The Complexities of Mortgage Options

JOSEPH R. PRENDERGAST

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Options on mortgage-backed securities are popular, useful, and liquid instruments. The options are typically written on TBA agency pass-throughs insured against default by Fannie Mae, Freddie Mac, or Ginnie Mae. A TBA contract is a forward agreement with a specified coupon, agency, and maturity, and an embedded cheapest to deliver option that allows flexibility, given the specified characteristics. Mortgage options are commonly available for a broad range of customer-specified strikes, maturities, and underlying coupons, and are traded over the counter by most major Wall Street broker-dealers.

Mortgage options have several natural end users (Averbukh and Teytel [2001] provide an overview). Money managers use them to hedge negatively convex mortgage positions while avoiding the basis risk that swaptions or U.S. Treasury options would entail. This strategy allows a portfolio manager to separate the decision to buy or sell the prepayment option embedded in a mortgage from the decision to buy or sell mortgages as an asset class.

Mortgage options can also be used to hedge basis risk directionality. For example, a duration-matched position that is long Treasury calls and short mortgage calls protects a portfolio against spread widening in a market rally, a situation that occurs frequently during refinancing waves.

Finally, mortgage originators often sell loans forward for which they have received an application but that are not yet closed. If mortgage rates rally in the interim, some homeowners will reapply at the new lower rates. The originator will then have to buy the loans at a premium to deliver into the forward contracts. For an up-front premium, a strategy of buying mortgage calls protects the originator from this risk.

Useful as they are, mortgage options, or any options on instruments that have significant negative or positive convexity, can be difficult to understand. The intuition imparted to many from the Black-Scholes model is not easily transferred. Attempts to understand the interplay between the negative convexity of the underlying mortgage and the positive gamma of the option often frustrate the intuition of portfolio managers and traders. Analytic modeling of the option price and risk sensitivities is also difficult, as the prepayment option embedded in the underlying mortgage must be addressed.

This article quantifies and explores the risk sensitivities of mortgage options, especially as they differ from risk sensitivities of options on zero-convexity underlying instruments. To do this, I develop a model that uses an empirical price-yield relationship for the underlying mortgage and a simple one-factor interest rate model for the yield. The model is embedded in a Monte Carlo simulation to obtain the risk sensitivities. I also use the simulation to obtain risk sensitivities for a benchmark option on a zero-convexity bond (referred to as a ZCB option).

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Following this approach, I show that the delta and gamma of the mortgage option differ in important ways from those of the ZCB option.[1] This is because the *price* distribution of the mortgage is negatively skewed relative to the price distribution of the ZCB. Consequently, accounting for skewness of the distribution is critical when hedging the negative convexity of MBS positions with mortgage options.

Using yield as a risk variable instead of price makes the associated risk sensitivities more comparable across all the assets in a fixed-income portfolio. When interest rates change and spreads stay fixed, for example, the durations and convexities across asset classes are directly comparable. As a result, portfolio managers are often interested in the first and second derivatives of price with respect to an interest rate or yield change rather than a price change. These measures, referred to as duration and convexity, are analogous to delta and gamma, with yield used as a risk variable in place of price.

The risk sensitivities obtained from the simulation offer powerful insights into how the negative convexity of the *underlying mortgage* and the naturally positive gamma of the *mortgage option* interact to determine the convexity of the mortgage option. I show that an expression for the convexity of a mortgage option can be partitioned into two terms: one dependent on the option gamma, and the other on the convexity of the underlying mortgage. Depending on the option specification and interest rate environment, either effect can dominate.

The result is that a mortgage call option is sometimes positively convex and sometimes negatively convex. A mortgage put option, on the other hand, is always positively convex.

Convexity is a key characteristic of option behavior. If a dealer or portfolio manager holds mortgage call options, knowing that they can potentially exhibit either positive or negative convexity and being able to predict and understand the sign and magnitude of that convexity is absolutely critical. A portfolio manager who does not take into account the negative convexity of the underlying mortgage and views the option in a more traditional sense will be quite surprised when a long position in a mortgage call option exhibits negative convexity and positive carry.

I. MORTGAGE OPTION PRICING MODEL

As with any European option, a mortgage option may be valued by taking the expectation of the discounted cash flow at expiration under a risk-neutral distribution. In theory, this could be done within the popular option-adjusted spread/Monte Carlo framework, given a prepayment and term structure model. Such an approach would be quite involved, as an entire *distribution* for the mortgage value at *option expiration* would need to be generated, not just an estimate of the current mortgage value.

As my purpose is to provide insight into the behavior of mortgage option prices, I opt for a more tractable and transparent model. The negative convexity of a mortgage pass-through has an important impact on the distribution of its future prices. To see this, consider the difference between the price behavior of a non-callable bond and a mortgage pass-through. As the mortgage rate drops below the pass-through coupon, loans in the pool tend to be refinanced (i.e., called) at an increasing rate. This limits the upside of the pass-through compared to the non-callable. Thus, given an interest rate distribution, the price distribution of the mortgage will be negatively skewed compared to the price distribution of the non-callable.

The challenge in developing a mortgage option pricing model is obtaining the distribution of the underlying mortgage price at expiration. Given 1) the empirically observed behavior of mortgage prices relative to mortgage rates, and 2) the distribution of mortgage rates, the empirical distribution of the mortgage price can be easily obtained.

Empirical Price-Yield Curve for a Mortgage Pass-Through

Casual observation of mortgage pass-through prices suggests that the duration of a mortgage pass-through graphed against mortgage rates traces an s-curve. This behavior is intuitive and well documented (see Breeden [1995]). When a mortgage pass-through is refinanceable, a reduction in mortgage rates increases the expected rate of future prepayments.[2] Thus, the expected cash flow stream of the mortgage shortens as rates drop, resulting in less of a price increase than would have occurred had the cash flows remained fixed. This makes the option-adjusted duration of the pass-through shorter than the static cash flow duration. As the mortgage rate drops further, the marginal impact on prepayments lessens, but remains positive.

To pose an extreme illustration of this declining marginal impact, consider that if homeowners do not prepay when their mortgage coupons are 500 basis points above the prevailing mortgage rate, they will not be much more likely to prepay if rates drop further. Certainly, the marginal impact of a drop in rates on expected prepayments would be less than if the mortgage coupon were

only 50 bp over the prevailing mortgage rate. Thus, durations shorten as mortgage rates rally, but at a declining rate as the mortgage rate continues to drop.

The first graph in Exhibit 1 shows an s-curve fit to recently observed conventional 30-year pass-through dollar durations.[3] The interest rate on the x-axis is the conventional 30-year par coupon minus the coupon of the mortgage, in basis points.[4] This *relative interest rate* places pass-throughs with different coupons on a similar footing with respect to the prepayment option. Hence, a pass-through with a 5.5% coupon should have a duration and price similar to a pass-through with an 8.5% coupon when rates are 300 bp higher.[5]

A relative rate of zero indicates a pass-through that is priced at par. This rate is henceforth referred to as the "relative rate," or simply as the "rate" *r*.

The functional form for the s-curve is chosen to be

$$D = -100[a + b/(1 + e^{-c(r-d)})] \quad (1)$$

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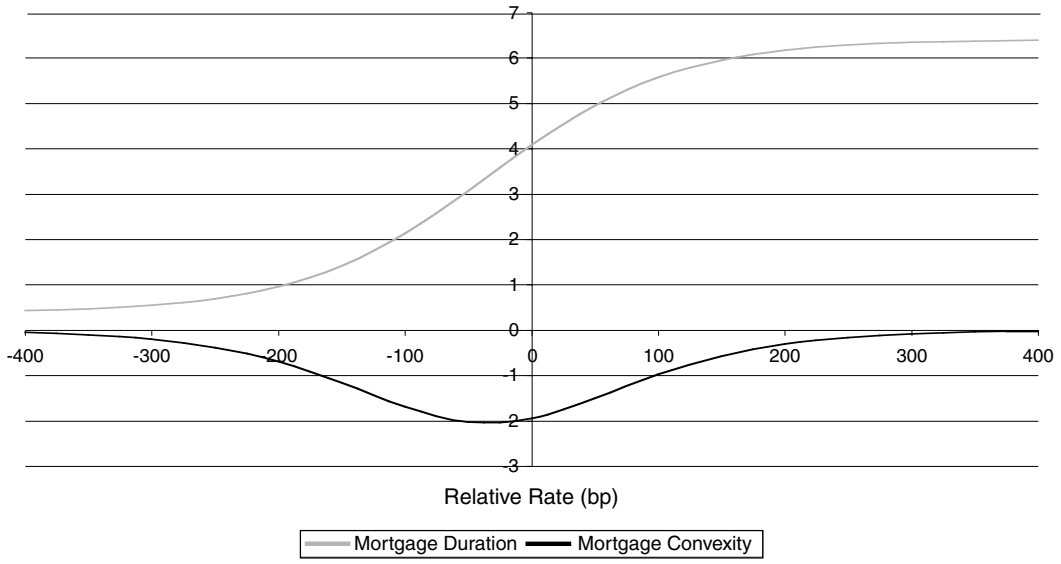
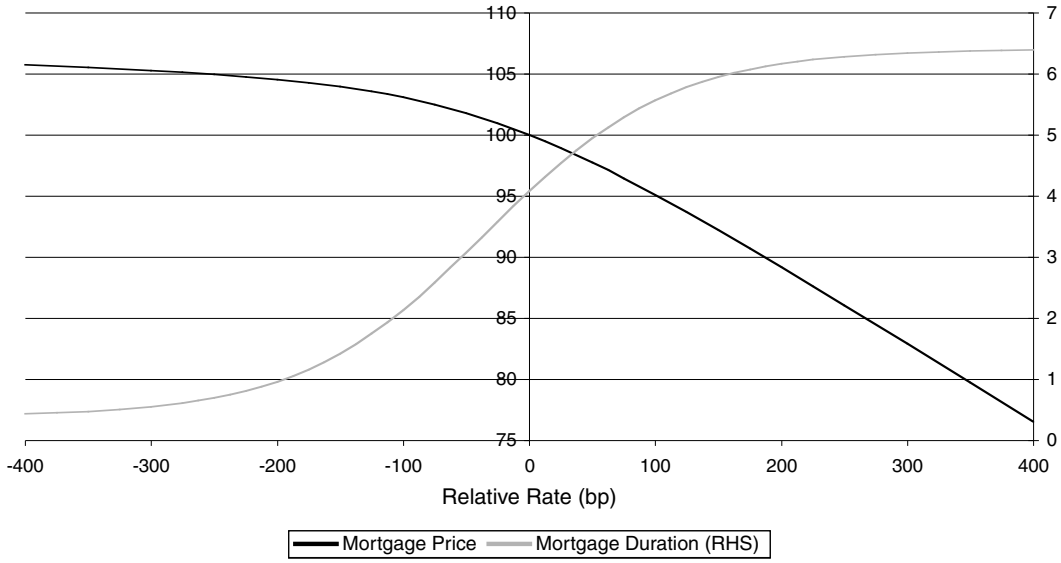
EXHIBIT 1
Mortgage Price, Duration, and Convexity





