

$$\lambda_1^N \leq \lambda_2^N \leq \dots \leq \lambda_N^N$$

$$L_N = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i} \frac{1}{\sqrt{N}}$$

$$\bar{L}_N = \mathbb{E} L_N$$

$$\langle \bar{L}_N, e^s \cdot \rangle = \int e^{sx} d\bar{L}_N$$

$$\langle \bar{L}_N, e^s \cdot \rangle = \sum_{k=0}^{\infty} \underbrace{\binom{2k}{k} \frac{1}{k+1} b_k^{(n)} \frac{s^{2k}}{(2k)!}}_{\langle \bar{L}_N, x^{2k} \rangle}$$

Harris - Zagier relations:

$$b_{k+1}^{(n)} = b_k^{(n)} + \frac{k(k+1)}{4n^2} b_{k-1}^{(n)} \geq 0$$

$$\left. \begin{aligned} b_k^{(n)} \leq b_{k+1}^{(n)} &\leq \left(1 + \frac{k(k+1)}{4n^2}\right) b_k^{(n)} \\ &\leq e^{\frac{k(k+1)}{4n^2}} b_k^{(n)} \end{aligned} \right\}$$

$$\begin{aligned} b_0^{(n)} &= 1 \\ b_k^{(n)} &\leq e^{\frac{1(1+1)}{4n^2} + \frac{2(2+1)}{4n^2} + \dots + \frac{(k-1)k}{4n^2}} b_0^{(n)} \\ &\leq e^{\frac{(1^2+2^2+\dots+k^2)}{4n^2}} \stackrel{j(j+1)}{\leq \binom{j+1}{j}^2} \\ &\leq e^{\frac{(k+1)^3}{3 \cdot 4n^2}} = e^{\frac{(k+1)^3}{12n^2}} \leq e^{ck^3/n^2} \quad c > \frac{1}{12} \end{aligned}$$

$$\frac{\lambda_N^N}{\sqrt{N}} > 2 + \epsilon \quad \frac{\lambda_N^N}{\sqrt{N}} \xrightarrow{P} 2$$

$$\text{r.v.s: } \left[\left(\frac{\lambda_N^N}{\sqrt{N}} - 2 \right) N^{2/3} \right] \text{ tight} \quad P\left(\frac{\lambda_N^N}{2\sqrt{N}} - 1 \geq \delta N^{-2/3} \right) \leq Ce^{-c\delta}$$

$$\Rightarrow P\left(\frac{\lambda_N^N}{2\sqrt{N}} \geq e^{\epsilon N^{-2/3}} \right) \leq Ce^{-c'\epsilon}$$

$$\langle \bar{L}_N, e^s \cdot \rangle = \sum_{k=0}^{\infty} \frac{b_k^{(n)}}{k+1} \binom{2k}{k} \frac{s^{2k}}{(2k)!} \quad b_k^{(n)} \leq e^{ck^3/n^2} \quad c > \frac{1}{12}$$

$$P\left(\frac{\lambda_N^N}{2\sqrt{N}} \geq e^{\epsilon N^{-2/3}} \right) \leq \mathbb{E} \left[\frac{\lambda_N^N}{2\sqrt{N} e^{\epsilon N^{-2/3}}} \right]^{2k} \quad (*)$$

Markov with exponent 21

$$= \frac{e^{-2\epsilon N^{-2/3} k}}{2^{2k}} \cdot \mathbb{E} \left[\frac{\lambda_N^N}{\sqrt{N}} \right]^{2k} \quad (i)$$

$$L_N = \frac{1}{N} \sum \delta_{\left(\frac{\lambda_i}{\sqrt{N}}\right)}$$

$$\langle L_N, x^{2k} \rangle = \frac{1}{N} \sum_{i=1}^N \left(\frac{\lambda_i}{\sqrt{N}} \right)^{2k}$$

$$\langle \bar{L}_N, x^{2k} \rangle = \mathbb{E} \langle L_N, x^{2k} \rangle = \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left[\left(\frac{\lambda_i}{\sqrt{N}} \right)^{2k} \right] \geq \frac{1}{N} \mathbb{E} \left[\left(\frac{\lambda_N^N}{\sqrt{N}} \right)^{2k} \right]$$

$$(i) \quad P\left(\frac{\lambda_N^N}{2\sqrt{N}} \geq e^{\epsilon N^{-2/3}} \right) \leq \frac{e^{-2\epsilon k N^{-2/3}}}{2^{2k}} \mathbb{E} \left[\left(\frac{\lambda_N^N}{\sqrt{N}} \right)^{2k} \right]$$

