$$\lambda_1^N \le \lambda_2^N \le \dots \le \lambda_N^N$$

$$L_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\frac{\lambda_i}{\sqrt N}}$$

$$\bar L_N = \mathbb{E} L_N$$

$$\langle \bar L_N, e^{s\cdot}\rangle = \int e^{sx}\, d\bar L_N$$

$$\langle \bar L_N, e^{s\cdot}\rangle = \sum_{k=0}^{\infty} \underbrace{\binom{2k}{k}\frac{1}{k+1} b_k^{(N)} \frac{s^{2k}}{(2k)!}}_{\langle \bar L_N, x^{2k}\rangle}$$

Harer - Zagier relations:

$$b_{k+1}^{(N)} = b_k^{(N)} + \frac{k(k+1)}{4N^2} \underbrace{b_{k-1}^{(N)}}_{\ge 0}$$

$$b_k^{(N)} \le b_{k+1}^{(N)} \le \left(1 + \frac{k(k+1)}{4N^2}\right) b_k^{(N)} \le e^{\frac{k(k+1)}{4N^2}} b_k^{(N)}$$

$b_0 = 1$

$$b_k^{(N)} \le e^{\frac{1(1+1)}{4N^2} + \frac{2(2+1)}{4N^2} + \dots + \frac{(k-1)k}{4N^2}} \underbrace{b_0^{(N)}}_{=1}$$

$$\le e^{(1^2+2^2+\dots+k^2)/4N^2} \qquad j(j+1) \le (j+1)^2$$

$$\le e^{\frac{(k+1)^3}{3\cdot 4N^2}} = e^{\frac{(k+1)^3}{12N^2}} \approx e^{ck^3/N^2} \qquad c > \frac{1}{12}$$

— x —

$$\frac{\lambda_N^N}{\sqrt N} > 2 + \epsilon \qquad \frac{\lambda_N^N}{\sqrt N} \xrightarrow{\mathbb{P}} 2$$

RVS: $\left(\frac{\lambda_N^N}{\sqrt N} - 2\right) N^{2/3}$ tight

$$\mathbb{P}\left(\frac{\lambda_N^N}{2\sqrt N} - 1 \ge \delta N^{-2/3}\right) \le Ce^{-c'\delta}$$

$$\Rightarrow \mathbb{P}\left(\frac{\lambda_N^N}{2\sqrt N} \ge e^{\epsilon N^{-2/3}}\right) \le C' e^{-c'' \epsilon}$$

$$\langle \bar L_N, e^{s\cdot}\rangle = \sum_{k=0}^{\infty} \frac{b_k^{(N)}}{k+1}\binom{2k}{k}\frac{s^{2k}}{(2k)!}$$

$$b_k^{(N)} \le e^{ck^3/N^2} \qquad c > \frac{1}{12}$$

$$\mathbb{P}\left(\frac{\lambda_N^N}{2\sqrt N} \ge e^{\epsilon N^{-2/3}}\right)$$

Markov with exponent 2l

$$\le \mathbb{E}\left[\frac{\lambda_N^N}{2\sqrt N\, e^{\epsilon N^{-2/3}}}\right]^{2k}$$

$$= \underbrace{\frac{e^{-2\epsilon N^{-2/3}k}}{2^{2k}} \cdot \mathbb{E}\left[\frac{\lambda_N^N}{\sqrt N}\right]^{2k}}_{(i)}$$

$$L_N = \frac{1}{N}\sum \delta\left(\frac{\lambda_i}{\sqrt N}\right)$$

$$\langle L_N, x^{2k}\rangle = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\lambda_i}{\sqrt N}\right)^{2k}$$

$$\langle \bar L_N, x^{2k}\rangle = \mathbb{E}\langle L_N, x^{2k}\rangle = \frac{1}{N}\sum_{i=1}^{n}\mathbb{E}\left[\frac{\lambda_i}{\sqrt N}\right]^{2k} \ge \frac{1}{N}\mathbb{E}\left[\frac{\lambda_N^N}{\sqrt N}\right]^{2k}$$