where a, b, c, and d are constants, and r is the relative rate. The s-curve function has some attractive properties for modeling durations. It is continuous, and continuous in all its derivatives. The function converges asymptotically to a finite value in both directions.

The first graph in Exhibit 1 also shows the dollar convexity of the mortgage which is simply the first derivative of (minus) the duration. Note that the convexity reaches its minimum value at a relative rate of –34 bp, the point at which transaction costs are overcome and it becomes economical to refinance the mortgage.

To obtain the distribution of the mortgage price at option expiration, a price-rate curve rather than a duration-rate curve is required. Integrating Equation (1) results in an expression for price as a function of the relative rate:

$$P = cons + (a+b)r + \frac{b\ln(1+e^{cd-cr})}{c} \tag{2}$$

where *cons* is set so that price is equal to $100 at a relative rate of zero (i.e., the bond is priced at par when its coupon equals the par coupon).

The second graph in Exhibit 1 shows the price-rate curve. The price-rate curve may now be used to map an interest rate distribution into a price distribution.

Interest Rate Distribution

In order to illuminate the risk characteristics of mortgage options, I choose a transparent, one-factor interest rate process where the rate follows a normal distribution with drift

$$dr = \mu dt + \sigma dz$$

where μ is the annualized drift in bp, σ is the annualized standard deviation in bp, and $dz \sim N(0, \sqrt{dt})$. Since the process is normal, this reduces to a simple expression for the distribution of the rate at a future time $t = T$ (with $t = 0$ at present):

$$r \sim N(r_0 + \mu T, \sigma\sqrt{T}) \tag{3}$$

Modeling an Option

The price-rate curve and interest rate distribution are sufficient to obtain the distribution of future mortgage prices, and hence to value a European mortgage option. The mortgage option is priced in a Monte Carlo framework.

First, the rate at the time of option expiration is simulated using Equation (3). Next, the simulated rates are converted to prices using the price-rate curve in Equation (2). Then, the prices are used to determine a set of simulated cash flows to the option at expiration. Finally, these cash flows are discounted at a risk-free rate and averaged to obtain the mortgage option value.

In actual practice, we implement this model by calibrating the interest rate model to a set of at-the-money forward mortgage options with varying maturities and varying underlying conventional 30-year coupons.[6] The calibration procedure estimates the (risk-neutral) parameters of the interest rate process in Equation (3).

Here, some simplifying assumptions are made for ease of exposition (they do not detract from exploration of the risk characteristics of mortgage options):

- The risk free rate for discounting is set to zero.
- For consistency, relative rates and all risk sensitivities are based on forward mortgage prices and rates. The forward mortgage price is defined as the expected mortgage price, under the empirical distribution described above, at option expiration. The forward relative rate is defined as the relative rate that will produce the forward mortgage price using Equation (2).
- In the base case, the forward par coupon is assumed to be equal to the mortgage coupon of 5.5%. This assumption matches the base case to a forward relative rate of zero and a forward par price for the mortgage, making the risk sensitivity plots easier to interpret.
- The options are struck at the money forward, henceforth referred to simply as at the money. That is, they are struck at the forward price. At this strike, by put-call parity, the price of a put equals the price of a call.
- The drift term in Equation (3) is set to zero.
- The rate volatility term in Equation (3) is set at a value consistent with recent volatility of the 30-year conventional par coupon rate, 140 bp per year.
- The option tenor is set to 90 days, which is representative of these options.

The values are thus:

Conventional 30-year par coupon (forward)	5.5%
Underlying mortgage pass-through coupon	5.5%
Mortgage option strike price (put and call)	$100.00
Zero-convexity bond option strike price (put and call)	$100.00
Rate drift (μ)	0 bp/year

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Rate standard deviation (σ)	140 bp/year
Option tenor	90 days (90/360)

Estimates of the risk sensitivities converge slowly as we increase the number of simulated rates. One million simulated rates provides reasonably good stability and does not impose onerous computing requirements.

II. CONCEPTUAL FRAMEWORK FOR MORTGAGE OPTION RISK CHARACTERISTICS

Options are often written on underlying instruments with little or no convexity. As a result, the intuition that most market participants have about option price behavior does not necessarily apply when the underlying itself has significant convexity, as is the case with mortgages. To extend the intuition to mortgage options, I develop some basic relationships among option durations, deltas, convexities and gammas, and durations and convexities of the underlying mortgage.

First, I define some variables and risk sensitivities. P_{MBS} is the dollar price of the mortgage pass-through, Y_{MBS} is the yield of the pass-through (or, equivalently, the rate or relative rate), and P_{Opt} is the mortgage option dollar price. *Mortgage* duration and convexity are defined as the negative of the first derivative and the second derivative of the mortgage dollar price with respect to the yield of the mortgage:

$$D_{MBS} = -\frac{\partial P_{MBS}}{\partial Y_{MBS}}$$

and

$$C_{MBS} = \frac{\partial^2 P_{MBS}}{\partial Y_{MBS}^2}$$

Mortgage option duration and convexity are defined as minus the first derivative and the second derivative of the mortgage option dollar price with respect to the yield of the mortgage:

$$D_{Opt} = -\frac{\partial P_{Opt}}{\partial Y_{MBS}}$$

and

$$C_{Opt} = \frac{\partial^2 P_{Opt}}{\partial Y_{MBS}^2}$$

Finally, *mortgage option* gamma and delta are defined as the first and second derivative of the mortgage option dollar price with respect to the dollar price of the mortgage pass-through:

$$\Delta_{Opt} = \frac{\partial P_{Opt}}{\partial P_{MBS}}$$

and

$$\Gamma_{Opt} = \frac{\partial^2 P_{Opt}}{\partial P_{MBS}^2}$$

In a fixed-income portfolio, option duration and convexity are of primary interest, since using yield as a risk variable makes the sensitivities more comparable across assets. The behavior and the determinants of mortgage option delta and gamma and the duration and convexity of the underlying mortgage are more intuitive. Therefore, it makes sense to view option duration and convexity as an outcome of the interaction among option delta and gamma and mortgage duration and convexity.

First, by the chain rule:

$$\frac{\partial P_{Opt}}{\partial Y_{MBS}} = \frac{\partial P_{Opt}}{\partial P_{MBS}} \frac{\partial P_{MBS}}{\partial Y_{MBS}}$$

or

$$D_{Opt} = \Delta_{Opt} D_{MBS} \tag{4}$$

This first-order relation is straightforward. The duration of the option is equal to the duration of the underlying mortgage, scaled up or down by the option delta.

Differentiating again with respect to Y_{MBS} and applying the chain rule yields:

$$\frac{\partial^2 P_{Opt}}{\partial Y_{MBS}^2} = \frac{\partial}{\partial Y_{MBS}}\left(\frac{\partial P_{MBS}}{\partial Y_{MBS}}\right) \times \frac{\partial P_{Opt}}{\partial P_{MBS}} + \frac{\partial}{\partial Y_{MBS}}\left(\frac{\partial P_{Opt}}{\partial P_{MBS}}\right) \times \frac{\partial P_{MBS}}{\partial Y_{MBS}}$$

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For ease of calculation, this is rearranged to obtain

$$\frac{\partial^2 P_{Opt}}{\partial Y_{MBS}^2} = \frac{\partial^2 P_{MBS}}{\partial Y_{MBS}^2}\frac{\partial P_{Opt}}{\partial P_{MBS}} + \frac{\partial^2 P_{Opt}}{\partial P_{MBS}^2}\left(\frac{\partial P_{MBS}}{\partial Y_{MBS}}\right)^2$$

or

$$C_{Opt} = C_{MBS}\Delta_{Opt} + \Gamma_{Opt}D_{MBS}^2 \quad (5)$$

Equation (5) suggests some interesting intuition. The first and second terms quantify two distinct effects that drive option convexity.

