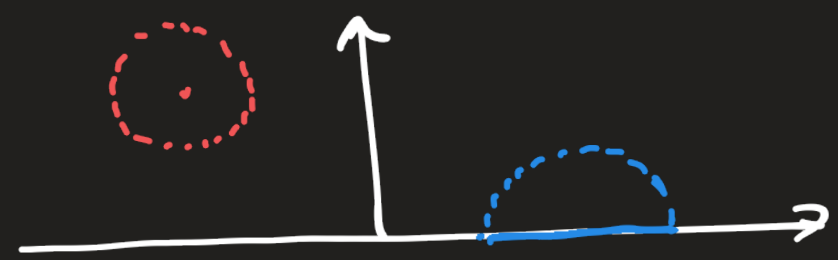
Manifold with boundary

The upper half space in $\mathbb{R}^n$ is the set $\mathbb{H}^n = \{(x^1..x^n) \in \mathbb{R}^n : x^n \geq 0\}$, with the subspace topology from $\mathbb{R}^n$

$n=2$



The boundary of $\mathbb{H}^n$ is $\partial \mathbb{H}^n = \{(x^1..x^n) \in \mathbb{H}^n : x^n = 0\}$.
Points in $\mathbb{H}^n - \partial \mathbb{H}^n$ are called interior points.

Def. An n-dim. top. manifold with boundary M is a topological space M such that

(i) For each $p \in M$, there exist a nbh U and a homeomorphism $\phi : U \to U' \subset \mathbb{H}^n$, where $U' = \phi(U)$ is an open subset of $\mathbb{H}^n$

(ii) M is second countable

(iii) M is Hausdorff.

Def Let M be an n-dim topological manifold with bdr.
An atlas $\{(U_\alpha, \phi_\alpha : U_\alpha \to U_\alpha' \subseteq \mathbb{H}^n)\}$ on M is a collection of charts such that $M = \bigcup_\alpha U_\alpha$. The atlas is C^∞ if for each α, β, the transition function $\phi_\beta \circ \phi_\alpha^{-1}$ is a diffeomorphism:

$$\begin{array}{ccc} & U_\alpha \cap U_\beta & \\ \swarrow^{\phi_\alpha} & & \searrow^{\phi_\beta} \\ \phi_\alpha(U_\alpha \cap U_\beta) & \xrightarrow{\phi_\beta \circ \phi_\alpha^{-1}} & \phi_\beta(U_\alpha \cap U_\beta) \end{array}$$

An n-dim. smooth manifold with boundary is an n-dim top. manifold with boundary together with a maximal C^∞ atlas.

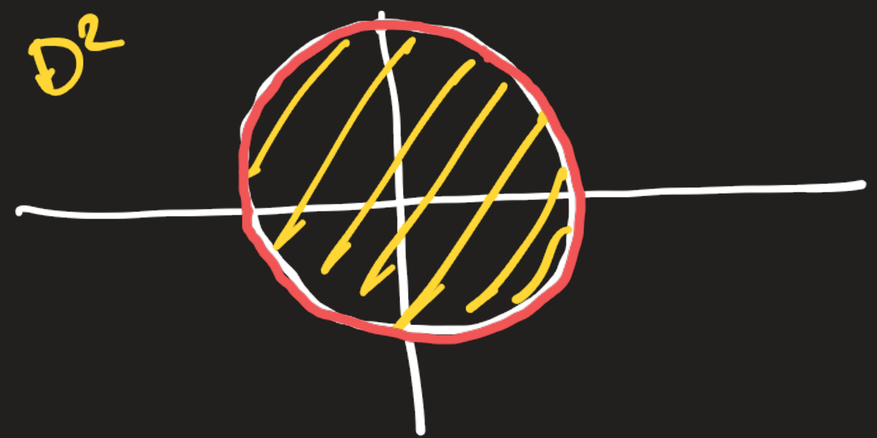
Def Let M be an n-dim smooth manifold with boundary. The boundary $\partial M \subseteq M$ is the subset of points $p \in M$ that are mapped to $\partial \mathbb{H}^n$ by the charts in the maximal atlas. (Independent of choice of chart).

Proposition Let M be a smooth n-dim manifold with boundary. Then ∂M is an $(n-1)$-dim smooth manifold (without boundary).

