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ON THE MAXIMUM DRAWDOWN OF A BROWNIAN MOTION

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Abstract

The maximum drawdown at time T of a random process on $[0, T]$ can be defined informally as the largest drop from a peak to a trough. In this paper, we investigate the behaviour of this statistic for a Brownian motion with drift. In particular, we give an infinite series representation of its distribution and consider its expected value. When the drift is zero, we give an analytic expression for the expected value, and for nonzero drift, we give an infinite series representation. For all cases, we compute the limiting ($T \to \infty$) behaviour, which can be logarithmic (for positive drift), square root (for zero drift) or linear (for negative drift).

Keywords: Random walk; asymptotic distribution; expected maximum drawdown

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1. Introduction

The maximum drawdown is commonly used in finance as a measure of risk for a stock that follows a particular random process. Here we consider the maximum drawdown of a Brownian motion. Let $W(t),\ 0 \le t \le T$, be a standard Wiener process and let $X(t)$ be the Brownian motion given by $X(t) = \sigma W(t) + \mu t$, where $\mu \in \mathbb{R}$ is the drift and $\sigma \ge 0$ is the diffusion parameter. The high H, the low L and the range R of X are defined by

$$H = \sup_{t\in[0,T]} X(t), \qquad L = \inf_{t\in[0,T]} X(t), \qquad R = H - L.$$

The maximum drawdown is defined by

$$\bar{D} = \bar{D}(T; \mu, \sigma) = \sup_{t\in[0,T]} \Big[\sup_{s\in[0,t]} X(s) - X(t) \Big].$$

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Denote the distribution function for $\bar{D}$ by $G_{\bar{D}}(h) = \mathrm{P}[\bar{D} \geq h]$. We will show that

$$G_{\bar{D}}(h) = 2\sigma^4 \sum_{n=1}^{\infty} \frac{\theta_n \sin\theta_n}{\sigma^4\theta_n^2 + \mu^2 h^2 - \sigma^2\mu h} \exp\left\{-\frac{\mu h}{\sigma^2}\right\}\left(1 - \exp\left\{-\frac{\sigma^2\theta_n^2 T}{2h^2}\right\}\exp\left\{-\frac{\mu^2 T}{2\sigma^2}\right\}\right) + M, \tag{1}$$

where, for $n \geq 1$, θ_n is the positive solution of the eigenvalue condition

$$\tan\theta_n = \frac{\sigma^2}{\mu h}\theta_n \tag{2}$$

and M is given by

$$M = \begin{cases} 0, & \mu < \dfrac{\sigma^2}{h}, \\ \dfrac{3}{\mathrm{e}}\left(1 - \exp\left\{-\dfrac{\mu^2 T}{2\sigma^2}\right\}\right), & \mu = \dfrac{\sigma^2}{h}, \\ \dfrac{2\sigma^4\eta \sinh\eta \exp\{-\mu h/\sigma^2\}}{\sigma^4\eta^2 - \mu^2 h^2 + \sigma^2\mu h}\left(1 - \exp\left\{-\dfrac{\mu^2 T}{2\sigma^2}\right\}\exp\left\{\dfrac{\sigma^2\eta^2 T}{2h^2}\right\}\right), & \mu > \dfrac{\sigma^2}{h}, \end{cases} \tag{3}$$

where η is the unique positive solution of

$$\tanh\eta = \frac{\sigma^2}{\mu h}\eta. \tag{4}$$

Using the identity $\mathrm{E}[\bar{D}] = \int_0^\infty \mathrm{d}h\, G_{\bar{D}}(h)$ and defining $\alpha = \mu\sqrt{T/2\sigma^2}$, we find that

$$\mathrm{E}[\bar{D}] = \frac{2\sigma^2}{\mu} Q_{\bar{D}}(\alpha^2), \tag{5}$$

where

$$Q_{\bar{D}}(x) = \begin{cases} Q_{\mathrm{p}}(x), & \mu > 0, \\ \gamma\sqrt{2x}, & \mu = 0, \\ -Q_{\mathrm{n}}(x), & \mu < 0, \end{cases}$$

with $\gamma = \sqrt{\pi/8} \approx 0.6267$ a constant, and Q_{p} and Q_{n} functions whose exact expressions are given in (12) and (10) respectively. These functions can be evaluated numerically (a comprehensive list of values is given in Table 1 below), and their asymptotic behaviour is given by

$$Q_{\mathrm{p}}(x) \to \begin{cases} \gamma\sqrt{2x}, & x \to 0^+, \\ \frac{1}{4}\log x + 0.490\,88, & x \to \infty, \end{cases}$$

$$Q_{\mathrm{n}}(x) \to \begin{cases} \gamma\sqrt{2x}, & x \to 0^+, \\ x + \frac{1}{2}, & x \to \infty. \end{cases}$$

The asymptotic behaviour is logarithmic for $\mu > 0$, linear for $\mu < 0$ and square root for $\mu = 0$. A similar result in the asymptotic case was obtained by Berger and Whitt [1]. For positive μ,

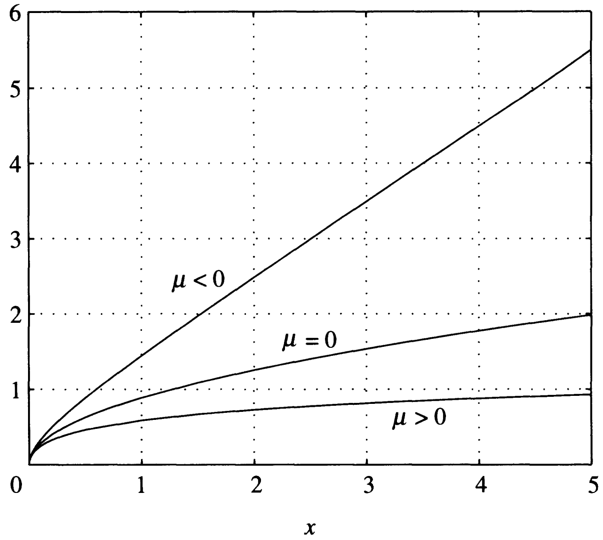

FIGURE 1: Behaviour of $Q_{\bar{D}}(\cdot)$ for positive, negative and zero μ.

using a reflected Brownian motion [8], they considered the asymptotic distribution of queueing processes and obtained a Gumbel distribution, which is consistent with our findings. Douady *et al.* [5] considered the maximum drawdown for a Brownian motion when the drift is zero and obtained similar results to ours in this special case. The behaviour of $Q_{\bar{D}}(\cdot)$ is shown in Figure 1.

