Differential 1-forms and the cotangent bundle

Let M be a smooth manifold, $p \in M$.

Tangent space at p:

$$T_pM = \{ \text{point derivations } v : C_p^\infty(M) \to \mathbb{R} \}.$$

The cotangent space at p is

$$T_p^*M = \mathrm{Hom}_{\mathbb{R}}(T_pM, \mathbb{R})$$

<u>Def</u> A 1-form on M is a collection of cotangent vectors $\omega = \{ \omega_p \in T_p^*M : p \in M \}$

<u>Def</u> A smooth function $f: M \to \mathbb{R}$ defines a 1-form:

$$df = \{ df_p \in T_p^* M : p \in M \}$$

Here $df_p : T_p M \to \mathbb{R}$ is defined by $df_p(v) = v(f)$.
This is the <u>differential</u> of f.

Let $\phi = (x^1, \dots, x^n) : U \to \mathbb{R}^n$ be a chart on M.
There are associated 1-forms on U:

$$dx^i = \{ dx^i_p : p \in U \}.$$

<u>Prop</u> The cotangent vectors $dx^1_p, \dots, dx^n_p$ define the dual basis to the basis $\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p$ for $T_p M$.

Proof We calculate

$$dx^i_p\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \frac{\partial}{\partial x^j}\Big|_p(x^i) = \frac{\partial x^i}{\partial x^j}(p) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j. \end{cases}$$

$$\left(\text{By definition: } \frac{\partial x^i}{\partial x^j}(p) = \frac{\partial(x^i\circ\phi^{-1})}{\partial r^j}(\phi(p)) = \frac{\partial r^i}{\partial r^j}(\phi(p))\right) \quad \square$$

Consequence: Locally in a chart domain $(U, x^1, \ldots, x^n)$, a 1-form ω can be written uniquely

$$\omega_p = \sum_{i=1}^n a_i(p)\, dx^i_p \quad , \quad a_i(p) \in \mathbb{R}.$$

Hence the restriction $\omega|_U$ can be written

$\omega|_U = \sum_{i=1}^n a_i\, dx^i$ for unique functions $a_1, \ldots, a_n : U \to \mathbb{R}$.

Notice: $a_i(p) = \omega_p\left(\frac{\partial}{\partial x^i}\Big|_p\right)$.

The cotangent bundle

Let $T^*M = \bigcup_{p \in M} T_p^* M$

We must define a topology on T^*M.

Let $\phi = (x^1, \dots, x^n) : U \to \mathbb{R}^n$ be a chart on M.

Write $T^*U = \bigcup_{p \in U} T_p^* M$

Bijection

$$\tilde{\phi} : T^*U \xrightarrow{\sim} \phi(U) \times \mathbb{R}^n$$

$$\omega_p \longmapsto \left(\phi(p), \omega_p\left(\frac{\partial}{\partial x^1}\Big|_p\right), \dots, \omega_p\left(\frac{\partial}{\partial x^n}\Big|_p\right) \right)$$

$$\sum_{i=1}^n c_i \, dx^i_{\phi^{-1}(u)} \longleftarrow (u, c_1, \dots, c_n)$$

We use this to transfer the standard topology on $\phi(U) \times \mathbb{R}^n$ to T^*U, so that $\tilde{\phi}$ becomes a homeomorphism.

Let (U, ϕ) and (V, ψ) be charts on M.

Then $T^*U \cap T^*V = T^*(U \cap V)$

<u>Lemma</u> The subspace topology on $T^*(U \cap V)$ inherited from T^*U is the same as the subspace topology inherited from T^*V.

<u>Proof</u> The commutative diagram

$$\begin{tikzcd} & U \cap V \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ \phi(U \cap V) \arrow[rr, "\psi \circ \phi^{-1}", "\text{diffeo}"'] & & \psi(U \cap V) \end{tikzcd} \quad \overset{\text{gives}}{\rightsquigarrow} \quad \begin{tikzcd} & T^*(U \cap V) \arrow[dl, "\tilde{\phi}"'] \arrow[dr, "\tilde{\psi}"] & \\ \phi(U \cap V) \times \mathbb{R}^n \arrow[rr, "\tilde{\psi} \circ \tilde{\phi}^{-1}", "\text{diffeo}"'] & & \psi(U \cap V) \times \mathbb{R}^n \end{tikzcd}$$

$$(u, v) \longmapsto \left(\psi \circ \phi^{-1}(u), \left[(\phi \circ \psi^{-1})_{*, \psi \circ \phi^{-1}(u)} \right]^T (v)\right)$$

