Ex. 3.7 V vector space, $\beta^1, \dots, \beta^k$ and $\gamma^1, \dots, \gamma^k$ covectors on V.

$$\beta^i = \sum_{j=1}^{k} a^i_j \gamma^j, \text{ for } i = 1, \dots, k, \quad [a^i_j] \ k \times k \text{ matrix.}$$

Claim $\beta^1 \wedge \dots \wedge \beta^k = \det[a^i_j]\, \gamma^1 \wedge \dots \wedge \gamma^k$.

$$\beta^1 \wedge \dots \wedge \beta^k = \left(\sum_{j=1}^{k} a^1_j \gamma^j \right) \wedge \left(\sum_{j=1}^{k} a^2_j \gamma^j \right) \wedge \dots \wedge \left(\sum_{j=1}^{k} a^k_j \gamma^j \right)$$

$$= \sum_{1 \le j_1, \dots, j_k \le k} a^1_{j_1} a^2_{j_2} \cdots a^k_{j_k} \, \gamma^{j_1} \wedge \gamma^{j_2} \wedge \dots \wedge \gamma^{j_k}$$

$$= \sum_{\sigma \in S_k} a^1_{\sigma(1)} a^2_{\sigma(2)} \cdots a^k_{\sigma(k)} \, \gamma^{\sigma(1)} \wedge \gamma^{\sigma(2)} \wedge \dots \wedge \gamma^{\sigma(k)}.$$

$$= \sum_{\sigma \in S_k} a^1_{\sigma(1)} a^2_{\sigma(2)} \cdots a^k_{\sigma(k)} \, \mathrm{sgn}(\sigma) \, \gamma^1 \wedge \dots \wedge \gamma^k$$

$$= \det[a^i_j]\, \gamma^1 \wedge \dots \wedge \gamma^k$$

Ex 3.8 $\quad f \in A_k(V), \quad f : V \to \mathbb{R}$

Suppose $\quad u_j = \sum_{i=1}^{k} a_j^i v_i \quad$ for $j = 1, \ldots k$.

Claim: $\quad f(u_1, \ldots, u_k) = \det[a_j^i] \, f(v_1, \ldots v_k)$.

$$f(u_1, \ldots, u_k) = f\left(\sum_{i=1}^{k} a_1^i v_i, \sum_{i=1}^{k} a_2^i v_i, \ldots, \sum_{i=1}^{k} a_k^i v_i\right)$$

$$= \sum_{1 \le i_1, \ldots, i_k \le k} a_1^{i_1} \cdots a_k^{i_k} f(v_{i_1}, v_{i_2}, \ldots, v_{i_k})$$

$$= \sum_{\sigma \in S_k} a_1^{\sigma(1)} \cdots a_k^{\sigma(k)} f(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(k)})$$

$$= \sum_{\sigma \in S_k} a_1^{\sigma(1)} \cdots a_k^{\sigma(k)} \operatorname{sgn}(\sigma) f(v_1, \ldots, v_k)$$

$$= \det[a_j^i] \, f(v_1, \ldots v_k).$$

Ex 3.10 Let V be an n-dim. real vector space.

Let $\alpha^1, \dots, \alpha^k$ be covectors on V.

Claim: $\alpha^1 \wedge \dots \wedge \alpha^k \neq 0 \iff \alpha^1, \dots, \alpha^k$ are lin. indep. in $V^\vee$.

$\Rightarrow$ Suppose $\alpha^1 = a_2 \alpha^2 + \dots + a_k \alpha^k$.

Then $\alpha^1 \wedge \dots \wedge \alpha^k = (a_2 \alpha^2 + \dots + a_k \alpha^k) \wedge \alpha^2 \wedge \dots \wedge \alpha^k$

$= a_2 \alpha^2 \wedge \alpha^2 \wedge \dots \wedge \alpha^k + a_3 \alpha^3 \wedge \alpha^2 \wedge \dots \wedge \alpha^k + \dots + a_k \alpha^k \wedge \alpha^2 \wedge \dots \wedge \alpha^k = 0.$

$\Leftarrow$

$\alpha^1 \wedge \wedge \alpha^k (v_1, \dots, v_k) = \det [\alpha^i (v_j)] = 0$ for all $(v_1, \dots, v_k)$

then $\alpha^1, \dots, \alpha^k$ are linearly dependent.

Let $e_1, \dots, e_n$ be a basis for V,

Let $\gamma^1, \dots, \gamma^n$ be the dual basis

$\alpha^i = \sum_{j=1}^{n} a^i_j \gamma^j, \quad i = 1, \dots, k$

Extend $\alpha^1, \dots, \alpha^k$ to a basis $\alpha^1, \dots, \alpha^k, \alpha^{k+1}, \dots, \alpha^n$ for $V^\vee$.

Claim: There exist $v_1, \dots, v_n$ in V st. $\alpha^i(v_j) \overset{(*)}{=} \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$.

$V \xrightarrow{\simeq} (V^\vee)^\vee \quad v \mapsto \{\alpha \mapsto \alpha(v)\}$ isomorphism since V is finite dimensional. (Suffices to check injectivity since V and $(V^\vee)^\vee$ have the same dimension).

We know $\alpha^1, \dots, \alpha^n$ in $V^\vee$ have a dual basis in $(V^\vee)^\vee$ - this give $v_1 \dots v_n$ in V st. $(*)$ holds.

Then $\alpha^1 \wedge \dots \wedge \alpha^k(v_1, \dots, v_k) = \det[\alpha^i(v_j)] = \det \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = 1.$

So $\alpha^1 \wedge \dots \wedge \alpha^k \neq 0$.