Ex 8.2 Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map.

Show there is a commutative diagram

$$\begin{tikzcd} \partial/\partial x^i, \ D_v & X_p = c'(0) \\ T_p(\mathbb{R}^n) \arrow[r, "L_{*,p}"] \arrow[d, leftrightarrow, "\cong"] & T_{L(p)}(\mathbb{R}^m) \arrow[d, leftrightarrow, "\cong"] & (L\circ c)'(0) \\ \mathbb{R}^n \arrow[r, "L"] & \mathbb{R}^m \\ e_i, \ v \end{tikzcd}$$

$\partial/\partial x^i \leftrightarrow e_i$, $D_v \leftrightarrow v$

By definition

$c'(0) : C^\infty_p(\mathbb{R}^n) \to \mathbb{R}$

$c'(0)(f) = \frac{d}{dt}\Big|_{t=0} (f \circ c)$

Represent $X_p = c'(0) \in T_p(\mathbb{R}^n)$, where $c: (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ has $c(0) = p$. $c'(0)$ corresponds to $\left(\frac{dc^1}{dt}(0), \dots, \frac{dc^n}{dt}(0)\right) = (\dot{c}^1(0) \dots \dot{c}^n(0))$ in $\mathbb{R}^n$

$$L_{*,p}(X_p) = L_{*,p}(c'(0)) = (L\circ c)'(0) \mapsto L \begin{bmatrix} \dot{c}^1(0) \\ \vdots \\ \dot{c}^n(0) \end{bmatrix}$$ in $\mathbb{R}^n$, since

$$L \circ c : t \mapsto L \begin{bmatrix} c^1(t) \\ \vdots \\ c^n(t) \end{bmatrix} = \begin{bmatrix} a_{11} c^1(t) + \dots + a_{1n} c^n(t) \\ a_{i1} c^1(t) + \dots + a_{in} c^n(t) \\ a_{m1} c^1(t) + \dots + a_{mn} c^n(t) \end{bmatrix} \qquad L = [a_{ij}]$$

and evaluating the derivative is

$$\begin{bmatrix} a_{11} \dot{c}^1(t) + \dots + a_{1n} \dot{c}^n(t) \\ \vdots \\ a_{m1} \dot{c}^1(t) + \dots + a_{mn} \dot{c}^n(t) \end{bmatrix}$$

Alternatively: $v \in \mathbb{R}^n \longmapsto D_v \in T_p(\mathbb{R}^n)$, where

$$D_v(f) = \frac{d}{dt}\Big|_{t=0} \left(f(p+tv) \right)$$

$$L_{*,p}(D_v) \in T_{L(p)}(\mathbb{R}^m),$$

$$L_{*,p}(D_v)(f) = D_v(f \circ L) = \frac{d}{dt}\Big|_{t=0} f \circ L(p+tv)$$

$$= \frac{d}{dt}\Big|_{t=0} f\left(L(p) + tL(v) \right) = D_{L(v)}(f)$$

Hence $L_{*,p}(D_v) = D_{L(v)}$.

Ex 8.7 $\pi_1 : M\times N \to M$, $\pi_2 : M\times N \to N$

Claim $(\pi_{1*}, \pi_{2*}) : T_{(p,q)}(M\times N) \to T_p(M)\times T_q(N)$.

Choose charts
$$M \supseteq U \xrightarrow{\phi} \mathbb{R}^m$$
$$N \supseteq V \xrightarrow{\psi} \mathbb{R}^n \quad \text{s.t. } (p,q)\in U\times V.$$

Write $\phi = (x^1 .. x^m)$, $\psi = (y^1, .., y^n)$, $\phi\times\psi = (z^1, .. z^m, z^{m+1} .. z^{m+n})$

Basis $T_{(p,q)}(M\times N) = \{ \partial/\partial z^i|_{p,q} \}_{i=1,..,m+n}$.

Claim
$$\pi_{1*}\, \partial/\partial z^i|_{(p,q)} = \begin{cases} \partial/\partial x^i|_p & \text{for } 1\le i\le m \\ 0 & \text{for } m+1\le i\le m+n. \end{cases}$$

$$\pi_{1*}\left(\frac{\partial}{\partial z^i}\Big|_{(p,q)}\right)(f) = \frac{\partial}{\partial z^i}\Big|_{(p,q)} (f\circ\pi_1)$$
$$= \frac{\partial(f\circ\pi_1\circ(\phi^{-1}\times\psi^{-1}))}{\partial r^i}\Bigg|(\phi(p),\psi(q)) = \begin{cases} \dfrac{\partial(f\circ\phi^{-1})}{\partial r^i}\phi(p), & 1\le i\le m \\ 0, & m+1\le i\le m+n. \end{cases}$$

Similarly for $\pi_{2*}(\partial/\partial z^i)$.

<u>Consequence</u> $(\pi_{1,*}, \pi_{2,*})$ maps the basis $\{\partial/\partial z^i\}_{i=1..m+n}$

to the bases

$$\left\{\begin{array}{l} (\partial/\partial x^1|_p, 0), \ldots, (\partial/\partial x^m|_p, 0) \\ (0, \partial/\partial y^1|_q), \ldots, (0, \partial/\partial y^n|_q) \end{array}\right\} \quad \text{for} \quad T_p(M) \times T_q(N)$$