Differential k-forms

Let M be a smooth manifold, $p \in M$.

- The vector space of k-tensors on T_pM is denoted $L_k(T_pM)$:

$$L_k(T_pM) = \{ f : \underbrace{T_pM \times \dots \times T_pM}_{k} \to \mathbb{R},\ k\text{-linear} \}$$

- $A_k(T_pM)$ is the subspace of alternating k-tensors:

$$f \in A_k(T_pM) \text{ if } f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma) f(v_1, \dots, v_k),\ \sigma \in S_k.$$

<u>Def</u> A (differential) k-form on M is a family of alternating k-tensors parametrized by the points of M:

$$\omega = \{ \omega_p \in A_k(T_pM) : p \in M \}$$

Lokal expression of k-forms

Let $\phi = (x^1, .., x^n) : U \to \mathbb{R}^n$ be a chart on M.

- $\frac{\partial}{\partial x^1}|_p, .., \frac{\partial}{\partial x^n}|_p$ is a basis of T_pM.
- $dx^1_p, .., dx^n_p$ is the dual basis of T^*_pM.
- $\{dx^{i_1}_p \wedge .. \wedge dx^{i_k}_p : 1 \leq i_1 < .. < i_k \leq n\}$ basis of $A_k(T_pM)$.

Hence $\dim A_k(T_pM) = \binom{n}{k}$

Write $dx^I_p = dx^{i_1}_p \wedge .. \wedge dx^{i_k}_p$, where $I = (1 \leq i_1 < .. < i_k \leq n)$

If $\omega = \{\omega_p \in A_p(T_pM) : p \in M\}$ is a k-form, then for each $p \in U$: $\omega_p = \sum_I a_I(p)\, dx^I_p$, $a_I(p) \in \mathbb{R}$,

where the $a_I(p)$ are uniquely determined

Hence the restriction $\omega|_U$ can be written uniquely

$$\omega|_U = \sum_I a_I \, dx^I \text{, where } a_I : U \to \mathbb{R} \text{ are functions.}$$

Notice $a_I(p) = \omega_p\left(\frac{\partial}{\partial x^{i_1}}\Big|_p, \dots, \frac{\partial}{\partial x^{i_k}}\Big|_p\right)$, if $I = (i_1 < \dots < i_k)$.

Goal: Define a smooth vector bundle $\pi : \Lambda_k(M) \to M$ such that k-forms can be identified with sections.

The vector bundle of alternating k-tensors

Def $A_k(TM) = \bigcup_{p \in M} A_k(T_pM)$

Given a chart $\phi = (x^1, \ldots, x^n) : U \to \mathbb{R}^n$, let $A_k(TU) = \bigcup_{p \in U} A_k(T_pM)$

Bijective correspondence

$$A_k(TU) \xrightarrow{\tilde{\phi}} \phi(U) \times \mathbb{R}^{\binom{n}{k}}$$

$$\omega_p \longmapsto \left(\phi(p), \left\{\omega_p\left(\frac{\partial}{\partial x^{i_1}}\Big|_p, \ldots, \frac{\partial}{\partial x^{i_k}}\Big|_p\right)\right\}_{(i_1 < \ldots < i_k)}\right)$$

$$\sum c_I \, dx^I_{\phi^{-1}(a)} \longleftarrow \left(a, \{c_I\}_{I = (i_1 < \ldots < i_k)}\right)$$

Use $\tilde{\phi}$ to define a topology on $A_k(TU)$

Using all charts this give a topology on $A_k(TM)$

(The details are as for TM and T^*M).

Let $\phi = (x^1, \ldots x^n) : U \to \mathbb{R}^n$ and $\psi = (y^1, \ldots y^n) : V \to \mathbb{R}^n$ be charts on M.

Given $I = (i_1 < \ldots < i_k)$, $J = (j_1 < \ldots < j_k)$, write

$$\frac{\partial(x^{i_1}, \ldots, x^{i_k})}{\partial(y^{j_1}, \ldots, y^{j_k})}(p) = \begin{vmatrix} \frac{\partial x^{i_1}}{\partial y^{j_1}}(p), & \cdots & \frac{\partial x^{i_1}}{\partial y^{j_k}}(p) \\ \vdots & & \\ \frac{\partial x^{i_k}}{\partial y^{j_1}}(p), & \cdots, & \frac{\partial x^{i_k}}{\partial y^{j_k}}(p) \end{vmatrix}$$

Notice that $A_k(TU) \cap A_k(TV) = A_k(T(U \cap V))$

<u>Lemma</u> There is a commutative diagram

$$\begin{tikzcd} & A_k(T(U\cap V)) \arrow[dl, "\tilde{\phi}"'] \arrow[dr, "\tilde{\psi}"] & \\ \phi(U\cap V) \times \mathbb{R}^{\binom{n}{k}} \arrow[rr] & & \psi(U \cap V) \times \mathbb{R}^{\binom{n}{k}} \end{tikzcd}$$

$$(u, \{a_I\}) \longmapsto (\psi \circ \phi^{-1}(u), \{b_J\})$$

where for $J = (j_1 < \ldots < j_k)$,

$$b_J = \sum_{I = (i_1 < \ldots < i_k)} \frac{\partial(x^{i_1}, \ldots, x^{i_k})}{\partial(y^{j_1}, \ldots, y^{j_k})}(p) \cdot a_I$$

Hence $\tilde{\psi} \circ \tilde{\phi}^{-1}$ is C^∞ and similarly for $\tilde{\phi} \circ \tilde{\psi}^{-1}$.

