1. Smooth Functions on a Euclidean Space

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Euclidean Space

- $\mathbb{R}^n$ prototype of all manifolds
- generalize differential and integral calculus to manifolds
- coordinates good for computations
- some computations are invariant under change of coordinates
- manifolds do not have preferred coordinates
- the "dx" in integration is not invariant under change of coordinates

Smooth Functionctons on a Euclidean Space

Supersctipt Convention

Point p in $\mathbb{R}^n$ written $p = (p^1, \ldots, p^n)$ with coordinates $x^1, \ldots, x^n$.

Definition

The *open ball* $B(p, \varepsilon)$ around p of radius ε is

$$B(p, \varepsilon) = \{x \in \mathbb{R}^n \mid |x - p| < \varepsilon\}$$

Definition

$U \subseteq \mathbb{R}^n$ is open if for every $p \in U$, there exists $\varepsilon > 0$ such that $B(p, \varepsilon) \subseteq U$.

Definition

Suppose $p \in U$ with $U \subseteq \mathbb{R}^n$ open. A function $f\colon U \to \mathbb{R}$ is

- C^0 if it is continuous
- C^1 if $\frac{\partial f}{\partial x^i}(p)$ exists for all $p \in U$ and $\frac{\partial f}{\partial x^i}\colon U \to \mathbb{R}$ is continuous for $i = 1, \ldots, n$.
- C^k for $k \geq 2$ if $\frac{\partial f}{\partial x^i}(p)$ exists for all $p \in U$ and $\frac{\partial f}{\partial x^i}\colon U \to \mathbb{R}$ is C^{k-1} for $i = 1, \ldots, n$.
- C^∞ if it is C^k for all k.
- *smooth* if it is C^∞.
- $C^\infty(U)$ is the set of smooth functions from U to $\mathbb{R}$.

Exercise

Find the biggest k such that $f(x) = |x^3|$ is C^k.

Tangent Vectors in $\mathbb{R}^n$ as Derivations

Directional Derivatives

- $T_p\mathbb{R}^n$ is the tangent space at $p \in \mathbb{R}^n$.
- $T_p\mathbb{R}^n \cong \{p\} \times \mathbb{R}^n \cong \mathbb{R}^n$.
- $v \in T_p\mathbb{R}^n$ gives a directional derivative $D_v \colon C^\infty(\mathbb{R}^n) \to \mathbb{R}$

Definition of Directional Derivative (calculus)

Given $v \in \mathbb{R}^n$, the function $D_v \colon C^\infty(U) \to \mathbb{R}$ takes $f \in C^\infty(U)$ to the real number

$$D_v f = \lim_{t \to 0} \frac{f(p + tv) - f(p)}{t}$$

Exercise

Show that given $f, g \in C^\infty(U)$, we have

- $D_v(f + g) = D_v f + D_v g$
- $D_v(fg) = (D_v f)g(p) + f(p)D_v g$

Standard basis

We write $e_1, \ldots, e_n$ for the standard basis on $T_p\mathbb{R}^n$. A vector $v \in T_p\mathbb{R}^n$ is written

$$v = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = \sum_i v^i e_i$$

where e_i is the i-th standard basis vector.

Directional Derivative and Partial Derivatives

Let $c(t) = \sum_i c^i(t) e_i = p + tv$. The chain rule says that

$$D_v f = \lim_{t \to 0} \frac{f(p+tv) - f(p)}{t} = \frac{d(f \circ c)}{dt}(0) = \sum_i \frac{dc^i}{dt}(0) \frac{\partial f}{\partial x^i}(p) = \sum_i v^i \frac{\partial f}{\partial x^i}(p)$$

Conclusion: at the point p,

$$D_v = \sum_i v^i \frac{\partial}{\partial x^i}$$

Germs of Functions

Let U and V be open sets in $\mathbb{R}^n$ both containing a point p.

Observation

Let $f \in C^\infty(U)$ and $g \in C^\infty(V)$. If there exists an $\varepsilon > 0$ such that f and g agree on $B(p, \varepsilon) \subseteq U \cap V$, then $D_v f = D_v g$ for every $v \in T_p\mathbb{R}^n$.

Definition

Given $f \in C^\infty(U)$ and $g \in C^\infty(V)$, we say that (f, U) and (g, V) are *equivalent* if there exists an $\varepsilon > 0$ such that f and g agree on $B(p, \varepsilon) \subseteq U \cap V$. Show that this is an equivalence relation $\sim_p$. (That is, reflexive, symmetric and transitive.)

