Orientations

Orientations of vector spaces

Let V be an n-dim vector space

Consider all the ordered bases $(v_1, \dots, v_n)$ for V

Given ordered bases $(v_1 \dots v_n)$ and $(w_1 \dots w_n)$, we can write

$$v_j = \sum_{i=1}^{n} a_{ij} w_i \quad \text{for } j = 1, \dots, n,$$

where $A = (a_{ij}) \in GL_n(\mathbb{R})$ is the change of basis matrix.

__Rem__ Given an ordered bases $(v_1, \dots, v_n)$, we get an iso $\phi_{(v_1 \dots v_n)} : V \longrightarrow \mathbb{R}^n$, $v = \sum_{i=1}^{n} r_i v_i \mapsto (r_1, \dots, r_n)$. The change of basis matrix A makes the diagram

$$\begin{tikzcd} & V \arrow[dl, "\phi_{(v_1,\dots,v_n)}"'] \arrow[dr, "\phi_{(w_1,\dots,w_n)}"] & \\ \mathbb{R}^n \arrow[rr, "A"] & & \mathbb{R}^n \end{tikzcd}$$

commutes

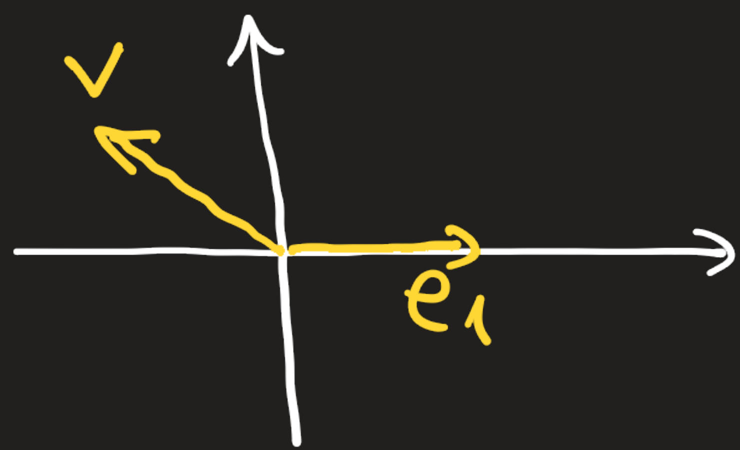
Define an equivalence relation on ordered bases for V:

$(v_1, \dots v_n) \sim (w_1, \dots w_n)$ if the change of basis matrix A has $\det(A) > 0$.

<u>Def.</u> An orientation of V is an equivalence class of ordered bases.

<u>Ex</u> $V = \mathbb{R}^2$. The standard basis $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ gives the counterclockwise orientation $[(e_1, e_2)]$ of $\mathbb{R}^2$.

Given a vector $v = \begin{bmatrix} a \\ b \end{bmatrix}$, $b \neq 0$, then $(e_1, e_2) \sim (e_1, v)$ iff $b > 0$. $\begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}$



