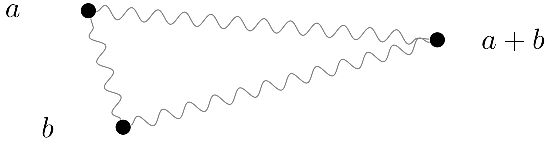
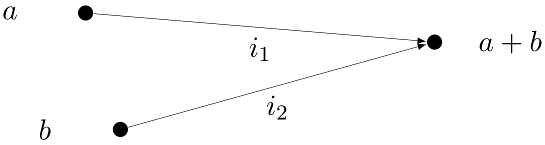
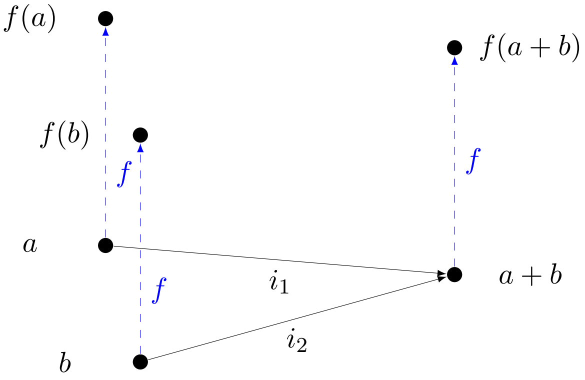
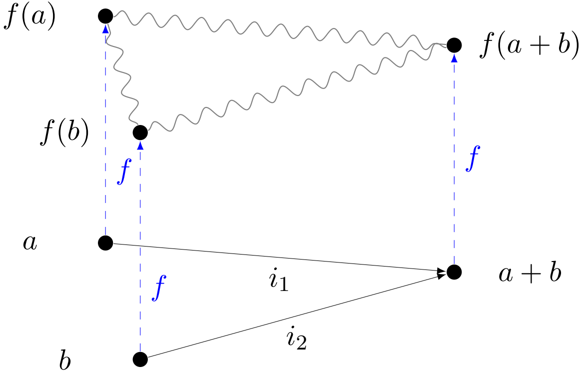

a
b
$a + b$


a
$a+b$
b

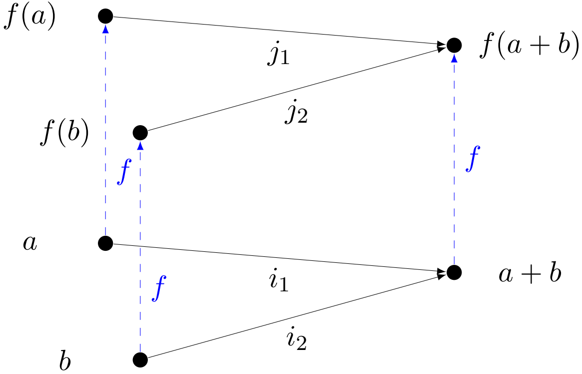

a
b
i_1
i_2
$a + b$



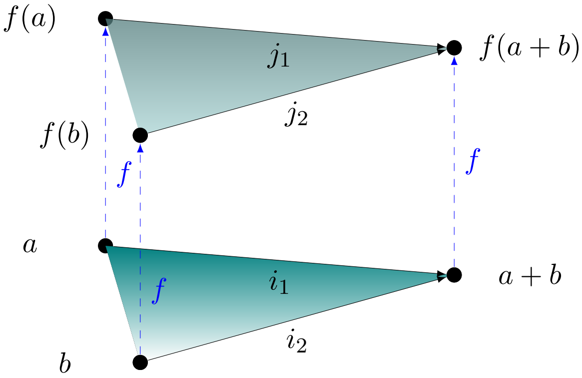
f(a)
f(a + b)
f(b)
f
f
a
a + b
i1
f
i2
b



$f(a)$
$f(a+b)$
$f(b)$
f
f
a
i_1
$a+b$
f
i_2
b



$f(a)$
$f(a+b)$
j_1
j_2
$f(b)$
f
f
a
f
i_1
$a+b$
i_2
b



$f(a)$
$f(a+b)$
j_1
j_2
$f(b)$
f
f
a
i_1
f
$a+b$
i_2
b

Natural Deduction

Definition: Logical Calculus

$$\boxed{C = \langle \Sigma, L_\Sigma, A, R \rangle}$$

$$\begin{array}{lcl} \Sigma & : & \text{Alphabet} \\ L_\Sigma & : & \text{Formal Language over } \Sigma \\ A & : & \text{Axioms} \\ R & : & \text{Inference Rules} \end{array}$$

