<u>Recall</u> An n-dimensional topological manifold is a topological space M such that

(i) For each $p \in M$, there exists a neighborhood $p \in U \subseteq M$ and a homeomorphism $\phi : U \to U' \subseteq \mathbb{R}^n$, where $U' = \phi(U)$ is an open subset of $\mathbb{R}^n$

(ii) M is second countable

(iii) M is Hausdorff.

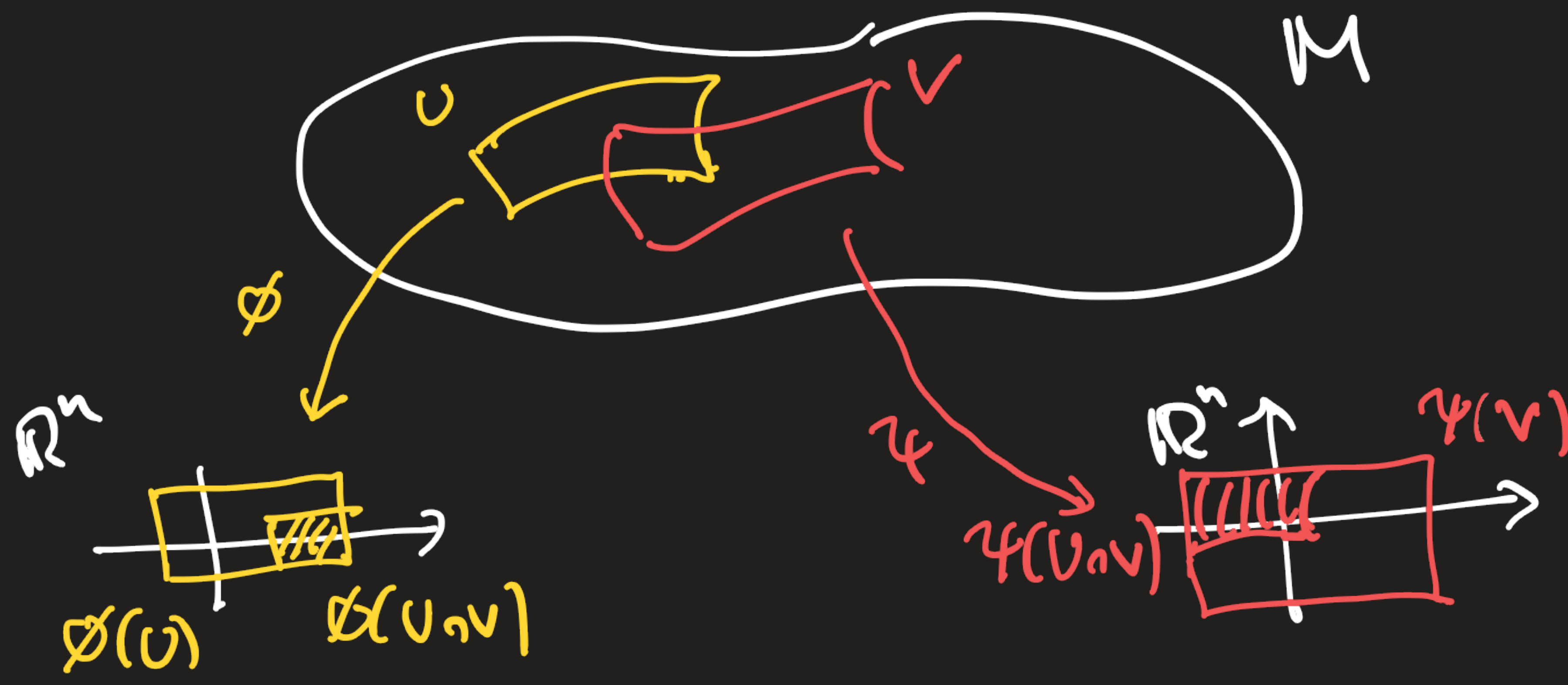
<u>Recall</u> A function $f : X \to Y$ between topological spaces X and Y is continuous if:

$$V \subseteq Y \text{ open} \implies f^{-1}(V) \subseteq X \text{ open}$$

A continuous function is a homeomorphism if it is a bijection and the inverse is also continuous.

Let M be topological manifold.

Consider charts (U,ϕ), (V,ψ) on M.



