AMIA
Guía 3

G3

3) Calcular la SVD $\rightarrow A = U \cdot \Sigma \cdot V^T$

V aves $\{A^T A\}$
$D\{\sigma_i = \sqrt{\lambda_i}\}$
U aves $\{A \cdot A^T\}$

a) $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

→ A es 2x3
→ U es 2x2
→ Σ es 2x3
→ V es 3x3

a) $A^T \cdot A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

→ Calculo avals: $\det(A^T A - \lambda I) = \begin{vmatrix} 9-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = (9-\lambda)(1-\lambda)(-\lambda)$

$\Rightarrow \begin{cases} \lambda_1 = 9 \\ \lambda_2 = 1 \\ \lambda_3 = 0 \end{cases}$

→ Calculo avecs: E1) $\begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 9 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{cases} x = x \\ y = 0 \\ z = 0 \end{cases} \Rightarrow E_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

E2) $\begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{cases} x = 0 \\ y = y \\ z = 0 \end{cases} \Rightarrow E_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

E3) $\begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ z = z \end{cases} \Rightarrow E_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

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$$\Rightarrow \boxed{V=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}}$$

$$\boxed{\Sigma=\begin{pmatrix}\sqrt{9}&0&0\\0&\sqrt{1}&0\end{pmatrix}=\begin{pmatrix}3&0&0\\0&1&0\end{pmatrix}}$$

<u>Calculo U</u>

$$u_1=\frac{1}{9}\begin{pmatrix}3&0&0\\0&-1&0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\frac{1}{9}\begin{pmatrix}3\\0\end{pmatrix}=\begin{pmatrix}1/3\\0\end{pmatrix}\xrightarrow{\text{Normalizo}}\begin{pmatrix}1\\0\end{pmatrix}$$

$$u_2=1\cdot\begin{pmatrix}3&0&0\\0&-1&0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\-1\end{pmatrix}$$

$$\Rightarrow \boxed{U=\begin{pmatrix}1&0\\0&-1\end{pmatrix}}$$

$$\Rightarrow A=U\cdot\Sigma\cdot V^T=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}3&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

b) $\underset{3\times2}{B}=\begin{pmatrix}-2&0\\0&1\\0&-1\end{pmatrix}=\underset{3\times3}{U}\cdot\underset{3\times2}{\Sigma}\cdot\underset{2\times2}{V^T}$

1) $B^T\cdot B=\begin{pmatrix}-2&0&0\\0&1&-1\end{pmatrix}\cdot\begin{pmatrix}-2&0\\0&1\\0&-1\end{pmatrix}=\begin{pmatrix}4&0\\0&2\end{pmatrix}$

→ Calculo avals: $\det(B^T B-\lambda I)=\begin{vmatrix}4-\lambda&0\\0&2-\lambda\end{vmatrix}=(4-\lambda)(2-\lambda)$

$$\Rightarrow\begin{cases}\lambda_1=4\\\lambda_2=2\end{cases}$$

→ Calculo avecs: $\begin{pmatrix}4&0\\0&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=4\begin{pmatrix}x\\y\end{pmatrix}\Rightarrow\begin{matrix}4x=4x\\2y=4y\end{matrix}\Rightarrow\begin{matrix}x=x\\y=0\end{matrix}\Rightarrow v_1=\begin{pmatrix}1\\0\end{pmatrix}$

$\begin{pmatrix}4&0\\0&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=2\begin{pmatrix}x\\y\end{pmatrix}\Rightarrow\begin{matrix}4x=2x\\2y=2y\end{matrix}\Rightarrow\begin{matrix}x=0\\y=y\end{matrix}\Rightarrow v_2=\begin{pmatrix}0\\1\end{pmatrix}$

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$$V=\begin{pmatrix}1&0\\0&1\end{pmatrix}\qquad \Sigma=\begin{pmatrix}\sqrt{4}&0\\0&\sqrt{2}\\0&0\end{pmatrix}=\begin{pmatrix}2&0\\0&\sqrt{2}\\0&0\end{pmatrix}$$

Calculo U

$$u_1=\frac{1}{\sqrt{4}}\begin{pmatrix}-2&0\\0&1\\0&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}-1\\0\\0\end{pmatrix}$$

$$u_2=\frac{1}{\sqrt{2}}\begin{pmatrix}-2&0\\0&1\\0&-1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}$$

Para hallar u_3, aplico Gram-Schmidt:

Sea $e_3=\begin{pmatrix}0\\0\\1\end{pmatrix} \Rightarrow u_3=e_3-P_{u_1}(e_3)-P_{u_2}(e_3)$

$$P_{u_1}(e_3)=\frac{\langle u_1,e_3\rangle}{\|u_1\|^2}\cdot u_1=0$$

$$P_{u_2}(e_3)=\frac{\langle u_2,e_3\rangle}{\|u_2\|^2}u_2=\frac{-1/\sqrt{2}}{1}\cdot\begin{pmatrix}0\\1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}=\begin{pmatrix}0\\-1/2\\-1/2\end{pmatrix}$$

$$u_3=\begin{pmatrix}0\\1/2\\3/2\end{pmatrix}\Rightarrow u_3^*=\frac{u_3}{\underbrace{\|u_3\|}_{\sqrt{1/4+9/4}=\frac{\sqrt{10}}{2}}}=\frac{1}{\sqrt{10}}\begin{pmatrix}0\\1\\3\end{pmatrix}$$

