Volatility Skew and the Valuation of Mortgages

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Mortgage-backed securities (MBS) contain embedded options and are thus exposed to the uncertainty of interest rates. Market participants usually distill this uncertainty into a single number—volatility—using the Black formula. In doing so, the shape of the distribution of future uncertain rates is assumed to be lognormal. If the actual distribution differs from the assumed lognormal distribution, then volatility will have its own dynamics. Understanding the dynamic of volatility is thus crucial for pricing and hedging MBS.

The market's expectation of interest rate volatility is mainly expressed in prices of liquid interest rate derivatives, swaptions, and caps. A swaption is the option on the right to receive (or pay) a fixed coupon while paying (or receiving) a floating rate starting on the expiry date of the option and ending on a maturity date. A cap (floor) is an option to pay (receive) a fixed coupon versus receiving (paying) three-month LIBOR.

Swaption and cap prices are often quoted in yield volatility. Consider a 5×10 swaption struck at-the-money at 5% quoted with a yield volatility of 20%. This volatility implies that a one-standard-deviation move over one year of the five-year forward ten-year swap rate would be $5\% \times 0.2 = 1\%$. The pricing of these swaptions is then given by the celebrated Black formula (Black [1976]).

The Black formula assumes the same yield volatility for all strikes. In recent years, however, this assumption has been shown to be faulty. Swaptions struck at rates below the at-the-money rate consistently trade at volatilities higher than at-the-money swaptions, and swaptions struck at rates above the at-the-money rate trade at lower volatilities. This feature is called the OTM (off-the-money) volatility skew.

At the same time, at-the-money swaptions trade at higher volatilities in a low-rate environment than they do in a high-rate environment. This observation is usually called the ATM (at-the-money) volatility skew. Together, these features have been collectively dubbed the "volatility skew."

A common misconception in the market is that the OTM and ATM volatility skews are identical, but we will argue that this can lead to significant errors.

All term-structure models used for valuing interest rate contingent claims will have an embedded OTM and ATM volatility skew. If investors are using a term-structure model to value MBS, it is crucial that they understand which volatility skew the model expresses.

An understanding of volatility skewness in the fixed-income market is essential to proper pricing and hedging of securities. In the following sections we discuss various term-structure models and their apprehension of volatility skew.

WHAT IS VOLATILITY SKEW?

Measuring Volatility

We don't measure interest rate volatility with a yardstick. Although we have a strong sense of the difference between high volatility (rates will likely change a lot) and low volatility (rates will likely change a little), a precise measurement must be based on some chosen distribution of future rates.

Distributions of future interest rates are intuitively simple. Investors agree that large changes in rates will be less likely than small changes, so that the general distribution consists of a single hump centered near the most common expectation with narrow tails to account for the slim possibilities of unforeseen significant changes. But the specific nature of the distribution is more problematical.

A commonly used distribution of interest rates is the lognormal distribution, which presumes that future changes in interest rates will be proportional to their level. Another distribution is the normal distribution, which presumes that future changes in interest rates will be independent of their level. Both the lognormal and normal distributions can be completely described by specifying a mean and standard deviation. In Exhibit 1, we show these two distributions with the same mean and standard deviation. There are many other possible distributions, but they are more difficult to describe and specify.

Exhibit 1 also shows a shifted lognormal distribution. Because the normal distribution assumes that changes in rates do not vary with rate level, it assumes that the change is proportional to a constant. The lognormal distribution assumes that the change is proportional to the rate level. We can find something between these two if we assume that changes in rates are proportional to the rate level plus a constant. If the constant is small, the distribution is very close to lognormal. If the constant is large, then the distribution is very close to normal. This constant is the "shift" in the shifted lognormal distribution.

The distribution most commonly used in measuring interest-rate volatility is the lognormal distribution because it is the one assumed by the Black option model. The Black volatility is commonly used to quote interest-rate derivatives, such as caps and swaptions.[1] The Black formula converts this volatility to a price.

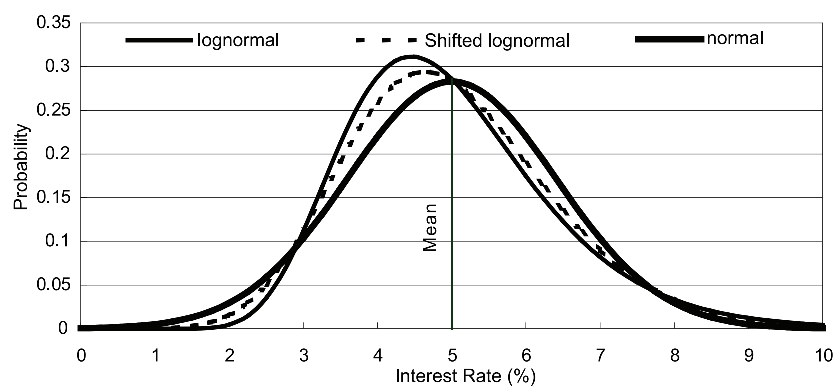
Swaptions, Caps, and Moneyness

Although the pricing of almost all fixed-income securities, with the exception of fixed-term bullet bonds and floaters without caps or floors, is based on the market's expectations for the volatility of interest rates, certain derivative instruments have prices that are very directly correlated to volatility. The most liquid of these are swaptions and caps.

A receiver (payer) swaption is the option on the right to receive (pay) a preset fixed-rate coupon (the strike) in exchange for a floating-rate coupon (i.e., the option to enter into a standard swap transaction at some future time). Most commonly, swaptions are entered into at

EXHIBIT 1

Interest Rate Distributions with Same Mean and Standard Deviation



Source: Citigroup.

strikes that are *at-the-money* (i.e., the fixed-rate coupon is equivalent to the forward rate with a tenor equal to that of the swaption at the time of expiration). Thus, a standard 1×5 swaption struck at the money would be an option with one year to expiration on a five-year swap contract with a strike equivalent to the five-year rate one year hence. This strike is called at-the-money because if at the expiry of the option the current rate is equal to the forward rate, the value of the swaption is nil. If the swap is struck at a strike different from the forward rate, the strike is *off-the-money*. As time progresses and rates change, at-the-money swaptions become off-the-money swaptions.

A cap is a contract to receive over a fixed period of time any positive difference between a floating rate and a preset fixed rate. As with swaptions, caps and floors become off-the-money as time progresses and rates change.