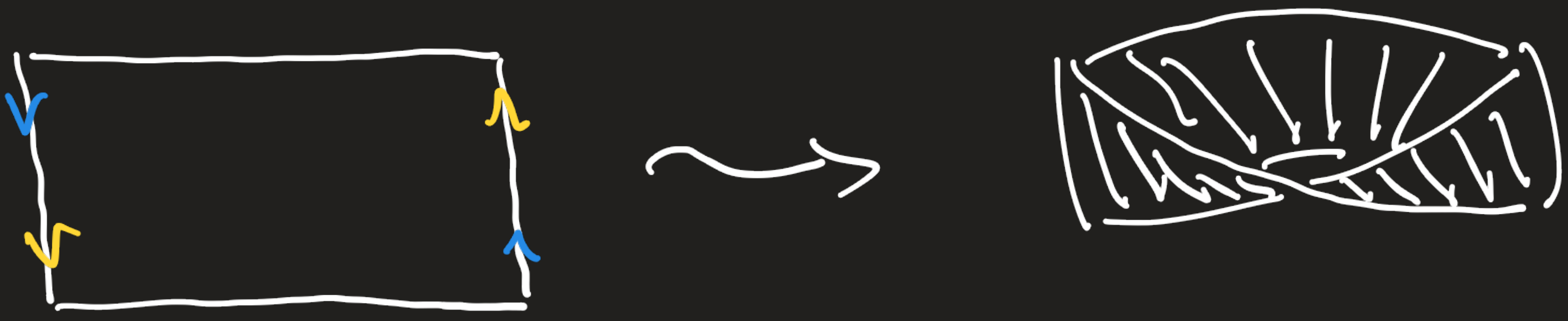
Orientation of manifolds

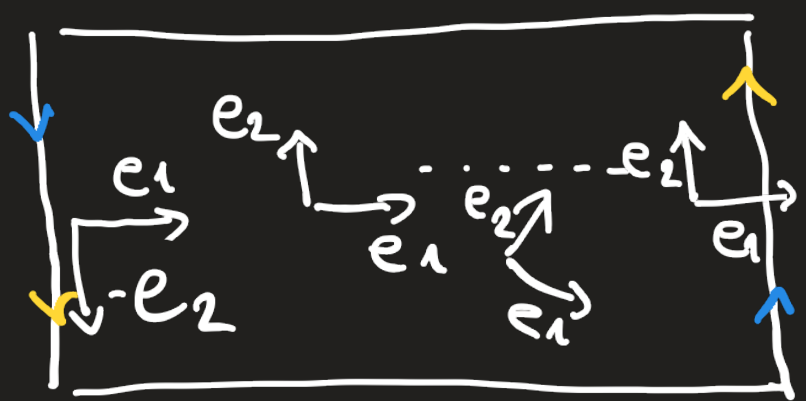
Let M be an n-dim. manifold.

- A pointwise orientation of M is a choice of orientation of each of the tangent spaces T_pM, $p \in M$
- A pointwise orientation is continuous if for each $p \in M$ there exists a nbh. U and a continuous local frame $(X_1, \dots X_n)$ on U st. $(X_1(q), \dots, X_n(q))$ represents the chosen orientation on T_qM for all $q \in U$.
- A manifold is orientable if there exists a continuous pointwise orientation.

Ex. $M = \mathbb{R}^n$, the canonical basis $\left(\frac{\partial}{\partial x^1}\Big|_p, \dots \frac{\partial}{\partial x^n}\Big|_p\right)$ for $T_p\mathbb{R}^n$ specifies an orientation of $\mathbb{R}^n$.

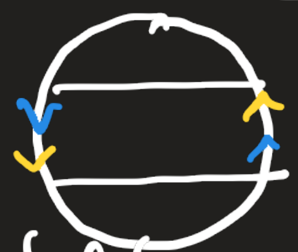
Ex The Moebius band is not orientable:





Ex $\mathbb{RP}^2$ is not orientable: $\mathbb{RP}^2 = D^2/\sim$ $\quad p \sim -p$ if $p \in S^1 = \partial D^2$.

There is an embedding of the Moebius band in $\mathbb{RP}^2$



If $\mathbb{RP}^2$ were orientable, then the Möbius band would also be orientable

Orientations via n-covectors (= alternating n-tensors)

Let V be an n-dim. vector space. Recall that

$$A_n(V) = \{ f : \underbrace{V \times \dots \times V}_{n} \to \mathbb{R} , \text{ n covectors} \}$$

has $\dim A_n(V) = 1$. Let $\beta \in A_n(V)$.

Lemma Given bases $(v_1, \dots, v_n)$ and $(w_1, \dots, w_n)$, write $v_j = \sum_{i=1}^n a_{ij} w_i$. Then $\beta(v_1, \dots v_n) = \det(a_{ij}) \beta(w_1, \dots, w_n)$.

Proof

$$\beta(v_1, \dots v_n) = \beta\left(\sum_{i=1}^n a_{i1} w_i, \dots, \sum_{i=1}^n a_{in} w_i\right) = \sum_{i_1, \dots, i_n} a_{i_1 1} \cdots a_{i_n n} \beta(w_{i_1}, \dots, w_{i_n})$$

$$= \sum_{\sigma \in S_n} a_{\sigma(1)1} \cdots a_{\sigma(n)n} \beta(w_{\sigma(1)}, \dots, w_{\sigma(n)})$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n} \beta(w_1, \dots w_n) = \det(a_{ij}) \beta(w_1, \dots, w_n)$$

$\square$

<u>Consequence</u> Suppose $\beta \in A_n(V)$. $\beta \neq 0$, Then

$$(v_1, \ldots v_n) \sim (w_1, \ldots, w_n) \Leftrightarrow \begin{cases} \beta(v_1 \ldots v_n) \text{ and } \beta(w_1 \ldots w_n) \text{ have} \\ \text{the same sign} \end{cases}$$

Hence $\beta \neq 0$ determines an orientation of V.

