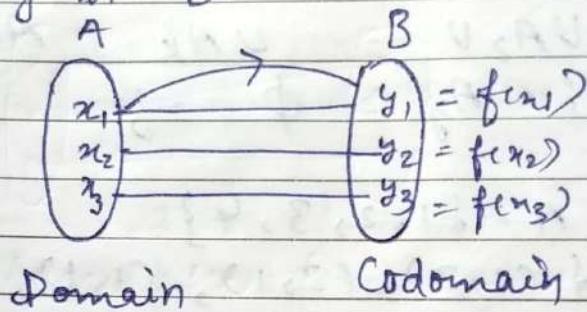


#

Function :-

Let A and B be two non-empty sets.
 A function $f: A \rightarrow B$ is a rule that assigns to each element x in A a unique element y in B.

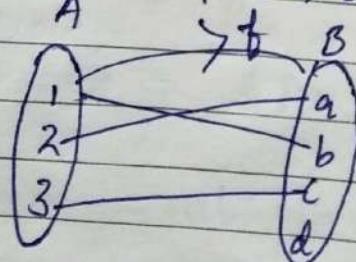


→ if $y = f(x)$ then y is called image of x and x is called preimage of y .

Range of a function - The set consisting of all the images of the elements of A under the function f is called the range of f . It is denoted by $f(A)$.

$f(A) = \{f(x) : \text{for all } x \in A\}$
 and $f(A) \subseteq B$.

Ex:- Let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$



So Range $f = \{a, b, c\}$

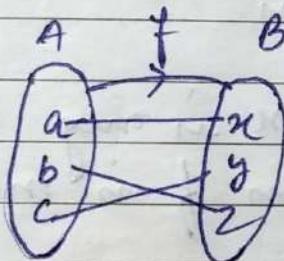
(1) Types of Functions: →

1.) Injective / one-to-one function: →

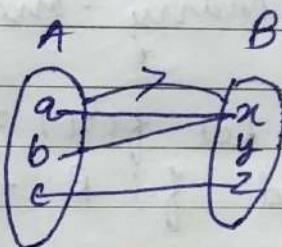
A function $f: A \rightarrow B$ is called one-one if no two elements in A corresponds to the same element in B .

$$\text{i.e. } x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

$$\text{or } f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$



One-one



Not one-one

Ex:- ① $f(x) = 3x - 1$ is a one-one function
since $f: R \rightarrow R$

$$f(x_1) = f(x_2)$$

$$\Rightarrow 3x_1 - 1 = 3x_2 - 1$$

$$\Rightarrow 3x_1 = 3x_2$$

$$\Rightarrow x_1 = x_2$$

2.) $f(x) = x^2$ is not one-one

where $f: R \rightarrow R$

since

$$-1 \neq 1 \text{ but } f(-1) = f(1)$$

so $f(x)$ is not one-one fn.

2.) Many-one function :-

A function $f: A \rightarrow B$ is said to be many one if and only if two or more elements of A have same image in B .



Many-one function

Ex! - Let $f(x) = x^2$, x is any real no. and $f: R \rightarrow R$, then f is many-one.

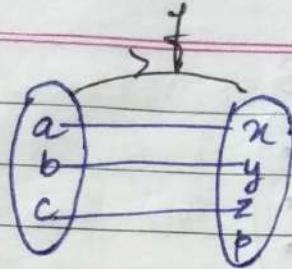
$$\text{for } x=1, f(1) = 1^2 = 1$$

$$\text{for } x=-1, f(-1) = (-1)^2 = 1$$

Thus $f(1) = f(-1)$ which shows that two distinct numbers 1 and -1 are assigned to the same no. 1 under f .
∴ f is many-one.

3.) Into function :-

A function f from A to B is called into function if and only if there exists at least one element in B which is not the image of any element in A , i.e. the range of f is a proper subset of codomain of f .

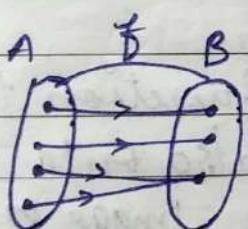


into function.

