Program for this week: Multilinear algebra

Dual vector spaces

Let V be a finite dimensional real vector space.
The dual vector space is the vector space

$$V^\vee = \mathrm{Hom}(V, \mathbb{R}) = \{ \text{Linear functions } V \to \mathbb{R} \}$$

Elements $f: V \to \mathbb{R}$ in $V^\vee$ are called covectors.
Given $f, g: V \to \mathbb{R}$, then $(f+g)(v) = f(v) + g(v)$
c scalar, $cf: V \to \mathbb{R}$, $(cf)(v) = c \cdot f(v)$

Suppose V has a basis $e_1, .., e_n$.
Define covectors $\alpha^1, .., \alpha^n : V \to \mathbb{R}$, by $\alpha^i(v) = v^i$ if
$v = v^1 e_1 + .. + v^n e_n$
In particular $\alpha^i(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

proposition The covectors $\alpha^1, \dots \alpha^n$ form a basis for $V^\vee$.

proof Linearly independent:

Suppose $c_1 \alpha^1 + \dots + c_1 \alpha^n = 0$, for $c_1, \dots c_n \in \mathbb{R}$.

Evaluate at e_i to get

$$0 = (c_1 \alpha^1 + \dots + c_n \alpha^n)(e_i) = c_i \alpha^i(e_i) = c_i$$

$\alpha^1, \dots, \alpha^n$ generate $V^\vee$

Given a covector $f: V \to \mathbb{R}$, let $c_i = f(e_i)$

Claim $f = c_1 \alpha^1 + \dots + c_n \alpha^n$.

For each $v = \sum_{i=1}^n v^i e_i$ in V we have

$$f(v) = \sum_{i=1}^n v^i f(e_i) = \sum_{i=1}^n v^i c_i$$

$$(c_1 \alpha^1 + \dots + c_n \alpha^n)(v) = \sum_{i=1}^n c_i \alpha^i(v) = \sum_{i=1}^n c_i v^i$$

$\square$

Consequence: $V^\vee$ has the same dimension as V

Rmk For $V = \mathbb{R}^n$ this is well known from linear algebra: Linear functions $\mathbb{R}^n \to \mathbb{R}$ can be identified with $1 \times n$ matrices and α^i (with respect to the standard basis) corresponds to $[0, \dots 0, 1, 0 \dots 0]$.

Let V be a vector space. Write $V^k = \underbrace{V \times \dots \times V}_{k}$

A function $f : V^k \to \mathbb{R}$ is k-linear if it is linear in each of the k-variables:

- $f(v_1, \dots, v_{i-1}, X+Y, v_{i+1} \dots v_k) = f(v_1, \dots, v_{i-1}, X, v_{i+1} \dots v_k) + f(v_1, \dots, v_{i-1}, Y, v_{i+1}, \dots v_k)$

- $f(v_1 \dots, v_{i-1}, c \cdot X, v_{i+1}, \dots, v_k) = c f(v_1 \dots, v_{i-1}, X, v_{i+1} \dots, v_k)$.

Ex. The scalar product on $\mathbb{R}^n$ is bilinear = (2-linear)

- $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $x \cdot y = \sum_{i=1}^{n} x^i y^i$

Ex. The determinant $\det : \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{n} \to \mathbb{R}$

is n-linear

Write $L_k(V)$ for the vector space of k-linear functions.

$f : V^k \to \mathbb{R}$, $g : V^k \to \mathbb{R}$, $(f+g)(v_1 \ldots v_k) = f(v_1 \ldots v_k) + g(v_1 \ldots v_k)$.

$(cf)(v_1 \ldots v_k) = c\, f(v_1 \ldots v_k)$.

(k-linear functions $f : V^k \to \mathbb{R}$ are also called k-tensors)

Notice: For $k=1$, $L_1(V) = V^\vee$.

Exercise: If V has dimension n, then $L_k(V)$ has dimension n^k.

Review of permutations

A permutation of the set $\{1, .. n\}$ is a bijective function
$\sigma : \{1, .. n\} \to \{1, .. n\}$

Let S_n be the set of all permutations of $\{1, .. n\}$.

Permutations can be composed $\sigma \circ \tau$.

This makes S_n a group with unit element the identity function.

A permutation σ can be described by a diagram

$$\begin{bmatrix} 1 & 2 & , \dots , & n \\ \sigma(1) & \sigma(2), & & , \sigma(n) \end{bmatrix}$$

Let σ be a permutation. An <u>inversion</u> in σ is a pair $(\sigma(i), \sigma(j))$, where $i < j$ but $\sigma(i) > \sigma(j)$.

<u>Example</u> $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 3 & 2 \end{bmatrix}$

Inversions in σ : $(4,1), (4,3), (4,2), (5,1), (5,3), (5,2), (3,2)$

There are 7 inversions in σ.

<u>Def</u> A permutation σ is <u>even</u> if it has an even number of inversions, it is <u>odd</u> if it has an odd number of inversions.

The sign of σ is defined by

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

A permutation σ is a <u>transposition</u> if there exist $a \neq b$ in $\{1, \dots n\}$ such that $\sigma(a) = b$, $\sigma(b) = a$ and $\sigma(i) = i$ for $i \notin \{a, b\}$. Write $\sigma = (a, b)$.

