Introducing the Citi LMM Term Structure Model for Mortgages

Yakov Karpishpan, Ozgur Turel, and Alexander Hasha

Yakov Karpishpan
is a director at Citigroup Global Markets in New York, NY.
yakov.karpishpan@citi.com

Ozgur Turel
was a vice president at Citigroup Global Markets in New York, NY.

Alexander Hasha
is an associate at Citigroup Global Markets in New York, NY.
alex.hasha@citi.com

Valuation of mortgage-backed securities (MBS) proceeds by applying a prepayment model over a Monte Carlo simulation of interest rates generated by a term-structure model. While the prepayment model is crucial, the choice of the term-structure model matters as well. This article presents Citigroup's newly developed Libor Market Model (LMM) for MBS valuation and compares it to the term-structure model currently in production (known as, "2fskew"). There are structural differences: the LMM works with market-observable forward rates, whereas the 2fskew model belongs to the short-rate class of models. But the differences that impact mortgage pricing and hedging are more subtle.

The LMM term-structure model improves control over the volatility and correlations structure of forward LIBORs, leading to more realistic correlations of swap rates. Even more importantly, the new model achieves a more accurate calibration to the entire volatility surface. This is something that may not appear immediately significant to MBS. After all, the mortgage rate is primarily driven by the 10-year swap rate, which seems to suggest that shorter-tenor swaption volatilities are not very relevant here. One of the article's main observations is that short volatility affects the embedded mortgage option via the serial correlation among the forward 10-year swap rates at different horizons. The authors explain how higher short volatility raises the mortgage option's value by lowering the serial correlation of 10-year swap rates.

Volatility skew is important to mortgages: the off-the-money (OTM) skew affects pricing, and the at-the-money (ATM) skew directly contributes to duration (see Bhattacharjee, Radak, and Russell [2006]). Citigroup's implementation of LMM provides several skew choices, which the article discusses.

The authors focus on the practical implications of term-structure modeling for the pricing and hedging of mortgage-backed securities. The article covers in detail the valuation impact for different types of MBS. Compared to the current production model, the LMM produces somewhat wider spreads and longer durations for pass-throughs. The LMM performs similarly for IOs, with wider spreads and less negative duration, while POs, naturally, follow the reverse pattern with tighter spreads and mostly shorter duration. The average life is shorter for all types of securities under the LMM. All of these effects are consequences of the more accurate calibration to the shorter-expiry, shorter-tenor swaptions, driving the differences in the serial correlation of the 10-year swap rates mentioned earlier.

The Mortgage Option-Adjusted Term Structure (MOATS) model, Citi's approach to arbitrage-free mortgage rate modeling, has been re-implemented to work with the LMM, retaining all desirable properties.

The authors provide a unified explanation of the effect of MOATS on the partial durations and vegas of different MBS types, expanding on the earlier work by Bhattacharjee and Hayre [2006].

The article concludes with two appendices on the more technical aspects of the exposition. The first describes the relationship between short volatility and the serial correlation of forward 10-year swap rates, and the second builds useful intuition through a Black–Scholes analysis of a simplified version of the embedded mortgage option.

THE LIBOR MARKET MODEL

The Citi Libor Market Model (LMM) is a term-structure model that deals directly with market-observable objects, rather than the infinitesimal short rate, as in the current production (2fskew) model. It belongs to a class of term-structure models previously known as Brace–Gatarek–Musiela (BGM), but "Market Model" is the more current terminology. The new model has two random factors, as does the 2fskew, but the meaning of these factors is different. In the LMM the factors essentially constitute the principal components of the forward LIBOR movements. The 2fskew stochastic factors drive the short rate and the slope of the curve.

- *The main appeal of the LMM is its control over the volatility and correlation structure of forward rates.*

The LMM defines an evolution of forward LIBOR rates spanning the entire forward curve, making it easy to obtain other quantities of interest, such as the swap rates. Concretely, the term structure of interest rates at time t can be expressed in terms of the forward rates $L_i(t) = L(t, T_i, T_{i+1})$ from T_i to T_{i+1}, for a fixed set of maturities T_i. The terminal value $L_i(T_i)$ is just a LIBOR of tenor $T_{i+1} - T_i$. Each $L_i(t)$ is a stochastic process governed by a law of type

$$\frac{dL_i}{f(L_i)} = \mu_i(t)dt + \sigma_i(t)dW_i \tag{1}$$

where the W_i are Brownian motions, not necessarily independent, one for each LIBOR maturity.

A crucial advantage of the LMM lies in its ability to specify the instantaneous correlations $\rho_{ij}(t)$ among the forward rates, as well as the general shape of the volatility structure, which defines the local volatilities σ_i during the model's calibration. These, in turn, determine the drift terms μ_i through a no-arbitrage condition. The only remaining item is the function f, which controls the skew; for example, $f(L) = L$ defines the log-normal case and $f(L) = 1$ yields normally distributed rates.

- *The correlation and volatility structure of forward LIBORs determines the corresponding quantities for the forward swap rates. A good choice of this structure for LIBORs results in more realistic swap correlations.*

This is because the forward swap rates are naturally linear combinations of forward LIBOR rates,

$$S^{N\times M}(t) = \sum_{i=\alpha}^{\beta} w_i(t)L_i(t) \tag{2}$$

and a well-established LMM approximation[1] allows one to freeze the weights and obtain swap volatilities and correlations in terms of the ρ_{ij}'s and σ_i's.

The following examples highlight LMM's control over correlations.

- The LMM maintains correlations between the 2-year and 10-year swap rates and between one-month LIBOR and the 10-year swap rate close to their long-term historical levels (94% and 86%, respectively).[2]
- The model is consistent with historical observations of forward swap rates, which retain a substantial degree of correlation even at increasingly distant maturities: In the LMM, the correlation between the forward LIBORs of different maturities falls to a positive floor, as the maturity gap grows.
- The model's volatility parameterization is flexible enough to enable a close fit to the entire volatility surface. This feature, together with better control over the forward LIBOR correlation and volatility structure, is the main source of differences with the current production model.

There is a choice of several realistic skews; here we focus on the LMM Skew, representing the long-term historical average of the ATM skew. The LMM produces LIBOR and swap rates that are non-negative by design with every choice of the skew.

As with most market model implementations, the Citi LMM is entirely Monte Carlo–based, unlike the 2fskew,

EXHIBIT 1
LMM vs. 2fskew

LMM	2fskew
Built on market-observable forward LIBORs, with a straight-forward connection to swap rates.	Built on the less intuitive short rate.
Calibrates to full swaption volatility surface, achieving a close overall fit.	Calibrates to 10-year-tenor swaptions and caps. In the current rates environment, typically, elevates short volatility, leading to lower serial correlations among the 10-year swap rates.
Ability to specify realistic volatility and correlation structure of simulated rates.	Less control over correlations and the shape of forward volatility. Broadly speaking, 2fskew inflates volatilities and undervalues correlations.
More stable calibration over time.	
Greater flexibility in the skew implementation.	
Non-negative rates with every choice of the skew.	Some negative rates are possible, although this is not a significant effect.

