Pricing Agency MBS under Quadratic Gaussian Models

Xu Bai

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Interest rate modeling is an integral part of the mortgage-backed security (MBS) pricing mechanism. The particular model choice can have a significant impact on both the MBS valuation and its risk metrics. The market-implied interest rate volatility skew suggests that the interest rate distribution is often more normal than log-normal. A normal model tends to shorten the MBS durations while a log-normal model prevents the rates from going negative. We show in this article how QGM models can have the best of both worlds.

Quadratic Gaussian models (QGMs) have become increasingly popular for pricing interest rate derivatives due to their tractability, flexibility in handling the volatility skew/smile, and ease of implementation through the Markov functional approach (see Piterbarg [2009] and McCloud [2010], among others). To the best of my knowledge, quadratic Gaussian models have not been applied to the mortgage space. This article discusses their application to agency MBS pricing and focuses in particular on the simplest securitization structure of an agency MBS, the pass-throughs. Understanding the interest rate model impact on pass-throughs serves as an important first step because more-complex CMOs (collateralized mortgage obligations) and strips are all derived out of pass-throughs.

QGM FRAMEWORK

The QGM framework is rather flexible and can have many factors. We present in this section a 1-factor CEV-like parameterization.[1] A more-general two-factor stochastic volatility parameterization can be found in the Appendix.

Define a one-dimensional Markov process:

$$dX_t = \sigma(t)dW_t \tag{1}$$

with the cumulative variance defined as $V(t) = \int_0^t \sigma^2(s)\,ds$.

The numeraire-rebased zero-coupon bond can be modeled as an exponential quadratic function of the underlying driver (Hagan [2002]):

$$\begin{aligned}\tilde{P}(t,T,X_t) &= \frac{P(t,T,X_t)}{N(t,X_t)}\\ &= P(0,T)\exp\left[-\phi(t,T,X_t)\right]\end{aligned} \tag{2}$$

with

$$\phi(t,T,X_t) = Q(t,T)X_t^2 + B(t,T)X_t + A(t,T) \tag{3}$$

where $N(t,X_t)$ is the numeraire and its initial value $N(0,0)$ is restricted to be 1.

The numeraire can be readily derived from Equations (2) and (3) as

$$N(t,X_t)=\frac{1}{P(0,t)\exp\left[-\phi(t,t,X_t)\right]} \tag{4}$$

Applying Ito's lemma and the martingale property leads to

$$\frac{\partial\phi}{\partial t}=\frac{1}{2}V'(t)\left[\left(\frac{\partial\phi}{\partial X_t}\right)^2-\frac{\partial^2\phi}{\partial X_t^2}\right] \tag{5}$$

By solving Equation (5), one can obtain the general forms for the coefficients:

$$Q(t,T)=\frac{1}{2}\frac{C(T)}{\left[1-C(T)V(t)\right]}$$

$$B(t,T)=\frac{H(T)}{1-C(T)V(t)} \tag{6}$$

$$A(t,T)=\frac{1}{2}\frac{H^2(T)V(t)}{\left[1-C(T)V(t)\right]}+\frac{1}{2}\log\left[1-C(T)V(t)\right]$$

The short rate $r(t)$ in the QGM model is also quadratic in X:[2]

$$r(t)\approx\frac{1}{2}c(t)X_t^2+h(t)X_t+f(0,t) \tag{7}$$

where $c(t) = C'(t)$, $h(t) = H'(t)$ and $f(0,t)$ is the instantaneous forward rate.

To value a deal, one can use the martingale property with Q^t being the measure associated with the numeraire:

$$\tilde{\gamma}(s,X_s)=E_{Q^t}\left[\tilde{\gamma}(t,X_t)\Big|_{X_s}\right], s<t \tag{8}$$

where $\tilde{\gamma}(s,X_s)$ is the numeraire-rebased pay-off function.

In particular,

$$\gamma(0,0)=\tilde{\gamma}(0,0)=E_{Q^t}\left[\tilde{\gamma}(t,X_t)\Big|_{X_0}\right] \tag{9}$$

The zero-coupon bond itself, of course, verifies the martingale condition:

$$E_{Q^t}\left[\tilde{P}(t,T,X_t)\Big|_{X_0}\right]=P(0,T) \tag{10}$$

The one-factor QGM model is fully defined by the three curves: $H(T)$, $C(T)$, and $V(t)$. The calibration can be directly carried out on these curves in practice. Nevertheless, we show here a CEV-like parameterization in order to express the QGM skewness in a more intuitive fashion.

The linear coefficient $H(T)$ and the quadratic coefficient $C(T)$ can be set up as follows so that they control respectively the mean-reversion and the skew:

$$H(T)=\int_0^T \exp\left(-\int_0^s k(u)\,du\right)ds \tag{11}$$

$$C(T)=\int_0^T \frac{\beta}{f(0,s)}\exp\left(-2\int_0^s k(u)\,du\right)ds \tag{12}$$

where k is the mean-reversion parameter and β is the CEV-type exponent for the skew.

Differentiating Equation (7) leads to the stochastic differential equation for $r(t)$:

$$\begin{aligned} dr(t)=&\left[f'(0,t)+h'(t)X_t+\frac{1}{2}c'(t)X_t^2+\frac{1}{2}c(t)\sigma^2(t)\right]dt\\ &+\sigma(t)h(t)\sqrt{1+2\beta\frac{r-f(0,t)}{f(0,t)}}\,dW_t \end{aligned} \tag{13}$$

Ignoring higher-order terms, the drift becomes:

$$drift\approx f'(0,t)+k(t)\left[f(0,t)-r(t)\right] \tag{14}$$

Clearly $k(t)$ controls the mean-reverting speed. As to the diffusion term, it can be expanded around the forward curve,

$$diffusion\approx\sigma(t)h(t)\left(1+\beta\frac{r-f(0,t)}{f(0,t)}\right)dW_t \tag{15}$$

Comparing this to a CEV-type model expansion,

$$\begin{aligned} dr&=\left[drift\right]dt+\sigma(t)r^\beta dW_t\\ &\approx\left[drift\right]dt+\sigma(t)f^\beta(0,t)\left(1+\beta\frac{r(t)-f(0,t)}{f(0,t)}\right)dW_t \end{aligned} \tag{16}$$

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one can see that β is the skew parameter, which can be used to change the skewness *à la* CEV.

The numeraire-rebased swaption price in the QGM model is calculated as follows:

$$SWP\tilde{T}_t = E_{Q_{t_e}}\left[\left(o\sum_{i=1}^{N}\Delta_i K\tilde{P}(t_e,t_i,X_{t_e}) + o\tilde{P}(t_e,t_N,X_{t_e}) - o\tilde{P}(t_e,t_0,X_{t_e})\right)^+ \Big| X_t\right] \tag{17}$$

where K is strike; t_e is expiry; t_0 is swap start date; t_i is the ith fixed-leg payment date; Δ_i is day count fraction; and $o = \pm 1$ depending on the swaption type (receiver or payer).

THREE DISTRIBUTIONAL MODELS

For comparison, this section presents three one-factor distributional short-rate models in a similar Markov functional form. The three models are equivalent respectively to the Hull–White (HW) model, the Gaussian-squared (GS) model, and the Black–Karasinski (BK) model.[3] They have a common mean-reverting Gaussian Markov process:

$$X_t = h(t)W\left(\sqrt{V(t)}\right) \tag{18}$$

with

$$h(t) = \exp\left[\int_0^t -k(s)\,ds\right], V(t) = \int_0^t \frac{\sigma^2(s)}{h^2(s)}ds$$

and $W(\cdot)$ is a time-scaled Brownian motion.

The underlying Markov driver X_t is mapped to the short rate via the following three different functional forms:

1. HW model:

$$r(t) = \theta(t) + X_t \tag{19}$$

2. GS model:

$$r(t) = \left[\theta(t) + X_t\right]^2 \tag{20}$$

3. BK model:

$$r(t) = \theta(t)\exp(X_t) \tag{21}$$

where $\theta(t)$ is determined by calibration.

The zero-coupon bond and the swaption price are calculated respectively in the risk-neutral measure as follows:

$$P(t,T) = E_t\left[\exp\left(-\int_t^T r(s)\,ds\right)\Big| X_t\right] \tag{22}$$

$$SWPT_t = E_t\left[\exp\left(-\int_t^{t_e} r(s)\,ds\right) \times \left(o\sum_{i=1}^{N}\Delta_i KP(t_e,t_i) + oP(t_e,t_N) - oP(t_e,t_0)\right)^+ \Big| X_t\right] \tag{23}$$

where K is strike; t_e is expiry; t_0 is swap start date; t_i is the i^{th} fixed-leg payment date; Δ_i is day count fraction; and $o = \pm 1$ depending on the swaption type (receiver or payer).

The three one-factor short-rate models have different underlying distributions:

- The short rate in the HW model is normally distributed and the absolute volatility is rate-independent;
- the short rate in the GS model follows a non-central chi-squared distribution and has a square-root volatility function; and
- the BK model generates log-normal short-rate and the absolute volatility is proportional to the rate level.

