Calculating differentials using curves

Let M be a smooth manifold

Recall: A smooth curve $c:(-\varepsilon,\varepsilon)\to M$ define a tangent vector $c'(0)\in T_{c(0)}(M)$:

$$c'(0): C^\infty_{c(0)}(M)\to \mathbb{R}, \quad c'(0)(f)=\frac{d(f\circ c)}{dt}(0)$$

Notice: $c'(0)=c_{*,0}\left(\frac{d}{dt}\Big|_0\right)$, where $c_{*,0}: T_0(\mathbb{R})\to T_{c(0)}(M)$ and $\frac{d}{dt}\Big|_0$ is the basis vector for $T_0(\mathbb{R})$

Have proved that for each $X_p\in T_p(M)$, there exist $c:(-\varepsilon,\varepsilon)\to M$ s.t. $c(0)=p$ and $c'(0)=X_p$

<u>Recall</u> : Give $F: N \to M$ a smooth map, the differential at $p \in N$ is a linear map $F_{*,p}: T_p(N) \to T_{F(p)}(M)$.

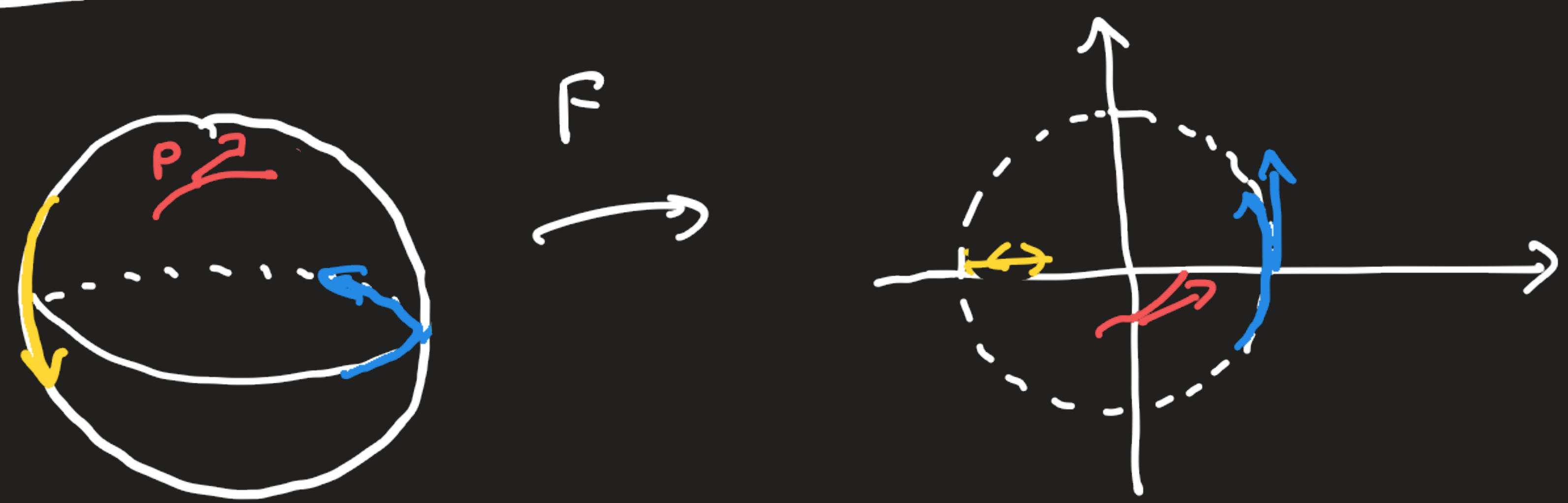
<u>Proposition</u> Suppose $X_p = c'(0)$ for a smooth curve $c: (-\varepsilon, \varepsilon) \to M$ with $c(0) = p$.

Then $F_{*,p}(X_p) = (F \circ c)'(0) \in T_{F(p)}(M)$

<u>Proof</u> $F_{*,p}(X_p) = F_{*,p}(c'(0)) = F_{*,p} \circ c_{*,0}\left(\frac{d}{dt}\Big|_0\right)$

$= (F \circ c)_{*,0}\left(\frac{d}{dt}\Big|_0\right) = (F \circ c)'(0)$, by the chain rule $\square$

<u>Ex</u> Let $F: S^2 \to \mathbb{R}^2$ be the projection on the xy-plane



$\underline{\text{Ex}}$ $M = GL_n(\mathbb{R})$, open subset of $M_n(\mathbb{R}) = \mathbb{R}^{n \times n}$.

Let $g \in GL_n(\mathbb{R})$, define $l_g : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $A \mapsto gA$ smooth

What is $(l_g)_{*,I} :$

$$
\begin{array}{ccc}
T_I(GL_n(\mathbb{R})) & \longrightarrow & T_g(GL_n(\mathbb{R})) \\
\wr\!\shortparallel & & \wr\!\shortparallel \\
M_n(\mathbb{R}) & \overset{?}{\dashrightarrow} & M_n(\mathbb{R})
\end{array}
$$

Given $X_p \in T_I(GL_n(\mathbb{R}))$ represented by $X_p = c'(0)$, where $c : (-\varepsilon, \varepsilon) \to GL_n(\mathbb{R})$, $c(0) = I$.

Then $(l_g)_{*,I}(X_p) = (l_g \circ c)'(0) = \dfrac{d(g\,c(t))}{dt}(0)$

$= g\,c'(0) = g X_p$.

