Ex 3.4 Let $f \in L_k(V)$. Show that the following are equivalent: (i) f is alternating

(ii) $f(\dots, v_i, v_{i+1}, \dots) = -f(\dots, v_{i+1}, v_i, \dots)$ for each neighbour pair.

(i) $\Rightarrow$ (ii) OK since a neighbour transposition has sign -1.

(ii) $\Rightarrow$ (i). We can write each $\sigma \in S_k$ as a product $\sigma = \tau_1 \cdots \tau_n$, where each τ_i is a neighbour transposition.

$\sigma \cdot f = (\tau_1 \cdots \tau_n) f = (-1)^n f = \operatorname{sgn}(\sigma) f$.

Ex 3.5 Let $f \in L_k(V)$. Show

(i) f is alternating

(ii) $f(v_1, \dots v_k) = 0$ whenever $v_i = v_j$ for some $i \neq j$.

(i) $\Rightarrow$ (ii) If $\sigma = (i,j)$ $\qquad \sigma f(v_1 \dots v_k) = f(v_1 \dots v_k)$

$\qquad\qquad \overset{\shortparallel}{-f(v_1, \dots v_k)}$ sing $\operatorname{sgn}(\sigma) = -1$.

(ii) $\Rightarrow$ (i)

$0 = f(\ldots, v_i + v_{i+1}, v_i + v_{i+1}, \ldots) =$

$= \cancel{f(\ldots v_i, v_i \ldots)} + f(\ldots, v_i, v_{i+1} \ldots)$

$+ f(\ldots, v_{i+1}, v_i, \ldots) + \cancel{f(\ldots v_{i+1}, v_{i+1} \ldots)}$

$\Rightarrow f(\ldots v_i, v_{i+1}, \ldots) = - f(\ldots, v_{i+1}, v_i, \ldots)$

Conclusion by Ex. 3.4.

<u>Ex. 3.2</u> (i) $\dim V = n$, $f: V \to \mathbb{R}$ non-zero

Show $\dim \operatorname{Ker} f = n-1$. Know $V/\operatorname{Ker} f \cong \mathbb{R}$ and

$1 = \dim(V/\operatorname{Ker} f) = \dim V - \dim \operatorname{Ker} f$.

Alternatively: $n = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$.

$\boxed{\text{Choose inner product on } V}$

(ii) Given $f, g: V \to \mathbb{R}$ s.t. $\operatorname{Ker} f = \operatorname{Ker} g$ show there exists a scalar c s.t. $cf = g$. Let w be a basis for $\operatorname{Ker} f^{\perp} = \operatorname{Ker} g^{\perp}$. (Suppose $f, g \neq 0$) Choose $c \in \mathbb{R}$ s.t. $cf(w) = g(w)$. Every vector x in V can be written $x = rw + w'$, where $w' \in \operatorname{Ker}(f) = \operatorname{Ker}(g)$. Then

$cf(x) = cf(rw) = crf(w) = rg(w) = g(rw) = g(x)$ for all $x \in V$.