OTM Volatility Skew and ATM Volatility Skew

The market volatility of interest rates is directly correlated to the value of caps and swaptions. The more volatile the market expects interest rates to be, the more valuable these securities are. To avoid constant repricing during the trading day as rates move up or down, the prices of these securities are often quoted as yield volatilities, which are more stable. The conversion between price and volatility is done with the Black formula, which has its attendant assumptions.

The market for caps and swaptions is highly liquid because of the large volume of trading between entities with low credit risk. Because we have accurate prices for liquid instruments, we can examine these quoted volatilities, which show a skewness pattern in two different ways, one based on historical prices for the most widely traded at-the-money strikes and the other based on current prices for a variety of strikes. These two features have been collectively dubbed the *volatility skew*.

The Black volatilities of caps and swaptions show distinctive patterns. To obtain a price from the Black option model we require three things: the volatility, and the mean of the distribution and of the strike price. We normally refer to the mean as the at-the-money strike, and a derivative for which the mean and strike are the same is struck at the money. Any other strike is off-the-money. On any given trading day, we can observe the Black volatility of the at-the-money derivative, and also the volatilities for a variety of off-the-money strikes. As we shall see, these volatilities are not the same. There is a strong correlation between the Black volatility and the strike price, and the pattern they reveal is called OTM volatility skew.

But there is another pattern that we can observe by looking at the volatilities of at-the-money derivatives over a period of time. We can observe a strong correlation between the level of interest rates and the Black volatility of at-the-money derivatives. We call this pattern ATM volatility skew. In the next section we look at the empirical evidence for both OTM and ATM volatility skew.

EMPIRICAL EVIDENCE OF VOLATILITY SKEW

We have noted that there are observable patterns for both OTM volatility skew and ATM volatility skew. Let us look at some hard evidence.

Evidence of OTM Volatility Skew

OTM derivatives are not particularly liquid, but trades do occur regularly. Exhibit 2 shows the volatility the market assigns to different strikes; obviously, the lower the strike, the higher the Black volatility. This is the OTM volatility skew.

Evidence of ATM Volatility Skew

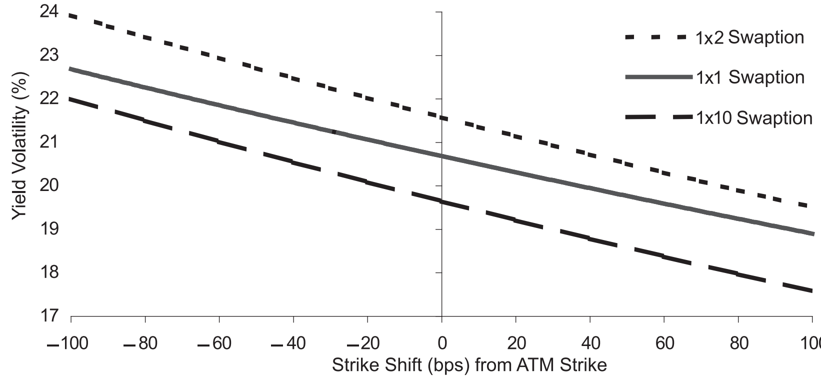
In Exhibits 3 and 4 we show data that can be obtained easily from The Yield Book™ historical data page. We have plotted ATM Black volatility (The Yield Book calls it "yield volatility") against the applicable ATM strike over a period of five years. The fitted trend line in these exhibits is logarithmic, since the volatility cannot go below zero as rates increase, and this curve fits better than a straight line; the lower the rate level, the higher the ATM Black volatility.

But remember that the volatility skew shown in Exhibits 3 and 4 depends on a lognormal distribution of interest rates. These exhibits tell us only that the skew is observable for that distribution. If we plotted the volatility of a different distribution, we could see a different effect.

Is the Magnitude of Interest Rate Changes Proportional to Rate Levels?

We have one more piece of empirical evidence regarding volatility skew, and again this chart can be

EXHIBIT 2
OTM Market Volatility Skews (at close on Sep 15, 05)



Source: Citigroup.

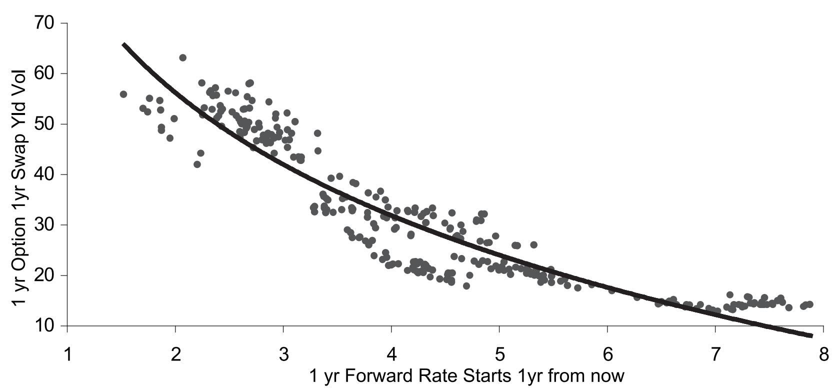
obtained easily from the historical data page of The Yield Book. As we mentioned earlier in this article, a lognormal distribution of interest rates assumes that the size of changes in interest rates is proportional to the level of rates. But is this what we observe in the marketplace? Exhibit 5 shows absolute changes over the prior one-month period of the five-year swap rate plotted against the rate itself. Is the change proportional to the level of the rate? For this 15-year sample, it appears that this is not the case.

In fact, the trend line shows a slight negative correlation. The simple conclusion from this chart is that 1) changes in this interest rate were independent of the rate level for the period in question and 2) a normal distribution of expected interest rates would likely be a more accurate predictor than a lognormal distribution, as the size of changes in interest rates is independent of the rate level for a normal distribution.

