Partition of unity

Let M be a smooth manifold, let $U \subseteq M$ be an open set, and let $p \in U$ be a point

Recall A smooth bump function at p supported in U is a smooth function $\rho: M \to \mathbb{R}$ such that

(i) $\rho(M) \subseteq [0,1]$

(ii) $\mathrm{supp}(\rho) \subseteq U$

(iii) There exists a nbh. $p \in V \subseteq U$ st. $\rho|_V = 1$.

Have proved:

Theorem Given $p \in U \subseteq M$, there exists a smooth bump function at p supported in U

Application : Extension of smooth functions

Let $U \subseteq M$ be open. Given a smooth function $f: U \to \mathbb{R}$, it is not always possible to extend f to a smooth function $\tilde{f}: M \to \mathbb{R}$.

Ex $M = \mathbb{R}$, $U = (-\frac{\pi}{2}, \frac{\pi}{2})$, $f = \tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$

<u>Thm</u> Given $U \subseteq M$ and a smooth function $f: U \to \mathbb{R}$. For each $p \in U$, there exists a smooth function $\tilde{f}: M \to \mathbb{R}$ that agrees with f in a neighborhood of p.

<u>Proof</u> Choose a smooth bump function $\rho: M \to \mathbb{R}$ at p supported in U. Then define $\tilde{f}: M \to \mathbb{R}$, $\tilde{f}(q) = \begin{cases} \rho(q) \cdot f(q), & q \in U \\ 0 & \text{if } q \notin U \end{cases}$

Notice : if $q \notin U$, then $\tilde{f}$ vanish in a nbh. of q.

$\square$

Partition of unity

Let $\{U_\alpha : \alpha \in A\}$ be an open covering of a manifold M.

<u>Def</u> A smooth partition of unity subordinate to the covering $\{U_\alpha\}$ is a collection of non-negative smooth functions $\{\rho_\alpha : M \to \mathbb{R}\}_{\alpha \in A}$ such that

(i) $\operatorname{supp}(\rho_\alpha) \subseteq U_\alpha$

(ii) for each $q \in M$, there exists a nbh $q \in V_q$ such that V_q only intersects $\operatorname{supp}(\rho_\alpha)$ for finitely many α.

(iii) $\sum_{\alpha \in A} \rho_\alpha(q) = 1$ for all $q \in M$.

<u>Rmk</u>: If (ii) holds, then the collection $\{\operatorname{supp}(\rho_\alpha)\}_{\alpha \in A}$ is said to be locally finite. This implies that the sum in (iii) is well-defined.

Proposition Let $\{U_\alpha : \alpha \in A\}$ be an open covering of M. Then there exists a smooth partition of unity subordinate to $\{U_\alpha\}$.

Proof We shall only prove this when M is compact. (General argument is in Appendix C).

For each $p \in M$, choose a bump function $\sigma_p : M \to \mathbb{R}$ at p such that $\operatorname{supp}(\sigma_p) \subseteq U_\alpha$ for some α.

Let W_p be a nbh. of p s.t. $\sigma_p(q) = 1$ for $q \in W_p$.

Then $\{W_p : p \in M\}$ open covering of M, so by compactness $M = W_{p_1} \cup W_{p_2} \cup \dots \cup W_{p_k}$

Let $\tau_i : M \to \mathbb{R}$, $\tau_i(q) = \dfrac{\sigma_{p_i}(q)}{\sum_{j=1}^k \sigma_{p_j}(q)}$, $i = 1, .., k$.

These are smooth functions and $\sum_{i=1}^k \tau_i(q) = 1$.

We must reindex the functions:

For each $i = 1, \dots, k$, choose $\alpha(i) \in A$ s.t. $\mathrm{supp}(\tau_i) = \mathrm{supp}(\sigma_{p_i}) \subseteq U_{\alpha(i)}$.

Let $\rho_\alpha : M \to \mathbb{R}$, $\rho_\alpha = \sum_{\{i : \alpha(i) = \alpha\}} \tau_i$ smooth

If $\{i : \alpha(i) = \alpha\} = \emptyset$, then $\rho_\alpha = 0$ by definition

<u>Exercise</u> $\mathrm{supp}(\rho_\alpha) \subseteq \bigcup_{\{i : \alpha(i) = \alpha\}} \mathrm{supp}(\tau_i) \subseteq U_\alpha$.

Finally we observe that

$$\sum_{\alpha \in A} \rho_\alpha(p) = \sum_{\alpha \in A} \sum_{\{i : \alpha(i) = \alpha\}} \tau_i(p) = \sum_{i=1}^{k} \tau_i(p) = 1. \qquad \square$$

__Example__ Suppose $A \subseteq U \subseteq M$, where A is closed and U is open. Then there exists a smooth function $f: M \to \mathbb{R}$, such that $f|A \equiv 1$ and $\mathrm{supp}(f) \subseteq U$.

__Reason__: Let $V = M - A$ and consider the open covering $\{U, V\}$ of M. Choose a corresponding portion of unity $\{\rho_U, \rho_V\}$. Let $f = \rho_U : M \to \mathbb{R}$.

Then $\rho_U + \rho_V \equiv 1$. and $\rho_V | A = 0$ implies $f|A \equiv 1$.