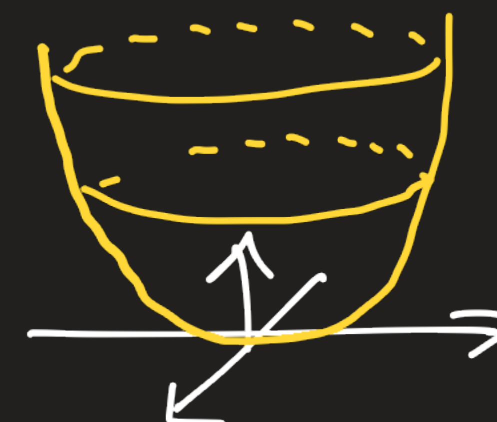
Examples of manifolds

$S^1 =$  $\subseteq \mathbb{R}^2$ circle


spiral

1-dimensional manifold = curve

$S^2 =$  sphere



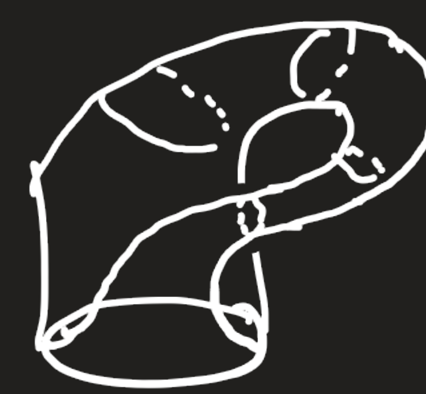
2-dimensional manifolds = surfaces.

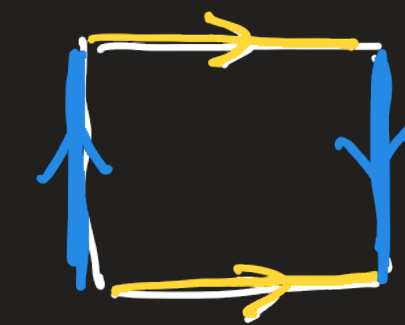
$S^n = \{ x \in \mathbb{R}^{n+1} : \|x\| = 1 \}$ n-sphere n-dim manifold.

$\mathbb{R}^n$ is the prototype of an n-dim. manifold.
locally an n-dim. manifold "looks like" as an open subset of $\mathbb{R}^n$.

<u>Rmk</u> Not all manifolds are subspaces of $\mathbb{R}^n$ in a natural way.

<u>Ex</u> Klein bottle





2-dim. manifold.

<u>Program for first weeks</u>

- Review differentiation in $\mathbb{R}^n$
- Give coordinate free description of the tangent space $T_p\mathbb{R}$ for $p \in \mathbb{R}^n$. - Shall use this to define the tangent space at a point on a manifold.
- Introduce the exterior algebra of a vector space - Use this to define differential forms (eg. $dx + dy + dz$)

<u>Homework</u>: Read up on point set topology in App A.

Differentiation in $\mathbb{R}^n$

$p \in \mathbb{R}^n$, $\varepsilon > 0$

$B(p, \varepsilon) = \{ x \in \mathbb{R}^n \mid \|x - p\| < \varepsilon \}$ open ball with centre p.

A subset $U \subseteq \mathbb{R}^n$ is open if for every $p \in U$, there exists $\varepsilon > 0$ such that $B(p, \varepsilon) \subseteq U$.

A neighbourhood of p is an open set U s.t. $p \in U$.

Notation: Given $x \in \mathbb{R}^n$, write $x = (x^1, x^2, \ldots, x^n)$, coordinates of x.

Definition $U \subseteq \mathbb{R}^n$ open, $f : U \to \mathbb{R}$ function

- f is C^0 if f is continuous.
- f is C^1 if $\frac{\partial f}{\partial x^i}(p)$ exists for all $p \in U$ and $\frac{\partial f}{\partial x^i} : U \to \mathbb{R}$ is continuous for $i = 1, \ldots, n$
- Inductively: f is C^k if $\frac{\partial f}{\partial x^i}(p)$ exists for all $p \in U$ and $\frac{\partial f}{\partial x^i} : U \to \mathbb{R}$ C^{k-1} for $i = 1, \ldots, n$.

- f is C^∞ on U if f is C^k for all $k \geq 0$. This means: All iterated partial derivatives $\frac{\partial^k f}{\partial x^{i_1} \ldots \partial x^{i_k}}$ exist and are continuous on U.

Terminology: f smooth $\iff$ f is C^∞

The tangent space $T_p\mathbb{R}^n$, $p \in \mathbb{R}^n$

$T_p\mathbb{R}^n$ can be identified with $\mathbb{R}^n$

will give an alternative coordinate free description in terms of *point derivations*

Let $U \subseteq \mathbb{R}^n$ be an open set containing p.

$C^\infty(U)$ is the real vector space of C^∞ functions $f: U \to \mathbb{R}$.

$C^\infty(U)$ has a multiplication $(f \cdot g)(x) = f(x) \cdot g(x)$.

The constant function 1 is the multiplicative unit.

This makes $C^\infty(U)$ an <u>$\mathbb{R}$-algebra</u> (see book).