<u>Lemma</u> Let M be an n-dim manifold. The following conditions on a pointwise orientation are equivalent:

(i) The pointwise orientation is continuous

(ii) For each $p \in M$, there exists a chart $\phi = (x^1 \ldots x^n) : U \to \mathbb{R}^n$, $p \in U$, such that $(dx^1 \wedge \ldots \wedge dx^n)_q (v_1, \ldots, v_n) > 0$ for each $q \in U$ and each positively oriented basis $(v_1 \ldots v_n)$ for $T_q M$.

(ii) $\Rightarrow$ (i) Given $p \in M$, choose $\phi = (x^1, \dots, x^n)$ as in (ii).
Then $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$ is a smooth local frame that represents the orientation of $T_q M$ for all $q \in U$ - since

$$(dx^1 \wedge \dots \wedge x^n)_q \left(\frac{\partial}{\partial x^1}\Big|_q, \dots, \frac{\partial}{\partial x^n}\Big|_q \right) = 1 > 0 .$$

(i) $\Rightarrow$ (ii) Suppose the pointwise orientation is continuous.
Given $p \in M$, choose a nbh. V and a continuous local frame $(X_1, \dots, X_n)$ on V st. $(X_1(q), \dots, X_n(q))$ is a positively oriented basis for $T_q M$ for all $q \in V$.
Choose a chart $\phi = (x^1, \dots, x^n) : U \to \mathbb{R}^n$ st. $p \in U \subseteq V$ and st. U is connected.
Write $X_j = \sum_{i=1}^{n} a_{ij} \frac{\partial}{\partial x^i}$, where $a_i : U \to \mathbb{R}$ are continuous.

Then

$$(dx^1 \wedge \dots \wedge dx^n)|_q (X_1(q), \dots, X_n(q)) = \det(a_{ij})(dx^1 \wedge \dots \wedge dx^n)_q \left(\frac{\partial}{\partial x^1}\Big|_q, \dots, \frac{\partial}{\partial x^n}\Big|_q\right)$$

$$= \det(a_{ij}) \neq 0$$

The function $\det(a_{ij}) : U \to \mathbb{R}$ is continuous, so is either positiv or negativ.

If $\det(a_{ij}) > 0$, then

$(dx^1 \wedge \dots \wedge dx^n)_q (v^1, \dots, v^n) > 0$ if $(v_1, \dots, v_n)$ is pos. oriented.

If $\det(a_{ij}) < 0$, then use the chart $(-x^1, x^2, \dots, x^n) : U \to \mathbb{R}^n$ and proceed in the same way.

$\square$

Def Let M be an n-dim. manifold. An n-form $\omega \in \Omega^n(M)$ is said to be nowhere-vanishing if $\omega_p \neq 0$ in $A_n(T_pM)$ for all $p \in M$.

Theorem An n-dim. manifold M is orientable iff there exists a nowhere-vanishing smooth n-form ω on M.

Proof Suppose first that we have chosen an orientation on M. Using the previous lemma, choose an atlas $\{(U_\alpha, x^1_\alpha .. x^n_\alpha)\}$ st. for α, $(dx^1_\alpha \wedge .. \wedge dx^n_\alpha)_p (v_1 .. v_n) > 0$ if $(v_1 .. v_n)$ pos. orient. basis for T_pM, $p \in U_\alpha$.

Choose a partition of unity $\{\rho_\alpha : M \to [0,1]\}$ subordinate to the covering $\{U_\alpha\}$ of M. Let $\omega = \sum_\alpha \rho_\alpha \, dx^1_\alpha \wedge \ldots \wedge dx^n_\alpha \in \Omega^n(M)$.

Claim. ω is non-vanishing: Given $p \in M$ and a pos. oriented basis $(v_1 .. v_n)$ for T_pM, $\omega_p(v_1 .. v_n) = \sum \rho_\alpha(p) (dx^1_\alpha \wedge .. \wedge dx^n_\alpha)_p (v_1 .. v_n) > 0$.