Although all three models can be calibrated to match the observed ATM (at-the-money) swaption volatility, their different distribution profiles underpin the skewness they can generate for OTM (out-of-the-money) and ITM (in-the-money) swaptions. Exhibit 1 shows the implied volatility skews generated by the three models for a 5×5 receiver swaption.[4] Whereas the log-normal BK model generates a flat line for the implied volatilities, the normal HW model generates a heavily skewed implied volatility curve. In particular, for the OTM swaptions, the HW model dominates the BK

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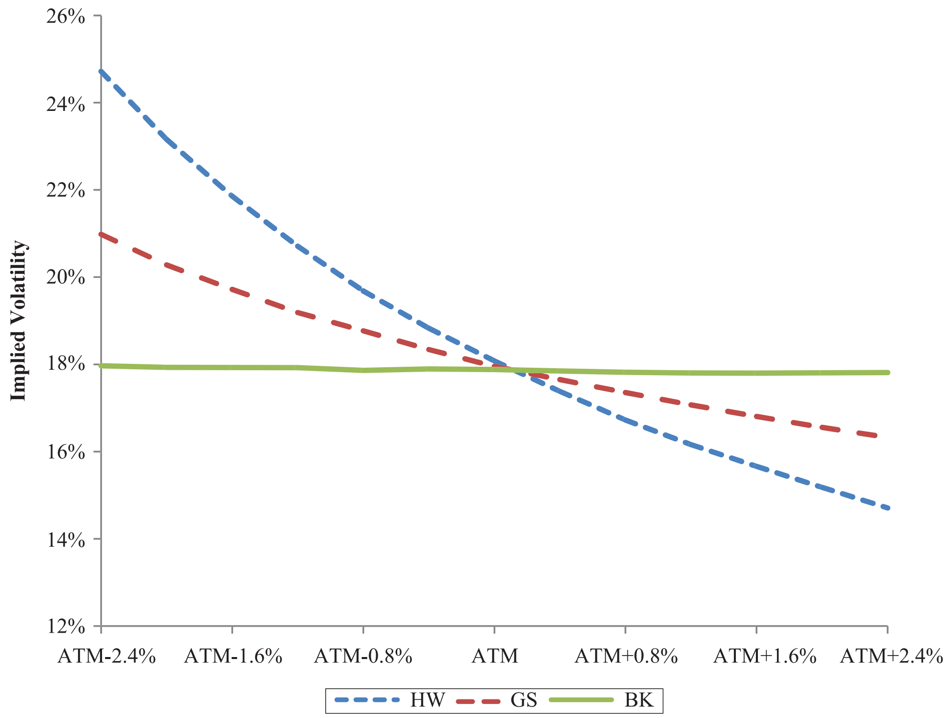
EXHIBIT 1
Short-Rate Model Skew (5 × 5 swaption)



model, whereas for the ITM swaptions, the BK model dominates the HW model; the GS model is in between for both cases.

GENERATING VOLATILITY SKEW IN QGM MODEL

Compared with the three one-factor distributional short-rate models, the one-factor QGM model is more flexible in terms of generating different degrees of volatility skew through the appropriate choice of the beta parameter. This flexibility is illustrated in Exhibit 2, where the 5 × 5 receiver swaption implied volatility skew is generated from the QGM model with three different beta values.

When beta = 0, the negatively sloped skew is very pronounced as in a normal model; when beta = 1, the implied volatility curve is mostly flat, exhibiting the property of a lognormal model; the implied volatility curve with beta = 0.5 is somewhat in-between and mimics the behavior of a square-root model.

However, we should note that the distributional short-rate model skews are generated by the dependency of absolute volatility on the rate level defined by their underlying distributions, whereas in the QGM framework, this dependency is approximated by a quadratic function with an underlying Gaussian distribution.

The normal "backbone" in the QGM model can be further evidenced by looking at the skew dynamics as shown in Exhibits 3, 4, and 5: The absolute ATM volatility does not move with the forwards and the implied volatility curve floats along the normal "backbone" for all beta values when shifting the forwards. This differentiates the QGM model from a true log-normal or square-root model.

AGENCY PREPAYMENT MODELING AND OPTION THEORETIC PRICING OF MBS

In an agency MBS, the timely payment of principal and interest is guaranteed by Fannie Mae, Freddie Mac, or Ginnie Mae. Defaulted loans are bought out of the pool at par by the servicer and can be regarded as "involuntary" prepayment. Consequently, an agency MBS is only subject to the prepayment risk. Therefore prepayment modeling is the heart of the agency MBS pricing

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Exhibit 2
QGM1F Skew (5 × 5 swaption)

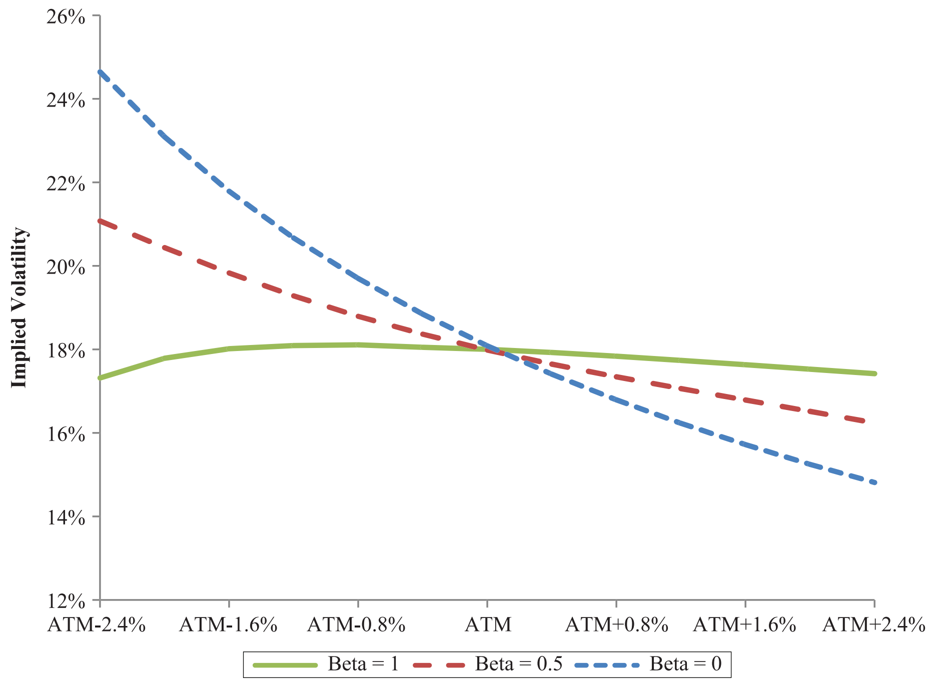

Exhibit 3
QGM1F Skew Dynamics: Beta = 1 (5 × 5 swaption)

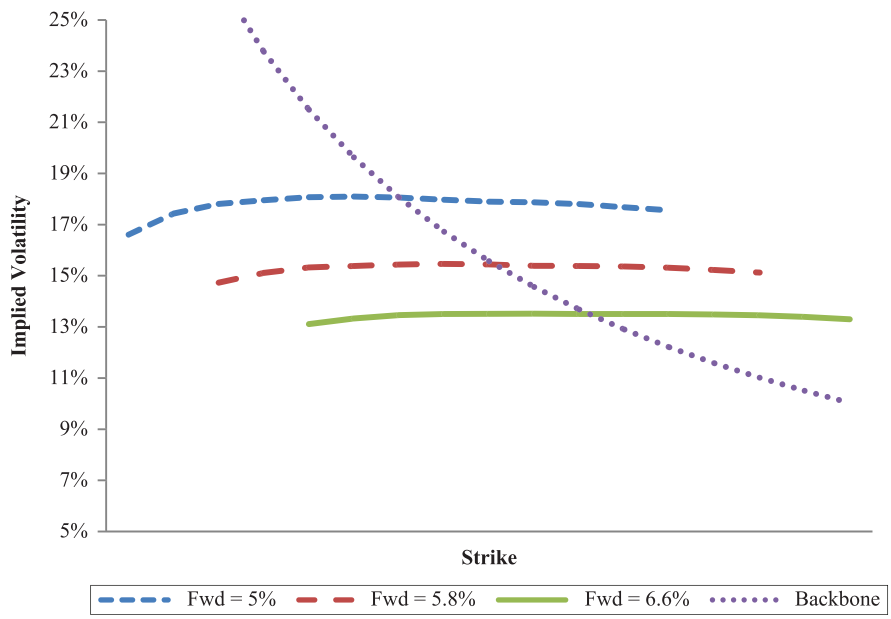

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Exhibit 4

QGM1F Skew Dynamics: Beta = 0.5 (5 × 5 swaption)

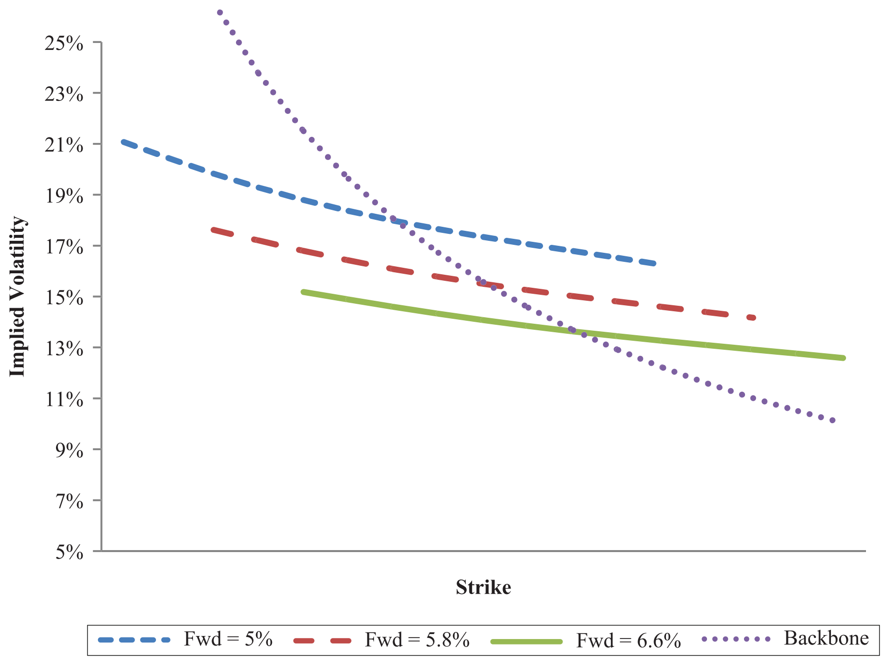
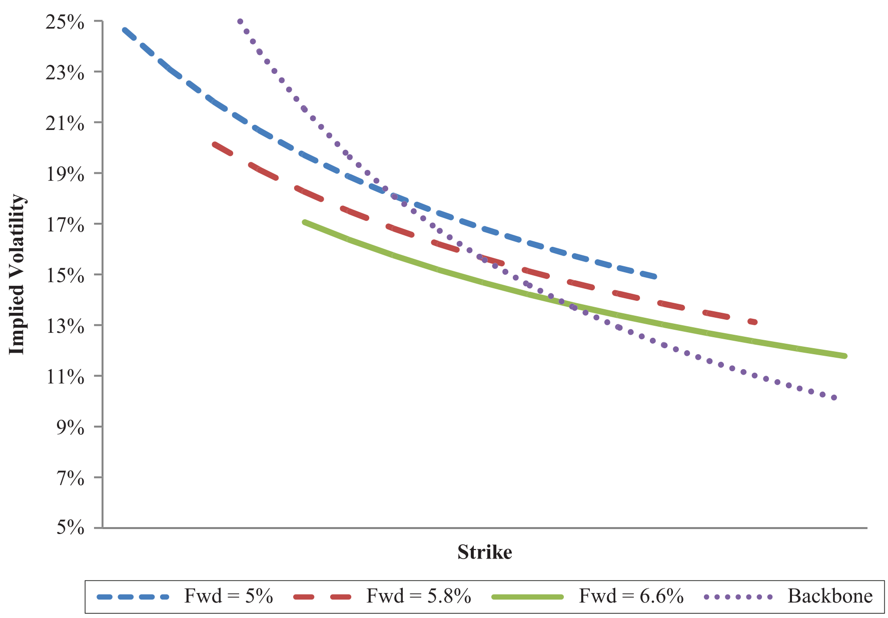


