The exterior derivative

Let $U \subseteq \mathbb{R}^n$ be an open set

$\Omega^k(U)$ vector space of C^∞ k-forms

$\omega \in \Omega^k(U)$ can be uniquely written $\omega = \sum_I a_I \, dx^I$, where

$dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_k}$, $I = (1 \le i_1 < \dots < i_k \le n)$ and $a_I : U \to \mathbb{R}$

are C^∞ functions

wedge product $\wedge : \Omega^k(U) \times \Omega^l(U) \to \Omega^{k+l}(U)$

makes $\Omega^*(U) = \{ \Omega^k(U) : k \ge 0 \}$ an anticommutative

graded $\mathbb{R}$-algebra.

Exterior derivative $d : \Omega^k(U) \to \Omega^{k+1}(U)$, $\mathbb{R}$-liner

$$d\left(\sum_I a_I \, dx^I \right) = \sum_I da^I \wedge dx^I = \sum_I \sum_{i=1}^n \frac{\partial a_I}{\partial x^i} dx^i \wedge dx^I.$$

Proposition (Uniqueness of exterior derivative)

Let $D : \Omega^k(U) \to \Omega^{k+1}(U)$ be a collection of $\mathbb{R}$-linear functions for $k \geq 0$ and suppose that

(i) $D(\omega \wedge \tau) = D(\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge D(\tau)$

(ii) $D \circ D = 0$

(iii) If X is a C^∞ vector field and $f \in \Omega^0(U) = C^\infty(U)$, then $(Df)_p(X_p) = X_p(f)$ for all $p \in U$.

Then $D = d$ exterior derivative

__Rmk__ We have proved that d satisfies (i), (ii), (iii).

Proof. By linearity suffices to show $D\omega = d\omega$ for $\omega = f\, dx^I$.

By (i) $D\omega = D(f) \wedge dx^I + f\, D(dx^I)$.

By (iii) $Df = df$, so

$D(f) \wedge dx^I = df \wedge dx^I = d(f \wedge dx^I) = d\omega$.

Remains to show $D(dx^I) = 0$

$k=1$ $\quad D(dx^i) = DDx^i = 0$ by (ii)

$k=2$ $\quad D(dx^{i_1} \wedge dx^{i_2}) = \underbrace{D(dx^{i_1})}_{=0} \wedge dx^{i_2} - dx^{i_1} \wedge \underbrace{D(dx^{i_2})}_{=0} = 0$

$k=3$ $\quad D(dx^{i_1} \wedge dx^{i_2} \wedge dx^{i_3})$

$= \underbrace{D(dx^{i_1} \wedge dx^{i_2})}_{=0} \wedge dx^{i_3} + dx^{i_1} \wedge dx^{i_2} \wedge \underbrace{D(dx^{i_3})}_{=0} = 0$

General case follows by induction $\square$

Application to vector calculus

Let $U \subseteq \mathbb{R}^3$. A C^∞ vector field on U can be written in the form $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$, $P, Q, R : U \to \mathbb{R}$ C^∞.

We can identify X with the vector function

$$\begin{bmatrix} P \\ Q \\ R \end{bmatrix} : U \to \mathbb{R}^3.$$

Let $\mathfrak{X}(U)$ be the vector space of C^∞ vector fields on U.

Linear transformations

$$\text{grad} : C^\infty(U) \to \mathfrak{X}(U), \quad f \mapsto \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{bmatrix}$$

$$\text{curl} : \mathcal{H}(U) \to \mathcal{H}(U), \quad \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \mapsto \text{"} \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \text{"} = \begin{bmatrix} \partial R/\partial y - \partial Q/\partial z \\ -(\partial R/\partial x - \partial P/\partial z) \\ \partial Q/\partial x - \partial P/\partial y \end{bmatrix}$$

$$\text{div} : \mathcal{H}(U) \to C^\infty(U), \quad \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \longmapsto \partial P/\partial x + \partial Q/\partial y + \partial R/\partial z.$$

<u>Claim</u>: There is a commutative diagram:

$$\begin{tikzcd}
\Omega^0(U) \arrow[r, "d"] \arrow[d, equal] & \Omega^1(U) \arrow[r, "d"] \arrow[d, leftrightarrow, "\cong"] \arrow[dr, phantom, "\mathrm{II}"] & \Omega^2(U) \arrow[r, "d"] \arrow[d, leftrightarrow, "\cong"] & \Omega^3(U) \arrow[d, leftrightarrow, "\cong"] \\
C^\infty(U) \arrow[r, "\text{grad}"] & \mathcal{H}(U) \arrow[r, "\text{curl}"] & \mathcal{H}(U) \arrow[r, "\text{div}"] & C^\infty(U) \\
 & \begin{bmatrix} P \\ Q \\ R \end{bmatrix} & &
\end{tikzcd}$$

