The wedge product of alternating multilinear functions

$\wedge : A_k(V) \times A_\ell(V) \longrightarrow A_{k+\ell}(V)$, V finite dim $\mathbb{R}$ vector space

Recall: $f \wedge g = \frac{1}{k!\,\ell!} A(f \otimes g)$

Explicitly: $f \wedge g (v_1, \dots v_{k+\ell}) = \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma)\, \sigma(f \otimes g)(v_1, \dots v_{k+\ell})$

$$= \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma)\, f(v_{\sigma(1)}, \dots v_{\sigma(k)})\, g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}).$$

Prop The wedge product is associative:

$$(f \wedge g) \wedge h = f \wedge (g \wedge h)$$

for $f \in A_m(V)$, $g \in A_\ell(V)$, $h \in A_m(V)$.

<u>Lemma</u> For $f \in L_k(V)$ and $g \in L_\ell(V)$, we have

(i) $A(A(f) \otimes g) = k!\, A(f \otimes g)$

(ii) $A(f \otimes A(g)) = \ell!\, A(f \otimes g)$.

<u>Proof</u> check (i)

$$A(A(f) \otimes g) = A\Big(\sum_{\tau \in S_k} (\operatorname{sgn}(\tau)\, \tau f) \otimes g\Big) = \sum_{\tau \in S_k} A((\operatorname{sgn}\tau\ \tau f) \otimes g).$$

For each $\tau \in S_k$:

$$A((\operatorname{sgn}(\tau)\, \tau f) \otimes g) = \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma)\, \sigma((\operatorname{sgn}(\tau)\, \tau f) \otimes g)$$

$$= \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau)\, \sigma(\tau \cup 1_\ell)(f \otimes g)$$

$$= \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma(\tau \cup 1_\ell))\, \sigma(\tau \cup 1_\ell)(f \otimes g)$$

$$= \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma)\, \sigma(f \otimes g) = A(f \otimes g)$$

The proof of (ii) is similar $\square$

Proof that $(f \wedge g) \wedge h = f \wedge (g \wedge h)$

$$(f \wedge g) \wedge h = \frac{1}{(k+l)!\, m!} A((f \wedge g) \otimes h) = \frac{1}{(k+l)!\, m!} A\left(\frac{1}{k!\, l!} A(f \otimes g) \otimes h \right)$$

$$= \frac{1}{(k+l)!\, m!} \frac{1}{k!\, l!} A(A(f \otimes g) \otimes h) \quad (\text{by lemma})$$

$$= \frac{1}{\cancel{(k+l)!}\, m!} \frac{1}{k!\, l!} \cancel{(k+l)!}\, A((f \otimes g) \otimes h) = \frac{1}{k!\, l!\, m!} A(f \otimes g \otimes h).$$

similarly, $f \wedge (g \wedge h) = \frac{1}{k!\, l!\, m!} A(f \otimes g \otimes h)$ $\square$

Rem We have proved that

$$f \wedge g \wedge h = \frac{1}{k!\, l!\, m!} A(f \otimes g \otimes h).$$

Prop Suppose $f_1 \in A_{k_1}(V), \dots, f_r \in A_{k_r}(V)$.

Then $f_1 \wedge \dots \wedge f_r = \frac{1}{k_1! \dots k_r!} A(f_1 \otimes \dots \otimes f_r)$ in $A_{k_1+\dots+k_r}(V)$.

Proof By induction:

$$f_1 \wedge \dots \wedge f_r = (f_1 \wedge \dots \wedge f_{r-1}) \wedge f_r = \frac{1}{(k_1+\dots+k_{r-1})!\, k_r!} A((f_1 \wedge \dots \wedge f_{r-1}) \otimes f_r)$$

$$= \frac{1}{(k_1+\dots+k_{r-1})!\, k_r!} A\left(\frac{1}{k_1! \dots k_{r-1}!} A(f_1 \otimes \dots \otimes f_{r-1}) \otimes f_r \right]$$

$$= \frac{1}{(k_1+\dots+k_{r-1})!\, k_r!} \frac{1}{k_1! \dots k_{r-1}!} A(A(f_1 \otimes \dots \otimes f_{r-1}) \otimes f_r]$$

$$= \frac{1}{k_1! \cdots k_r!} A(f_1 \otimes \dots \otimes f_r),$$

again using the lemma. $\square$

Ex Given covectors $\alpha^1, \dots, \alpha^k \in A_1(V) = V^\vee$ ($\alpha^i : V \to \mathbb{R}$).