The first term is the contribution from the change in the underlying mortgage duration due to a change in mortgage yield (holding the option delta fixed). This term is negative for a call option on a mortgage, since the mortgage itself is negatively convex and a call option's delta is always positive. The first term is positive for a put option since a put option's delta is always negative.

The second term is the contribution from the change in the option delta due to a change in mortgage yield (holding the mortgage duration fixed). This term is always positive since the option gamma and the squared mortgage duration are both positive.

Finally, note that if the underlying has zero convexity, the first term in Equation (5) drops out, and the option will always have positive convexity.

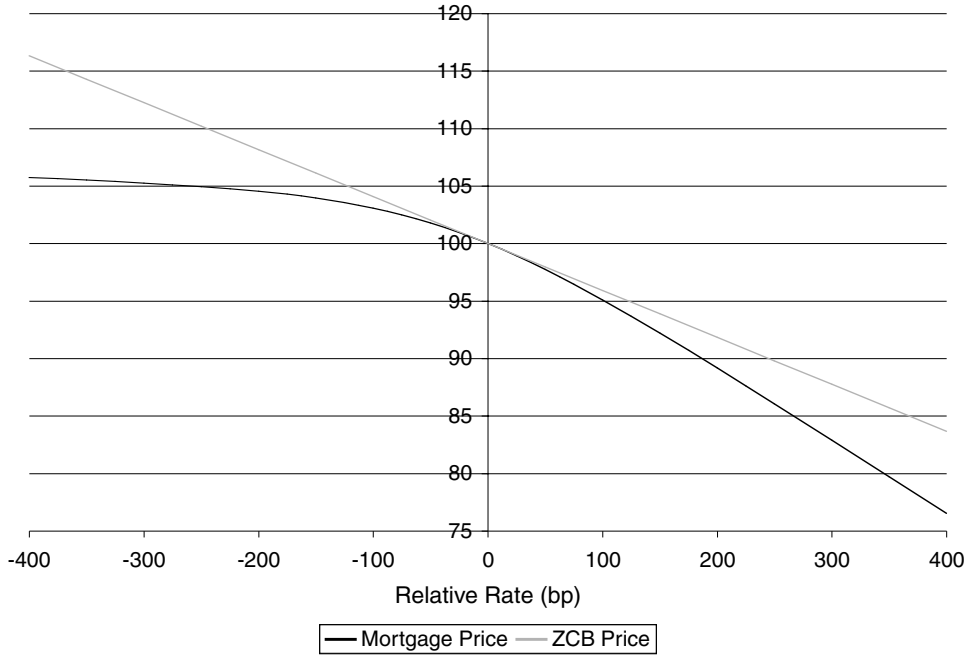
This discussion suggests one of my main points. The convexity of a mortgage call option is driven by two competing effects: 1) the negative convexity of the underlying mortgage, and 2) the naturally positive gamma of a long option position.

III. ESTIMATING OPTION RISK SENSITIVITIES

To demonstrate the impact of the negative convexity of the underlying mortgage on option characteristics, I quantify Equation (5) for a representative mortgage call and put option and for benchmark call and put options on a zero-convexity bond (ZCB). For ease of comparison, I set the duration of the ZCB equal to the duration of the mortgage at par, which also corresponds to the base case scenario.

Exhibit 2 graphs the underlying price of the mortgage and the ZCB versus relative rate. Note that the price of the mortgage is less than or equal to the price of the ZCB for all rate scenarios, as the ZCB and the mortgage are both priced at par in the base case, and the mortgage is negatively convex at all rate levels.

EXHIBIT 2
Mortgage and ZCB Price-Rate Curves



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Mortgage Call Option Delta and Gamma

Exhibits 3, 4, and 5 show prices and risk sensitivities for the mortgage call option and ZCB call option graphed against relative rate. Recall that a relative rate of zero represents the base case in which the underlying is priced at par and the option is struck at the money.

To obtain the curves, the two options are repriced, and risk sensitivities are recalculated at 5 bp rate intervals using the model developed previously. The option strike prices and the rest of the structure remain unchanged — the rate change just sets the initial price of the underlying. A downward shift in rate moves the options in the money, and an upward shift moves them out of the money.

Exhibit 3 shows the model price of the mortgage call option and ZCB option graphed against relative rate. As the relative falls below zero, the options move into the money, and the mortgage option underperforms the ZCB option. This is primarily because the duration of the underlying mortgage is shortening as rates drop and expected prepayments rise. The price of the ZCB option is higher than the price of the mortgage option for all rates. This is to be expected, since both the options are struck at par, and the ZCB is worth less than the mortgage in all rate scenarios (as shown in Exhibit 2).

The first graph in Exhibit 4 shows the deltas of the two call options. The delta of the ZCB option is ½ at a relative rate of zero (i.e., when the option is at the money), but the delta of the mortgage call option is only 0.44. As one would expect, the deltas both converge to a value of 1.0 as the rate drops, and to a value of zero as the rate rises.

In the second graph of Exhibit 4, the mortgage option gamma peaks in the money at –37 bp, while the ZCB option gamma peaks at the money, as is typically expected. This behavior is related to the observation that, at-the-money, the delta of the mortgage option is lower than that of the ZCB option, but that both converge to the same value for extreme rate shifts in either direction. As the options move more in the money and both deltas increase, the delta of the mortgage option catches up with the delta of the ZCB option, resulting in a gamma peak for the mortgage option that is in the money.

The delta and gamma of the ZCB option are consistent with the typical expectations of market participants. The at-the-money delta is ½, and gamma peaks at the money and is symmetric about that point. Why, then, is the delta of the mortgage option not ½, when the option is at the money?

This is the delta of the option and not the duration. Thus, first-order effects of underlying duration differences, as shown in Equation (4), do not explain the difference in deltas between the two options. Further, the