INTEREST RATE DISTRIBUTIONS EXPLAIN VOLATILITY SKEW

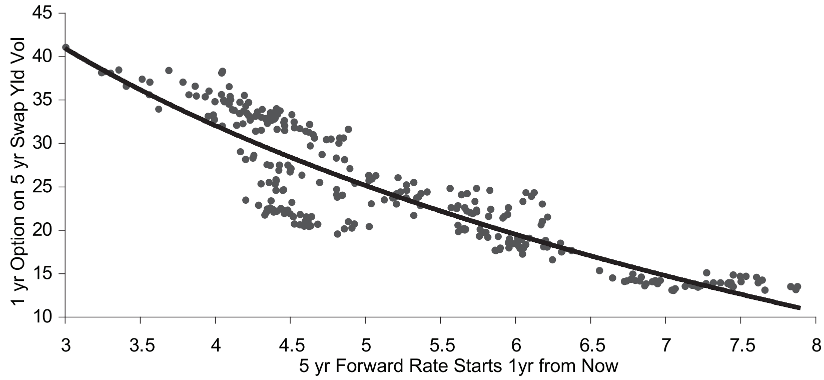
Some (but not the sophisticated readers of this publication) might jump to the conclusion that volatility skew indicates that the market has different expectations for interest rate distributions depending upon the strike (OTM skew) or the level of rates (ATM skew). But there is an

EXHIBIT 3
1 × 1 Swaption Yield Volatility versus 1 × 1 Forward Rate, Weekly Data, Jan 00–Sep 05



Source: Citigroup.

EXHIBIT 4
OTM Market Volatility Skews (at close on Sep 15, 05)



Source: Citigroup.

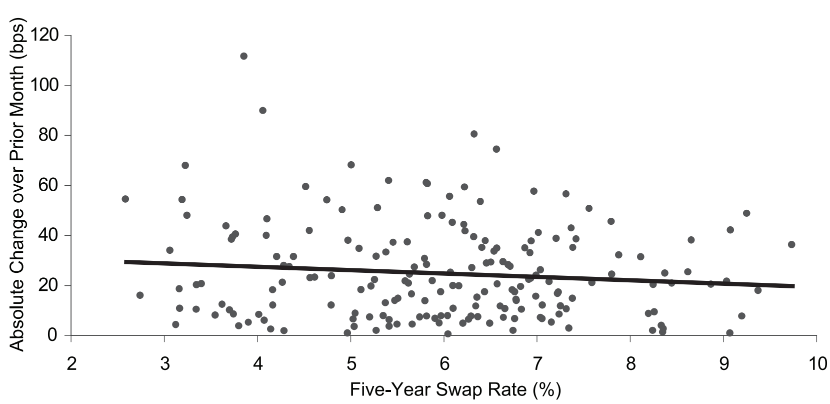
alternative explanation—the market does not expect a lognormal distribution of interest rates. We shall show how the assumption of an alternative distribution can cause both OTM and ATM volatility skew.

OTM Volatility Skew

Exhibit 6 shows how an alternative expected distribution of interest rates can give rise to OTM volatility skew. The lognormal and normal distributions shown are set to have the same mean value and to price an at-the-money receiver swaption at the same value. The derivative price is proportional to the integral of the weighted area beneath the distribution curve to the left of the strike barrier, with the weight of each area fragment determined by its distance from the strike price.[2] For higher rates near the strike price, the lognormal curve is higher, while for lower rates farther from the strike price, the normal curve is higher. It is easy to see that these excess areas are not equal, but they balance out because the smaller one is farther from the strike price and thus has a higher weight.

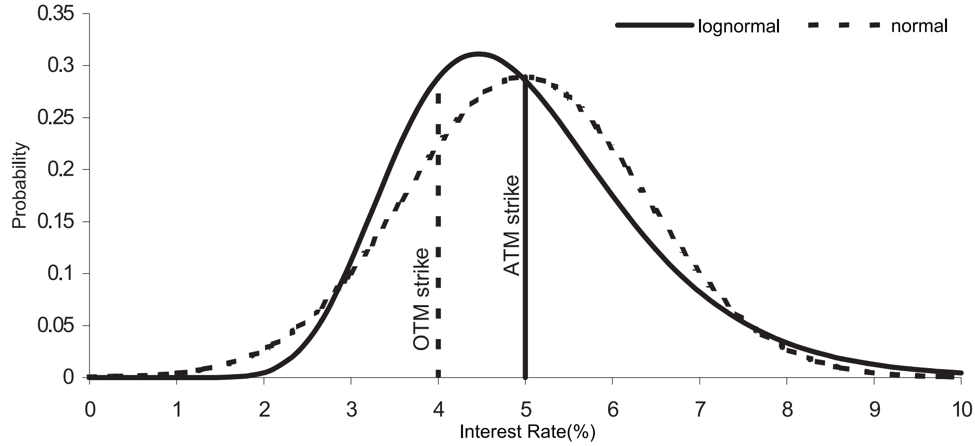
Although the ATM derivative prices are the same, if we look at the value of an OTM receiver swaption with

EXHIBIT 5
Absolute Change in 5-Year Swap Rate versus Level, Monthly Data, Jan 90–Aug 05



Source: Citigroup.

EXHIBIT 6
Comparable Distributions for an At-the-Money Receiver Swaption



Source: Citigroup.

a strike of 4%, the swaption values will both be smaller, but they will also be quite different. First, the excess lognormal area between 4% and 5% is eliminated entirely. Second, the remaining excess lognormal area moves proportionately closer to the new strike (4%) than does the excess normal area at the lower rate values. Thus, the weight of the lognormal surplus is reduced much more than the weight of the normal surplus. The result of these two effects is that the value of the lognormal OTM option is significantly less than the value of the normal OTM option if the normal distribution is used. In order to make the OTM prices the same, the lognormal volatility must be increased, the very effect that we observe as OTM volatility skew.

Thus, if the market perceives the distribution of rates to be normal, the OTM swaption price struck at a low rate will be much higher than the Black value with its implied lognormal distribution. To compensate, the Black volatility must be increased for a lower strike, and this is the OTM volatility skew that we observe. The market's expectation of a rate distribution that is more like the normal distribution than the lognormal distribution will give rise to OTM volatility skew.

The skew effect for a receiver swaption also obtains when we increase the strike above the ATM level, although it is difficult to calculate this from Exhibit 6. Although the normal distribution is above the lognormal distribution for rates just above the ATM strike, this advantage for the price of the normal integral is outweighed by the increase in weight for the area where the lognormal distribution is higher at rates just below the ATM strike. So, when the OTM strike is increased, the Black volatility must be reduced if we assume a normal distribution of rates.

This analysis shows that OTM volatility skew can be fully explained by the market's expectation of a rate distribution that is different from lognormal. Although the Black volatility must change, we can forego any change in rate distribution if we employ the distribution that the market expects.[3] It is not our task at this point to determine the "correct" distribution. We are only trying to explain the phenomenon of volatility skew as resulting from a specific choice of rate distribution.