Source: Citi.

which employs Monte Carlo sampling from a lattice. This has some advantages by preventing unwanted periodicities, although it also creates a challenge for the Mortgage Option-Adjusted Term Structure (MOATS) model, Citi's approach to arbitrage-free mortgage rate modeling, which is lattice-based in 2fskew. The new version reproduces all of the desirable features of MOATS in a pure Monte Carlo setting.

COMPARISON WITH THE PRODUCTION MODEL

Exhibit 1 summarizes the main advantages of the LMM over the current production model. These features are discussed in detail in the rest of the article. We conclude this section by mentioning which differences are most relevant to the model's end user.

- *The main sources of valuation differences between the LMM and 2fskew are the lower serial correlation among the 10-year swap rates in 2fskew and, to a lesser extent, the somewhat different skews in the two models.*

CALIBRATION FIT AND STABILITY

The LMM is calibrated to the entire swaption volatility surface and achieves a close fit (under 1% volatility RMSE on a typical day); see Exhibit 2. By contrast, the 2fskew model in production is calibrated to the 10-year-tenor swaptions only, plus a set of caps. Short volatilities in 2fskew can widen significantly from their market values, partly due to incomplete swaption volatility surface information, and partly due to a misalignment between the cap and swaption markets.

Note the relatively balanced nature of the LMM Skew fit, with the over- and under-calibration distributed fairly evenly.

Inflated short volatility in 2fskew calibration must have consequences, since an earlier Citi study[3] shows that MBS are exposed to the entire volatility surface. The next section explains the mechanism by which the inflated short volatility impacts the mortgage option embedded in a pass-through.

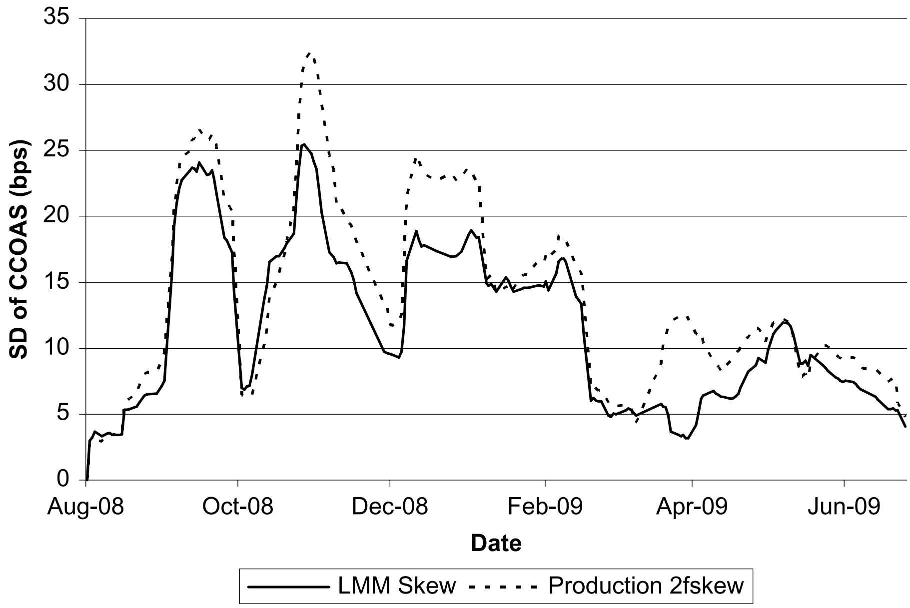
EXHIBIT 2
LMM Skew Calibration Fit on February 16, 2010

Vol Diff (%)	Tenor						
Expiry	1	2	3	4	5	7	10
1	-1.3	-0.2	0.0	0.2	0.0	0.1	-0.2
2	-0.2	-0.1	0.2	0.3	0.3	0.1	-0.2
3	-1.2	-0.4	0.0	0.2	0.1	-0.1	-0.2
4	-0.3	0.0	0.1	0.1	0.0	-0.3	-0.4
5	0.0	0.1	0.1	0.0	-0.1	-0.3	-0.4
7	0.0	-0.1	-0.1	-0.1	-0.1	-0.1	-0.2
10	0.0	-0.1	0.0	0.0	0.0	-0.1	-0.2

Source: Citi.

EXHIBIT 3

Stability of Current Coupon OAS in the Two Models: 30-Day Rolling Standard Deviation of CCOAS



Source: Citi.

It is also important to know how stable the calibration is over time. Exhibit 3 compares the two models by calculating the 30-day rolling standard deviation of the current coupon OAS (CCOAS), an essential ingredient in the MOATS calculations.

The CCOAS is observed over a year that included the most dramatic period in the market history, and the LMM Skew shows materially lower volatility than 2fskew across the entire period.

SERIAL CORRELATION OF THE 10-YEAR SWAP RATE AND THE MORTGAGE OPTION

In this section we will explain how short volatility influences the mortgage option embedded in a pass-through.

- *Accurate calibration to short volatility is important even when the mortgage rate is modeled as a constant spread over 10-year swap rate, because it impacts the serial correlation of the 10-year swap rate—an underappreciated factor affecting all MBS. In turn, serial correlation affects the mortgage option.*
- *It is even more important with MOATS, which makes the mortgage rate depend on shorter-tenor rates.*
- *Prepayment models include features that magnify the effects of serial correlation, and of the short volatility itself.*

To focus on the key issues, assume that the mortgage rate is at a fixed spread to the 10-year swap rate. Then one may think that an error in calibrating the swaptions of other tenors should not seriously affect MBS valuation. A pass-through has a set of embedded prepayment options, exercisable at multiple dates. Each option taken individually depends only on the mortgage rate, i.e., on the 10-year swap rate, at the exercise date. And the volatilities of the 10-year rates achieve a close fit to the market in both 2fskew and the LMM. But calibrating to the 10-year-tenor swaptions does not fix the ***serial correlation***, or the *autocorrelation*, of the simulated 10-year swap rates at different horizons. In Appendix A we show that

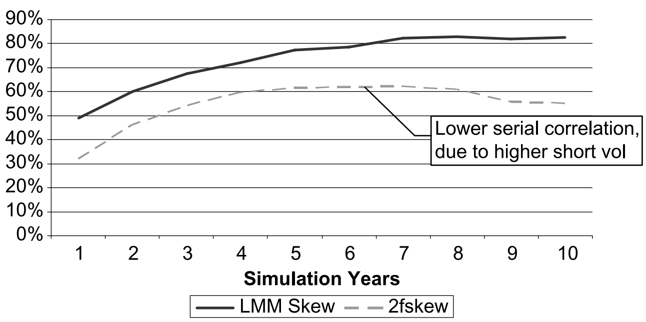
- *Shorter-expiry, shorter-tenor swaption volatility lowers the serial correlation of the 10-year swap rate.*

Elevated short volatility in 2fskew leads to commensurate reduction of the 10-year swap rate's serial correlation. For example, setting the lag to five years, the correlation difference between the two models can grow to 20%–25% (see Exhibit 4).

