Recall V finite dim. real vectorspace

$L_k(V)$ vectorspace of k-linear functions $f: V^k \to \mathbb{R}$.

$A_k(V)$ subspace of k-linear alternating functions:

$$f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma) f(v_1, \dots v_k) \text{ for all } \sigma \in S_k.$$

Alternating operator $A: L_k(V) \to A_k(V)$, $A(f) = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \, \sigma f$,

where $(\sigma f)(v_1, \dots, v_k) = f(v_{\sigma(1)}, \dots, v_{\sigma(k)})$.

Tensor product $\otimes : L_k(V) \times L_\ell(V) \to L_{k+\ell}(V)$, where

$$f \otimes g \, (v_1, \dots, v_k, v_{k+1}, \dots, v_{k+\ell}) = f(v_1, \dots v_k) \, g(v_{k+1}, \dots, v_{k+\ell}).$$

Notice: $\otimes$ is bilinear and associative:

$$(f \otimes g) \otimes h = f \otimes (g \otimes h).$$

Def. The wedge product of alternating multilinear functions $\wedge : A_k(V) \times A_\ell(V) \to A_{k+\ell}(V)$.

$$f \wedge g = \frac{1}{k!\,\ell!} A(f \otimes g)$$

Explicitly $f \wedge g\,(v_1, \dots v_{k+\ell}) = \frac{1}{k!\,\ell!} A(f \otimes g)\,(v_1, \dots v_{k+\ell})$

$$= \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma)\, \sigma(f \otimes g)\,(v_1, \dots v_{k+\ell})$$

$$= \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma)\, f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \cdot g(v_{\sigma(k+1)}, \dots v_{\sigma(k+\ell)})$$

For $k=0$ or $\ell=0$, $c \in A_0(V) = \mathbb{R}$ $\quad c \wedge f = f \wedge c = c f$

Ex. $\wedge : A_1(V) \times A_1(V) \to A_2(V)$

$$f \wedge g\,(v_1, v_2) = f(v_1)\, g(v_2) - f(v_2)\, g(v_1).$$

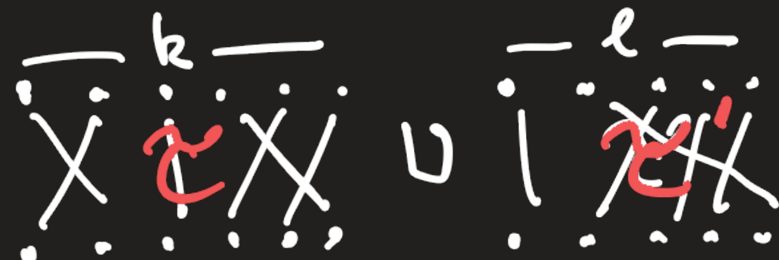
Reason for the factor $\frac{1}{k!\,\ell!}$: The terms in the sum

$$\sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma)\, \sigma(f\otimes g)$$

can be partitioned into classes with $k!\,\ell!$ equal terms:

Consider $S_k \times S_\ell \longrightarrow S_{k+\ell}$, $(\tau, \tau') \mapsto \tau \sqcup \tau'$,

where τ acts on $\{1, .. k\}$ and τ' acts on $\{k+1, .. k+\ell\}$



Notice, if $f \in A_k(V)$ and $g \in A_\ell(V)$, then

$$\operatorname{sgn}(\tau \sqcup \tau') \cdot (\tau \sqcup \tau')(f \otimes g) = \operatorname{sgn}(\tau)\operatorname{sgn}(\tau')\,(\tau f) \otimes (\tau' g)$$

$$= \operatorname{sgn}(\tau)\operatorname{sgn}(\tau')\,(\operatorname{sgn}(\tau)\cdot f) \otimes (\operatorname{sgn}(\tau') g) = f \otimes g$$

Consequence $\operatorname{sgn}(\sigma)\,\sigma(f\otimes g) = \operatorname{sgn}(\sigma(\tau\sqcup\tau'))\cdot \sigma(\tau\sqcup\tau')(f\otimes g)$

for all $\tau \in S_k$ and $\tau' \in S_\ell$.

This gives $k! \cdot \ell!$ equal terms.

A permutation $\sigma \in S_{k+l}$ is a (k,l)-shuffle if

$$\sigma(1) < \sigma(2) < \ldots < \sigma(k) \quad \text{and} \quad \sigma(k+1) < \sigma(k+2) < \ldots < \sigma(k+l).$$

Fact: The (k,l)-shuffles give a representative for each class of equal terms such that

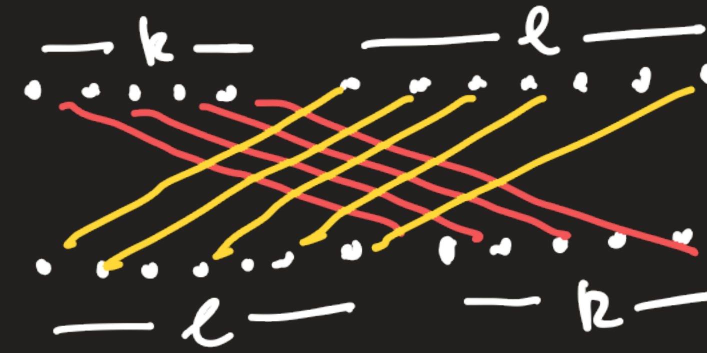
$$(f \wedge g)(v_1, \ldots, v_{k+l}) = \sum_{(k,l)\text{-shuffles } \sigma} \operatorname{sgn}(\sigma)\, f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdot g(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)})$$

Prop. The wedge product is anticommutative:

$$f \wedge g = (-1)^{k \cdot l} g \wedge f \quad \text{for } f \in A_k(V) \text{ and } g \in A_l(V).$$

Proof

Let $\tau = \begin{bmatrix} 1, \dots, l, & l+1, \dots & l+k \\ k+1, \dots, k+l, & 1, \dots, & k \end{bmatrix}$



Claim $\tau(g \otimes f) = f \otimes g$. Reason:

$$\tau(g \otimes f)(v_1, \dots, v_{k+l}) = (g \otimes f)(v_{\tau(1)}, \dots v_{\tau(k+l)})$$

$$= g(v_{k+1}, \dots v_{k+l}) \cdot f(v_1, \dots v_k) = (f \otimes g)(v_1, \dots v_{k+l})$$

Hence $f \wedge g = \frac{1}{k!\,l!} A(f \otimes g) = \frac{1}{k!\,l!} A(\tau(g \otimes f))$

$$= \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\, \sigma\tau(g \otimes f) = \frac{1}{k!\,l!} \operatorname{sgn}(\tau) \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma\tau)\, \sigma\tau(g \otimes f)$$

$$= \frac{1}{k!\,l!} \operatorname{sgn}(\tau) \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\, \sigma(g \otimes f) = \frac{1}{k!\,l!} \operatorname{sgn}(\tau) \cdot A(g \otimes f)$$

$= \operatorname{sgn}(\tau)\, g \wedge f = (-1)^{k \cdot l} g \wedge f$, since τ has $k \cdot l$ inversions. $\square$

Corollary If k is odd and $f \in A_k(V)$, then $f \wedge f = 0$.

Proof: $f \wedge f = (-1)^{k^2} f \wedge f = -f \wedge f \Rightarrow f \wedge f = 0$ $\square$