The vertical isomorphism are:

$$\Omega^1(U) \xleftrightarrow{\cong} \mathfrak{X}(U) \,;\; P\,dx + Q\,dy + R\,dz \longleftrightarrow \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

$$\Omega^2(U) \xleftrightarrow{\cong} \mathfrak{X}(U) \,;\; P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy \longleftrightarrow \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

$$\Omega^3(U) \xleftrightarrow{\cong} C^\infty(U) \qquad f\,dx\wedge dy\wedge dz \longleftrightarrow f$$

<u>Check II</u>: $d(P\,dx + Q\,dy + R\,dz)$

$$= \partial P/\partial y \, dy\wedge dx + \partial P/\partial z \, dz\wedge dx$$
$$+ \partial Q/\partial x \, dx\wedge dy + \partial Q/\partial z \, dz\wedge dy$$
$$+ \partial R/\partial x \, dx\wedge dz + \partial R/\partial y \, dy\wedge dz$$
$$= (\partial R/\partial y - \partial Q/\partial z)\, dy\wedge dz + (\partial P/\partial z - \partial R/\partial x)\, dz\wedge dx + (\partial Q/\partial x - \partial P/\partial y)\, dx\wedge dy.$$

Exercise check the other squares.

Topological manifolds

Def An n-dimensional topological manifold is a topological space M such that

(i) For each $p \in M$, there exists a neighborhood $p \in U \subseteq M$ and a homeomorphism $\phi : U \to U' \subseteq \mathbb{R}^n$, where $U' = \phi(U)$ is an open subset of $\mathbb{R}^n$.

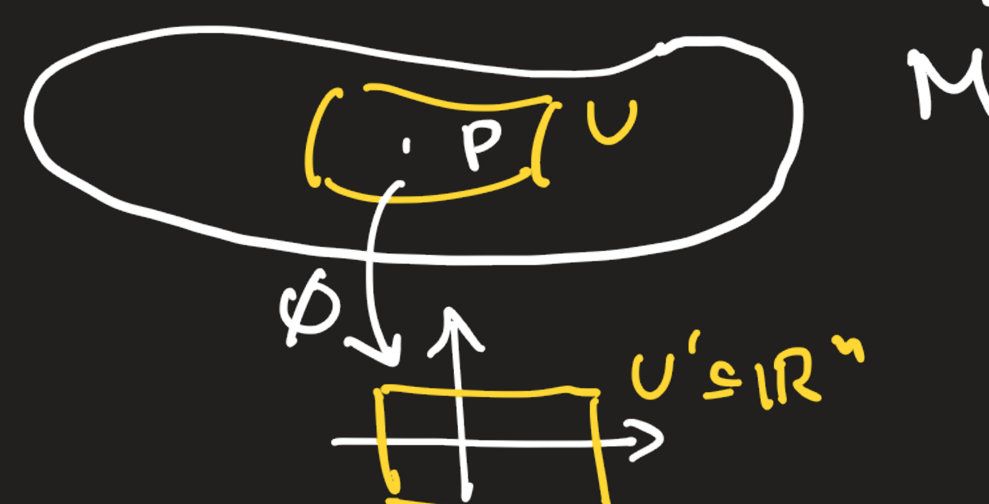
(ii) M second countable

(iii) M is Hausdorff.

(we recall topological notions later)

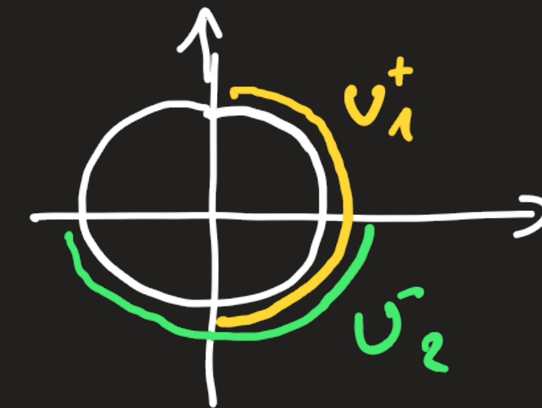
A topological space satisfying (i) is said to be locally Euclidian of dimension n.

(U, ϕ) is a chart on M near p.