<u>Fact from algebra</u> Let $\sigma \in S_n$

- σ can be written as a product of an even number of transpositions iff $\mathrm{sgn}(\sigma) = 1$.

- σ can be written as a product of an odd number of transpositions iff $\mathrm{sgn}(\sigma) = -1$.

<u>Consequence</u>: $\mathrm{sgn}(\sigma \circ \tau) = \mathrm{sgn}(\sigma) \cdot \mathrm{sgn}(\tau)$.

Def Let $f: V^k \to \mathbb{R}$ be a k-linear function.

- f is symmetric if $f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = f(v_1, \dots, v_k)$, all $\sigma \in S_k$.
- f is alternating if $f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma) f(v_1 \dots v_k)$, all $\sigma \in S_k$.

$S_k(V)$ is the vector space of all symmetric k-linear functions - subspace of $L_k(V)$.

$A_k(V)$ is the vector space of all alternating k-linear functions - subspace of $L_k(V)$

By definition: $A_1(V) = S_1(V) = L_1(V) = V^\vee$.

We define $A_0(V) = S_0(V) = \mathbb{R}$.

Ex. • The scaler product $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is symmetric.

• $\det : \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{n} \to \mathbb{R}$ is an alternating n-linear function.

• Given covectors $f : V \to \mathbb{R}$, $g : V \to \mathbb{R}$, let $f \wedge g$ be the 2-linear function defined by

$$f \wedge g : V \times V \to \mathbb{R}, \quad (f \wedge g)(v,w) = f(v) \cdot g(w) - f(w) \cdot g(v).$$

This is alternating: $(f \wedge g)(v,w) = -(f \wedge g)(w,v)$.

A permutation $\sigma \in S_k$ defines a linear function

$$\sigma : L_k(V) \to L_k(V), \quad (\sigma f)(v_1, \ldots, v_k) = f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

f is symmetric iff $\sigma f = f$ for all $\sigma \in S_k$

f is alternating iff $\sigma f = \mathrm{sgn}(\sigma) \cdot f$ for all $\sigma \in S_k$.

Lemma: $(\tau\sigma)f = \tau(\sigma f)$ for $\tau, \sigma \in S_k$ and $f \in L_k(V)$.

proof $\tau(\sigma f)(v_1, \dots v_k) = (\sigma f)(v_{\tau(1)}, \dots, v_{\tau(k)})$ Let $w_i = v_{\tau(i)}$

$= (\sigma f)(w_1, \dots, w_k) = f(w_{\sigma(1)}, \dots, w_{\sigma(k)}) = f(v_{\tau\sigma(1)}, \dots, v_{\tau\sigma(k)})$

$= (\tau\sigma)\cdot f(v_1, \dots v_k)$ $\square$

Hence $(\sigma, f) \mapsto \sigma \cdot f$ defines an <u>action</u> of S_k on $L_k(V)$.
[The notion of an "action" is described in the book].

For $f \in L_k(V)$ define

$$S(f) = \sum_{\sigma \in S_k} \sigma f \quad \text{and} \quad A(f) = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\, \sigma f$$

Proposition:

(i) $S(f)$ is symmetric
(ii) $A(f)$ is alternating.

Consequence

symmetrization operator $S : L_k(V) \to S_k(V)$

Alternating operator $A : L_k(V) \to A_k(V)$.

proof of Proposition Check for A

Must show $\tau A(f) = \mathrm{sgn}(\tau) A(f)$ for all $\tau \in S_k$.

$$\tau A(f) = \tau \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, \sigma f = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, \tau(\sigma f)$$

$$= \mathrm{sgn}(\tau) \sum_{\sigma \in S_k} \mathrm{sgn}(\tau\sigma)\, (\tau\sigma) f$$

$$\overset{(*)}{=} \mathrm{sgn}(\tau) \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, \sigma f = \mathrm{sgn}(\tau) A(f).$$

Reason for $(*)$: $\{\tau\sigma : \sigma \in S_k\} = \{\sigma : \sigma \in S_k\}$ since $\sigma = \tau(\tau^{-1}\sigma)$. $\square$

Exercise: Check for S.

Remark The composition $A_k(V) \to L_k(V) \xrightarrow{A} A_k(V)$ is multiplication by $k!$: If $f \in A_k(V)$,

$$A(f) = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\, \sigma f = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \operatorname{sgn}(\sigma)\, f$$

$$= \sum_{\sigma \in S_k} f = k!\, f.$$

The wedge product

First define a tensor product of multilinear functions:

$$\otimes : L_k(V) \times L_\ell(V) \to L_{k+\ell}(V), \quad (f, g) \mapsto f \otimes g$$

$$(f \otimes g)(v_1, \dots v_k, v_{k+1}, \dots v_{k+\ell}) = f(v_1, \dots v_k)\, g(v_{k+1}, \dots, v_{k+\ell})$$

For $k=0$ or $\ell=0$, $c \in L_0(V) = \mathbb{R}$, $c \otimes f = f \otimes c = cf$.

Notice that $\otimes$ is bilinear and associative:

$$(f \otimes g) \otimes h = f \otimes (h \otimes g).$$

<u>Definition</u> The wedge product of alternating multilinear functions is defined by

$$\wedge : A_k(V) \times A_\ell(V) \to A_{k+\ell}(V)$$

$$f \wedge g = \frac{1}{k!\,\ell!} A(f \otimes g)$$