Exhibit 5

QGM1F Skew Dynamics: Beta = 0 (5 × 5 swaption)



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and risks.[5] In order to project the prepayments, one needs a prepayment model that is essentially an econometric/statistical model composed of various functional forms and fitted to historical data. The independent variables in a prepayment model can be pool characteristics, such as WALA, WAC, WAM, CLTV, SATO, FICO, loan size, and so on. There are also some exogenous factors, such as current coupon rates and HPI. In an agency prepayment model, the main stochastic driver is usually the current coupon rate, which is simulated by an interest rate model after calibrating to the swaption volatilities.

The prepayment can be generally categorized into the following four sub-groups:

1. Curtailments
2. Defaults
3. Turnover
4. Refinancings

The curtailments are extra payments that mortgagors put in to accelerate the repayment of their loans and may include full payoffs. The defaults represent the loans that are delinquent for three to four consecutive months and removed from the pool by the servicer. Because the agencies guarantee full and timely payment of principal and interest, the defaults are actually treated as prepayments and often called involuntary prepayments. The turnover describes the prepayments due to home sales or relocations that result in sale of a mortgaged house. The refinancings are prepayments resulting from the mortgagors refinancing their loans when mortgage rates decrease. There may be other types of refinancing, such as cash-out, credit-curing, and so on.

Note that the curtailments and defaults are generally not rate related; the turnover is weakly linked to interest rate movements because higher-rate environments hinder the mortgagors' mobility. The refinancings are mainly driven by interest rate movements because mortgagors seek to lower their borrowing costs. Refinancings constitute the most important component of prepayments and are the most difficult to capture.

The pricing uncertainty for an MBS bond resides in both the interest rates (discounting) and the cash flows and they are interlinked. The expectation is often calculated via Monte Carlo simulations because of the path dependency of cash flows.[6] This can be illustrated in the Equation (24) with the following notions: PX = price, DF = discount factor, L = LIBOR, OAS = option-adjusted-spread, CF = cash flow, P = principal payment, I = interest payment, PPM = prepayment, CC = mortgage current coupon.

$$PX_{MBS} = E_Q\left[\sum_i \prod_{j=1}^{i} DF_j(L_j + OAS)CF_i(P + I + PPM)\right]$$

$$\boxed{\text{IR Model}} \xrightarrow{CC} \boxed{\text{PPM Model}} \tag{24}$$

Not only does the interest rate model project the LIBOR for discounting purposes, it also generates the current coupon rate that feeds into the prepayment model in order to forecast the prepayments, especially the refinancings. The prepayments in turn alter the principal payments and interest payments. In a Monte Carlo simulation, the MBS price is simply the average of pathwise discounted cash flows for all paths. Depending on the model setup, the OAS can have an impact on the prepayment, which adds another layer of complexity in the calculations, especially when the OAS is calculated via iteration given the market price.

The pricing and risks of an MBS can be better understood if looked from an option-theoretic point of view. A typical example of a fixed-income security with embedded optionality is a callable bond, which is a corporate bond with an embedded call feature that allows the issuing company to call its debt when it becomes too expensive (Boyce and Kalotay [1979]). An MBS bond is analogous to a callable bond in that a mortgagor issues effectively a callable mortgage bond to the lender (indirectly to the MBS bond holder) to finance the acquisition of a home. When interest rates drop, the mortgagor has the option to refinance to lower the borrowing costs. Thus, an MBS bond can be regarded as a long position on an amortizing bond (including deterministic prepayments) plus a short position on a refinancing option. The latter can also be regarded as a put option on the current coupon rate, with the strike being the WAC (weighted average coupon), or a call option on the mortgage bond with the strike being the par value of the bond.

Nevertheless, rarely do prepayment models explicitly model refinancing as an option.[7] Instead, they generally

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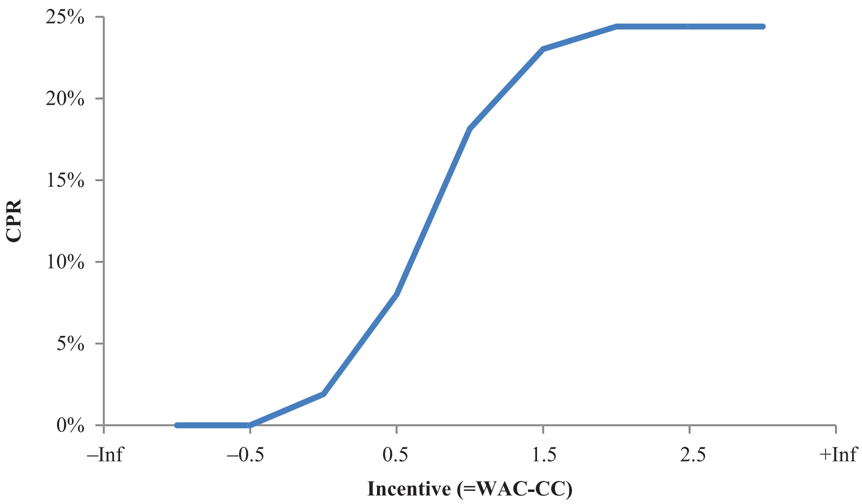
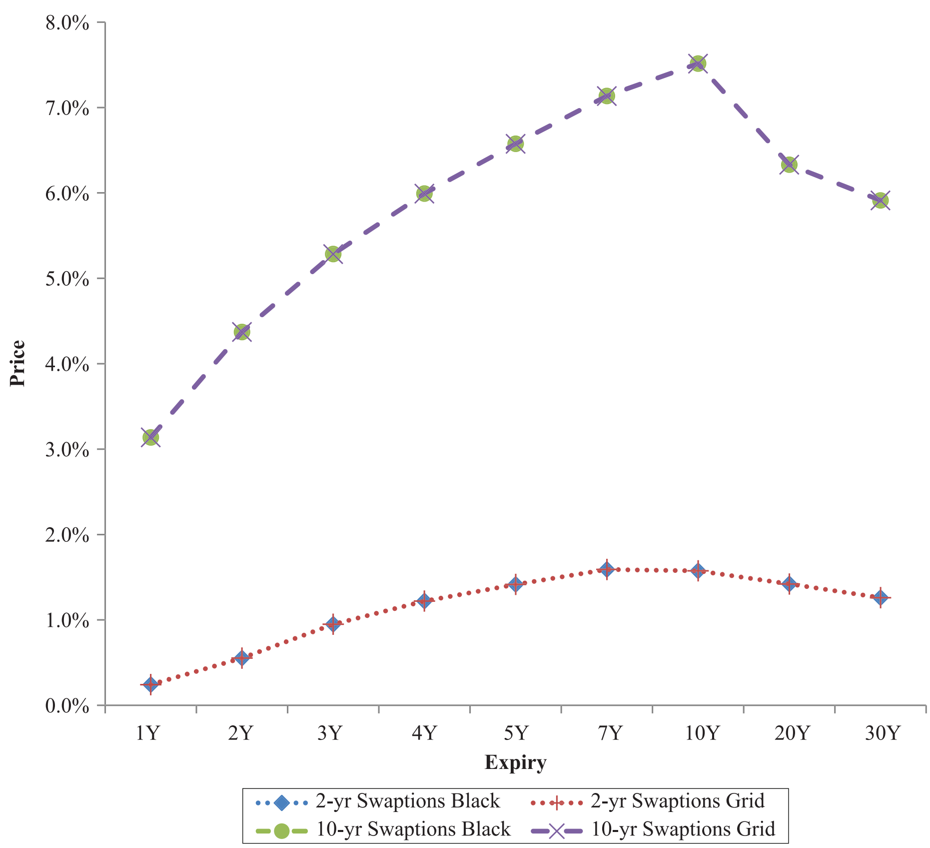
EXHIBIT 6
Refi S-Curve



model the refinancing CPR (conditional prepayment rate) as a function of incentive (WAC–current coupon)[8] as illustrated in Exhibit 6. It is commonly called the refinancing S-curve. The refinancing S-curve actually resembles the pay-off function of a put option on the current coupon rate if we flip the S-curve, even though the y-axis is the prepayment rate instead of the pay-off value. The refinancing option value is a complex function of the refinancing S-curve.

Theoretically, the refinancing option can be OTM, ATM, and ITM. This means the choice of the interest rate model plays a critical role in determining the option value and its risk characteristics.

EXHIBIT 7
QGM Model Calibration: Beta = 1



Note: The lines for 2-yr Swaptions Black and 2-yr Swaptions Grid overlap; 10-yr Swaptions Black and 10-yr Swaptions Grid also overlap.

CALIBRATION

In order to price 30-year TBA (to-be-announced) trades—for instance, July 2012 FNCL TBAs with coupons ranging from 3.0 to 6.5—the one-factor QGM model and the three distributional models are calibrated to the LIBOR/swap curve and two strips of two-year and 10-year ATM swaptions[9] with 1, 2, 3, 4, 5, 7, 10, 20, and 30-year expiries as of 6/19/2012.