EXHIBIT 3
Mortgage Call and ZCB Call Price

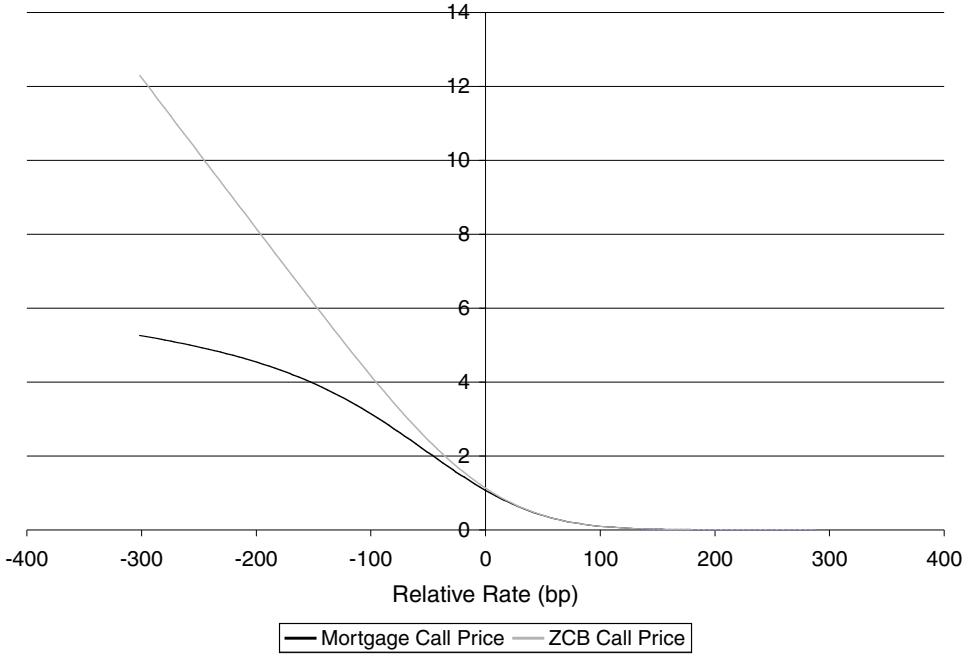

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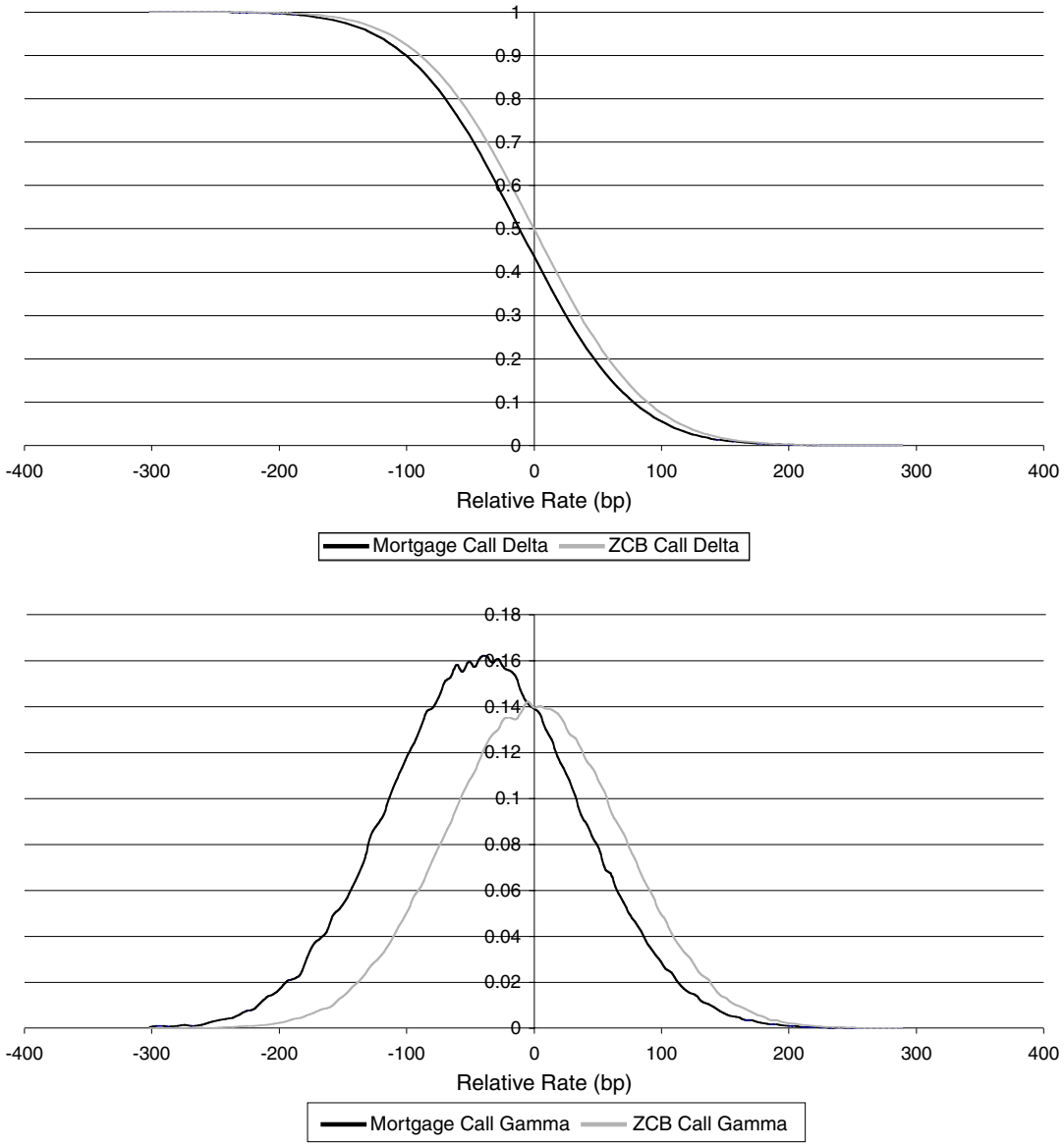
EXHIBIT 4
Mortgage Call and ZCB Call Delta and Gamma



ZCB and the mortgage both have the same duration at a relative rate of zero.

To explain, it is useful to review why the delta of the ZCB option is ½ when the option is at the money. The ZCB price at expiration is a linear function of the normally distributed rate at expiration and is therefore normally distributed itself. An option delta measures the sensitivity of the option price to a price change in the underlying. For the ZCB, changing the underlying price amounts to shifting the entire distribution, while retaining its normality and its original standard deviation (i.e., it retains its shape). Consequently, for purposes of determining delta, shifting the distribution is equivalent to shifting the strike price in the opposite direction. The appendix shows that the delta in this case is 1 – CDF(k), where CDF is the cumulative density function of the ZCB price at option expiration, and k is the strike price. When the option is struck at the money, the mean of the price distribution at expiration is also equal to the median (the distribution being normal) and 1 – CDF(k) = ½.

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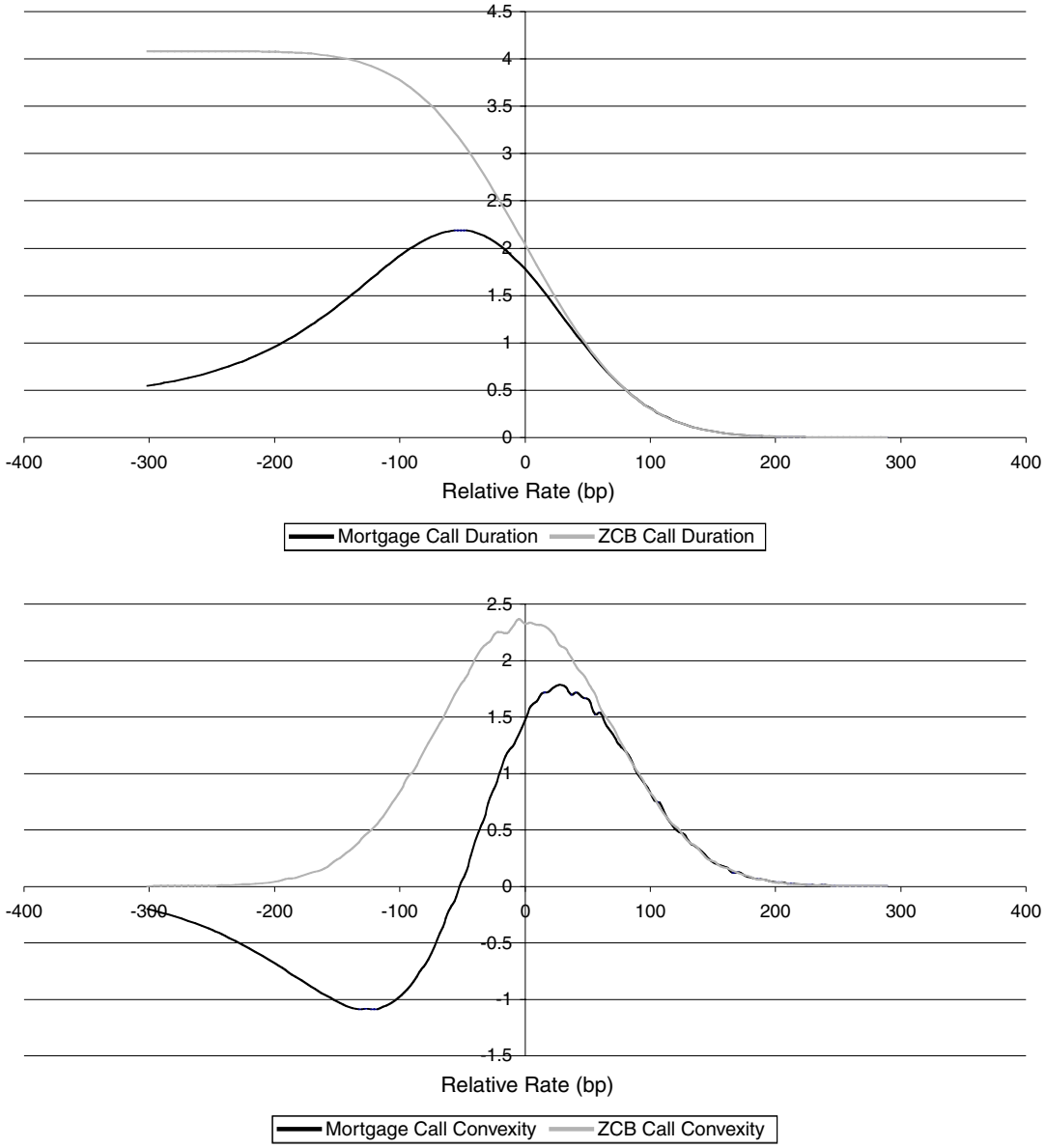
EXHIBIT 5
Mortgage Call and ZCB Call Duration and Convexity



If the delta of the *mortgage* option were in fact equal to 1 – CDF(k), the at-the-money delta would be greater than ½ because the mortgage price distribution is negatively skewed. Since the option is struck at the money forward, the strike price is the mean of the mortgage price distribution at expiration. Due to the negative skewness, the mean will lie below the median and 1 – CDF(k) will be more than 0.5. As discussed, however, the delta of the at-the-money mortgage call is 0.44, which is less than 0.5, not more. This is because the delta of a mortgage call option is *not* equal to 1 – CDF(k).

The result in the appendix that delta is equal to 1 – CDF(k) is based on the assertion that shifting the ZCB price distribution has the same effect on the option price as shifting the strike price the same amount in the opposite direction. This redefinition of delta works because the ZCB price distribution is normal and maintains its shape as it is shifted.