Ex $U \subseteq \mathbb{R}^n$ open subset. Let $M = U$ n-dimensional topological manifold. $\mathrm{id} : M \to U$ is a chart.

Ex $M = S^1 = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$



Let $U_1^+ = \{ (x,y) \in S^1 : x > 0 \}$

$U_1^- = \{ (x,y) \in S^1 : x < 0 \}$

$U_2^+ = \{ (x,y) \in S^1 : y > 0 \}$

$U_2^- = \{ (x,y) \in S^1 : y < 0 \}$

Let $\varphi_1^+ : U_1^+ \to (-1,1)$, $\varphi_1^+(x,y) = y$ homeomorphism with inverse $(\varphi_1^+)^{-1} : (-1,1) \to S^1$, $t \mapsto (\sqrt{1-t^2}, t)$.

Similar charts for U_1^-, U_2^+, U_2^-.

Ex A similar argument shows that $S^n = \{ x \in \mathbb{R}^{n+1} : \|x\| = 1 \}$ is locally Euclidian of dimension n, for all $n \geq 1$.

Topological notions:

A topological space is a set X together with a family of subsets called the <u>open sets</u>.

- $\emptyset, X$ are open
- Arbitrary unions of open sets are open
- Finite intersections of open sets are open.

A <u>neighborhood</u> of a point $p \in X$ is an open set U such that $p \in U$.

A family $\mathcal{B}$ of open subsets in X is a basis for the topology if: For each $x \in X$ and each neighborhood $x \in U$, there exists $B \in \mathcal{B}$ s.t. $x \in B \subseteq U$. Then every open set in X is a union of elements from $\mathcal{B}$.

<u>Ex</u> $X = \mathbb{R}^n$, let $\mathcal{B}$ be the collection of all open balls $B(x, \varepsilon) = \{ y \in \mathbb{R}^n : \| y - x \| < \varepsilon \}$ for $x \in \mathbb{R}^n$, $\varepsilon > 0$.

Def. A topological space X is second countable if there exists a countable basis for the topology.

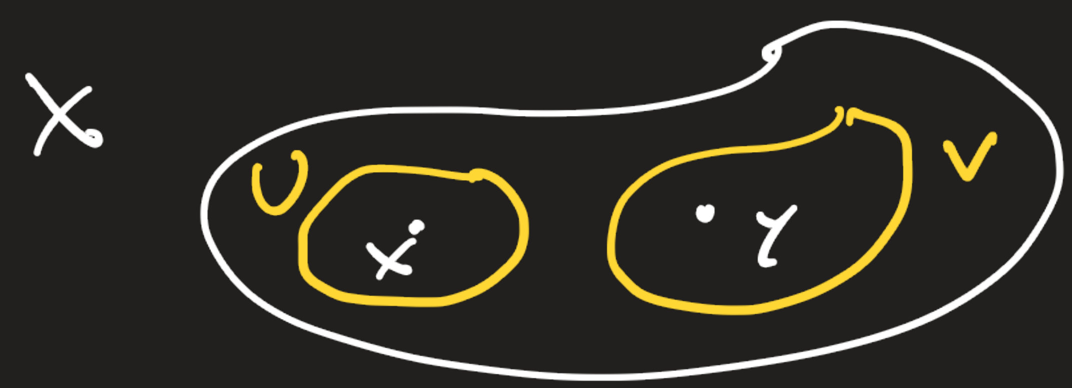
(Recall: A set B is countable if there exists a surjective function $\mathbb{N} \to B$.)

<u>Example</u> $X = \mathbb{R}^n$. Let $\mathcal{B}$ be the collection of all open balls $B(x, \varepsilon)$, where $x \in \mathbb{R}^n$ has rational coordinates and $\varepsilon \in \mathbb{Q}_+$. Then $\mathcal{B}$ is a countable basis for the standard topology on $\mathbb{R}^n$. Hence $\mathbb{R}^n$ is second countable.

<u>Rmk</u> If X is second countable, then every subspace $A \subseteq X$ is also second countable.

<u>Consequence</u> Any subspace $A \subseteq \mathbb{R}^n$ is second countable.

Def A topological space X is Hausdorff if for each pair of distinct point $x, y \in X$, there exist neighborhoods $x \in U$, $y \in V$ such that $U \cap V = \emptyset$.



Ex $\mathbb{R}^n$ is Hausdorff.

Rmk If X is Hausdorff, then every subspace $A \subseteq X$ is also Hausdorff.

Consequence Any subspace $A \subseteq \mathbb{R}^n$ is Hausdorff.