Vector bundles

Def An n-dim smooth vector bundle is a tripple (E, M, π), where E and M are smooth manifolds, and $\pi: E \to M$ is a smooth map, such that

- $\pi^{-1}(p)$ is an n-dim vector space for all $p \in M$.
- For each $p \in M$, there exists an open nbh $p \in U$ and a diffeomorphism $\phi: \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that

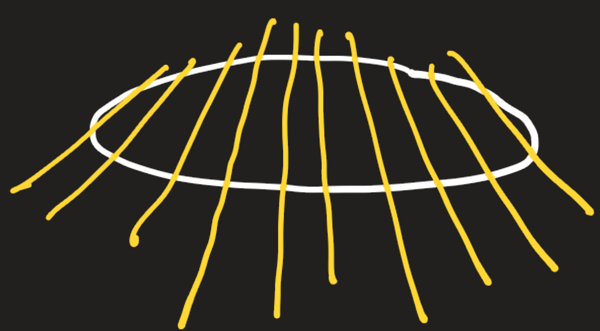
$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\phi"] \arrow[dr, "\pi"'] & U \times \mathbb{R}^n \arrow[d, "\text{proj}"] \\
& U
\end{tikzcd}$$

is commutative and the restriction $\phi_q: \pi^{-1}(q) \to \{q\} \times \mathbb{R}^n \cong \mathbb{R}^n$ linear isomorphism for each $q \in U$.

Terminology

- E is the total space
- M is the base space
- $E_p := \pi^{-1}(p)$ is the fiber of $p \in M$
- $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ is a local trivialization of E over U.

Ex $M = S^1$



<u>Ex</u> If M is an n-dim smooth manifold, then the tangent bundle

$$\pi: TM = \bigcup_{p\in M} T_pM \longrightarrow M \quad . \quad (p,v) \longmapsto p$$

is an n-dim smooth vector bundle.

We can check that π is smooth locally: For a chart $\phi: U \to \mathbb{R}^n$ on M and $TU = \bigcup_{p\in U} T_pM$, we have

$$\begin{tikzcd} TU \arrow[r, "\pi"] \arrow[d, "\tilde{\phi}"'] & U \arrow[d, "\phi"] \\ \phi(U)\times\mathbb{R}^n \arrow[r, "\text{proj.}"] & \phi(U) \end{tikzcd}$$

Local trivializations $\quad TU \cong U\times\mathbb{R}^n, \ (p,v) \mapsto (p, \phi_{*,p}(v))$

A smooth map $f: N \to M$ induces a smooth map of tangent bundles

$$f_*: TN \longrightarrow TM$$

by setting $f_*(p,v) = (f(p), f_{*,p}(v)) \in T_{f(p)}M$.

Then there is a comm. diagram of smooth maps

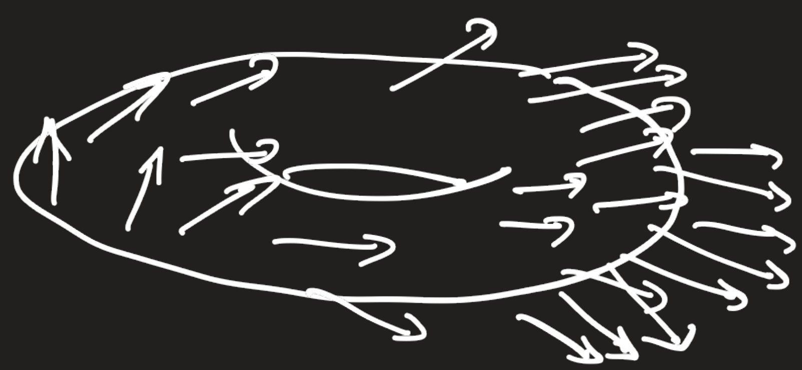
$$\begin{tikzcd} TN \arrow[r, "f_*"] \arrow[d, "\pi_N"'] & TM \arrow[d, "\pi_M"] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

Locally $f_{*,p}$ is given by the Jacobian matrix, hence f_* is a smooth map.

Let $\pi : E \to M$ be a smooth vector bundle

<u>Def</u> A smooth section of π is a smooth function $s : M \to E$ such that $\pi \circ s = id_M$.

<u>Def</u> A smooth vector field on a manifold M is a smooth section of the tangent bundle $\pi : TM \to M$.



Proposition Let $\pi: E \to M$ be a smooth vector bundle

(i) If $s, t: M \to E$ are smooth sections, then

$s+t : M \to E$, $(s+t)(p) = s(p) + t(p)$, is a smooth section

(ii) If $s: M \to E$ is a smooth section and $f: M \to \mathbb{R}$ is a smooth function, then

$f \cdot s : M \to E$, $(f \cdot s)(p) = f(p)\, s(p)$, is a smooth section.