Then $\alpha^1 \wedge \dots \wedge \alpha^k \in A_k(V)$ is given by

$$\alpha^1 \wedge \dots \wedge \alpha^k = A(\alpha^1 \otimes \dots \otimes \alpha^k) = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, \sigma(\alpha^1 \otimes \dots \otimes \alpha^k).$$

Given $v_1, \dots, v_k \in V$, we get

$$\alpha^1 \wedge \dots \wedge \alpha^k (v_1, \dots, v_k) = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, \alpha^1 \otimes \dots \otimes \alpha^k (v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

$$= \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, \alpha^1(v_{\sigma(1)}) \cdot \alpha^2(v_{\sigma(2)}) \cdot \ldots \cdot \alpha^k(v_{\sigma(k)})$$

$$= \det \begin{pmatrix} \alpha^1(v_1) & \alpha^1(v_2) & \cdots & \alpha^1(v_k) \\ \vdots & & & \vdots \\ \alpha^k(v_1) & \alpha^k(v_2) & \cdots & \alpha^k(v_k) \end{pmatrix} = \det\left(\alpha^i(v_j)\right)$$

Def A graded $\mathbb{R}$-algebra $A = \{ A(k) : k \geq 0 \}$ is a collection of $\mathbb{R}$ vector spaces $A(k)$, together with

- unit $1 \in A(0)$
- bilinear multiplication $\bullet : A(k) \times A(\ell) \to A(k+\ell)$, $(a,b) \mapsto a \cdot b$,

such that the multiplication is

- unital: $a \cdot 1 = a$, $1 \cdot a = a$.
- associative:

$$\begin{tikzcd} A(k) \times A(\ell) \times A(m) \arrow[r, "\bullet \times id"] \arrow[d, "id \times \bullet"] & A(k+\ell) \times A(m) \arrow[d, "\bullet"] \\ A(k) \times A(\ell+m) \arrow[r, "\bullet"] & A(k+\ell+m) \end{tikzcd}$$

that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

A is anticommutative if $a \cdot b = (-)^{k\ell} b \cdot a$, $a \in A(k)$, $b \in A(\ell)$.

We proved: $A_*(V) = \{ A_k(V) : k \geq 0 \}$ is an anticomm. graded algebra. This is the <u>exterior algebra</u> of V.

Suppose V is finite dimensional with basis $e_1, \dots, e_n$.

Goal: Find a basis for each $A_k(V)$.

Let $\alpha^1, \dots, \alpha^k \in A_1(V) = V^\vee$ be the dual basis: $\alpha^i(e_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j. \end{cases}$

Given $I = (i_1, \dots, i_k)$ with each $i_s \in \{1, \dots, n\}$.

Write $\alpha^I = \alpha^{i_1} \wedge \dots \wedge \alpha^{i_k}$, $e_I = (e_{i_1}, \dots, e_{i_k})$

<u>Lemma</u> Let $I = (i_1, \dots, i_k)$, $1 \leq i_1 < \dots < i_k \leq n$, $J = (j_1, \dots, j_k)$, $1 \leq j_1 < \dots < j_k \leq n$.

Then $\alpha^I(e_J) = \begin{cases} 1 & \text{if } I = J \\ 0 & \text{if } I \neq J \end{cases}$

<u>Proof</u> Have checked

$$\alpha^I(e_J) = \alpha^{i_1} \wedge \dots \wedge \alpha^{i_k}(e_{j_1}, \dots, e_{j_k}) = \det \begin{pmatrix} \alpha^{i_1}(e_{j_1}) & \alpha^{i_1}(e_{j_2}) & \cdots & \alpha^{i_1}(e_{j_k}) \\ \vdots & & & \vdots \\ \alpha^{i_k}(e_{j_1}) & \alpha^{i_k}(e_{j_2}) & \cdots & \alpha^{i_k}(e_{j_k}) \end{pmatrix}$$

For $I = J$ this is the identity matrix, so $\det = 1$.