<u>Proof</u> Know we can write $dx^I = \sum_J c_J \, dy^J$ on $U \cap V$

Then $c_J(p) = dx^I_p \left(\frac{\partial}{\partial y^{j_1}}\Big|_p, \ldots, \frac{\partial}{\partial y^{j_k}}\Big|_p \right)$

$$\underset{\text{Prop. 3.27}}{=} \det \begin{bmatrix} dx^{i_1}_p\left(\frac{\partial}{\partial y^{j_1}}\Big|_p\right) & \cdots & dx^{i_1}_p\left(\frac{\partial}{\partial y^{j_k}}\Big|_p\right) \\ \vdots & & \\ dx^{i_k}_p\left(\frac{\partial}{\partial y^{j_1}}\Big|_p\right) & \cdots & dx^{i_k}_p\left(\frac{\partial}{\partial y^{j_k}}\Big|_p\right) \end{bmatrix}$$

$$= \frac{\partial(x^{i_1}, \ldots, x^{i_k})}{\partial(y^{j_1}, \ldots, y^{j_k})}(p).$$

Consequence:

$$\sum_I a_I(p) \, dx^I_p = \sum_I a_I(p) \sum_J \frac{\partial(x^{i_1}, \ldots, x^{i_k})}{\partial(y^{j_1}, \ldots, y^{j_k})}(p) \cdot dy^J_p$$

$$= \sum_J \left(\sum_I a_I(p) \cdot \frac{\partial(x^{i_1}, \ldots, x^{i_k})}{\partial(y^{j_1}, \ldots, y^{j_k})}(p) \right) \cdot dy^J_p.$$

Hence $b_J(p) = \sum_I a_I(p) \frac{\partial(x^{i_1}, \ldots, x^{i_k})}{\partial(y^{j_1}, \ldots, y^{j_k})}(p).$ $\square$

Theorem Let M be a smooth n-dim manifold. Then $A_k(TM)$ is a smooth manifold of dimension $n+\binom{n}{k}$

Proof As for TM and T^*M, using the lemma above. $\square$

Theorem The projection $\pi: A_k(TM) \to M$ is a smooth vector bundle of dimension $\binom{n}{k}$.

Proof A chart $\phi=(x^1,\dots x^n): U \to \mathbb{R}^n$ gives rise to a local trivialization

$$A_k(TU) \xrightarrow{\sim} U \times \mathbb{R}^{\binom{n}{k}}$$
$$\omega_p = \sum c_I \, dx^I_p \mapsto (p, \{c_I\}) \quad \square$$

Rmk A chart $\phi=(x^1..x^n)$ defines a local frame $\{dx^I\}_{I=(i_1<\dots<i_k)}$ on U.

<u>Def</u> A smooth k-form on M is a smooth section

$$\omega : M \to A_k(TM).$$

The vector space of smooth k-forms is denoted $\Omega^k(M)$

<u>Rmk</u> For $k=0$, $A_0(TM) = M \times \mathbb{R}$ (since $A_0(T_pM) = \mathbb{R}$)

Therefore a smooth section $\omega : M \to A_0(TM)$ amounts to a smooth map $M \to \mathbb{R}$.

We shall identify $\Omega^0(M) = C^\infty(M)$.

Prop Let $\omega = \{\omega_p \in A_k(T_pM) : p \in M\}$ be a k-form on M. Then the following conditions are equivalent:

(i) ω is smooth

(ii) For every $p \in M$, there exists a chart $\phi = (x^1..x^n) : U \to \mathbb{R}^n$ on M such that $p \in U$, and $\omega|_U = \sum_I a_I \, dx^I$, where each $a_I : U \to \mathbb{R}$ is smooth.

(iii) For every chart $\phi = (x^1..x^n) : U \to \mathbb{R}^n$, we have $\omega|_U = \sum_I a_I \, dx^I$, where each $a_I : U \to \mathbb{R}$ is smooth.

Proof As for T^*M, using the local trivialization (or that $\{dx^I\}$ is a local frame). $\square$

Wedge product of differential forms

Let $\omega \in \Omega^k(M)$ and $\tau \in \Omega^l(M)$.

Define $\omega \wedge \tau = \{ \omega_p \wedge \tau_p \in A_{k+l}(T_pM) : p \in M \}$.

Prop $\omega \wedge \tau$ is a smooth $k+l$-form

Proof suffices to check this locally in a coordinate nbh. $\phi = (x^1, \dots, x^n) : U \to \mathbb{R}^n$.