2. Derivations

Let $D(t)$, $t \in [0, T]$, be the random process defined by

$$D(t) = \sup_{s \in [0,t]} X(s) - X(t).$$

Then $D(t)$ is the drawdown from the previous maximum value at time t. It is well known that $D(t)$ is a reflected Brownian motion on $[0, T]$,

$$\mathrm{d}D(t) = \begin{cases} -\mathrm{d}X(t), & D(t) > 0, \\ \max\{0, -\mathrm{d}X(t)\}, & D(t) = 0. \end{cases} \tag{6}$$

(If $X(t)$ has drift and diffusion parameters μ_X and σ, then $D(t)$ is a Brownian motion with drift $-\mu_X$ and diffusion parameter σ; $D(0) = 0$ and $D(t)$ has a reflective barrier at 0.) A rigorous justification of (6) can be found in [8]. From $D(t)$, we can get the maximum drawdown, $\bar{D} = \sup_{t \le T} D(t)$. Let $h > 0$ be an absorbing barrier, let τ be the absorption time and let f_τ be the absorbtion time density, i.e. $f_\tau(t \mid h)\mathrm{d}t$ is the probability that $\tau \in [t, t + \mathrm{d}t]$. Let $G_{\bar{D}}(h)$ be the probability that $\bar{D} \ge h$ in the interval $[0, T]$. Then

$$G_{\bar{D}}(h) = \int_0^T \mathrm{d}t\, f_\tau(t \mid h). \tag{7}$$

The absorption time density $f_\tau(t \mid h)$ can be computed from a more general result for a Brownian motion between two partly absorbing barriers given in [4], and is given by

$$
\begin{aligned}
f_\tau(t \mid h) &= \exp\left\{-\frac{\mu^2 t}{2\sigma^2}\right\} \\
&\quad \times \left[\frac{\sigma^2}{h^2}\sum_{n=0}^{\infty}\frac{(\sigma^4\theta_n^2+\mu^2h^2)\theta_n\sin\theta_n}{\sigma^4\theta_n^2+\mu^2h^2-\sigma^2\mu h}\exp\left\{-\frac{\mu h}{\sigma^2}\right\}\exp\left\{-\frac{\sigma^2\theta_n^2 t}{2h^2}\right\}+K\right], \qquad (8)
\end{aligned}
$$

where θ_n is the positive solution to the eigenvalue condition (2), K is given by

$$
K = \begin{cases} 0, & \mu < \dfrac{\sigma^2}{h}, \\[2ex] \dfrac{3\sigma^2}{2eh^2}, & \mu = \dfrac{\sigma^2}{h}, \\[2ex] \dfrac{\sigma^2}{h^2}\dfrac{(\mu^2h^2-\sigma^4\eta^2)\eta\sinh\eta}{\sigma^4\eta^2-\mu^2h^2+\sigma^2\mu h}\exp\left\{-\dfrac{\mu h}{\sigma^2}\right\}\exp\left\{\dfrac{\sigma^2\eta^2 t}{2h^2}\right\}, & \mu > \dfrac{\sigma^2}{h}, \end{cases} \qquad (9)
$$

and η is the unique positive solution to the eigenvalue condition (4). Alternatively, in Section 3, we will obtain the same result by taking the continuous limit of the discrete random walk. Substituting (8) into (7), and after an integration, we arrive at (1). To get the expectation (5), we use the identity $\mathrm{E}[\bar{D}] = \int_0^\infty \mathrm{d}h\, G_{\bar{D}}(h)$, which is valid for positive-valued random variables.

2.1. Case $\mu = 0$

In this case, the eigenvalue condition (2) is solved by $\theta_n = (n - \frac{1}{2})\pi$. Thus,

$$
\begin{aligned}
G_{\bar{D}}(h) &= 2\sum_{n=1}^{\infty}\frac{\sin(n-\frac{1}{2})\pi}{(n-\frac{1}{2})\pi}\left(1-\exp\left\{-\frac{\sigma^2(n-\frac{1}{2})^2\pi^2 T}{2h^2}\right\}\right) \\
&= \frac{2}{\pi}\, g\left(\frac{h}{\pi\sigma\sqrt{T}}\right),
\end{aligned}
$$

where

$$
g(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+\frac{1}{2}}\left(1-\exp\left\{-\frac{(n+\frac{1}{2})^2}{2x^2}\right\}\right).
$$

The expected value of $\bar{D}$ is then given by

$$
\begin{aligned}
\mathrm{E}[\bar{D}] &= \int_0^\infty \mathrm{d}h\, G_{\bar{D}}(h) \\
&= \frac{2}{\pi}\int_0^\infty \mathrm{d}h\, g\left(\frac{h}{\pi\sigma\sqrt{T}}\right) \\
&= 2\gamma\sigma\sqrt{T},
\end{aligned}
$$

where

$$
\gamma = \int_0^\infty \mathrm{d}h\, g(h) = \sqrt{\pi/8} \approx 0.6267.
$$

The exact computation of γ from the integral

$$\int_0^\infty \mathrm{d}h\, g(h)$$

is challenging. An alternate route to the same expression using reflected Brownian motion was obtained by Bond [3], who showed that $\gamma = \sqrt{\pi/8}$. A useful comparison is to the expected value of the range R, the difference between the two extrema of the motion. The range R is computed for $\mu = 0$ in [6], and the generalization to nonzero μ can be obtained from (29) in Appendix A and the identity $\mathrm{E}[R] = \mathrm{E}[H \mid \mu] + \mathrm{E}[H \mid -\mu]$, where H is the maximum of the motion. For $\mu = 0$, $\mathrm{E}[R] = 2\sqrt{2/\pi}\sigma\sqrt{T}$ and $\sqrt{2/\pi} \approx 0.798 > \gamma$; thus the expected range is considerably larger than the expected maximum drawdown for $\mu = 0$.