Definition: Logical Calculus

$$\boxed{C = \langle \Sigma, L_\Sigma, A, R \rangle}$$

$$\begin{array}{lcl}
\Sigma & : & \{P, Q, R, \ldots, \wedge, \vee, \neg, \implies, \exists, \forall, \ldots\} \\
L_\Sigma & : & \{P \wedge Q, P \vee Q \implies R, \ldots\} \\
A & : & \{\vdash \top, P \vdash P, \ldots\} \\
R & : & \{P \wedge Q \vdash Q, \textit{Modus Ponens}, \ldots\}
\end{array}$$

Definition: Logical Calculus

$$\boxed{C = \langle \Sigma, L_\Sigma, A, R \rangle}$$

$$\begin{array}{lll} \Sigma & : & \{P, Q, R, \ldots, \wedge, \vee, \neg, \Rightarrow, \exists, \forall, \vdash, (,), \ldots\} \\ L_\Sigma & : & \{P \wedge Q, P \vee Q \implies R, \ldots\} \\ A & : & \{\vdash \top, P \vdash P, \ldots\} \\ R & : & \{P \wedge Q \vdash Q, \textit{Modus Ponens}, \ldots\} \end{array}$$

Note: Different Calculi can be based on same logical language

Definition: Predicate Logic — Syntax

- Let $\mathcal{V}$ denote a set of *variable* symbols
- Let $\mathcal{F}$ denote set of *function* symbols including
- Let $\mathcal{P}$ denote a set of *predicate* symbols.

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Definition: Predicate Logic — Syntax

- Let $\mathcal{V}$ denote a set of *variable* symbols
- Let $\mathcal{F}$ denote set of *function* symbols including
- Let $\mathcal{P}$ denote a set of *predicate* symbols.
- Let t be a *term*, then $t \in \mathcal{V}$ or $t \in \mathcal{F}$, where the arity of t is zero.
- Let f be a *formula*, then either $f \in \mathcal{P}$, where the arity of f is zero, or $f \equiv P(t_1, \ldots, t_n)$ where $P \in \mathcal{P}$ or $f \equiv t_1 = t_2$ where t_i are terms or $f \equiv \neg P$, $P \wedge Q$, $P \vee Q$, $P \Rightarrow Q$, $\forall x.P$, $\exists x.P$, (P) where $x \in \mathcal{V}$.

Definition: Predicate Logic — Semantics

A given formula is evaluated by applying an interpretation function $[\![]\!]$ to it.

$$\begin{aligned}[\![]\!] : &\mathcal{V} \to \mathcal{U}\\ &\mathcal{F}^n \to \mathcal{U}^n \to \mathcal{U}\\ &\mathcal{P}^n \to \mathcal{U}^n \to \{\top, \bot\}\end{aligned}$$

where $\mathcal{U}$ is a set of *semantic units*, the semantics of which further depend on their implementation (e.g. $3 \xrightarrow{i} \overline{3} \xrightarrow{\textsf{impl}} \mathrm{s}(\mathrm{s}(\mathrm{s}(0)))$) and $\mathcal{F}^n$ and $\mathcal{P}^n$ denote functions and predicates of arity n.

$$
\begin{aligned}
[\![f(t_1,\ldots,t_n)]\!] &\mapsto [\![f]\!]([\![t_1]\!],\ldots,[\![t_n]\!]) && \texttt{where } f \in F^n \\
[\![P(t_1,\ldots,t_n)]\!] &\mapsto [\![P]\!]([\![t_1]\!],\ldots,[\![t_n]\!]) && \texttt{where } P \in P^n \\
[\![(A)]\!] &\mapsto [\![A]\!] && \texttt{where } A \texttt{ is a formula} \\
[\![t_1 = t_2]\!] &\mapsto \begin{cases} \top & [\![t_1]\!] = [\![t_2]\!] \\ \bot & \texttt{else} \end{cases} && \texttt{where } t_i \texttt{ are a terms} \\
[\![\neg A]\!] &\mapsto \begin{cases} \top & \neg[\![A]\!] \\ \bot & \texttt{else} \end{cases} && \texttt{where } A \texttt{ is a formula} \\
[\![A \wedge B]\!] &\mapsto \begin{cases} \top & [\![A]\!] \wedge [\![B]\!] \\ \bot & \texttt{else} \end{cases} && \texttt{where } A, B \texttt{ are formulas} \\
[\![A \Rightarrow B]\!] &\mapsto \begin{cases} \top & [\![A]\!] \Rightarrow [\![B]\!] \\ \bot & \texttt{else} \end{cases} && \texttt{where } A, B \texttt{ are formulas}
\end{aligned}
$$

$$\llbracket A \Leftrightarrow B \rrbracket \mapsto \begin{cases} \top & \llbracket A \rrbracket \Leftrightarrow \llbracket B \rrbracket \\ \bot & \texttt{else} \end{cases} \qquad \texttt{where } A, B \texttt{ are formulas}$$