This is not the case for the skewed distribution of the mortgage price. As the normally distributed rate drops, the dispersion in the mortgage price distribution tightens as the duration of the mortgage shortens. The distribution

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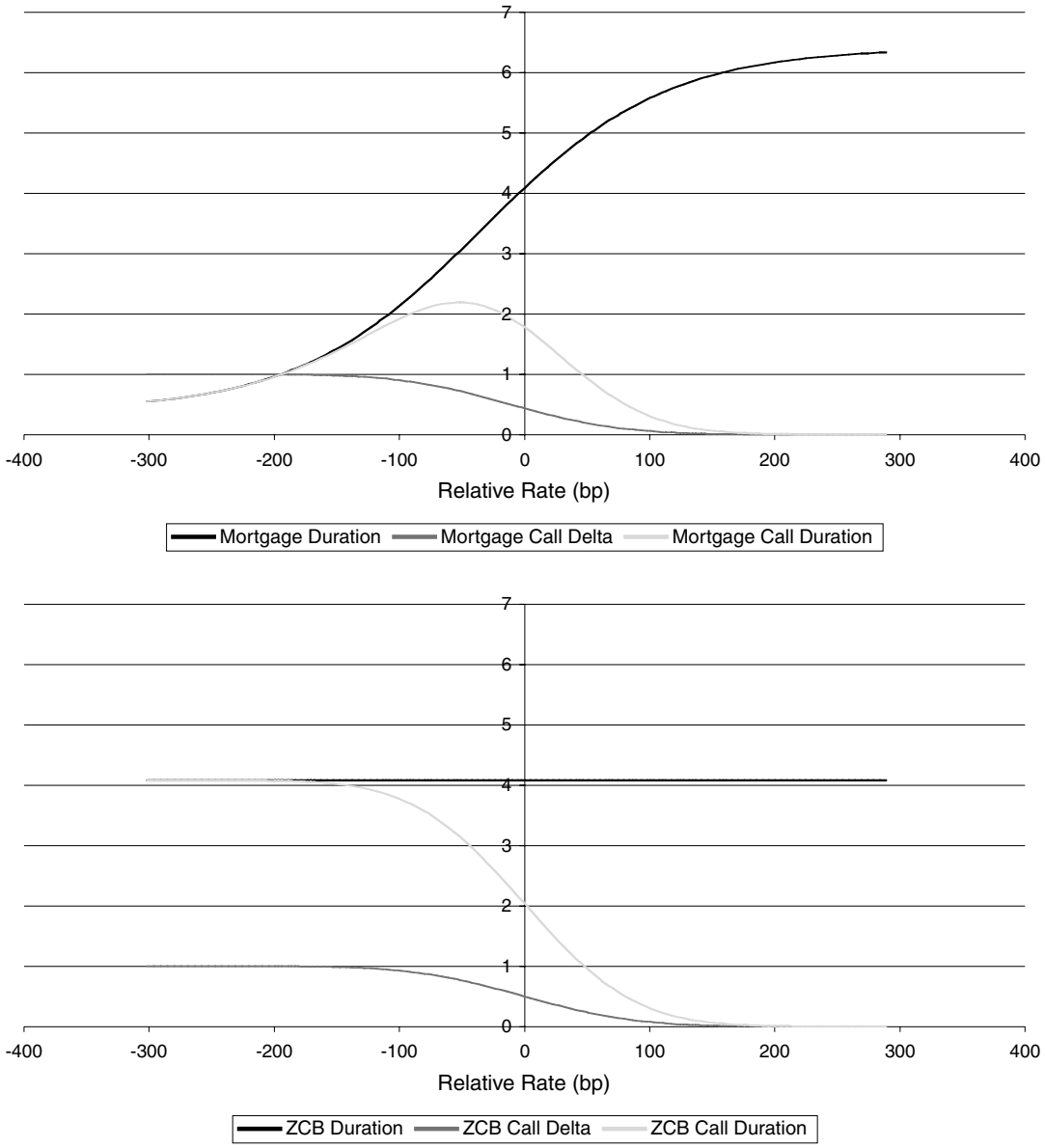
EXHIBIT 6

Call Option Duration Decomposition—Mortgage and ZCB



of the future price of a pass-through priced at $95 will exhibit much more dispersion than the distribution of a premium priced at $105. Therefore, the mortgage price distribution changes shape as it shifts, implying that shifting the mortgage price distribution is not equivalent to shifting the strike price.

This reshaping effect overwhelms the effect from the mean-median dichotomy, and the delta of the mortgage call option is always less than the delta of the ZCB call option.

Mortgage Call Option Duration and Convexity

Expressing risk sensitivities with respect to rate changes rather than price changes allows better comparability across assets. Accordingly, option duration and convexity are of more interest than delta and gamma in evaluation of the contribution of an option position to the overall risk of a fixed-income portfolio. As Equation (5) shows, the option delta and gamma interact with the duration and convexity of the underlying security to produce

option duration and convexity. Some additional analysis will quantify this interaction and provide more intuition about the determinants of option duration and convexity.

The first graph in Exhibit 5 shows the duration of the mortgage and ZCB call option as a function of rate level. The duration is simply the delta of the option multiplied by the duration of the underlying as expressed in Equation (4). The duration peaks because of the countervailing effects of option delta and mortgage duration.

The first graph in Exhibit 6 shows the mortgage duration, the mortgage call delta, and the resulting option duration in the same graph. The mortgage duration shortens as rates rally and prepayments increase, but the option delta lengthens as the option moves more in the money. These opposing effects cause the mortgage call option duration to peak in the money at –50 bp.

The second graph in Exhibit 6 shows the same decomposition for the ZCB option. The ZCB has a constant duration of 4.08, so the ZCB option duration is simply the option delta multiplied by 4.08. It does not reach a maximum and declines monotonically with relative rate, mirroring the delta.

As I have noted, the convexity of the mortgage call option may be either positive or negative, depending on whether the negative convexity of the underlying mortgage offsets the positive option gamma. Equation (5) decomposes the components of the mortgage option convexity (and is repeated here for convenience):

$$C_{Opt} = C_{MBS}\Delta_{Opt} + \Gamma_{Opt}D^2_{MBS} \tag{5}$$

The first term represents the contribution to option convexity due to the *convexity of the underlying mortgage,* and the second term represents the contribution to option convexity due to the *option gamma*. For a mortgage call option, the first term will be negative and the second term positive. The magnitude of each term will determine the overall convexity of the mortgage option. For the ZCB option, the first term is always zero, and the second term will be positive. Therefore, the convexity of the ZCB option is expected to be always positive.

The second graph in Exhibit 5 shows the convexity of the mortgage option and the ZCB option. The convexity of the mortgage option is positive above –50 bp and negative below. That is, for a moderately to well in-the-money mortgage call option, the option convexity is negative, but is positive everywhere else. This is just as Equation (5) predicts: The convexity can be of either sign for the mortgage option. The ZCB option convexity, on the other hand, is always positive as expected.

We gain more insight by graphing the components of Equation (5) separately. The graph in Exhibit 7 shows the first and second terms of Equation (5) as a function of relative rate, and their sum, mortgage option convexity. The first graph in Exhibit 8 shows the two