(i)

$$\mathbb{P}\left(\frac{\lambda_N^N}{2\sqrt N} \ge e^{\epsilon N^{-2/3}}\right) \le \frac{e^{-2\epsilon k N^{-2/3}}}{2^{2k}}\mathbb{E}\left[\frac{\lambda_N^N}{\sqrt N}\right]^{2k}$$

$$\le \frac{e^{-2\epsilon k N^{-2/3}}}{2^{2k}} N \times \frac{b_k^{(N)}}{(k+1)}\binom{2k}{k}$$

$$b_k^{(N)} \le e^{\frac{1}{12}k^3/N^2}$$

$$\binom{2k}{k} \simeq \frac{2^{2k}}{\sqrt k}$$

$$\frac{e^{-2\epsilon k N^{-2/3}}}{2^{2k}} N \frac{e^{\frac{1}{12}k^3/N^2}}{(k+1)}\binom{2k}{k}$$

$$C 2^{-2k}\frac{1}{(k+1)}\frac{2^{2k}N}{\sqrt k} e^{-2\epsilon k N^{-2/3} + \frac{1}{12}k^3/N^2}$$

$$= C N k^{-3/2} e^{-2\epsilon k N^{-2/3} + \frac{1}{12}k^3/N^2}$$

$$k = N^{2/3}, \qquad k = N^{2/3}\epsilon^{1/2}\beta$$

$$= C \frac{N \epsilon^{-3/4}}{N} e^{-2\epsilon N^{2/3}\epsilon^{1/2} + \frac{1}{12}N^2\epsilon^{3/2}/N^2}$$

$$= \frac{C}{\epsilon^{3/4}} e^{-\underbrace{\left(2 - \frac{1}{12}\right)}_{c'}\epsilon^{3/2}}$$

$$= \frac{C}{\epsilon^{3/4}} e^{-c'\epsilon^{3/2}}$$

$$\mathbb{P}\left(\frac{\lambda_N}{\sqrt N} \ge e^{\epsilon N^{-2/3}}\right) \le C\epsilon^{-3/4} e^{-c'\epsilon^{3/2}}$$

$$c' \approx -\frac{23}{12}$$

$$c^* = -\frac{1}{12}$$

$$\le C\epsilon^{-3/4} e^{-\frac{1}{12}\epsilon^{3/2}}$$

$$-2\epsilon k N^{-2/3} + \frac{1}{12}k^3/N^2$$

$$-2\epsilon N^{-2/3} + \frac{1}{4}k^2/N^2$$

$$\frac{1}{4}k^2/N^2 = 2\epsilon N^{-2/3}$$

$$k^2 = 8\epsilon N^{2-2/3}$$

$$= 8\epsilon N^{4/3}$$

$$k = \sqrt{8}\,\epsilon^{1/2} N^{2/3}$$

$$-2\epsilon N^{-2/3}\sqrt 8\, \epsilon^{1/2} N^{2/3} + \frac{1}{12} 8\epsilon^{3/2} N^2/N^2$$

$$= \left(-2\sqrt 8 + \frac{2}{3}\right)\epsilon^{3/2}$$

$$\mathbb{P}\left(\frac{\lambda_N}{2\sqrt N} \ge \underbrace{e^{\epsilon N^{-2/3}}}_{\approx 1 + \epsilon N^{-2/3}}\right) \le C \frac{e^{-c\epsilon_1^{3/2}}}{\epsilon^{3/4}}$$

— x —

Harer - Zagier Relations

$$e^{-t/2} \qquad -s^2/N$$

$$\langle \bar L_N, e^{s\cdot}\rangle = e^{s^2/2N}\sum_{k=0}^{N-1}\frac{1}{k+1}\binom{2k}{k}\frac{(N-1)^{\underline{k}}}{N^k}\frac{s^{2k}}{(2k)!}$$

$$N^{\underline{k}} = N(N-1)\dots(N-k+1)$$

$$\langle \bar L_N, e^{s\cdot}\rangle = \sum_{k=0}^{\infty}\frac{1}{k+1}\binom{2k}{k} b_k^{(N)}\frac{s^{2k}}{(2k)!}$$

$$F_n(t) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}\binom{n-1}{k} t^k \quad ☆$$

$$\left(t\frac{d^2}{dt^2} + (2-t)\frac{d}{dt} + (n-1)\right)F_n(t) = 0$$

$$\Phi_n(t) = e^{-t/2}F_n(t)$$

$$\Phi_n\left(-s^2/N\right) = \langle \bar L_N, e^{s\cdot}\rangle$$

$$\left(4t\frac{d^2}{dt^2} + 8\frac{d}{dt} + 4n - t\right)\Phi_n(t) = 0$$

$$\Phi_n(t) = \sum_{k=0}^{\infty} a_k t^k$$

$$4t\sum_{k=0}^{\infty} a_k k(k-1)t^{k-2} + 8\sum_{k=0}^{\infty} a_k k t^{k-1} + (4n-t)\sum_{k=0}^{\infty} a_k t^k$$

