Ex. 18.3

$F : N \to M$ smooth, $F^* : \Omega^*(M) \to \Omega^*(N)$.

Let $\omega \in \Omega^k(M)$, $\tau \in \Omega^l(M)$. claim: $F^*(\omega \wedge \tau) = F^*(\omega) \wedge F^*(\tau)$.

Let $p \in N$, $v_1, \dots, v_{k+l} \in T_pN$.

$(*)$ $F^*(\omega \wedge \tau)_p(v_1, \dots, v_{k+l}) = (\omega \wedge \tau)_{F(p)}\big(F_{*,p}(v_1), \dots, F_{*,p}(v_{k+l})\big)$

$$= (\omega_{F(p)} \wedge \tau_{F(p)})\big(F_{*,p}(v_1), \dots, F_{*,p}(v_{k+l})\big)$$

$$= \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\, \omega_{F(p)}\big(F_{*,p}(v_{\sigma(1)}), \dots, F_{*,p}(v_{\sigma(k)})\big) \cdot \tau_{F(p)}\big(F_{*,p}(v_{\sigma(k+1)}), \dots, F_{*,p}(v_{\sigma(k+l)})\big)$$

$(**)$ $(F^*(\omega) \wedge F^*(\tau))_p(v_1, \dots, v_{k+l}) = (F^*\omega)_p \wedge F^*(\tau)_p\,(v_1, \dots, v_{k+l})$

$$= \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\, F^*(\omega)_p(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \cdot F^*(\tau)_p(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)})$$

$$= \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\, \omega_{F(p)}\big(F_{*,p}(v_{\sigma(1)}), \dots, F_{*,p}(v_{\sigma(k)})\big) \cdot \tau_{F(p)}\big(F_{*,p}(v_{\sigma(k+1)}), \dots, F_{*,p}(v_{\sigma(k+l)})\big).$$

Ex. 18.4 $\quad \omega \in \Omega^k(M)$, $\operatorname{supp}(\omega) = \overline{\{p \in M : \omega_p \neq 0\}}$

(a) $\quad \operatorname{supp}(\omega + \tau) \subseteq \operatorname{supp}(\omega) \cup \operatorname{supp}(\tau)$, $\quad \omega, \tau \in \Omega^k(M)$.

$$\{p \in M : \omega_p + \tau_p \neq 0\} \subseteq \{p \in M : \omega_p \neq 0\} \cup \{p \in M : \tau_p \neq 0\} \subseteq \operatorname{supp}(\omega) \cup \operatorname{supp}(\tau)$$

$\Rightarrow \quad \operatorname{supp}(\omega + \tau) \subseteq \operatorname{supp}(\omega) \cup \operatorname{supp}(\tau)$.

(b) $\quad \operatorname{supp}(\omega \wedge \tau) \subseteq \operatorname{supp}(\omega) \cap \operatorname{supp}(\tau)$

$$\{p \in M : \omega_p \wedge \tau_p \neq 0\} \subseteq \{p \in M : \omega_p \neq 0\} \cap \{p \in M : \tau_p \neq 0\}$$
$$\subseteq \operatorname{supp}(\omega) \cap \operatorname{supp}(\tau)$$

$\Rightarrow \quad \operatorname{supp}(\omega \wedge \tau) \subseteq \operatorname{supp}(\omega) \cap \operatorname{supp}(\tau)$.

<u>Ex. 18.6</u> $\{\rho_\alpha : \alpha \in A\}$ collection of functions $\rho_\alpha : M \to \mathbb{R}$ that is locally finite. Let $\omega \in \Omega^k(M)$ have compact support.

<u>Claim</u> $\rho_\alpha \cdot \omega \equiv 0$ except for finitely many α.

Write $K = \text{supp}(\omega)$.

For each $p \in K$, choose nbh $U_p \subseteq M$ s.t. $U_p \cap \text{supp}(\rho_\alpha) = \emptyset$ except for finitely many α.

Then $K \subseteq U_{p_1} \cup \dots \cup U_{p_n}$ by compactness.

Hence $K \cap \text{supp}\, \rho_\alpha = \emptyset$ except for finitely many α.

This verifies the claim.

Ex. 18.8 $\quad \pi : \tilde{M} \to M$ is a surjective submersion.

Show $\pi^* : \Omega^k(M) \to \Omega^k(\tilde{M})$ is injective.

Given $\omega \in \Omega^k(M)$ such that $\pi^*\omega = 0$ in $\Omega^k(N)$

Must show that $\omega = 0 \in \Omega^k(M)$.

That is: $\omega_q(v_1, \ldots v_k) = 0$ for all $q \in M$ and $v_1, \ldots v_k$ in $T_q M$.

We know $\pi^*\omega = 0$, i.e., for all $p \in N$ and $w_1 \ldots w_k$ in $T_p N$,

$$0 = (\pi^*\omega)_p(w_1, \ldots w_k) = \omega_{\pi(p)}(\pi_{*,p}(w_1), \ldots, \pi_{*,p}(w_k)).$$

Choose $p \in N$ st. $\pi(p) = q$.

We know $\pi_{*,p} : T_p N \to T_{\pi(p)} M = T_q M$ is surjective.

Choose $w_1 \ldots w_k$ st. $\pi_{*,p}(w_i) = v_i$ for $i = 1..k$.

Then $\omega_q(v_1 \ldots v_k) = (F^*\omega)_p(w_1 \ldots w_k) = 0$.

Ex $\quad \pi : S^n \to \mathbb{R}P^n$ surjective submersion.