{\partial^2 f}{\partial x^j \partial x^i} = \dfrac{\partial^2 f}{\partial x^i \partial x^j}$ and $dx^j \wedge dx^i = -dx^i \wedge dx^j$.

(iii) Follows from the definition of $d: \Omega^0(U) \to \Omega^1(U)$:

$df_p(X_p) = X_p(f)$. $\square$