Ex. 19.10 $\quad S^1 \subseteq \mathbb{R}^2$, $U_x = \{(x,y) : x \neq 0\}$, $U_y = \{(x,y) : y \neq 0\}$

Know $x\,dx + y\,dy \equiv 0$ on S^1. (differential of $f(x,y) = \frac{1}{2}(x^2+y^2)$)

$$\omega = \begin{cases} \frac{1}{x}\,dy & \text{on } U_x \\ -\frac{1}{y}\,dx & \text{on } U_y \end{cases}$$

Claim: $\omega = -y\,dx + x\,dy \in \Omega^1(S^1)$

(Let $\iota : S^1 \to \mathbb{R}^2$, then $\iota^*(x\,dx + y\,dy) = \iota^* df = d(\iota^* f)$
$=$ d(constant function) $= 0$)

On U_x : $dx = -\frac{y}{x}\,dy \Rightarrow -y\,dx + x\,dy = (-y)\left(-\frac{y}{x}\,dy\right) + x\,dy$

$= \left(\frac{y^2}{x} + x\right) dy = \frac{y^2+x^2}{x}\,dy = \frac{1}{x}\,dy$ on $U_x \subseteq S^1$.

On U_y a similar argument applies. ($dy = -\frac{x}{y}\,dx$)

19.11 (a) $f: \mathbb{R}^2 \to \mathbb{R}$ C^∞ function. Suppose $0 \in \mathbb{R}$ regular value, so $M := f^{-1}(0)$ is a 1-dim. submanifold of $\mathbb{R}^2$.

Goal: Construct a nowhere vanishing 1-form $\omega \in \Omega^1(M)$.

Know $J(f)(p) = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right] \neq [0,0]$. Let $i: M \to \mathbb{R}^2$,

$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \in \Omega^1(\mathbb{R}^2)$, $f \circ i \equiv 0$, so

$$0 = d(f \circ i) = \left(\frac{\partial f}{\partial x} \circ i\right) \cdot i^* dx + \left(\frac{\partial f}{\partial y} \circ i\right) i^* dy. \; (*)$$

Let $U_x = \{ p \in M : \frac{\partial f}{\partial x}(p) \neq 0 \}$, $U_y = \{ p \in M : \frac{\partial f}{\partial y}(p) \neq 0 \}$.

$$\text{Let } \omega = \begin{cases} \frac{1}{(\partial f / \partial x)} i^*(dy) & \text{on } U_x \\ \frac{-1}{(\partial f / \partial y)} i^*(dx) & \text{on } U_y \end{cases}$$

well-defined on $U_x \cap U_y$ by $(*)$

Here we write $\partial f / \partial x$, $\partial f / \partial y$ also for the restrictions to M. These are automatically smooth since $M \subseteq \mathbb{R}^2$ is a submanifold.

Must check ω is nowhere-vanishin

Know $\jmath_{*,p} : T_pM \subseteq T_p\mathbb{R}^2 \cong \mathbb{R}^2$ injective and $\jmath_{*,p} T_pM \subseteq \mathbb{R}^2$

has normal vector grad $f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$, so

$$\jmath_{*,p}(T_pM) = \left\{ a \frac{\partial}{\partial x}\Big|_p + b \frac{\partial}{\partial y}\Big|_p : a \frac{\partial f}{\partial x}(p) + b \frac{\partial f}{\partial y}(p) = 0 \right\}$$

On U_x: $\frac{\partial f}{\partial x}(p) \neq 0$, so grad$(f)_p$ is not parallell to y-axis,

hence $T_pM \neq \mathbb{R}\left\{ \frac{\partial}{\partial x}\Big|_p \right\}$, hence $\jmath^*(dy_p) \neq 0$ on U_x.

Explicitly: let $v = \dfrac{-(\partial f/\partial y)(p)}{(\partial f/\partial x)(p)} \dfrac{\partial}{\partial x}\Big|_p + \dfrac{\partial}{\partial y}\Big|_p \in T_pM$

and $dy(v) = 1 \neq 0$.

Similar argument on U_y.

(b) $f:\mathbb{R}^3\to\mathbb{R}$, $0\in\mathbb{R}$ regular value, so $M=f^{-1}(0)\subseteq\mathbb{R}^3$ is a 2-dim. submanifold. Let $j: M\to\mathbb{R}^3$.

$U_x=\{p\in M: \frac{\partial f}{\partial x}(p)\neq 0\}$, $U_y=\{p\in M: \frac{\partial f}{\partial y}(p)\neq 0\}$, $U_z=\{p\in M: \frac{\partial f}{\partial z}(p)\neq 0\}$

$$\text{Let}\quad \omega=\begin{cases}\frac{1}{\partial f/\partial x}\, j^*(dy\wedge dz) & \text{on } U_x\\ \frac{-1}{\partial f/\partial y}\, j^*(dx\wedge dz) & \text{on } U_y\\ \frac{1}{\partial f/\partial z}\, j^*(dx\wedge dy) & \text{on } U_z\end{cases}$$

Must check well-defined.

Know $df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz\in\Omega^1(\mathbb{R}^3)$

$f\circ j\equiv 0$ so $0=d(f\circ j)=j^*df$

$df\wedge dz=\frac{\partial f}{\partial x}dx\wedge dz+\frac{\partial f}{\partial y}dy\wedge dz$, so

$0=j^*(df)\wedge j^*(dz)=j^*(df\wedge dz)\Rightarrow \frac{\partial f}{\partial x}j^*(dx\wedge dz)=-\frac{\partial f}{\partial y}j^*(dy\wedge dz)$

$\Longrightarrow -\frac{1}{\partial f/\partial y}dx\wedge dz=\frac{1}{\partial f/\partial x}dy\wedge dz$ on $U_x\cap U_y$.

Similarly for $U_x \cap U_z$ and $U_y \cap U_z$

To see that ω is nowhere-vanishing can use that

$$j_{*,p}(T_pM) = \left\{ a\frac{\partial}{\partial x}\Big|_p + b\frac{\partial}{\partial y}\Big|_p + c\frac{\partial}{\partial z}\Big|_p : a\frac{\partial f}{\partial x}(p) + b\frac{\partial f}{\partial y}(p) + c\frac{\partial f}{\partial z}(p) = 0 \right\}$$

and proceed as in (a).

- Eg on U_x look at $v_1 = -\left(\frac{\partial f/\partial y(p)}{\partial f/\partial x(p)}\right)\frac{\partial}{\partial x}\Big|_p + \frac{\partial}{\partial y}\Big|_p$

$$v_2 = -\left(\frac{\partial f/\partial z(p)}{\partial f/\partial x(p)}\right)\frac{\partial}{\partial x}\Big|_p + \frac{\partial}{\partial z}\Big|_p$$