The reason this is relevant to MBS pricing is straightforward: the bond's value depends not only on volatility, but also on the joint distribution of the mortgage rates at all possible refinancing dates. Additionally,

EXHIBIT 4

Serial Correlation of the 10-Year Swap Rate at 5-Year Lag in the Two Models on February 16, 2010



Source: Citi.

- *Serial correlation among the 10-year swap rates at different horizons "helps" the bond by reducing the overall value of the embedded option.*

A simple example illustrates this point. Suppose there are only two possible refinancing dates, t_1 and t_2. Assume also that the entire balance is either prepaid in full, if the prepayment option is in the money, or not exercised. If the 10-year swap rates S_1 and S_2 at these two dates are perfectly correlated, then those Monte Carlo paths that are in the money at t_1 are still in the money at t_2. But the balance remaining after the exercise at t_1 is 0. And the other paths have non-zero balance at t_2 but are not in the money. Thus, perfect correlation between S_1 and S_2 destroys the value of the option at t_2. On the other hand, if S_1 and S_2 are independent, then a Monte Carlo path not in the money at t_1 still has a chance of being in the money at t_2 with a non-zero balance, and so the total embedded option value is greater than just the option at t_1.

This behavior is aggregated over many loans. The result of lower correlation between the rates S_1 and S_2 is an increase in the notional with which the simulation paths enter in-the-money states at time t_2. In turn, the higher volume of prepayments shortens the average pathwise WAL of the bond.

Volatility

Continuing as above, with just two possible refinancing dates, the overall option value can be viewed as a sum of two European options exercising on these dates, with weights reflecting the notional to which the option applies. Evidently, the weight at t_2 must be lower for the model with higher serial correlation. If we simplify the total optionality into a single European put, the same weights can be used to estimate its effective volatility, since the option value is approximately linear in volatility, at least near at-the-money. Thus,

- *Higher serial correlation, as in the LMM vs. 2fskew, leads to lower effective volatility of the embedded mortgage option.*

More on Serial Correlation

Moving on from just two possible refinancing dates, the general case is similar, if one focuses on two horizons, say 5y and 6y. Exhibit 5 illustrates Monte Carlo paths for the 10-year swap rates simulated between these horizons, exhibiting higher or lower serial correlation.

Higher serial correlation means that the rates do not change much from 5y to 6y. Therefore, any path in the money at 6y was likely in the money already at 5y; hence, little new prepayment can be expected. Conversely, lower serial correlation makes it likelier that a high-rate out-of-the-money path at 5y drops in the money by 6y, creating new refinancing opportunities.

The Role of the Prepayment Model

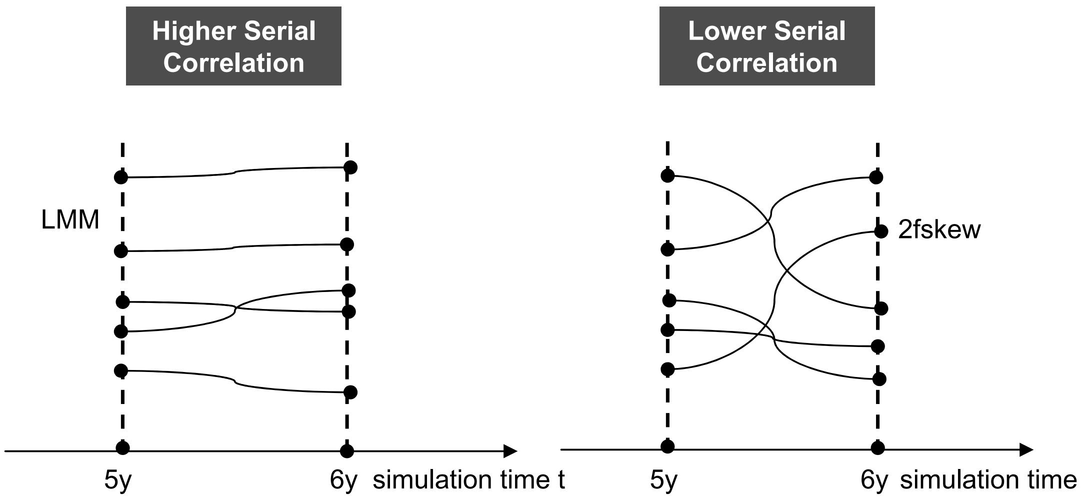
Before leaving the issue of the overcalibrated short volatility in 2fskew, we should mention that certain features of the prepayment model and of MOATS magnify its effects. Most important is the burnout, which postpones some of the prepayment, and hence gives the serial correlation among the 10-year swap rates more time to have an impact. Also, the Citi Prepayment Model uses the one-year rate alongside the 10-year swap rate, making the inflated one-year volatility directly contribute to the overpricing of the mortgage option. Finally, Citi's MOATS method of projecting mortgage rates involves all tenors, further boosting the effects of elevated short volatility.

THE VOLATILITY SKEW

Swaption markets have long recognized the existence of the volatility skew: the dependence of the implied volatility on the strike (OTM skew) and on the current level of the forward swap rate (ATM skew).

The importance of ATM and OTM volatility skews to mortgage valuation and hedging is well established.[4]

Exhibit 5
Visualizing Serial Correlation



Source: Citi.

In particular, the duration of a security is directly affected by the model's ATM skew:

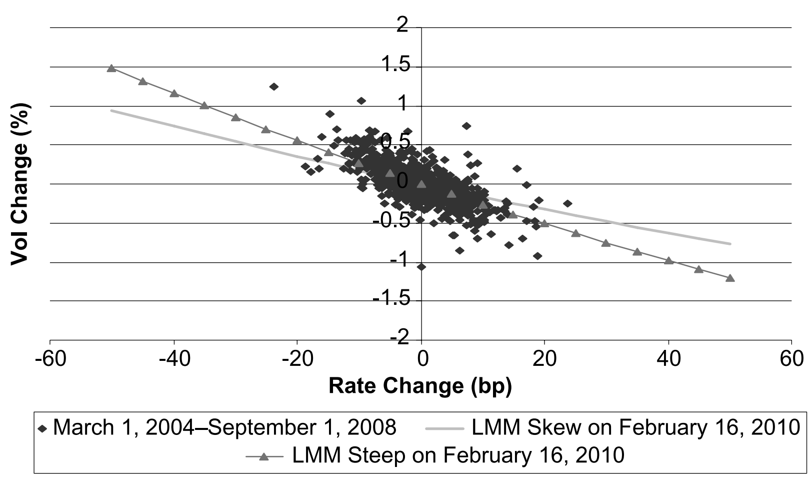
$$Dur_{\text{skew}} \approx Dur_{\text{log normal}} + (\text{ATM Skew Slope}) \times (Vol\ Dur) \qquad (3)$$

Both ATM and OTM skews arise from the same source: the skewness (in the standard statistical sense) of the rates distribution.

