Limiting Shapes of Random Young Diagrams

Mriganka Basu Roy Chowdhury

advised by

Dr. Subhamay Saha

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Outline of the talk

- I will first describe a few facts about Young Tableaux.
- Then I will speak about the Limit Shape Theorem.
- Finally I will recall some basic Representation Theory and describe the problem we wish to solve.

Young Diagrams

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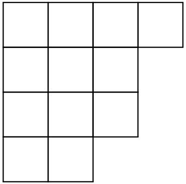

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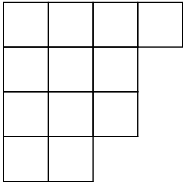

For any n, we define $YD_n := \{T \mid T \text{ is a Young Diagram of size } n\}$, where $\mathrm{size}(T) = n$ if T represents a partition of n.

If we fill a Young Diagram with *distinct* numbers from $\{1, 2, \ldots, n\} =: [n]$, such that each row and column is increasing, we call it a Standard Young Tableaux (SYT).

If we fill a Young Diagram with *distinct* numbers from $\{1, 2, \ldots, n\} =: [n]$, such that each row and column is increasing, we call it a Standard Young Tableaux (SYT). For the above diagram, this is a valid SYT:

12	10	7	3
11	9	5	
8	6	2	
4	1		

For any T, we write SYT_T to be the set of all SYTs of "shape" T. And for any n, we write $SYT_n = \bigcup_{T \in YD_n} SYT_T$.

For any permutation π of $[n]$, there is a way to associate a pair of SYTs with it. This is the content of the celebrated RSK Correspondence:

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This result is not just an existence result. There is an explicit algorithm that does this, called the *Schensted Algorithm*. As an example, if $\pi = \langle 4, 1, 5, 3, 2 \rangle$, then the associated pair is:

$$P = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1} \end{array} \quad \text{and} \quad Q = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1} \end{array}$$

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This is an extremely important fact in the Representation Theory of the Symmetric Group S_n (more later). It can also be seen from a Markov Chain Structure on the set of all Young Diagrams, called the *Bratelli Diagram*. This last idea is outlined in the report, but we do not go into this here.

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Definition (Plancherel Measure)

Let $\pi \in S_n$ be uniformly distributed. Then $\mathrm{shape}(RS(\pi)) = \lambda$ is an random element in YD_n. The distribution of λ is called the Plancherel Measure on YD_n. From the above discussion,

$$p_T = t_T^2/n!$$

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Note that, $\mathrm{shape}(RS(\pi))$ is the shape of either of the two SYTs produced (which is unambiguous because they have the same shape).

The Limit Shape Theorem

Now, suppose we take a Plancherel Distributed $\lambda \in YD_n$, for large n. Equivalently, we take a uniformly distributed $\pi \in S_n$, and take the shape of the diagrams formed under RSK. *What does it look like?*

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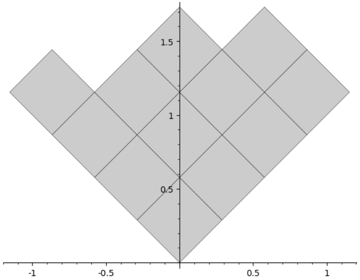
In the following, we do not draw Young Diagrams as above. Instead, we rotate it by 135° and scale everything appropriately (for reasons that will be clear):