EXHIBIT 7
Mortgage Call Option Convexity Decomposition

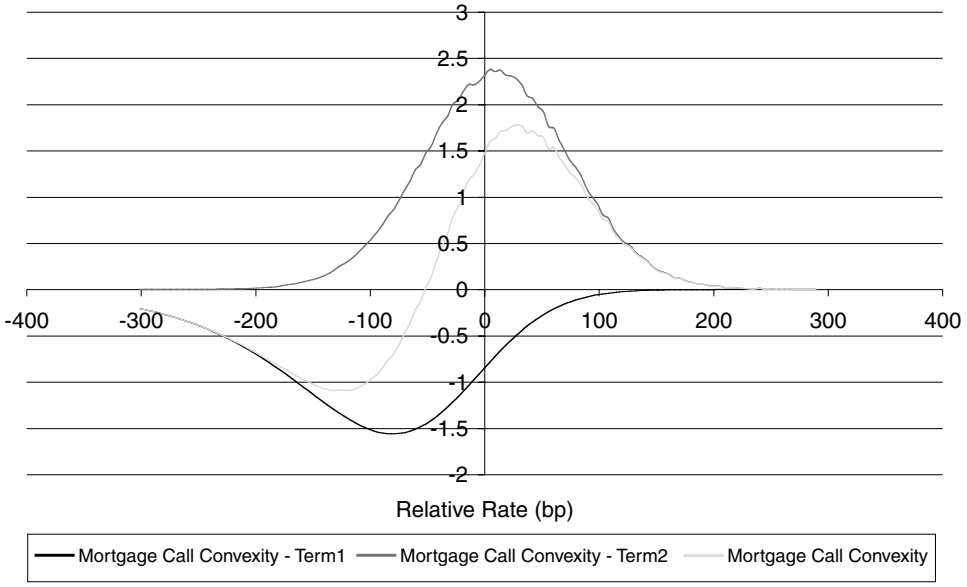


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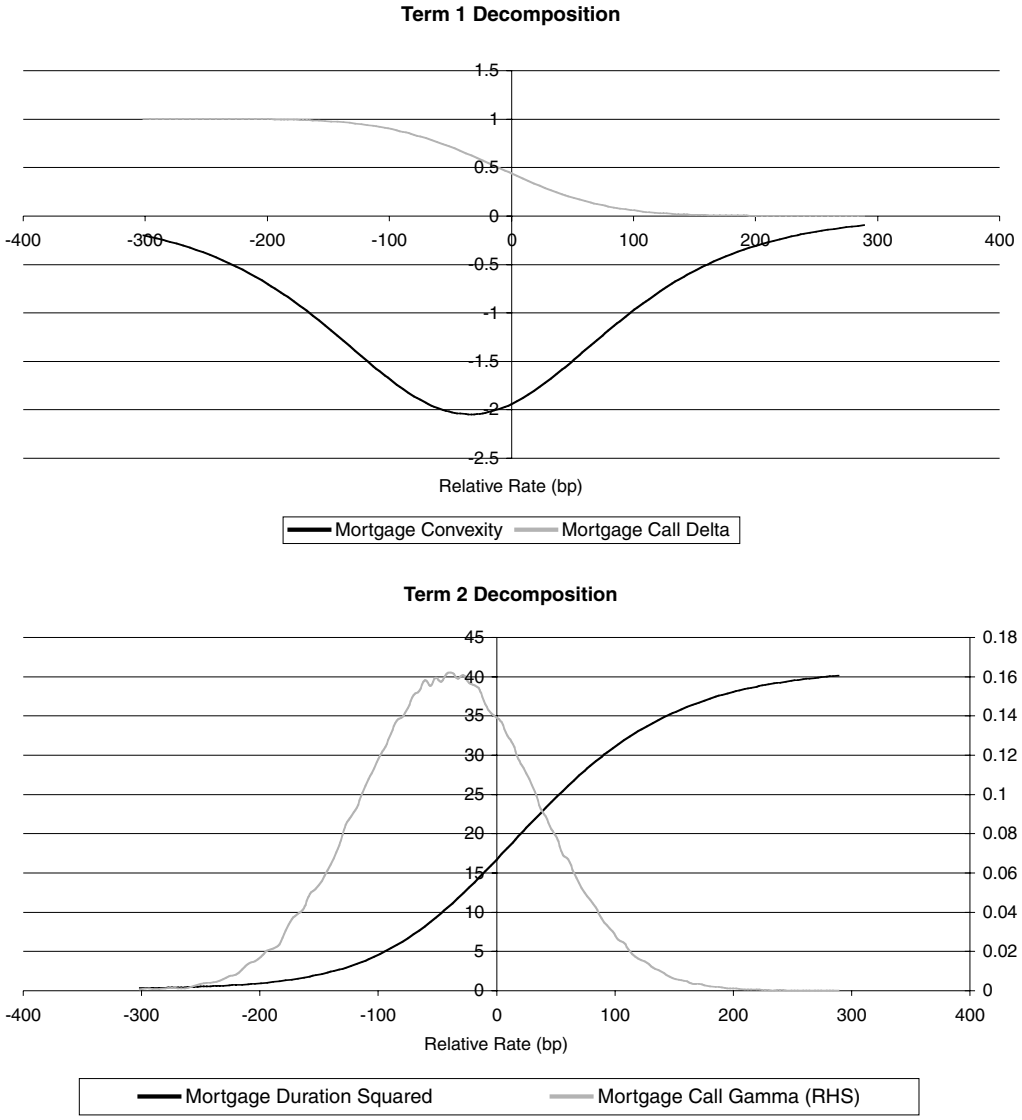
EXHIBIT 8
Mortgage Call Option Convexity Decomposition—Detailed



factors that constitute the first term, mortgage convexity and mortgage call delta. The second graph in Exhibit 8 shows the two factors that constitute the second term, mortgage duration squared and mortgage call gamma.

The first graph in Exhibit 8 indicates that mortgage convexity is symmetric around a –30 bp relative rate. To obtain term 1, this convexity is scaled up or down by the option delta. This results in an overall contribution to call option convexity that is always negative, but more negative for lower rates than for higher rates. Intuitively, the negative convexity of the mortgage translates into negative option convexity to a greater extent when the option value is more sensitive to changes in the value of the underlying asset (i.e., for higher deltas). At –80 bp, the delta is high enough and the mortgage convexity is negative enough to maximize the negative impact of term 1 on overall option convexity, as shown in Exhibit 7.

Term 2 is always positive and represents the impact of the option gamma on option convexity. As shown in the second graph in Exhibit 8, the option gamma itself is roughly symmetric around –37 bp. To obtain term 2, however, the gamma is scaled up or down by the square

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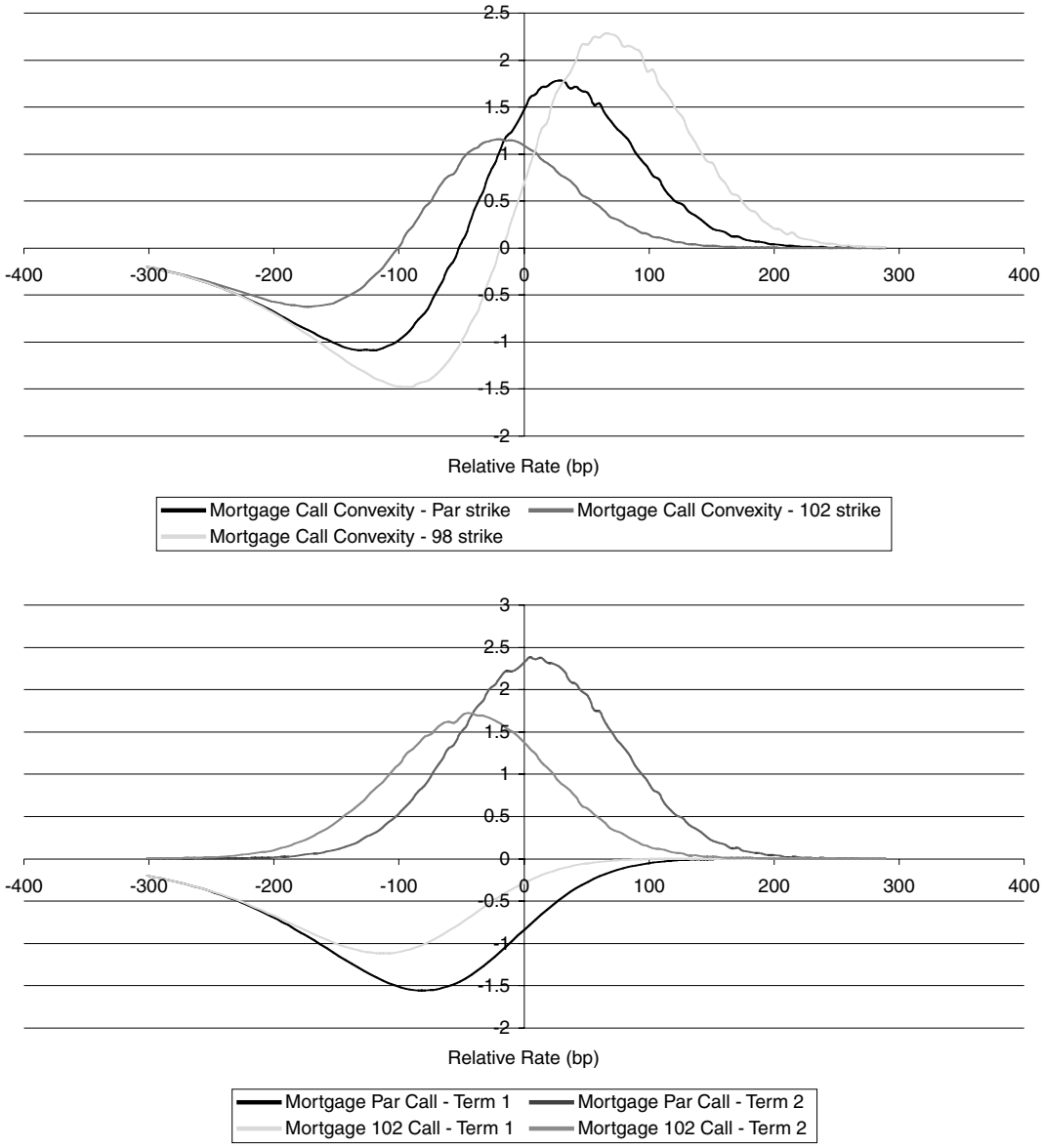
EXHIBIT 9
Effect of Varying Strike Price



of the mortgage duration. Thus, this term has an asymmetric (positive) impact on option convexity in up-rate scenarios as the mortgage lengthens.

Exhibit 7 shows that when term 1 and term 2 are summed, the result is an option convexity that tends to be negative for moderate to extreme in-the-money scenarios. Intuitively, for well in-the-money scenarios, the option becomes more like a position in the underlying mortgage, which has significant negative convexity. Thus, the option itself is negatively convex.

As rates rise, the option exhibits significant positive gamma, and the duration of the underlying mortgage becomes long enough for some of this gamma to translate into significant positive option convexity. As this effect begins to dominate, the option becomes positively convex. As rates rise further, the option goes far out of the money, and underlying rate changes have a negligible effect on the option value as the likelihood of the option expiring in the money becomes very low. At this point, the option becomes insensitive to rate moves, and the option convexity goes to zero.