The LMM is implemented with a choice of three skews:

- *The default choice,* ***LMM Skew****, and* ***LMM Steep*** *approximate the historical ATM skew. See Exhibit 6 for their current relationship to the 3.5 years preceding the crisis.*
- *One more choice,* ***LMM Flat****, can be used to gauge the skew impact, or to express a view.*

Exhibit 6
The 5 × 10 ATM Skew: Pre-Crisis History and LMM



Source: Citi.

Exhibit 6 presents some historical data that went into the choice of LMM Skew.

One may hope to regularly calibrate the term-structure model to the market (OTM) skew, but this is neither feasible nor desirable. The OTM skew is highly variable and poorly observable. However, a historically based choice of the ATM skew, as before, determines the internal model parameters that, in turn, define the OTM skew, as well.

Exhibit 7 displays the three OTM skews provided with the LMM, together with the production model.

LMM Skew is close to 2fskew, particularly in the money (on the right-hand side of Exhibit 7, where moneyness is positive). In the out-of-the-money region, 2fskew falls in between LMM Skew and LMM Steep.

The skews in the two models also differ in the details of their implementation. In 2fskew, the skew is achieved by using a shifted log-normal distribution, resulting in

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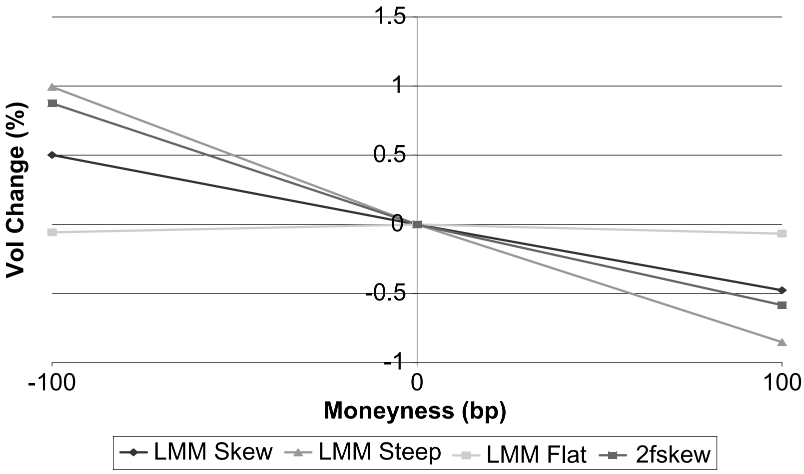
The 5 × 10 Swaption OTM Volatility Skew on February 16, 2010



Source: Citi.

some negative rates. While this is not a serious problem, it leads to some awkward moments, such as a floor struck at zero having a small positive value. By contrast, each skew provided with the Citi LMM model has its defining function $f(L)$ in the Equations (1) designed to ensure that the rates are non-negative.

IMPACT ON VALUATION: PASS-THROUGHS

This section discusses the differences between the production 2fskew term-structure model and the LMM from a user's perspective, starting with the comparison of the two models on the FNMA 30-year TBA coupon stack. First, the numbers.

Very little of the differences is due to MOATS, because the magnitude of the MOATS impact is similar in production and in the LMM. Accordingly, as we review each model output, we continue to model the mortgage rate as a fixed spread to the 10-year swap rate, leaving the MOATS effects for a later section.

OAS

The pass-through OAS under LMM Skew is close to 2fskew for discount coupons, but the difference widens to 15–17 bps for premiums; see Exhibit 9.

Tighter OAS in 2fskew vs. LMM Skew is an immediate consequence of the higher value of the mortgage option in 2fskew. As explained earlier, the root cause is the lower serial correlation of 10-year swap rates, deriving from the overcalibrated short volatility in 2fskew.

The OAS difference is greater for premium coupons in part because the spread duration declines with coupon, reflecting a shortening WAL.

It is instructive to note that running both models with a simplified prepayment model, with no burnout, and whose incentive function has only the 10-year swap rate as its input, the differences shrink dramatically. This highlights two additional details alluded to above.

1. Lower serial correlation of the 10-year swap rate in 2fskew is the most significant explanatory factor, but it is not the only one. Even with the mortgage rate modeled as a constant spread to the 10-year rate, the Citi prepayment model brings the one-year rate into the mix it uses for the mortgage rate. Higher volatility for the one-year rate in 2fskew directly contributes to a higher option cost, hence wider OAS gap with the LMM.
2. The prepayment model's burnout feature amplifies the significance of the serial correlation of the 10-year swap rate by shifting more prepayment from earlier to later dates, hence increasing the value of the refinancing option at later dates.

Duration

The LMM Skew duration is longer by 0.3–0.4 year than in 2fskew; see Exhibit 10.

Once again, the difference arises from the lower serial correlation of the 10-year swap rates in 2fskew, but the argument is more involved. A partial explanation goes as follows. Lower serial correlation spurs prepayment at later dates, hence shortens the average of path-wise WAL numbers in 2fskew, compared to LMM Skew. However, one cannot make a blank assumption that any model that shortens the average WAL automatically shortens durations as well. For example, if two term-structure models differ only in their skews, then the one with the flatter skew will produce shorter average WAL, yet longer pass-through durations.[5]

A more complete qualitative picture of duration differences between the two models comes from an analysis of the mortgage option embedded in a pass-through; see Appendix B.

Exhibit 8

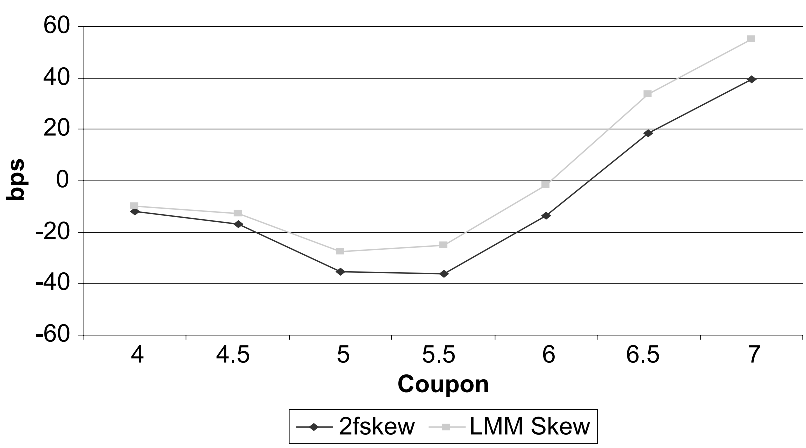
FNMA TBA OAS Measures in the Two Models, with MOATS, on February 16, 2010

Coupon	Price	Model	OAS	Duration	Convexity	Mean WAL
4	98–04	2fskew	-13	5.9	-1.7	8.2
		LMM Skew	-11	6.2	-1.9	8.6
4.5	101–02	2fskew	-17	4.7	-2.5	7.7
		LMM Skew	-12	5.1	-2.8	8.1
5	103–26	2fskew	-35	2.9	-3.5	6.1
		LMM Skew	-25	3.4	-3.9	6.6
5.5	105–18	2fskew	-35	2.0	-3.0	5.4
		LMM Skew	-20	2.4	-3.8	5.9
6	106–12	2fskew	-12	1.8	-2.0	4.8
		LMM Skew	3	2.1	-2.6	5.2
6.5	107–04+	2fskew	20	1.8	-1.4	4.6
		LMM Skew	38	2.0	-2.1	5.0
7	107–26	2fskew	41	1.4	-1.3	4.1
		LMM Skew	59	1.4	-2.0	4.5

Source: Citi.

