## Ex. 1.2

$$f(x) = \begin{cases} e^{-1/x} & , x > 0 \\ 0 & , x \leq 0 \end{cases}$$

(a) Show $f^{(n)}(x) = P_{2n}\left(\frac{1}{x}\right) e^{-1/x}$ for $x > 0$, where $P_{2n}(Y)$ is a polynomial of degree $2n$.

$\underline{n=0}$ $\quad f(x) = e^{-1/x}$, $\quad P_0(Y) = 1$.

$\underline{n=1}$ $\quad f'(x) = \frac{1}{x^2} e^{-1/x}$, let $P_2(Y) = Y^2$

<u>Induction:</u> suppose $f^{(n)}(y) = P_{2n}(1/x)\, e^{-1/x}$.

$$f^{(n+1)}(x) = -\frac{1}{x^2} P_{2n}'\left(\frac{1}{x}\right) e^{-1/x} + P_{2n}(1/x) \frac{1}{x^2} e^{-1/x}$$

$$= \underbrace{\left[-P_{2n}'(1/x) + P_{2n}(1/x)\right] 1/x^2}_{P_{2(n+1)}(1/x).} e^{-1/x}$$

(b) Claim $f$ is $C^\infty$ and $f^{(n)}(0) = 0$.

By induction, suppose $f^{(n)}$ exists and is cont, and $f^{(n)}(0)=0$.

$$\frac{f^{(n)}(x) - f^{(n)}(0)}{x} = \frac{P_{2n}(\frac{1}{x}) \cdot e^{-1/x}}{x} = \frac{P_{2n}(\frac{1}{x}) \cdot \frac{1}{x}}{e^{1/x}}$$

$$\overset{y=\frac{1}{x}}{=\!=} \frac{P_{2n}(y)\, y}{e^{y}} \to 0 \quad \text{as } y \to \infty.$$

Hence $f^{(n)}$ is differentiable at 0 and $f^{(n+1)}(0) = 0$.

Must prove that $f^{(n+1)}$ is continuous on $\mathbb{R}$.

Clearly continuous at all points $x \neq 0$.

At the point 0 we have for $x>0$:

$$f^{(n+1)}(x) = P_{2(n+1)}(1/x) \cdot e^{-1/x} = \frac{P_{2(n+1)}(1/x)}{e^{1/x}} \to 0 \quad \text{as } x \to 0_+$$

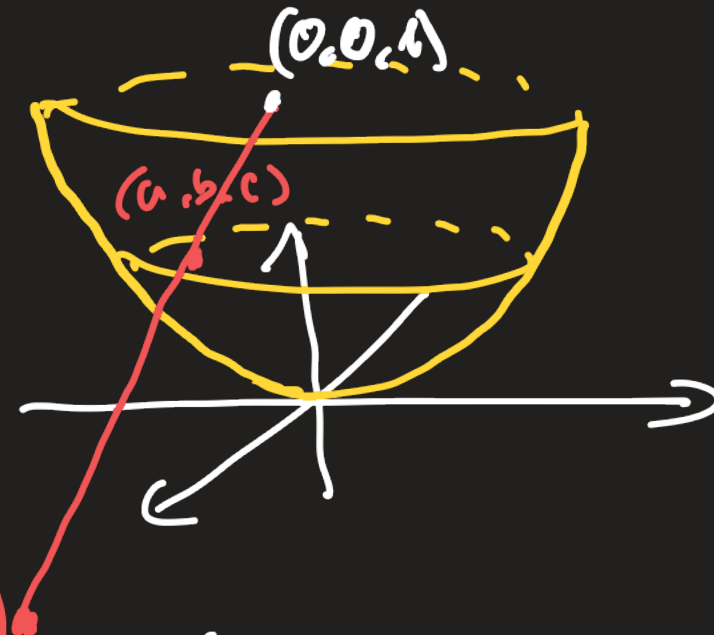
Hence $f^{(n+1)}$ is also continuous at 0.

Conclusion: $f$ is $C^{n+1}$.

Ex. 1.5 $\quad S = \{(x,y,z) : x^2+y^2+(z-1)^2 = 1\}$

Bijection $f : B(\underline{0},1) \to S$, $f(a,b) = (a, b, 1-\sqrt{1-(a^2+b^2)})$.

(a) $\quad g : S \to \mathbb{R}^2$




$$(0,0,1) + t\big((a,b,c) - (0,0,1)\big) = (ta, tb, 1+t(c-1)) = (u,v,0)$$

$$\Rightarrow t = \frac{-1}{c-1} = \frac{1}{1-c} \quad \text{and} \quad (u,v) = \left(\frac{a}{1-c}, \frac{b}{1-c}\right).$$

similarly, given $(u,v)$, have

$$(a,b,c) = \left(\frac{u}{\sqrt{1+u^2+v^2}}, \frac{v}{\sqrt{1+u^2+v^2}}, 1 - \frac{1}{\sqrt{1+u^2+v^2}}\right).$$

(b) Composition $h = g\circ f : B(\underline{0},1) \to S \to \mathbb{R}^2$,

$$h(a,b) = \left(\frac{a}{\sqrt{1-a^2-b^2}}, \frac{b}{\sqrt{1-a^2-b^2}}\right).$$

The inverse $(u,v) \mapsto \left(\frac{u}{\sqrt{1+u^2+v^2}}, \frac{v}{\sqrt{1+u^2+v^2}}\right)$

These are $C^\infty$ so $h$ is a diffeomorphism.

In general $h: B(\underline{0}, 1) \to \mathbb{R}^n$

$$h(a_1, \dots, a_n) = \left( \frac{a_1}{\sqrt{1-(a_1^2+\dots+a_n^2)}}, \dots, \frac{a_n}{\sqrt{1-(a_1^2+\dots+a_n^2)}} \right)$$

with inverse

$$(a_1, \dots, a_n) \mapsto \left( \frac{a_1}{\sqrt{1+a_1^2+\dots+a_n^2}}, \dots, \frac{a_n}{\sqrt{1+a_1^2+\dots+a_n^2}} \right)$$

Hence $h$ is a diffeomorphism.

<u>Ex 1.8</u> $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ $C^\infty$

$f$ is bijective with inverse $g(x) = x^{1/3}$.

Notice $g$ is <u>not</u> differentiable at $0$ since

$$\frac{g(x)-g(0)}{x} = x^{-2/3} \to \infty \text{ when } x \to 0.$$

Hence $f$ is not a diffeomorphism.

__Ex 2.2__ Check $C^\infty_p$ is an $\mathbb{R}$-algebra.

Define addition: $(f, U) + (g, V) = (f+g, U \cap V)$

by working with representatives of function germs.

Must check this is well-defined on equivalence classes:

$(f_1, U_1) \sim (f_2, U_2)$ so have $f_1|U_{12} = f_2|U_{12}$, $U_{12} \subseteq U_1 \cap U_2$

$(g_1, V_1) \sim (g_2, V_2)$ so have $g_1|V_{12} = g_2|V_{12}$, $V_{12} \subseteq V_1 \cap V_2$.

$(f_1 + g_1, U_1 \cap V_1) \sim (f_2 + g_2, U_2 \cap V_2)$ since

$f_1 + g_1 | U_{12} \cap V_{12} = f_2 + g_2 | U_{12} \cap V_{12}$ OK.