Review of point derivations

$p \in \mathbb{R}^n$. A function germ at p is an equivalence class of pairs (f, U), where U is a nbh of p and $f: U \to \mathbb{R}$ C^∞.

$(f, U) \sim (g, V)$ if there exists $p \in W \subseteq U \cap V$ s.t. $f|W = g|W$.

$C^\infty_p(\mathbb{R}^n)$: $\mathbb{R}$ algebra of C^∞ function germs at p.

A point derivation at p is a linear map $D: C^\infty(\mathbb{R}^n) \to \mathbb{R}$ such that $D(f \cdot g) = D(f)\, g(p) + f(p)\, D(g)$

Let $\mathcal{D}_p(\mathbb{R}^n)$ be the real vector space of point derivations

$D : C^\infty_p(\mathbb{R}^n) \to \mathbb{R}$ $\qquad \left((D + D')(f) = D(f) + D'(f) \right)$

There is a linear map $\phi: T_p\mathbb{R}^n \cong \mathbb{R}^n \longrightarrow \mathcal{D}_p(\mathbb{R}^n)$, $v \mapsto D_v$

where D_v is the directional derivative:

$$D_v(f) = \lim_{t \to 0} \frac{f(p+tv) - f(p)}{t} = \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}(p), \quad v = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$$

<u>Theorem</u> ϕ is an isomorphism of vector spaces

Proof later today.

<u>Consequence</u> The standard basis vectors $e_1, \dots, e_n$ for $\mathbb{R}^n$ correspond to the basis $\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^n}\big|_p$ for $\mathcal{D}_p(\mathbb{R}^n)$

Later we shall redefine $T_p\mathbb{R}^n$ to be $\mathcal{D}(\mathbb{R}^n)$

shall use similar construction to define T_pM for a point p on a manifold M.

(only depends on defining when a function in a neighborhood of p is C^∞ - can the define $C^\infty_p(M)$).

Def A subset $S \subseteq \mathbb{R}^n$ is <u>starshaped</u> wrt. $P \in S$ if for each $x \in S$, the line segment $\{P + t(x-P) : 0 \le t \le 1\}$ is contained in S

For instance any open ball $B(P, \varepsilon) = \{x \in \mathbb{R}^n : \|x - P\| < \varepsilon\}$ is star shaped wrt P.

<u>**Lemma**</u> Let $U \subseteq \mathbb{R}^n$ be an open set which is star shaped with respect to $P \in U$ and let $f : U \to \mathbb{R}$ be C^∞. Then there exist C^∞ functions $g_1, \ldots, g_n : U \to \mathbb{R}$ such that
$$f(x) = f(P) + \sum_{i=1}^{n} (x^i - p^i)\, g_i(x) \text{ for } x \in U, \text{ and } g_i(P) = \frac{\partial f}{\partial x^i}(P).$$

<u>**Proof**</u> Consider the function $(x,t) \mapsto f(P + t(x-P))$ defined on $U \times [0,1]$.

We use the chain rule to evaluate the derivative wrt. t :

$$\frac{d}{dt} f(P+t(x-P)) = \sum_{i=1}^{n} (x^i - p^i) \frac{\partial f}{\partial x^i}(P+t(x-P))$$

$$\int_0^1 \frac{d}{dt} f(P+t(x-P))\,dt = \Big[f(P+t(x-P)) \Big]_0^1 = f(x) - f(P)$$

$$\parallel$$

$$\int_0^1 \sum_{i=1}^{n} (x^i - p^i) \frac{\partial f}{\partial x^i}(P+t(x-P))\,dt = \sum_{i=1}^{n} (x^i - p^i) \int_0^1 \frac{\partial f}{\partial x^i}(P+t(x-P))\,dt$$

Let $g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(P+t(x-P))\,dt$, $g_i(P) = \frac{\partial f}{\partial x^i}(P)$.

<u>Claim</u> $g_i : U \to \mathbb{R}$ is C^∞.

That uses Leibniz rule for differentiation under the integral sign: $\frac{\partial g_i}{\partial x^j}(x) = \int_0^1 \frac{\partial}{\partial x^j} \frac{\partial f}{\partial x^i}(P+t(x-P))\,dt$

(see eg. Adams Calculus Section 13.5). $\square$

Lemma Let $D \in \mathcal{D}_p(\mathbb{R}^n)$ (so $D: C^\infty_p(\mathbb{R}^n) \to \mathbb{R}$). Then $D(c) = 0$ for any constant function c.

Proof D is linear, so $D(c) = D(c \cdot 1) = c \cdot D(1)$.

suffices to show $D(1) = 0$.

$$D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1\, D(1) = 2D(1) \Rightarrow D(1) = 0.$$

$\square$

Proof that $\phi: T_p(\mathbb{R}^n) = \mathbb{R}^n \to \mathcal{D}_p(\mathbb{R}^n)$ is an isomorphism.