Exhibit 9

Pass-Through OAS in the New and Old Models with Constant Mortgage Spread, on February 16, 2010



Source: Citi.

Duration and the Skew

Another factor contributing to the duration difference is visible in Exhibit 7: 2fskew falls half-way between LMM Skew and LMM Steep for near-at-the-money and the out-of-the-money strikes. As a result, the 2fskew durations are closer to LMM Steep (included in Exhibit 10). They get even closer if one recalibrates the LMM for the elevated ATM swaption volatilities implied by 2fskew.

Impact of the Prepayment Model

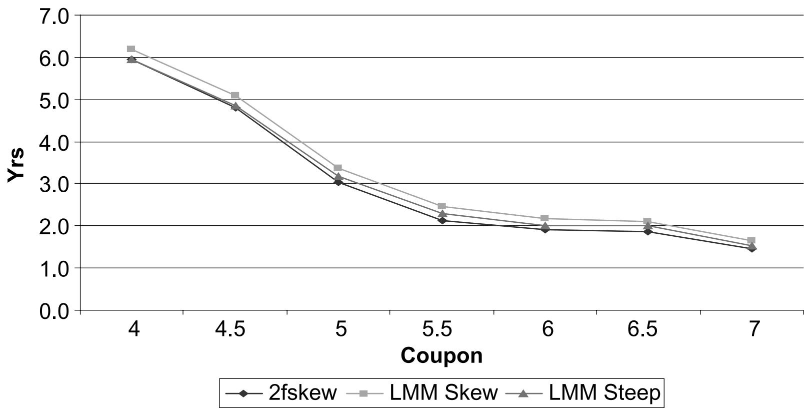
In line with the OAS observations, the duration effects of the short volatility overcalibration and the serial correlation of the 10-year swap rates are amplified by the prepayment model, through its use of the one-year rate and the burnout feature.

Convexity

Exhibit 11 compares pass-through convexity in the LMM and the production model. Pass-through (negative) convexity derives from the embedded put's gamma, which exhibits a similar pattern when the option's attributes are adjusted to reflect the

EXHIBIT 10

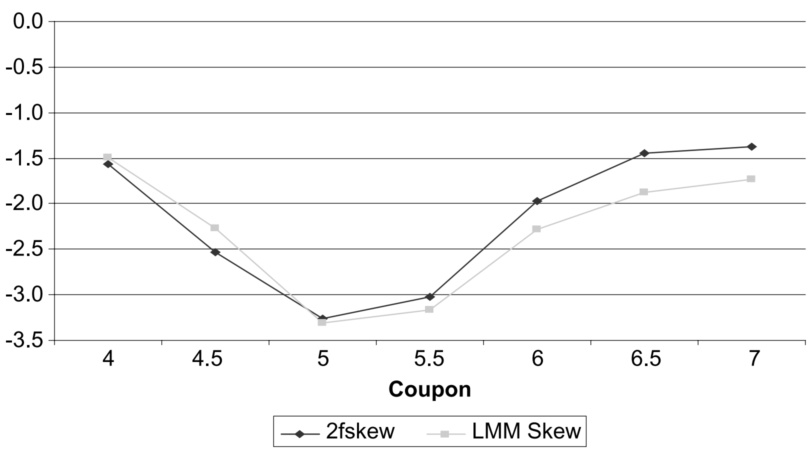
Pass-Through Duration in the New and Old Models with Constant Mortgage Spread, February 16, 2010



Source: Citi.

EXHIBIT 11

Pass-Through Convexity in the New and Old Models with Constant Mortgage Spread, on February 16, 2010



Source: Citi.

differences between the models. See Exhibit B2 in Appendix B for more details.

IMPACT ON VALUATION: IOs AND POs

Here we extend the previous comparisons to several representative trusts.

The most significant difference is in the IO OAS: the LMM numbers are substantially higher than in the production model. As with the pass-throughs, this is because the embedded option in IO is priced higher in 2fskew, due to the lower serial correlations of the 10Y swap rates. The PO OAS exhibits the reverse pattern, with the LMM tighter than 2fskew, because the PO is long its embedded option.

Most of the time, the LMM Skew durations are slightly longer (less negative) for IOs, and shorter for POs, than in 2fskew, but the differences are relatively small. It is interesting to note that even when this relationship begins to invert (the highest-coupon IO or the lower-coupon POs), the average path-wise WAL is still longer in the LMM than in 2fskew, as expected from the serial correlation discussion.

MOATS

MOATS is Citi's methodology projecting mortgage rates consistent with the pricing of mortgages.[6] Constant spread over swaps and other regression-type approaches create a serious discrepancy between the mortgage rate at some future time, and the current coupon interpolated from a coupon stack priced at the same time. With MOATS, the projected mortgage rate and the interpolated current coupon are in agreement throughout the simulation.

While the overall concept of MOATS in the LMM remains the same as in the current model, the execution has to be different. In 2fskew the MOATS algorithm is implemented on a lattice. The LMM is implemented via Monte Carlo simulation, and the MOATS method had to be adapted to this setting.

The properties of MOATS in the LMM and in 2fskew are similar. In fact, MOATS can be thought of as a natural way to establish proper dependence of the projected mortgage rate on the rates of different tenors:

- *MOATS redistributes some of the current coupon's rate dependence from 10-year to other tenors. It also induces positive volatility dependence.*

This reworking of the current coupon dependence is realized not only for projections in a Monte Carlo simulation but also in the instantaneous or horizon rates and

EXHIBIT 12
IO/PO in Old and New Models, with MOATS, on February 16, 2010

	MOATS			IO					PO			
CPN	Deal	Age	Model	Price	OAS	Dur	Conv	Mean WAL	Price	OAS	Dur	Conv
4.5	TR.396	10	2fskew	25–00	147	-26.8	-34.7	7.9	76–14	-64	15.7	8.0
			LMM Skew		213	-22.3	-36.5	8.3		-80	14.7	8.3
5	TR.377	50	2fskew	20–26	201	-36.3	-25.1	5.8	83–22+	-81	13.3	2.4
			LMM Skew		295	-32.4	-32.6	6.1		-99	12.9	4.1
5.5	TR.379	36	2fskew	18–14	547	-35.8	-16.4	5.4	87–22	-167	11.3	-0.4
			LMM Skew		659	-32.8	-24.3	5.7		-183	11.2	1.1
6	TR.391	23	2fskew	15–02	1593	-29.5	-10.9	5.5	91–15	-279	9.8	-1.6
			LMM Skew		1717	-27.5	-20.0	6.0		-287	10.0	-0.2
6.5	TR.380	41	2fskew	16–00+	1448	-23.3	0.1	5.0	91–22	-237	8.6	-2.1
			LMM Skew		1556	-23.0	-7.1	5.3		-244	8.9	-1.1

Source: Citi.

volatility scenarios, based on the idea of keeping the current coupon OAS constant. This has important consequences for vegas and for partial durations of all MBS, examined in the next section.