Definition Germs

The set $C_p^\infty(\mathbb{R}^n)$ of *germs* of functions at $p \in \mathbb{R}^n$ is the set of equivalence classes of the equivalence relation $\sim_p$ on the set of pairs (f, U) of an open set $U \subseteq \mathbb{R}^n$ and a function $f \in C^\infty(U)$

Definition

Given $f \in C^\infty(U)$ and $g \in C^\infty(V)$, we say that (f, U) and (g, V) are *equivalent* if there exists an $\varepsilon > 0$ such that f and g agree on $B(p, \varepsilon) \subseteq U \cap V$. Show that this is an equivalence relation $\sim_p$. (That is, reflexive, symmetric and transitive.)

Example

Let $p = 1$ and $U = V = \mathbb{R}$. Define $f(x) = e^{-1/x}$ and

$$g(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Then $f \sim_0 g$.

Definition Germs

The set $C_p^\infty(\mathbb{R}^n)$ of *germs* of functions at $p \in \mathbb{R}^n$ is the set of equivalence classes $[f, U]$ of the equivalence relation $\sim_p$ on the set of pairs (f, U) of an open set $U \subseteq \mathbb{R}^n$ and a function $f \in C^\infty(U)$

Exercise

Show that the following operations are well-defined for $[f, U], [g, V] \in C_p^\infty(\mathbb{R}^n)$ and $r \in \mathbb{R}$:

1. $r[f, U] = [rf, U]$
2. $[f, U] + [g, V] = [f + g, U \cap V]$
3. $[f, U][g, V] = [fg, U \cap V]$

Definition Germs

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Definition Point Derivations

The set $\mathcal{D}_p(\mathbb{R}^n)$ of *point-derivations at p* is the real vector space of linear maps $D\colon C_p^\infty(\mathbb{R}^n) \to \mathbb{R}$ satisfying the Leibniz rule

$$D(fg) = (Df)g(p) + f(p)Dg.$$

Definition Directional Derivations

For $v \in T_p(\mathbb{R}^n)$ we define $D_v \in \mathcal{D}_p(\mathbb{R}^n)$ by $D_v([f, U]) = D_v(f)$. We write $\phi\colon T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$ for the linear map given by $\phi(v) = D_v$.

After some preparation we will prove the following:

Theorem

$\phi\colon T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$, $v \mapsto \phi(v) = D_v$ is an isomorphism.

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Exercise

Explain why $\phi(e_i) = \frac{\partial}{\partial x^i}$.

Basis for $\mathcal{D}_p(\mathbb{R}^n)$

The vectors $\phi(e_1) = \frac{\partial}{\partial x^1}, \ldots, \phi(e_n) = \frac{\partial}{\partial x^n}$ form a basis for $\mathcal{D}_p(\mathbb{R}^n)$.

Remark

Later we will use $\mathcal{D}_p$ to define tangent spaces of manifolds. The advantage is that $\mathcal{D}_p$ is coordiante free.

Definition Point Derivations

The set $\mathcal{D}_p(\mathbb{R}^n)$ of *point-derivations at p* is the set of linear maps $D\colon C_p^\infty(\mathbb{R}^n) \to \mathbb{R}$ satisfying the Leibniz rule

$$D(fg) = (Df)g(p) + f(p)Dg.$$

Lemma

If $c \in C_p^\infty(\mathbb{R}^n)$ is represented by a constant function on $\mathbb{R}^n$, and $D \in \mathcal{D}_p(\mathbb{R}^n)$, then $D(c) = 0$.

Proof.

Let f be represented by the function with constant value 1. Then $c = rf$ for an $r \in \mathbb{R}$. Thus, $D(c) = D(rf) = rD(f)$. It suffices to show that $D(f) = 0$. Now, $f \cdot f = f$, so

$$D(f) = D(f \cdot f) = (Df)f(p) + f(p)Df = (Df) \cdot 1 + 1 \cdot D(f) = 2D(f).$$

Subtracting $D(f)$ from both sides we get $0 = D(f)$. $\square$

Lemma Taylor with remainder

Let $p \in \mathbb{R}^n$ and $U = B(p, \varepsilon)$. Given $f \in C^\infty(U)$, there exist $g_1, g_2, \ldots, g_n \in C^\infty(U)$ such that $g_i(p) = \frac{\partial f}{\partial x^i}(p)$ and

$$f(x) = f(p) + \sum_i (x^i - p^i) g_i(x)$$

for all $x \in U$.

Proof.

