Ex. 5.5 Let M be an m-dim. smooth manifold
Let N be an n-dim smooth manifold

We want to make $M \times N$ an $m+n$ dim. smooth manifold.

Check $M \times N$ is locally Euclidian:

Given $(x,y) \in M \times N$, choose charts $\phi : U \to U' \subseteq \mathbb{R}^m$ on M
$\psi : V \to V' \subseteq \mathbb{R}^n$ on N

s.t. $(x,y) \in U \times V$. Then

$\phi \times \psi : U \times V \longrightarrow U' \times V' \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is a chart on $M \times N$.

Suppose $\{(U_\alpha, \phi_\alpha)\}$ smooth atlas on M
$\{(V_\beta, \psi_\beta)\}$ smooth atlas on N.

Show $\{(U_\alpha \times V_\beta, \phi_\alpha \times \psi_\beta)\}$ is a smooth atlas on $M \times N$.

$$(U_{\alpha_1} \times V_{\beta_1}) \cap (U_{\alpha_2} \times V_{\beta_2}) = (U_{\alpha_1} \cap U_{\alpha_2}) \times (V_{\beta_1} \cap V_{\beta_2})$$

$$\begin{tikzcd}[column sep=small] & (U_{\alpha_1}\cap U_{\alpha_2})\times(V_{\beta_1}\cap V_{\beta_2}) \arrow[dl, "\phi_{\alpha_1}\times\psi_{\beta_1}"'] \arrow[dr, "\phi_{\alpha_2}\times\psi_{\beta_2}"] & \\ \phi_{\alpha_1}(U_{\alpha_1}\cap U_{\alpha_2})\times\psi_{\beta_1}(V_{\beta_1}\cap V_{\beta_2}) & & \phi_{\alpha_2}(U_{\alpha_1}\cap U_{\alpha_2})\times\psi_{\beta_2}(V_{\beta_1}\cap V_{\beta_2}) \end{tikzcd}$$

$$(\phi_{\alpha_2}\times\psi_{\beta_2})\circ(\phi_{\alpha_1}\times\psi_{\beta_1})^{-1} = (\phi_{\alpha_2}\circ\phi_{\alpha_1}^{-1})\times(\psi_{\beta_2}\circ\psi_{\beta_1}^{-1}) \quad C^\infty.$$

<u>Ex 6.1</u> charts on $\mathbb{R}$: $(\mathbb{R}, id:\mathbb{R}\to\mathbb{R})$

$\mathbb{R}'$: $(\mathbb{R}, \psi:\mathbb{R}\to\mathbb{R})$, $\psi(x)=x^{1/3}$.

(a) show that these smooth structures on $\mathbb{R}$ are distinct.
Since ψ is a smooth function in second smooth structure, but not in first smooth structure, these are distinct.

(b) show they are diffeomorphic.
In fact ψ gives a diffeomorphism from $\mathbb{R}'$ to $\mathbb{R}$.

$$\begin{tikzcd} \mathbb{R}' \arrow[r, "\psi"] \arrow[d, "\psi"'] & \mathbb{R} \arrow[d, "id"] \\ \mathbb{R} & \mathbb{R} \end{tikzcd}$$

$id\circ\psi\circ\psi^{-1} = id: \mathbb{R}^n\to\mathbb{R}^n$, hence ψ is smooth with this structure.

Ex. 6.2

Let M and N be smooth manifolds and $q_0 \in N$.

Show $i_{q_0} : M \to M \times N$, $i_{q_0}(p) = (p, q_0)$ is smooth.

Given $p \in M$, choose charts $\phi : U \to U' \subseteq \mathbb{R}^m$ on M

$\psi : V \to V' \subseteq \mathbb{R}^n$ on N

s.t. $(p, q_0) \in U \times V$.

$$\begin{array}{ccc}
M & \xrightarrow{i_{q_0}} & M \times N \\
\cup\!\vert & & \cup\!\vert \\
U & \longrightarrow & U \times V \\
\phi \downarrow & & \downarrow \phi \times \psi \\
U' & \dashrightarrow & U' \times V' \\
x & \longmapsto & (x, \psi(q_0))
\end{array}$$

C^∞.

Ex C

(a) Show S^1 is a Lie group.

(i) Show $\mu : S^1 \times S^1 \to S^1$ is smooth.

(ii) Show inverse $S^1 \to S^1$ $z \mapsto z^{-1}$ is smooth.

For (i) use charts $(x-\varepsilon, x+\varepsilon) \xrightarrow{\phi^{-1}} U = \{ e^{it} : t \in (x-\varepsilon, x+\varepsilon)\}$

Then $\phi : U \to (x-\varepsilon, x+\varepsilon)$, $e^{it} \mapsto t$ is a chart when $0 < \varepsilon < \pi$.

$$\begin{tikzcd}
e^{ix} \quad e^{iy} & & e^{i(x+y)} \\
S^1 \times S^1 \arrow[rr, "\mu"] & & S^1 \\
\cup\!\!\mid \quad \cup\!\!\mid & & \cup\!\!\mid \\
U \times V \arrow[d, "\phi \times \psi"'] & & W \arrow[d] \\
(x-\varepsilon, x+\varepsilon) \times (y-\delta, y+\delta) \arrow[rr] & & (x+y-(\varepsilon+\delta), x+y+(\varepsilon+\delta))
\end{tikzcd}$$

$(s,t) \longmapsto s+t$ $\quad C^\infty$.

Choose ε, δ s.t. $\varepsilon + \delta < \pi$.

For (ii), locally we have

$$e^{xi} \longmapsto e^{-xi}$$

$$\begin{tikzcd}
S^1 \arrow[r] & S^1 \\
U \arrow[u, phantom, sloped, "\subseteq"] \arrow[d] & V \arrow[u, phantom, sloped, "\subseteq"] \arrow[d] \\
(x-\varepsilon, x+\varepsilon) \arrow[r] & (-x-\varepsilon, x+\varepsilon)
\end{tikzcd}$$

$t \longmapsto -t$ is C^∞.