Oriented atlases

Let M be a smooth n-dim. manifold.

Given charts $\phi = (x^1, \ldots x^n) : U \to \mathbb{R}^n$, $\psi = (y^1, \ldots, y^n) : V \to \mathbb{R}^n$, then

$$\frac{\partial}{\partial x^j} = \sum_{i=1}^{n} \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i} \quad \text{on } U \cap V.$$

By definition, the change of basis matrix $\left(\frac{\partial y^i}{\partial x^j}(p) \right)$ is the Jacobian matrix $J(\psi \circ \phi^{-1})(\phi(p))$.

<u>Def</u> An atlas $\{(U_\alpha, \phi_\alpha = (x^1_\alpha, \ldots x^n_\alpha)\}$ is <u>oriented</u> if $\det\left(\frac{\partial x^i_\alpha}{\partial x^j_\beta} \right) > 0$ on $U_\alpha \cap U_\beta$ for all α, β.

<u>Theorem</u> A smooth manifold M is orientable iff it has an oriented atlas.

<u>Proof</u> Suppose first M is oriented. Then we can find an atlas $\{(U_\alpha, x^1_\alpha, \dots, x^n_\alpha)\}$ such that

$(dx^1_\alpha \wedge \dots \wedge dx^n_\alpha)_p (v_1, \dots, v_n) > 0$ for $(v_1 \dots v_n)$ pos. oriented basis for T_pM.

This is then an oriented atlas:

$$(dx^1_\alpha \wedge \dots \wedge dx^n_\alpha)_p (v_1 \dots v_n) = \det\left(\frac{\partial x^j_\alpha}{\partial x^i_\beta}(p)\right)(dx^1_\beta \wedge \dots \wedge dx^n_\beta)_p(v_1 \dots v_n)$$

for $p \in U_\alpha \cap U_\beta$ (by Cor. 18.4 in the textbook).

Next suppose M has an oriented atlas $\{(U_\alpha, x^1_\alpha \dots x^n_\alpha)\}$.

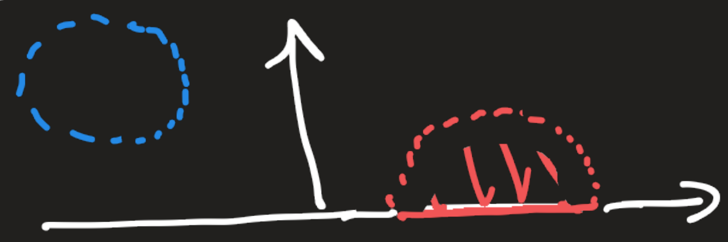
Then $\left(\frac{\partial}{\partial x^1_\alpha}\Big|_p, \dots, \frac{\partial}{\partial x^n_\alpha}\Big|_p\right)$ defines an orientation of T_pM for each $p \in U_\alpha$, independent of the choice of chart.

This is a continuous pointwise orientation of M $\square$

Manifold with boundary

The upper half-space in $\mathbb{R}^n$ is the set $\mathbb{H}^n = \{(x^1..x^n) \in \mathbb{R}^n : x^n \geq 0\}$ with the subspace topology from $\mathbb{R}^n$.

$n=2$



The boundary of $\mathbb{H}^n$ is the subspace $\partial \mathbb{H}^n = \{(x^1..x^n) \in \mathbb{H}^n : x^n = 0\}$

Points in $\mathbb{H}^n - \partial \mathbb{H}^n$ are called inner points in $\mathbb{H}^n$.

<u>Def</u> A n-dim. topological manifold with boundary M is a topological space such that

(i) For each $p \in M$, there exists a nbh U and a homeomorphism $\phi : U \to U' \subseteq \mathbb{H}^n$, where $U' = \phi(U)$ is an open subset of $\mathbb{H}^n$.

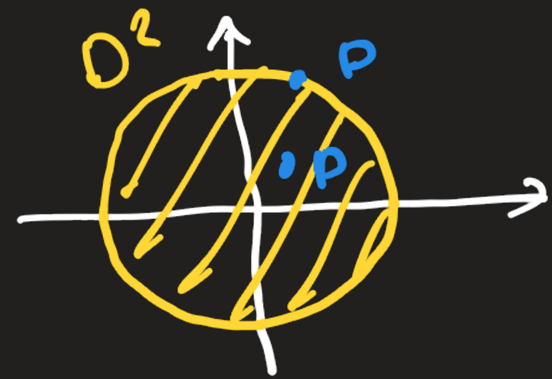
(ii) M is second countable

(iii) M is Hausdorff.