Given $x \in U$, let $c\colon [0,1] \to U$ be the smooth function with

$$c(t) = p + t(x - p)$$

The chain rule for $f \circ c$ states

$$\frac{d}{dt} f(p + t(x-p)) = \frac{d(f \circ c)}{dt}(t) = \sum_i \frac{dc^i}{dt}(t) \frac{\partial f}{\partial x^i}(c(t)).$$

Here $c^i(t) = p^i + t(x^i - p^i)$ and $\frac{dc^i}{dt}(t) = x^i - p^i$ □

proof (cont.)

The chain rule states

$$\frac{d}{dt}f(p+t(x-p))=\sum_i (x^i-p^i)\frac{\partial f}{\partial x^i}(p+t(x-p)).$$

Integrate with respect to t from 0 to 1 and define

$$g_i(x)=\int_0^1 \frac{\partial f}{\partial x^i}(p+t(x-p))dt.$$

The Leibniz integration rule states that

$$\frac{\partial g_i}{\partial x^j}=\int_0^1 \frac{\partial}{\partial x^j}\frac{\partial f}{\partial x^i}(p+t(x-p))dt.$$

Thus, $g_i(x)$ is in $C^\infty(U)$ and $g_i(p)=\frac{\partial f}{\partial x^i}(p)$. The equation on the top implies $f(x)-f(p)=\sum_i (x^i-p^i)g_i(x)$ $\square$

Let us summarize what we have proved:

Lemma

If $c \in C_p^\infty(\mathbb{R}^n)$ is represented by a constant function on $\mathbb{R}^n$, and $D \in \mathcal{D}_p(\mathbb{R}^n)$, then $D(c) = 0$.

Lemma Taylor with remainder

Let $p \in \mathbb{R}^n$ and $U = B(p, \varepsilon)$. Given $f \in C^\infty(U)$, there exist $g_1, g_2, \ldots, g_n \in C^\infty(U)$ such that $g_i(p) = \frac{\partial f}{\partial x^i}(p)$ and

$$f(x) = f(p) + \sum_i (x^i - p^i) g_i(x)$$

for all $x \in U$.

Proof that ϕ given by $v \mapsto D_v$ is an isomorphism.

We first show that if $\phi(v) = 0$, then $v = 0$. This implies that ϕ is injective. If $\phi(v) = D_v = 0$, then $D_v(x^j) = 0$ for all j. Since $D_v = \sum_i v^i \frac{\partial}{\partial x^i}$, this implies that $0 = D_v(x^j) = v^j$. Thus $v = 0$.

Let D be a derivation at p. In order to prove surjectivity, we apply Taylor with remainder. Let (f, V) be a representative of a germ in $C_p^\infty(\mathbb{R}^n)$. Pick $\varepsilon > 0$ such that $U = B(p, \varepsilon) \subseteq V$ and let $h = f|_U$. Then $[f, V] = [h, U]$. Choose g_i as in Taylor with remainder such that

$$f(x) = f(p) + \sum (x^i - p^i) g_i(x) \quad \text{for } x \in U$$

Using $D(f(p)) = 0$ and $D(p^i) = 0$ we get

$$Df = \sum (Dx^i) g_i(p) + \sum (p^i - p^i) Dg_i = \sum (Dx^i) \frac{\partial f}{\partial x^i}(p).$$

Since this is true for all (f, V), this shows that $D = D_v$ for v the vector with $v^i = Dx^i$. $\square$

Vector Fields

Definition Vector Field

A *vector field* on an open $U \subseteq \mathbb{R}^n$ is a collection $X = \{X_p \in T_p\mathbb{R}^n \mid p \in U\}$

For each p write $X_p = \sum a^i(p) \frac{\partial}{\partial x^i}\Big|_p$. Then we have functions $a^i : U \to \mathbb{R}$. The vector field X is *smooth* if the functions a^i are all smooth. Identifying each T_pU with $\mathbb{R}^n$, there is an identification

$$X = \sum a^i \frac{\partial}{\partial x^i} \longleftrightarrow \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix}$$

Exercise

Check that the set $\mathfrak{X}(U)$ of vector fields on U is a vector space.

Define $C^\infty(U) \times \mathfrak{X}(U) \to \mathfrak{X}(U)$ by $(f \cdot X)_p = f(p)X_p$ for $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$.