QGM Model

We use three beta values, 0, 0.5, and 1, respectively in the one-factor QGM model to mimic the skews produced by the three distributional short-rate models. The calibration is carried out on the underlying Gaussian variable grid by using the three curves $H(T)$, $C(T)$, and $V(t)$. The swaption formula in Equation (17) is very convenient for calibration because all numeraire-rebased zero-coupon bonds are analytic functions of the state variable X_t given the three aforementioned curves. In order to price a swaption, one only needs to integrate over X_t. In our implementation, the integration is very efficient and remains the same for all expiries because we use the same number of grid points for each time step and the grid covers the same variance range.

The QGM model can calibrate almost perfectly to two-year and 10-year ATM swaptions for all three beta values. Exhibit 7 shows the calibration quality with beta = 1. Not only does the QGM model calibrate well to the ATM swaptions, the model-implied skewness can also be fit to the market-observed skewness by adjusting the beta value. Exhibits 8 and 9 show the comparison between the model skews and market skews for 10 × 2

EXHIBIT 8
QGM Model: 10 × 2 Swaption

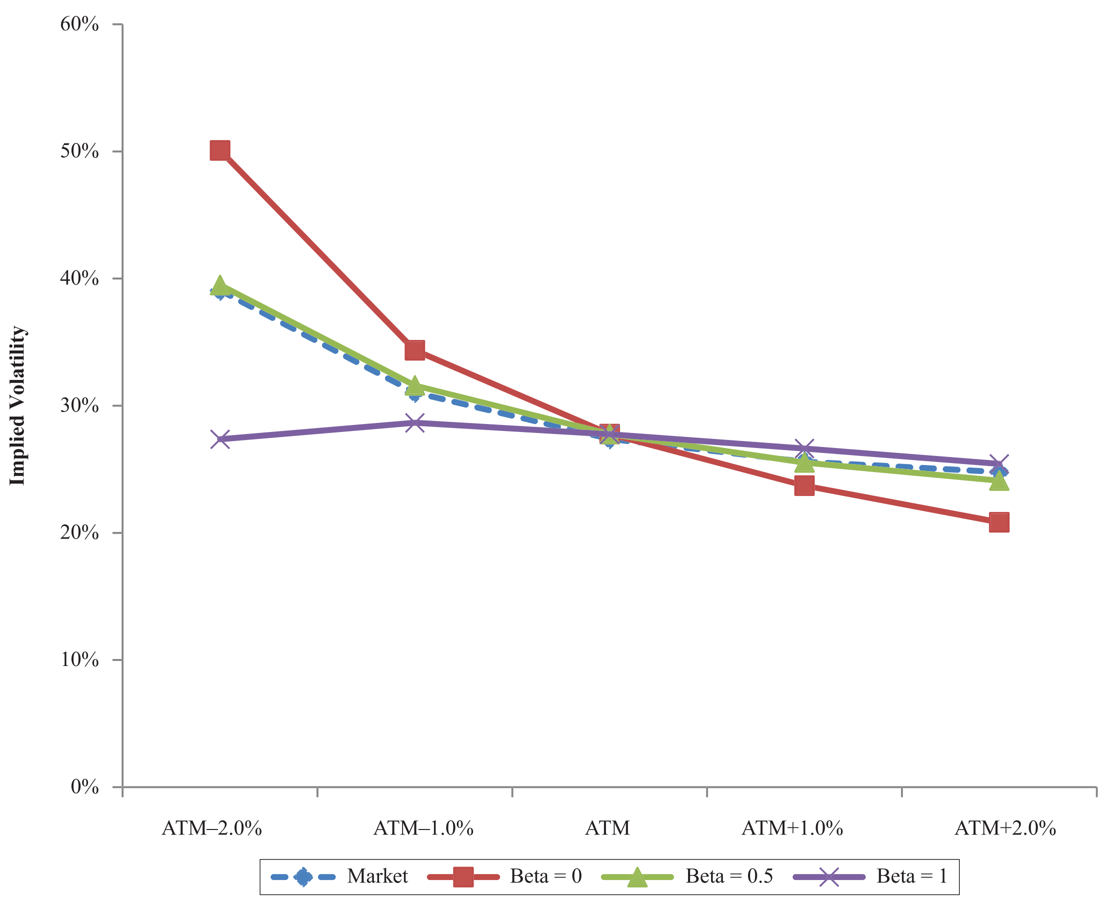


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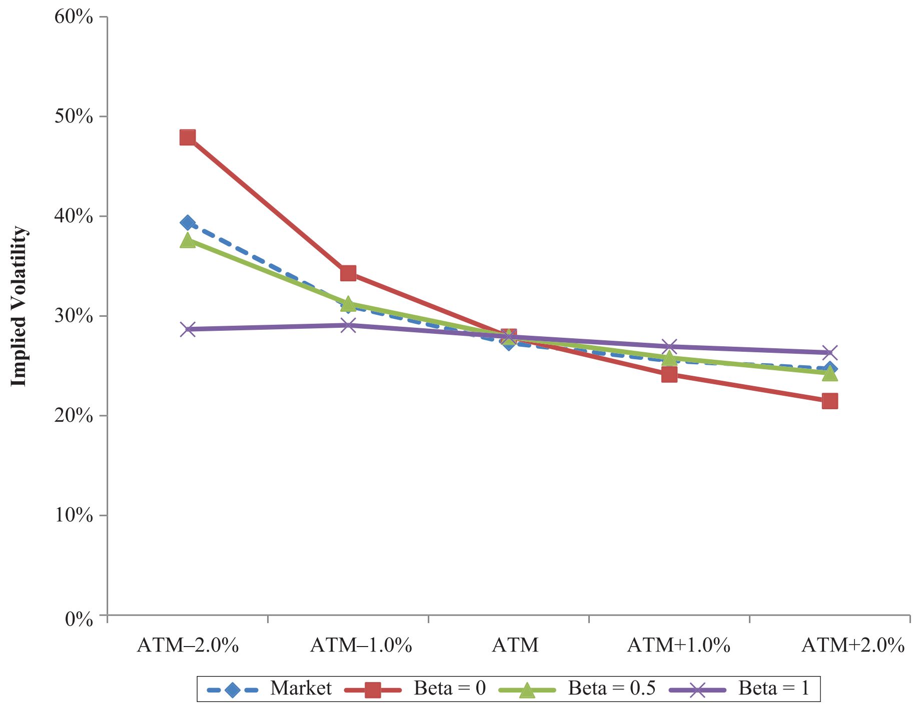
EXHIBIT 9
QGM Model: 10 × 10 Swaption



and 10 × 10 swaptions, respectively. The model skews with beta = 0.5 more closely fit the market quoted Black vols.

Distributional Short-Rate Models

For the three distributional short-rate models, the calibration is also performed on the underlying Gaussian variable grid using the $h(t)$ and $V(t)$ curves as defined in Equation (18). Given these two curves, two sweeps are needed to price a swaption: a forward induction to match the initial yield curve using Equation (22) and a backward induction using Equation (23) to price the swaption. Exhibits 10, 11, and 12 show the calibration quality. It can be observed that all three models calibrate relatively well to 10-year ATM swaptions and less well to two-year ATM swaptions.

As to the 10 × 2 and 10 × 10 swaptions skewness implied by the three distributional short-rate models, Exhibits 13 and 14 show that the market-observed skew is somewhat between those generated by HW model and BK model and the GS model's implied skew is closest to the market observed skewness among the three models, even though it does not fit as well as the QGM model does with beta = 0.5. The implied skew is even worse for 10 × 2 swaption because the ATM calibration is not exact.

The short-rate distribution will ultimately underpin the current coupon distribution, which will impact the refinancing option value and risks as explained earlier. Exhibits 15, 16, and 17 give an intuitive view of the distributions that the QGM model can mimic. These are the current-coupon rate grids with three beta values after calibration:[10] We can see that the distribution patterns generated by the QGM model can resemble normal, square-root, and log-normal distributions. In fact, the QGM model degenerates to the LGM (linear Gaussian) model with beta = 0 and is exactly equivalent to the HW model.

