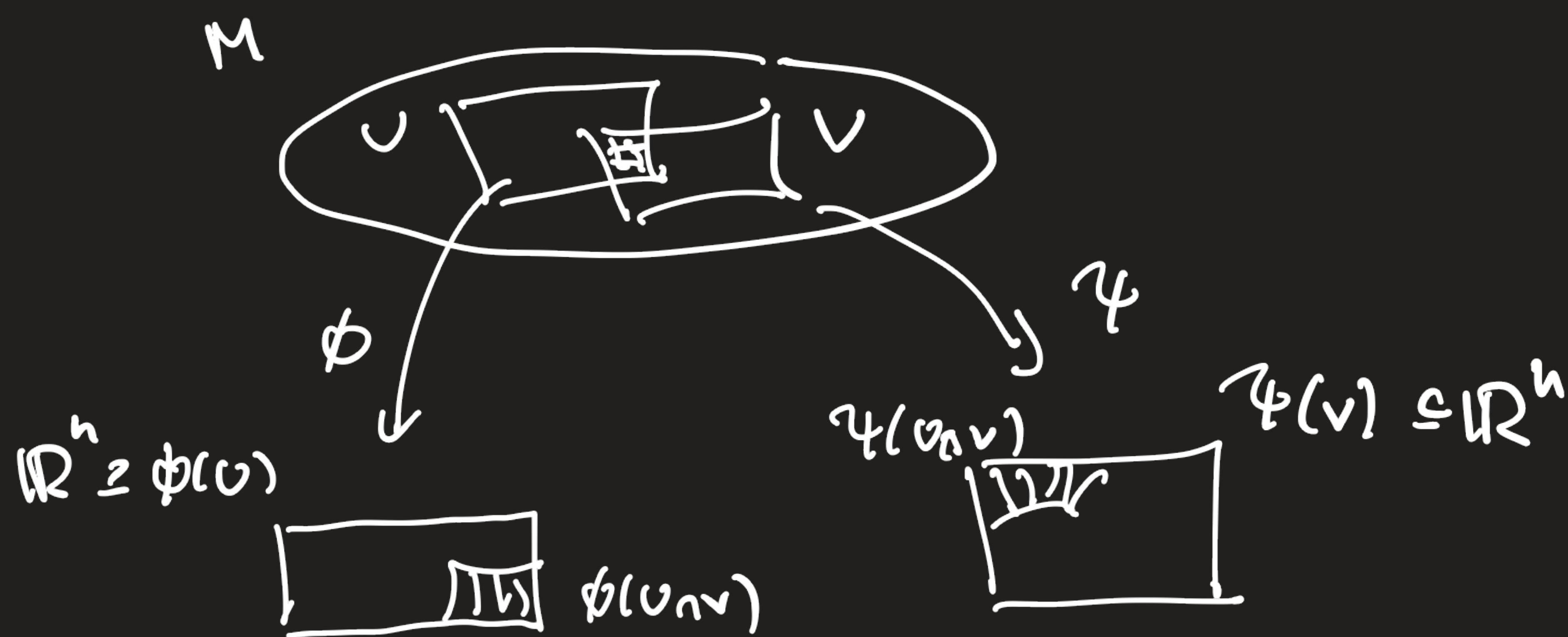
Smooth maps between smooth manifold

Recall A smooth manifold $(M, \mathcal{M})$ is a topological manifold M together with a maximal smooth atlas $\mathcal{M}$ (C^∞ = smooth).

Let (U, ϕ) and (V, ψ) be charts in the maximal atlas



Then $\phi(U \cap V) \xrightarrow{\phi^{-1}} U \cap V \xrightarrow{\psi} \psi(U \cap V)$ is a diffeomorphism (C^∞ map with C^∞ inverse)