Check formula for $\tilde{\psi} \circ \tilde{\phi}^{-1}$: $(u, c_1 .. c_n) \in \phi(U \cap V) \times \mathbb{R}^n$, let $p = \phi^{-1}(u)$

$$(u, c_1, .., c_n) \xrightarrow{\tilde{\phi}^{-1}} \sum_{i=1}^{n} c_i \, dx^i_p \in T^*_p M$$

write
$\phi = (x^1, .. x^n)$
$\psi = (y^1, .., y^n)$

$$\xrightarrow{\tilde{\psi}} \left(\psi \circ \phi^{-1}(u), \sum_{i=1}^{n} c_i \, dx^i_p \left(\frac{\partial}{\partial y^1}\Big|_p \right), .., \sum_{i=1}^{n} c_i \, dx^i_p \left(\frac{\partial}{\partial y^n}\Big|_p \right) \right)$$

$$= \left(\psi \circ \phi^{-1}(u), \sum_{i=1}^{n} c_i \frac{\partial x^i}{\partial y^1}(p), .., \sum_{i=1}^{n} c_i \frac{\partial x^i}{\partial y^n}(p) \right)$$

$$= \left(\psi \circ \phi^{-1}(u), \begin{bmatrix} \frac{\partial x^1}{\partial y^1}(p) & \cdots & \frac{\partial x^n}{\partial y^1}(p) \\ \vdots & & \vdots \\ \frac{\partial x^1}{\partial y^n}(p) & \cdots & \frac{\partial x^n}{\partial y^n}(p) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \right)$$

Transpose of the Jacobian for $\phi \circ \psi^{-1}$.

$\square$

Let $\mathcal{B}$ be the following collection of subsets of T^*M:

$$\mathcal{B} = \bigcup_{\{(U,\phi) \text{ charts on } M\}} \{ A \subseteq T^*U : \tilde{\phi}(A) \subseteq \phi(U) \times \mathbb{R}^n \text{ is open} \}$$

<u>Prop</u> The collection $\mathcal{B}$ is a basis for a topology on T^*M. This topology is second countable and Hausdorff.

<u>Proof</u>: As for TM, using the lemma. $\square$

<u>Theorem</u> Let M be a smooth n-dimensional manifold. Then T^*M is a smooth manifold of dimension $2n$.

<u>Proof</u> We already know T^*M is a topological manifold with charts $\tilde{\phi} : T^*U \longrightarrow \phi(U) \times \mathbb{R}^n$.

In order to see that the charts are C^∞ compatible, we again consider the commutative diagrams

$$\begin{tikzcd}
 & T^*(U \cap V) \arrow[dl, "\tilde{\phi}"'] \arrow[dr, "\tilde{\psi}"] & \\
\phi(U \cap V) \times \mathbb{R}^n \arrow[rr, "\tilde{\psi} \circ \tilde{\phi}^{-1}"] & & \psi(U \cap V) \times \mathbb{R}^n
\end{tikzcd}$$

$$(u, v) \longmapsto \left((\psi \circ \phi^{-1})(u), \left[(\phi \circ \psi^{-1})_{*, \psi \circ \phi^{-1}(u)} \right]^T (v) \right)$$

C^∞ map. $\square$

<u>Theorem</u> The cotangent bundle $\pi : T^*M \to M$, $\omega_p \mapsto p$, is a smooth vector bundle of dimension n.

<u>Proof</u> Every chart $\phi = (x^1, \dots, x^n)$ gives rise to a local trivialization $T^*U \to U \times \mathbb{R}^n$, $\omega_p = \sum c_i \, dx^i_p \mapsto (p, c_1, \dots, c_n)$ $\square$

Notice A 1-form $\omega = \{ \omega_p \in T_p^* M : p \in M \}$ gives rise to a section $M \to T^*M$ of the cotangent bundle.

Def A smooth 1-form on M is a smooth section $\omega : M \to T^*M$ of the cotangent bundle. The vector space of smooth 1-forms on M is denoted $\Omega^1(M)$.

Rem If $\phi = (x^1, \dots x^n) : U \to \mathbb{R}^n$ is a chart, then the 1-forms $dx^1, \dots, dx^n$ give a local frame of T^*M over U: $dx^1_p, \dots, dx^n_p$ is a basis of T_p^*M for all $p \in U$.

As we remarked earlier, if $\omega = \{ \omega_p \in T_p^* M : p \in M \}$ is a 1-form on U, then $\omega|_U = \sum_{i=1}^n a_i \, dx^i$, for unique functions $a_1, \dots a_n : U \to \mathbb{R}$.

<u>Prop</u> Let $\omega = \{\omega_p \in T_p^* M : p \in M\}$ be a 1-form on M.
Then the following conditions are equivalent:

(i) ω is smooth

(ii) For each $p \in M$, there exists a chart $(U, x^1, \dots, x^n)$ s.t. $p \in U$ and $\omega_{|U} = \sum_{i=1}^n a_i dx^i$ for smooth $a_1, \dots, a_n : U \to \mathbb{R}$.