Proof Every chart $\phi : U \to U' \subseteq \mathbb{H}^n$ on M induces a chart $\phi|_{U \cap \partial M} : U \cap \partial M \longrightarrow U' \cap \partial \mathbb{H}^n \subseteq \mathbb{R}^{n-1}$ on ∂M.
In this way a smooth atlas on M induces a smooth atlas on ∂M. $\square$

Ex For $M = D^n$ we have $\partial M = \partial D^n = S^{n-1}$

$n=2$



Tangent spaces

Consider first $T_p \mathbb{H}^n$ for $p \in \partial \mathbb{H}^n$.

$C_p^\infty(\mathbb{H}^n)$ is the $\mathbb{R}$-algebra of C^∞ function germs at p.

(i.e., equivalence classes of smooth functions $f : U \to \mathbb{R}$, $p \in U \subseteq \mathbb{H}^n$)

Let $T_p \mathbb{H}^n = \{$ point derivations $C_p^\infty(\mathbb{H}^n) \to \mathbb{R}\}$, $\mathbb{R}$-vector space

Notice : There is a surjective homomorphism of $\mathbb{R}$-algebras

$$C_p^\infty(\mathbb{R}^n) \longrightarrow C_p^\infty(\mathbb{H}^n)$$

This induces an isomorphism

$$\underbrace{\{\text{point derivations } C_p^\infty(\mathbb{H}^n) \to \mathbb{R}\}}_{=\, T_p \mathbb{H}^n} \longrightarrow \underbrace{\{\text{point derivations } C_p^\infty(\mathbb{R}^n) \to \mathbb{R}\}}_{=\, T_p \mathbb{R}^n}$$

Bases $\left\{ \frac{\partial}{\partial x^i}\Big|_p : i = 1..n \right\} \longleftrightarrow \left\{ \frac{\partial}{\partial x^i}\Big|_p : i = 1..n \right\}$

Def Let M be an n-dim smooth manifold with boundary.
The tangent space at $p \in M$ is defined by

$$T_pM = \{\text{point derivations } C_p^\infty(M) \to \mathbb{R}\},$$

where $C_p^\infty(M)$ is the $\mathbb{R}$-algebra of C^∞ function germs at p.

<u>Notice</u>: A chart $\phi = (x^1 .. x^n) : U \to U' \subseteq \mathcal{H}^n$ defines an isomorphism

$$\phi_{*,p} : T_pM \longrightarrow T_{\phi(p)}\mathcal{H}^n \cong T_{\phi(p)}\mathbb{R}^n \cong \mathbb{R}^n$$

$$\left\{ \frac{\partial}{\partial x^i}\Big|_p : i = 1..n \right\} \longmapsto \left\{ \frac{\partial}{\partial r^i}\Big|_{\phi(p)} : i = 1..n \right\}$$

The tangent bundle of M has total space $TM = \bigcup_{p \in M} T_pM$

This is again a smooth manifold with boundary.

$$\left(\tilde{\phi} : TU \longrightarrow \phi(U) \times \mathbb{R}^n \quad \left(\text{or } \underset{\mathcal{H}^{2n}}{\underset{\cap}{\mathbb{R}^n \times \phi(U)}} \right) \right).$$

Similarly, the vector bundle of alternating k-tensors (k-covectors) has total space

$$A_k(TM) = \bigcup_{p \in M} A_k(T_p M)$$

This is again a smooth manifold with boundary.

By definition $\Omega^k(M)$ is the vector space of smooth sections of $A_k(TM) \to M$.
(The vector of smooth k-forms on M).

Boundary orientations

Let M be an n-dim. smooth manifold with boundary. We say that M is orientable if there exists a continuous pointwise orientation of the vector spaces T_pM.

The following conditions are equivalent (as when $\partial M = \emptyset$)

(i) M is orientable

(ii) There exists a nowhere-vanishing n-form $\omega \in \Omega^n(M)$

(iii) There exists an oriented atlas on M.

Rem In order to have an oriented atlas, we sometimes have to replace a chart $\phi = (x^1 .. x^n)$ by $\psi = (-x^1, x^2 .. x^n)$

For $n=1$, this means we must allow charts

$$\psi : U \to U' \subseteq \mathcal{H}^1 = \{ x \in \mathbb{R} : x \leq 0 \}.$$

Since there is a diffeomorphism $\mathcal{H}^1 \to \mathcal{H}^1$, $x \mapsto -x$, this does not change the definition of a manifold with boundary.

Ex $D^n = \{ x \in \mathbb{R}^n : \|x\| \leq 1 \}$ is an orientable manifold with boundary. The nowhere-vanishing n-form $\omega = dx^1 \wedge \dots \wedge dx^n$ defines an orientation.

Given an oriented manifold with boundary M, we would like to have an induced orientation of ∂M.