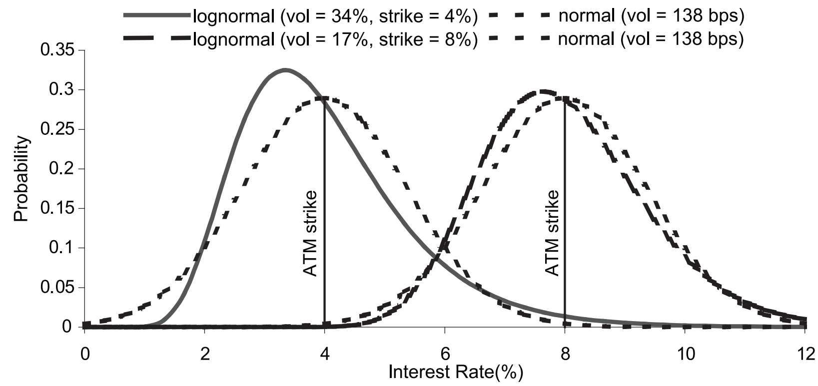
ATM Volatility Skew

In Exhibit 7 we show four interest rate distributions. The peaks, left to right, are

Lognormal distribution, mean of 4%, standard deviation of 34.64%;
Normal distribution, mean of 4%, standard deviation of 137.87 bps;
Lognormal distribution, mean of 8%, standard deviation of 17.25%; and
Normal distribution, mean of 8%, standard deviation of 137.87 bps.

Exhibit 7

Comparable Distributions for an At-the-Money Receiver Swaption (Strike = 4% and 8%)



Source: Citigroup.

The distributions with the same mean are calibrated to produce the same value. If we make the assumption that the level of interest rates does not influence the size of prospective changes in rates, then the normal distributions should have the same volatility, which they do. But if we want to get the same swaption price for a lognormal distribution, the Black volatility for the lower strike must be over twice that of the higher strike.

The higher Black volatility for lower rates in Exhibit 7 is comparable to the historical data that we have displayed in Exhibits 3 and 4, where Black volatilities for low rates are substantially higher than those for high rates. This again is evidence that the market does not see the magnitude of future changes in interest rates as proportional to the level of rates. If it did, the Black volatility would not change as significantly with a change in rate level.

Exhibit 7 is a correct representation if one assumes that interest rate levels do not affect the size of changes in interest rates. Thus, the normal distribution is exactly the same for both ATM strikes, except that the mean is shifted to coincide with the strike price. The lognormal volatility must change, however. When the strike goes down, the Black volatility must increase to compensate for the reduced extent from the strike to 0%. If the ATM strike were very small, the lognormal distribution with even a huge volatility could not achieve the same price as the normal distribution. The lognormal integral is bounded by 0%, while the normal distribution gains value from its negative interest rates.

Looking at Exhibit 7, we can also see that the skewness of ATM volatility is related to, but not the same as, the skewness of OTM volatility. *For ATM volatility, we are looking at the effect of changing the mean, while for OTM volatility we are looking at the effect of changing the strike.* In the ATM case, we require that the volatility increase with lower rates in order to compensate for the reduced distance to zero. In the OTM case, we require that the volatility increase with lower rates to compensate for the loss of the parts of the distribution that are advantageous for the lognormal distribution.

Just as we saw for OTM volatility skew, if the market makes a different assumption about interest rate distributions, we will observe an ATM volatility skew. If the market assumes that the size of changes in interest rates is independent of rate level, the Black model, which makes the very different assumption that the size of rate changes is proportional to the interest rate level, will show ATM volatility skew. Although the assumption of a normal distribution of future rates is likely to break down when we get to very low interest rate levels and the high likelihood of negative rates, the market does seem to reflect the view that the magnitude of rate changes is definitely not proportional to the level of rates, as we saw in Exhibit 5.

ATM VERSUS *OTM* SKEW

As we have seen, both ATM and OTM volatility skew arise from the same root cause—the market does not accept the lognormal distribution of expected interest rates that the Black model necessarily implies. However, the nature of the skewness in the two cases arises from different effects.

OTM skewness is caused by a difference in the tails of the distribution for the lognormal and normal models. The lognormal distribution is more concentrated near the mean for lower rates, and less concentrated near the mean for higher rates.

ATM volatility skewness is caused by the limited amount that rates can go down in a lognormal distribution as the mean declines, while in a normal distribution rates can go negative. To compensate, the lognormal distribution has to have higher comparative volatility for lower rate levels. This means that OTM and ATM skewness need not be the same. In both cases, we can observe the increase in volatility as the strike (for OTM skew) or the rate level (for ATM skew) declines, and vice versa. But the methodology resulting in the skewness is such that we cannot expect the skewness to be the same for both.

In Exhibit 8 we show the Black volatilities for a hypothetical situation in which the normal volatility is always the same. Each normal distribution of rates can be converted to a swaption price, which, in turn, can be converted to a Black volatility. Exhibit 8 shows that the ATM volatility skew is about double the OTM volatility skew.

To evaluate a portfolio using current rate expectations, one must take into account only OTM volatility skew; ATM volatility skew applies only when rates move. The OTM volatility skewness is required for fixed-income securities that have off-the-money options. Thus, if we want to evaluate a portfolio of swaptions that were once at-the-money, but now are not, we need to evaluate them as OTM options, and it helps to fit OTM volatility skew accurately to do so. ATM volatility skew will have no effect on this evaluation.

But if we are concerned about hedging against moves in interest rates and the interest rate duration of our portfolio, we must also consider ATM volatility skew. We cannot assume the same lognormal volatility for higher or lower rate levels because of ATM volatility skew.

VOLATILITY SKEW IN TERM-STRUCTURE MODELS

The choice of what distribution of rates to use for evaluating a derivative such as a cap or swaption applies also to a complex term-structure model. The model makes assumptions about how likely changes in rates are when the level of rates moves up or down. If it assumes that a certain change is just as likely regardless of the level of rates, it is like a normal model. If it assumes that the magnitude of changes in rates is proportional to the level of rates, it is like a lognormal model. It can also assume something between the two, in which case it is like a shifted lognormal model.