Exhibit 10
HW Model Calibration

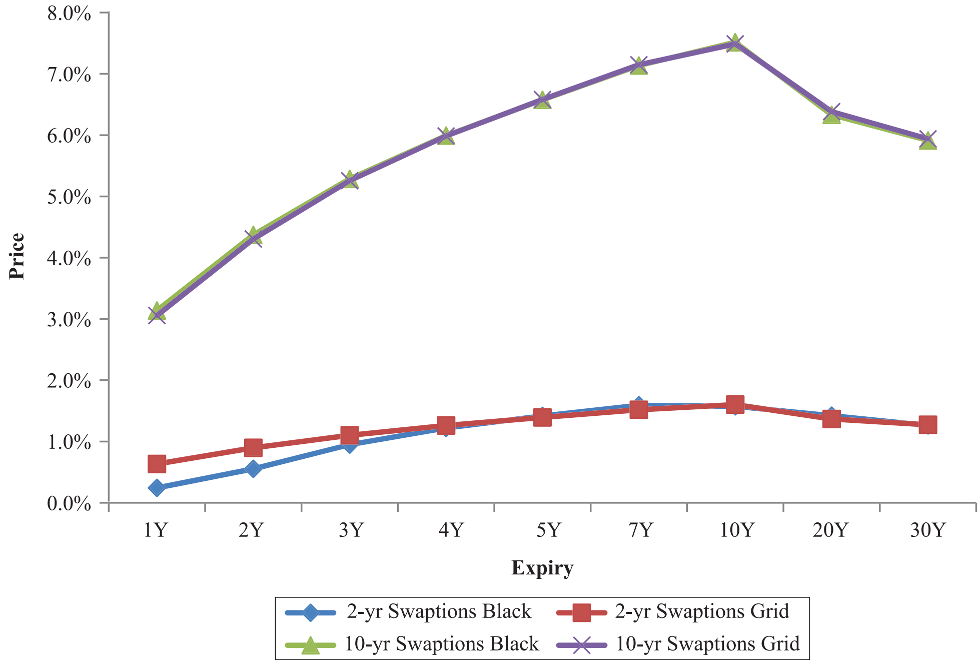


Exhibit 11
GS Model Calibration

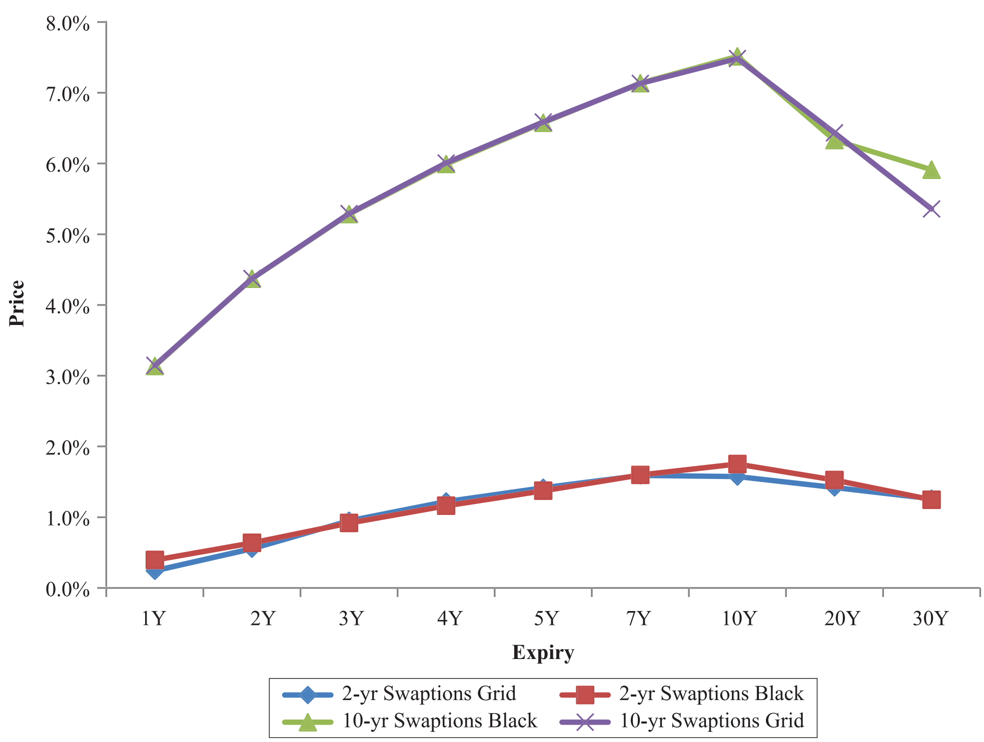


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EXHIBIT 12
BK Model Calibration

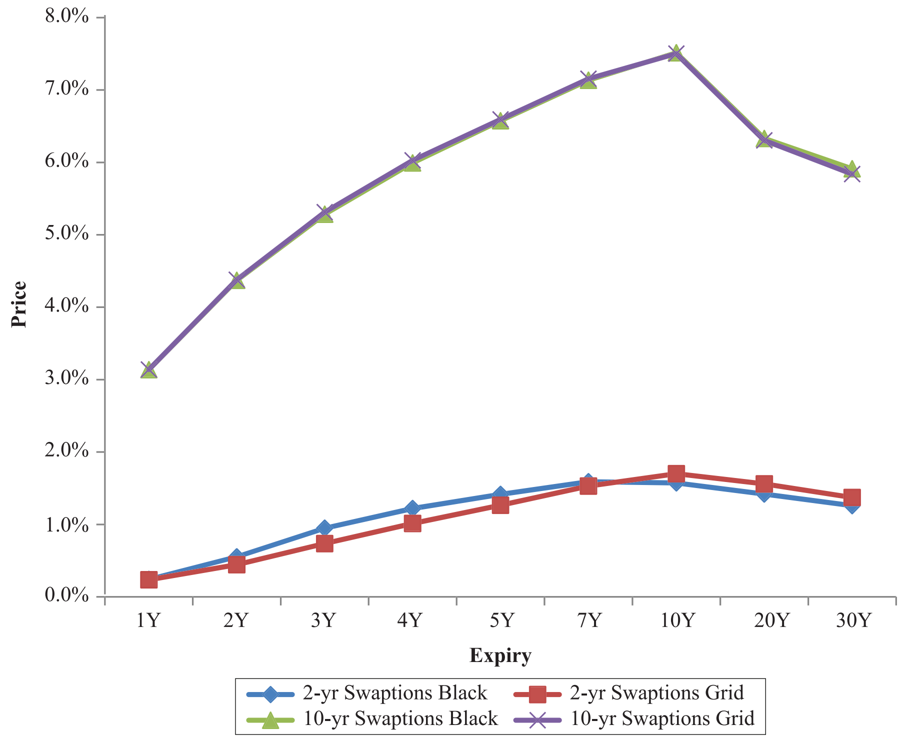

EXHIBIT 13
Distributional Short-Rate Models: 10 × 2 Swaption

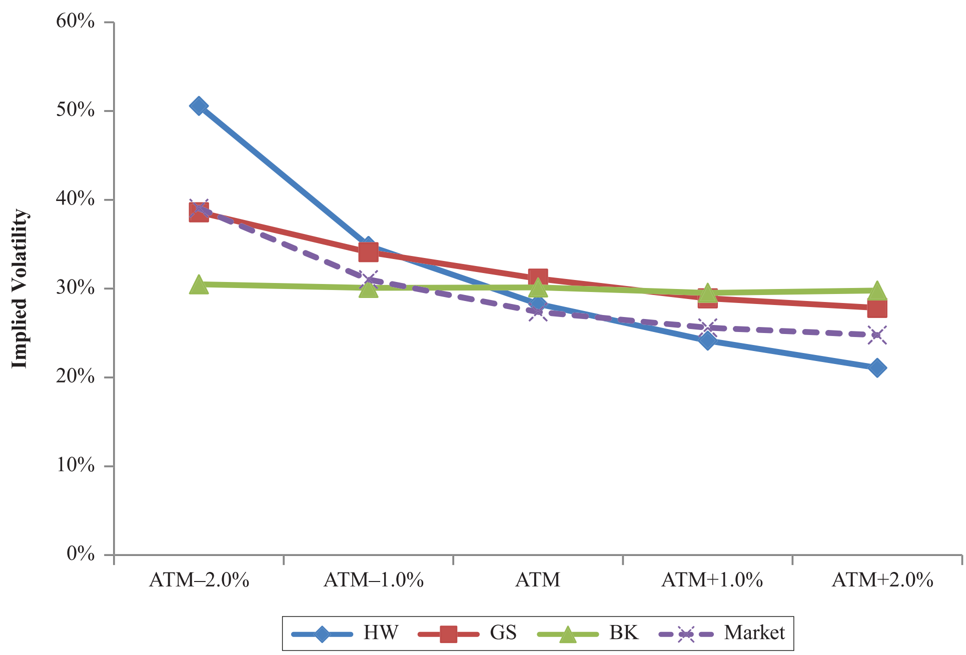

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EXHIBIT 14

Distributional Short-Rate Models: 10 × 10 Swaption

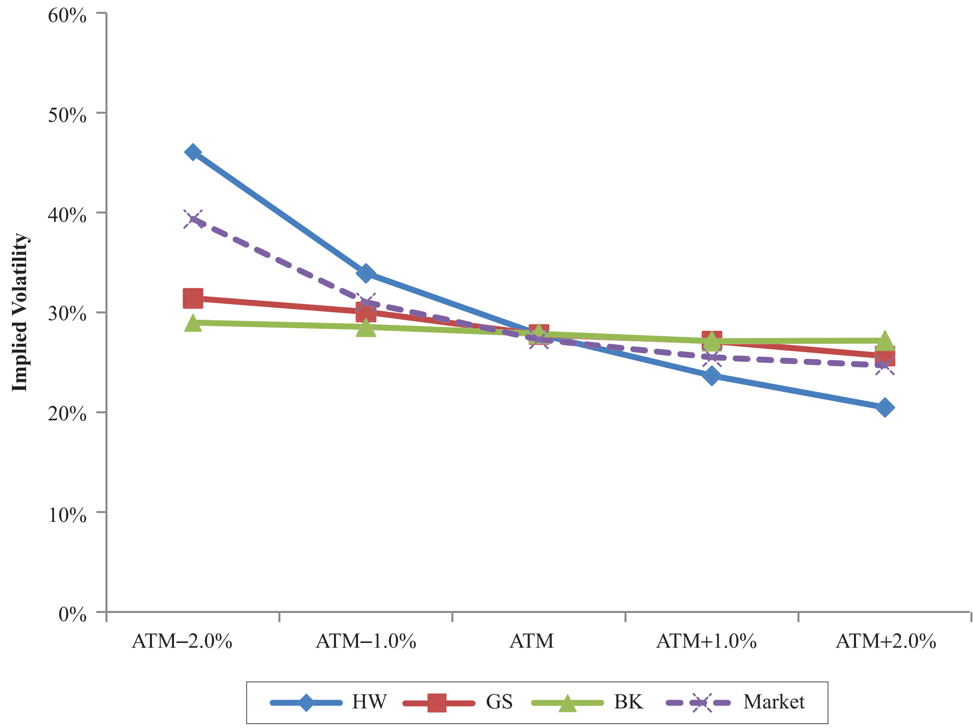

EXHIBIT 15

QGM1F Model CC Grid (beta = 0)

