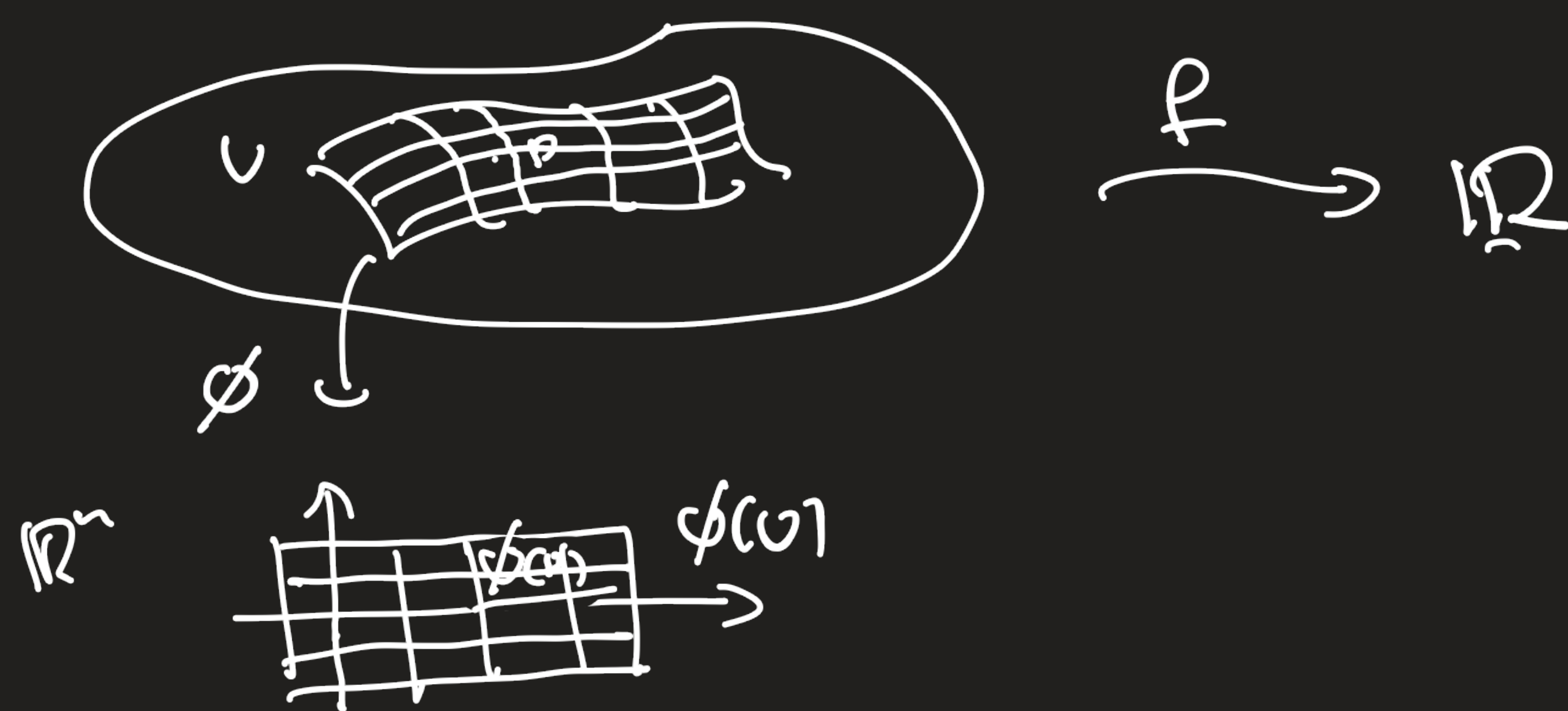
Partial derivatives in a chart domain

Let M be an n-dim. smooth manifold and let (U, ϕ) be a chart on M, $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$.

Let $r^1, \dots, r^n$ be the standard coordinates on $\mathbb{R}^n$, write $x^i = r^i \circ \phi$.

Then $\phi(p) = (x^1(p), \dots, x^n(p))$, $x^1, \dots, x^n : U \to \mathbb{R}$.



Def Let $f : M \to \mathbb{R}$ be a smooth map. The ith partial derivative of f wrt. (U, ϕ) at $p \in U$ is

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial (f \circ \phi^{-1})}{\partial r^i}(\phi(p))$$

Notice $\frac{\partial f}{\partial x^i}$ is a smooth function on U:

$$\frac{\partial f}{\partial x^i} \circ \phi^{-1} = \frac{\partial (f \circ \phi^{-1})}{\partial r^i} : \phi(U) \to \mathbb{R} \quad \text{smooth}$$

Given $F: M \to N$ smooth map, $\dim M = m$, $\dim N = n$

Choose charts (U, ϕ) on M and (V, ψ) on N such that

$F(U) \subseteq V$

$$\begin{tikzcd} M \arrow[r, "F"] & N \\ U \arrow[u, phantom, "\cup" rotate=90] \arrow[r, "F"] \arrow[d, "\phi"'] & V \arrow[u, phantom, "\cup" rotate=90] \arrow[d, "\psi"] \\ \mathbb{R}^m \supseteq \phi(U) & \psi(V) \subseteq \mathbb{R}^n \end{tikzcd}$$

Write $\phi = (x^1, .. x^m)$, $\psi = (y^1, .. y^n)$

Let $F^i = y^i \circ F : U \to \mathbb{R}$.