Injective: Suppose $\phi(v) = D_v = 0$ for $v = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$

We know $D_v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\Big|_p$.

Hence $0 = D_v(x^j) = \sum_{i=1}^{n} v^i \frac{\partial x^j}{\partial x^i}(p) = v^j$ for all j,

so $v = 0$.

Surjectivity Given $D \in \mathcal{D}_p(\mathbb{R}^n)$, let $v^i = D(x^i)$, $i=1..n$.

Claim: $D = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\Big|_p = D_v$.

Given a representative (f, U) for a function germ at p.
we may assume that U is star shaped w.r.t. p.

Then $f(x) = f(p) + \sum_{i=1}^{n} (x^i - p^i)\, g_i(x)$, $g_i(p) = \frac{\partial f}{\partial x^i}(p)$.

$$D(f) = \underset{\overset{\shortparallel}{0}}{D(f(p))} + \sum_{i=1}^{n} D((x^i - p^i)\, g_i(x))$$

$$= \sum_{i=1}^{n} D(x^i - p^i)\, g_i(p) + \underset{\overset{\shortparallel}{0}}{(p^i - p^i)} \cdot D(g_i)$$

$$= \sum_{i=1}^{n} D(x^i)\, g_i(p) = \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}(p).$$

This verifies the claim above. $\square$

Remember Homework: Read up on general topology in App A.

Vector fields on open subsets of $\mathbb{R}^n$

$U \subseteq \mathbb{R}^n$ open. A vector field X on U is a collection of tangent vectors $X = \{X_p \in T_p\mathbb{R}^n : p \in U\}$

Can write $X_p = \sum_{i=1}^{n} a^i(p) \frac{\partial}{\partial x^i}\Big|_p$, $a^i : U \to \mathbb{R}$.

We say that X is C^∞ if the coefficient functions $a^1, \ldots, a^n$ are C^∞ on U.

Rmk: If we identify $T_p\mathbb{R}^n$ with $\mathbb{R}^n$, then a C^∞ vector fied is the same as a C^∞ function

$$U \longrightarrow \mathbb{R}^n, \quad x \mapsto \begin{bmatrix} a^1(x) \\ \vdots \\ a^n(x) \end{bmatrix}$$

Rmk Later we shall consider vector fields on manifolds

$X = \{X_p \in T_pM : p \in M\}$. Since T_pM varies with p, we cannot identify X with a collection of C^∞ functions on M.

Given open subset $U \subseteq \mathbb{R}^n$, let $\mathfrak{X}(U)$ be the real vector space of C^∞ vector fields on U: For $p \in U$

$$(X+Y)_p = X_p + Y_p \quad , \quad (rX)_p = rX_p, \, r \in \mathbb{R}$$

Let $C^\infty(U)$ be the $\mathbb{R}$-algebra of C^∞ functions on U.

There is a function $C^\infty(U) \times \mathfrak{X}(U) \to \mathfrak{X}(U)$.

$$(f \cdot X)_p = f(p) \cdot X_p \quad \text{for } f \in C^\infty(U), \; X \in \mathfrak{X}(U).$$

This makes $\mathfrak{X}(U)$ a module over $C^\infty(U)$:

(i) associativity $(f \cdot g) X = f(gX)$

(ii) identity $1 \cdot X = X$ (1 is the constant function)

(iii) distributivity $(f+g) X = fX + gX$

$$f(X+Y) = fX + fY.$$

Def A derivation of $C^\infty(U)$ is an $\mathbb{R}$-linear map $D: C^\infty(U) \to C^\infty(U)$ such that $D(f \cdot g) = D(f) \cdot g + f \cdot D(g)$.

Write $\mathrm{Der}(C^\infty(U))$ for the real vector space of such derivations.

Proposition A vector field X defines a derivation $X: C^\infty(U) \to C^\infty(U)$ by $X(f)(p) = X_p(f)$.

(By definition $X_p: C^\infty_p(\mathbb{R}^n) \to \mathbb{R}$ point derivation)

Proof Must check $X(f \cdot g) = X(f) \cdot g + f\, X(g)$.

Check for each p:

$$X(fg)(p) = X_p(f \cdot g) = X_p(f)\, g(p) + f(p)\, X_p(g)$$
$$= \big(X(f) g + f X(g)\big)(p).$$

$\square$

This construction gives a linear function

$$\mathcal{X}(U) \longrightarrow \mathrm{Der}(C^\infty(U))$$

One can show this is an isomorphism.
(We shall not prove this now).