$$4(k+1)(k+2)a_{k+1} + 4na_k - a_{k-1} = 0$$

$n = N$

$$\frac{(-1)^k a_k (2k)!}{N^k} = \frac{b_k^{(N)}}{k+1}\binom{2k}{k}$$

Christoffel - Darboux formula

$X \to$ GUE

$\lambda_1 \le \dots \le \lambda_n$

shuffle them uniformly

$$\langle \bar L_N, f\rangle = \frac{1}{N}\mathbb{E}\left[\sum f\left(\frac{\lambda_i}{\sqrt N}\right)\right]$$

substitute $y = \frac{x}{\sqrt N}$

$$= N\frac{1}{N}\mathbb{E} f\left(\frac{\lambda_1}{\sqrt N}\right)$$

$$\rho(\lambda_1, \dots, \lambda_n) = \frac{1}{n!}\det K^{(N)}(\lambda_i, \lambda_j)$$

$$= \frac{1}{N}\int f\left(\frac{x}{\sqrt N}\right) K^{(N)}(x, x)\, dx$$

$$= \frac{1}{N}\int f(y)(\dots)\, dx$$

$$\rho(\lambda_1) = \frac{(n-1)!}{n!}K^{(N)}(\lambda, \lambda)$$

$$K^{(N)}(x, y) = \sum_{k=0}^{n-1}\psi_k(x)\psi_k(y)$$

$H_k(x)$ = monic polynomial of degree k

$$\int_{\mathbb{R}} H_k(x)H_l(x)e^{-x^2/2}\,dx = \sqrt{2\pi}\,k!\,\delta_{kl}$$

$$\psi_k(x) = e^{-x^2/4}H_k(x)\frac{1}{\sqrt{\sqrt{2\pi}\,k!}}$$

$$\int \psi_k(x)\psi_l(x)\,dx = \delta_{kl}$$

$$\int |\psi_k|^2 dx = 1$$

$$K^{(N)}(x, y) = \boxed{\sum_{k=0}^{n-1}\psi_k(x)\psi_k(y)}$$

$$\frac{1}{N}\int f\left(\frac{x}{\sqrt N}\right)K^{(N)}(x, x)\,dx \qquad y = \frac{x}{\sqrt N}$$

$$\int f(y)\frac{1}{N}K^{(N)}(y\sqrt N, y\sqrt N)\sqrt N\,dy$$

$$\langle \bar L_N, f\rangle = \int f(y)\underbrace{\left[\frac{1}{\sqrt N}K^{(N)}(y\sqrt N, y\sqrt N)\right]}_{\text{density of } \bar L_n} dy$$

Christoffel - Darboux

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)$$

$$H_n(y)H_{n+1}(x) = xH_n(y)H_n(x) - nH_n(y)H_{n-1}(x)$$

$$H_n(x)H_{n+1}(y) = yH_n(x)H_n(y) - nH_n(x)H_{n-1}(y)$$

$$H_n(y)H_{n+1}(x) - H_n(x)H_{n+1}(y) = (x-y)H_n(x)H_n(y) + n\left[H_n(y)H_{n-1}(x) + H_n(x)H_{n-1}(y)\right]$$

$$\frac{H_n(y)H_{n+1}(x) - H_n(x)H_{n+1}(y)}{n!\,(x-y)} =$$

$$\frac{1}{n!}H_n(x)H_n(y) + \frac{1}{(n-1)!}\left[H_{n-1}(y)H_n(x) - H_{n-1}(x)H_n(y)\right]$$

$$\sum_{k=0}^{N-1}\frac{1}{k!}H_k(x)H_k(y) = \frac{H_N(x)H_{N-1}(y) - H_{N-1}(x)H_N(y)}{(N-1)!\,(x-y)}$$

$$\psi_k(x) = \frac{e^{-x^2/4}H_k(x)(2\pi)^{-1/4}}{\sqrt{k!}}$$

$$\sqrt{N!}\,\sqrt{(N-1)!}$$

$$\sum_{k=0}^{N-1}\psi_k(x)\psi_k(y) = \frac{\psi_N(x)\psi_{N-1}(y) - \psi_{N-1}(x)\psi_N(y)}{(N-1)!\,(x-y)}$$

$$= \sqrt N\left(\frac{\psi_N(x)\psi_{N-1}(y) - \psi_{N-1}(x)\psi_N(y)}{x-y}\right)$$

$$\sum_{k=0}^{N-1}\psi_k(x)\psi_k(y) = \frac{\sqrt N}{x-y}\left[\psi_N(x)\psi_{N-1}(y) - \psi_{N-1}(x)\psi_N(y)\right]$$

$$f_n(x) = (A_N + B_N x) f_n(x$$

This is [illegible]