Exercise

Check that the set $\mathfrak{X}(U)$ of vector fields on U is a module over the $\mathbb{R}$-algebra $C^\infty(U)$, that is, that

1. associativity: $(f \cdot g)X = f(gX)$
2. identity: $1 \cdot X = X$
3. left distributivity: $(f + g)X = fX + gX$
4. right distributivity: $f(X + Y) = fX + fY$

Vector Fields as Derivations

Let U be an open subset of $\mathbb{R}^n$ and let $\mathfrak{X}^\infty(U)$ be the set of smooth vector fields on U. Let $f \in C^\infty(U)$. Given $X \in \mathfrak{X}^\infty(U)$, say $X = \sum a^i \partial/\partial x^i$, we define

$$(Xf)(p) = X_p f = \sum a^i(p) \frac{\partial f}{\partial x^i}(p).$$

Thus $Xf \in C^\infty(U)$ and we have defined a map

$$\mathfrak{X}^\infty(U) \times C^\infty(U) \to C^\infty(U), \quad (X, f) \mapsto Xf$$

Proposition

Given $X \in \mathfrak{X}^\infty(U)$ and $f, g \in C^\infty(U)$, the following Leibniz rule holds:

$$X(fg) = (Xf)g + fXg.$$

Proof.

Check equality after evaluating at $p \in U$. $\square$

Definition

A *derivation* of $C^\infty(U)$ is an $\mathbb{R}$-linear map $D\colon C^\infty(U) \to C^\infty(U)$ satisfying the Leibniz rule $D(fg) = (Df)g + fDg$. We write $\mathrm{Der}(C^\infty(U))$ for the real vectorspace of such derivations.

We have constructed a linear function $\mathfrak{X}^\infty(U) \to \mathrm{Der}(C^\infty(U))$. This is actually an isomorphism. We will not show this now.

Multilinear Algebra

We introduce the Exterior Algebra. This is where tensors in tensor calculus live.
The idea is that when linearizing we obatin tangent spaces, and we work with these instead of directly with mainfolds. This leads into multilinear algebra.

Dual Space

Notation

Unless otherwise stated, V will be a real vector space of dimension n, and $e_1, \dots, e_n$ will be a basis for V.

Definition

The *dual* vector space of V is
$$V^\vee = \mathrm{Hom}(V, \mathbb{R}),$$
the vector space of linear maps form V to $\mathbb{R}$. Elements of $V^\vee$ are called *covectors*.

Exercise

Explain how sum and scalar multiplication makes $V^\vee$ a vector space.

The covectors $\alpha^1, \ldots, \alpha^n$ are defined by $\alpha^i(v) = v^i$, where v^i is the i-th coordinate of $v = v^1 e_1 + \cdots + v^n e_n$.

Observation

$$\alpha^i(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Proposition

The covectors $\alpha^1, \ldots, \alpha^n$ form a basis for $V^\vee$.

Proof (Linearly independent:)

Assume $c_1\alpha^1 + \cdots + c_n\alpha^n = 0$ for $c_1, \ldots, c_n \in \mathbb{R}$. Evaluate at e_i to get

$$0 = (c_1\alpha^1 + \cdots + c_n\alpha^n)(e_i) = c_i\alpha^i(e_i) = c_i.$$

□

Proposition

The covectors $\alpha^1, \ldots, \alpha^n$ *form a basis for* $V^\vee$.

Proof (the α^i generate: $V^\vee$).

Let $f \in V^\vee$ and $v = v^1 e_1 + \cdots + v^n e_n \in V$. Then

$$f(v) = v^1 f(e_1) + \cdots + v^n f(e_n) = f(e_1)\alpha^1(v) + \ldots f(e_n)\alpha^n(v)$$

This reads $f = f(e_1)\alpha^1 + \cdots + f(e_n)\alpha^n$. $\square$

Consequence

$V^\vee$ has the same dimension as V.

Remark

In linear algebra vectors are column vectors and covectors are row vectors. Multiplication by a fixed row vector gives a linear map from column vectors to the real numbers.

Let $V^k = V \times \cdots \times V$ be the product of k copies of V.

Definition

A function $f\colon V^k \to \mathbb{R}$ is *k-linear* if it is linear in each or the k variables:

$$f(v_1, \ldots, x + y, \ldots, v_n) = f(v_1, \ldots, x, \ldots, v_n) + f(v_1, \ldots, y, \ldots, v_n)$$

and

$$f(v_1, \ldots, cx, \ldots, v_n) = cf(v_1, \ldots, x, \ldots, v_n)$$

Example

The scalar product on $\mathbb{R}^n$ is bilinear (2-linear),

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \quad (x, y) \mapsto x \cdot y = x^1 y^1 + \cdots + x^n y^n$$

Example

The determinant $\det : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ is n-linear.

Write $L_k(V)$ for the vector space of k-linear functions.

Exercise

Explain how sum and scalar multiplication makes $L_k(V)$ a vector space. Show that the dimension of $L_k(V)$ is n^k.

Convention $L_0(V) = \mathbb{R}$ and $L_1(V) = V^\vee$.