Valuation Impact

In both 2fskew and LMM Skew the MOATS pass-through durations are 0.2-0.3 year shorter than the constant spread version, with the duration gap between LMM Skew and 2fskew almost unaffected by MOATS. The OAS gap between LMM Skew and 2fskew widens by a couple of basis points for the cusp coupons.

The MOATS impact on IO and PO durations is also relatively small. The effect on the IO OAS is more pronounced, in part because of the high spread duration.

The real differences arise when we come to partial durations and vega, in the next section.

PARTIAL DURATIONS AND VEGA

In the LMM the MOATS impact on partial durations and vega is similar to the production model, but it is sufficiently complex to warrant a review.

MOATS Effects for Common MBS Types

- *For pass-throughs*[7] *MOATS brings greater concentration of partials in the 10-year bucket and lower volatility durations.*

For example, on February 16, 2010, the FN4.5 volatility duration was 0.34 with constant mortgage spread and 0.26 with MOATS. Its partial durations are displayed in Exhibit 13.

- *For POs, MOATS redistributes partial durations away from the 10-year bucket and increases the volatility duration.*

Thus, the TR 379 PO's LMM Skew volatility duration on February 16, 2010, went from 0.10 with constant mortgage spread, to 0.42 with MOATS. The partials are shown in Exhibit 14.

IO durations are negative, but the MOATS effect is similar to that on pass-throughs, in the absolute sense:

- *For IOs, MOATS makes the 10-year partial duration less negative, while other partials are (more) negative; see Exhibit 15.*

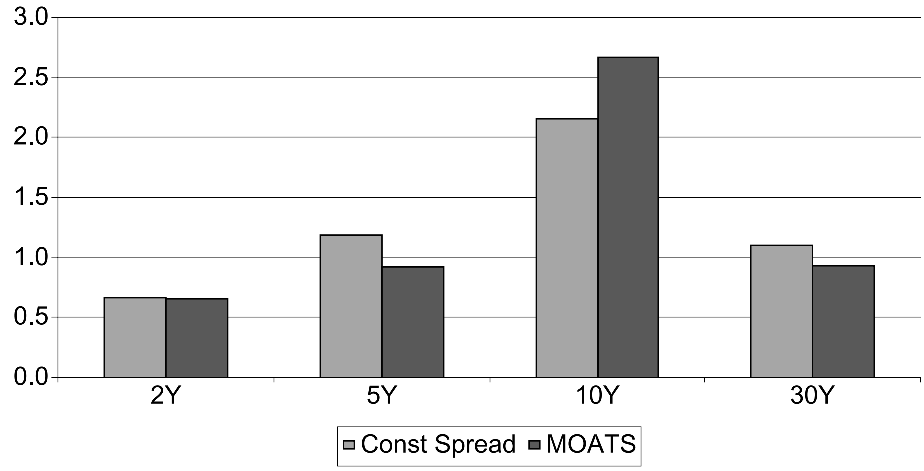
Importantly—*with MOATS*—both models assign negative volatility duration to IOs, i.e., *volatility helps IOs, even as it hurts POs and pass-throughs.* In particular, the LMM Skew volatility duration for TR 379 IO on February 16, 2010, was 1.1 with constant mortgage spread and –0.9 with MOATS.

Explanation

These effects may appear disparate, but they flow from the same cause. The price P of any mortgage-backed

EXHIBIT 13

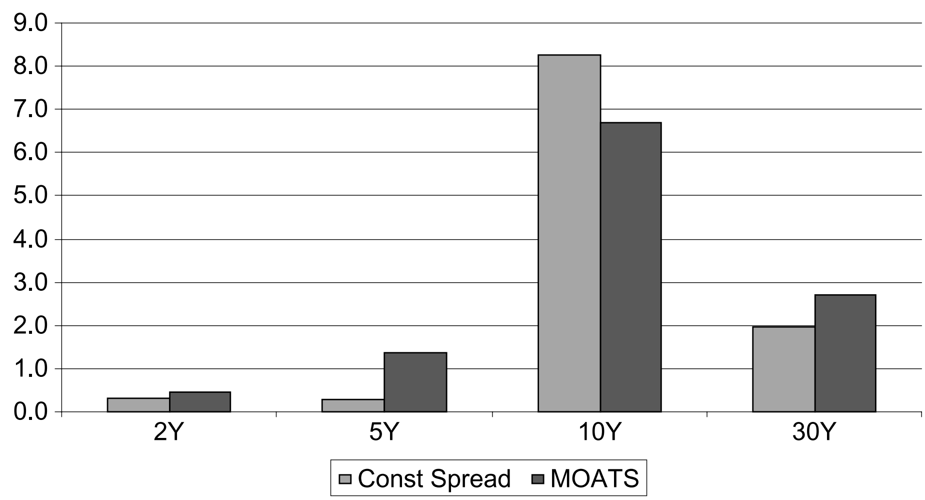
FN4.5 Partials in LMM Skew, with and without MOATS, on February 16, 2010



Source: Citi.

EXHIBIT 14

TR 379 PO Partials in LMM Skew, with and without MOATS, on February 16, 2010



Source: Citi.

security is a function of the current coupon spread *ccs*, volatility *vol*, and the rates of various tenors, e.g., y_2, y_5, y_{10}, and y_{30}, among other market factors:

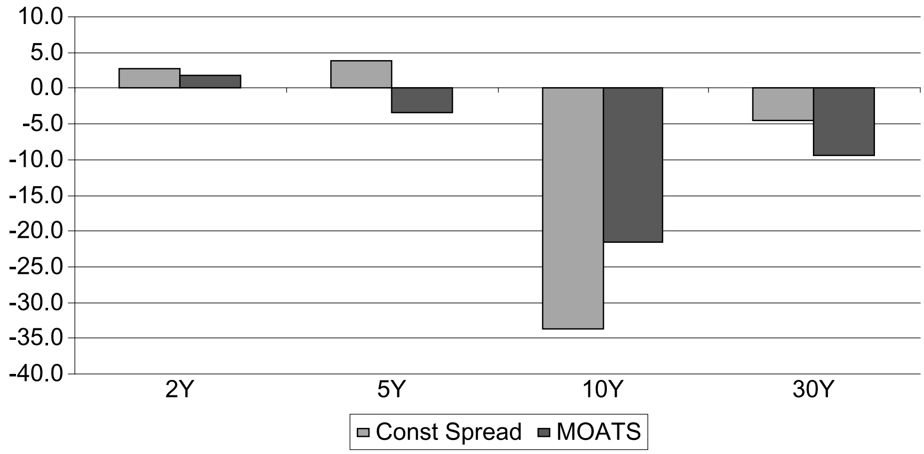
$$P = f(y_2, y_5, y_{10}, y_{30}, vol, css, \ldots) \tag{4}$$

Importantly, the OAS is assumed constant, as expected when calculating partial and volatility durations.

