Computer Science 597A

Fall 2014

Homework 2

Due at 3pm on Tue, Nov 25, 2014

In this assignment we will explore a number of problems related to auctions, and Myerson's result.

Problem 1

Below is a theorem due to Bulow and Klemperer.

Theorem 1 *(Bulow-Klemperer) Let $OPT(F, n)$ denote the revenue of Myerson's auction with n i.i.d. bidders sampled from F and $VCG(F, n)$ denote the revenue of the second price auction (with no reserve) with n i.i.d. bidders sampled from F. Then if F is regular, $VCG(F, n+1) \geq OPT(F, n)$.*

- Prove the theorem.
- Provide an example showing that it's necessary to assume F is regular.

Note: See e.g. Chapter 3.3 in Hartline's book (http://users.eecs.northwestern.edu/ hartline/amd.pdf) for a refresher on the optimal auction.

Hint: Define $OPT^*(F, n)$ to be the optimal revenue obtained by an auction that always allocates the item. Prove that $OPT^*(F, n+1) \geq OPT(F, n)$ and that $OPT^*(F, n) = VCG(F, n)$ for all n. Think why the second claim fails when F is irregular (this is very subtle! pay special attention to the details in Lemma 3.24 about ironed virtual values). Note that while the latter point is good to understand, it is possible to come up with an example directly, e.g. when $n = 1$.

Problem 2

Below is a theorem due to Hartline and Roughgarden about auctions with independent but not identical bidders. Let F_i be the distribution for v_i, and r_i denote the optimal reserve for F_i (i.e. $r_i = \bar{\phi}_i^{-1}(0)$). Consider the following mechanism: each bidder reports a bid. Let b_1 be the highest bid and b_2 the second highest. If $b_1 > r_1$, then bidder 1 gets the item and pays $\max\{b_2, r_1\}$. Otherwise, no one gets the item. Denote the revenue of this mechanism by $VCG_r(\vec{F})$.

Theorem 2 *(Hartline-Roughgarden 2009) Let $OPT(\vec{F})$ denote the revenue of Myerson's auction with n independent bidders with v_i drawn from F_i. If each F_i is regular, then $VCG_r(\vec{F}) \geq OPT(\vec{F})/2$.*

- Prove the theorem.

Hint: Break $OPT(\vec{F})$ into $OPT_1(\vec{F}) + OPT_2(\vec{F})$. OPT_1 denotes the expected virtual value of the winner *if he is the highest bidder*, and OPT_2 denotes the expected virtual value of the winner otherwise. Show that $VCG_r(\vec{F}) \geq OPT_1(\vec{F})$, and that $VCG_r(\vec{F}) \geq VCG(\vec{F}) \geq OPT_2(\vec{F})$.

Problem 3

The following are lemmas about setting reserves for a single bidder due to Daskalakis and Pierrakos, and Dhangwatnotai, Roughgarden, and Yan.

Lemma 1 *(Daskalakis-Pierrakos) Let $Rev_r(F)$ denote the revenue obtained by selling to a single bidder with value sampled from F using price r, and $OPT(F)$ denote the revenue when using the optimal reserve. If F is regular and r is the* median *of F, show that $Rev_r(F) \geq OPT(F)/2$.*

Lemma 2 *(Dhangwatnotai-Roughgarden-Yan) If F is regular and r is a* sample *from F, show that $\mathbb{E}_{r \leftarrow F}[Rev_r(F)] \geq OPT(F)/2$.*

- Prove Lemma 1.
- Prove Lemma 2.
- Provide an example showing that it's necessary to assume that F is regular (in both lemmas).

Hint: Use revenue curves.

Problem 4

Below is a Theorem due to Birkhoff and Von Neumann. An $n \times n$ matrix M is *doubly-stochastic* if $M_{ij} \geq 0$ for all i, j, $\sum_j M_{ij} = 1$ for all i, and $\sum_i M_{ij} = 1$ for all j. M is a *permutation* matrix if in addition each $M_{ij} \in \{0, 1\}$.

Theorem 3 *(Birkhoff-von Neumann) Every doubly-stochastic matrix can be written as a convex combination of permutation matrices. That is, for all doubly stochastic M, there exists a finite k and permutation matrices $M_1, \ldots, M_k$ and non-negative multiplies $c_1, \ldots, c_k$ such that $\sum_i c_i = 1$ and $\sum_i c_i M_i = M$.*

- Prove the theorem. (*Hint:* There are several inductive proofs, where the basic premise is that a doubly-stochastic matrix with T non-zero entries where $T > n$, can be represented as a convex combination of matrices with fewer than T non-zero entries.)