Proof This follows by using local trivializations; see Prop. 12.7 in [Tu].

Consequence: The set $\Gamma(E)$ of smooth sections of $\pi: E \to M$ is a vector space and a module over the $\mathbb{R}$-algebra $C^\infty(M)$.

Let $\pi: E \to M$ be an n-dim smooth vector bundle.

<u>Def</u> A local frame over an open set $U \subseteq M$ is a collection of smooth sections $s_1, \dots, s_n : U \to \pi^{-1}(U)$, such that $\{s_1(p), \dots, s_n(p)\}$ is a basis for the vector space $\pi^{-1}(p)$ for all $p \in U$.

<u>Rem</u>. Let $\phi: \pi^{-1}(U) \to U \times \mathbb{R}^n$ be a local trivialization. Let $s_i : U \to \pi^{-1}(U)$, $s_i(p) = \phi^{-1}(p, e_i)$ (where $e_i \in \mathbb{R}^n$ is the ith basis vector), for $i = 1, \dots, n$. Then $s_1, \dots, s_n$ is a local frame over U.

Proposition Let $s_1, .., s_n : U \to \pi^{-1}(U)$ be a local frame over $U \subseteq M$. Then $s_1 .. s_n$ define a local trivialization
$\psi : U \times \mathbb{R}^n \longrightarrow \pi^{-1}(U)$, $(p, c^1 .. c^n) \mapsto c^1 s_1(p) + .. + c^n s_n(p)$.

Proof Clearly ψ is C^∞, bijective and a linear isomorphism on each fiber. We must show that ψ is a diffeomorphism. Given $p \in U$, choose a nbh $p \in V \subseteq U$ and a local trivialization $\phi : \pi^{-1}(V) \to V \times \mathbb{R}^n$.
It suffices to show that the composition

$$V \times \mathbb{R}^n \xrightarrow{\psi} \pi^{-1}(V) \xrightarrow{\phi} V \times \mathbb{R}^n$$ is a diffeomorphism.

This has the form $(p, v) \longmapsto (p, G(p) v)$, where $G(p) \in GL_n(\mathbb{R})$, and $G : V \to GL_n(\mathbb{R})$ is smooth. By Cramers rule, the function $V \to GL_n(\mathbb{R})$, $p \mapsto G(p)^{-1}$ is also smooth. $\square$

Bump functions

Let M be a smooth manifold. The support of a smooth function $f: M \to \mathbb{R}$ is the closed subspace

$$\operatorname{supp}(f) = \overline{\{ p \in M : f(p) \neq 0 \}}$$

Def Let $U \subseteq M$ be an open subset and let $p \in U$ be a point. A smooth bump function at p supported in U is a smooth function $\rho : M \to \mathbb{R}$ such that

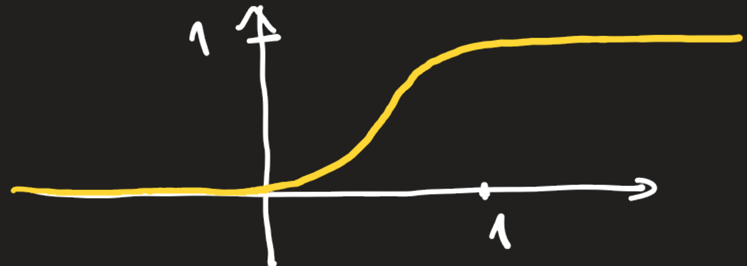
(i) $\rho(M) \subseteq [0,1]$

(ii) $\operatorname{supp}(\rho) \subseteq U$

(iii) There exists a nbh $p \in V \subseteq U$ s.t. $\rho|_V = 1$.

Theorem Given $p \in U \subseteq M$, there exists a bump function at p supported in U.

Proof **Step 1**: Define a smooth step function $g: \mathbb{R} \to \mathbb{R}$ such that $g(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t \geq 1. \end{cases}$



We know $f: \mathbb{R} \to \mathbb{R}$, $f(t) = \begin{cases} e^{-1/t}, & t > 0 \\ 0, & t \leq 0 \end{cases}$ is smooth (Exercise 1.2).

Set $g(t) = \dfrac{f(t)}{f(t) + f(1-t)}$ smooth

For $t \leq 0$: $g(t) = 0$

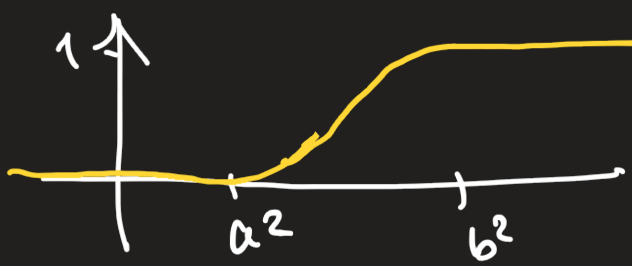
For $t \geq 1$, $f(1-t) = 0$, so $g(t) = 1$

Notice $f(\mathbb{R}) \subseteq [0,1] \Rightarrow g(\mathbb{R}) \subseteq [0,1]$.