The Yield Book currently offers two choices of two-factor term-structure models, the current default 2F-Skew model and the older 2F model (Bhattacharjee, Russell, and Walter [2003]; Chan and Russell [1997]). The older model, 2F, assumes a lognormal distribution of rates (i.e., it assumes that changes in rates are proportional to

EXHIBIT 8

OTM versus ATM Volatility Skew for 2 × 10 Payer Swaption Assuming 93.2 bps Normal Volatility, Oct 21, 05

OTM Volatility Skew							
Change in Strike	−75	−50	−25	0	25	50	75
Normal Distr. Price	702.1850	580.4707	471.2901	375.3337	292.8401	223.5707	166.8350
Implied Black Vol	20.2382	19.6777	19.1571	18.6718	18.2180	17.7923	17.3920
Vol Skew	1.5665	1.0059	0.4853	0.0000	−0.4538	−0.8795	−1.2797
ATM Volatility Skew							
Change in Rates (bps)	−75	−50	−25	0	25	50	75
Norm Mod. ATM Price	394.8418	388.2164	381.6962	375.3337	369.1290	363.0294	357.0350
BS Implied ATM Vol	21.9868	20.7578	19.6594	18.6718	17.7788	16.9676	16.2273
Realized Vol Skew	3.3150	2.0861	0.9877	0.0000	−0.8929	−1.7042	−2.4444

Source: Citigroup.

their level). Because of this, it cannot accurately price OTM derivatives at their market prices, and it does not increase Black volatility when rates go down or decrease Black volatility when rates go up.

The current production model, the 2F-Skew model, assumes a shifted lognormal distribution of rates with a shift of 3%. This distribution lies in between a lognormal distribution and a normal distribution, as is illustrated in Exhibit 1. This model does increase Black volatility when rates go down and decreases Black volatility when rates go up, and it does a better job of pricing OTM derivatives than the older 2F model.

In the following paragraphs we will examine two additional models that have even more volatility skew than 2F-Skew. While the shift parameter in the 2F-Skew model is set to 3%, we will consider a shifted lognormal two-factor model with a shift of 7%. The increase in the shift parameter makes the model more like a normal model. Finally, we will consider a normal two-factor model that is designed to avoid negative rates by placing a boundary at zero.

All of these two-factor term-structure models are arbitrage free and can accurately price liquid interest-rate derivatives such as at-the-money caps and swaptions. They all produce correlations between rates of different tenors that reasonably match historical correlations. In addition, the 2F-Skew model prices OTM options more accurately than the older 2F model. The normal model does this even better, and it also eliminates any negative interest rates.

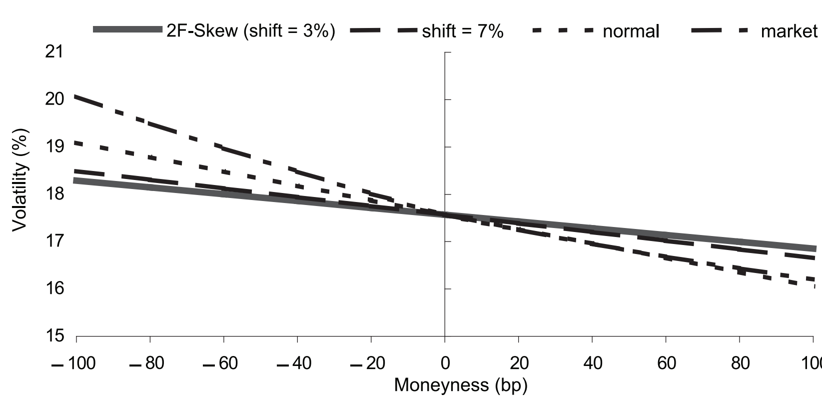
Exhibit 9 shows the OTM skew for these models as well as the market volatility skew. As one can see in Exhibit 9, the normal two-factor model is the most accurate of the three models in matching OTM volatility skew.

Exhibit 10 shows the ATM volatility skew that arises from the three models. It is scaled the same as Exhibit 9, and it shows that the ATM skew is about twice as great as the OTM skew for each model. These results should be compared against the historical 5×10 ATM volatility skew, using the technique applied to 1×1 and 1×5 swaptions in Exhibits 3 and 4.[4] There is no accurate "market" ATM skew, as both volatilities and rate levels change over time, and one cannot ascribe price changes in interest rate derivatives to just one of these factors. The best we can do is look at the historical correlation between rate level and volatility, as we have in Exhibits 3 and 4.

Finally, in Exhibits 11 and 12 we show the volatility surface generated for the 2F-Skew model and the normal model. This volatility surface can be obtained from The Yield Book for the 2F-Skew model and it shows in one exhibit both the ATM and the OTM skew. The exhibit is separated into three parts. The middle part contains options struck at different strikes using closing yield curve levels, while the top and bottom parts show the same for the two scenarios of shifting the curve up and down. In order to obtain the ATM skew, one simply compares vols in a column, whereas the OTM skew is found in the rows. Note again that the ATM volatility skew differs significantly from the OTM volatility skew for each model.

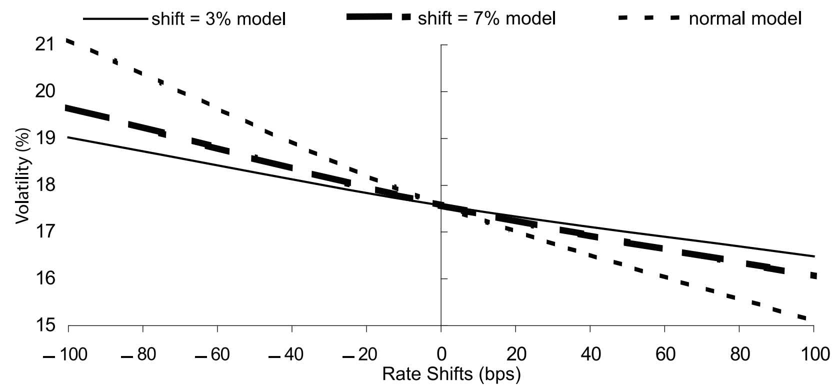
EXHIBIT 9

OTM Volatility Skew for 5×10 Swaption (at close on Sep 30, 05)



Source: Citigroup.

EXHIBIT 10
ATM Volatility Skews for the 5 × 10 Swaption (at close on Sep 30, 05)



Source: Citigroup.

