Compact spaces (App. A)

An open covering of a top. space X is a collection of open subsets $\{U_j \subseteq X : j \in J\}$ s.t. $X = \bigcup_{j \in J} U_j$

Def A top space X is compact if every open covering of X has a finite subcovering.

Heine-Borel Theorem

A subset $A \subseteq \mathbb{R}^n$ is compact (with the subspace topology) iff. A is closed and bounded.

Ex The n-sphere $S^n \subseteq \mathbb{R}^{n+1}$ is compact.

Lemma The following are equivalent for a subspace $A \subseteq X$. (i) A is compact in the subspace topology
(ii) Given a collection of open subsets $\{U_j \subseteq X : j \in J\}$ s.t. $A \subseteq \bigcup_{j \in J} U_j$
Then $A \subseteq U_{j_1} \cup \dots \cup U_{j_n}$ for a finite subset $\{j_1, \dots, j_n\} \subseteq J$.

Prop If $f: X \to Y$ is cont. and X is compact, then also $f(X) \subseteq Y$ is compact.

Proof Suppose $f(X) \subseteq \bigcup_{j \in J} V_j$, where $V_j \subseteq Y$ is open.

Then $X = \bigcup_{j \in J} f^{-1}(V_j) \Rightarrow X = f^{-1}(V_{j_1}) \cup .. \cup f^{-1}(V_{j_n})$

$\Rightarrow f(X) \subseteq V_{j_1} \cup .. \cup V_{j_n}$ $\square$

Prop

(i) A closed subspace of a compact space is compact

(ii) A compact subspace of a Hausdorff is closed.

Proof (See App. A).

Theorem Let $f: X \to Y$ be a bijective cont. map. suppose that X is compact and Y is Hausdorff. Then f is a homeomorphism.

Proof of Theorem

suffices to show that $A \subseteq X$ is closed, then $f(A) \subseteq Y$ is also closed. :

A closed $\Longrightarrow$ A is compact $\Longrightarrow$ $f(A)$ compact $\Rightarrow$ $f(A)$ closed

$\square$

Ex $f: (0, 2\pi] \longrightarrow S^1$, $f(t) = (\cos t, \sin t)$

This is a continuous bijection, but not a homeomorphism

- since $(0, 2\pi]$ is not compact.

Quotient spaces

Let X be a topological space with an equivalence relation $\sim$.

For $x \in X$, let $[x] = \{ y \in X : y \sim x \}$, equivalence class of x.

X is the disjoint union of the equivalence classes.

Let $X/\sim$ be the set of equivalence classes.

Let $p : X \to X/\sim$ be the quotient map $p(x) = [x]$.

Def The quotient topology on $X/\sim$ is defined by

$$U \subseteq X/\sim \text{ is open} \iff p^{-1}(U) \subseteq X \text{ is open.}$$

Then $p : X \to X/\sim$ is continuous by definition.

Let X be a top. space with an equivalence relation $\sim$ and let $p: X \to X/\sim$ be the quotient map.

If $f: X \to Y$ is a function such that

$$x \sim x' \implies f(x) = f(x').$$

then there is an induced function $\tilde{f}: X/\sim \longrightarrow Y$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \\ X/\sim \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

is commutative

<u>Lemma</u> (Universal mapping property)

In this situation, f is continuous iff $\tilde{f}$ is continuous.

<u>Proof</u> Easy exercise (see App. A) $\square$

Ex $X = [0, 2\pi]$ with the equivalence relation that identifies 0 and 2π, $0 \sim 2\pi$.

Then $X/_{\sim} = [0, 2\pi]/_{0 \sim 2\pi}$ is a compact top. space.

(since $p : [0, 2\pi] \to [0, 2\pi]/_{0 \sim 2\pi}$ is cont.).

$f = (\cos t, \sin t) : [0, 2\pi] \to S^1$ induces a continuous function $\tilde{f} : [0, 2\pi]/_{0 \sim 2\pi} \to S^1$ by the lemma:

$$\begin{tikzcd}
{[0,2\pi]} \arrow[r, "f"] \arrow[d] & S^1 \\
{[0,2\pi]/_{\sim}} \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$

This is a continuous bijection, hence a homeomorphism.

Def. $X = \mathbb{R}^{n+1} - \{0\}$. Define an equivalence relation:

$x \sim y$ if $x = \lambda y$ for some $\lambda \in \mathbb{R} - \{0\}$.

The n-dimensional projective space is the quotient space $\mathbb{R}P^n := \mathbb{R}^{n+1} - \{0\}/\sim$

We can think of $\mathbb{R}P^n$ as the space of lines through 0 in $\mathbb{R}^{n+1}$.

For $n=2$, this is the projective plane $\mathbb{R}P^2 = \mathbb{R}^3 - \{0\}/\sim$ of lines through 0 in $\mathbb{R}^3$.