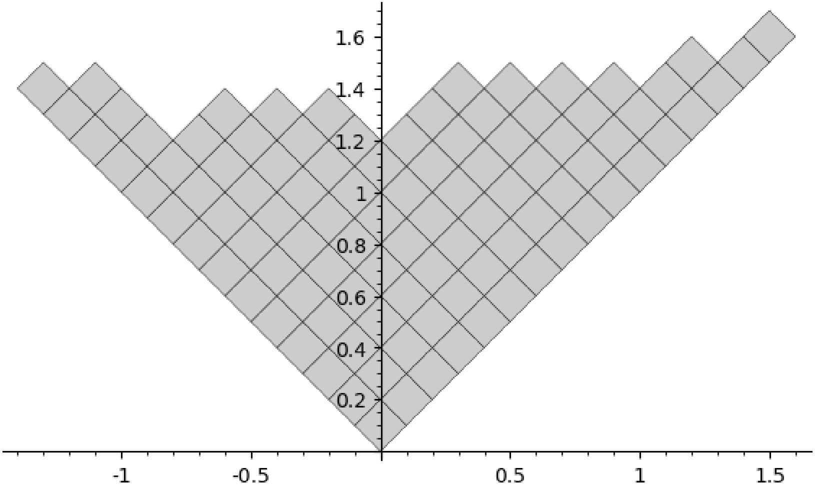


Figure: $n = 100$

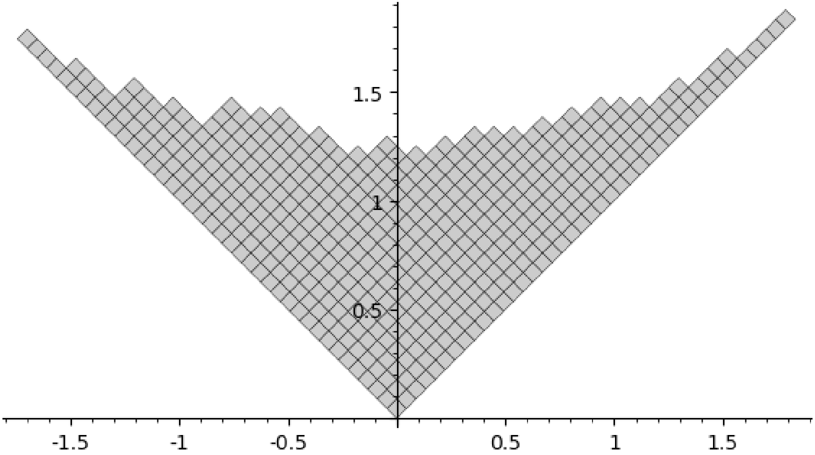


Figure: $n = 500$

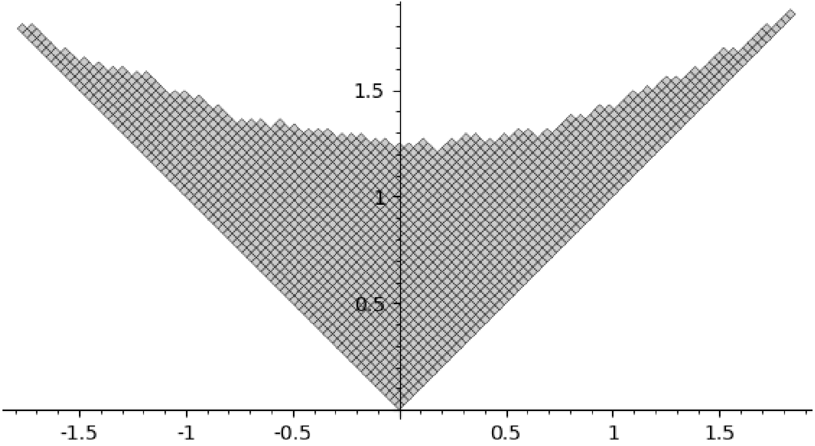


Figure: $n = 2000$

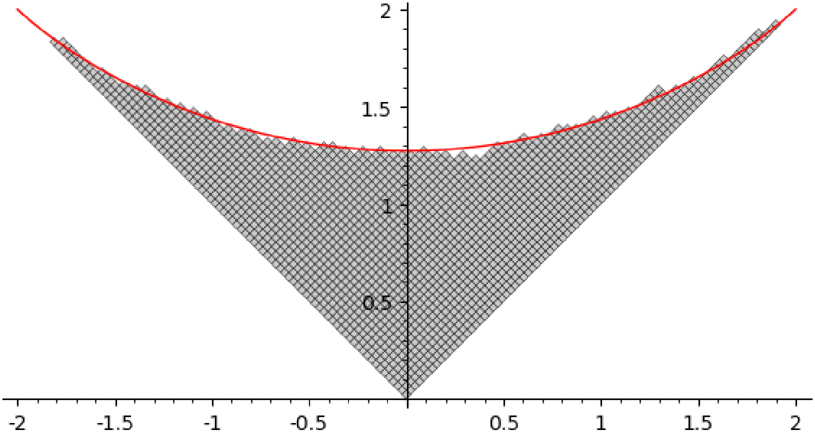


Figure: $n = 2000$ again

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Theorem (Logan-Shepp-Vershik-Kerov, 1977)

Under appropriate scaling, and with an appropriate metric on the space of all 1-Lipschitz compactly supported functions with area 2 *between the graph of the function and* $y = |x|$ *(see the report for details), the function corresponding to a Plancherel-random Young Diagram converges to*

$$\Omega(u) = \begin{cases} \frac{2}{\pi}\left(u \sin^{-1}\left(\frac{u}{2}\right) + \sqrt{4 - u^2}\right) & |u| \leq 2 \\ |u| & |u| > 2 \end{cases}$$

which is the red line above.

Some Representation Theory

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- Two representations $\phi : G \to GL(V)$, $\psi : G \to GL(W)$ are *equivalent* if there exists an invertible map $T : V \to W$ such that $T^{-1}\psi(g)T = \phi(g)$ for all $g \in G$.

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- For a given group G, there are only finitely many distinct irreducible (defined later) representations upto equivalence.

