Boundary orientations

Let M be a manifold with boundary ∂M. The restriction of TM to ∂M defines a smooth vector bundle $TM|_{\partial M} \to \partial M$.

<u>Def</u> A smooth outward-pointing vector field along ∂M is a smooth section $X : \partial M \to TM|_{\partial M}$, such that $X_p \in T_pM$ is outward pointing for all $p \in \partial M$.

Have proved

<u>Proposition</u> On every manifold with boundary there exists a smooth outward pointing vector field along the boundary.

Let V be an n-dim. vector space. For $v \in V$, <u>contraction with v</u> is the linear map

$$\iota_v : A_n(V) \longrightarrow A_{n-1}(V), \quad \iota_v(\alpha)(v_2, \dots, v_n) = \alpha(v, v_2, \dots, v_n).$$

<u>Lemma</u> For $\alpha^1, \dots, \alpha^n \in A_1(V)$, the contraction of $\alpha^1 \wedge \dots \wedge \alpha^n \in A_n(V)$ is given by

$$\iota_v(\alpha^1 \wedge \dots \wedge \alpha^n) = \sum_{i=1}^{n} (-1)^{i-1} \alpha^i(v)\, \alpha^1 \wedge \dots \wedge \hat{\alpha}^i \wedge \dots \wedge \alpha^n$$

<u>Proof</u>

$$\iota_v(\alpha^1 \wedge \dots \wedge \alpha^n)(v_2, \dots, v_n) = \alpha^1 \wedge \dots \wedge \alpha^n (v, v_2, \dots, v_n)$$

$$= \det \begin{bmatrix} \alpha^1(v) & \alpha^1(v_2) & \cdots & \alpha^1(v_n) \\ \alpha^2(v) & \alpha^2(v_2) & \cdots & \alpha^2(v_n) \\ & \vdots & & \\ \alpha^n(v) & \alpha^n(v_2) & \cdots & \alpha^n(v_n) \end{bmatrix} = \sum_{i=1}^{n} (-1)^{i-1} \alpha^i(v) \det \begin{bmatrix} \alpha^1(v) & \cdots & \alpha^1(v_n) \\ \alpha^i(v) & \cdots & \alpha^i(v_n) \\ \alpha^n(v) & \cdots & \alpha^n(v_n) \end{bmatrix}$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \alpha^i(v)\, \alpha^1 \wedge \dots \wedge \hat{\alpha}^i \wedge \dots \wedge \alpha^n (v_2, \dots, v_n) \qquad \square$$

<u>Recall</u> If M is an oriented n-dim manifold with boundary, then an orientation form $\omega \in \Omega^n(M)$ is a nowhere-vanishing n-form such that $\omega_p(v_1 .. v_n) > 0$ if $(v_1 . v_n)$ is a positively oriented basis for T_pM.

Let $\omega \in \Omega^n(M)$ be an orientation form and X an outward-pointing vector field along ∂M

We define an $(n-1)$-form $\iota_X(\omega)$ on ∂M by setting

$\iota_X(\omega)_p(v_2, .. v_n) = \omega_p(X_p, v_2 . v_n)$ for $p \in \partial M$, $v_2 .. v_n \in T_p \partial M$.

Given a chart $\phi = (x^1 .. x^n) : U \to \phi(U) \subseteq \mathbb{R}^n$, we have

$\omega|_U = f \, dx^1 \wedge .. \wedge dx^n$, so by the previous lemma

$\iota_X(\omega)|_U = \sum_{i=1}^{n} (-1)^{i-1} f \, dx^i(X) \, dx^1 \wedge .. \wedge \widehat{dx^i} \wedge .. \wedge dx^n$ (Restricted to vectors in $T\partial M$)

This shows that $\iota_X(\omega)$ is a smooth $(n-1)$-form on ∂M.

Prop $\iota_X(\omega)$ is a nowhere-vanishing $(n-1)$-form on ∂M.

Proof Let $p \in \partial M$ and let $v_2 \ldots v_n$ be a basis for $T_p \partial M$.

Then $X_p, v_2 \ldots v_n$ is a basis for $T_p M$, so

$$\iota_X(\omega)_p(v_2, \ldots v_n) = \omega_p(X_p, v_2, \ldots v_n) \neq 0,$$

since ω is nowhere-vanishing. $\square$

Def If $\omega \in \Omega^n(M)$ is an orientation form, then we give ∂M the orientation determined by $\iota_X(\omega)$, where X is an outward-pointing vector field along ∂M.

This means that $(v_1 \ldots v_{n-1})$ is a positively oriented basis for $T_p \partial M$ iff. $(X_p, v_1 \ldots v_{n-1})$ is a positively oriented basis for $T_p M$.

Ex Consider $M = D^3$ with standard coordinates x, y, z. Then

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

is an outward pointing vector field along $\partial M = S^2$.

On D^3 we have the orientation form $\omega = dx \wedge dy \wedge dz$.

The boundary orientation of S^2 is given by

$$\iota_X(dx \wedge dy \wedge dz) = dx(X)\, dy \wedge dz - dy(X)\, dx \wedge dz + dz(X)\, dx \wedge dy$$

$$= x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy.$$

Rmk The orientation of ∂M is independent of the choice of outward-pointing vector field: If X and Y are outward-pointing then so is $tX + (1-t)Y$ for $t \in [0,1]$. Hence $\iota_X(\omega)$ and $\iota_Y(\omega)$ are connected by a path of nowhere-vanishing $(n-1)$ form in $\Omega^{n-1}(\partial M)$, therefore determine the same orientations.

Next week: Integration of forms

Let M be an n-dim manifold (possible with boundary).

The support of $\omega \in \Omega^k(M)$ is defined by

$$\operatorname{supp} \omega = \overline{\{ p \in M : \omega_p \neq 0 \}}$$

Let $\Omega^k_c(M) \subseteq \Omega^k(M)$ be the subspace of k-forms with compact support. This is a subspace because:

- $\operatorname{supp}(\omega + \tau) \subseteq \operatorname{supp}(\omega) \cup \operatorname{supp}(\tau)$ for $\omega, \tau \in \Omega^k(M)$
- $\operatorname{supp}(c \cdot \omega) \subseteq \operatorname{supp}(\omega)$ for $\omega \in \Omega^k(M)$ and $c \in \mathbb{R}$.

<u>Goal</u>:
- Define the integral $\int_M \omega$ for M an oriented n-dim manifold (possibly with boundary) and $\omega \in \Omega^n_c(M)$
- Prove Stokes theorem in this general manifold setting.