IMPACT OF THE SKEW ON VALUATION OF *MBS*

A mortgaged-back security is long an amortizing bond and is short a call option on the bond. The strike on this option is below the coupon of the mortgage. A more complete way of looking at a mortgage-backed security would be to assume that there are many options, with different strikes, as homeowners choose different levels at which to refinance. In the absence of the skew, all these options have the same volatility. Once the skew is introduced, each option gets its own volatility. The differences in these volatilities grow as we keep increasing the skew. Because MBS are short the option, discount MBS (low coupon, therefore, low-option strike) will have tighter option-adjusted spreads (OAS). Similarly, premium MBS (high coupons and high-option strikes) will have wider OAS.

In the following exhibits we show the differences in OAS and durations for a shifted lognormal model with a shift of 7% (Exhibit 13) and a normal model

EXHIBIT 11
Volatility Surface for the 2F-Skew Model

		Change in Strike (bps)				
Swaption	**ATM Rate**	**– 100**	**– 50**	**0**	**50**	**100**
Rate Shift Down 100 bps						
3x10	3.963	21.621	21.053	20.485	19.917	19.349
5x10	4.073	20.102	19.563	19.024	18.485	17.947
7x10	4.171	18.305	17.821	17.337	16.853	16.369
No Rate Shift						
3x10	4.969	19.583	19.186	18.791	18.394	17.997
5x10	5.085	18.288	17.929	17.571	17.211	16.852
7x10	5.191	16.665	16.401	16.151	15.892	15.635
Rate Shift Up 100 bps						
3x10	5.975	18.116	17.823	17.528	17.235	16.942
5x10	6.098	16.962	16.72	16.479	16.238	15.996
7x10	6.211	15.478	15.347	15.215	15.083	14.952

Source: Citigroup.

EXHIBIT 12
Volatility Surface for the Normal Model

		Change in Strike (bps)				
Swaption	**ATM Rate**	**−100**	**−50**	**0**	**50**	**100**
Rate Shift Down 100 bps						
3x10	3.963	25.161	23.926	22.693	21.459	20.226
5x10	4.073	23.197	22.147	21.097	20.047	18.997
7x10	4.171	20.898	20.064	19.231	18.397	17.564
No Rate Shift						
3x10	4.969	20.509	19.649	18.791	17.931	17.071
5x10	5.085	19.072	18.321	17.571	16.819	16.068
7x10	5.191	17.412	16.781	16.151	15.519	14.888
Rate Shift Up 100 bps						
3x10	5.975	17.261	16.628	15.997	15.365	14.734
5x10	6.098	16.212	15.654	15.096	14.538	13.979
7x10	6.211	14.899	14.421	13.943	13.465	12.987

Source: Citigroup.

(Exhibit 14) as compared with the 2F-Skew model, which has a shift of 3%.

Duration changes can be understood in an analogous way. The duration calculation requires price estimation in two scenarios—the interest rate curve shifted up and shifted down. But to those two scenarios we associate different volatilities. When rates rally, the volatility increases, and when they go up, the volatility decreases. In the first case, additional skew reduces the price (assuming the same OAS), and in the second case, increases the price. This will reduce duration. The effect will be bigger for the model that predicts the higher ATM skew. Hence, the normal model has more pronounced duration shortening than the model with a shift of 7%.

The results shown in Exhibits 13 and 14 are consistent with historical OAS. In Exhibit 15 we plot the average (in time) OAS of each model versus the relative coupon for pass-throughs.[5] Then, we subtract the lognormal OAS from each model's OAS to discover how OAS vary. Hence, the *y*-axis represents the difference from the lognormal case. Exhibit 16 repeats this for interest-only (IO) securities.

So, if in hypothetical market conditions the current coupon is 5% and an investor holds a to-be-announced (TBA) security with a 6% coupon, one can see from Exhibit 15 that when going from a lognormal model to a normal model, the OAS will widen by about 13 bps, on average.

EXHIBIT 13
OAS and Durations for Conventional 30-Year MBS, Shift = 7% versus 2F-Skew

			OAS (bps)			Duration		
Coupon (%)	**WAM**	**Price[a]**	**2F-Skew**	**shift = 7%**	**Diff**	**2F-Skew**	**shift = 7%**	**Diff**
4.5	354	95-24	−3	−4	−1	5.3	5.2	−0.1
5.0	356	98-10	−11	−11	0	4.4	4.3	−0.1
5.5	352	100-10	−10	−10	0	3.3	3.3	0.0
6.0	343	101-29	−10	−8	2	2.2	2.2	0.0
6.5	330	103-05	18	20	2	2.1	2.1	0.0
7.0	312	104-23	27	28	1	1.9	1.9	0.0
7.5	303	106-00	27	29	2	1.5	1.5	0.0

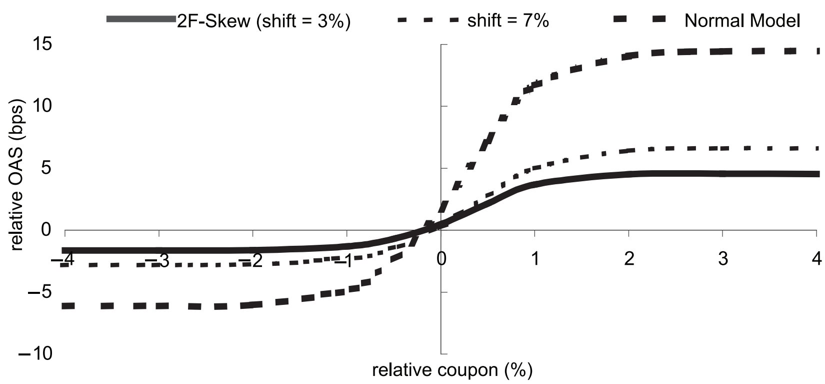
[a]*Prices from close of September 19, 2005.*
Source: Citigroup.