$$U=\begin{pmatrix}-1&0&0\\0&1/\sqrt{2}&1/\sqrt{10}\\0&-1/\sqrt{2}&3/\sqrt{10}\end{pmatrix}$$

$$\Rightarrow B=\begin{pmatrix}-1&0&0\\0&1/\sqrt{2}&1/\sqrt{10}\\0&-1/\sqrt{2}&3/\sqrt{10}\end{pmatrix}\begin{pmatrix}2&0\\0&\sqrt{2}\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\overset{2\times3}{\uparrow} \qquad \overset{2\times2}{\uparrow} \quad \overset{2\times3}{\uparrow} \quad \overset{3\times3}{\uparrow}$$

c) $C = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \end{pmatrix} = U \cdot \Sigma \cdot V^T$

1) $C^T \cdot C = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \text{avals}(C^T \cdot C) = \begin{cases} \lambda_1 = 4 \\ \lambda_2 = 4 \\ \lambda_3 = 0 \end{cases}$

avecs: $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Longrightarrow \begin{matrix} 4x = 4x \\ 4y = 4y \\ z = 0 \end{matrix} \Longrightarrow v_1 = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Longrightarrow \begin{matrix} x = 0 \\ y = 0 \\ z = z \end{matrix} \Longrightarrow v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\Longrightarrow V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$

<u>Calculo U</u>

$u_1 = \frac{1}{\sqrt{4}}\begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$u_2 = \frac{1}{\sqrt{4}}\begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\Longrightarrow C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

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d) $\underset{3\times 3}{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{pmatrix} = \underset{3\times 3}{U} \cdot \underset{3\times 3}{\Sigma} \cdot \underset{3\times 3}{V^T}$

1) $D^T \cdot D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{pmatrix}$

→ Calculo avals

$\det(D^T \cdot D - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & -2 \\ 0 & -2 & 4-\lambda \end{vmatrix} = (1-\lambda)\left[(1-\lambda)(4-\lambda)-4\right] =$

$= (1-\lambda)(4-\lambda-4\lambda+\lambda^2-4) =$

$= (1-\lambda)(\lambda^2-5\lambda) = \lambda(1-\lambda)(\lambda-5)$

$\Rightarrow \begin{cases} \lambda_1 = 5 \\ \lambda_2 = 1 \\ \lambda_3 = 0 \end{cases}$

→ Calculo avecs

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{matrix} x = 5x \\ y-2z = 5y \\ -2y+4z = 5z \end{matrix} \Rightarrow \begin{matrix} x = 0 \\ z = -2y \\ -2y = z \end{matrix} \Rightarrow v_1 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{matrix} x = x \\ y-2z = y \\ -2y+4z = z \end{matrix} \Rightarrow \begin{matrix} x = x \\ z = 0 \\ y = 0 \end{matrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \begin{matrix} x = 0 \\ y-2z = 0 \\ -2y+4z = 0 \end{matrix} \Rightarrow \begin{matrix} x = 0 \\ y = 2z \\ z = 0 \end{matrix} \Rightarrow v_3 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

$\Rightarrow V = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ -2 & 0 & 1 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sqrt{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$e)\ \underset{3\times 2}{E} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \underset{3\times 3}{U} \cdot \underset{3\times 2}{\Sigma} \cdot \underset{2\times 2}{V^T}$$

$$1)\ E^T \cdot E = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \rightarrow \underline{\text{Calculo avals}}: \det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 1 =$$

$$= ((2-\lambda)+1)\cdot((2-\lambda)-1)$$

$$= (3-\lambda)(1-\lambda)$$

$$\boxed{\begin{matrix} \lambda_1 = 3 \\ \lambda_2 = 1 \end{matrix}}$$

↳ Calculo avecs:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} 2x+y = 3x \Rightarrow y = x \\ x+2y = 3y \Rightarrow x = y \end{matrix}$$

$$\bullet\ v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \longrightarrow v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} 2x+y = x \\ x+2y = y \end{matrix} \Rightarrow \begin{matrix} y = -x \\ x = -y \end{matrix} \Rightarrow \bullet\ v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \longrightarrow v_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$\Rightarrow V = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\cdot\frac{1}{\sqrt{2}} \qquad \Sigma = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Calculo U

$$u_1 = \frac{1}{\sqrt{3}}\cdot\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{\text{normalizo}} u_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$u_2 = \frac{1}{\sqrt{1}}\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \xrightarrow{\text{normalizo}} u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Para hallar u_3, aplico Gram-Schmidt:

$$u_3 = e_3 - P_{u_1}(e_3) - P_{u_2}(e_3) \quad \text{con } e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$a)\ P_{u_1}(e_3) = \frac{\langle u_1, e_3\rangle}{\|u_1\|^2}\cdot u_1 = \frac{1}{\sqrt{6}}u_1$$

$$b)\ P_{u_2}(e_3) = \frac{\langle u_2, e_3\rangle}{\|u_2\|^2}\cdot u_2 = \frac{-1}{\sqrt{2}}u_2$$

$$\Rightarrow u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 1/3 \\ 1/3 \end{pmatrix} \xrightarrow{\text{normalizo}} u_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

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$$\Rightarrow E = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{6}}{3} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$