$$\llbracket \forall x. A \rrbracket \mapsto \begin{cases} \top & \forall x \in \mathcal{U}. \llbracket A \rrbracket \\ \bot & \texttt{else} \end{cases} \qquad \texttt{where } A \texttt{ is a formula}$$

$$\llbracket \exists x. A \rrbracket \mapsto \begin{cases} \top & \exists x \in \mathcal{U}. \llbracket A \rrbracket \\ \bot & \texttt{else} \end{cases} \qquad \texttt{where } A \texttt{ is a formula}$$

$$[\![A \Leftrightarrow B]\!] \mapsto \begin{cases} \top & [\![A]\!] \Leftrightarrow [\![B]\!] \\ \bot & \texttt{else} \end{cases} \qquad \texttt{where } A, B \texttt{ are formulas}$$

$$[\![\forall x.A]\!] \mapsto \begin{cases} \top & \forall x \in \mathcal{U}.[\![A]\!] \\ \bot & \texttt{else} \end{cases} \qquad \texttt{where } A \texttt{ is a formula}$$

$$[\![\exists x.A]\!] \mapsto \begin{cases} \top & \exists x \in \mathcal{U}.[\![A]\!] \\ \bot & \texttt{else} \end{cases} \qquad \texttt{where } A \texttt{ is a formula}$$

Note: Neither axioms nor inference rules — no logical calculus

$$[\![A \Leftrightarrow B]\!] \mapsto \begin{cases} \top & [\![A]\!] \Leftrightarrow [\![B]\!] \\ \bot & \texttt{else} \end{cases} \quad \texttt{where } A, B \texttt{ are formulas}$$

$$[\![\forall x.A]\!] \mapsto \begin{cases} \top & \forall x \in \mathcal{U}.[\![A]\!] \\ \bot & \texttt{else} \end{cases} \quad \texttt{where } A \texttt{ is a formula}$$

$$[\![\exists x.A]\!] \mapsto \begin{cases} \top & \exists x \in \mathcal{U}.[\![A]\!] \\ \bot & \texttt{else} \end{cases} \quad \texttt{where } A \texttt{ is a formula}$$

Note: Neither axioms nor inference rules — no logical calculus (yet).

Definition: Natural Deduction

Natural deduction builds upon a given logical language and adds to it structural axioms and inference rules.

$$\langle \boxed{\Sigma, G}, \boxed{A, R} \rangle$$

logical language

extension

Definition: Natural Deduction — Formation Rules (A)

Definition: Natural Deduction — Formation Rules (A)

$\wedge_{\mathsf{F}}$

$$\frac{P \quad Q}{P \wedge Q}$$

Definition: Natural Deduction — Formation Rules (A)

$$\frac{P \quad Q}{P \wedge Q}\ \wedge_{\mathsf{F}} \qquad\qquad \frac{P \quad Q}{P \vee Q}\ \vee_{\mathsf{F}}$$

Definition: Natural Deduction — Formation Rules (A)

$\wedge_{\mathsf{F}}$

$$\frac{P \quad Q}{P \wedge Q}$$

$\vee_{\mathsf{F}}$

$$\frac{P \quad Q}{P \vee Q}$$

$\Rightarrow_{\mathsf{F}}$

$$\frac{P \quad Q}{P \Rightarrow Q}$$

Definition: Natural Deduction — Formation Rules (A)

$\wedge_{\mathsf{F}}$

$$\frac{P \quad Q}{P \wedge Q}$$

$\vee_{\mathsf{F}}$

$$\frac{P \quad Q}{P \vee Q}$$

$\Rightarrow_{\mathsf{F}}$

$$\frac{P \quad Q}{P \Rightarrow Q}$$

$\top_{\mathsf{F}}$

$$\frac{}{\top}$$

Definition: Natural Deduction — Formation Rules (A)

$$\frac{P \quad Q}{P \wedge Q}\ \wedge_{\mathsf{F}} \qquad \frac{P \quad Q}{P \vee Q}\ \vee_{\mathsf{F}} \qquad \frac{P \quad Q}{P \Rightarrow Q}\ \Rightarrow_{\mathsf{F}}$$

$$\frac{}{\top}\ \top_{\mathsf{F}} \qquad \frac{}{\bot}\ \bot_{\mathsf{F}}$$

Definition: Natural Deduction — Formation Rules (A)

$$\wedge_{\mathsf{F}}\quad \frac{P \quad Q}{P \wedge Q}$$

$$\vee_{\mathsf{F}}\quad \frac{P \quad Q}{P \vee Q}$$

$$\Rightarrow_{\mathsf{F}}\quad \frac{P \quad Q}{P \Rightarrow Q}$$

$$\top_{\mathsf{F}}\quad \frac{}{\top}$$

$$\bot_{\mathsf{F}}\quad \frac{}{\bot}$$

$$\neg_{\mathsf{F}}\quad \frac{P}{\neg P}$$

Definition: Natural Deduction — Inference Rules (R)