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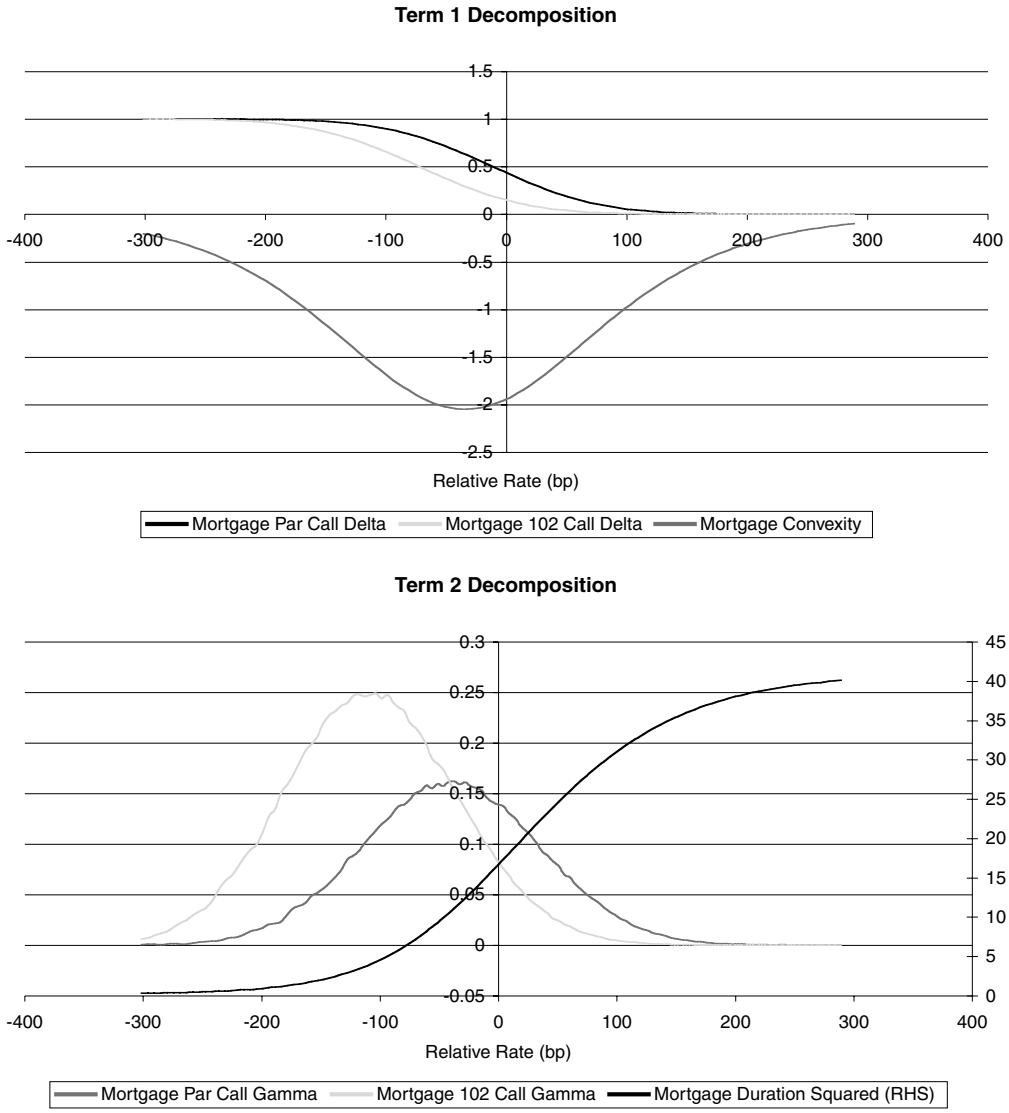
EXHIBIT 10
Effect of Varying Strike Price—Detailed Decomposition



The overall result is a peak in positive convexity when the option is out of the money and a peak in negative convexity when the option is in the money. The option convexity goes to zero for extreme rate moves of either sign.

Decoupling the Prepayment Option from the Mortgage Option

The mortgage call option we have been analyzing is struck at the money forward in the base case (i.e., at a relative rate of zero), resulting in a strike price of par. The *prepayment* option embedded in the underlying mortgage is, in a manner of speaking, slightly out of the money in the base case. Prepayments begin accelerating when rates drop by about 35 bp from the base case.

In any event, the relationship between the two options is fixed. When the rate drops by 200 bp, for example, the prepayment option and the mortgage option will both be well in the money. Relaxing this constraint provides some additional insights.

The first graph in Exhibit 9 shows convexities for the option struck at par, \$102 and \$98. The underlying mortgage is the same for all three options; it is always priced

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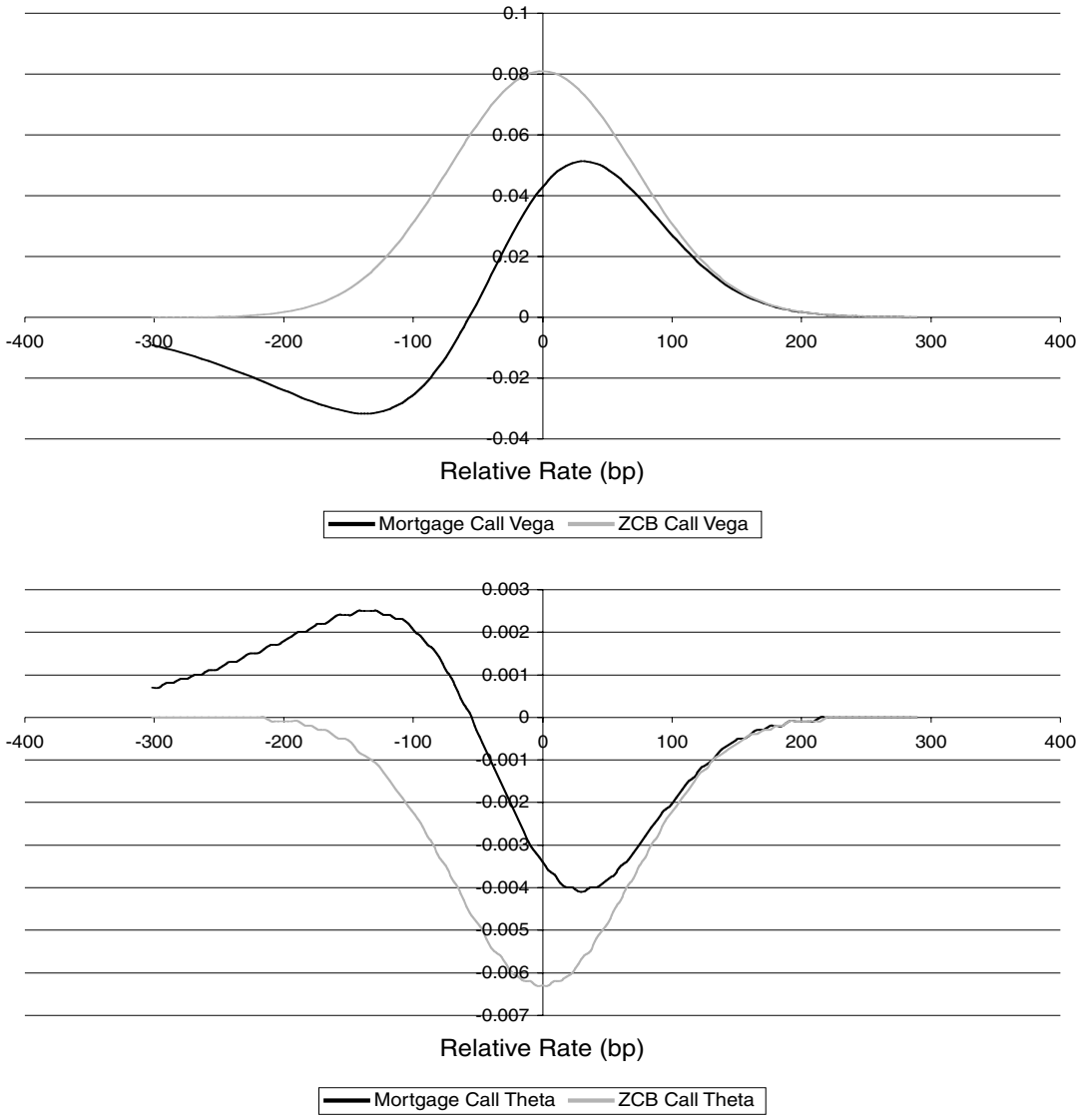
EXHIBIT 11
Call Option Vega and Theta



at par when the relative rate is zero. Effectively, this analysis allows the strike of the mortgage option to change without changing the strike of the prepayment option.

Note that, as the option strike rises, the convexity pattern becomes more attenuated. That is, the maximum and minimum convexities are closer to zero. This pattern makes intuitive sense—in the extreme, an option that is far out of the money will have little sensitivity to yield changes in the underlying, and hence little convexity.

More insight can again be gained by decomposing the convexity pattern using Equation (5). For a given relative rate, changing the strike price does not affect the mortgage duration or the mortgage convexity, as only the strike price of the option has changed. The option gamma and delta, however, do change.

The second graph in Exhibit 9 shows the first and the second term of Equation (5) for the par strike and the 102 strike mortgage calls. Term 1, which captures the impact of mortgage convexity on option convexity, is everywhere smaller in absolute value for the 102 strike option than the par strike option.