$$\left. \begin{aligned} b_k^{(n)} &\leq e^{\frac{1}{12} k^3 / N} \\ \binom{2k}{k} &\approx \frac{2^{2k}}{\sqrt{k}} \end{aligned} \right\} \leq \frac{e^{-2\epsilon k N^{-2/3}}}{2^{2k}} N^x$$

$$\frac{e^{-2\epsilon k N^{-2/3}}}{2^{2k}} \frac{1}{N^x} \frac{e^{\frac{1}{12} k^3 / N} \binom{2k}{k}}{\binom{2k}{k+1} \binom{2k}{k}} \leq \frac{e^{-2\epsilon k N^{-2/3}}}{2^{2k}} N^x$$

$$C 2^{2k} \frac{1}{(k+1)} \frac{2^{2k}}{\sqrt{k}} N^{-x} e^{-2\epsilon k N^{-2/3} + \frac{1}{12} k^3 / N} \leq \frac{e^{-2\epsilon k N^{-2/3}}}{2^{2k}} N^x$$

$$= C N^x k e^{-2\epsilon k N^{-2/3} + \frac{1}{12} k^3 / N}$$

$$\left(\begin{aligned} k &= N^{2/3}, & k &= N^{2/3} e^{\sqrt{2}\beta} \\ & & & e^{-3/4} - 2\epsilon N^{2/3} e^{\sqrt{2}\beta} + \frac{1}{12} N e^{1/2} \end{aligned} \right)$$

$$= C \frac{N^x}{N} e^{-3/4 - 2\epsilon N^{2/3} e^{\sqrt{2}\beta} + \frac{1}{12} N e^{1/2}}$$

$$= C \frac{e^{-(2-\frac{1}{12})e^{3/2}}}{e^{3/4}} e^{C' e^{3/2}} = C e^{-3/4} e^{-C' e^{3/2}}$$

$$P\left(\frac{\lambda_N^N}{\sqrt{N}} \geq e^{\epsilon N^{-2/3}} \right) \leq \frac{C e^{-3/4} e^{-C' e^{3/2}}}{C e^{-3/4} e^{-\frac{1}{12} e^{3/2}}} \quad \left| \begin{aligned} C' &\approx -\frac{23}{12} \\ C^* &= -\frac{1}{12} \end{aligned} \right.$$

$$\leq C e^{-3/4} e^{-\frac{1}{12} e^{3/2}} \quad -2\epsilon k N^{-2/3} + \frac{1}{12} k^3 / N$$

$$-2\epsilon N^{-2/3} + \left(\frac{1}{4} \right) k^2 / N^2$$

$$\frac{1}{4} k^2 / N^2 = 2\epsilon N^{-2/3} \quad k^2 = 8\epsilon N^{2-2/3} = 8\epsilon N^{4/3}$$

$$k = \sqrt{8\epsilon} N^{2/3} \quad -2\epsilon N^{-2/3} \sqrt{8\epsilon} N^{2/3} + \frac{1}{12} 8\epsilon^{3/2} N^{2/3} / N$$

$$= \left(-2\sqrt{8\epsilon} + \frac{2}{3} \right) \epsilon^{3/2} \quad P\left(\frac{\lambda_N^N}{2\sqrt{N}} \geq e^{\epsilon N^{-2/3}} \right) \leq C e^{-\frac{1}{4}\epsilon} e^{-\frac{1}{12}\epsilon^{3/2}}$$

Harris - Zagier Relations

$$\langle \bar{L}_N, e^s \cdot \rangle = \frac{e^{s^2/2N}}{e^{s^2/2N} \sum_{k=0}^{N-1} \frac{1}{k+1} \binom{2k}{k} \frac{(N-1)^k}{N^k} \frac{s^{2k}}{(2k)!}}$$

$$\frac{N^k}{N^k} = N(N-1) \dots (N-k+1) \quad \langle L_N, e^s \cdot \rangle = \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{2k}{k} b_k^{(n)} \frac{s^{2k}}{(2k)!}$$

$$F_n(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \binom{n-1}{k} t^k \quad (*)$$

$$\left(t \frac{d^2}{dt^2} + (2-t) \frac{d}{dt} + (n-1) \right) F_n(t) = 0$$

$$\Phi_n(t) = e^{-t/2} F_n(t) \quad \Phi_n(-s^2/N) = \langle \bar{L}_N, e^s \cdot \rangle$$

$$\left(4t \frac{d^2}{dt^2} + 8 \frac{d}{dt} + (n-t) \right) \Phi_n(t) = 0$$

$$\Phi_n(t) = \sum_{k=0}^{\infty} a_k t^k \quad 4t \sum_{k=0}^{\infty} a_k k(k-1) t^{k-2} + 8 \sum_{k=0}^{\infty} a_k k t^{k-1} + (n-t) \sum_{k=0}^{\infty} a_k t^k$$

$$4(k+1)(k+2) a_{k+1} + 4n a_k - a_{k-1} = 0$$

$$n \approx N \quad \frac{(-1)^k a_k (2k)!}{N^k} = \frac{b_k^{(n)}}{k+1} \binom{2k}{k}$$