EXHIBIT 14
OAS and Durations for Conventional 30-Year MBS, Normal Model versus 2F-Skew

			OAS (bps)			Duration		
Coupon (%)	WAM	Price[a]	2F-Skew	Normal Model	Diff	2F-Skew	Normal Model	Diff
4.5	354	94-05	2	–2	–4	5.6	5.2	–0.4
5.0	356	96-28	–6	–9	–3	4.8	4.5	–0.3
5.5	352	99-06	–10	–10	0	4.0	3.8	–0.2
6.0	343	101-06	–13	–10	3	2.8	2.7	–0.1
6.5	330	102-26	9	11	2	2.6	2.5	–0.1
7.0	312	104-20	10	14	4	2.2	2.1	–0.1
7.5	303	105-18	22	27	5	1.8	1.8	0.0

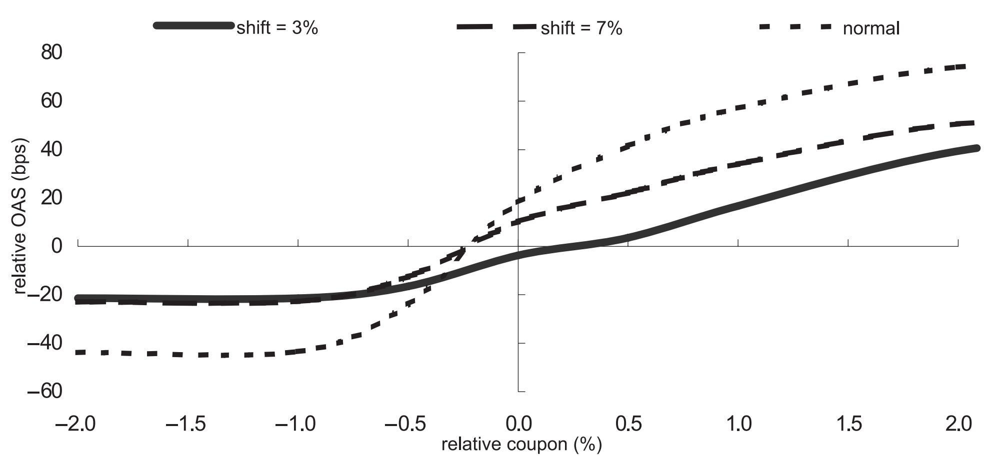
[a]*Price from close of October 17, 2005.*
Source: Citigroup.

EXHIBIT 15
Relative OAS versus Relative Coupon for Pass-Throughs, Monthly Data, Jan 00–Sep 05



Source: Citigroup.

EXHIBIT 16
Relative OAS versus Relative Coupon for IOs, Monthly Data, Jan 00–Sep 05



Source: Citigroup.

IMPACT OF THE SKEW ON HEDGING OF *MBS*

It is clear that the distributional assumptions have an impact on hedging, too. The durations get shorter as we increase the skew. Therefore, the 2F-Skew model will have shorter durations than the 2F (lognormal) model, and the normal model will have the shortest durations. In order to calculate duration, one has to shift the level of interest rates, but, as we have seen, that also changes volatilities.

If we denote the price of a TBA pass-through by P and assume that the volatility is independent of the level of interest rates (2F model), then a change in interest rates of dy and in volatility of $d\sigma$ will cause a change in the price of

$$\frac{dP}{P} = -(EffDur) \times dy - (VolDur) \times d\sigma$$

Their analytical forms are given in appendix B. Both of these quantities are positive for pass-throughs.

Let us now assume that the volatility σ of our pass-through TBA is a function of the level of interest rates y. We now have

$$P = P(y, \sigma(y))$$

If interest levels change by dy, then the change in price up to the second order is given in appendix B. In terms of duration this can be rewritten as

$$\frac{dP}{P} = -(VolSkewEffDur) \times dy$$

where the skewed duration is given by the last equation in Appendix B, or

$$\begin{aligned} VolSkewEffDur \approx\ & LognormalEffDur + \\ & LognormalVolDur \times (ATMVolatilitySkew \\ & \qquad - OTMVolatilitySkew) \end{aligned}$$

In this last equation the skewed partial duration on the left side is obtained by the 2F-Skew model for the given parameter β (shift), or by the normal model. The partial durations on the right are produced by the 2F (lognormal) model. Since the ATM skew tells us that volatilities rise when rates fall, we see that the skewed duration must be shorter than the lognormal one.

In Exhibits 17 and 18 we calculate the durations for pass-throughs and IOs as a function of the relative coupon. We see that, in absolute value, the normal model has the shortest duration.

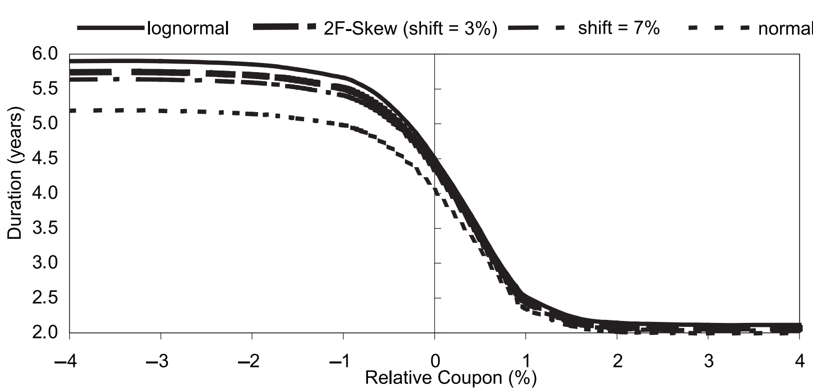
WHICH DISTRIBUTION SHOULD INVESTORS CHOOSE?

As we have shown, choosing different distributions of rate changes has a profound impact on the valuation and hedging of mortgages. Investors might naturally ask the question: What distribution should I choose? The answer is based on how the valuation model is to be used.

One could argue that the market is signaling the OTM skew, hence one should use this information in finding the correct distribution. There are two problems with this approach, however. First, even though market

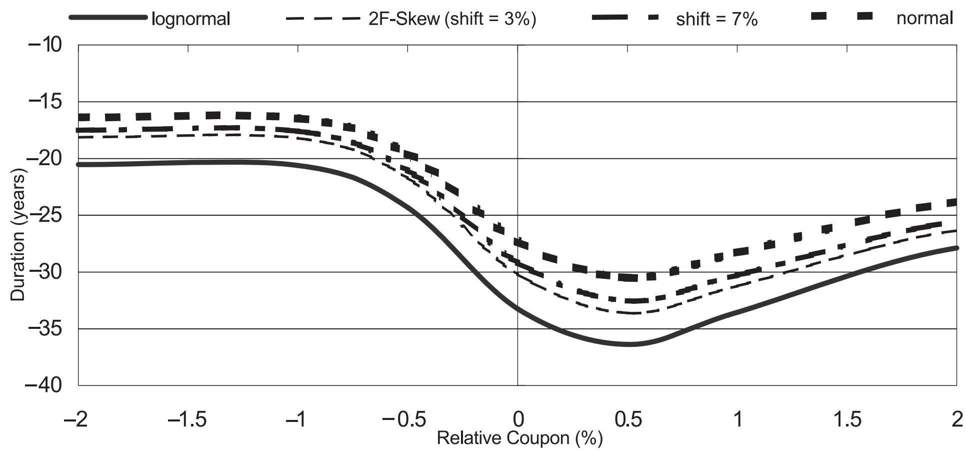
EXHIBIT 17

Duration versus Relative Coupon for Pass-Throughs, Monthly Data, Jan 01–Sep 05



Source: Citigroup

EXHIBIT 18
Duration of IOs Versus Relative Coupon, Monthly Data, Jan 01–Sep 05



Source: Citigroup.

