Constant rank theorems

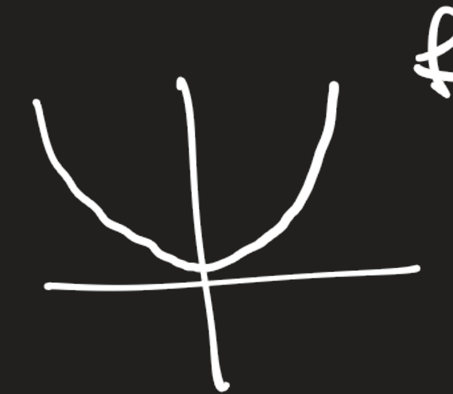
Let A be an $m \times n$ matrix.
The rank of A, $\operatorname{rk} A$, is the dimension of the image of the linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$, $x \mapsto Ax$.

Recall: $\operatorname{rk}(A)$ = dim of the column space of A
= number of lin. independent columns
= number of lin. independent rows.

Let $U \subseteq \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^m$ smooth.

Def
- The rank of f at $p \in U$ is the rank of the Jacobian matrix $\left[\partial f^i / \partial x^j (p) \right]$.
- f has constant rank in a nbh. of p if there exists $p \in U_0 \subseteq U$ s.t. $\operatorname{rk} \left[\partial f^i / \partial x^j (q) \right] = \operatorname{rk} \left[\partial f^i / \partial x^j (p) \right]$ for all $q \in U_0$

Ex. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$,



Rank of $Jf(x) = [2x] = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Hence f has constant rank in a nbh. of x if $x \neq 0$.

The constant rank theorem for Euclidian spaces

Let $U \subseteq \mathbb{R}^n$ be open, $f: U \to \mathbb{R}^m$ smooth with constant rank k in a nbh. of $p \in U$. Then there exist neighborhood $p \in U_1 \subseteq U$ and $f(p) \in V_1 \subseteq \mathbb{R}^m$ and diffeomorphisms

$G: U_1 \to U_1' \subseteq \mathbb{R}^n$, $G(p) = 0$

$F: V_1 \to V_1' \subseteq \mathbb{R}^m$, $F(f(p)) = 0$, such that $f(U_1) \subseteq V_1$

$$\begin{tikzcd} (U_1, p) \arrow[r, "f"] \arrow[d, "G"'] & (V_1, f(p)) \arrow[d, "F"] \\ (U_1', 0) \arrow[r] & (V_1', 0) \end{tikzcd}$$

and

$F \circ f \circ G^{-1}(r^1, .., r^k, r^{k+1}, .., r^n)$
$= (r^1, .., r^k, 0, .., 0)$
for $(r^1, .., r^n) \in U_1'$.

Rmk The linear version is as follows:

Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with rank k.

Then there exist linear isomorphisms

$$G: \mathbb{R}^n \to \mathbb{R}^n, \qquad F: \mathbb{R}^m \to \mathbb{R}^m,$$

such that

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "f"] \arrow[d, "G", "\cong"'] & \mathbb{R}^m \arrow[d, "F", "\cong"'] \\
\mathbb{R}^n \arrow[r, "F f G^{-1}"] & \mathbb{R}^m
\end{tikzcd}$$

$$(u^1, \dots, u^n) \longmapsto (u^1, \dots, u^k, 0, \dots, 0).$$

Proof of the constant rank theorem

Suppose $m = n = 2$ and $k = 1$ (General case analogous)

$U \subseteq \mathbb{R}^2$, $f : U \to \mathbb{R}^2$ constant rank 1 in a nbh of $p \in U$.

May assume $p = 0$ and $f(p) = 0$ by translating if necessary. Write $f(x,y) = (f^1(x,y), f^2(x,y))$

Since $J(f)(0)$ has rank 1, may assume $\partial f^1/\partial x(0) \neq 0$, by permuting coordinates if necessary.

Let $G : U \to \mathbb{R}^2$, $G(x,y) = (f^1(x,y), y)$

$$J(G)(0) = \begin{bmatrix} \frac{\partial f^1}{\partial x}(0) & \frac{\partial f^1}{\partial y}(0) \\ 0 & 1 \end{bmatrix} \quad \text{non singular}$$

IFT: There exists a nbh $0 \in U_0 \subseteq U$ s.t.

$G : U_0 \to G(U_0) =: U_0'$ is a diffeomorphism.

May assume that f has constant rank 1 on U_0.

Consider $f \circ G^{-1} : U_0' \longrightarrow \mathbb{R}^2$, constant rank 1.

$(u,v) = G \circ G^{-1}(u,v) = (f^1 G^{-1}(u,v), \, y \circ G^{-1}(u,v)) \Rightarrow f^1 \circ G^{-1}(u,v) = u$

Let $h(u,v) = f^2 \circ G^{-1}(u,v)$, $h : U_0' \longrightarrow \mathbb{R}$.