In the constant spread case, by definition, $ccs = ccs_0$, the initial current coupon spread. With MOATS it changes in any rates or volatility scenario. Thus the MOATS *ccs* can be thought of as some more complicated function of ccs_0, *vol*, and of all the rates:

$$ccs = g(y_2, y_5, y_{10}, y_{30}, vol, \ldots) + ccs_0 \tag{5}$$

EXHIBIT 15
TR 379 IO Partials in LMM Skew, with and without MOATS, on February 16, 2010



Source: Citi.

EXHIBIT 16
MOATS Impact on Partial Durations and Volatility Durations for Pass-Throughs and Trusts

		MOATS Effect on Duration with Respect to		
MBS Type	**CC Spread Dur**	**10Y**	**2Y, 5Y, 30Y**	**vol**
Pass-through	−	up	down	down
IO	−	up	down	down
PO	+	down	up	up

Source: Citi.

Total price sensitivity to any variable z is given by

$$\frac{dP}{dz} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial ccs} \cdot \frac{\partial ccs}{\partial z} \tag{6}$$

with the second summand becoming non-zero only under MOATS. This gives a way to relate the duration with respect to any market factor z under MOATS and under the constant spread model:

$$Dur_z^{MOATS} = Dur_z^{\text{Const Spread}} + Dur_{ccs} \cdot \frac{\partial ccs}{\partial z} \tag{7}$$

Now, $\partial ccs/\partial y_{10} < 0$ and $\partial ccs/\partial y_i > 0$ for $i = 2, 5$, and 30, and $\partial ccs/\partial vol > 0$, because of the principal property of MOATS highlighted in the previous section. And the current coupon spread duration Dur_{ccs} is negative for pass-throughs[8] and IOs and positive for POs. Going through $z = y_2, y_5, y_{10}, y_{30}$, and vol, and keeping track of the Dur_{ccs} sign for each MBS type, produces Exhibit 16, which is nothing but a summary of the previously highlighted MOATS effects.

LMM Partials and Vegas vs. the Production Model

Compared with 2fskew, the LMM Skew partials are slightly shifted toward longer maturities, which is consistent with the slightly longer average WAL.

Volatility duration for pass-throughs and POs is almost indistinguishable in the two models, while the IO volatility duration is lower (more negative) by 0.1–0.2.

APPENDIX A

SHORT VOLATILITY AND THE SERIAL CORRELATION OF THE 10-YEAR SWAP RATE

We will sketch how for most shorter-tenor/shorter-expiry swaptions,

- *Increasing short volatility decreases the serial correlation of the 10-year swap rates.*

Concretely, we focus on the 10-year spot swap rates starting in five and six years and show how their correlation is linked to the 5×1 swaption volatility.

Some Stochastic Processes

Write $S(t, T_1, T_2)$ for the forward swap rate from T_1 to T_2, as known at time t ($t \le T_1 < T_2$). In this notation, the five- and six-year starting spot 10-year swap rates are $S(5, 5, 15)$ and $S(6, 6, 16)$. To relate the two rates, one may think of a stochastic process whose value at time t is the spot 10-year swap rate $R_t = S(t, t, t + 10)$, i.e., the swap rate from t to $t + 10$, observed at t. However, R_t is difficult to analyze directly. Instead, the Market Model approach suggests introducing the processes, $X_t = S(t, 5, 15)$ and $Y_t = S(t, 6, 16)$. Their terminal values are precisely the two rates we are interested in: the 10-year spot swap rate starting in 5 years is X_5, and the 10-year rate starting in 6 years is Y_6.

To connect X_t and Y_t, bring in two more processes: $X_t' = S(t, 5, 16)$ and $Z_t = S(t, 5, 6)$. Now, $Y_0 = S(0, 6, 16)$ is just the ordinary six-year-forward 10-year swap rate ("6×10"). It can be decomposed into the forward swap rates $X_0' = S(0, 5, 16)$ and $Z_0 = S(0, 5, 6)$ ("5×11" and "5×1" respectively) with suitable weights w_1 and w_2 (see Exhibit A1),

$$Y_0 = w_1 X_0' - w_2 Z_0 \tag{A-1}$$

The decomposition remains approximately valid for all $t \le 5$ with the weights frozen.[9] And X_t and X_t' are very close, as co-initial swap rates with similar tenors. Therefore,

$$Y_5 \approx w_1 X_5 - w_2 Z_5 \tag{A-2}$$

Covariance

The correlation we are interested in is

$$corr(X_5, Y_6) = \frac{\mathrm{Cov}(X_5, Y_6)}{\sigma(X_5)\sigma(Y_6)} \tag{A-3}$$

The volatilities in the denominator are assumed fixed by calibration. Therefore, the correlation in question moves together with the covariance in the numerator.

We make a simplifying assumption that the process (X_t, Y_t) has independent increments. Then

$$\mathrm{Cov}(X_5, Y_6) = \mathrm{Cov}(X_5, Y_5) \tag{A-4}$$

This, together with the linearity of covariance applied to the decomposition of Y_5, implies

$$\mathrm{Cov}(X_5, Y_6) \approx w_1 \mathrm{Var}(X_5) - w_2 \mathrm{Cov}(X_5, Z_5) \tag{A-5}$$

But co-initial forward start rates, such as $X_5 = S(5, 5, 15)$ and $Z_5 = S(5, 5, 6)$, have correlations close to 1 in both models. Hence, $\mathrm{Cov}(X_5, Z_5) \approx \sigma(X_5)\sigma(Z_5)$ and, putting it all together,

$$\mathrm{Cov}(X_5, Y_6) \approx w_1 \sigma^2(X_5) - w_2 \sigma(X_5)\sigma(Z_5) \tag{A-6}$$

The 5×10 volatility $\sigma(X_5)$ is fixed by calibration. Then, if the 5×1 volatility $\sigma(Z_5)$ goes up, the correlation between five- and six-year starting 10-year spot swap rates must go down, which is what we wanted to show.

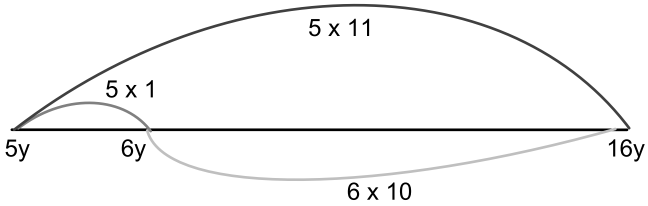
Variance of Change in the 10-Year Swap Rate

There is another way to express the consequences of higher short volatility for the 10-year swap rate. Declining serial correlation increases the variance of the change in the 10-year swap rate between two horizons,

$$\mathrm{Var}(S_2 - S_1) = \mathrm{Var}(S_2) + \mathrm{Var}(S_1) - 2\mathrm{Cov}(S_1, S_2) \tag{A-7}$$

since the 10-year swap rate's volatility is fixed at every horizon when each model is calibrated to the 10-year-tenor swaption volatilities.