$\underline{\text{Ex}}$. Let $X = S^n = \{ x \in \mathbb{R}^{n+1} : \|x\| = 1 \}$

Define an equivalence relation on S^n:

$$x \sim y \iff x = y \text{ or } x = -y.$$

$\underline{\text{Claim}}$ $S^n/\sim$ is homeomorphic to $\mathbb{R}P^n$.

$\underline{\text{Define}}$ $S^n \longrightarrow \mathbb{R}^{n+1} - \{0\} \longrightarrow \mathbb{R}^{n+1} - \{0\}/\sim$

By the lemma this gives a continuous map $S^n/\sim \;\to \mathbb{R}^{n+1} - \{0\}/\sim$

$\underline{\text{Define}}$ $\mathbb{R}^{n+1} - \{0\} \longrightarrow S^n \longrightarrow S^n/\sim$, $x \mapsto \frac{x}{\|x\|}$

By the lemma this gives a continuous map $\mathbb{R}^{n+1} - \{0\}/\sim \;\to S^n/\sim$

These are inverse maps:

$\mathbb{R}^{n+1} - \{0\}/\sim \;\longrightarrow S^n/\sim \;\longrightarrow \mathbb{R}^{n+1} - \{0\}/\sim$, $x \mapsto \frac{x}{\|x\|} \sim x$

similarly, the other composition is the identity.

$\underline{\text{Consequence}}$: $\mathbb{R}P^n$ is compact, since S^n is compact.

Let $P: X \to X/\sim$ be a quotient map. and let $B \subseteq X/\sim$. Then $P^{-1}(B)$ is a union of equivalence classes.

<u>Lemma</u> Let $B \subseteq X/\sim$ be an open subset and let $Y = P^{-1}(B) \subseteq X$. Then the subspace topology on B is the same as the quotient topology on $B = Y/\sim$.

<u>Proof</u> $U \subseteq B$ open in the subspace topology

$\Updownarrow$

$U \subseteq X/\sim$ is open

$\Updownarrow$

$P^{-1}(U) \subseteq X$ is open

$\Updownarrow$

$P^{-1}(U) \subseteq Y$ is open in subspace topologies.

$\square$

Lemma $\mathbb{RP}^n$ is locally Euclidian of dimension n.

Proof Let $U_i = \{ [a^0, .., a^n] \in \mathbb{RP}^n : a^i \neq 0 \}$ $\left(\mathbb{RP}^n = \mathbb{R}^{n+1} - \{0\} \big/ \sim \right)$

Then $\mathbb{RP}^n = \bigcup_{i=0}^{n} U_i$ and each U_i is open since

$p^{-1}(U_i) = \{ (a^0, .. a^n) \in \mathbb{R}^{n+1} - \{0\} : a^i \neq 0 \}$ is open.

Define $\phi_i : U_i \longrightarrow \mathbb{R}^n$, $\phi_i([a^0 .. a^n]) = \frac{1}{a^i}(a^0, .., \hat{a}^i, .., a^n)$.

Well-defined: $(a^0, .., a^n) \sim (\lambda a^0, .., \lambda a^n)$, $\lambda \neq 0$, then

$$\frac{1}{a^i}(a^0, .., \hat{a}^i, .., a^n) = \frac{1}{\lambda a^i} \cdot (\lambda a^0, .. \widehat{\lambda a^i}, .., \lambda a^n)$$

Therefore ϕ_i is continuous by the lemma, since

$p^{-1}(U_i) \to U_i \xrightarrow{\phi_i} \mathbb{R}^n$ is continuous.

The inverse $\phi_i^{-1} : \mathbb{R}^n \to U_i$, $(x^1 .., x^n) \mapsto [x^1, .., x^i, 1, x^{i+1} .., x^n]$

is also continuous.

$$\begin{tikzcd} \mathbb{R}^n \arrow[dr] & & U_i \\ & p^{-1}(U_i) \arrow[ur] & \end{tikzcd}$$

Check inverse functions:

$$U_i \xrightarrow{\phi_i} \mathbb{R}^n \xrightarrow{\phi_i^{-1}} U_i$$

$$[a^0, \dots a^n] \overset{\phi}{\longmapsto} \frac{1}{a^i}(a^0, \dots, \hat{a}^i, \dots a^n) = \left(\frac{a^0}{a^i}, \dots \frac{\hat{a}^i}{a^i}, \dots \frac{a^n}{a^i}\right)$$

$$\overset{\phi^{-1}}{\longmapsto} \left[\frac{a^0}{a^i}, \dots, \frac{a^{i-1}}{a^i}, 1, \frac{a^{i+1}}{a^i}, \dots, \frac{a^n}{a^i}\right]$$

$\cdot a^i$

$$= [a^0, \dots a^{i-1}, a^i, \dots, a^n].$$

Similarly for the other composition $\square$