The tangent bundle

Let M be a smooth manifold, $\dim M = n$.

Def The tangent bundle of M is the (disjoint) union of the tangent space $TM = \bigcup_{p \in M} T_p M$

Notice: If $p \neq q$, then $T_p M \cap T_q M = \emptyset$

Rmk If $M \subseteq \mathbb{R}^N$ is a submanifold, then

$$i_{*,p} : T_p M \longrightarrow T_p \mathbb{R}^N \cong \mathbb{R}^N$$

is injective for all $p \in M$. The images of $T_p M$ and $T_q M$ in $\mathbb{R}^N$ will intersect even though $T_p M$ and $T_q M$ do not.

Ex $M = S^2 \subseteq \mathbb{R}^3$

We often write elements of $TM = \bigcup_{p \in M} T_pM$ in the form (p, v) for $v \in T_pM$.

Goal: Show TM is a smooth manifold of dimension $2n$.

First we need to define a topology on TM.

Let $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$ be a chart on M.

Write $TU = \bigcup_{p \in U} T_p U = \bigcup_{p \in U} T_p M$

For each $p \in U$ we have an isomorphism

$$\phi_{*,p} : T_p M \longrightarrow T_{\phi(p)} \phi(U) \cong \mathbb{R}^n$$

Hence we get a bijective map

$$\tilde{\phi} : TU \longrightarrow \phi(U) \times \mathbb{R}^n, \quad (p, v) \mapsto (\phi(p), \phi_{*,p}(v))$$

<u>Def</u> A subset $A \subseteq TU$ is open iff $\tilde{\phi}(A)$ is open in $\phi(U) \times \mathbb{R}^n$

With this topology $\tilde{\phi}$ is a homeomorphism

Let (U, ϕ) and (V, ψ) be charts on M.
Then $TU \cap TV = T(U \cap V)$ by definition

<u>Lemma</u> The subspace topology on $T(U \cap V)$ inherited from TU is the same as the subspace topology inherited from TV.

<u>Proof</u> The commutative diagram

$$\begin{tikzcd} & U \cap V \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ \phi(U \cap V) \arrow[rr, "\psi \circ \phi^{-1}", "\text{diffeo}"'] & & \psi(U \cap V) \end{tikzcd}$$

gives $\rightsquigarrow$

$$\begin{tikzcd} & T(U \cap V) \arrow[dl, "\tilde{\phi}"'] \arrow[dr, "\tilde{\psi}"] & \\ \phi(U \cap V) \times \mathbb{R}^n \arrow[rr] & & \psi(U \cap V) \times \mathbb{R}^n \\ (u, v) \arrow[rr, mapsto, "\text{diffeo}"'] & & (\psi \circ \phi^{-1}(u), (\psi \circ \phi^{-1})_{*,u}(v)) \end{tikzcd}$$

The commutativity of the last diagram gives the result.
($\mathbb{R}^n \xrightarrow{\phi_{*,p}^{-1}} T_p(U \cap V) \xrightarrow{\psi_{*,p}} \mathbb{R}^n$ equals $(\psi \circ \phi^{-1})_{*,\phi(p)}$ by the chain rule) $\square$

Let $\mathcal{B}$ be the following collection of subsets of TM:

$$\mathcal{B} = \bigcup_{\{(U,\phi) \text{ charts on } M\}} \{A \subseteq TU : \tilde{\phi}(A) \subseteq \phi(U) \times \mathbb{R}^n \text{ is open}\}$$

<u>Proposition</u> The collection $\mathcal{B}$ is a basis for a topology on TM. This topology is second countable and Hausdorff.

<u>Proof</u> It follows from the previous lemma that $\mathcal{B}$ is a basis for a topology.

Next check that the topology is second countable:

Let $\mathcal{C}$ be a countable basis for the topology on M.

For each chart (U_α, ϕ_α) and each $p \in U_\alpha$ choose $B_{\alpha,p} \in \mathcal{C}$ such that $p \in B_{\alpha,p} \subseteq U_\alpha$. Then $\{B_{\alpha,p}\} \subseteq \mathcal{C}$ is countable.

Write $\{B_{d,p}\} = \{U_i : i \geq 1\}$. This is a basis for the topology on M by construction. For each i there is a chart $\phi_i : U_i \to \phi_i(U_i) \subseteq \mathbb{R}^n$. Hence there is a countable atlas $\{(U_i, \phi_i)\}$ on M.

For each i have $\tilde{\phi}_i : TU_i \xrightarrow{\cong} \phi_i(U_i) \times \mathbb{R}^n$

Choose for each i a countable basis $\{B_{i,j} : j \geq 1\}$ for TU_i.

Then $\{B_{i,j} : i,j \geq 1\}$ is a countable basis for the topology on TM.

<u>Exercise for next week</u>: TM is Hausdorff. $\square$

__Theorem__ Let M be a smooth n-dimensional manifold. Then TM is a smooth manifold of dimension $2n$.

__Proof__ We already know that TM is a topological manifold with charts $\tilde{\phi} : TU \to \phi(U) \times \mathbb{R}^n$.
In order to see the charts are C^∞ compatible we again consider the commutative diagrams

$$\begin{tikzcd}
& TU \cap TV \arrow[dl, "\tilde{\phi}"'] \arrow[dr, "\tilde{\psi}"] & \\
\phi(U \cap V) \times \mathbb{R}^n \arrow[rr] & & \psi(U \cap V) \times \mathbb{R}^n
\end{tikzcd}$$

$$(u, v) \longmapsto (\psi \circ \phi^{-1}(u), (\psi \circ \phi^{-1})_{*,u}(v))$$

diffeomorphism

□