2.2. Case $\mu < 0$

After applying the eigenvalue conditions and taking the integral of $G_{\bar{D}}(h)$ to get the expectation, we find that

$$\begin{aligned}\mathrm{E}[\bar{D}] &= \int_0^\infty \mathrm{d}h\, G_{\bar{D}}(h)\\ &= 2\int_0^\infty \mathrm{d}h\, \exp\left\{-\frac{\mu h}{\sigma^2}\right\}\sum_{n=1}^{\infty}\frac{\sin^3\theta_n}{\theta_n - \cos\theta_n\sin\theta_n}\left(1 - \exp\left\{-\frac{\mu^2 T}{2\sigma^2\cos^2\theta_n}\right\}\right).\end{aligned}$$

Making a change of variables to $u = -\mu h/\sigma^2$, we find that

$$\mathrm{E}[\bar{D}] = -2\frac{\sigma^2}{\mu} Q_{\mathrm{n}}(\alpha^2),$$

where, for $x > 0$,

$$Q_{\mathrm{n}}(x) = \int_0^\infty \mathrm{d}u\, \mathrm{e}^u \sum_{n=1}^{\infty}\frac{\sin^3\theta_n}{\theta_n - \cos\theta_n\sin\theta_n}\left(1 - \exp\left\{-\frac{x}{\cos^2\theta_n}\right\}\right). \tag{10}$$

Here, each θ_n satisfies $\tan\theta_n = -\theta_n/u$ and $\alpha = \mu\sqrt{T/2\sigma^2}$. The numerical computation of $Q_{\mathrm{n}}(x)$ is not a straightforward task. The sum in the integrand is a function of u that decreases faster than e^{-u}. Since the magnitude of the nth term in the sum is approximately $1/n$, we need to take $\Omega(\mathrm{e}^u)$ terms in the sum to make sure that the next term left out has magnitude less than the size of the sum. Thus, efficient computation of this integral is computationally nontrivial. Table 1 gives approximate values of $Q_{\mathrm{n}}(x)$ for various values of x, computed using an extensive numerical integration. Intermediate values can be obtained using interpolation and the asymptotic behaviour is discussed below.

When $x \to 0^+$, $Q_{\mathrm{n}}(x) \to \gamma\sqrt{2x}$ since, in this limit, we must recover the behaviour as $\mu \to 0$. We get the behaviour in the limit as $x \to \infty$ by noting that $R \geq \bar{D} \geq -L$. Taking expectations and using (29), we see that, for all α,

$$\frac{\alpha^2}{2} + \frac{Q_R(-\alpha)}{2} \leq Q_{\mathrm{n}}(\alpha^2) \leq Q_R(-\alpha). \tag{11}$$

where

$$Q_R(x) = \operatorname{erf}(x)(\tfrac{1}{2} + x^2) + \frac{1}{\sqrt{\pi}} x\mathrm{e}^{-x^2} \quad\text{and}\quad \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{d}u\, \mathrm{e}^{-u^2}.$$

TABLE 1: Numerical values for $Q_{\mathrm{p}}(\cdot)$ (for positive μ) and $Q_{\mathrm{n}}(\cdot)$ (for negative μ).

x	$Q_{\mathrm{p}}(x)$, $\mu > 0$	x	$Q_{\mathrm{n}}(x)$, $\mu < 0$
$x \to 0$	$\gamma\sqrt{2x}$	$x \to 0$	$\gamma\sqrt{2x}$
0.0005	0.019 690	0.0005	0.019 965
0.0010	0.027 694	0.0010	0.028 394
0.0015	0.033 789	0.0015	0.034 874
0.0020	0.038 896	0.0020	0.040 369
0.0025	0.043 372	0.0025	0.045 256
0.0050	0.060 721	0.0050	0.064 633
0.0075	0.073 808	0.0075	0.079 746
0.0100	0.084 693	0.0100	0.092 708
0.0125	0.094 171	0.0125	0.104 259
0.0150	0.102 651	0.0150	0.114 814
0.0175	0.110 375	0.0175	0.124 608
0.0200	0.117 503	0.0200	0.133 772
0.0225	0.124 142	0.0225	0.142 429
0.0250	0.130 374	0.0250	0.150 739
0.0275	0.136 259	0.0275	0.158 565
0.0300	0.141 842	0.0300	0.166 229
0.0325	0.147 162	0.0325	0.173 756
0.0350	0.152 249	0.0350	0.180 793
0.0375	0.157 127	0.0375	0.187 739
0.0400	0.161 817	0.0400	0.194 489
0.0425	0.166 337	0.0425	0.201 094
0.0450	0.170 702	0.0450	0.207 572
0.0500	0.179 015	0.0475	0.213 877
0.0600	0.194 248	0.0500	0.220 056
0.0700	0.207 999	0.0550	0.231 797
0.0800	0.220 581	0.0600	0.243 374
0.0900	0.232 212	0.0650	0.254 585
0.1000	0.243 050	0.0700	0.265 472
0.2000	0.325 071	0.0750	0.276 070
0.3000	0.382 016	0.0800	0.286 406
0.4000	0.426 452	0.0850	0.296 507
0.5000	0.463 159	0.0900	0.306 393
1.5000	0.668 992	0.0950	0.316 066
2.5000	0.775 976	0.1000	0.325 586
3.5000	0.849 298	0.1500	0.413 136
4.5000	0.905 305	0.2000	0.491 599
10.0000	1.088 998	0.2500	0.564 333
20.0000	1.253 794	0.3000	0.633 007
30.0000	1.351 794	0.3500	0.698 849
40.0000	1.421 860	0.4000	0.762 455
50.0000	1.476 457	0.5000	0.884 593
150.0000	1.747 485	1.0000	1.445 520
250.0000	1.874 323	1.5000	1.970 740
350.0000	1.958 037	2.0000	2.483 960
450.0000	2.020 630	2.5000	2.990 940
1000.0000	2.219 765	3.0000	3.492 520
2000.0000	2.392 826	3.5000	3.995 190
3000.0000	2.494 109	4.0000	4.492 380
4000.0000	2.565 985	4.5000	4.990 430
5000.0000	2.621 743	5.0000	5.498 820
$x \to \infty$	$\frac{1}{4}\log x + 0.490\,88$	$x \to \infty$	$x + \frac{1}{2}$