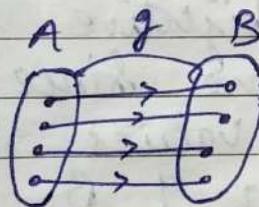
4.) Onto function :-

A function $f: A \rightarrow B$ is called onto or surjective if every element of B is the image of some element of A i.e. if $B = f(A)$ (Range of f)

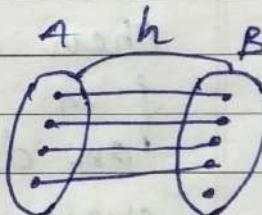
Note :- In order to check, whether $y = f(x)$ from set A to B is onto or not. write x in terms of y and see if for every $y \in B$, $x \in A$. if so it is onto. otherwise, it is not onto.



surjective



surjective

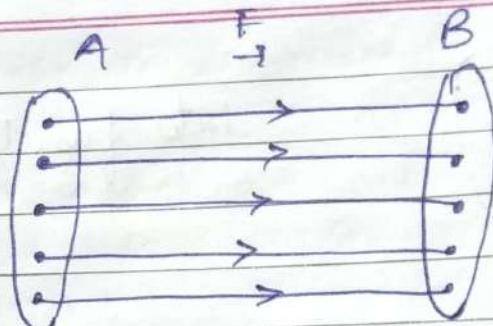


Not Surjective

Ex :- Let $f(n) = n^2$, n is any real number and $f: R \rightarrow R$. then f is not onto. The reason is that we can find a real no. whose square is negative then Range of f can not be equal to R .

5. Bijective function :-

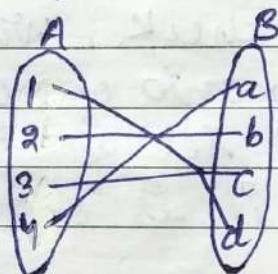
A function f from A to B is said to be Bijective if f is both injective and surjective i.e. both one-to-one and onto.



Bijective.

Ex:- Let f be a function from A to B ,

$A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$ with
 $f(1) = d, f(2) = b, f(3) = c, f(4) = a$



then f is bijective function.

f is one-one since the function takes on different values. It is also onto since every element of B is the image of some element in A . Hence f is a bijective f .



Equal Functions :-

Two functions $f: A \rightarrow B$ and $g: X \rightarrow Y$ are said to be equal if $A = X, B = Y$, and

$f(x) = g(x)$, for all $x \in A (= X)$

Ques. Let A and B be two sets such that $|A|=|B|$. Prove that a function $f: A \rightarrow B$ is an injective function if and only if f is a surjective function.

Soln:-

In general, if $f: A \rightarrow B$ is injective Then $|A| \leq |B|$. However, since $|A|=|B|$
 $\Rightarrow \text{Range } f = B$

Thus f is a surjective function.

Conversely, if $f: A \rightarrow B$ is surjective we have $\text{Range } f = B$ so that $|A| \geq |B|$

However, since $|A|=|B|$, so no element $b \in B$ has more than one pre-image.

Hence, f is an injective function.

Ques. Let A and B be finite sets and $f: A \rightarrow B$

Then show that

i.) If f is one-to-one, then $|A| \leq |B|$

ii.) If f is onto, then $|B| \leq |A|$

iii.) If f is a bijection then $|A|=|B|$.

Solution:-

Let $A = \{a_1, a_2, \dots, a_n\}$, $B = \{b_1, b_2, \dots, b_m\}$
 So that $|A|=n$ and $|B|=m$

i.) Suppose f is one-to-one function.

Then the n images of the elements of A i.e. $f(a_1), f(a_2), \dots, f(a_n)$ are all different

Since all these n images belong to B ,

So B must have at least n elements

i.e. $|B| \geq n$ But $n=|A|$

so $|B| \geq |A|$ i.e. $|A| \leq |B|$. Hence proved

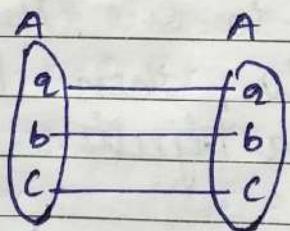
ii) Suppose f is onto. Then every element of B is the image of some element in A . Since two or more elements (distinct) in A may have the same image in B , then $|B| \leq n$. i.e $|B| \leq |A|$

iii) Suppose f is a bijection.
Then f is both one-to-one and onto.
 $\Rightarrow |A| \leq |B|$ and $|B| \leq |A|$
Therefore $|A| = |B|$

#

Identity function : →

A function $f: A \rightarrow A$ defined by $f(x) = x$ for every $x \in A$ is called the Identity function of A . It is denoted by I_A .



#

Constant function : →

A function $f: R \rightarrow R$ defined by $f(x) = c$, $\forall x \in R$

where $c \in R$ is a constant, is called Constant fn.

