

Computer Science 597A

Fall 2014

Homework 1

Due at 3pm on Thu, Oct 9, 2014

Problem 1

Let M be any game with n players who each have m strategies. Suppose that the players play M repeatedly, such that at the end of T rounds, each player has *no regret*.

- Show that the distribution that samples the action profile played in one of the T rounds uniformly at random is a *coarse correlated equilibrium*.
- If instead each player has *no swap regret*¹ at the end of T rounds, show that the distribution that samples the action profile played in one of the T rounds uniformly at random is a *correlated equilibrium*.

Problem 2

An airline loses two identical bags with identical contents belonging to different passengers. The airline is liable for up to the maximum of \$100 and the value of the contents of the luggage. In order to learn the value of the contents, the airline asks each passenger separately to report an integer dollar value (at least \$2, at most \$100) for the contents of the luggage, with the following rules: If both passengers report the same amount, that is treated as the luggage's value. If they report different amounts, the lower report is treated as the luggage's value. In addition, whoever reports the lower value will receive \$2 from the other traveler as a reward for "telling the truth."

- Formulate this as a game where the players are the passengers (i.e. state the strategy space and payoffs), find a Nash equilibrium, and show that it is unique.
- Show that in fact it is also the unique correlated equilibrium.
- Is it the unique *coarse* correlated equilibrium?

Problem 3

Consider the following two player zero-sum game: the strategy space of each player is $[0, 1]$, with payoffs defined as follows. If player one reports $x = .x_1x_2x_3 \dots$ written in binary and player two reports $y = .y_1y_2y_3 \dots$ written in binary, let b be the smallest index such that at least one of $\{x_b, y_b\}$ is non-zero. if $x_b = y_b = 1$, then both players receive zero payoff. Otherwise, player one receives a payoff of 1 and player two receives a payoff of -1 . If both players report 0 (i.e. no b exists), then both players receive zero payoff. For example, if $x = .000101101$ and $y = .01000101$, then $b = 2$, $x_b = 0$, $y_b = 1$, so player one gets a payoff of 1 and player two gets a payoff of -1 . Show that this game has no Nash equilibrium (even with mixed strategies). What about correlated or coarse correlated equilibria?

¹Recall that having no regret means it is *not* the case that the player wishes they had played some fixed strategy s at every round. Informally, having no swap regret means it is *not* the case that the player wishes they had played some fixed strategy s at every round that they had played s' instead. Writing the definition formally is a good portion of the problem.

Hint: show that player one has a strategy guaranteeing a payoff of $1 - \epsilon$ for all $\epsilon > 0$, but no strategy guaranteeing a payoff of 1.

Problem 4

A (mixed) strategy profile s^* of a simultaneous-move game is said to be an ϵ -Nash equilibrium if for all agents i ,

$$u_i(a_i, s_{-i}^*) \geq u_i(a'_i, s_{-i}^*) - \epsilon,$$

for all $a_i \in \sigma(s_i)$ (the support of s_i), where $u_i(a_i, s_{-i}^*)$ is the expected utility of agent i for action a_i given the mixed strategy of other agents. [cf. formula (2.6) in the text.] In other words, in an ϵ -NE, no player gains more than ϵ from deviating from her current action. In particular, a 0-NE is just a NE.

Computing a precise Nash equilibrium of a game can be computationally difficult. The purpose of this exercise is to show that an ϵ -Nash equilibrium for a small constant ϵ can be computed much faster than in time exponential in the size of the game.

Fix a two-player game G , where each player has n actions available to her. Further, we assume that all payoffs are between 0 and 1. The payoffs of the two players can therefore be represented by two $n \times n$ matrices $P_1, P_2 \in [0, 1]^{n \times n}$. So if the first player plays action $j \in \{1, \dots, n\}$, and second player plays action $k \in \{1, \dots, n\}$, the first player's payoff is $u_1(j, k) = P_1[j, k] \in [0, 1]$ and the second player's payoff is $u_2(j, k) = P_2[j, k] \in [0, 1]$.

- (a) Let $s^* = (s_1, s_2)$ be a (mixed-strategy) Nash equilibrium of the game G . Let $L_1 = (a_1, \dots, a_t)$ be a list of actions of player 1 obtained by sampling actions from s_1 (with repetition). Let \tilde{s}_1 be a mixed strategy for player 1 given by selecting a uniformly random action from L_1 . Define \tilde{s}_2 to be the similarly obtained mixed strategy for player 2.

Prove that for a sufficiently large n , if $t = (5 \log n)/\epsilon^2$, we have that

$$|u_2(\tilde{s}_1, k) - u_2(s_1^*, k)| < \epsilon,$$

for all actions $k \in \{1, \dots, n\}$ of the second player with probability $> 1 - 1/n$ (where the probability is over the random choice of the list L_1).

- (b) Prove that for t as above, $\tilde{s} = (\tilde{s}_1, \tilde{s}_2)$ is an (2ϵ) -Nash equilibrium with probability $> 1 - 2/n$.
- (c) Give an algorithm that given a 2-player game as above finds an ϵ -Nash equilibrium in running time $n^{O((\log n)/\epsilon^2)}$. For a constant $\epsilon > 0$ this becomes $n^{O(\log n)}$ – which is worse than polynomial but is much better than exponential. Such an algorithm is called a *quasi-polynomial* algorithm.

Note: You may use a part in a later part of the problem, even if you haven't solved it.