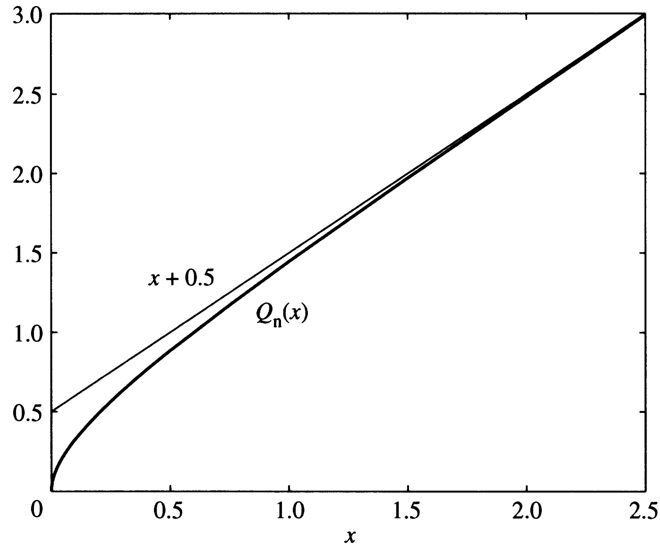

FIGURE 2: Asymptotic behaviour of $Q_{\mathrm{n}}(\cdot)$ for $\mu < 0$ (thick line) and its asymptote (thin line).

Asymptotically, as $\alpha \to -\infty$, this yields that

$$\alpha^2 + \tfrac{1}{4} \le Q_{\mathrm{n}}(\alpha^2) \le \alpha^2 + \tfrac{1}{2},$$

from which we deduce that $Q_{\mathrm{n}}(x) \to x + \varepsilon(x)$, where $\frac{1}{4} \le \varepsilon(x) \le \frac{1}{2}$. We now argue that, asymptotically, $\mathrm{E}[\bar{D}] = \mathrm{E}[R]$ when $\mu < 0$. By the definition of $\bar{D}$, when the maximum occurs before the minimum, $\bar{D} = R$, so

$$\mathrm{E}[\bar{D}] \ge \mathrm{E}[R \mid H \to L]\,\mathrm{P}[H \to L],$$

where $A \to B$ denotes the event 'A occurs before B'. Since $\mu < 0$, $\mathrm{P}[H \to L] \to 1$ as $T \to \infty$. Considering the range, we have

$$\mathrm{E}[R] = \mathrm{E}[R \mid H \to L]\,\mathrm{P}[H \to L] + \mathrm{E}[R \mid L \to H]\,\mathrm{P}[L \to H].$$

The term $\mathrm{E}[R \mid L \to H]$ is slowly growing with T, but, for $\mu < 0$, $\mathrm{P}[L \to H] \to 0$ exponentially fast, hence the second term asymptotically approaches 0, and so we conclude that $\mathrm{E}[R] \to \mathrm{E}[R \mid H \to L]$. Thus, we see that, asymptotically, $\mathrm{E}[\bar{D}] \ge \mathrm{E}[R]$, hence that $\mathrm{E}[\bar{D}] = \mathrm{E}[R]$, and so $\lim_{x\to\infty}\varepsilon(x) = \frac{1}{2}$. Collecting the results together, we have

$$Q_{\mathrm{n}}(x) \to \begin{cases} \gamma\sqrt{2x}, & x \to 0^+, \\ x + \frac{1}{2}, & x \to \infty. \end{cases}$$

The asymptotic behaviour is illustrated in Figure 2, which shows the numerical computation of $Q_{\mathrm{n}}(\cdot)$ along with the asymptote.

2.3. Case $\mu > 0$

In this case, for $h > \sigma^2/\mu$ in the integral (7), the expression for M in (3) adds another term. Thus, we find that

$$\begin{aligned}
\mathrm{E}[\bar{D}] &= \int_0^\infty \mathrm{d}h\, G_{\bar{D}}(h), \\
&= 2\int_0^\infty \mathrm{d}h\, \exp\left\{-\frac{\mu h}{\sigma^2}\right\}\sum_{n=1}^{\infty}\frac{\sin^3\theta_n}{\theta_n - \cos\theta_n\sin\theta_n}\left(1-\exp\left\{-\frac{\mu^2 T}{2\sigma^2\cos^2\theta_n}\right\}\right) \\
&\quad - 2\int_{\sigma^2/\mu}^\infty \mathrm{d}h\, \exp\left\{-\frac{\mu h}{\sigma^2}\right\}\frac{\sinh^3\eta}{\eta - \cosh\eta\sinh\eta}\left(1-\exp\left\{-\frac{\mu^2 T}{2\sigma^2\cosh^2\eta}\right\}\right).
\end{aligned}$$

The second integral can be reduced by a change of variables $u = \eta(h)$ as follows. Since $\tanh u = \sigma^2 u/\mu h$,

$$\frac{\mathrm{d}h}{\mathrm{d}u} = \frac{\sigma^2}{\mu}\frac{\cosh u\sinh u - u}{\sinh^2 u};$$

hence, the second integral reduces to

$$-\frac{\sigma^2}{\mu}\int_0^\infty \mathrm{d}u\, \exp\left\{-\frac{u}{\tanh u}\right\}\sinh u\left(1-\exp\left\{-\frac{\mu^2 T}{2\sigma^2\cosh^2 u}\right\}\right).$$

Changing variable in the first integral to $u = \mu h/\sigma^2$, we arrive at

$$\mathrm{E}[\bar{D}] = \frac{2\sigma^2}{\mu}Q_{\mathrm{p}}(\alpha^2),$$

where $\alpha = \mu\sqrt{T/2\sigma^2}$ once again and, for $x > 0$,

$$\begin{aligned}
Q_{\mathrm{p}}(x) = \int_0^\infty \mathrm{d}u\,\Bigg[&\mathrm{e}^{-u}\sum_{n=0}^{\infty}\frac{\sin^3\theta_n}{\theta_n - \cos\theta_n\sin\theta_n}\left(1-\exp\left\{-\frac{x}{\cos^2\theta_n}\right\}\right) \\
&+ \exp\left\{-\frac{u}{\tanh u}\right\}\sinh u\left(1-\exp\left\{-\frac{x}{\cosh^2 u}\right\}\right)\Bigg].
\end{aligned} \tag{12}$$