Definition: Natural Deduction — Inference Rules (R)

$\wedge_{\mathsf{I}_1}$

$$\frac{P \quad Q}{P \wedge Q}$$

$\wedge_{\mathsf{I}_2}$

$$\frac{P \quad Q}{Q \wedge P}$$

Definition: Natural Deduction — Inference Rules (R)

$$\wedge_{\mathsf{I}_1} \quad \frac{P \quad Q}{P \wedge Q}$$

$$\wedge_{\mathsf{I}_2} \quad \frac{P \quad Q}{Q \wedge P}$$

$$\wedge_{\mathsf{E}_1} \quad \frac{P \wedge Q}{P}$$

$$\wedge_{\mathsf{E}_2} \quad \frac{P \wedge Q}{Q}$$

Definition: Natural Deduction — Inference Rules (R)

$\wedge_{I_1}$

$$\frac{P \quad Q}{P \wedge Q}$$

$\wedge_{I_2}$

$$\frac{P \quad Q}{Q \wedge P}$$

$\wedge_{E_1}$

$$\frac{P \wedge Q}{P}$$

$\wedge_{E_2}$

$$\frac{P \wedge Q}{Q}$$

$\vee_{I_1}$

$$\frac{P}{P \vee Q}$$

$\vee_{I_2}$

$$\frac{P}{Q \vee P}$$

Definition: Natural Deduction — Inference Rules (R)

$\wedge_{I_1}$

$$\frac{P \quad Q}{P \wedge Q}$$

$\wedge_{I_2}$

$$\frac{P \quad Q}{Q \wedge P}$$

$\wedge_{E_1}$

$$\frac{P \wedge Q}{P}$$

$\wedge_{E_2}$

$$\frac{P \wedge Q}{Q}$$

$\vee_{I_1}$

$$\frac{P}{P \vee Q}$$

$\vee_{I_2}$

$$\frac{P}{Q \vee P}$$

$\vee_{E}$

$$\frac{P \vee Q \quad \begin{matrix}[P] \\ R\end{matrix} \quad \begin{matrix}[Q] \\ R\end{matrix}}{R}$$

Definition: Natural Deduction — Inference Rules (R)

$\wedge_{I_1}$

$$\frac{P \quad Q}{P \wedge Q}$$

$\wedge_{I_2}$

$$\frac{P \quad Q}{Q \wedge P}$$

$\forall_I$

$$\frac{P_{[a/x]}}{\forall x.P}$$

$\wedge_{E_1}$

$$\frac{P \wedge Q}{P}$$

$\wedge_{E_2}$

$$\frac{P \wedge Q}{Q}$$

$\vee_{I_1}$

$$\frac{P}{P \vee Q}$$

$\vee_{I_2}$

$$\frac{P}{Q \vee P}$$

$\vee_E$

$$\frac{P \vee Q \quad \begin{array}{c}[P]\\ R\end{array} \quad \begin{array}{c}[Q]\\ R\end{array}}{R}$$

a does not occur in premises and undischarged assumptions

Definition: Natural Deduction — Inference Rules (R)

$\wedge_{I_1}$

$$\frac{P \quad Q}{P \wedge Q}$$

$\wedge_{I_2}$

$$\frac{P \quad Q}{Q \wedge P}$$

$\forall_I$

$$\frac{P_{[a/x]}}{\forall x.P}$$

$\wedge_{E_1}$

$$\frac{P \wedge Q}{P}$$

$\wedge_{E_2}$

$$\frac{P \wedge Q}{Q}$$

$\forall_E$

$$\frac{\forall x.P}{P_{[a/x]}}$$

$\vee_{I_1}$

$$\frac{P}{P \vee Q}$$

$\vee_{I_2}$

$$\frac{P}{Q \vee P}$$

$\vee_E$

$$\frac{P \vee Q \quad \begin{array}{c}[P]\\ R\end{array} \quad \begin{array}{c}[Q]\\ R\end{array}}{R}$$

Definition: Natural Deduction — Inference Rules (R)