Def ϕ and ψ are C^∞ compatible if the transition functions

$$\phi(U\cap V) \xrightarrow{\phi^{-1}} U\cap V \xrightarrow{\psi} \psi(U\cap V)$$
$$\psi(U\cap V) \xrightarrow{\psi^{-1}} U\cap V \xrightarrow{\phi} \phi(U\cap V)$$

are C^∞. (hence diffeomorphisms)

Def A C^∞ atlas on a topological manifold M is a collection of charts $\{(U_\alpha,\phi_\alpha)\}$ such that $M = \bigcup U_\alpha$ and the charts are pairwise C^∞ compatible.

Ex $M = S^1$

$U_1^+ = \{(x,y) : x > 0\}$, $U_1^- = \{(x,y) : x < 0\}$

$U_2^+ = \{(x,y) : y > 0\}$, $U_2^- = \{(x,y) : y < 0\}$.

$$(0,1) = \phi_1^+(U_1^+ \cap U_2^+) \xrightarrow{(\phi_1^+)^{-1}} U_1^+ \cap U_2^+ \xrightarrow{\phi_2^+} \phi_2^+(U_1^+ \cap U_2^+) = (0,1)$$

$$t \longmapsto (\sqrt{1-t^2}, t) \longmapsto \sqrt{1-t^2} \quad C^\infty.$$

Similarly, one can check that the other transition functions are C^∞.

We have thus defined a C^∞ atlas on S^1.

A C^∞ atlas $\mathcal{M} = \{(U_\alpha, \phi_\alpha)\}$ is <u>maximal</u> if for any other C^∞ atlas $\mathcal{M}'$ on M such that $\mathcal{M} \subseteq \mathcal{M}'$, we have $\mathcal{M} = \mathcal{M}'$.

<u>Def</u> • A smooth (or C^∞) n-dimensional manifold is an n-dimensional topological manifold together with a maximal C^∞ atlas.

• A maximal C^∞ atlas is also called a differentiable structure.

<u>Prop</u> Any C^∞ atlas on a topological manifold is contained in a unique maximal C^∞ atlas.

<u>Proof</u> Let $\{(U_\alpha, \phi_\alpha)\}$ be a C^∞ atlas on M.

Let $\mathcal{M}$ be the set of all charts (U, ϕ) that are C^∞ compatible with all the charts (U_α, ϕ_α).

<u>Claim</u> $\mathcal{M}$ is a C^∞ atlas.

Given charts (U, ϕ) and (V, ψ) that are C^∞ compatible with all the charts (U_α, ϕ_α). Must show that

$$\phi(U \cap V) \xrightarrow{\phi^{-1}} U \cap V \xrightarrow{\psi} \psi(U \cap V) \quad \text{is } C^\infty.$$

Let $p \in U \cap V$, choose a chart (U_α, ϕ_α) s.t. $p \in U_\alpha$.

Then $p \in U \cap V \cap U_\alpha$

$$\begin{tikzcd}
\phi(U \cap V \cap U_\alpha) & U \cap V \cap U_\alpha \arrow[l, "\phi"'] \arrow[r, "\psi"] \arrow[d, "\phi_\alpha"] & \psi(U \cap V \cap U_\alpha) \\
& \phi_\alpha(U \cap V \cap U_\alpha) &
\end{tikzcd}$$

Then $\psi \circ \phi^{-1} = (\psi \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \phi^{-1})$ C^∞ on $\phi(U \cap V \cap U_\alpha)$

Clearly $\mathcal{M}$ is maximal by construction.

If $\mathcal{M}'$ is another maximal atlas that contains $\{(U_\alpha, \phi_\alpha)\}$ then $\mathcal{M}' \subseteq \mathcal{M}$, hence $\mathcal{M}' = \mathcal{M}$ by maximality $\square$

Consequence: We can specify a smooth manifold by finding some C^∞ atlas

Ex: $U \subseteq \mathbb{R}^n$ be open, $f: U \to \mathbb{R}$ a C^∞ function.

Graph of f: $\Gamma(f) = \{(x, f(x)) \in \mathbb{R}^{n+1} : x \in U\}$

Let $\phi: \Gamma(f) \to U$, $(x, f(x)) \mapsto x$ homeomorphism

Hence $\{\phi\}$ defines an atlas on $\Gamma(f)$ with a single chart. This makes $\Gamma(f)$ a C^∞ manifold.

Rmk: Kervaire proved (~ 1960) that there exist topological manifolds for which there is no C^∞ compatible atlas - Kervaire's example has dimension 10.