- Given two representations $\phi : G \to GL(V), \psi : G \to GL(W)$, there is a representation $\phi \oplus \psi : G \to GL(V \oplus W)$, called the *direct sum*, defined by

$$(\phi \oplus \psi)(g)(v \oplus w) = (\phi(g)(v)) \oplus (\psi(g)(w))$$

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- Also, with the above notation, there is a representation $\phi \otimes \psi : G \to GL(V \otimes W)$, called the *tensor product*, defined by

$$(\phi \otimes \psi)(g)(v \otimes w) = (\phi(g)(v)) \otimes (\psi(g)(w))$$

and extended by linearity.

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- Given any representation ρ, it is possible find finitely many irreducible representations $\psi_1, \psi_2, \ldots, \psi_k$ such that

$$\rho \sim \psi_1 \oplus \psi_2 \oplus \psi_3 \oplus \cdots \oplus \psi_k$$

  where $\sim$ denotes equivalence.

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- Thus, given $T \in YD_n$, we write $[T]$ for the irreducible representation corresponding to T.

Transition Measures

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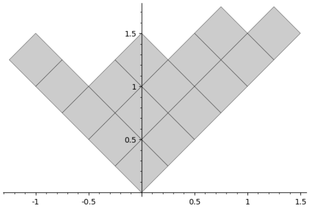

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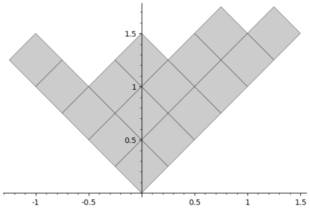

and consider the "interlacing" sequence $x_1 < y_1 < x_2 < \ldots < y_{k-1} < x_k$ of minima and maxima.

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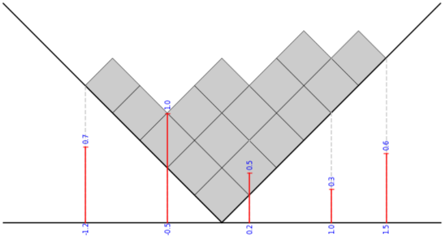

Figure: The transition measure. Here we report $q_k = p_k / \max p_k$.

Now we can look at our problem

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Definition (Biane's combination)

Given two Young Diagrams $T, T' \in YD_n$, consider the corresponding representations $[T], [T']$ of S_n. We define Biane's combination as:

$$\mathrm{BianeComb}(T, T') = \{S \in YD \mid [S] \textit{ is an irreducible component of } [T] \otimes [T']\}$$

Under some technical conditions, he proved:

Theorem

Given two Young Diagrams $T, T' \in YD_n$, a large fraction of the elements λ of $\mathrm{BianeComb}(T, T')$ satisfies:

$$\mu_T \boxplus \mu'_T \approx \mu_\lambda \text{ as } n \to \infty$$

where μ_T is the transition measure associated with T, for some appropriate meaning of $\approx$.

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Definition (Alternate Shuffle)

Given two Young Diagrams $T, T' \in YD_n$, *define the random Young Diagram* $\lambda = \lambda_{T,T'}$ *as follows:*

$$\lambda = \mathrm{shape}(RS(\mathrm{Shuffle}(\pi, \pi')))$$

where π *and* π' *are uniformly random permutations with* $\mathrm{shape}(RS(\pi)) = T$ *and* $\mathrm{shape}(RS(\pi')) = T'$ *respectively.*

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where π and π' are uniformly random permutations with $\mathrm{shape}(RS(\pi)) = T$ and $\mathrm{shape}(RS(\pi')) = T'$ respectively. Also $\mathrm{Shuffle}(\pi \in S_n, \pi' \in S_m) = \tau$ is a random variable defined as follows: let $S \subseteq [n+m]$ be a uniformly random subset of size n. Then $\tau \in S_{n+m}$ has the positions in S filled by π (in order), and the positions in $[n+m] - S$ filled by $\pi' + n$.

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1. Generate two independent SYT $P, Q \in SYT_n$, with $\mathrm{shape}(P) = \mathrm{shape}(Q) = T$, uniformly.
2. Construct $\pi = RS^{-1}(P, Q)$.

The step for generating uniform SYT with given shape is nontrivial. The celebrated Greene-Nijenhuis-Wilf algorithm does this via an elegant "hook walk" procedure.