Next suppose $\omega \in \Omega^n(M)$ is nowhere-vanishing. Then ω specifies a pointwise orientation of T_pM for all p:

$(v_1, .. v_n)$ pos. oriented basis for $T_pM \Leftrightarrow \omega_p(v_1, .. v_n) > 0$.

<u>Claim</u>. This is a continuous pointwise orientation:

Given $p \in M$, choose a chart $(U, x^1 .. x^n)$ s.t. $p \in U$ and U is connected. Then

$dx^1 \wedge .. \wedge dx^n = f \cdot \omega|_U$, $f: U \to \mathbb{R}$ smooth nowhere vanishing.

If $f > 0$ ok

If $f < 0$, use the $(U, -x^1, x^2, .. x^n)$ instead.

It then follows from the lemma the pointwise orientation determined by ω is continuous. $\square$

<u>Rem</u> If M has the orientation specified by a nowhere-vanishing n-form ω, then ω is called an <u>orientation form</u> on M.

Example $M = S^2 = \{(x,y,z) : x^2+y^2+z^2=1\}$. We shall define a nowhere-vanishing 2-form $\omega \in \Omega^2(S^2)$

Let $\imath : S^2 \to \mathbb{R}^3$ be the inclusion.

$U_x = \{(x,y,z) : x \neq 0\}$, $U_y = \{(x,y,z) : y \neq 0\}$, $U_z = \{(x,y,z) : z \neq 0\}$.

Define $\omega = \begin{cases} \imath^*\left(\frac{1}{x}\, dy \wedge dz\right) & \text{on } U_x \\ \imath^*\left(-\frac{1}{y}\, dx \wedge dz\right) & \text{on } U_y \\ \imath^*\left(\frac{1}{z}\, dx \wedge dy\right) & \text{on } U_z. \end{cases}$ Notice that $S^2 = U_x \cup U_y \cup U_z$

Must check that ω is well-defined on $U_x \cap U_y$, $U_x \cap U_z$, $U_y \cap U_z$.

Let $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = \frac{1}{2}(x^2+y^2+z^2-1)$, then $f^{-1}(0) = S^2$.

$df = x\,dx + y\,dy + z\,dz$ in $\Omega^1(\mathbb{R}^3)$

$\Rightarrow \quad 0 = d(f \circ \imath) = \imath^*(df) = \imath^*(x\,dx + y\,dy + z\,dz)$.

Notice $df \wedge dz = (x\,dx + y\,dy + z\,dz) \wedge dz = x\,dx \wedge dz + y\,dy \wedge dz$.

Hence $0 = \imath^*(df) \wedge \imath^* dz = \imath^*(df \wedge dz) = \imath^*(x\,dx \wedge dz) + \imath^*(y\,dy \wedge dz)$

This implies $\imath^*\left(\frac{1}{x}\,dy \wedge dz\right) = \imath^*\left(-\frac{1}{y}\,dx \wedge dz\right)$ on $U_x \cap U_y$.

Similar arguments apply for $U_x \cap U_y$ and $U_y \cap U_z$.

It remains to check that ω is nowhere-vanishing.

For $p = (a,b,c) \in S^2$, we know $\imath_{*,p}(T_p S^2) \subseteq T_p \mathbb{R}^3$ is

$$\imath_{*,p}(T_p S^2) = \left\{ r \frac{\partial}{\partial x}\Big|_p + s \frac{\partial}{\partial y}\Big|_p + t \frac{\partial}{\partial z}\Big|_p : ar + bs + c\cdot z = 0 \right\}$$

(This is Ex. 11.17)

If $a \neq 0$, then $v_1 = -\frac{b}{a} \frac{\partial}{\partial x}\Big|_p + \frac{\partial}{\partial y}\Big|_p$, $v_2 = -\frac{c}{a} \frac{\partial}{\partial x}\Big|_p + \frac{\partial}{\partial z}\Big|_p$ is a basis for $T_p S^2$. Since

$(dy \wedge dz)(v_1, v_2) = dy \wedge dz \left(\frac{\partial}{\partial y}\Big|_p, \frac{\partial}{\partial z}\Big|_p\right) = 1$, it follows that ω is nowhere-vanishing on U_x. Similarly for U_y, U_z.