Review of permutations

Let $A = \{1, \ldots, k\}$. A *permutation* of A is a bijective function $\sigma \colon A \to A$.
Let S_k be the set of all permutations of A.
Permutations can be composed $\tau \circ \sigma$
This makes S_k a group with unit element the identity function.
A permutation σ can be described by its matrix

$$\sigma = \begin{bmatrix} 1 & 2 & \ldots & k \\ \sigma(1) & \sigma(2) & \ldots & \sigma(k) \end{bmatrix}$$

An *inversion* in σ is a pair $(\sigma(i), \sigma(j))$ with $\sigma(i) > \sigma(j)$ and $i < j$.

Example

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{bmatrix}$$

Inversions in σ are $(3,1), (4,1), (5,1), (2,1), (3,2), (4,2)$ and $(5,2)$. There are 7 inversions in σ.

Definition

A permutation σ is *even* if it han an even number of inversions. Otherwise it is odd. The sign of σ is

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

Fact from algebra

$\operatorname{sgn}(\tau \circ \sigma) = \operatorname{sgn}(\tau) \cdot \operatorname{sgn}(\sigma)$

Multilinear Functions

Let $f\colon V^k \to \mathbb{R}$ be a k-linear function.

Definition

f is *symmetric* if $f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = f(v_1, \dots, v_k)$ for every $\sigma \in S_k$.

$$S_k(V) = \{f \in L_k(V) \mid f \text{ is symmetric}\}$$

Definition

f is *alternating* if $f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma) f(v_1, \dots, v_k)$ for every $\sigma \in S_k$.

$$A_k(V) = \{f \in L_k(V) \mid f \text{ is alternating}\}$$

By convention $A_0(V) = S_0(V) = L_0(V) = \mathbb{R}$ and $A_1(V) = S_1(V) = L_1(V) = V^\vee$

Example

The scalar product on $\mathbb{R}^n$ is symmetric

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \quad (x, y) \mapsto x \cdot y = x^1 y^1 + \cdots + x^n y^n$$

Example

The determinant $\det : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ is alternating

Example

Given covectors $f\colon V \to \mathbb{R}$ and $g\colon V \to \mathbb{R}$, we can define an alternating 2-linear function $f \wedge g\colon V^2 \to \mathbb{R}$ by

$$(f \wedge g)(v, w) = f(v)g(w) - f(w)g(v)$$

There is a function

$$S_k \times L_k(V) \to L_k(V), \quad (\sigma, f) \mapsto \sigma f$$

where

$$(\sigma f)(v_1, \ldots, v_k) = f(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

Observation

f is symmetric if and only if $\sigma f = f$ for all $\sigma \in S_k$
f is alternating if and only if $\sigma f = \mathrm{sgn}(\sigma) f$ for all $\sigma \in S_k$

Lemma

$(\tau\sigma)f = \tau(\sigma f)$ for all $\tau, \sigma \in S_k$ and $f \in L_k(V)$.

Proof.

Let $w_i = v_{\tau(i)}$. Then $w_{\sigma(i)} = v_{\tau(\sigma(i))} = v_{(\tau\sigma)(i)}$ and

$$\tau(\sigma f)(v_1, \ldots, v_k) = (\sigma f)(w_1, \ldots, w_k) = f(w_{\sigma(1)}, \ldots, w_{\sigma(k)}) = f(v_{(\tau\sigma(1))}, \ldots, v_{(\tau\sigma)(k)})$$

□

The above lemma states that $(\sigma, f) \mapsto \sigma f$ is an action of the group S_k on the set $L_k(V)$.

Definition

For $f \in L_k(V)$ we define

$$S(f) = \sum_{\sigma \in S_k} \sigma f \quad \text{and} \quad A(f) = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\sigma f$$

Proposition

$S(f)$ is symmetric and $A(f)$ is alternating.

Proof that $A(f)$ is alternating.

$$\begin{aligned}\tau(Af) &= \sum_{\sigma} \tau(\operatorname{sgn}(\sigma)\sigma f) = \sum_{\sigma} \operatorname{sgn}(\sigma)\tau(\sigma f)\\ &= \sum_{\sigma} \operatorname{sgn}(\sigma)(\tau\sigma)f = \operatorname{sgn}(\tau)\sum_{\sigma} \operatorname{sgn}(\tau\sigma)(\tau\sigma)f = \operatorname{sgn}(\tau)Af\end{aligned}$$

□

Lemma

If $f \in A_k(V)$, *then* $A(f) = (k!)f$.