#

Real Valued function : →

A function f from R to R i.e $f: R \rightarrow R$ is called Real valued fn.

Some Important functions :-

1.) Floor Function :-

An. Integer-valued function $f: \mathbb{R} \rightarrow \mathbb{Z}$ given by

$$\lfloor x \rfloor = n, \text{ if } x \geq n, x \in \mathbb{R}$$

is called the floor function

It is also known as Greatest Integer funct.

e.g. $\lfloor 1.2 \rfloor = 1, \lfloor 0.5 \rfloor = 0$

2.) Ceiling Function :-

An Integer-valued function $f: \mathbb{R} \rightarrow \mathbb{Z}$ given by

$$\lceil x \rceil = n, \text{ where } x \leq n, x \in \mathbb{R}$$

is called the ceiling function.

e.g. $\lceil 1.2 \rceil = 2, \lceil 0.5 \rceil = 1$

Theorem :-

For any $x \in \mathbb{R}$

1.) $\lfloor x \rfloor = n \Leftrightarrow n \leq x < n+1$

$$\lceil x \rceil = n \Leftrightarrow n-1 < x \leq n$$

2.) $\lfloor x \rfloor = n \Leftrightarrow x-1 < n \leq x$

$$\lceil x \rceil = n \Leftrightarrow x \leq n < x+1$$

3.) $x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$

4.) $\lfloor -x \rfloor = -\lceil x \rceil \text{ and } \lceil -x \rceil = -\lfloor x \rfloor$

5.) $\lfloor x+n \rfloor = \lfloor x \rfloor + n$

$$\lceil x+n \rceil = \lceil x \rceil + n$$

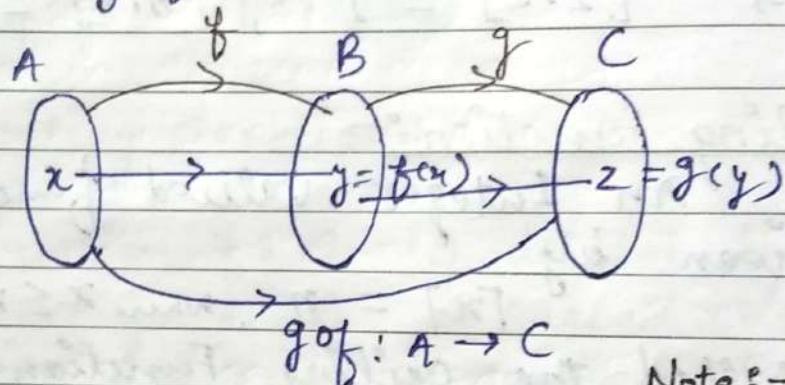


composite function | composition of Functions

Let A, B , and C be three non-empty sets and $f: A \rightarrow B$ & $g: B \rightarrow C$

Then the composition of f and g is a function denoted by $gof: A \rightarrow C$ and is defined by

$$(gof)(x) = g[f(x)] \quad \forall x \in A$$

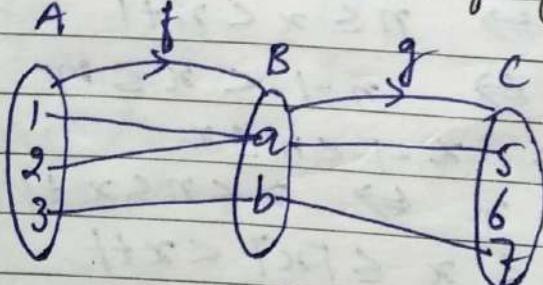


Note :- $fog \neq gof$

Ex:-1 Let $A = \{1, 2, 3\}$, $B = \{a, b\}$, $C = \{5, 6, 7\}$
define $f: A \rightarrow B$, $f = \{(1, a), (2, a), (3, b)\}$
and $g: B \rightarrow C$, $g = \{(a, 5), (b, 7)\}$

Find the composition of f and g .

