The real projective space $\mathbb{RP}^n$

Recall $\mathbb{RP}^n = \mathbb{R}^{n+1} - \{0\}/\sim$, $x \sim y$ if $x = \lambda y$, for some $\lambda \in \mathbb{R} - \{0\}$.

Have checke there is a homeomorphism $\mathbb{RP}^n \cong S^n/\sim$, $x \sim y$ if $\begin{matrix} x = y \\ \text{or} \\ x = -y \end{matrix}$.

Have checked $\mathbb{RP}^n$ is locally Euclidian of dimension n.

Must show $\mathbb{RP}^n$ is Hausdorff and second countable.

<u>Def</u> A continuous function $f : X \to Y$. between topological spaces X and Y, is open if

$$U \subseteq X \text{ open} \implies f(U) \subseteq Y \text{ is open.}$$

<u>Lemma</u> $p : S^n \to S^n/\sim$ is open

<u>Proof</u> Let $U \subseteq S^n$ be open, must check $p^{-1} p(U) \subseteq S^n$ is open.
Notice $p^{-1} p(U) = U \cup -U$, where $-U = \{-p \in S^n : p \in U\}$.
The function $-1 : S^n \to S^n$ is a homeomorphism.
Hence U open implies $-U$ is open, so also $U \cup -U$ is open

$\square$

Lemma $\mathbb{RP}^n$ is Hausdorff.

Proof Think of $\mathbb{RP}^n = S^n/\sim$

Given a, b in $\mathbb{RP}^n$, choose $a', b' \in S^n$ st. $P(a') = a$, $P(b') = b$.

Want to find nbhs $a' \in U$, $b' \in V$ st. $P(U) \cap P(V) = \emptyset$.

(Then $P(U)$ and $P(V)$ a disjoint open sets in $\mathbb{RP}^n$).

$P(U) \cap P(V) = \emptyset$ is equivalent to

$U \cap V = \emptyset$ and $U \cap -V = \emptyset$.

Let $\varepsilon = \min\{\|a' - b'\|, \|a' - (-b')\|\}$, let

$U = B(a'; \varepsilon/2) \cap S^n$, $V = B(b', \varepsilon/2) \cap S^n$ $\square$

Lemma $\mathbb{RP}^n$ is second countable.

Proof Let $\mathcal{B} = \{B \subseteq S^n\}$ be a countable basis for the topology on S^n.

Claim $\{p(B) \subseteq \mathbb{RP}^n : B \in \mathcal{B}\}$ is a countable basis for the topology on $\mathbb{RP}^n$.

Must check this is a basis:

Given open set U in $\mathbb{RP}^n$ and $x \in U \subseteq \mathbb{RP}^n$.

Choose $x' \in S^n$ st. $p(x') = x$. Then $x' \in p^{-1}(U) \subseteq S^n$.

Choose $B \in \mathcal{B}$ st. $x' \in B \subseteq p^{-1}(U)$.

Then $x \in p(B) \subseteq U$.

$\square$

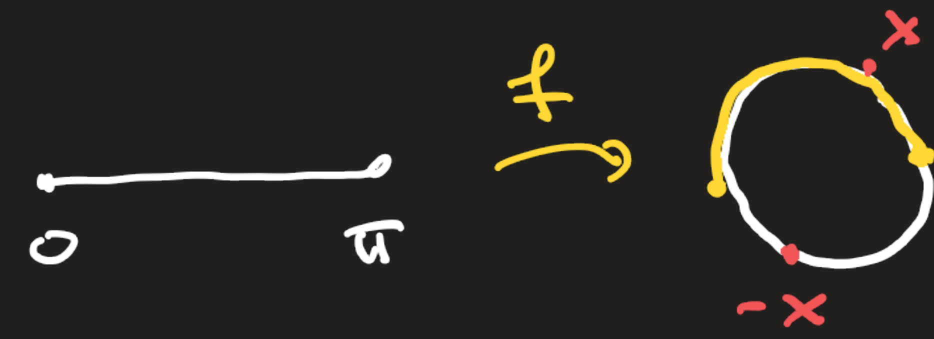
Conclusion: $\mathbb{RP}^n$ is an n-dim. topological manifold.

Ex $n=1$ Claim: There is a homeomorphism $\mathbb{R}P^1 \cong S^1$.

Define
$$\begin{tikzcd}
{[0,\pi]} \arrow[r, "t \mapsto (\cos t, \sin t)"] \arrow[d] & S^1 \arrow[r] & S^1/x\sim -x = \mathbb{R}P^1 \\
{[0,\pi]/0\sim\pi} \arrow[urr] & &
\end{tikzcd}$$

$S^1 \xleftarrow{\cong} [0,\pi]/0\sim\pi$

$(\cos 2t, \sin 2t) \mapsfrom t$



The induced map $[0,\pi]/0\sim\pi \to \mathbb{R}P^1$ is a cont. bijection. Since $[0,\pi]/0\sim\pi$ is compact and $\mathbb{R}P^1$ is Hausdorff, this is a homeomorphism. Also $[0,\pi]/0\sim\pi \cong S^1$.

Fact: For $n \geq 2$, $\mathbb{R}P^n$ is not homeomorphic to S^n.

Ex $n=2$

$\mathbb{R}P^2 \cong$  $/\sim$ $\cong$  $/\sim$ $\cong$  $/ x\sim -x$ for $x \in S^1$.

shall later see that S^2 is "orientable" but $\mathbb{R}P^2$ is not.

The standard atlas on $\mathbb{RP}^n = \mathbb{R}^{n+1} - \{0\} / \sim$

$U_i = \{ [a^0, .., a^n] : a^i \neq 0 \}$, $\quad \mathbb{RP}^n = \bigcup_{i=0}^{n} U_i$

$\phi_i : U_i \longrightarrow \mathbb{R}^n$, $\quad \phi_i [a^0, .. a^n] = \frac{1}{a^i} (a^0, .., \hat{a}^i, .., a^n)$

Have checked ϕ_i is a homeomorphism with inverse

$\phi_i^{-1} : \mathbb{R}^n \to U_i$, $\phi_i^{-1}(x^1, .., x^n) = [x^1, .., x^i, 1, x^{i+1}, .., x^n]$

Transition functions $\quad \{ [a^0, .., a^n] : a^i \neq 0, a^j \neq 0 \} = U_i \cap U_j \quad$ Suppose $i < j$.

$$\begin{tikzcd}
 & U_i \cap U_j \arrow[dl, "\phi_i"'] \arrow[dr, "\phi_j"] & \\
\{(x^1 .. x^n) : x^j \neq 0\} = \phi_i(U_i \cap U_j) & & \phi_j(U_i \cap U_j) = \{(x^1 .. x^n) : x^{i+1} \neq 0\}
\end{tikzcd}$$

$$\phi_j \circ \phi_i^{-1} : (x^1, .., x^n) \xmapsto{\phi_i^{-1}} [\overset{a^0}{x^1}, .. \overset{a^{i-1}}{x^i}, \overset{a^i}{1}, \overset{a^{i+1}}{x^{i+1}}, .. x^n]$$

$$\xmapsto{\phi_j} \frac{1}{x^j} (x^1, .. x^i, 1, x^{i+1}, .. \hat{x}^j, ..., x^n) \quad \text{this is } C^\infty.$$

similar argument for $i > j$.

<u>Conclusion</u> $\mathbb{RP}^n$ with standard atlas is a smooth manifold.