<u>Step 2</u> Modify g to get a bump function on $\mathbb{R}$.

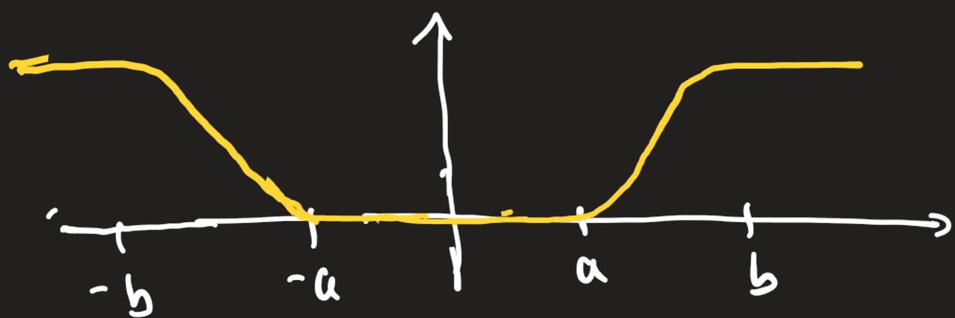
Given $0 < a < b$, linear bijection $[a^2, b^2] \to [0,1]$, $t \mapsto \frac{t-a^2}{b^2-a^2}$

Let $h: \mathbb{R} \to \mathbb{R}$, $h(t) = g\left(\frac{t-a^2}{b^2-a^2}\right)$



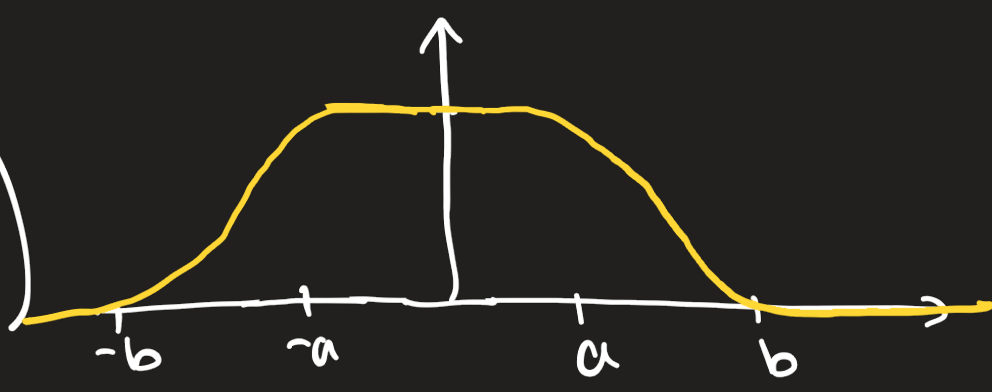
Make h symmetric:

$$k(t) = h(t^2) = g\left(\frac{t^2-a^2}{b^2-a^2}\right)$$



<u>Get bump function on $\mathbb{R}$</u>:

$$p(t) = 1 - k(t) = 1 - g\left(\frac{t^2-a^2}{b^2-a^2}\right)$$



Step 3 Bump function on $\mathbb{R}^n$

$\sigma : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\sigma(x) = \rho(\|x\|) = 1 - g\left(\frac{\|x\|^2 - a^2}{b^2 - a^2}\right)$

Then $\sigma(x) = 1$ for $\|x\| \leq a$, $\sigma(x) = 0$ for $\|x\| \geq b$.

Step 4 Given $p \in U \subseteq M$. Must define bump function at p supported in U.

Choose a chart $\phi : U_0 \longrightarrow \phi(U_0) \subseteq \mathbb{R}^n$ st. $p \in U_0 \subseteq U$ and $\phi(p) = 0$. Choose $b > 0$ such that $\overline{B(0,b)} \subseteq \phi(U_0)$

Let $a = b/2$, $\sigma : \mathbb{R}^n \to \mathbb{R}$, $\sigma(x) = 1 - g\left(\frac{\|x\|^2 - a^2}{b^2 - a^2}\right)$.

(This is a bump function at 0 supported in $\phi(U_0)$)

Define $\rho : M \to \mathbb{R}$, $\rho(q) = \begin{cases} \sigma(\phi(q)) & \text{if } q \in U_0 \\ 0 & \text{if } q \notin U_0. \end{cases}$

Notice: $\operatorname{supp}(\rho) \subseteq \phi^{-1}(\overline{B(0,b)}) \subseteq U_0$ and $\rho(q) = 1$ if $q \in \phi^{-1}(B(0,a))$ $\square$