Here, each θ_n satisfies $\tan\theta_n = \theta_n/u$. The bound (11) is still valid, but not very useful. The numerical computation of $Q_{\mathrm{p}}(\cdot)$ is relatively straightforward, as the e^{-u} term in the integrand makes it well behaved for the purposes of numerical integration. We know that $Q_{\mathrm{p}}(x) \to \gamma\sqrt{2x}$ when $x \to 0^+$. We now consider the other asymptotic limit, namely $\alpha \to \infty$. We will evaluate the two contributions to $Q_{\mathrm{p}}(\cdot)$ separately. Consider the first part of the integral in (12), which we denote by $I_1(x)$,

$$I_1(x) = \int_0^\infty \mathrm{d}u\,\mathrm{e}^{-u}\sum_{n=0}^{\infty}\frac{\sin^3\theta_n}{\theta_n - \cos\theta_n\sin\theta_n}\left(1-\exp\left\{-\frac{x}{\cos^2\theta_n}\right\}\right).$$

Since $0 \le \cos^2\theta_n \le 1$ and $x \to \infty$, the term in parentheses is rapidly approaching 1. Since e^{-u} is rapidly decreasing, we interchange the summation with the integration and, after changing variables in the integral to $v = \theta_n(u)$ and using the identity

$$\mathrm{d}u = \frac{\cos v\sin v - v}{\sin^2 v}\,\mathrm{d}v,$$

we find that

$$I_1(x) = \sum_{n=0}^{\infty} \int_{n\pi}^{n\pi+\pi/2} \mathrm{d}v \, \exp\left\{-\frac{v}{\tan v}\right\} \sin v \left(1 - \exp\left\{-\frac{x}{\cos^2 v}\right\}\right).$$

After translating each integral by $n\pi$ and bringing the summation back into the integral, the sum is a geometric progression, which implies that

$$I_1(x) = \int_0^{\pi/2} \mathrm{d}v \, \frac{\exp\{-v/\tan v\} \sin v (1 - \exp\{-x/\cos^2 v\})}{1 + \exp\{-\pi/\tan v\}}.$$

Thus, $(1 - \mathrm{e}^{-x})\beta_1 \le I_1(x) \le \beta_1$, where

$$\beta_1 = \int_0^{\pi/2} \mathrm{d}v \, \frac{\exp\{-v/\tan v\} \sin v}{1 + \exp\{-\pi/\tan v\}},$$

and so we see that $I_1(x)$ rapidly converges to β_1. The constant β_1 can be evaluated numerically to give $\beta_1 = 0.4575$. Now consider the second term in (12), which we denote by $I_2(x)$,

$$I_2(x) = \int_0^{\infty} \mathrm{d}u \, \exp\left\{-\frac{u}{\tanh u}\right\} \sinh u \left(1 - \exp\left\{-\frac{x}{\cosh^2 u}\right\}\right). \tag{13}$$

When x is large, the term in parentheses is very close to 1 until u gets so large that $\cosh u \sim x$, from which point this term rapidly decreases to 0. The first term in the integrand is always less than $\frac{1}{2}$ and rapidly increases from 0 to $\frac{1}{2}$. Thus, we can write

$$\begin{aligned} I_2(x) &= \int_0^{\infty} \mathrm{d}u \left(\exp\left\{-\frac{u}{\tanh u}\right\} \sinh u - \frac{1}{2} + \frac{1}{2}\right)\left(1 - \exp\left\{-\frac{x}{\cosh^2 u}\right\}\right) \\ &= \frac{1}{2}\int_0^{\infty} \mathrm{d}u \left(1 - \exp\left\{-\frac{x}{\cosh^2 u}\right\}\right) - \int_0^{\infty} \mathrm{d}u \left(\frac{1}{2} - \exp\left\{-\frac{u}{\tanh u}\right\} \sinh u\right) \\ &\quad + \int_0^{\infty} \mathrm{d}u \, \exp\left\{-\frac{x}{\cosh^2 u}\right\}\left(\frac{1}{2} - \exp\left\{-\frac{u}{\tanh u}\right\} \sinh u\right). \end{aligned} \tag{14}$$

We first show that the third integral approaches zero as $x \to \infty$. Since this integral is monotonically decreasing in x, it suffices to consider the integral for $x \in \mathbb{N}$. Let

$$g(u) = \frac{1}{2} - \exp\left\{-\frac{u}{\tanh u}\right\} \sinh u$$

and let

$$f_n(u) = g(u) \exp\left\{-\frac{n}{\cosh^2 u}\right\}.$$

Then $|f_n| < g$ almost everywhere and $f_n \to 0$ almost everywhere. Therefore, by the Lebesgue dominated convergence theorem, the third integral in (14) converges to 0. The second integral is a constant, denoted by β_2, independent of x and can be evaluated numerically to give $\beta_2 = 0.4575$, which apparently is (numerically) equal to β_1. The authors suspect that $\beta_2 = \beta_1$ but the proof has been elusive.