EXHIBIT A1
Decomposing Forward Swap Rates



Intuitively, because of the way in which swaption volatility scenarios are defined, the increased short-rate volatility spills into increased variance of the 10-year rate change.

Appendix B

THE MORTGAGE OPTION IN THE BLACK–SCHOLES WORLD

The option embedded in a pass-through bond can be approximated by a single vanilla put on the mortgage rate. Its analysis helps understand the bond's duration and convexity.

Pass-Through Duration and the Embedded Option's Delta

Breaking up the full pass-through value $V = V_0 - Put$ into its optionless part V_0 and the embedded short put, and letting y denote the level of rates, one can write[10]

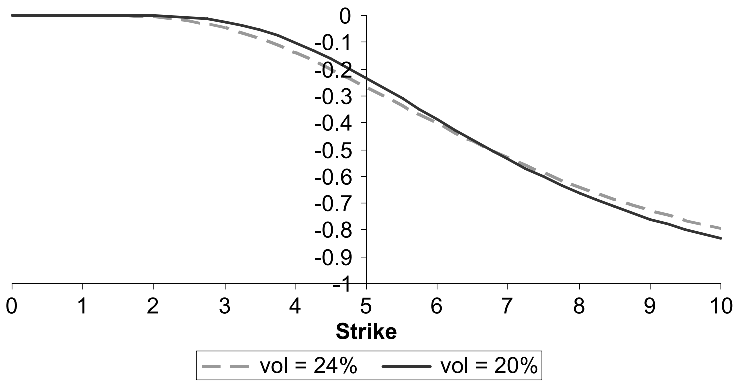
$$Dur(\text{Bond}) = -\frac{100}{V}\frac{dV}{dy} = \frac{100}{V}\left(-\frac{dV_0}{dy} + \frac{dPut}{dy}\right)$$
$$= \frac{100}{V}\left(-\frac{dV_0}{dy} + \Delta_{Put}\right) \quad \text{(B-1)}$$

So comparing durations in the LMM vs. 2fskew reduces to comparing deltas for two options, which comes down to their basic characteristics:

- As discussed, the put's volatility is higher in 2fskew.
- Higher serial correlation slows prepayment, on average. Therefore, the simplified put's exercise time should be longer in the LMM than in 2fskew.
- Exercise delay also implies a higher forward rate, since the yield curve is upward-sloping.

Exhibit B1

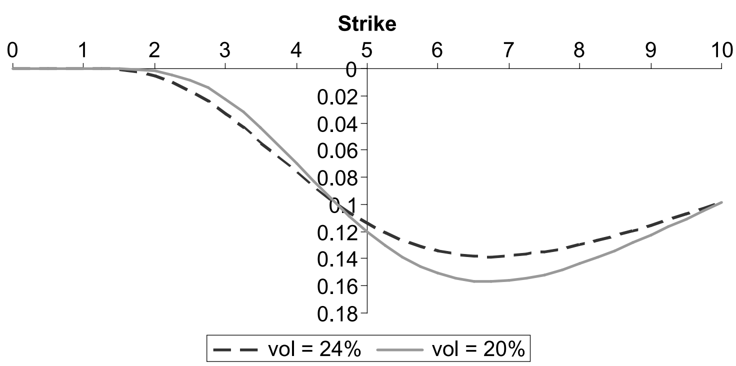
Delta of Two Puts Over a Range of Strikes; the Low–Vol Option Has Longer Exercise Time and Higher Forward Rate



Source: Citi.

Exhibit B2

Gamma of Two Puts (Inverted Axis) Over a Range of Strikes; the Low–Vol Option Has Longer Exercise Time and Higher Forward Rate



Source: Citi.

This motivates the following comparison: one put with the volatility of 20%, expiry in 4.4 years, and the forward rate of 6.2%; and another with volatility 24%, expiry 4 years, and the forward rate 6%. The example will illustrate how, with suitable choice of expiries and forward rate values, the higher volatility of the embedded option in 2fskew vs. LMM Skew shortens the pass-through duration and results in a less negative convexity for much of the coupon stack.

The higher-volatility put in Exhibit B1 has lower (more negative) delta for all but the highest strikes. This directly compares with the duration chart in Exhibit 10, which shows shorter duration in 2fskew vs. LMM Skew for most coupons.

Convexity and the Embedded Option's Delta

A pass-through's negative *convexity* arises from (the negative of) the embedded option's *gamma*.[11] Exhibit B2 compares the gamma of the same two puts as in Exhibit B1, with the gamma axis inverted to make the relationship to the convexity chart in Exhibit 10 easier to observe.

The two graphs cannot match exactly, since each coupon's embedded option requires its own translation into a single put.[12] Even taking the bond's coupon to be the put's strike, as we do, is clearly a simplification. But in Exhibit B2, the gammas of the higher- and the lower-volatility puts, restricted to the strikes between 4% and 7%, display the same relationship as the convexity in 2fskew vs. LMM Skew in Exhibit 11.

ENDNOTES

The authors thank Ranjit Bhattacharjee and Mikhail Teytel for many vigorous discussions.

[1]See Rebonato's formula on p. 248 of Brigo and Mercurio [2001].

[2]In 2fskew the first correlation is close, but the second is substantially lower for the first five years of simulation.

[3]See Bhattacharjee and Radak [2005].

[4]See Bhattacharjee, Radak, and Russell [2006].

[5]This holds both in production and LMM. The skew effect on pass-through durations follows from Formula (3) in the Volatility Skew section. And WAL shortens because a flatter skew shifts more probability to the rates paths that are slightly in the money, which is precisely the region where prepayment reacts to the incentive most strongly. And out of the money, or deeply in the money, prepayment is relatively insensitive to volatility differences.

[6]See Bhattacharjee and Hayre [2006].

[7]We may need to exclude deep discounts, which imposes no restrictions in the current rates environment.

[8]It is here that we may have to exclude deep discounts, because of the turnover effects. Also, the constant OAS assumption is critical for a pass-through: if the pass-through OAS is allowed to move with CC spread, as it normally does, the (positive) spread duration contribution would reverse the overall CC spread effect.

[9]See the reference in Note 1.

[10]The constant mortgage spread assumption is used here to identify the put's derivative with respect to the rates level with its delta, i.e., the sensitivity to the (forward) mortgage rate.

[11]Analogously to (B-1) for duration, and with the same notation, $Conv(\text{Bond}) = \frac{100}{V}\left(\frac{d^2V_0}{dy^2} - \Gamma_{put}\right)$, at least under the constant mortgage spread assumption.

[12]Even closer fit can be achieved at the cost of added complexity. Explicitly recognizing the S-curve-driven nature of the prepayment option suggests modeling it as bear put spread or, more precisely, as a put struck at the bond's coupon, minus a multiple of the put struck at a somewhat lower strike.

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