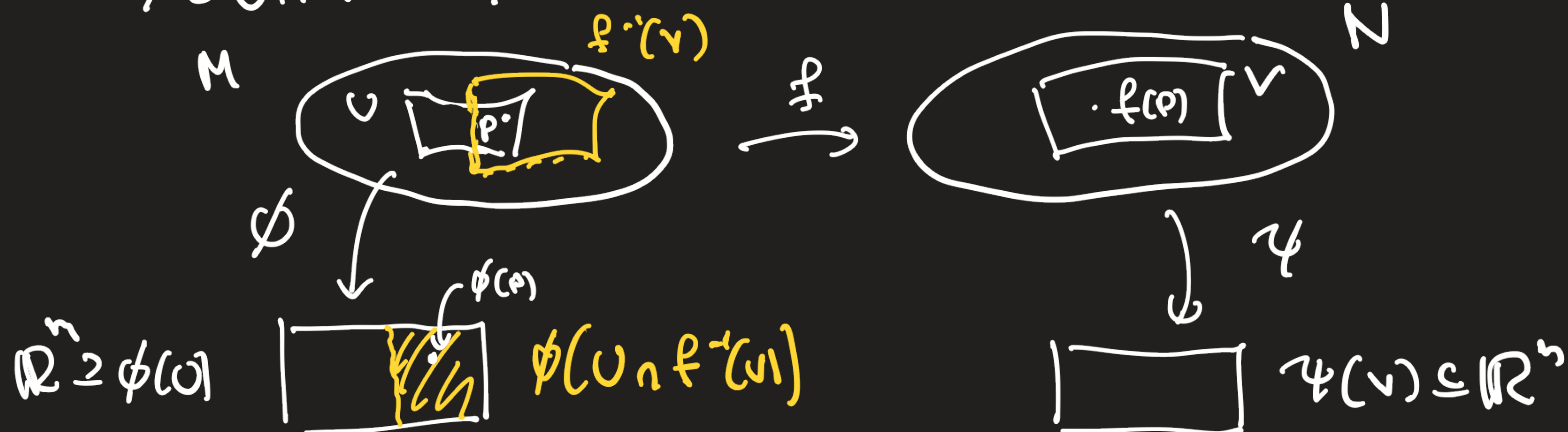
(If $U \cap V = \emptyset$, we say that $\psi \circ \phi^{-1} : \emptyset \to \emptyset$ is C^∞)

Let $(M, \mathfrak{m}_M)$ and $(N, \mathfrak{m}_N)$ be smooth manifolds.
Let $f : M \to N$ be a continuous map. $\left(\begin{array}{l} \dim M = m \\ \dim N = n \end{array} \right)$

<u>Def</u> • f is smooth at a point $p \in M$ if the following hold:

For each chart (U, ϕ) on M st. $p \in U$ and each chart (V, ψ) on N st. $f(p) \in V$, the composition

$$\mathbb{R}^m \supseteq \phi(U \cap f^{-1}(V)) \xrightarrow{\phi^{-1}} U \cap f^{-1}(V) \xrightarrow{f} V \xrightarrow{\psi} \psi(V) \subseteq \mathbb{R}^n$$



is smooth in a neighborhood of $\phi(p)$. (<u>Rmk</u> f cont. $\Rightarrow f^{-1}(V) \subseteq M$ is open $\Rightarrow U \cap f^{-1}(V)$ open $\Rightarrow \phi(U \cap f^{-1}(V)) \subseteq \mathbb{R}^m$ open)

• f is smooth if it is smooth at all points $p \in M$.

<u>Lemma</u> Let $(M, \mathcal{M}_M)$ and $(N, \mathcal{M}_N)$ be smooth manifolds

A function $f : M \to N$ is smooth iff:

For each $p \in M$ there exist a chart (U, ϕ) on M st. $p \in U$ and a chart (V, ψ) on N st. $f(U) \subseteq V$ and the composition $\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ is smooth:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \cup\!\!\mid & & \cup\!\!\mid \\ U & \xrightarrow{f} & V \\ \phi \downarrow & & \downarrow \psi \\ \mathbb{R}^n \supseteq \phi(U) & & \psi(V) \subseteq \mathbb{R}^n \end{array}$$

<u>Proof</u> Suppose such charts (U, ϕ) and (V, ψ) exist for all points $p \in M$. We first check that f is continuous: suffices to show that each $p \in M$ has a neighborhood $p \in U$ st. $f|_U$ is continuous. OK since

$f|_U = \psi^{-1} \circ (\psi \circ f \circ \phi^{-1}) \circ \phi$ composition of cont. functions.

$C^\infty \Rightarrow$ continuous

Next check that f is smooth. Let $p \in M$.

Given any charts (U_1, ϕ_1) on M with $p \in U_1$ and (V_1, ψ_1) on N with $f(p) \in V_1$. Must check

$$\phi_1(U_1 \cap f^{-1}(V_1)) \xrightarrow{\phi_1^{-1}} U_1 \cap f^{-1}(V_1) \xrightarrow{f} V_1 \xrightarrow{\psi} \psi_1(V_1) \subseteq \mathbb{R}^n$$

is smooth in a neighborhood of $\phi_1(p)$.

Choose charts (U, ϕ), $p \in U$ and (V, ψ), $f(U) \subseteq V$ as in the statement of the lemma. ($\phi(U) \xrightarrow{\phi^{-1}} U \xrightarrow{f} V \xrightarrow{\psi} \psi(U)$ C^∞)

$$\begin{tikzcd}
 & U \cap U_1 \cap f^{-1}(V_1) \arrow[dl, "\phi_1"'] \arrow[d, "\phi"] \arrow[rr, "f"] & & V \cap V_1 \arrow[d, "\psi"'] \arrow[dr, "\psi_1"] & \\
\phi_1(U \cap U_1 \cap f^{-1}(V_1)) & \phi(U \cap U_1 \cap f^{-1}(V_1)) & & \psi(V \cap V_1) & \psi_1(V \cap V_1)
\end{tikzcd}$$

Then $\psi_1 \circ f \circ \phi_1^{-1} = (\psi_1 \circ \psi^{-1}) \circ (\psi \circ f \circ \phi^{-1}) \circ (\phi \circ \phi_1^{-1})$

This is a composition of smooth functions, hence smooth. $\square$

<u>Ex</u> $f : \mathbb{R} \to S^1$, $f(t) = (\cos t, \sin t)$

<u>Claim:</u> f is smooth

Consider first $t = 0 \in \mathbb{R}$.