The reason can be seen by decomposing term 1 into its two components, as shown in the first graph of Exhibit 10. For each relative rate, the delta of the more out-of-the

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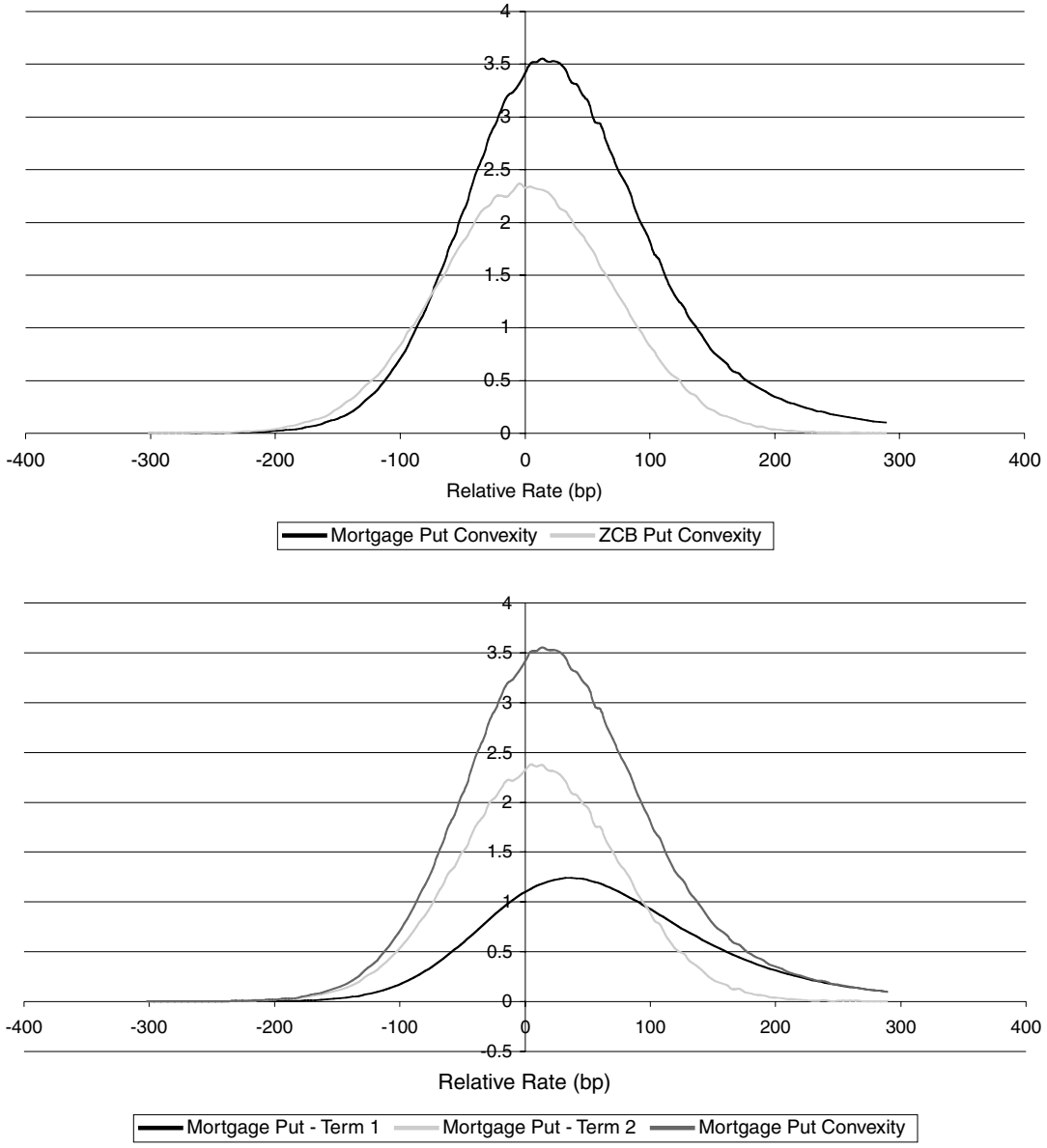
EXHIBIT 12
Mortgage Put Option Convexity



money 102 call is smaller than the par call delta. Thus, the mortgage convexity is not translated into option convexity as strongly for the 102 call as for the par call.

The second graph in Exhibit 9 shows that term 2, which captures the impact of option gamma on option convexity, peaks lower for the 102 call than for the par call, with the peak shifted to a lower rate level. The second graph in Exhibit 10 shows that term 2 is impacted by two offsetting effects as the strike of the option increases from par to 102. The option gamma term peaks higher and at a lower rate for the 102 strike than for the par strike. The mortgage duration (and hence the square of the mortgage duration) does not vary with strike. Yet, because the peak of the 102 strike option gamma occurs at a lower rate than the peak of the par strike option, the squared duration term scales down the gamma peak for the 102 strike option. Thus, even though the 102 strike option has a higher peak gamma, much less of it is transferred to option convexity.

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Mortgage Call Option Vega and Theta

A key issue has been distinguishing the effect of underlying yield changes from the effect of underlying price changes. This is not an issue for vega, the sensitivity of the option price to a change in volatility, and theta, sensitivity to the passage of time.

Vega measures the impact of a 10 bp change in annualized rate volatility [σ in Equation (3)] on the option price and is shown in the first graph of Exhibit 11. Comparing this graph to the second graph in Exhibit 5 shows that vega as a function of rate closely mirrors convexity as a function of rate, as expected. Thus, as is the case with convexity, the vega of a mortgage call option can be negative as well as positive, and for similar reasons. When the negative mortgage convexity dominates the option convexity, the vega is negative. This occurs in low rate scenarios. When the positive option gamma dominates the option convexity, as happens in high rate scenarios, the vega is positive.

Vega as discussed here is a partial vega, since it does not take into account the impact of a volatility change on the price of the underlying mortgage. Since the mortgage is short a prepayment option, an increase in volatility will result in a reduction in the mortgage price and a reduction in the value of the call (and an increase in the value of the put).

Theta, the rate of time decay of the option, is measured as the change in option price per day. Intuitively, theta can be viewed as yield compensation (penalty) for negative (positive) convexity. As the second graph of Exhibit 11 shows, theta is positive (negative) when the option is negatively (positively) convex.

Mortgage Put Option Risk Sensitivities

So far we have focused almost exclusively on call options. Prices of mortgage put options behave quite differently from prices of mortgage call options. The most striking difference is that mortgage put option convexities are always positive. The option delta is always negative, and the other sensitivities in Equation (5) maintain the same signs as for their call option counterparts. Thus, both the first and second term will always be positive for the put option; as a result, the put option convexity will always be positive.

The first graph in Exhibit 12 shows that the convexity of the mortgage put option is even more positive than the convexity of the zero-convexity bond option in most scenarios. Again, Equation (5) helps explain. For a ZCB option, the first term is always zero, but it is positive for a mortgage put. There are some differences between the ZCB and mortgage option in the second term. However, the main reason the mortgage put is more positively convex than the ZCB put lies in the first term which captures the transmission of the mortgage convexity through the delta of the option.

The second graph in Exhibit 12 breaks out the first and second term for the mortgage put.

IV. CONCLUSION

The main finding of this research is that the convexity of an option is determined by two mechanisms: the impact of the convexity of the underlying security, and the impact of the positive option gamma. The positive option gamma may be viewed as the source of an option's natural positive convexity. For an option on a zero-convexity underlying, option gamma is the sole driver of option convexity.

A mortgage call option is a long position in a negatively convex underlying. The negative convexity of the underlying and the gamma-driven positive convexity of the option act in opposing directions. Depending on the particular option specification and the interest rate environment, either effect can dominate, and the call option can be either positively or negatively convex.

A mortgage put option, on the other hand, is a *short* position in a negatively convex underlying. In this case, the convexity of the underlying adds to the natural positive convexity of the option. This results in a mortgage put option convexity that is always positive.

Armed with the insights provided here, market participants who have been reluctant to use mortgage options are better informed. The findings are especially relevant for those who have natural uses for these options but are not sure about their impact on the overall risk of their portfolio.

There are several possible avenues for further research. First, it would be informative to know whether mortgage options really trade as implied by the model laid out here. Empirical analysis would be useful. Second, there is more work to be done in modeling mortgage options. It would be interesting to see the prices and risk sensitivities from a mortgage option pricing model that uses an OAS simulation framework.

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APPENDIX

ZCB Delta

This appendix shows that the delta of the ZCB option is 1 – CDF(k), where CDF is the cumulative distribution function of the price of the underlying at option expiration. The discussion in Section III explains why, for an option on an underlying security with a normally distributed price, delta can be expressed as the negative of the derivative of the option price with respect to the strike price.

For a call option, this is expressed as:

$$\Delta_{Opt} = -\frac{\partial}{\partial k}\int_k^\infty (s-k)\,\mathrm{f}(s)ds$$

where k is the strike price, s is a dummy variable, and f( ) is the probability density function of the terminal price of the underlying at option expiration. Note that the equation assumes a risk-free rate of zero. By Leibniz's rule for the differentiation of integrals, the solution to this is:

$$\Delta_{Opt} = \int_k^\infty \mathrm{f}(s)ds$$

which is equal to 1 – CDF(k).

QED.

ENDNOTES

The author thanks Daniel R. Adler, Robert W. Aufdenspring, Michael J. Brennan, Sue Y. Chow, Timothy J. Cunneen, Daniel C. Dektar, Stanley J. Kon, Eduardo S. Schwartz, and Gagan Singh for helpful discussion and comments.

[1]Delta and gamma are the first and second derivative of an option's price with respect to the price of the underlying. Duration and convexity are the first and second derivatives of a security's price with respect to a yield or interest rate. It is appropriate to talk about the duration or convexity of both the underlying security and of the option itself. Delta and gamma are always associated with an option.

[2]This happens when the prevailing mortgage rate is far enough below the average coupon in the pass-through pool to overcome fixed refinancing costs.

[3]The s-curve applies to TBA pass-throughs which tend to be relatively new. Seasoned pass-through premiums would be expected to exhibit longer durations as a result of burnout. The *dollar duration* is the instantaneous dollar change in the pass-through price per 100 bp change in the rate.

[4]The 30-year par coupon is an estimate of the yield of a pass-through. It is the linearly interpolated coupon, given the two coupons on either side of par, that would result in a price of par.

[5]See Breeden [1994] for the initial presentation of this normalization.

[6]This particular method of calibration is useful for evaluating the relative mispricing of mortgage options to each other. It also suffices for risk management applications.

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