Soln:-



The composition of f and g is gof

$$(gof)(x) = g[f(x)]$$

$$(gof)(1) = g[f(1)] = g(a) = 5$$

$$(gof)(2) = g[f(2)] = g(a) = 5$$

$$(gof)(3) = g[f(3)] = g(b) = 7$$

$$\therefore g \circ f = \{f(1, 5), f(2, 5), f(3, 7)\}$$

Ex:-2 Let $f: R \rightarrow R$ and $g: R \rightarrow R$ be two R.V.fns given by $f(x) = x^2$ and $g(x) = x+4$
Then

Find i.) gof ii.) fog iii.) fof iv.) gog

Sol:-

$$\text{Given } f(x) = x^2, g(x) = x+4$$

$$1) (gof)(x) = g[f(x)] = g(x^2) \\ = x^2 + 4$$

$$2) (fog)(x) = f[g(x)] = f(x+4) \\ = (x+4)^2$$

$$3) (fof)(x) = f[f(x)] = f(x^2) \\ = (x^2)^2 \\ = x^4$$

$$4) (gog)(x) = g[g(x)] = g(x+4) \\ = (x+4)+4 \\ = x+8.$$

Ex:-3 Find the composition of functions

$f: N \rightarrow Z$ and $g: Z \rightarrow Z$ resp. given by
 $f(n) = n-2$ and $g(n) = n^2$.

Sol:- composition of f and g is

$$(gof)(n) \\ = g[f(n)] \\ = g(n-2) \\ = (n-2)^2 \\ = n^2 - 4n + 4.$$

we can also find composition of g and f
i.e fog , However codomain $g \neq$ domain of f

since $\text{Range}(g) = \{n^2 \mid n \in \mathbb{Z}\} \subset \mathbb{N}$

$$\begin{aligned}(fog)(cn) &= f[g(cn)] \\ &= f(n^2) \\ &= n^2 - 2\end{aligned}$$

Note: - $fog \neq gof$

because of two reasons

- i.) their values f and g are different
- ii.) the domain of fog and gof are not the same yet.

Ques. Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$,

$g: \mathbb{R} \rightarrow \mathbb{R}$ and $h: \mathbb{R}^* \rightarrow \mathbb{R}$ given by

$$f(n) = 3n^2 + 2, g(n) = 7n - 5,$$

$$h(n) = \frac{1}{n}.$$

Compute the composite functions $fogoh$, gog , goh , $hogf$.

Sol: -

$$\text{i.) } (fogoh)(cn) = f[g[h(cn)]]$$

$$= (fog)\left(\frac{1}{n}\right)$$

$$= f\left[g\left(\frac{1}{n}\right)\right]$$

$$= f\left(\frac{7}{n} - 5\right)$$

$$= 3\left(\frac{7}{n} - 5\right)^2 + 2$$

$$= \frac{147}{n^2} + 77 - \frac{210}{n}$$

$$\begin{aligned}
 \text{i.) } (g \circ g)(x) &= g[g(x)] \\
 &= g(7x - 5) \\
 &= 7(7x - 5) - 5 \\
 &= 49x - 35 - 5 \\
 &= 49x - 40
 \end{aligned}$$

$$\begin{aligned}
 \text{ii.) } (g \circ h)(x) &= g[h(x)] \\
 &= g\left[\frac{1}{x}\right] \\
 &= \frac{7}{x} - 5
 \end{aligned}$$

$$\text{iii.) } (g \circ h)(x) = g[h(x)]$$

$$\begin{aligned}
 \text{iv.) } (h \circ g \circ f)(x) &= h[g[f(x)]] \\
 &= h[g(3x^2 + 2)] \\
 &= h[g(3x^2 + 2)] \\
 &= h[7(3x^2 + 2) - 5] \\
 &= h(21x^2 + 9) \\
 &= \frac{1}{21x^2 + 9}
 \end{aligned}$$

Th^{ry} - Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two functions such that the composite function $gof: A \rightarrow C$ is a bijective function. Then f is injective and g is surjective function.

Proof :-

Given that the composite function $gof: A \rightarrow C$ is a bijective function i.e one-one and onto.

Since gof is one-one / Injective
 \Rightarrow for $x_1, x_2 \in A$,
 $gof(x_1) = gof(x_2) \Rightarrow x_1 = x_2$ —①

Now to show $f: A \rightarrow B$ is one-one
we have to show that

$$\text{for } x_1, x_2 \in A, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

$$\begin{aligned} f(x_1) &= f(x_2) \\ \text{taking } g \text{ function on both the sides} \\ gof(x_1) &= gof(x_2) \\ \Rightarrow x_1 &= x_2 \quad \text{from(1)} \end{aligned}$$

Hence f is one-one / Injective function.