$\wedge_{I_1}$ $$\frac{P \quad Q}{P \wedge Q}$$

$\wedge_{I_2}$ $$\frac{P \quad Q}{Q \wedge P}$$

$\wedge_{E_1}$ $$\frac{P \wedge Q}{P}$$

$\wedge_{E_2}$ $$\frac{P \wedge Q}{Q}$$

$\vee_{I_1}$ $$\frac{P}{P \vee Q}$$

$\vee_{I_2}$ $$\frac{P}{Q \vee P}$$

$\vee_E$ $$\frac{P \vee Q \quad \begin{matrix}[P]\\ R\end{matrix} \quad \begin{matrix}[Q]\\ R\end{matrix}}{R}$$

$\forall_I$ $$\frac{P_{[a/x]}}{\forall x.P}$$

$\forall_E$ $$\frac{\forall x.P}{P_{[a/x]}}$$

$\exists_I$ $$\frac{P_{[a/x]}}{\exists x.P}$$

Definition: Natural Deduction — Inference Rules (R)

$\wedge_{I_1}$ $$\frac{P \quad Q}{P \wedge Q}$$

$\wedge_{I_2}$ $$\frac{P \quad Q}{Q \wedge P}$$

$\wedge_{E_1}$ $$\frac{P \wedge Q}{P}$$

$\wedge_{E_2}$ $$\frac{P \wedge Q}{Q}$$

$\vee_{I_1}$ $$\frac{P}{P \vee Q}$$

$\vee_{I_2}$ $$\frac{P}{Q \vee P}$$

$\vee_E$ $$\frac{P \vee Q \quad \begin{matrix}[P]\\ R\end{matrix} \quad \begin{matrix}[Q]\\ R\end{matrix}}{R}$$

$\forall_I$ $$\frac{P_{[a/x]}}{\forall x.P}$$

$\forall_E$ $$\frac{\forall x.P}{P_{[a/x]}}$$

$\exists_I$ $$\frac{P_{[a/x]}}{\exists x.P}$$

$\exists_E$ $$\frac{\exists x.P \quad \begin{matrix}\left[P_{[a/x]}\right]\\ Q\end{matrix}}{Q}$$

Given: $\exists x.P(x)$. Prove for some unspecified a that $P(a) \vdash C$. But then certainly if $P(x)$ is true for some x, then also $P(x)$ is a proof of C, as every x is also an unspecified a.

a does not occur in premises, undischarged assumptions and Q

Definition: Natural Deduction — Inference Rules (R)

$\wedge_{I_1}$

$$\frac{P \quad Q}{P \wedge Q}$$

$\wedge_{I_2}$

$$\frac{P \quad Q}{Q \wedge P}$$

$\wedge_{E_1}$

$$\frac{P \wedge Q}{P}$$

$\wedge_{E_2}$

$$\frac{P \wedge Q}{Q}$$

$\vee_{I_1}$

$$\frac{P}{P \vee Q}$$

$\vee_{I_2}$

$$\frac{P}{Q \vee P}$$

$\vee_E$

$$\frac{P \vee Q \quad \begin{array}{c}[P]\\ R\end{array} \quad \begin{array}{c}[Q]\\ R\end{array}}{R}$$

$\forall_I$

$$\frac{P_{[a/x]}}{\forall x.P}$$

$\forall_E$

$$\frac{\forall x.P}{P_{[a/x]}}$$

$\exists_I$

$$\frac{P_{[a/x]}}{\exists x.P}$$

$\exists_E$

$$\frac{\exists x.P \quad \begin{array}{c}\left[P_{[a/x]}\right]\\ Q\end{array}}{Q}$$

$\Rightarrow_I$

$$\frac{\begin{array}{c}[P]\\ Q\end{array}}{P \to Q}$$

Definition: Natural Deduction — Inference Rules (R)

$\wedge_{I_1}$

$$\frac{P \quad Q}{P \wedge Q}$$

$\wedge_{I_2}$

$$\frac{P \quad Q}{Q \wedge P}$$

$\wedge_{E_1}$

$$\frac{P \wedge Q}{P}$$

$\wedge_{E_2}$

$$\frac{P \wedge Q}{Q}$$

$\vee_{I_1}$

$$\frac{P}{P \vee Q}$$

$\vee_{I_2}$

$$\frac{P}{Q \vee P}$$

$\vee_{E}$

$$\frac{P \vee Q \quad \begin{array}{c}[P]\\ R\end{array} \quad \begin{array}{c}[Q]\\ R\end{array}}{R}$$

$\forall_{I}$

$$\frac{P_{[a/x]}}{\forall x.P}$$

$\forall_{E}$

$$\frac{\forall x.P}{P_{[a/x]}}$$

$\exists_{I}$

$$\frac{P_{[a/x]}}{\exists x.P}$$

$\exists_{E}$

$$\frac{\exists x.P \quad \begin{array}{c}\left[P_{[a/x]}\right]\\ Q\end{array}}{Q}$$