Christoffel - Darboux formula

$$\langle \bar{L}_N, f \rangle = \frac{1}{N} \mathbb{E} \left[\sum_{i=1}^N f\left(\frac{\lambda_i}{\sqrt{N}}\right) \right] \quad X \rightarrow GUE \quad \lambda_1 \leq \dots \leq \lambda_N \quad \text{christoffel-darboux uniformly}$$

$$= \frac{1}{N} \int f\left(\frac{x}{\sqrt{N}}\right) K^{(n)}(x, x) dx = \frac{1}{N!} \det K^{(n)}(\lambda_i, \lambda_j) \quad S(\lambda_1, \dots, \lambda_N)$$

$$= \frac{1}{N} \int f(y) \left(\frac{1}{\sqrt{N}} K^{(n)}(y\sqrt{N}, y\sqrt{N}) \right) dy \quad S(\lambda) = \frac{(n-1)!}{n!} K^{(n)}(\lambda, \lambda)$$

$$K^{(n)}(x, y) = \sum_{k=0}^{n-1} \psi_k(x) \psi_k(y) \quad H_k(x) = \text{monic polynomial of degree } k$$

$$\int_{\mathbb{R}} H_k(x) H_\ell(x) e^{-x^2/2} dx = \sqrt{2\pi} k! \delta_{k\ell} \quad \psi_k(x) = e^{-x^2/4} H_k(x) \frac{1}{\sqrt{2\pi} k!}$$

$$\int \psi_k(x) \psi_\ell(x) dx = \delta_{k\ell} \quad \int \psi_k^2 dx = 1$$

$$K^{(n)}(x, y) = \left[\sum_{k=0}^{n-1} \psi_k(x) \psi_k(y) \right]$$

$$\frac{1}{N} \int f\left(\frac{x}{\sqrt{N}}\right) K^{(n)}(x, x) dx \quad y = \frac{x}{\sqrt{N}} \quad \int f(y) \frac{1}{N} K^{(n)}(y\sqrt{N}, y\sqrt{N}) \sqrt{N} dy$$

$$\langle \bar{L}_N, f \rangle = \int f(y) \left[\frac{1}{\sqrt{N}} K^{(n)}(y\sqrt{N}, y\sqrt{N}) \right] dy \quad \text{density of } \bar{L}_N$$

Christoffel - Darboux

$$H_{n+1}(x) = x H_n(x) - n H_{n-1}(x)$$

$$H_n(y) H_{n+1}(x) = x H_n(y) H_n(x) - n H_n(y) H_{n-1}(x)$$

$$H_n(x) H_{n+1}(y) = y H_n(x) H_n(y) - n H_n(x) H_{n-1}(y)$$

$$H_n(y) H_{n+1}(x) - H_n(x) H_{n+1}(y) = (x-y) H_n(x) H_n(y) + n [H_n(y) H_{n-1}(x) + H_n(x) H_{n-1}(y)]$$

$$\frac{H_n(y) H_{n+1}(x) - H_n(x) H_{n+1}(y)}{n!(x-y)} = \frac{\frac{1}{n!} H_n(x) H_n(y) + \frac{1}{(n-1)!} [H_{n-1}(y) H_n(x) - H_{n-1}(x) H_n(y)]}{(n-1)!(x-y)}$$

$$\psi_k(x) = e^{-x^2/4} H_k(x) \frac{1}{\sqrt{2\pi} k!} \quad \sum_{k=0}^{n-1} \psi_k(x) \psi_k(y) = \frac{\psi_n(x) \psi_{n-1}(y) - \psi_{n-1}(x) \psi_n(y)}{(n-1)!(x-y)} = \sqrt{N} \frac{\psi_n(x) \psi_{n-1}(y) - \psi_{n-1}(x) \psi_n(y)}{x-y}$$

$$\sum_{k=0}^{n-1} \psi_k(x) \psi_k(y) = \frac{\sqrt{N}}{x-y} [\psi_n(x) \psi_{n-1}(y) - \psi_{n-1}(x) \psi_n(y)]$$

$$f_n(x) = (A_n + B_n x) f_n(x)$$