We get bounds for the first integral using the inequalities $\cosh u \geq \frac{1}{2}\mathrm{e}^u$ and, for $u \geq A$, $\cosh u \leq \frac{1}{2}\mathrm{e}^{\lambda(A)u}$ where $\lambda(A) = 1 + \mathrm{e}^{-2A}/A$. Denoting the first integral in (14) by $F(x)$, we immediately get the following bounds, which hold for arbitrary fixed A:

$$A\left(1 - \exp\left\{-\frac{x}{\cosh^2 A}\right\}\right) + \int_A^\infty \mathrm{d}u\,(1 - \exp\{-4x\mathrm{e}^{-2\lambda(A)u}\}) \\ \leq 2F(x) \leq \int_0^\infty \mathrm{d}u\,(1 - \exp\{-4x\mathrm{e}^{-2u}\}).$$

A change of variables to $v = x\mathrm{e}^{-2\lambda(A)u}$ in the lower bound and $v = x\mathrm{e}^{-2u}$ in the upper bound then leads to the following bounds:

$$A\left(1 - \exp\left\{-\frac{x}{\cosh^2 A}\right\}\right) + \frac{1}{2\lambda}\int_0^{x\mathrm{e}^{-2\lambda A}} \frac{\mathrm{d}u}{u}(1 - \mathrm{e}^{-4u}) \\ \leq 2F(x) \leq \frac{1}{2}\int_0^x \frac{\mathrm{d}u}{u}(1 - \mathrm{e}^{-4u}). \quad (15)$$

We can get an asymptotic form as follows. Suppose that $z > 1$. Then

$$\int_0^z \frac{\mathrm{d}u}{u}(1 - \mathrm{e}^{-4u}) = \int_0^1 \frac{\mathrm{d}u}{u}(1 - \mathrm{e}^{-4u}) + \log z - \int_1^z \frac{\mathrm{d}u}{u}\mathrm{e}^{-4u}.$$

The first term is a constant and, when $z \to \infty$, the third term converges to Ei(-4) (see for example [7]). Thus, as $z \to \infty$,

$$\int_0^z \frac{\mathrm{d}u}{u}(1 - \mathrm{e}^{-4u}) \to \log z + C, \quad (16)$$

where

$$C = \int_0^1 \frac{\mathrm{d}u}{u}(1 - \mathrm{e}^{-4u}) + \mathrm{Ei}(-4) \approx 1.9635.$$

Substituting (16) into (15), we find that, for fixed A,

$$\frac{1}{2\lambda(A)}(\log x + C) - A\exp\left\{-\frac{x}{\cosh^2 A}\right\} \leq 2F(x) \leq \frac{1}{2}\log x + \frac{C}{2}.$$

Since A is arbitrary, it can be chosen to grow with x, for example $\frac{1}{2}(1 + \varepsilon)\log x$, in which case $\lambda(A) \to 1$ and the second term in the lower bound approaches 0, so the upper and lower bounds approach each other. Thus, we can conclude that, as $x \to \infty$,

$$F(x) \to \tfrac{1}{4}\log x + D,$$

where $D = C/4$. Collecting the results, and remembering that $Q_\mathrm{p}(x) = I_1(x) + I_2(x)$, we have

$$Q_\mathrm{p}(x) \to \begin{cases} \gamma\sqrt{2x}, & x \to 0^+, \\ \frac{1}{4}\log x + D, & x \to \infty, \end{cases}$$

where $D \approx 0.490\,88$, and we have used the fact that $\beta_1 \approx \beta_2$. The asymptotic behaviour is illustrated in Figure 3.

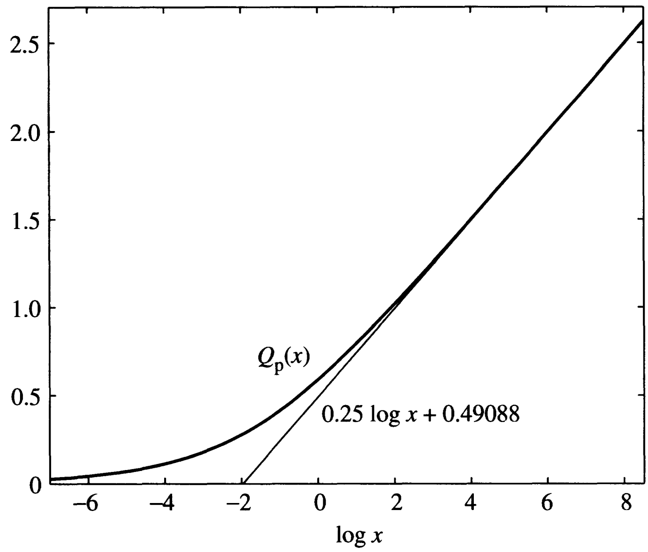

FIGURE 3: Asymptotic behaviour of $Q_{\mathrm{p}}(\cdot)$ for $\mu > 0$ (thick line) and its asymptote (thin line).

3. Discrete random walk

We model the Brownian motion as a discrete random walk at the times $t_i = i\Delta t$, where $\Delta t = T/n$ and $i = 0, 1, \dots, n$. Define $X_i = X(t_i)$, the position of the random walk at time t_i. We assume that X_i has the following dynamics:

$$X_{i+1} = \begin{cases} X_i + \delta & \text{with probability } p, \\ X_i - \delta & \text{with probability } q = 1 - p. \end{cases}$$

Defining

$$\begin{aligned} \delta &= \sqrt{\sigma^2 \Delta t + \mu^2 \Delta t^2}, \\ p &= \frac{1}{2}\left(1 + \frac{\mu\sqrt{\Delta t}}{\sqrt{\sigma^2 + \mu^2 \Delta t}}\right), \\ q &= \frac{1}{2}\left(1 - \frac{\mu\sqrt{\Delta t}}{\sqrt{\sigma^2 + \mu^2 \Delta t}}\right), \end{aligned} \tag{17}$$

in the limit $\Delta t \to 0$, the random walk converges in distribution to a Brownian motion on $[0, T]$ with drift μ and diffusion parameter σ.

