Ex. 12.2 charts $\phi = (x^1 .. x^n) : U \to \mathbb{R}^n$, $\psi = (y^1 .. y^n) : V \to \mathbb{R}^n$ on the manifold M.

$$\tilde{\phi} : TU \longrightarrow \phi(U) \times \mathbb{R}^n \quad \sum a_i(p) \frac{\partial}{\partial x^i}\Big|_p \longmapsto (\phi(p), a_1(p) .. a_n(p)).$$

$$\begin{tikzcd}
 & T(U \cap V) \arrow[dl, "\tilde{\phi}"'] \arrow[dr, "\tilde{\psi}"] & \\
\phi(U \cap V) \times \mathbb{R}^n \arrow[rr, "\tilde{\psi} \circ \tilde{\phi}^{-1}"] & & \psi(U \cap V) \times \mathbb{R}^n
\end{tikzcd}$$

$(x, v) \mapsto ((\psi \circ \phi^{-1})(x), (\psi \circ \phi^{-1})_{*,x}(v))$

(a) Find the Jacobian matrix for $\tilde{\psi} \circ \tilde{\phi}^{-1}$.

Consider in general $W \subseteq \mathbb{R}^n$ open, $f = \begin{bmatrix} f^1 \\ \vdots \\ f^n \end{bmatrix} : W \to W' \subseteq \mathbb{R}^n$

$\tilde{f} : W \times \mathbb{R}^n \longrightarrow W' \times \mathbb{R}^n$, $\tilde{f}(x, v) = (f(x), f_{*,x}(v)) = (f(x), J(f)(x) \cdot v)$

$$\begin{bmatrix} [J(f)(x)] & [0] \\ C & D \end{bmatrix}$$

$$\mathcal{F}(x,v) = (f(x),\ J(f)(x)\cdot v), \quad x \in W,\ v \in \mathbb{R}^n$$

$$J(f)(x)\cdot v = \begin{bmatrix} \frac{\partial f^1}{\partial x^1}(x)\cdot v^1 + \dots + \frac{\partial f^1}{\partial x^n}(x)\cdot v^n \\ \vdots \\ \frac{\partial f^n}{\partial x^1}(x)\cdot v^1 + \dots + \frac{\partial f^n}{\partial x^n}(x)\cdot v^n \end{bmatrix}$$

The C matrix has columns:

$$C = \left(\left[\frac{\partial^2 f^j}{\partial x^1 x^i}(x) \right] \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \left[\frac{\partial^2 f^i}{\partial x^2 \partial x^j}(x) \right] \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}, \dots, \left[\frac{\partial^2 f^i}{\partial x^n \partial x^j}(x) \right] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \right)$$

$$D = J(f)(x).$$

(b) Show the determinant is $[\det J(f)(x)]^2$.

The Jacobian has the form $\begin{bmatrix} J(f)(x) & 0 \\ C & J(f)(x) \end{bmatrix}$

The determinant is the product of the diagonal entries, i.e., $[\det J(f)(x)]^2$

Ex. 17.2 charts $(U, x^1 .. x^n)$ and $(V, y^1 .. y^n)$ on M.

Let $\omega \in \Omega^1(M)$ then $\omega|_{U \cap V} = \sum a_j dx^j = \sum b_i dy^i$

Find a_j in terms of $\{b_i\}$:

$$a_j = \omega\left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^{n} b_i dy^i\left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^{n} b_i \frac{\partial y^i}{\partial x^j}$$

Hence
$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^2}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^1} \\ & \vdots & & \\ \frac{\partial y^1}{\partial x^n} & \frac{\partial y^2}{\partial x^n} & \cdots & \frac{\partial y^n}{\partial x^n} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

transpose of the Jacobian matrix for $\psi \circ \phi^{-1}$.