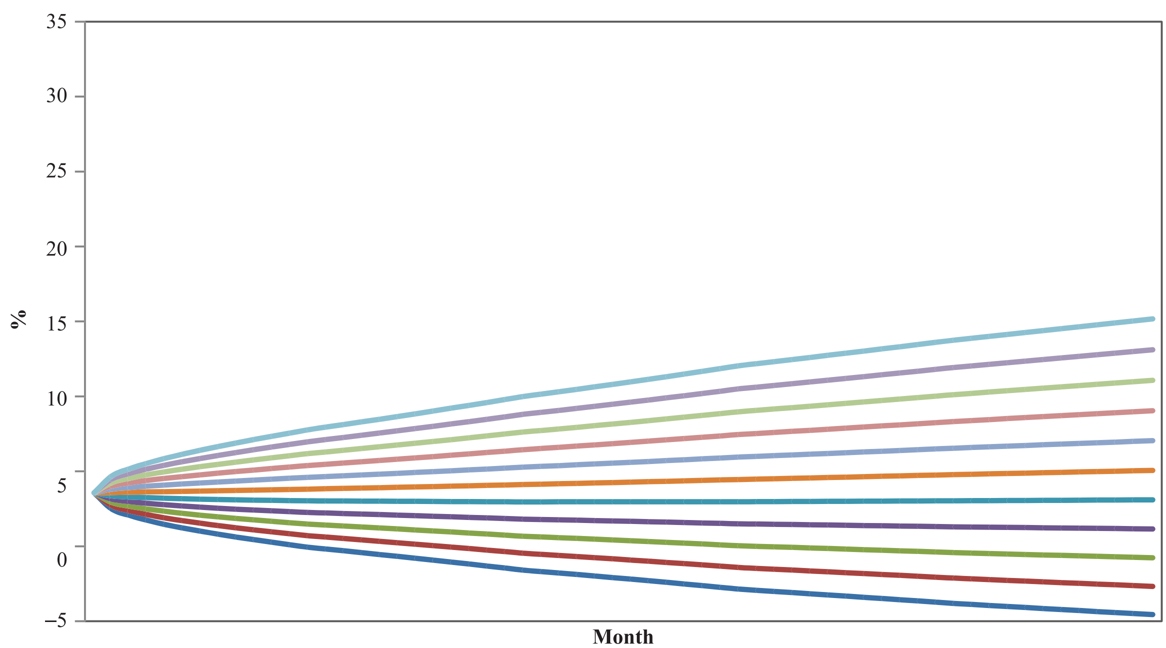

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EXHIBIT 16
QGM1F Model CC Grid (beta = 0.5)

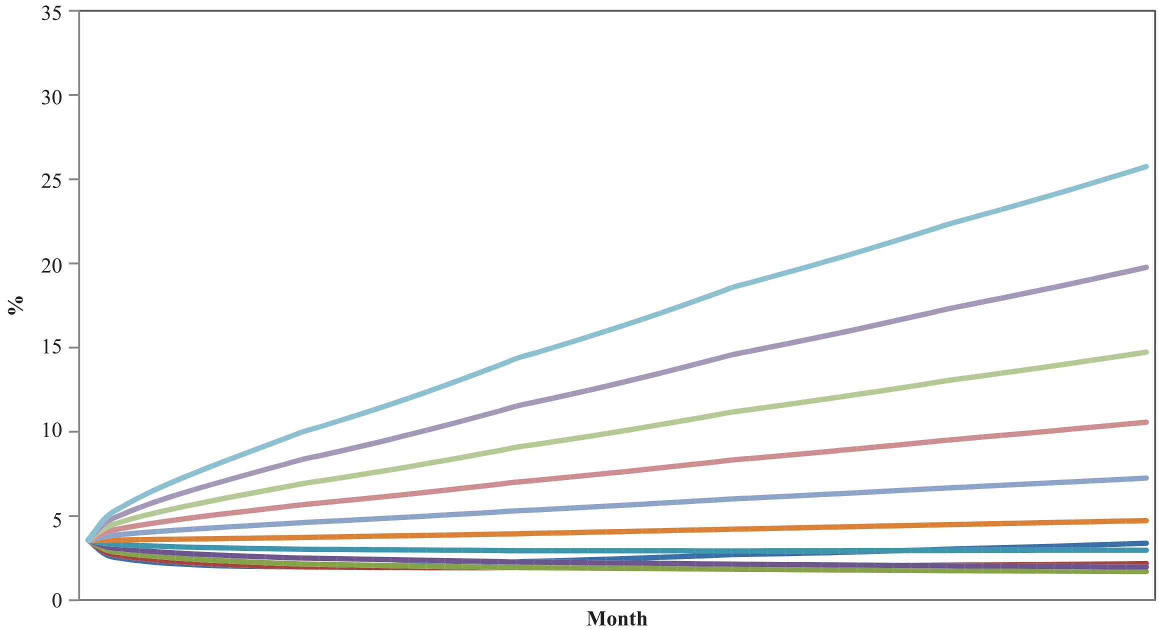
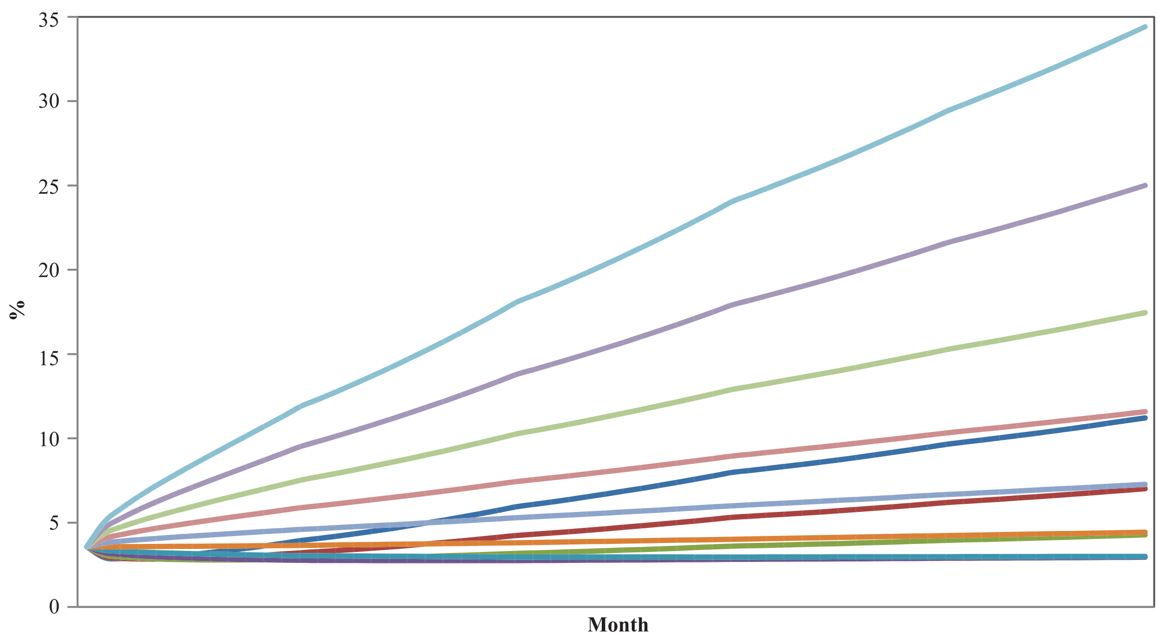


EXHIBIT 17
QGM1F Model CC Grid (beta = 1)



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TBA VALUATION UNDER DISTRIBUTIONAL MODELS AND QGM MODEL

Given that the refinancing option can be considered as a put option on the current coupon rate, which is similar to a receiver swaption, we can approximately treat the OTM section for the swaption as the discount sector for MBS and the ITM section as the premium sector even though the implied volatility skew "switching point" in the MBS market generally will not coincide with that in the swaption market. The skewness will impact the refinancing option pricing and risks and ultimately the MBS pricing and risks.

Exhibits 18, 19, and 20 summarize the valuation results for TBAs under these three different distributional short-rate models. Although all TBAs become premium bonds given the historical low rates, the "ATM" coupon is approximately 3.5% based on the prepayment model setup, at which the TBAs are priced with very close option-adjusted spreads by all three models. Similar to the swaption market, for the discount sector, however, the HW model implies a greater prepay option value than the BK model and therefore this sector looks relatively rich under the HW model whereas the premium sector is relatively rich under the BK model. The GS model results are kind of in-between for both sectors.[11]

As to the durations, the option-adjusted durations (OADs) from the HW model are the shortest across the coupon stack. This can be explained by the changes in refinancing option value when shifting the yield curve to calculate the OAD: Given that the HW model is a Gaussian model, when shifting the yield curve up and down to calculate the duration, the shift size does not depend on the rate levels as opposed to the BK model.

EXHIBIT 18
Libor OAS and OAD for Short-Rate Models

	Libor OAS (bps)			OAD		
Model	HW	GS	BK	HW	GS	BK
FNCL3	27	35	40	5.74	6.79	6.94
FNCL3.5	2	4	4	3.37	4.00	4.17
FNCL4	–12	–17	–22	1.74	2.10	2.25
FNCL4.5	–22	–32	–40	1.30	1.54	1.70
FNCL5	–2	–13	–20	2.06	2.36	2.60
FNCL5.5	41	32	27	2.59	2.92	3.10
FNCL6	64	57	53	3.20	3.52	3.70
FNCL6.5	45	39	35	3.51	3.79	3.93