Since gof is onto for
So for all $z \in C$, $\exists x \in A$ such that
 $gof(x) = z$

Now to show $g: B \rightarrow C$ is onto
we have to show that
for all $z \in C$, $\exists y \in B$ such that
 $g(y) = z$

let $x \in A$, $y \in B$ and $z \in C$
since $g \circ f$ is onto

$\Rightarrow \exists x \in A$ such that $g \circ f(x) = z$

$$\Rightarrow g(f(x)) = z$$

$$\Rightarrow g(y) = z \quad \therefore y = f(x)$$

\Rightarrow for every $z \in C$, $\exists y \in B$ such that $g(y) = z$

Hence g is onto / Surjective f .

QED

Theorem:- Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be one-one and onto functions. Then the composite function $g \circ f: A \rightarrow C$ is also one-one and onto function and

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

Proof:-

Since f is one-one, for $x_1, x_2 \in A$
 $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

again g is one-one, for $y_1, y_2 \in B$
 $g(y_1) = g(y_2) \Rightarrow y_1 = y_2$

Now $(g \circ f)(x_1) = (g \circ f)(x_2)$
 $\Rightarrow g[f(x_1)] = g[f(x_2)]$

$$\Rightarrow f(x_1) = f(x_2) \quad \text{since } g \text{ is one-one}$$

$$\Rightarrow x_1 = x_2 \quad \text{since } f \text{ is one-one}$$

Therefore $(g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow x_1 = x_2$

Hence $g \circ f$ is one-one function.

since g is onto, so for each $z \in C$, $\exists y \in B$
such that $g(y) = z$

also, f is onto, so for each $y \in B$, $\exists x \in A$
such that $f(x) = y$

Now, for each $z \in C$,

$$\begin{aligned} gof(x) &= g[f(y)] \\ &= g(y) \\ &= z \end{aligned}$$

So $\forall z \in C$, $\exists x \in A$ s.t. $gof(x) = z$
every element $z \in C$ has preimage
under gof
Hence gof is an onto function.

Now gof is one-one and onto f'

so $(gof)^{-1}$ exists

By the def. of composite functions, if

$$gof: A \rightarrow C, \text{ so } (gof)^{-1}: C \rightarrow A$$

$$\text{Also } f^{-1}: B \rightarrow A \text{ and } g^{-1}: C \rightarrow B$$

$$\text{Then by def. of composite fns}$$

$$f^{-1}og^{-1}: C \rightarrow A$$

Therefore, domain of $(gof)^{-1} = \text{domain of } f^{-1}og^{-1}$

$$\text{Now } (gof)^{-1}(z) = x$$

$$\Leftrightarrow (gof)(x) = z$$

$$\Leftrightarrow g[f(x)] = z$$

$$\Leftrightarrow g(y) = z \quad \text{where } y = f(x)$$

$$\Leftrightarrow y = g^{-1}(z)$$

$$\Leftrightarrow f^{-1}(y) = f^{-1}[g^{-1}(z)] = (f^{-1}og^{-1})(z)$$

$$\Leftrightarrow x = (f^{-1}og^{-1})(z)$$

Thus $(gof)^{-1}(z) = (f^{-1}og^{-1})(z)$, so $(gof)^{-1} = f^{-1}og^{-1}$

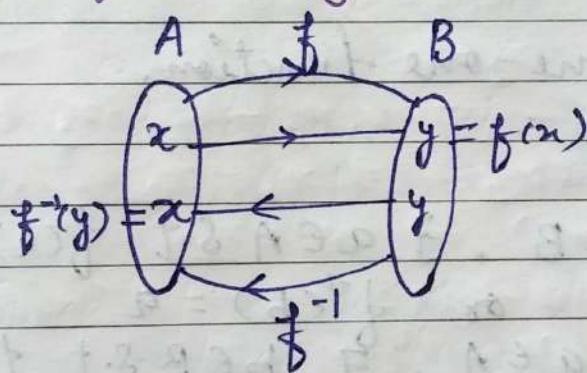
Inverse of a function :-

Let $f: A \rightarrow B$ be a function. Then $g: B \rightarrow A$ is called the inverse of f if
 $gof = I_A$ and $fog = I_B$
i.e. $g[f(x)] = x, \forall x \in A$ and $f[g(y)] = y, \forall y \in B$

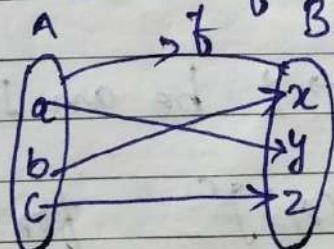
It is denoted by f^{-1} .

Thus $f(x) = y \Leftrightarrow x = f^{-1}(y)$.