Write $\omega = \sum_I a_I dx^I$, $\tau = \sum_J b_J dx^J$, then

$$\omega \wedge \tau = \sum_{I,J} a_I b_J dx^I \wedge dx^J$$

$$= \sum_{\{k+l \text{ multiindex } K\}} \Big(\sum_{\substack{I,J \text{ st.} \\ I \cup J = K \\ I \cap J = \emptyset}} \pm a_I b_J dx^K \Big) \quad \text{smooth}$$

$\square$

The wedge product gives a bilinear map

$$\wedge : \Omega^k(M) \times \Omega^l(M) \longrightarrow \Omega^{k+l}(M)$$

This is associative and anticommutative (i.e., $\omega \wedge \tau = (-1)^{kl}\, \tau \wedge \omega$.

Consequence $\Omega^*(M) = \{\Omega^k(M) : k \geq 0\}$ is an anticommutative graded $\mathbb{R}$-algebra.

Pullback of k-forms

Let $F : N \to M$ be a smooth map.

For each $p \in N$, we have the differential $F_{*,p} : T_pN \to T_{F(p)}M$

Let $\omega = \{ \omega_q \in A_k(T_qM) : q \in M \}$ be a k-form on M.

Define $F^*(\omega) = \{ F^*(\omega)_p \in A_k(T_pN) : p \in N \}$, k-form on N,

where $F^*(\omega)_p(v_1, .., v_k) = \omega_{F(p)}(F_{*,p}(v_1), .., F_{*,p}(v_k))$

for $v_1, .., v_k \in T_pN$.

Exercise

- $F^*(\omega + \tau) = F^*(\omega) + F^*(\tau)$
- $F^*(\omega \wedge \tau) = F^*(\omega) \wedge F^*(\tau)$

Prop Let $F: N \to M$ be a smooth map. If ω is a smooth k-form on M, then $F^*(\omega)$ is a smooth k-form on N.

Proof Choose charts $\phi = (x^1, \ldots, x^n): U \to \mathbb{R}^n$ on N and $\psi = (y^1, \ldots, y^m): V \to \mathbb{R}^m$ on M, such that $p \in U$ and $F(U) \subseteq V$.

Then $\omega|_V = \sum_I a_I \, dy^I$, where $a_I: V \to \mathbb{R}$ are smooth.

$$F^*(\omega)|_U = F^*(\omega|_V) = F^*\Big(\sum_I a_I \, dy^I\Big)$$

$$= \sum_I F^*(a_I \, dy^I) = \sum_{I=(i_1 < i_k)} F^*(a_I) \cdot F^*(dy^{i_1}) \wedge \ldots \wedge F^*(dy^{i_k})$$

$$= \sum_{i_1 < i_k} (a_I \circ F) \, d(y^{i_1} \circ F) \wedge \ldots \wedge d(y^{i_k} \circ F) \quad \text{smooth} \quad \square$$

<u>Consequence</u> A smooth map $F: N \to M$ induces a linear map $F^*: \Omega^k(M) \to \Omega^k(N)$ for each $k \geq 0$.

This gives a map of graded $\mathbb{R}$-algebras

$$F^*: \Omega^*(M) \to \Omega^*(N).$$

<u>Def</u> An $\mathbb{R}$-linear map of degree 1, $D: \Omega^*(M) \to \Omega^*(M)$ is a sequence of $\mathbb{R}$-linear maps

$$D: \Omega^k(M) \to \Omega^{k+1}(M), \quad k \geq 0.$$

D is an antiderivation of degree 1 if

$$D(\omega \wedge \tau) = (D\omega) \wedge \tau + (-1)^k \omega \wedge D(\tau)$$

for $\omega \in \Omega^k(M)$ and $\tau \in \Omega^\ell(M)$.

Def An $\mathbb{R}$-linear map $D: \Omega^*(M) \to \Omega^*(M)$ of deg. 1. is a local operator if given $\omega \in \Omega^k(M)$ and an open set $U \subseteq M$ s.t. $\omega|_U = 0$, then also $(D\omega)|_U = 0$

Equivalently: $\omega|_U = \tau|_U \Rightarrow D\omega|_U = D\tau|_U$

Prop. Any anti derivation of deg 1. $D: \Omega^*(M) \to \Omega^*(M)$ is a local operator.

Proof Given $\omega \in \Omega^k(M)$ such that $\omega|_U = 0$, must show that $(D\omega)_p = 0$ for all $p \in U$. Choose a bump function $\rho: M \to [0,1]$ at p supported in U. Then $\rho\omega = 0$.

Since $\rho\omega = 0$, we get

$$0 = D(0) = D(\rho\omega) = D(\rho)\,\omega + \rho\, D\omega$$

$$\Longrightarrow \quad 0 = D(\rho)_p \wedge \overset{0}{\overset{\shortparallel}{\omega_p}} + \overset{1}{\overset{\shortparallel}{\rho(p)}}\,(D\omega)_p = (D\omega)_p .$$

$$\Longrightarrow \quad (D\omega)_p = 0. \qquad \square$$