$\Rightarrow_{I}$

$$\frac{\begin{array}{c}[P]\\ Q\end{array}}{P \rightarrow Q}$$

$\Rightarrow_{E}$

$$\frac{P \quad P \Rightarrow Q}{Q}$$

Definition: Natural Deduction — Inference Rules (R)

$\wedge_{I_1}$
$$\frac{P \quad Q}{P \wedge Q}$$

$\wedge_{I_2}$
$$\frac{P \quad Q}{Q \wedge P}$$

$\wedge_{E_1}$
$$\frac{P \wedge Q}{P}$$

$\wedge_{E_2}$
$$\frac{P \wedge Q}{Q}$$

$\vee_{I_1}$
$$\frac{P}{P \vee Q}$$

$\vee_{I_2}$
$$\frac{P}{Q \vee P}$$

$\vee_E$
$$\frac{P \vee Q \quad \begin{array}{c}[P]\\ R\end{array} \quad \begin{array}{c}[Q]\\ R\end{array}}{R}$$

$\forall_I$
$$\frac{P_{[a/x]}}{\forall x.P}$$

$\forall_E$
$$\frac{\forall x.P}{P_{[a/x]}}$$

$\exists_I$
$$\frac{P_{[a/x]}}{\exists x.P}$$

$\exists_E$
$$\frac{\exists x.P \quad \begin{array}{c}\left[P_{[a/x]}\right]\\ Q\end{array}}{Q}$$

$\Rightarrow_I$
$$\frac{\begin{array}{c}[P]\\ Q\end{array}}{P \to Q}$$

$\Rightarrow_E$
$$\frac{P \quad P \Rightarrow Q}{Q}$$

$\neg_I$
$$\frac{\begin{array}{c}[P]\\ \bot\end{array}}{\neg P}$$

Definition: Natural Deduction — Inference Rules (R)

$\wedge_{I_1}$: $$\frac{P \quad Q}{P \wedge Q}$$

$\wedge_{I_2}$: $$\frac{P \quad Q}{Q \wedge P}$$

$\wedge_{E_1}$: $$\frac{P \wedge Q}{P}$$

$\wedge_{E_2}$: $$\frac{P \wedge Q}{Q}$$

$\vee_{I_1}$: $$\frac{P}{P \vee Q}$$

$\vee_{I_2}$: $$\frac{P}{Q \vee P}$$

$\vee_{E}$: $$\frac{P \vee Q \quad \begin{matrix}[P] \\ R\end{matrix} \quad \begin{matrix}[Q] \\ R\end{matrix}}{R}$$

$\forall_{I}$: $$\frac{P_{[a/x]}}{\forall x.P}$$

$\forall_{E}$: $$\frac{\forall x.P}{P_{[a/x]}}$$

$\exists_{I}$: $$\frac{P_{[a/x]}}{\exists x.P}$$

$\exists_{E}$: $$\frac{\exists x.P \quad \begin{matrix}\left[P_{[a/x]}\right] \\ Q\end{matrix}}{Q}$$

$\Rightarrow_{I}$: $$\frac{\begin{matrix}[P] \\ Q\end{matrix}}{P \rightarrow Q}$$

$\Rightarrow_{E}$: $$\frac{P \quad P \Rightarrow Q}{Q}$$

$\neg_{I}$: $$\frac{\begin{matrix}[P] \\ \bot\end{matrix}}{\neg P}$$

$\neg_{E}$: $$\frac{P \quad \neg P}{Q}$$

Laws: Idempotence of Conjunction

1	P	
2	P	R, 1
3	$P \wedge P$	$\wedge_{\mathsf{I}}$ 1, 2

Laws: Commutativity of Conjunction

1	$P \wedge Q$	
2	P	$\wedge_{\mathsf{E}}$ 1
3	Q	$\wedge_{\mathsf{E}}$ 1
4	$Q \wedge P$	$\wedge_{\mathsf{I}}$ 2, 3

Laws: Associativity of Conjunction

1	$(P \wedge Q) \wedge R$	
2	$P \wedge Q$	$\wedge_E$ 1
3	P	$\wedge_E$ 2
4	Q	$\wedge_E$ 2
5	R	$\wedge_E$ 1
6	$P \wedge R$	$\wedge_I$ 3, 5
7	$Q \wedge (P \wedge R)$	$\wedge_I$ 4, 5

$Q \wedge (P \wedge R)$	
$\vdash (P \wedge R) \wedge Q$	`Commutativity of Conjunction`
$\vdash R \wedge (P \wedge Q)$	`Associativity of Conjunction`
$\vdash (P \wedge Q) \wedge R$	`Commutativity of Conjunction`