Def. The Jacobian of F at $p \in U$ wrt. (U, ϕ) and (V, ψ)

is $\left[\frac{\partial F^i}{\partial x^j}(p) \right] = \left[\frac{\partial (y^i \circ F \circ \phi^{-1})}{\partial r^j} (\phi(p)) \right]$

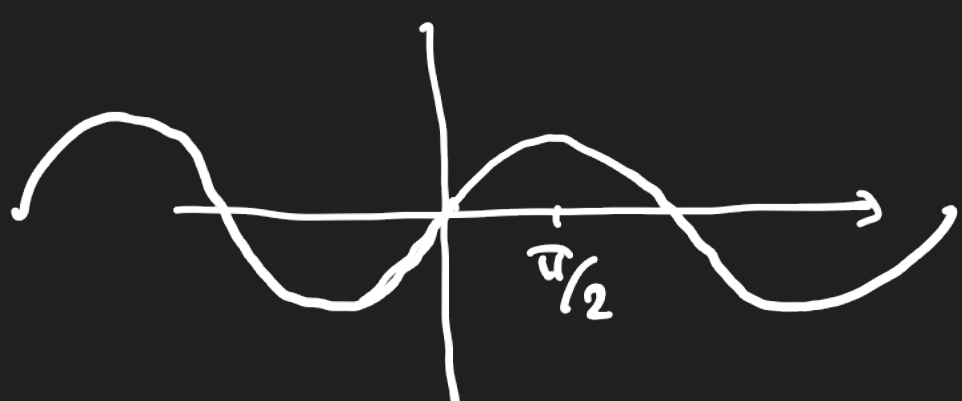
Ex Suppose $F = id : M \to M$.

The this is the usual Jacobian of the transition function $\psi \circ \phi^{-1}$: $\left[\frac{\partial y^i}{\partial x^j} \right]$

The inverse function theorem

Def A smooth function $F : M \to N$ is a local diffeomorphism at a point $p \in M$ if there exists a neighborhood $p \in U \subseteq M$ st. $F|U : U \to F(U) \subseteq N$ is a diffeomorphism to an open set $F(U)$ in N.

Ex $\sin : \mathbb{R} \to \mathbb{R}$ is a local diffeomorphism at $0 \in \mathbb{R}$ but not at $\pi/2$.



Inverse function theorem for $\mathbb{R}^n$

Let $U \subseteq \mathbb{R}^n$ be open. A smooth function $F: U \to \mathbb{R}^n$ is a local diffeomorphism at $p \in U$ iff. the Jacobian matrix is invertible, that is $\det \left[\frac{\partial F^i}{\partial x^j}(p) \right] \neq 0$ (see eg. Rudin: Principles of Mathematical analysis).

Inverse function theorem for smooth manifolds

Let M and N be smooth manifolds of dimension n and let $F: M \to N$ be a smooth map. Given $p \in M$ and choose charts $(U, x^1, \ldots x^n)$ on M and $(V, y^1, \ldots y^n)$ on N such that $p \in U$ and $F(U) \subseteq V$.

Then F is a local diffeomorphism at p iff the Jacobian matrix $\left[\frac{\partial F^i}{\partial x^j}(p) \right]$ is invertible.

Proof

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ \cup\!\!\mid & & \cup\!\!\mid \\ U & \xrightarrow{F} & V \\ \phi \swarrow & & \searrow \psi \\ \mathbb{R}^n \supseteq \phi(U) & & \psi(V) \subseteq \mathbb{R}^n \end{array}$$

We know $\left[\dfrac{\partial F^i}{\partial x^j}(p) \right] := \left[\dfrac{\partial (\psi^i \circ F \circ \phi^{-1})}{\partial r^j} (\phi(p)) \right]$

is invertible. This is the Jacobian of $\psi \circ F \circ \phi^{-1}$ at $\phi(p)$. Hence $\psi \circ F \circ \phi^{-1}$ is a local diffeomorphism at $\phi(p)$ by the inverse function theorem for $\mathbb{R}^n$.

Result follows since ϕ and ψ are diffeomorphisms.

$$F = \psi^{-1} \circ (\psi \circ F \circ \phi^{-1}) \circ \phi \qquad \square$$

Lie groups

Recall: A group is a set G together with an associative multiplication $\cdot\ G \times G \to G$ and a unit element $e \in G$ such that every $g \in G$ has an inverse, i.e., there exists $g^{-1} \in G$ s.t. $g \cdot g^{-1} = e = g^{-1} \cdot g$.

Def A Lie group is a group G which is also a smooth manifold such that $G \times G \to G$ and the inverse map $G \to G$, $g \mapsto g^{-1}$ are smooth maps.

Ex
- $G = \mathbb{R}^n$ with addition $+ : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$
- $G = \mathbb{C}^* = \mathbb{C} - \{0\}$ with complex multiplication.
- $G = S^1$ with complex multiplication
- $G = GL_n(\mathbb{R})$ with matrix multiplication (see book).

  Notice $GL_n(\mathbb{R}) = \{ A \in M_{n \times n}(\mathbb{R}) : \det A \neq 0 \}$ open subset of $\mathbb{R}^{n^2}$.