EXHIBIT 19
Short-Rate Model OAS

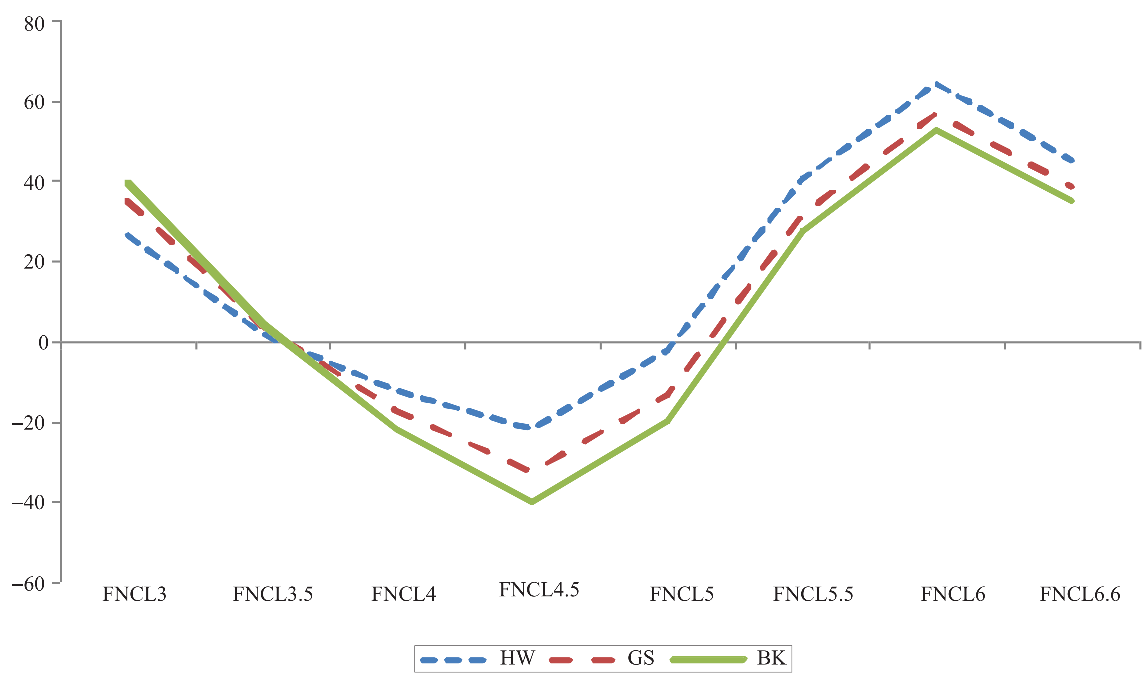
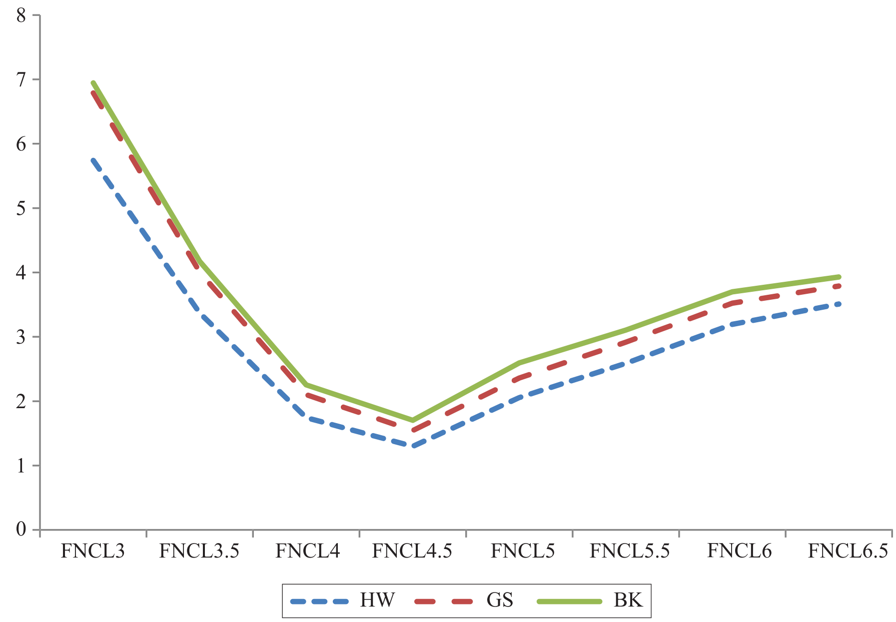


Exhibit 20
Short-Rate Model OAD



Consequently, the HW model implies a greater change in the absolute value of the refinancing option than the BK model. Given that an MBS bond is essentially a long position on a amortizing bond and a short position on the refinancing option, the greater the refinancing option change, the shorter the duration. This means the duration under the HW model is shorter than that under the BK model. The duration for the GS model is in between.

The valuation results under the QGM model are summarized in Exhibits 21, 22, and 23.

It can be observed that there is approximately the same "switching effect" for OAS as seen in the distributional models. This is due to the fact that the QGM model produces similar skews for swaptions and therefore the similar switching point for the OAS.

However, it is very interesting to observe that the duration is shortest across the board when beta = 1 and longest when beta = 0, with beta = 0.5 being in the middle. This is exactly the reverse situation compared with the distributional model case and seems counterintuitive. The remaining question is: Why does the QGM model with beta = 1 produce the shortest duration while its rate distribution clearly resembles a lognormal distribution?

Exhibit 21
LIBOR OAS and OAD for One-Factor QGM Model

	LIBOR OAS (bps)			OAD		
	Beta = 0	Beta = 0.5	Beta = 1	Beta = 0	Beta = 0.5	Beta = 1
FNCL3	38	40	42	6.09	5.43	4.77
FNCL3.5	15	12	10	3.45	2.62	1.78
FNCL4	0	–8	–16	1.54	0.84	0.19
FNCL4.5	–14	–24	–33	0.95	0.62	0.31
FNCL5	1	-8	–15	1.82	1.68	1.55
FNCL5.5	42	35	30	2.50	2.45	2.38
FNCL6	65	60	56	3.16	3.12	3.04
FNCL6.5	45	42	39	3.50	3.49	3.43

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Exhibit 22
QGM1F OAS

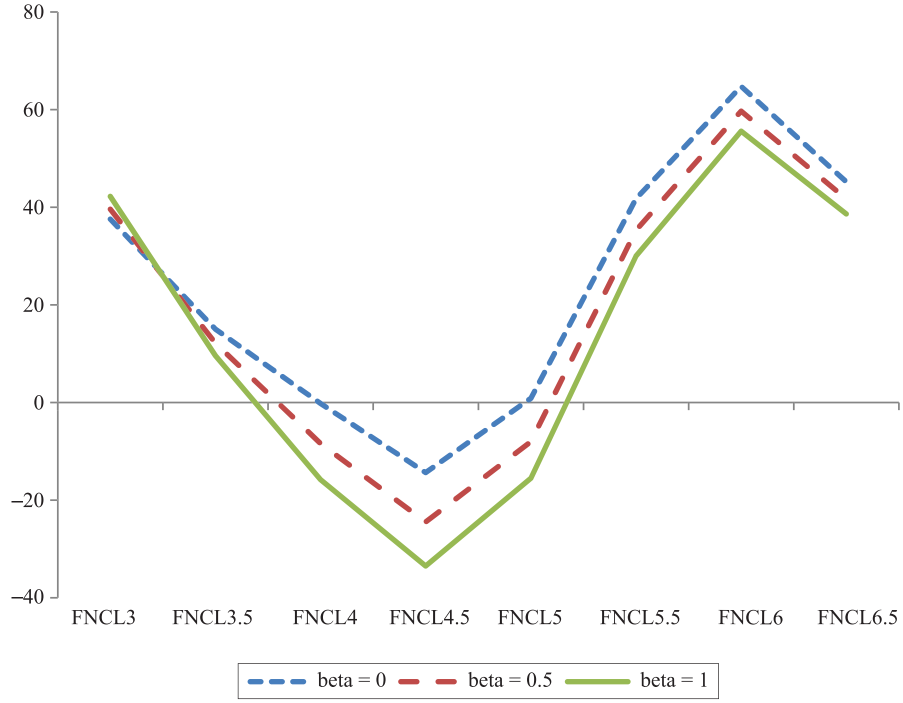
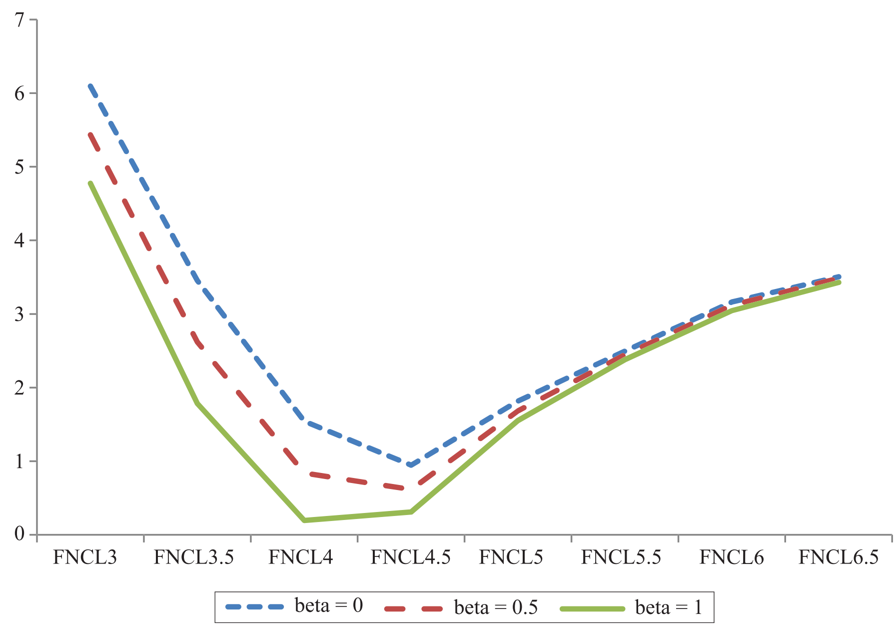


Exhibit 23
QGM1F OAD



The answer resides in the fact that the short rate in the QGM model framework is linear in the instantaneous forward rate for all skew situations despite its quadratic form. Consequently, when shifting the yield curve to calculate the durations, the shift size is independent of rate levels regardless the beta value as opposed to the real log-normal or square-root models in which the shift size does depend on the rate levels. Hence, the QGM model with beta = 1 implies greater refinancing option changes when shifting the yield curve compared with the case of beta = 0. This will shorten the duration. The case with beta = 0.5 is in between, as we can see in Exhibits 21 and 23.

CONCLUSION

Practitioners are hesitant to use a normal model when it comes to MBS valuation because the rates can go negative. They do not like the log-normal model either, however, because it gives too much probability to unrealistic high-rate scenarios and lengthens the duration. The popular alternative is a shifted log-normal model that is a hybrid of log-normal and normal models.[12] Nevertheless, such a model suffers from the fact that the calibration to swaptions relies on inefficient trial-and-error iterations and the model skew is hard to change to reflect what the market implies.

This article analyzes the application of the QGM model to agency MBS pricing. Besides the benefits of its ease of implementation, tractability, control over volatility skew, and fast calibration, a very interesting feature we found in the QGM model is that it can produce at the same time a log-normal-like rate distribution and shorter durations, as normally seen in a Gaussian model. These two features are the most sought-after among the practitioners in terms of a term structure model used in mortgage analytics.