The homeomorphism $\phi : U \to U' \subseteq \mathbb{H}^n$ is a <u>chart</u> on M.

<u>Ex</u> $D^n = \{ x \in \mathbb{R}^n : \|x\| \leq 1 \}$ is an n-dim topological manifold with boundary

$n=2$



<u>Def</u> • Let $S \subseteq \mathbb{R}^n$ be an arbitrary subset. A map $f : S \to \mathbb{R}^m$ is said to be C^∞ if there exist an open set $U \subseteq \mathbb{R}^n$ and a C^∞ map $\tilde{f} : U \to \mathbb{R}^m$ such that $S \subseteq U$ and $\tilde{f}|_S = f$.

• Two arbitrary subsets $S, T \subseteq \mathbb{R}^n$ are <u>diffeomorphic</u> if there exist C^∞ functions $f : S \to T \subseteq \mathbb{R}^n$ and $g : T \to S \subseteq \mathbb{R}^n$ such that $g \circ f = id_S$ and $f \circ g = id_T$.

<u>Lemma</u> Let $S \subseteq \mathbb{R}^n$ be a subset. Suppose there exist an open set $U \subseteq \mathbb{R}^n$ and a diffeomorphism $f : U \to S$. Then S is also open in $\mathbb{R}^n$.

<u>Proof</u> We have a C^∞ map $g : S \to U$ st. $g \circ f = id_U$. By assumption, there exist an open set $S \subseteq V$ and a C^∞ map $\tilde{g} : V \to \mathbb{R}^n$ st. $\tilde{g}|_S = g$. Apply the chain rule to $\tilde{g} \circ f =$ inclusion $U \to \mathbb{R}^n$ to get

$$\tilde{g}_{*, f(p)} \circ f_{*, p} = id : T_p U \to T_p U \quad \text{for } p \in U.$$

Hence $f_{*,p} : T_p U \to T_{f(p)} \mathbb{R}^n$ is an isomorphism. By the inverse function theorem there are nbhs $p \in U_p \subseteq U$, $f(p) \in V_{f(p)} \subseteq \mathbb{R}^n$ st. $f : U_p \to V_{f(p)} \subseteq S$ is a diffeomorphism.

<u>Conclusion</u>: For each $f(p) \in S$ there exists an open set $V_{f(p)} \subseteq \mathbb{R}^n$ such that $f(p) \in V_{f(p)} \subseteq S$. $\square$

Proposition Let $U, V \subseteq \mathbb{H}^n$ be open subsets and let $f: U \to V$ be a diffeomorphism. Then f takes boundary points to boundary points and interior points to interior points.

Proof Suppose $p \in U$ is an interior point. Then there exists an open ball B in $\mathbb{R}^n$ s.t. $p \in B \subseteq U$. By the lemma $f(B)$ is open in $\mathbb{R}^n$, so $f(p) \in f(B) \subseteq \mathbb{H}^n$ is an interior point. Now suppose $p \in U$ is a boundary point. Then $f^{-1}(f(p)) = p$, so by the first part applied to $f^{-1}: V \to U$, $f(p)$ cannot be an interior point. Hence $f(p) \in \mathbb{H}^n$ is a boundary point. $\square$

Def Let M be an n-dim top. manifold with boundary.
An atlas $\{(U_\alpha, \phi_\alpha : U_\alpha \to U_\alpha' \subseteq \mathbb{H}^n)\}$ on M is a collection of charts such that $M = \bigcup_\alpha U_\alpha$. The atlas is C^∞ if for each pair α, β, the transition function $\phi_\beta \circ \phi_\alpha^{-1}$ is a diffeomorphism

$$\begin{tikzcd}[column sep=small] & U_\alpha \cap U_\beta \arrow[dl, "\phi_\alpha"'] \arrow[dr, "\phi_\beta"] & \\ \phi_\alpha(U_\alpha \cap U_\beta) \arrow[rr, "\phi_\beta \circ \phi_\alpha^{-1}"] & & \phi_\beta(U_\alpha \cap U_\beta) \end{tikzcd}.$$

An n-dim smooth manifold with boundary is an n-dim topological manifold with boundary, together with a maximal C^∞ atlas.

Def Let M be a smooth manifold with boundary. The boundary $\partial M \subseteq M$ is the subset of points $p \in M$ that are mapped to $\partial \mathbb{H}^n \subseteq \mathbb{H}^n$ by the charts in the maximal atlas.