Analogously to $D(t)$ in Section 2, define D_t to be the drawdown from the previous maximum at time-step t, with $D_0 = 0$. The maximum drawdown is given by $\bar{D} = \max_t D_t$. The drawdown D_t is a random walk with probability $\tilde{p} = 1 - p$ and a reflective barrier at 0. Let $h > 0$ be an absorbing barrier and let $f(i \mid h)$ be the probability that the random walk gets absorbed at time-step i. Then

$$\mathrm{P}[\bar{D} > h] = \mathrm{P}[\text{absorption in } [0, T]] = \sum_{i=0}^{T/\Delta t} f(i \mid h). \tag{18}$$

The probability $f(i \mid h)$ was initially computed in [9] for $p/q < (1+1/N)^2$. The more general case was given in [2], which, after the correction of some typographic errors, is given by

$$f(i \mid h) = \begin{cases} \tilde{f}(1), & \dfrac{p}{q} < \left(1+\dfrac{1}{N}\right)^2, \\ \tilde{f}(2) + \dfrac{3}{2}\dfrac{2^i p^{(1/2)(i-N)} q^{(1/2)(i+N)}}{(N+1)(N+\frac{1}{2})}, & \dfrac{p}{q} = \left(1+\dfrac{1}{N}\right)^2, \\ \tilde{f}(2) + \dfrac{2^i p^{1/2(i-N)} q^{1/2(i+N)} q^{1/2} \cosh^{i-1}\beta \sinh^2\beta}{(N+1)q^{1/2}\cosh(N+1)\beta - Np^{1/2}\cosh N\beta}, & \dfrac{p}{q} > \left(1+\dfrac{1}{N}\right)^2, \end{cases} \tag{19}$$

where $N = h/\delta$ and

$$\tilde{f}(k) = -2^i p^{1/2(i-N)} q^{1/2(i+N)} \sum_{v=k}^{N} \frac{q^{1/2}\cos^{i-1}\alpha_v \sin^2\alpha_v}{(N+1)q^{1/2}\cos(N+1)\alpha_v - Np^{1/2}\cos N\alpha_v}, \tag{20}$$

with $\alpha_v \in (v\pi/(N-1), (v+1)\pi/(N-1))$ satisfying

$$q^{1/2}\sin(N+1)\alpha_v - p^{1/2}\sin N\alpha_v = 0, \tag{21}$$

and β satisfying

$$q^{1/2}\sinh(N+1)\beta - p^{1/2}\sinh N\beta = 0.$$

Using (19) and (20) in (18) gives the distribution function in the discrete case.

Using δ and p as given in (17), and taking the limit as $\Delta t \to 0$ gives the continuous case as follows. The sum in (18) is the Riemann sum approximation to the integral $\int_0^T \mathrm{d}t\, f_\tau(t \mid h)$, where $f_\tau(t \mid h)$ is the absorption time density given by the limit of $f(i \mid h)/\Delta t$ where $i\Delta t = t$. It thus remains to take the limit as $\Delta t \to 0$ of $f(i \mid h)/\Delta t$. Since $p = \frac{1}{2}(1+\lambda)$, where $\lambda \to \mu\sqrt{\Delta t}/\sigma$ and $\delta \to \sigma\sqrt{\Delta t}$, the three cases in (19) reduce to $\mu < \sigma^2/h$, $\mu = \sigma^2/h$ and $\mu > \sigma^2/h$ in the limit, analogous to the three cases in (9). Using the identity $\lim_{x\to\infty}(1+1/x)^x = \mathrm{e}$, we find that

$$2^i p^{1/2(i-N)} q^{1/2(i+N)} = (1-\lambda^2)^{i/2}\left(\frac{1-\lambda}{1+\lambda}\right)^{N/2} \to \exp\left\{-\frac{\mu^2 t}{2\sigma^2}\right\}\exp\left\{-\frac{\mu h}{\sigma^2}\right\}. \tag{22}$$

Expanding the eigenvalue condition for α_v in (21) to first order in λ and using some trigonometric identities gives

$$\tan(N+\tfrac{1}{2})\alpha_v \cos\alpha_v = \frac{2}{\lambda}\sin\frac{\alpha_v}{2}.$$

Since $\alpha_v \in (v\pi/(N-1), (v+1)\pi/(N-1))$, let $\theta_v = (N+\frac{1}{2})\alpha_v$. For fixed v, $\alpha_v \to 0$, so we take the first-order expansion in α_v to get

$$\tan\theta_v = \frac{\sigma^2}{\mu h}\theta_v, \tag{23}$$

with

$$\theta_v \in \left(v\pi\frac{N+\frac{1}{2}}{N-1}, (v+1)\pi\frac{N+\frac{1}{2}}{N-1}\right) \to (v\pi, (v+1)\pi].$$

In an identical manner, we can analyse the eigenvalue condition for β to get

$$\tanh(N+\tfrac{1}{2})\beta\cosh\beta = \frac{2}{\lambda}\sinh\frac{\beta}{2}.$$

Defining $\eta = (N+\frac{1}{2})\beta$ and taking the limit, we find that

$$\tanh\eta = \frac{\sigma^2}{\mu h}\eta$$

We now analyse the summands in the expression (20) for $\tilde{f}$. Since $\sin\alpha_v \to \theta_v/(N+\frac{1}{2})$, these summands become

$$\frac{\theta_v^2\cos^{i-1}\alpha_v}{(N+\frac{1}{2})^2(N+1)[\cos(\theta_v+\frac{1}{2}\alpha_v) - A\cos(\theta_v-\frac{1}{2}\alpha_v)]}, \tag{24}$$

where $A = (1+\lambda)^{1/2}/(1+1/N)(1-\lambda)^{1/2}$. The identity $\lim_{x\to 0}\cos^{1/x^2}x = \mathrm{e}^{-1/2}$ implies that $\cos^{i-1}\alpha_v \to \exp\{-\sigma^2\theta_v^2 t/2h^2\}$. Thus, after using some double-angle formulae and taking the limit, the summands in $\tilde{f}$ become

$$\frac{\Delta t\sigma^2}{h^2}\frac{\theta_v^2\exp\{-\sigma^2\theta_v^2 t/2h^2\}}{(1-\mu h/\sigma^2)\cos\theta_v - \theta_v\sin\theta_v} = \frac{-\theta_v\sin\theta_v[\sigma^4\theta_v^2+\mu^2h^2]}{\sigma^4\theta_v^2+\mu^2h^2-\mu h\sigma^2}, \tag{25}$$

where the second equality follows by use of (23) and some trigonometric identities. Substituting all these results back into (20) and dividing by Δt, we finally arrive at the continuous limit of $\tilde{f}$:

$$\tilde{f}(t) \to \exp\left\{-\frac{\mu^2 t}{2\sigma^2}\right\}\exp\left\{-\frac{\mu h}{\sigma^2}\right\}\frac{\sigma^2}{h^2}\sum_{v=0}^{\infty}\frac{\theta_v\sin\theta_v[\sigma^4\theta_v^2+\mu^2h^2]\exp\{-\sigma^2\theta_v^2 t/2h^2\}}{\sigma^4\theta_v^2+\mu^2h^2-\mu h\sigma^2}, \tag{26}$$

where each θ_v is a positive solution to the eigenvalue condition $\tan\theta_v = \sigma^2\theta_v/\mu h$. The remaining two cases are handled in exactly analogous ways. For the second case, $p/q = (1+1/N)^2$, the additional term can be computed using (22) and, on dividing by Δt, we get

$$\exp\left\{-\frac{\mu^2 t}{2\sigma^2}\right\}\frac{3\sigma^2}{2\mathrm{e}h^2}, \qquad \mu = \frac{\sigma^2}{h}. \tag{27}$$

