Ex 7.5 G topological group that acts on a topological space S from the right $S \times G \to S$, $(x,g) \mapsto x \cdot g$

Equivalence relation on S: $x \sim y$ if $x = y \cdot g$ for some $g \in G$.

Show that $\pi : S \to S/G$ is open, that is,

let $U \subseteq S$ be open, show $\pi(U) \subseteq S/G$ is open.

$$\pi^{-1}\pi(U) = \bigcup_{g \in G} U \cdot g$$

$\cdot g : S \to S$ homeomorphism with continuous inverse

$\cdot g^{-1} : S \to S$, $x \mapsto x \cdot g \mapsto (x \cdot g) g^{-1} = x(g \cdot g^{-1}) = x \cdot 1 = x$

<u>Ex. 7.6</u> $2\pi\mathbb{Z}$ acts on $\mathbb{R}$ by $x \cdot 2\pi n = x + 2\pi n$, $n \in \mathbb{Z}$

Claim: $\mathbb{R}/2\pi\mathbb{Z}$ is a smooth manifold.

Know $\pi : \mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$ is open

(i) Locally Euclidian:

Let $U_1 = \pi((0, 2\pi))$, $U_2 = \pi(-\pi, \pi)$

$\phi_1 : U_1 \to (0, 2\pi)$, $[x] \mapsto y \in (0, 2\pi)$, where $[x] = [y]$

$\phi_2 : U_2 \to (-\pi, \pi)$, $[x] \mapsto y \in (-\pi, \pi)$, where $[x] = [y]$

homeomorphisms

(ii) Hausdorff: Either do this directly or use thm 7.7:

$R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x = y + 2\pi n$ for some $n \in \mathbb{Z}\}$ is closed

(iii) second countable since $\mathbb{R}$ is second countable and π is open

Check smooth atlas:

$$\begin{tikzcd}
 & U_1 \cap U_2 \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & \\
(0,\pi) \sqcup (\pi, 2\pi) \arrow[rr, "\phi_2 \circ \phi_1^{-1}"] & & (-\pi, 0) \sqcup (0,\pi)
\end{tikzcd}$$

$$(0,\pi) \ni t \longmapsto t$$

$$(\pi, 2\pi) \ni t \longmapsto t - 2\pi$$

smooth.

$$t \longleftarrow\!\shortmid\ t \in (0,\pi)$$

$$t + 2\pi \longleftarrow\!\shortmid\ t \in (-\pi, 0)$$

Ex. 7.7

(a) $U_1 = \{ e^{it} \in S^1 : -\pi < t < \pi \} \xrightarrow{\phi_1} (-\pi, \pi)$, $e^{it} \mapsto t$

$U_2 = \{ e^{it} \in S^1 : 0 < t < 2\pi \} \xrightarrow{\phi_2} (0, 2\pi)$, $e^{it} \mapsto t$

smooth atlas on S^1.

Let $\bar{\phi}_1 : \overset{e^{it}}{U_1} \xrightarrow{\phi_1} (-\pi, \pi) \to \overset{[t]}{\mathbb{R}/2\pi\mathbb{Z}}$

$\bar{\phi}_2 : \overset{e^{it}}{U_2} \xrightarrow{\phi_2} (0, 2\pi) \to \overset{[t]}{\mathbb{R}/2\pi\mathbb{Z}}$

These glue together to define $\phi : S^1 \to \mathbb{R}/2\pi\mathbb{Z}$

show ϕ is smooth:

$$
\begin{tikzcd}
S^1 \arrow[r, "\phi"] & \mathbb{R}/2\pi\mathbb{Z} \\
U_1 \arrow[u, phantom, "\cup" description] \arrow[r] \arrow[d, "\phi_1"'] & \pi(-\pi, \pi) \arrow[u, phantom, "\cup" description] \arrow[d] \\
(-\pi, \pi) \arrow[r, "id"] & (-\pi, \pi)
\end{tikzcd}
$$

similarly with U_2.

(b)
$$\begin{tikzcd} \mathbb{R} \arrow[r, "t \mapsto e^{it}"] \arrow[d] & S^1 \\ \mathbb{R}/2\pi\mathbb{Z} \arrow[ur, "F"'] & \end{tikzcd}$$

gives $F: \mathbb{R}/2\pi\mathbb{Z} \to S^1$, $F([t]) = e^{it}$

Then F is automatically continuous.

Prove F is C^∞:

$$\begin{tikzcd} \mathbb{R}/2\pi\mathbb{Z} \arrow[r] & S^1 \\ \pi(-\pi,\pi) \arrow[u, phantom, "\cup"] \arrow[r] \arrow[d] & U_1 \arrow[u, phantom, "\cup"] \arrow[d] \\ (-\pi,\pi) \arrow[r, "id"] & (-\pi,\pi) \end{tikzcd}$$

(c) Conclusion: $F: \mathbb{R}/2\pi\mathbb{Z} \to S^1$ is a diffeomorphism.

Ex. D

(1) Show $f: S^2 \to \mathbb{R}P^2$ is smooth

Let $U = \{(x,y,z) \in S^2 : z > 0\}$

$\phi: U \to \mathbb{R}^2$, $\phi(x,y,z) = (x,y)$ chart on S^2.

Let $U_2 = \{[a^0, a^1, a^2] : a^2 \neq 0\}$.

$\phi_2: U_2 \to \mathbb{R}^2$, $\phi_2([a^0, a^1, a^2]) = \left(\frac{a^0}{a^2}, \frac{a^1}{a^2}\right)$ chart on $\mathbb{R}P^2$

$$\begin{tikzcd}
S^2 \arrow[r, "f"] & \mathbb{R}P^2 \\
U \arrow[u, phantom, "\cup" rotate=90] \arrow[r] \arrow[d, "\phi"'] & U_2 \arrow[u, phantom, "\cup" rotate=90] \arrow[d, "\phi_2"] \\
\{(x,y) : x^2+y^2<1\} \arrow[r] & \mathbb{R}^2
\end{tikzcd}$$

$(x,y) \mapsto (x,y,\sqrt{1-(x^2+y^2)}) \mapsto [x,y,\sqrt{1-(x^2+y^2)}] \overset{\phi_2}{\mapsto} \left(\frac{x}{\sqrt{1-(x^2+y^2)}}, \frac{y}{\sqrt{1-(x^2+y^2)}}\right)$

smooth.

similarly for other charts on S^2.

(2) To check that f is a local diffeomorphism, one can either use the inverse function theorem, or one can check directly that $f : U \to U_2$ is a diffeomorphism.