Due: Wednesday November 13, in class (3pm).

1. Let $p \in (0,1)$ be a parameter. Prove that for any function f, the following relationship between the randomized and distributional communication complexity of f holds:
$$p \cdot \max_{\mu} \bar{D}^{\mu}_{\varepsilon/(1-p)}(f) \leq \bar{R}^{pub}_{\varepsilon}(f) \leq \frac{1}{p} \cdot \max_{\mu} \bar{D}^{\mu}_{(1-p)\varepsilon}(f).$$

2. Let $M \in \{0,1\}^{n \times n}$ be a matrix. It is meaningful to view M as a matrix over the reals $\mathbb{R}$ or over any finite field $\mathbb{F}_p$. Recall that the log rank bound for deterministic communication complexity applies over any field. Let $rank(M)$ be the rank of M over $\mathbb{R}$, and $rank_p(M)$ be its rank over $\mathbb{F}_p$ for a prime integer p.
   (a) Prove that $rank(M) \geq rank_p(M)$ for all p; you may use the basic linear algebra machinery.
   (b) Give an example of a $0-1$ matrix M_1 such that $rank_2(M) < rank_3(M)$, and a matrix M_2 such that $rank_2(M) > rank_3(M)$.

3. Consider the following communication problem $FirstDisagreement(x, y)$: Alice and Bob are each given a string $x, y \in \{0,1\}^n$, respectively, such that $x \neq y$. They need to output the smallest index i such that $x_i \neq y_i$ using a deterministic communication protocol. In other words, the last message of the protocol π should be the index $i := \min\{j : \ x_j \neq y_j\}$. It is very easy to give a protocol with communication $n + \log n$: Alice sends x to Bob, and Bob responds with i. Give a protocol with communication $n + \log^* n + O(1)$.
   - Note: the function $\log^* n$ is defined is the number of iterated applications of log it would take to make $\log\log\ldots\log n < 1$. For example, $\log^* 5 = 3$, $\log^* 84 = 4$, $\log^* 1000 = 4$ — it is an extremely slowly growing function.
   - Hint: one solution has Alice send $n - \log n$ bits in the first round.

4. (a) Prove that for any valid prior distribution μ the *information complexity* $IC^{\mu}_0(FirstDisagreement) = O(\log n)$. You may use any fact from the paper "Interactive information complexity" (Braverman, 2012) by citing it without proof.
   (b) Give an example of a distribution μ such that $IC^{\mu}_0(FirstDisagreement) = \Omega(\log n)$. Prove your answer.