Proof.

$$A(f) = \sum_{\sigma} (\operatorname{sgn} \sigma)\sigma f = \sum_{\sigma} (\operatorname{sgn} \sigma)(\operatorname{sgn} \sigma) f = (k!)f$$

$\square$

The Wedge Product

First define the tensor product of multilinear functions

$$\otimes \colon L_k(V) \times L_l(V) \to L_{k+l}(V), \quad (f, g) \mapsto f \otimes g$$

where

$$(f \otimes g)(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+l}) = f(v_1, \ldots, v_k) \cdot g(v_{k+1}, \ldots, v_{k+l})$$

If $k = 0$, then $f = c \in \mathbb{R}$ and $f \otimes g = c \otimes g = cg$.

Observation

$\otimes$ is bilinear and associative, $(f \otimes g) \otimes h = f \otimes (g \otimes h)$.

Definition

The *wedge product* is the bilinear function

$$\wedge \colon A_k(V) \times A_l(V) \to A_{k+l}(V), \quad (f, g) \mapsto f \wedge g = \frac{1}{k!l!} A(f \otimes g)$$

Definition

The *wedge product* is the bilinear function

$$\wedge\colon A_k(V) \times A_l(V) \to A_{k+l}(V), \quad (f, g) \mapsto f \wedge g = \frac{1}{k!l!} A(f \otimes g)$$

Note that we know that $f \wedge g$ is alternating. Explicitly:

$$(f \wedge g)(v_1, \ldots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)})$$

For $k = 0$, $f \in A_0 V = \mathbb{R}$ is a constant c, and

$$(c \wedge g)(v_1, \ldots, v_l) = \frac{1}{l!} \sum_{\sigma \in S_l} cg(v_{\sigma(1)}, \ldots, v_{\sigma(l)}) = cg(v_1, \ldots, v_l)$$

so $c \wedge g = cg$ for $c \in \mathbb{R}$ and $g \in A_l(V)$. Note that the coefficient $\frac{1}{l!}$ compensates for repetition. So does the general coefficient $\frac{1}{k!l!}$.

Example

$f \in A_2(V)$, $g \in A_1(V)$. Then

$$\begin{aligned} A(f \otimes f)(v_1, v_2, v_3) = \quad & f(v_1, v_2)g(v_3) - f(v_1, v_3)g(v_2) + f(v_2, v_3)g(v_1) \\ & - f(v_2, v_1)g(v_3) - f(v_3, v_1)g(v_2) + f(v_3, v_2)g(v_1) \\ & = 2(f(v_1, v_2)g(v_3) - f(v_1, v_3)g(v_2) + f(v_2, v_3)g(v_1)) \end{aligned}$$

Definition

$\sigma \in S_{k+l}$ is a (k, l)-shuffle if both

$$\sigma(1) < \cdots < \sigma(k) \quad \text{and} \quad \sigma(k+1) < \cdots < \sigma(k+l)$$

Proposition

$$(f \wedge g)(v_1, \dots, v_{k+l}) = \sum_{(k,l)-shuffle\ \sigma} f(v_{\sigma(1)}, \dots, v_{\sigma(k)})g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)})$$

The wedge product is anticommutative:

$$f \wedge g = (-1)^{kl} g \wedge f \quad \text{for } f \in A_k(V) \text{ and } g \in A_l(V)$$

Proof.

Let

$$\tau = \begin{bmatrix} 1 & \dots & l & l+1 & \dots l+k \\ k+1 & \dots & k+l & 1 & \dots k \end{bmatrix}$$

Then

$$\sigma(1) = \sigma\tau(l+1), \dots, \sigma(k) = \sigma\tau(l+k)$$

and

$$\sigma(k+1) = \sigma\tau(1), \dots, \sigma(k+l) = \sigma\tau(l)$$

□

Proof (cont.)

For $v_1, \ldots, v_{k+l} \in V$ we have

$$
\begin{aligned}
&A(f \otimes g)(v_1, \ldots, v_{k+l}) \\
&= \sum_\sigma \operatorname{sgn}(\sigma) f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \ldots, g_{\sigma(k+l)}) \\
&= \sum_\sigma \operatorname{sgn}(\sigma) f(v_{\sigma\tau(l+1)}, \ldots, v_{\sigma\tau(l+k)}) g(v_{\sigma\tau(1)}, \ldots, g_{\sigma\tau(l)}) \\
&= \sum_\sigma \operatorname{sgn}(\sigma) g(v_{\sigma\tau(1)}, \ldots, g_{\sigma\tau(l)}) f(v_{\sigma\tau(l+1)}, \ldots, v_{\sigma\tau(l+k)}) \\
&= \operatorname{sgn}(\tau) \sum_\sigma \operatorname{sgn}(\sigma\tau) g(v_{\sigma\tau(1)}, \ldots, g_{\sigma\tau(l)}) f(v_{\sigma\tau(l+1)}, \ldots, v_{\sigma\tau(l+k)}) \\
&= \operatorname{sgn}(\tau) A(g \otimes f)(v_1, \ldots, v_{k+l})
\end{aligned}
$$

$\square$

Proof (cont.)