Laws: Law of Excluded Middle

1	$\neg(P \vee \neg P)$	Assumption for Indirect Proof
2	$\quad P$	Assumption for Indirect Proof
3	$\quad P \vee \neg P$	$\vee_{\mathsf{I}}$ 2
4	$\neg P$	$\neg_{\mathsf{I}}$ 2, 3
5	$P \vee \neg P$	$\vee_{\mathsf{I}}$ 4
6	$\neg\neg(P \vee \neg P)$	$\neg_{\mathsf{I}}$ 1, 5
7	$\neg(P \vee \neg P)$	$\neg\neg_{\mathsf{E}}$ 6

Laws: Double Negation

1	P	
2	$\quad \neg P$	Assumption
3	$\quad P \land \neg P$	$\land_I$ 1, 2
4	$\neg\neg P$	$\neg_I$ 1, 2

Laws: Double Negation

1	P	
2	$\quad \neg P$	Assumption
3	$\quad P \land \neg P$	$\land_I$ 1, 2
4	$\neg\neg P$	$\neg_I$ 1, 2

1	$\neg\neg P$	
2	$\quad \neg P$	Assumption
3	$\quad P \lor \neg P$	Law of Excluded Middle
4	$\quad \ldots$	

Laws: Conjunction Distributes over Disjunction

1	$P \wedge (R \vee Q)$	
2	$R \vee Q$	$\wedge_E$ 1
3	P	$\wedge_E$ 1
4	$\quad Q$	Assumption
5	$\quad P \wedge Q$	$\wedge_I$ 3, 4
6	$\quad (P \wedge Q) \vee (P \wedge R)$	$\vee_I$ 5
7		

Laws: Conjunction Distributes over Disjunction

1	$P \wedge (R \vee Q)$	
2	$R \vee Q$	$\wedge_E$ 1
3	P	$\wedge_E$ 1
4	$\quad Q$	Assumption
5	$\quad P \wedge Q$	$\wedge_I$ 3, 4
6	$\quad (P \wedge Q) \vee (P \wedge R)$	$\vee_I$ 5
7	$\quad R$	Assumption
8	$\quad P \wedge R$	$\wedge_I$ 3, 7
9	$\quad (P \wedge Q) \vee (P \wedge R)$	$\vee_I$ 8
10	$(P \wedge Q) \vee (P \wedge R)$	$\vee_E$ 2, 4-6, 7-9

Laws: Disjunction Distributes over Conjunction

1	$P \vee (Q \wedge R)$
2	

Laws: Disjunction Distributes over Conjunction

1	$P \vee (Q \wedge R)$	
2	P	Assumption
3		

Laws: Disjunction Distributes over Conjunction

1	$P \lor (Q \land R)$	
2	$\quad P$	Assumption
3	$\quad P \lor Q$	$\lor_I$ 2
4		

Laws: Disjunction Distributes over Conjunction

1	$P \vee (Q \wedge R)$	
2	$\quad P$	Assumption
3	$\quad P \vee Q$	$\vee_\mathsf{I}$ 2
4	$\quad P \vee R$	$\vee_\mathsf{I}$ 2
5		

Laws: Disjunction Distributes over Conjunction

1	$P \vee (Q \wedge R)$	
2	P	Assumption
3	$P \vee Q$	$\vee_I$ 2
4	$P \vee R$	$\vee_I$ 2
5	$(P \vee Q) \wedge (P \vee R)$	$\wedge$3, 4
6		

Laws: Disjunction Distributes over Conjunction

1	$P \lor (Q \land R)$	
2	$\quad P$	Assumption
3	$\quad P \lor Q$	$\lor_{\mathsf{I}}$ 2
4	$\quad P \lor R$	$\lor_{\mathsf{I}}$ 2
5	$\quad (P \lor Q) \land (P \lor R)$	$\land$3, 4
6	$\quad Q \land R$	Assumption
7		

Laws: Disjunction Distributes over Conjunction

1	$P \vee (Q \wedge R)$	
2	$\quad P$	Assumption
3	$\quad P \vee Q$	$\vee_I$ 2
4	$\quad P \vee R$	$\vee_I$ 2
5	$\quad (P \vee Q) \wedge (P \vee R)$	$\wedge$3, 4
6	$\quad Q \wedge R$	Assumption
7	$\quad Q$	$\wedge_E$ 6
8		

Laws: Disjunction Distributes over Conjunction

1	$P \vee (Q \wedge R)$	
2	$\quad P$	Assumption
3	$\quad P \vee Q$	$\vee_{\mathsf{I}}$ 2
4	$\quad P \vee R$	$\vee_{\mathsf{I}}$ 2
5	$\quad (P \vee Q) \wedge (P \vee R)$	$\wedge$3, 4
6	$\quad Q \wedge R$	Assumption
7	$\quad Q$	$\wedge_{\mathsf{E}}$ 6
8	$\quad P \vee Q$	$\vee_{\mathsf{I}}$ 7
9		