For $I \neq J$ the matrix has a zero row, so $\det = 0$.

$\square$

Prop The wedge products $\{\alpha^I : 1 \le i_1 < \dots < i_k \le n\}$ form a basis of $A_k(V)$

Proof Linearly independent: suppose $\sum_I c_I \alpha^I = 0$, $c_I \in \mathbb{R}$.

Apply to e_J to get $0 = \sum_I c_I \alpha^I(e_J) = c_J$ for all J.

Generators Given $f \in A_k(V)$, let $c_I = f(e_I)$

Claim: $f = \sum_I c_I \alpha^I$.

Rmk An alternating k-linear function f is uniquely determined by the values $f(e_J)$, $1 \le j_1 < \dots < j_k \le n$.

we have $\sum_I c_I \alpha^I(e_J) = c_J = f(e_J)$ for all J, by the previous lemma. $\square$

Corollary Suppose $\dim V = n$.

- $\dim A_k(V) = \binom{n}{k}$ for $0 \le k \le n$
- $A_k(V) = 0$ for $k > n$.

Proof The basis elements for $A_k(V)$ $d^{i_1} \wedge \dots \wedge d^{i_k}$ are defined by choosen a subset $\{i_1, \dots, i_k\}$ of $\{1, \dots n\}$.

Rem Given $d^{i_1} \wedge \dots \wedge d^{i_k}$, $1 \le i_1 < \dots < i_k \le n$ for $k > n$. Then $i_s = i_{s+1}$ and hence $d^{i_s} \wedge d^{i_{s+1}} = 0$ by anti-commutativity.

Differential forms

Recall We have identified $T_p\mathbb{R}^n$ with the vector space of point derivations $D: C_p^\infty(\mathbb{R}^n) \to \mathbb{R}$.

Canonical basis $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p \right\}$.

Let $U \subseteq \mathbb{R}^n$ be open

A C^∞ vector field on U has the form $X = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i}$ where $a^i : U \to \mathbb{R}$ are C^∞ functions, $X_p = \sum_{i=1}^n a^i(p) \frac{\partial}{\partial x^i}\Big|_p$

Def A covector field is a collection of covectors

$$\omega = \{ \omega_p \in (T_p\mathbb{R}^n)^\vee : p \in U \}$$

For each $p \in U$, there is a bilinear function

$$T_p\mathbb{R}^n \times C_p^\infty(\mathbb{R}^n) \to \mathbb{R}, \quad (X_p, f) \mapsto X_p(f).$$

Hence a C^∞ function $f: U \to \mathbb{R}$ defines a covector field

$$df = \{ df_p \in (T_p\mathbb{R}^n)^\vee : p \in U \}, \text{ where } df_p(X_p) = X_p(f)$$

This is the <u>differential of f</u>

Let $x^1, \dots, x^n$ be the coordinate functions on U

Gives covector fields $dx^1, \dots, dx^n$ on U

<u>Prop</u> For each $p \in U$, $dx^1_p, \dots, dx^n_p$ is the dual basis associated to the basis $\{ \frac{\partial}{\partial x^1}|_p, \dots, \frac{\partial}{\partial x^n}|_p \}$ for $T_p\mathbb{R}^n$.

<u>proof</u>

$$(dx^i_p)\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \frac{\partial}{\partial x^j}\Big|_p (x^i) = \frac{\partial x^i}{\partial x^j}(p) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$\square$

Consequence A covector field ω on U can be written uniquely $\omega = \sum_{i=1}^{n} a_i \, dx^i$, where $a_i : U \to \mathbb{R}$ are functions for $i = 1, \dots, n$.

Def ω is a C^∞ covector field if each a_i is C^∞.