We conclude that

$$A(f \otimes g)(v_1, \ldots, v_{k+l}) = \mathrm{sgn}(\tau) A(g \otimes f)(v_1, \ldots, v_{k+l})$$

Dividing by $k!l!$ on both sides gives $f \wedge g = \mathrm{sgn}(\tau) g \wedge f$. Now, $\mathrm{sgn}(\tau) = (-1)^{kl}$ because τ has kl inversions. □

Corollary

If k is odd and $f \in A_k(V)$, then $f \wedge f = 0$.

Proof.

$$f \wedge f = (-1)^{k^2} f \wedge f = -f \wedge f \Rightarrow f \wedge f = 0$$

□

Proposition

The wedge product is associative:

$$(f \wedge g) \wedge h = f \wedge (g \wedge h)$$

for $f \in A_k(V)$, $g \in A_l(V)$ *and* $h \in A_m(V)$.

Lemma

For $f \in A_k(V)$ and $g \in A_l(V)$, we have

(i) $A(A(f) \otimes g) = k! A(f \otimes g)$

(ii) $A(f \otimes A(g)) = k! A(f \otimes g)$

Proof of (i).

First note that

$$A(A(f) \otimes g) = A(\sum_{\tau \in S_k} (\operatorname{sgn}(\tau)\tau f) \otimes f) = \sum_{\tau \in S_k} A((\operatorname{sgn}(\tau)\tau f) \otimes g)$$

Given $\tau \in S_k \subseteq S_{k+l}$ we have $(\tau f) \otimes g = \tau(f \otimes g)$. Therefore

$$\begin{aligned} A((\operatorname{sgn}(\tau)\tau f) \otimes g) &= \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\sigma((\operatorname{sgn}(\tau)\tau f) \otimes g) = \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau)\sigma(\tau(f \otimes g)) \\ &= \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma\tau)(\sigma\tau)(f \otimes g)) = \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)(\sigma)(f \otimes g)) = A(f \otimes g) \end{aligned}$$

$\square$

Proposition

The wedge product is associative:

$$(f \wedge g) \wedge h = f \wedge (g \wedge h)$$

for $f \in A_k(V)$, $g \in A_l(V)$ *and* $h \in A_m(V)$.

Proof.

This is a direct calculation. Part (i) of the lemma gives:

$$\begin{aligned}(f \wedge g) \wedge h &= \frac{1}{(k+l)!m!} A((f \wedge g) \wedge h) \\ &= \frac{1}{(k+l)!m!k!l!} A(A(f \otimes g) \otimes h) \\ &= \frac{(k+l)!}{(k+l)!m!k!l!} A((f \otimes g) \otimes h) = \frac{1}{k!l!m!} A((f \otimes g) \otimes h)\end{aligned}$$

Similarly, part (ii) of the lemma gives $f \wedge (g \wedge h) = \frac{1}{k!l!m!} A(f \otimes (g \otimes h))$. $\square$

Note that we have proved that $f \wedge g \wedge h = \frac{1}{k!l!m!} A(f \otimes g \otimes h)$ for $f \in A_k(V)$, $g \in A_l(V)$ and $h \in A_m(V)$.
Let $k \in A_n(V)$. Then

$$\begin{aligned} & f \wedge g \wedge h \wedge k = (f \wedge g \wedge h) \wedge k \\ =& \frac{1}{(k+l+m)!n!} A((f \wedge g \wedge h) \otimes k) = \frac{1}{(k+l+m)!n!} A(\frac{1}{k!l!m!} A(f \otimes g \otimes h) \otimes k) \\ =& \frac{1}{k!l!m!n!} A(\frac{1}{(k+l+m)!n!} A(f \otimes g \otimes h) \otimes k) = \frac{1}{k!l!m!n!} A(f \otimes g \otimes h \otimes k) \end{aligned}$$

By induction, if $f_1 \in A_{k_1}(V), \ldots, f_r \in A_{k_r}(V)$, then

$$f_1 \wedge \cdots \wedge f_r = \frac{1}{k_1! \ldots k_r!} A(f_1 \otimes \cdots \otimes f_r)$$

Example

Let $\alpha^1, \dots, \alpha^k \in A_1(V)$. Then $\alpha^1 \wedge \cdots \wedge \alpha^k \in A_k(V)$ is given by

$$\alpha^1 \wedge \cdots \wedge \alpha^k = A(\alpha^1 \otimes \cdots \otimes \alpha^k) = \sum_{\sigma_i \in S_k} \operatorname{sgn}(\sigma)\sigma(\alpha^1 \otimes \cdots \otimes \alpha^k)$$