Outward-pointing vector fields

If M is an n-dim manifold with boundary, then ∂M is an $(n-1)$-dim. manifold.

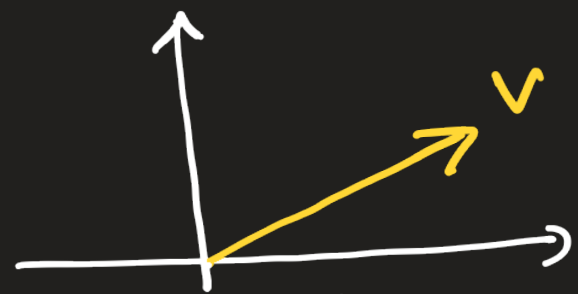
For $p \in \partial M$, we can view $T_p \partial M$ as an $(n-1)$-dim subspace of $T_p M$: The surjective algebra homomorphism $C_p^\infty(M) \to C_p^\infty(\partial M)$ induces $\{\text{point derivations } C_p^\infty(\partial M) \to \mathbb{R}\} \hookrightarrow \{\text{point derivations } C_p^\infty(M) \to \mathbb{R}\}$

<u>Def</u>. Let $p \in \partial M$. A vector $X_p \in T_p M$ is inward pointing if $X_p \notin T_p \partial M$ and there exists a smooth curve $c: [0, \varepsilon) \to M$ such that $c(0) = p$ and $c(0, \varepsilon) \subseteq M - \partial M$, and $c'(0) = X_p$.

We say that $X_p \in T_p M$ is outward-pointing if $-X_p$ is inward pointing.

Ex Let $M = \mathbb{H}^n$ with standard coordinates $x^1, \dots, x^n$.

Given $p \in \partial \mathbb{H}^n$, we have the standard basis $\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p$ for $T_p \mathbb{H}^n$. It is easy to see that $v = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i}\Big|_p$ is inward pointing iff $a^n > 0$.



Rem. Consider a manifold with boundary M and a point $p \in \partial M$. Let $\phi = (x^1, \dots, x^n) : U \to U' \subseteq \mathbb{H}^n$ be a chart s.t. $p \in U$. Then $\phi(p) \in \partial \mathbb{H}^n$ and $\phi_{*,p} : T_p M \to T_{\phi(p)} \mathbb{H}^n$ takes inward pointing tangent vectors to inward-pointing tangent vectors: If $v \in T_p M$ is the velocity vector $v = c'(0)$, then $\phi_{*,p}(v) = (\phi \circ c)'(0)$. For a general tangent vector $v = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i}\Big|_p$, we have that $\phi_{*,p}(v) = \sum_{i=1}^n a^i \frac{\partial}{\partial r^i}\Big|_{\phi(p)}$. Hence v is inward pointing iff $a^n > 0$. It follows that v is outward-pointing iff $a^n < 0$.

Ex Let $M = D^3$ such that $\partial M = S^2$.

For $p = (x,y,z)$, the tangent vector $X_p = x\frac{\partial}{\partial x}\Big|_p + y\frac{\partial}{\partial y}\Big|_p + z\frac{\partial}{\partial z}\Big|_p$ is outward point

In general, for a manifold with boundary M, the restriction of TM to ∂M gives a smooth vector bundle $TM|_{\partial M} \to \partial M$.

Def A smooth outward pointing vector field along ∂M is a smooth section $X : \partial M \to TM|_{\partial M}$, such that $X_p \in T_pM$ is outward pointing for each $p \in \partial M$.

Prop On every manifold with boundary M, there exists a smooth outward pointing vector field along ∂M.

Proof Let $\{(U_\alpha, \phi_\alpha = (x_\alpha^1, \dots, x_\alpha^n) : U_\alpha \to U_\alpha' \subseteq \mathbb{H}^n)\}$ be a collection of charts on M such that $\partial M \subseteq \bigcup_\alpha U_\alpha$.

On each $U_\alpha \cap \partial M$, we have the outward-pointing vector field $-\frac{\partial}{\partial x_\alpha^n}$. Let $\{\rho_\alpha : \partial M \to [0,1]\}$ be a partition of unity subordinate to the open covering $\{U_\alpha \cap \partial M\}$.

Each $\rho_\alpha \cdot \left(-\frac{\partial}{\partial x_\alpha^n}\right)$ extends by 0 to a smooth vector field along ∂M. Then $X = \sum_\alpha \rho_\alpha \left(-\frac{\partial}{\partial x_\alpha^n}\right)$ is a smooth outward pointing vector field along ∂M. $\square$