APPENDIX

A TWO-FACTOR STOCHASTIC VOLATILITY PARAMETERIZATION UNDER A QGM FRAMEWORK

Start with a two-dimensional Markov process,

$$dX(t) = \Sigma(t)dW(t) \tag{A-1}$$

with

$$X(t) = \begin{bmatrix} X_1(t) & X_2(t) \end{bmatrix}^{\mathrm{T}},\ \Sigma(t) = \sigma(t)\cdot I,$$

$$W(t) = \begin{bmatrix} W_1(t) & W_2(t) \end{bmatrix}^{\mathrm{T}},\ \langle dW_1 \cdot dW_2 \rangle = 0$$

and

$$V(t) = \int_0^t \sigma^2(s)\,ds \cdot I$$

where I is the identity matrix.

The numeraire-rebased zero-coupon bond keeps the general exponential quadratic form:

$$\tilde{P}(t,T,X(t)) = P(0,T)\exp\left[-\phi(t,T,X(t))\right] \tag{A-2}$$

where

$$\tilde{P}(t,T,X(t)) = \frac{P(t,T,X(t))}{N(t,X(t))}$$

$$\mathrm{N}(t,X(t)) = \frac{1}{P(0,t)\exp\left[-\phi(t,t,X(t))\right]}$$

$$\phi(t,T,X(t)) = X^{\mathrm{T}}(t)\cdot Q(t,T)\cdot X(t) + B(t,T)\cdot X(t) + A(t,T)$$

$$Q(t,T) = \frac{1}{2}(I - C(T)V(t))^{-1}C(T)$$

$$B(t,T) = H(T)(I - C(T)V(t))^{-1}$$

$$A(t,T) = \frac{1}{2}H(T)(I - C(T)V(t))^{-1}V(t)H^{\mathrm{T}}(T)$$

$$+\frac{1}{2}\log\left[\det(I - C(T)V(t))\right]$$

and $Q(t,T)$ is a 2×2 symmetric matrix, $B(t,T)$ is a 1×2 vector, $A(t,T)$ is a scalar, and *det* denotes the determinant of a matrix.

The two-factor QGM model can be re-parameterized in a stochastic volatility framework (Tezier [2007]). Not only can the second factor be used for de-correlation, it can also be set in a stochastic volatility model framework to accommodate both the volatility skew and the V-shaped volatility smile. Formally, let

$$P_{\theta(t)} = \begin{pmatrix} \cos\theta(t) & \sin\theta(t) \\ -\sin\theta(t) & \cos\theta(t) \end{pmatrix} \tag{A-3}$$

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$$C(T) = \int_0^T c(s)\,ds \tag{A-4}$$

with

$$c(s) = P_{\theta(s)}^{\mathrm{T}} \begin{pmatrix} \varepsilon(s)\rho(s) & \tfrac{1}{2}\varepsilon(s)\sqrt{1-\rho(s)^2} \\ \tfrac{1}{2}\varepsilon(s)\sqrt{1-\rho(s)^2} & 0 \end{pmatrix} P_{\theta(s)}$$

$$H(T) = \int_0^T h(s)\,ds \tag{A-5}$$

with

$$h(s) = \begin{bmatrix} b(s) & 0 \end{bmatrix} P_{\theta(s)}$$

The short rate $r(t)$ is quadratic in X vector:

$$r(t) \approx \frac{1}{2} X^{\mathrm{T}}(t) c(t) X(t) + h(t) X(t) + f(0,t) \tag{A-6}$$

It can be shown that the correlation between the two consecutive forwards depends approximately on the angle between them:

$$\rho \approx \cos(\theta_1 - \theta_2) \tag{A-7}$$

To show the stochastic volatility effect, one can rotate the two factors as follows:

$$Z = \begin{pmatrix} Z_1(t) \\ Z_2(t) \end{pmatrix} = P_{\theta(t)} W(t) = \begin{pmatrix} \cos\theta(t) W_1 + \sin\theta(t) W_2 \\ -\sin\theta(t) W_1 + \cos\theta(t) W_2 \end{pmatrix} \tag{A-8}$$

While the two rotated factors remain independent with $\langle dZ_1(t) \cdot dZ_2(t) \rangle = 0$, the short-rate function can be rewritten as

$$r(t) \approx \alpha(t) \cdot Z_1(t) + f(0,t) \tag{A-9}$$

with

$$\alpha(t) = \frac{1}{2}\sigma^2(t)\varepsilon(t)\left[\rho(t) Z_1(t) + \sqrt{1-\rho(t)^2} Z_2(t)\right] + \sigma(t) b(t)$$

Thus, $\alpha(t)$ represents the stochastic volatility with $\varepsilon(t)$ being the volatility of volatility and $\rho(t)$ the correlation between the short rate and the volatility.

Exhibit A1
QGM2F Skew (1 × 1 cap)

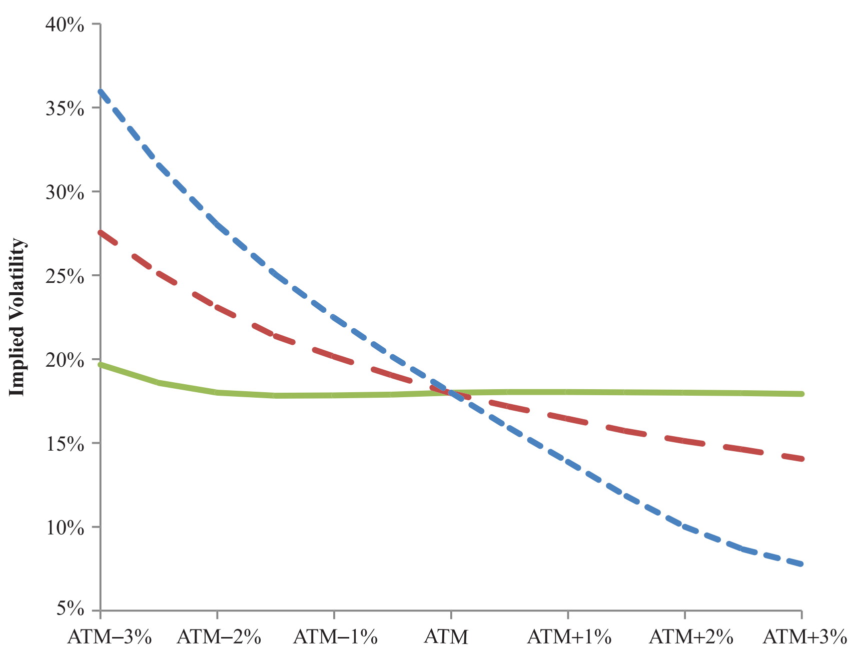

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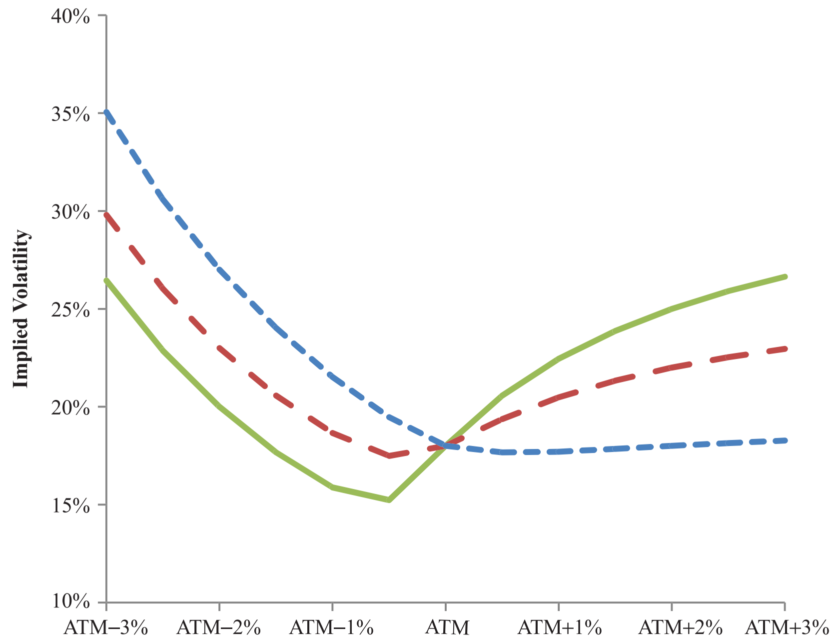
Exhibit A2
QGM2F Smile (1 × 1 cap)



One can still use the martingale property to value a deal,

$$\tilde{y}(s, X(s)) = E_{Q^t}\left[\tilde{y}(t, X(t))\right], s < t \qquad \text{(A-10)}$$

and the zero-coupon bond verifies the martingale condition:

$$E_{Q^t}\left[\tilde{P}(t, T, X(t))\right] = P(0, T) \qquad \text{(A-11)}$$

The swaption can still be priced using the same formula as in the one-factor case knowing that the X is a two-dimensional Gaussian variable.

While the one-factor QGM model can readily handle different degrees of volatility skew, a second factor can add further flexibility to match the curvature of the volatility smile. Exhibits A1 and A2 show the different curvatures a two-factor QGM model can generate. Because refinancing is mainly driven by long-term rates, capturing the smile for MBS pricing is not as important as capturing the skew. Thus, a one-factor QGM model can already go a long way for MBS pricing.

ENDNOTES

The author thanks Ren-Raw Chen and Liuren Wu for their valuable comments on this article.

[1]CEV = constant elasticity of variance.

[2]Some very small convexity terms are purposely ignored in the equation.

[3]See Levin [2004, 2008] for the analyses on the three one-factor short-rate models we denote by distributional short-rate models in this article.

[4]For the examples in this article, we use a flat forward curve at 5% and 18% black ATM volatility unless otherwise specified.

[5]We do not intend to get into much detail about the prepayment modeling, because that is not the purpose of this article.

[6]A fixed-rate MBS can actually be priced on a tree or grid using the backward induction after some transformations as opposed to the forward sampling in a Monte Carlo simulation. We use the backward induction in our examples.

[7]See Kalotay et al. [2004] for an exception.

[8]There may be some adjustments, such as, for example, a SATO adjustment to account for the borrower's credibility at origination.

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[9]The current coupon for the prepayment model is modeled as a linear combination of two-year and 10-year swap rates.

[10]Please note that these are grid nodes for integration, not actual paths.

[11]We refer here to the premium sector as bonds with coupons above the "ATM" coupon and the discount sector as bonds with coupons below that.

[12]Citigroup's MOATS model is in fact a shifted two-factor lognormal model. Please see Bhattacharjee et al. [2006].

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