Given $v_1, \dots, v_k \in V$, we get

$$\begin{aligned}\alpha^1 \wedge \cdots \wedge \alpha^k(v_1, \dots, v_k) &= \sum_{\sigma_i \in S_k} \operatorname{sgn}(\sigma)\sigma(\alpha^1 \otimes \cdots \otimes \alpha^k)(v_1, \dots, v_k) \\ &= \sum_{\sigma_i \in S_k} \operatorname{sgn}(\sigma)\alpha^1 \otimes \cdots \otimes \alpha^k(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= \sum_{\sigma_i \in S_k} \operatorname{sgn}(\sigma)\alpha^1(v_{\sigma(1)}) \otimes \cdots \otimes \alpha^k(v_{\sigma(k)})\end{aligned}$$

This is the determinant $\det(\alpha^i(v_j))$ of the matrix with (i, j)-entry $\alpha^i(v_j)$

Definition

A *graded* $\mathbb{R}$*-algebra* $A = \{A(k) \mid k \geq 0\}$ is a collection of $\mathbb{R}$-vector spaces $A(k)$, together with

1. unit $1 \in A(0)$
2. bilinear multiplication $A(k) \times A(l) \to A(k+l)$, $(a, b) \mapsto a \cdot b$

such that the multiplication is

(i) unital: $a \cdot 1 = a$, $1 \cdot a = a$

(ii) associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

A is *anticommutative* if $ab = (-1)^{kl}$ for $a \in A(k)$ and $b \in A(l)$.

Summary

$A_*(V) = \{A_k(V) \mid k \geq 0\}$ is an anticommutative graded algebra when $a \cdot b = a \wedge b$. This is the *exterior algebra* of V.

A basis for $A_k(V)$

Let $e_1, \ldots, e_n$ be a basis for V. We write $\alpha^1, \ldots, \alpha^n$ for the dual basis for $A_1(V) = V^\vee$ determined by $\alpha^i(e_j) = \delta^i_j$. Given $I = (i_1, \ldots, i_k)$ with each $i_s \in \{1, \ldots, n\}$ we write

$$\alpha^I = \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} \quad \text{and} \quad e_I = (e_{i_1}, \ldots, e_{i_k})$$

Proposition

The wedge products $\{\alpha^I = \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$ *form a basis for* $A_k(V)$.

We will prove this very soon.

Lemma

Given $I = (i_1, \dots, i_k)$ *with* $1 \leq i_1 < \dots < i_k \leq n$ *and* $J = (j_1, \dots, j_k)$ *with* $1 \leq j_1 < \dots < j_k \leq n$ *we have*

$$\alpha^I(e_J) = \begin{cases} 1 & \textit{if } I = J \\ 0 & \textit{if } I \neq J \end{cases}$$

Proof.

We know that

$$\alpha^I(e_J) = \alpha^{i_1} \wedge \dots \wedge \alpha^{i_k}(e_{j_1}, \dots, e_{j_k}) = \det[\alpha^{i_r}(e_{j_s})]$$

For $I = J$, this is an identity matrix, so $\det[\alpha^{i_r}(e_{j_s})] = 1$. For $I \neq J$ we can find an r such that i_r is not equal to any j_s. Thus the matrix has a zero row, and the determinant is zero. □

Proposition

The wedge products $\{\alpha^I = \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} \mid 1 \le i_1 < \cdots < i_k \le n\}$ *form a basis for* $A_k(V)$.

Proof.

Linearly independent: suppose $\sum_I c_I \alpha^I = 0$, $c_I \in \mathbb{R}$. Evaluate at any e_J:

$$0 = \sum_I c_I \alpha^I(e_J) = c_J$$

Generators: Given $f \in A_k(V)$, let $c_I = f(e_I)$.
Claim: $f = \sum_I c_I \alpha^I$.
Since both sides are alternating it suffices to show that evaluating f and $\sum_I c_I \alpha^I$ at each e_J with $J = (j_1 < \cdots < j_k)$ gives the same result.
By the lemma,

$$\sum_I c_I \alpha^I(e_J) = c_J = f(e_J)$$

□

Proposition

The wedge products $\{\alpha^I = \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$ *form a basis for* $A_k(V)$.

Corollary

Suppose $\dim V = n$.

1. *if* $0 \leq k \leq n$, *then* $\dim A_k V = \binom{n}{k}$
2. *if* $k > n$, *then* $A_k V = 0$.