For the third case, $p/q > (1+1/N)^2$, define $\eta = (N+\frac{1}{2})\beta$. Since $\lim_{x\to 0}\cosh^{1/x^2}x = \mathrm{e}^{1/2}$, so we get that $\cosh^{i-1}\beta \to \exp\{\sigma^2\eta^2 t/2h^2\}$. Thus, as with (24), the additional term in (19) is

$$\frac{\Delta t\sigma^2}{h^2}\frac{\eta^2\exp\{\sigma^2\eta^2 t/2h^2\}}{N(1-A)\cosh\eta\cosh\frac{1}{2}\beta - N(1+A)\sinh\eta\sinh\frac{1}{2}\beta},$$

which, upon using manipulations similar to those that led to (25), becomes

$$\frac{\eta\sinh\eta(\mu^2h^2-\sigma^4\eta^2)}{\sigma^4\eta^2-\mu^2h^2+\sigma^2\mu h}.$$

Using (22) and dividing by Δt, this additional term then becomes

$$\frac{\sigma^2}{h^2}\frac{(\mu^2h^2-\sigma^4\eta^2)\eta\sinh\eta}{(\sigma^4\eta^2-\mu^2h^2+\sigma^2\mu h)}\exp\left\{-\frac{\mu^2 t}{2\sigma^2}\right\}\exp\left\{-\frac{\mu h}{\sigma^2}\right\}\exp\left\{\frac{\sigma^2\eta^2 t}{2h^2}\right\}, \qquad \mu > \frac{\sigma^2}{h}. \tag{28}$$

Using (26), (27) and (28) in (19), we arrive at the continuous limit of the discrete-time density, in agreement with (8).

Appendix A. Expected value of the high, low and range

We derive the expected value of R. The authors have not been able to find this explicit result in the literature for the general asymmetric Brownian motion, though Feller [6] gives the result for the symmetric case. Consider the Brownian motion with an absorbing barrier at h. The probability density for the absorption time λ is known (see for example [4]) and is given by the inverse Gaussian distribution

$$f_\lambda(t) = \frac{h}{(2\pi\sigma^2 t^3)^{1/2}} \exp\left\{-\frac{(h-\mu t)^2}{2\sigma^2 t}\right\}.$$

Thus, $G_H(h) = \mathrm{P}[H \geq h]$ is given by

$$G_H(h) = \int_0^T \mathrm{d}t\, f_\lambda(t) = h\int_0^T \frac{\mathrm{d}t}{t}\frac{1}{(2\pi\sigma^2 t)^{1/2}} \exp\left\{-\frac{(h-\mu t)^2}{2\sigma^2 t}\right\}.$$

The expected value of H is given by $\mathrm{E}[H] = \int_0^\infty \mathrm{d}h\, G_H(h)$, so we find (after a change of variables to $u = (h-\mu t)/(2\sigma^2 t)^{1/2}$) that

$$\mathrm{E}[H] = \frac{1}{\sqrt{\pi}}\int_0^T \mathrm{d}t \left(\frac{\sigma^2}{2t}\right)^{1/2} \mathrm{e}^{-\alpha^2(t)} + \frac{\mu T}{2} + \frac{\mu}{\sqrt{\pi}}\int_0^T \mathrm{d}t \int_0^{\alpha(t)} \mathrm{d}u\, \mathrm{e}^{-u^2},$$

where $\alpha(t) = \mu t^{1/2}/(2\sigma^2)^{1/2}$. The following identity is useful in evaluating the second integral:

$$\int_0^A \mathrm{d}x\, x\int_0^x \mathrm{d}u\, \mathrm{e}^{-u^2} = \frac{\sqrt{\pi}}{4}\mathrm{erf}(A)(A^2 - \tfrac{1}{2}) + \frac{A}{4}\mathrm{e}^{-A^2}.$$

The first integral (after a change of variables) can be reduced to $(\sigma^2/\mu)\mathrm{erf}(\alpha(T))$. The final result is then given by

$$\mathrm{E}[H] = \frac{\mu T}{2} + \frac{\sigma^2}{\mu}\left[\mathrm{erf}(\alpha)(\tfrac{1}{2} + \alpha^2) + \frac{\alpha \mathrm{e}^{-\alpha^2}}{\sqrt{\pi}}\right], \tag{29}$$

where $\alpha = \alpha(T)$. Expectations for the low and the range can be obtained from the identities $\mathrm{E}[L] = -\mathrm{E}[H \mid -\mu]$ and $\mathrm{E}[R] = \mathrm{E}[H \mid \mu] + \mathrm{E}[H \mid -\mu]$. Using asymptotic forms for $\mathrm{erf}(x)$ [7], we find that, when $\mu = 0$, $\mathrm{E}[R] = 2\sqrt{2\sigma^2 T/\pi}$, reproducing the result in [6]. Asymptotically,

$$\mathrm{E}[R] = \begin{cases} \dfrac{2\sigma^2}{\mu}\left(\dfrac{2\alpha}{\sqrt{\pi}} + \dfrac{2\alpha^3}{\sqrt{\pi}} + \cdots\right), & \alpha \to 0, \\ \dfrac{2\sigma^2}{\mu}\left(\alpha^2 + \dfrac{1}{2} - \dfrac{\mathrm{e}^{-\alpha^2}}{\alpha^3} + \cdots\right), & \alpha \to \infty. \end{cases}$$

Thus, two different kinds of behaviour emerge at the different limits.

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