Laws: Disjunction Distributes over Conjunction

1	$P \vee (Q \wedge R)$	
2	$\quad P$	Assumption
3	$\quad P \vee Q$	$\vee_{\mathsf{I}}$ 2
4	$\quad P \vee R$	$\vee_{\mathsf{I}}$ 2
5	$\quad (P \vee Q) \wedge (P \vee R)$	$\wedge$3, 4
6	$\quad Q \wedge R$	Assumption
7	$\quad Q$	$\wedge_{\mathsf{E}}$ 6
8	$\quad P \vee Q$	$\vee_{\mathsf{I}}$ 7
9	$\quad R$	$\wedge_{\mathsf{E}}$ 6
10		

Laws: Disjunction Distributes over Conjunction

1	$P \vee (Q \wedge R)$	
2	P	Assumption
3	$P \vee Q$	$\vee_{\mathsf{I}}$ 2
4	$P \vee R$	$\vee_{\mathsf{I}}$ 2
5	$(P \vee Q) \wedge (P \vee R)$	$\wedge$3, 4
6	$Q \wedge R$	Assumption
7	Q	$\wedge_{\mathsf{E}}$ 6
8	$P \vee Q$	$\vee_{\mathsf{I}}$ 7
9	R	$\wedge_{\mathsf{E}}$ 6
10	$P \vee R$	$\vee_{\mathsf{I}}$ 9
11		

1	$P \vee (Q \wedge R)$	
2	P	Assumption
3	$P \vee Q$	$\vee_{\mathsf{I}}$ 2
4	$P \vee R$	$\vee_{\mathsf{I}}$ 2
5	$(P \vee Q) \wedge (P \vee R)$	$\wedge$3, 4
6	$Q \wedge R$	Assumption
7	Q	$\wedge_{\mathsf{E}}$ 6
8	$P \vee Q$	$\vee_{\mathsf{I}}$ 7
9	R	$\wedge_{\mathsf{E}}$ 6
10	$P \vee R$	$\vee_{\mathsf{I}}$ 9
11	$(P \vee Q) \wedge (P \vee R)$	$\wedge_{\mathsf{I}}$ 8, 10
12		

Laws: Disjunction Distributes over Conjunction

1	$P \lor (Q \land R)$	
2	$\quad P$	Assumption
3	$\quad P \lor Q$	$\lor_I$ 2
4	$\quad P \lor R$	$\lor_I$ 2
5	$\quad (P \lor Q) \land (P \lor R)$	$\land$3, 4
6	$\quad Q \land R$	Assumption
7	$\quad Q$	$\land_E$ 6
8	$\quad P \lor Q$	$\lor_I$ 7
9	$\quad R$	$\land_E$ 6
10	$\quad P \lor R$	$\lor_I$ 9
11	$\quad (P \lor Q) \land (P \lor R)$	$\land_I$ 8, 10
12	$(P \lor Q) \land (P \lor R)$	$\lor_E$ 1, 2-5, 6-11

Laws: De Morgan

1	$\neg(P \wedge Q)$	
2	$\neg(\neg P \vee \neg Q)$	Assumption
3	$\neg P$	Assumption
4	$\neg P \vee \neg Q$	$\vee_{\mathsf{I}}$ 3
5	$(\neg P \vee \neg Q) \wedge \neg(\neg P \vee \neg Q)$	$\wedge_{\mathsf{I}}$ 2, 4
6	$\neg\neg P$	$\neg_{\mathsf{I}}$ 3, 5
7	P	$\neg\neg_{\mathsf{E}}$ 6
8	$\neg Q$	Assumption
9	$\neg P \vee \neg Q$	$\vee_{\mathsf{I}}$ 7
10	$(\neg P \vee \neg Q) \wedge \neg(\neg P \vee \neg Q)$	$\wedge_{\mathsf{I}}$ 2, 8
11	$\neg\neg Q$	$\neg_{\mathsf{I}}$ 7, 9
12	Q	$\neg\neg_{\mathsf{E}}$ 10
13	$P \wedge Q$	$\wedge_{\mathsf{I}}$ 6, 10
14	$(P \wedge Q) \wedge \neg(P \wedge Q)$	$\wedge_{\mathsf{I}}$ 1, 11
15	$\neg\neg(\neg P \vee \neg Q)$	$\neg_{\mathsf{I}}$ 2, 12
16	$\neg P \vee \neg Q$	$\neg\neg_{\mathsf{E}}$ 13