$\underline{\text{Conclusion}}$: $(l_g)_{*,I}$ is also given by multiplication by $g \in GL_n(\mathbb{R})$.

Submanifolds (= regular submanifolds)

Let M be a manifold of $\dim(M) = n$.

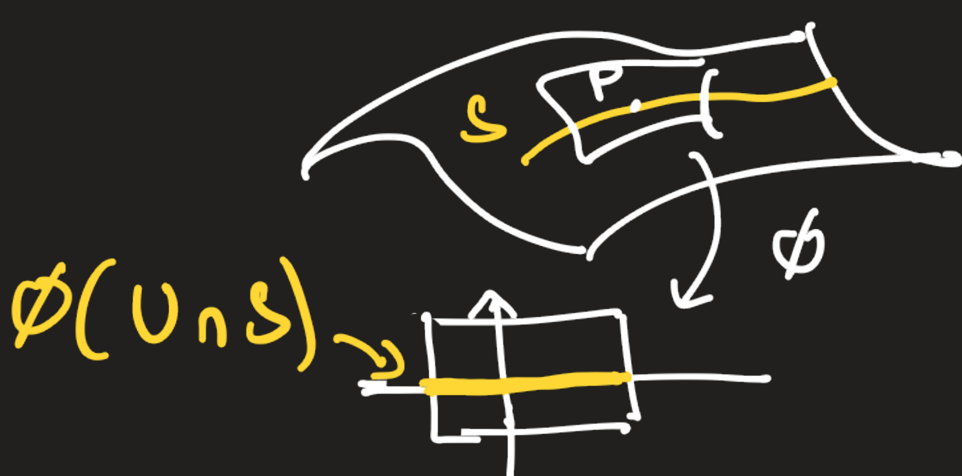
<u>Def</u> A subset $S \subseteq M$ is a submanifold of dimension k if for any $p \in S$, there exists a chart $\phi : U \to \mathbb{R}^n$ on M s.t. $p \in U$ and $\phi(U \cap S) = \phi(U) \cap (\mathbb{R}^k \times \{0\})$

$(\mathbb{R}^k \times \{0\} = \{(x^1, \dots, x^k, 0, \dots, 0) \in \mathbb{R}^n\})$

Equivalently, writing $\phi = (x^1, \dots, x^n)$, then

$$U \cap S = \{ q \in U : x^{k+1}(q) = \dots = x^n(q) = 0 \}.$$

Such a chart ϕ is said to be <u>adapted</u> to S.



<u>Rmk</u> Suppose $\phi = (x^1, \dots, x^n)$ is a chart on M s.t. $U \cap S = \{ q \in U : x^{j_1}(q) = \dots = x^{j_k}(q) = 0 \}$ for $1 \leq j_1 < \dots < j_k \leq n$. Then we can permute the coordinates to get an adapted chart.

Proposition Let $S \subseteq M$ be a k-dim. submanifold. Then the adapted charts define a smooth atlas on S such that S is a k-dim. smooth manifold.

Proof M Hausdorff and second countable implies that S is also Hausdorff and second countable.

Let $p \in S$ and choose and adapted chart on M, $\phi : U \to \mathbb{R}^n$ s.t. $\phi(U \cap S) = \phi(U) \cap (\mathbb{R}^k \times \{0\})$

Write $\phi = (x^1, \dots, x^n)$. Then the restriction

$$\phi_{|S} = (x^1, \dots, x^k) : U \cap S \to \phi(U) \cap \mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^k \times \{0\} \cong \mathbb{R}^k.$$

is a homeomorphism, so $\phi_{|S}$ is a chart on S at p.

Conclusion: S is a k-dim. topological manifold.

Notice: M has a smooth atlas $\{(U_\alpha, \phi_\alpha)\}$ such that each $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ is adapted to S.

<u>Claim</u>: Then $\{(\phi_\alpha|_S, U_\alpha \cap S)\}$ is a smooth atlas on S:

$$
\begin{tikzcd}
 & U_\alpha \cap U_\beta \cap S \arrow[dl, "\phi_\alpha|_S"'] \arrow[dr, "\phi_\beta|_S"] & \\
\phi_\alpha(U_\alpha \cap U_\beta \cap S) \arrow[d, phantom, "\parallel"] & & \phi_\beta(U_\alpha \cap U_\beta \cap S) \arrow[d, phantom, "\parallel"] \\
\phi_\alpha(U_\alpha \cap U_\beta) \cap (\mathbb{R}^k \times \{0\}) \arrow[rr, dashed] & & \phi_\beta(U_\alpha \cap U_\beta) \cap (\mathbb{R}^k \times \{0\})
\end{tikzcd}
$$

Must check $(\phi_\beta|_S) \circ (\phi_\alpha|_S)^{-1}$ is smooth:

$$(t^1, \dots, t^k) \mapsto \phi_\alpha^{-1}(t^1, \dots, t^k, 0, \dots, 0) \mapsto \phi_\beta(\phi_\alpha^{-1}(t^1, \dots, t^k, 0, \dots, 0))$$

$$= (Y^1(\phi_\alpha^{-1}(t^1 \dots t^k, 0 \dots 0)), \dots, Y^k(\phi_\alpha^{-1}(t^1 \dots t^k, 0 \dots 0)), 0, \dots, 0),$$

where $\phi_\beta = (Y^1, \dots, Y^n)$. This is smooth $\square$