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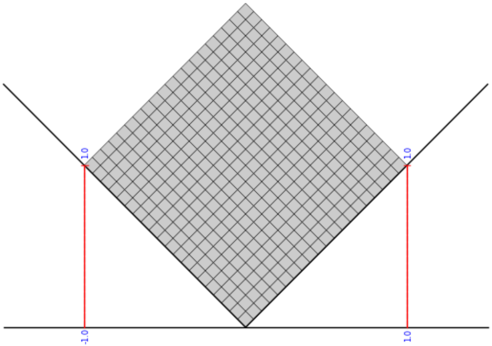

with the transition measure $\tau = \frac{1}{2}\left(\delta_{-1} + \delta_1\right)$

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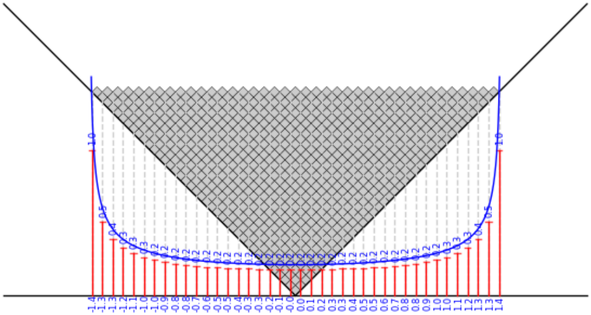

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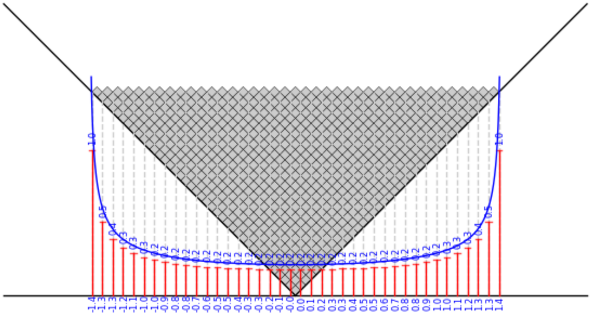

As we can see, there is significant match between the computed transition measure, and the *arcsine* distribution, plotted in blue.

Inspired from this, we formulate the following conjecture:

Conjecture

The "typical" transition measure of the Alternate Shuffle of two Young Diagrams with transition measures μ and ν is the free convolution $\mu \boxplus \nu$ of the two transition measures (upto scaling along the real line).

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Of course, we will need to precisely define "typical". But right now, our primary focus is to understand related constructions, and thinking about how we can handle such objects better.

Thank You

Intuition for the Free Convolution

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for the *Empirical Spectral Distribution* (ESD) of A. This is a fundamental object of study in Random Matrix Theory.

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Now, suppose A_n, B_n are two families of matrices such that $\mu_{A_n} \xrightarrow{w} \mu$ and $\mu_{B_n} \xrightarrow{w} \nu$. Then consider the random matrices $C_n := Q_n^T A_n Q_n, D_n := S^T B_n S$, where Q_n, S_n are independent families of uniform orthogonal matrices of size n.

Given a Symmetric real matrix A, let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be its eigenvalues. We write,

$$\mu_A = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i} \in \mathcal{P}(\mathbb{R})$$

for the *Empirical Spectral Distribution* (ESD) of A. This is a fundamental object of study in Random Matrix Theory. If we have a family of matrices A_n, we say that it has the μ as the *limiting* ESD if $\mu_A \xrightarrow{w} \mu$.

Now, suppose A_n, B_n are two families of matrices such that $\mu_{A_n} \xrightarrow{w} \mu$ and $\mu_{B_n} \xrightarrow{w} \nu$. Then consider the random matrices $C_n := Q_n^T A_n Q_n, D_n := S^T B_n S$, where Q_n, S_n are independent families of uniform orthogonal matrices of size n. Clearly, A_n and C_n have the same ESD, and similarly for B_n and D_n.

The question now is, *what is the limiting ESD for* $C_n + D_n$? That is, given two "typical" random matrices with given limiting ESD, what is the limiting ESD for their sum?

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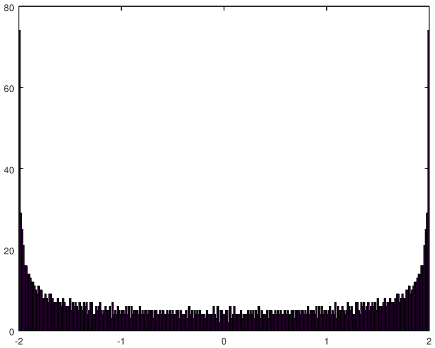

Figure: ESD for $C_n + D_n$, $n = 1000$

As we can see, this is very close to the arcsine$[-2, 2]$ distribution, which was seen to be the free convolution of $\frac{1}{2}(\delta_1 + \delta_{-1})$ with itself.

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This is indeed a general theorem. The limiting ESD for $C_n + D_n = \mu \boxplus \nu$. Of course, we need to interpret the limit in a suitable sense. In most cases, it is a limit in probability, with a metric which metrizes the weak topology on $\mathcal{P}(\mathbb{R})$.

This is the intuition for the free convolution. The non-commutativity arises in some sense from the noncommutativity of the matrices.