Frame-Stewart

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1 Defining the Solutions

- Let $H_K(n)$ be the minimal number of moves necessary to move n Hanoi discs with k pegs.
- sequence of moves S is a minimal sequence if it has $H_K(n)$ moves
- We designate the portion of sequence of S through the triple (n, ∞, ∞) as the demolishing phase, and the portion following the triple through the end as the reconstruction phase. In a minimal sequence we must have the triple (n, ∞, ∞) exactly once.
- If S is a minimal solution, its demolishing phase must have length $\frac{H_K(n)+1}{2}$ and its reconstruction phase must have length $\frac{H_K(n)-1}{2}$. This is because a demolishing phase can be used to construct a reconstruction phase, and a reconstruction phase can be used to construct a demolishing phase. For example, given a demolishing phase where the lth move is (n, ∞, ∞), for every $u < l$ let (r, s, y) be the $l - u$th move, we set the $l + u$th move to be (r, y, s). Such a sequence is called symmetric
- Since a minimal demolishing phase can generate a minimal sequence, we may focus on finding minimal demolishing phases.

2 Stacks

2.1 Number of Stacks

- Suppose S is a minimal demolishing sequence of n discs with k pegs.
- At the end of S, that is when we make the move (n, ∞, ∞) we may observe the stacks of discs.
- We let $j_1 < j_2 < ..., < j_{k-3} < n - 1 < n$ denote the stack bottoms.
- As long as $n > k - 1$ the only empty peg will be the source, as disc n has just moved off of it. N.B. When $n <= k - 1$ there is a

- minimal demolishing phase of length n that is easy to calculate.
- I will prove that $n > k-1$ implies $k-1$ stacks at the end of a minimal demolishing sequence S:
- Suppose S is a minimal demolishing sequence.
- If some there is some disc d that moves to the empty stack via $(d, d+1, \infty)$ when it is cleared, it must also leave the stack,
- otherwise it wouldn't be empty. So we must have $(d, \inf, x)$ for some x, possibly x a stack bottom.
- If we remove $(d, \inf, x)$ from S to form S', we see S' is shorter than S, hence S can't be minimal. The contradiction implies we have $k-1$ stacks.
- We also consider the case where a peg is never occupied at all.
- In this case we can derive a minimal demolishing sequence shorter than S by taking d some non-stack bottom, occurring in triples
- $(d, d+1, \inf)$,...,$(d, \inf, x)$ where there are at least two triples since d isn't a stack-bottom.
- Removing all of the above triples from S and instead sending d to the empty stack as soon as it's freed via $(d, d+1, \inf)$ we have constructed a minimal demolishing sequence shorter than S. The contradiction implies we must certainly have $k-1$ stacks.

2.1.1 Distribution of Stacks

- When n is freed, the $k-1$ stacks are distributed as follows:
  $j_1, j_1 - 1, ...2, 1$
  $j_2, j_2 - 1, ...j_1 + 2, j_1 + 1$
  $j_3, j_3 - 1, ...j_2 + 2, j_2 + 1$
  ....
  $j_{k-3}, j_{k-3} - 1, ...j_{k-2} + 1$
  $n-1, n-2, ...j_{k-3} + 1$
  n
- To see that this is true, first recall that a disc can only be stacked on a larger disk, so we couldn't have j_{k-3} on j_1's stack, for example.
- Now suppose S is a minimal demolishing sequence and we have a disk $j_{i-1} \le d \le j_i$, the smallest element on j_r's stack, where $r > i$
- We know d's first move is $(d, d+1, \infty)$
- We could have d moving before it moves to j_r's stack, e.g. $(d, \infty, x), (d, x, x'), (d, x', x'')$ etc.

- Eventually we have $(d, y, j_{r-1} + 1)$ for some y
- We may construct a minimal demolishing sequence S' by sending d to its proper stack as soon as $d + 1$ gets pushed onto that stack, which will free up the later moves to use more pegs and shorten the sequence.

3 Subsequences

- For a minimal demolishing sequence $S = (1, 2, \infty), (2, 3, \infty), ...(n, \infty, \infty)$, we must have $|S| = \frac{H_K(n)+1}{2}$
- First we move j_1 disks to a non-target peg using k pegs.
- Let $S_1 = S_1^D, S_1^R = (1, 2, \infty)(2, 3, \infty).(j_1, j_1 + 1, \infty)(j_1 - 1, \text{inf}, j_1)...(1, \infty, 2)$
- Since S is a minimal demolishing sequence, S_1 must be a minimal sequence of j_1 discs using k pegs–if S_1 wasn't minimal, we could find S_1' shorter than S_1 and use it to contradict minimality of S. Let S_1^* be the symmetric reflection of
- S_1 with similar notation for the following subproblems
- $|S_1| = H_K(j_1) = |S_1^D| + |S_1^R| = \frac{H_K(j_1)+1}{2} + \frac{H_K(j_i)-1}{2}$
- So S_1^D must be a minimal demolishing sequence of j_1 disks using k pegs. If $|S_1|$ isn't symmetric, we can set S_1^R as the symmetric reflection of s_1^D to generate a minimal symmetric sequence. We see that existence of a minimal demolishing sequence ensures existence of a minimal symmetric sequence.
- Also note that when we generate the entire n disk demolishing and reconstructing solution we will calculate the reconstruction phase by symmetry. In the reconstruction phase, moving the j_1 disks to the target peg using k pegs is the last step.
- Next we move $j_2 - j_1$ disks using $k - 1$ pegs
- We can define $S_2 = S_2^D, S_2^R$. Analogous to the above subproblem, S_2^D must be a minimal demolishing sequence, we use symmetry to get the reconstruction phase, and let S_2^* denote the symmetric reflection of S_2
- Moving $(n - 1, n - 2, ...j_{k-3} + 1)$ using 3 pegs generates S_{k-2}
- Moving the bottom disk with $k = 2$ gives S_{k-1}
- We can see $(S, S*) = (S_1, S_2, .., S_{k-2}, S_{k-1}, S_{k-2}, S_{k-3}, S_{k-2}*, ..., S_2*, S_1*)$

$$\begin{aligned}|(S, S*)| = H_K(n) = 1 + 2H_K(j_1) + 2 * H_{K-1}(j_2 - j_1) +& \\ 2 * H_{K-2}(j_3 - j_2) + \ldots +& \\ 2 * H_4(j_{k-3} - j_{k-2}) +& \\ 2 * H_3(n - 1 - j_{k-3})&\end{aligned}$$

4 Complexity

- Next define sufficient symmetry conditions, and demonstrate why the example in Hinz et al.is degenerate
- Derive the optimal bound