Then $f \circ G^{-1}(u,v) = \begin{bmatrix} u \\ h(u,v) \end{bmatrix}$, Jacobian $\begin{bmatrix} 1 & 0 \\ \partial h / \partial u & \partial h / \partial v \end{bmatrix}$

Rank 1 $\Longrightarrow$ $\partial h / \partial v = 0$ on U_0'.

Hence may assume $h(u,v)$ only depend on u.

(May shrink U_0' to a convex set if needed)

Let $F : U_0' \longrightarrow \mathbb{R}^2$, $F(x,y) = \begin{bmatrix} x \\ y - h(x,y) \end{bmatrix}$, $J(F) = \begin{bmatrix} 1 & 0 \\ -\frac{\partial h}{\partial x} & 1 \end{bmatrix}$

IFT: There exists nbh $0 \in V_1 \subseteq U_0' : F : V_1 \to F(V_1) = V_1'$ diffeo.

Let $U_1 = U_0 \cap f^{-1}(V_1)$, $G : U_1 \to G(U_1) =: U_1'$

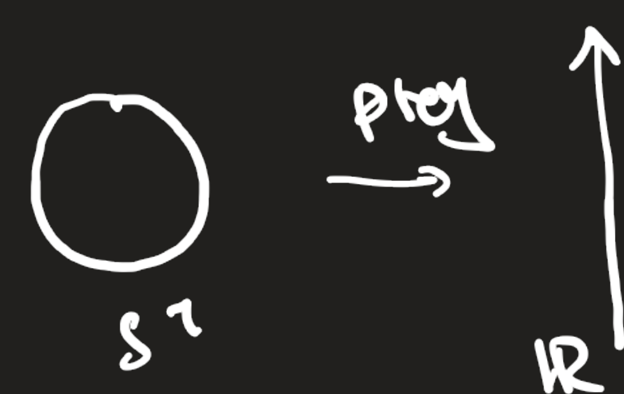
$F \circ f \circ G^{-1}(u,v) = F \begin{bmatrix} u \\ h(u,v) \end{bmatrix} = \begin{bmatrix} u \\ h(u,v) - h(u, h(u,v)) \end{bmatrix} = \begin{bmatrix} u \\ 0 \end{bmatrix}$ since h only depends on u. $\square$

Let $f: N \to M$ be smooth, $\dim N = n$, $\dim M = m$

<u>Def</u> • The rank of f at a point $p \in N$ is the rank of the linear transformation $f_{*,p}: T_p(N) \to T_{f(p)}(M)$

• f has constant rank in a nbh. of p if there exists $p \in U \subseteq N$ st $\mathrm{rk}\, f_{*,q} = \mathrm{rk}\, f_{*,p}$ for all $q \in U$.

<u>Ex</u>



constant rank in a neighborhood of each point except the top and bottom points on S^1.

Constant rank theorem for manifolds

Let $f : N \to M$ be smooth with constant rank in a nbh of $p \in N$. Then there are charts

$$\phi : U \to \mathbb{R}^n \text{ on } N \text{ about } p, \quad \phi(p) = 0$$
$$\psi : V \to \mathbb{R}^m \text{ on } M \text{ about } f(p), \quad \psi(f(p)) = 0$$

such that $f(U) \subseteq V$ and

$$\psi \circ f \circ \phi^{-1}(r^1, \dots, r^k, r^{k+1}, \dots, r^n) = (r^1, \dots, r^k, 0, \dots, 0).$$

$$\begin{array}{ccc}
(N, p) & \longrightarrow & (M, f(p)) \\
\cup & & \cup \\
(U, p) & \overset{f}{\dashrightarrow} & (V, f(p)) \\
\phi \downarrow & & \psi \downarrow \\
(\phi(U), 0) & & (\psi(V), 0) \\
\cap & & \cap \\
\mathbb{R}^n & & \mathbb{R}^m
\end{array}$$

proof First choose any charts ϕ_1 and ψ_1:

$$\begin{tikzcd}
N \arrow[r] & M \\
U \arrow[u, phantom, "\cup"] \arrow[r, "f"] \arrow[d, "\phi_1"'] & V \arrow[u, phantom, "\cup"] \arrow[d, "\psi_1"] \\
\mathbb{R}^n \supseteq \phi_1(U) \arrow[r, dashed, "\psi_1 f \phi_1^{-1}"] \arrow[d, "G"'] & \psi_1(V) \subseteq \mathbb{R}^m \arrow[d, "F"] \\
G\phi_1(U) \arrow[r, dashed] & F\psi_1(V)
\end{tikzcd}$$

$$(u^1, \dots, u^k, u^{k+1}, \dots, u^n) \longmapsto (u^1, \dots, u^k, 0, \dots, 0).$$

Now apply the constant rank theorem to $\psi_1 \circ f \circ \phi_1^{-1}$ to get G, F and let

$$\phi = G \circ \phi_1, \quad \psi = F \circ \psi_1.$$

$\square$