Let $V = \{(x,y) \in S^1 : x > 0\}$, $\psi : V \to (-1,1)$, $\psi(x,y) = y$.

(V, ψ) is a chart on S^1.

Let $U = (-\frac{\pi}{2}, \frac{\pi}{2}) \subseteq \mathbb{R}$. Then $f(U) \subseteq V$ and

$$\begin{tikzcd}
 & \mathbb{R} \arrow[r] & S^1 & \\
 & U \arrow[u, phantom, sloped, "\subseteq"] \arrow[r] \arrow[d, "\mathrm{id}"'] & V \arrow[u, phantom, sloped, "\subseteq"] \arrow[d, "\psi"] & \\
(-\frac{\pi}{2}, \frac{\pi}{2}) = \phi(U) & & (-1,1) &
\end{tikzcd}$$

$t \mapsto (\cos t, \sin t)$

$t \longmapsto \sin t$ smooth

Similar arguments for other points in $\mathbb{R}$.

Ex $M = U \subseteq \mathbb{R}^m$, $N = V \subseteq \mathbb{R}^n$ open subsets.

smooth atlases $\{ id_U : U \to U \}$, $\{ id_V : V \to V \}$.

Hence a function $f : M \to N$ is smooth iff it is smooth in the ordinary sense.

Lemma Let $f : M \to N$ and $g : N \to P$ be smooth maps. Then $g \circ f : M \to P$ is also smooth.

Proof Locally we have charts

$$\begin{array}{ccccc} M & \xrightarrow{f} & N & \xrightarrow{g} & P \\ \cup & & \cup & & \cup \\ U & \xrightarrow{f} & V & \xrightarrow{g} & W \\ \phi \downarrow & & \psi \downarrow & & \sigma \downarrow \\ \phi(U) & & \psi(V) & & \sigma(W) \end{array}$$

$\sigma \circ (g \circ f) \circ \phi^{-1} = (\sigma \circ g \circ \psi^{-1}) \circ (\psi \circ f \circ \phi^{-1})$ composition of smooth functions between open subsets of Euclidean space, hence again smooth

$\square$

<u>Ex</u> Suppose that $f: \mathbb{R}^2 \to \mathbb{R}$ is smooth.

<u>Claim</u> The restriction to $S^1 \subseteq \mathbb{R}^2$ is a smooth map $f|_{S^1}: S^1 \longrightarrow \mathbb{R}$.

By "composition lemma" it suffices to show that the inclusion $S^1 \to \mathbb{R}^2$ is smooth.

Must show that S^1 has a smooth atlas $\{(U_\alpha, \phi_\alpha)\}$ s.t. each composition

$$\phi_\alpha(U_\alpha) \xrightarrow{\phi_\alpha^{-1}} U_\alpha \xrightarrow{\text{inclusion}} \mathbb{R}^2$$

is smooth.

Here we can use the atlas from last week.

For instance $U = \{(x,y) \in S^1 : x > 0\}$, $\phi : U \to (-1,1)$, $\phi(x,y) = y$

$\phi^{-1}(t) = (\sqrt{1-t^2}, t)$ smooth.

Similarly for the other charts.

Def A smooth map $f: M \to N$ is a diffeomorphism if there exists a smooth map $g: N \to M$ st. $g \circ f = id_M$, $f \circ g = id_N$.

Ex Let M be an n-dim smooth manifold. A chart (U, ϕ) on M defines a diffeomorphism

$$\phi : U \to \phi(U) \subseteq \mathbb{R}^n.$$

Thus, a smooth n-dim. manifold is locally diffeomorphic to an open subset of $\mathbb{R}^n$.
(In fact locally diffeomorphic to $\mathbb{R}^n$ itself).

Theorem (Stalling ~1960) If $n \neq 4$, the any two smooth structures on $\mathbb{R}^n$ are diffeomorphic.

Theorem (Donaldson ~1980) There exists a smooth structure on $\mathbb{R}^4$ which is not diffeomorphic to the standard smooth structure (in fact there are uncountably many distinct smooth structures).

For the spheres S^n, there is a unique smooth structure for $n \leq 3$ and for $n \geq 5$ there are at most finitely many distinct smooth structures.

It is not known whether every smooth structure on S^4 is diffeomorphic to the standard one.