<u>Definition</u> A point derivation at $p \in U \subseteq \mathbb{R}^n$ is an $\mathbb{R}$-linear function $D: C^\infty(U) \longrightarrow \mathbb{R}$ such that

$$D(f \cdot g) = D(f)\, g(p) + f(p)\, D(g) \quad , \text{for all } f, g \in C^\infty(U).$$

<u>Recall</u> Given $v \in \mathbb{R}^n$. The directional derivative of P wrt. v is the point derivation $D_v : C^\infty(U) \to \mathbb{R}$,

$$D_v(f) = \lim_{t \to 0} \frac{f(P+tv) - f(P)}{t}.$$

(i.e., $D_v(f)$ is the derivative of $t \mapsto f(P+tv)$ at $t=0$.)

- $D_v(f+g) = D_v(f) + D_v(g)$, $D_v(rf) = rD_v(f)$, $r \in \mathbb{R}$.
- $D_v(f \cdot g) = D_v(f) \cdot g(P) + f(P)\, D_v(g)$

Check: $\frac{d}{dt}\Big|_{t=0} \left(t \mapsto f(P+tv) \cdot g(P+tv)\right)$

$$= \frac{d}{dt}\Big|_{t=0} \left(t \mapsto f(P+tv)\right) \cdot g(P) + f(P) \frac{d}{dt}\Big|_{t=0} \left(t \mapsto g(P+tv)\right)$$

<u>Ex</u> $v = e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i$, $D_{e_i}(f) = \frac{\partial f}{\partial x^i}(P)$, write $\frac{\partial}{\partial x^i}\Big|_P := D_{e_i}$

For $v = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$, $D_v(f) = \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}(P)$ (chain rule)

There is a function $T_p(\mathbb{R}^n) \cong \mathbb{R}^n \longrightarrow \left\{ \begin{array}{l} \text{Point derivations} \\ \text{at } p,\ C^\infty(U) \to \mathbb{R} \end{array} \right\}$

$$v \longmapsto D_v$$

Will modify this to become independent of U.

Function germs

Consider the collection of pairs (f, U), where U is a nbh. of p and $f : U \to \mathbb{R}$ is C^∞.

Define an equivalence relation: $(f, U) \sim (g, V)$ if there exists a nbh. $W \subseteq U \cap V$ of p st. $f|W = g|W$.

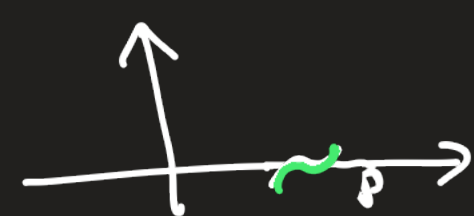
Conditions for equivalence relation:

- $(f, U) \sim (f, U)$ reflexive
- $(f, U) \sim (g, V) \Rightarrow (g, V) \sim (f, U)$ symmetry
- $(f, U) \sim (g, V)$ and $(g, V) \sim (h, W) \Rightarrow (f, U) \sim (h, W)$ transitivity.

The equivalence relation defines a partition of the pairs (f,U) into equivalence classes: (f,U) and (g,V) are in the same class iff $(f,U) \sim (g,V)$.

An equivalence class is called a **function germ at p**.

Ex



For now write $[f,U]$ for the function germ represented by (f,U)

The set of equivalence classes (= function germs) is denoted $C_p^\infty(\mathbb{R}^n)$ (or C_p^∞).

Define addition and scalar multiplication in $C_p^\infty(\mathbb{R}^n)$:

$$[f,U] + [g,V] = [f+g, U \cap V], \quad r[f,U] = [rf, U], \quad r \in \mathbb{R}$$

Exercise check this is well-defined and makes $C_p^\infty(\mathbb{R}^n)$ a real vector space. $0 = [0, \mathbb{R}^n]$

There is also an associative and commutative multiplication on $C^\infty_p(\mathbb{R}^n)$: $[f,U]\cdot[g,V] = [f\cdot g, U\cap V]$

This makes $C^\infty_p(\mathbb{R}^n)$ an $\mathbb{R}$-algebra. (see book).

<u>Def</u> $\mathcal{D}_p(\mathbb{R}^n)$ is the real vector space of point derivations

$D: C^\infty_p(\mathbb{R}^n) \to \mathbb{R}$. (i.e., $D(f\cdot g) = D(f)\, g(p) + f(p)\, D(g)$)

• $(D+D')(f) = D(f) + D'(f)$, $(rD)(f) = r\, D(f)$

<u>Exercise</u>: Check vector space axiom

There is a linear function $\phi: T_p\mathbb{R}^n \cong \mathbb{R}^n \longrightarrow \mathcal{D}_p(\mathbb{R}^n)$
$v \longmapsto \phi(v) = D_v$.

(since $D_v(f)$ only depends on the function germ of f at p)

For $v = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$, then $D_v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\Big|_p$ (chain rule)

This shows ϕ is linear in v.

<u>Theorem</u> $\phi : T_p(\mathbb{R}^n) \cong \mathbb{R}^n \to \mathcal{D}_p(\mathbb{R}^n)$ is an isomorphism.

(Proof Wednesday).