volatility skew has become readily available in the market, it is still a very noisy quantity. This is primarily because of supply and demand issues that arise when the majority in the market has exposures to similar strikes. Second, it is not clear that the information contained in the OTM skew is a good predictor of realized ATM skew. In fact, we have argued in the past that 2F-Skew is a good predictor of ATM volatility skew, while at the same time being a somewhat poor predictor of OTM volatility skew. Buy-and-hold investors should be more concerned about ATM volatility skew because the value of a buy-and-hold portfolio will come from hedging rate exposure, and ATM volatility impacts mainly hedging rate exposure.

Instead of using the OTM volatility skew, investors should instead formulate their beliefs for a future realized ATM volatility skew, perhaps obtained from historical data, and select a distribution that matches this belief.

Given the choice of fitting the OTM volatility skew or the ATM volatility skew, investors should choose to match the ATM volatility skew.

CONCLUSION

Volatility skew is principally a result of the market not perceiving volatilities of interest rates as proportional to their level. As we have shown, the term volatility skew refers to two effects—the OTM volatility skew and the ATM volatility skew.

There is significant empirical evidence for the existence of both the OTM and the ATM skew, and we have argued that ATM and OTM skews are not identical. This means that investors cannot just look at the OTM skew observed in the market and conclude that ATM volatility will have the same behavior.

All term-structure models used for valuing interest rate contingent claims will have embedded OTM and ATM volatility skew. Investors must verify whether the implicit assumptions of the model they are using are consistent with their own subjective views on the volatility skew behavior.

We have looked at several models with varying volatility skew, and we can conclude that, with increasing volatility skew, OASs of discount TBAs tighten, while OASs of premium TBAs widen. A similar effect is observed for IOs. As one steepens the volatility skew, durations also shorten for TBAs, while lengthening for IOs.

While the market volatility skew changes from day to day, we believe that investors should choose that volatility skew that best reflects their estimate of future realized volatility skew.

APPENDIX A
Black Cap and Swaption Formulas

We define a caplet to be one of the payments of a cap. Its present value is given by the formula

$$C = D\int_K^{\infty}(r-K)p(r)dr$$

where $p(r)$ is the probability distribution of the rate r, and K is the strike of the caplet. The variable D is the discount value from the time the caplet is paid times the period of the caplet. If $p(r)$ is a lognormal distribution (an essential requirement of the Black method) with standard deviation $\sigma\sqrt{t}$, where t is the length of time until the caplet payment is determined, we have

$$C = D(FN(d_1) - KN(d_2))$$

F is the forward rate at time t, and $N(d)$ is the integral of the standard normal distribution for all values less than or equal to d.

Also,

$$d_1 = \frac{\log\left(\frac{F}{K}\right) + \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}, \quad d_2 = d_1 - \sigma\sqrt{t}$$

Then, the value of the cap is simply the sum of the caplets

$$\sum_i C_i$$

For the receiver swaption, we have

$$RS = A\int_{-\infty}^{K}(K-r)p(r)dr$$

where A is the current value of the annuity that pays at the rate of \$1 per year during the term of the swap. After integrating over the lognormal distribution we get

$$RS = A(KN(-d_2) - FN(-d_1))$$

F is the forward swap rate, d_1 and d_2 are defined above, with t now being the time to expiration of the swaption.

APPENDIX B

Partial Durations

The partial durations (interest rate duration D_y and volatility duration D_σ) are negative normalized derivatives with respect to the independent variables y and σ, respectively. The negative signs are taken to ensure that the duration measures will be positive, such that

$$D_y = -\frac{1}{P}\frac{\partial P}{\partial y}, \quad D_\sigma = -\frac{1}{P}\frac{\partial P}{\partial \sigma}$$

The following equation gives the TBA price change due to the changes of interest rates and volatilities up to the second order:

$$\begin{aligned} dP = {} & \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial \sigma}\frac{d\sigma}{dy}dy \\ & + \frac{1}{2}\left(\frac{\partial^2 P}{\partial y^2}dy^2 + 2\frac{\partial^2 P}{\partial y \partial \sigma}dy\frac{d\sigma}{dy}dy \right. \\ & \left. + \frac{\partial^2 P}{d\sigma^2}\left(\frac{d\sigma}{dy}\right)^2 dy^2\right) + \cdots \end{aligned}$$

The explicit form of the volatility skew effective duration is given by

$$D_y^s = -\frac{1}{P}\frac{\partial P}{\partial y} - \frac{1}{P}\frac{\partial P}{\partial \sigma}\frac{d\sigma}{dy}$$

ENDNOTES

[1]The actual quote is the *effective annual volatility,* which is the standard deviation of the distribution of the logs of the rates (for a lognormal model) at a future time divided by the square root of the time in years from now until that future time.

[2]See appendix A for detailed information on evaluating the derivative price.

[3]If we had highly accurate OTM prices for a spectrum of strikes, we could actually construct a distribution that would satisfy all of these prices, although the Black volatilities would still be skewed, as they use the lognormal distribution.

[4]The vol skews in Exhibits 9 and 10 are for the 5 × 10 swaption, which is of great importance to MBS investors.

[5]The relative coupon is the difference between the coupon of the evaluated security and the current coupon.

REFERENCES

Bhattacharjee, R., Robert A. Russell, and Stephan Walter. "2F-Skew—A Two-Factor Term Structure Model with Volatility Skew." Citigroup, October 2003.

Black, F. "The Pricing of Commodity Contracts." *Journal of Financial Economics*, March 3, 1976, pp. 167–79.

Chan, Y.K., and Robert A. Russell. "A Term Structure Model and the Pricing of Fixed-Income Securities." Salomon Brothers, June 1997.