(iii) For any chart $(U, x^1, \dots, x^n)$, $\omega_{|U} = \sum_{i=1}^n a_i dx^i$ for smooth functions $a_1, \dots, a_n : U \to \mathbb{R}$.

<u>Proof</u> For any chart $(U, x^1, \dots, x^n)$, if $\omega_{|U} = \sum_{i=1}^n a_i dx^i$, then there is a comm. diagram

$$\begin{tikzcd}
T^*U \arrow[rr, "\tilde{\phi}_*"] & & \phi(U) \times \mathbb{R}^n \\
& U \arrow[ul, "\omega_{|U}"] \arrow[ur, "(\phi(u), a_1(u), \dots, a_n(u))"'] &
\end{tikzcd}$$

$u \mapsto (\phi(u), a_1(u), \dots, a_n(u))$

This gives the result. $\square$

Example Let $f: M \to \mathbb{R}$ be a smooth function. For any chart $(U, x^1 \dots x^n)$, we have $df_{|U} = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i$.

Reason: If $df_{|U} = \sum_{i=1}^{n} a_i \, dx^i$, then

$$a_i(p) = df_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial}{\partial x^i}\Big|_p(f) = \frac{\partial f}{\partial x^i}(p).$$

Consequence df is a smooth 1-form on M.

Pull back of smooth functions and smooth 1-forms

Let $F: N \to M$ be a smooth function.

Pull back of functions:

$F^*: C^\infty(M) \to C^\infty(N)$, $F^*(f) = f \circ F : N \to M \xrightarrow{f} \mathbb{R}$.

We want to define a linear transformation

$F^*: \Omega^1(M) \longrightarrow \Omega^1(N)$.

Given $\omega = \{\omega_q \in T_q^*M : q \in M\}$ must define

$F^*\omega = \{(F^*\omega)_p \in T_p^*N : p \in N\}$.

For $p \in N$, we have $F_{*,p}: T_pN \to T_{F(p)}M$

__Def__ $(F^*\omega)_p : T_pN \to \mathbb{R}$ is defined by

$(F^*\omega)_p(X_p) = \omega_{F(p)}(F_{*,p}(X_p))$, for $X_p \in T_pN$.

We shall later see that $F^*\omega$ is smooth.

Lemma Let $f: M \to \mathbb{R}$ be smooth. Then $F^*(df) = d(F^*f)$

Proof Must show $F^*(df)_p = d(F^*f)_p \in T_p^*N$ for $p \in N$.

Let $X_p \in T_pN$.

$F^*(df)_p(X_p) = (df)_{F(p)}(F_{*,p}(X_p)) = F_{*,p}(X_p)(f) = X_p(f \circ F)$

$d(F^*f)_p(X_p) = X_p(F^*(f)) = X_p(f \circ F)$. $\square$

Exercise

- $F^*(\omega + \tau) = F^*(\omega) + F^*(\tau)$ for $\omega, \tau \in \Omega^1(M)$
- $F^*(f\omega) = F^*(f) \cdot F^*(\omega)$ for $f \in C^\infty(M)$ and $\omega \in \Omega^1(M)$.

Prop Let $F: N \to M$ be smooth. Given a smooth 1-form $\omega \in \Omega^1(M)$, the pullback $F^*\omega$ is a smooth 1-form $F^*\omega \in \Omega^1(N)$.

Proof Let $p \in N$. Choose charts $\psi = (y^1, \dots, y^m): V \to \mathbb{R}^m$ on M, $\phi = (x^1, \dots, x^n): U \to \mathbb{R}^n$ on N such that $F(U) \subseteq V$.

$$\begin{array}{ccc} N & \xrightarrow{F} & M \\ \cup & & \cup \\ U & \xrightarrow{F} & V \\ \phi \downarrow & & \psi \downarrow \\ \mathbb{R}^n \supseteq \phi(U) & & \psi(V) \subseteq \mathbb{R}^m \end{array}$$

We know $\omega|_V = \sum_{i=1}^m a_i \, dy^i$, for smooth functions $a_1, \dots, a_m : V \to \mathbb{R}$. The restriction of $F^*\omega$ to U is

$$(F^*\omega)|_U = F^*\left(\sum_{i=1}^m a_i \, dy^i\right) = \sum_{i=1}^m F^* a_i \, F^*(dy^i) = \sum_{i=1}^m F^* a_i \cdot d(y^i \circ F)$$

write $F^i = y^i \circ F$

$$= \sum_{i=1}^m F^*(a_i) \sum_{j=1}^n \frac{\partial F^i}{\partial x^j} dx^j = \sum_{i,j} F^*(a_i) \frac{\partial F^i}{\partial x^j} dx^j$$

which is smooth on U. $\square$