Ex. 11.1 $S^n \subseteq \mathbb{R}^{n+1}$, $S^n = \{ x \in \mathbb{R}^{n+1} : (x^1)^2 + \dots + (x^{n+1})^2 = 1 \}$

$i : S^n \to \mathbb{R}^{n+1}$, $i_{*,p} : T_p S^n \to T_p \mathbb{R}^{n+1}$

Suppose $X_p = \sum_{i=1}^{n+1} a^i \frac{\partial}{\partial x^i}\Big|_p \in T_p \mathbb{R}^{n+1}$

What is the condition for X_p to be in the image of $i_{*,p}$?

Claim: if $p = (p^1, \dots, p^{n+1})$, the condition is that $\sum_{i=1}^{n+1} a^i p^i = 0$

Suppose $X_p = c'(0)$, $c : (-\varepsilon, \varepsilon) \to S^n$, $c(0) = p$.

We know $\sum_{i=1}^{n+1} c^i(t)^2 = 1 \Rightarrow 0 = \frac{d}{dt}\Big|_{t=0} \sum_{i=1}^{n+1} c^i(t)^2 = \sum_{i=1}^{n+1} 2\, c^i(0)\, \dot{c}^i(0)$

$= \sum_{i=1}^{n+1} 2\, p^i \cdot a^i \Longrightarrow \sum_{i=1}^{n+1} p^i a^i = 0 .$

Ex. 11.3 Show that a smooth function $f: N \to \mathbb{R}^m$ from a compact manifold N has a critical point.

Suppose f has no critical points. Then f is a submersion.

Corollary 11.6: Every submersion $f: N \to \mathbb{R}^m$ is an open map.

(since locally

$$\begin{array}{ccc} N & \xrightarrow{f} & \mathbb{R}^m \\ \cup\!\mid & & \cup\!\mid \\ U & \xrightarrow{f} & V \\ \phi \downarrow & & \downarrow \psi \\ \mathbb{R}^n \supseteq \phi(U) & \longrightarrow & \psi(V) \subseteq \mathbb{R}^m \\ (r^1, .. r^n) & \longmapsto & (r^1, .., r^m) \end{array}$$

This is a contradiction since $f(N)$ is compact.

[By definition $p \in N$ is a critical point if $f_{*,p}: T_pN \to T_{f(p)}\mathbb{R}^m$ is not surjective. If p is not critical, then $f_{*,p}$ is surjective and therefore $\dim N \geq m$.

Ex 11.4 $\quad i : S^2 \to \mathbb{R}^3$, x, y, z coordinates of $\mathbb{R}^3$.

$U = \{ p \in S^2 : z(p) > 0 \}$, $\phi : U \to \mathbb{R}^2$, $\phi = (u, v)$. where

$u(a,b,c) = a$, $\quad v(a,b,c) = b$.

we know $\phi^{-1} : \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \} \to S^2$, $(x,y) \mapsto (x, y, \sqrt{1-x^2-y^2})$

$\frac{\partial}{\partial u}\Big|_p, \frac{\partial}{\partial v}\Big|_p$ basis for $T_p S^2$. $\quad i_{*,p} : T_p S^2 \to T_p \mathbb{R}^3$.

Can write $i_{*,p}\left(\frac{\partial}{\partial u}\Big|_p\right) = \alpha^1 \frac{\partial}{\partial x}\Big|_p + \beta^1 \frac{\partial}{\partial y}\Big|_p + \gamma^1 \frac{\partial}{\partial z}\Big|_p$

we want to find $\alpha^1, \beta^1, \gamma^1$. Write $p = (a, b, c)$

$$i_{*,p}\left(\frac{\partial}{\partial u}\Big|_p\right)(x) = \frac{\partial}{\partial u}\Big|_p (x \circ i) = \frac{\partial (x \circ i \circ \phi^{-1})}{\partial x} \phi(p) = \frac{\partial x}{\partial x}(a,b) = 1.$$

$$i_{*,p}\left(\frac{\partial}{\partial u}\Big|_p\right)(y) = \text{----} = \frac{\partial (y \circ i \circ \phi^{-1})}{\partial x} \phi(p) = \frac{\partial y}{\partial x}(a,b) = 0.$$

$$i_{*,p}\left(\frac{\partial}{\partial u}\Big|_p\right)(z) = \text{---} = \frac{\partial (z \circ i \circ \phi^{-1})}{\partial x} \phi(p) = \frac{\partial \sqrt{1-x^2-y^2}}{\partial x}(a,b)$$

$$= \frac{-2x \cdot \frac{1}{2}}{\sqrt{1-x^2-y^2}}(a,b) = \frac{-a}{\sqrt{1-a^2-b^2}} = \frac{-a}{c}.$$

$$\boxed{\begin{aligned} \alpha^1 &= 1 \\ \beta^1 &= 0 \\ \gamma^1 &= -\frac{a}{c} \end{aligned}}$$

__Ex. 11.5__ : If N is compact, then an injective immersion $f: N \to M$ is an embedding.

- Follows since N is compact and $f(N) \subseteq M$ Hausdorff implies that $f: N \to f(N)$ is a homeomorphism.

__Ex. 12.1__ M smooth manifold. Show TM is Hausdorff.

Let $\pi: TM \to M$, $\pi(p,v) = p$.

Given (p,v) and (q,w) in TM. $(p,v) \neq (q,w)$.

(i) $p \neq q$: $p \in U$, $q \in V$, $U \cap V = \emptyset$.

Then $(p,v) \in TU$, $(q,w) \in TV$. $TU \cap TV = \emptyset$.

(ii) $p = q$. $v \neq w$. Let $\phi: U \to \mathbb{R}^n$ be a chart st. $p=q \in U$.

Then $TU \xrightarrow[\cong]{\tilde{\phi}} \phi(U) \times \mathbb{R}^n$ $(p,v) \mapsto (\phi(p), \phi_{*,p}(v))$

$(p,w) \mapsto (\phi(p), \phi_{*,p}(w))$.

Then use that $\mathbb{R}^n$ is Hausdorff and that $\tilde{\phi}$ is a homeomorphism.