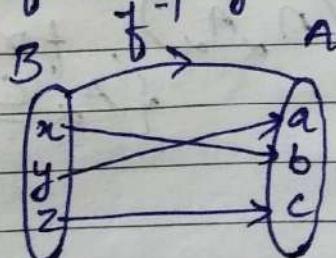
Note:- The Inverse of $f: A \rightarrow B$ exists if and only if f is bijective i.e one-one and onto f .



Ex:- Let function $f: A \rightarrow B$ be defined by



Then f is one-one and onto function
Therefore, f^{-1} exists and $f^{-1}: B \rightarrow A$
is defined by



Theorem 8 - If a mapping $f: A \rightarrow B$ is one-one and onto, then the inverse mapping $f^{-1}: B \rightarrow A$ is also one-one and onto.

Proof :-

$f: A \rightarrow B$ is one-one and onto $\Rightarrow f$ exists

Let $a_1, a_2 \in A$ and $b_1, b_2 \in B$ so that

$$f(a_1) = b_1 \text{ and } f(a_2) = b_2$$

$$\Rightarrow a_1 = f^{-1}(b_1) \text{ and } a_2 = f^{-1}(b_2)$$

$$\text{Now } f(a_1) = f(a_2) \Rightarrow a_1 = a_2 (\because f \text{ is 1-1})$$

$$\text{or } b_1 = b_2 \Leftrightarrow f^{-1}(b_1) = f^{-1}(b_2)$$

$$\text{i.e. } f^{-1}(b_1) = f^{-1}(b_2) \Rightarrow b_1 = b_2$$

$\therefore f^{-1}$ is one-one function.

As f is onto

\Rightarrow for each $b \in B$, $\exists a \in A$ s.t. $f(a) = b$

$$\Rightarrow a = f^{-1}(b) \text{ or } f^{-1}(b) = a$$

so for any $a \in A$, $\exists b \in B$ s.t. $f^{-1}(b) = a$
hence f^{-1} is onto.

Theorem 9 - Let $f: X \rightarrow Y$ be an invertible f . Then

$$1. \quad f^{-1}(A^c) = [f^{-1}(A)]^c \quad \text{for all } A \subseteq X$$

$$2. \quad \bigcup_{\alpha} f^{-1}(A_{\alpha}) = f^{-1}\left[\bigcup_{\alpha} A_{\alpha}\right], \quad \text{for every class } \{A_{\alpha} \subseteq X : \alpha \in I\}$$

$$3. \quad \bigcap_{\alpha} f^{-1}(A_{\alpha}) = f^{-1}\left(\bigcap_{\alpha} A_{\alpha}\right), \quad \text{for every class } \{A_{\alpha} \subseteq X : \alpha \in I\}$$

Ex:- Let $f: R \rightarrow R$ is defined by
 $f(x) = ax + b$ where $a, b, x \in R$ & $a \neq 0$
show that f is invertible and find the
Inverse of f .

Sol:-

A function is invertible iff it is one-one
and onto

Given $f: R \rightarrow R$ s.t $f(x) = ax + b$, $a, b, x \in R$, $a \neq 0$

—①

One-one: \rightarrow Let $x_1, x_2 \in R$

$$f(x_1) = f(x_2)$$

$$\Rightarrow ax_1 + b = ax_2 + b$$

$$\Rightarrow ax_1 = ax_2 \quad \because a \neq 0$$

$$\Rightarrow x_1 = x_2$$

$$\text{So } f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

Hence f is one-one

onto: \rightarrow $\forall y \in R$, \exists an element $x \in R$ such

that $f(x) = y$

$$\exists ax + b = y \quad \text{from (1)}$$

$$\Rightarrow ax = y - b$$

$$\Rightarrow x = \frac{y-b}{a} \in R$$

—②

$\therefore \forall y \in R \exists$ an element $x = \frac{y-b}{a} \in R$

such that $f(x) = f\left(\frac{y-b}{a}\right)$

$$= a\left(\frac{y-b}{a}\right) + b$$

$$= y$$

$$\Rightarrow f(x) = y$$

Hence f is onto function.

To find f^{-1} : - $x = f^{-1}(y)$
 $f^{-1}(y) = \frac{y-b}{a}$ from (1)

Hence $f^{-1}(y) = \frac{y-b}{a}$

Ques (2022)

Ques. Let a fn $f: R - \{3\} \rightarrow R - \{1\}$ be defined as
 $f(x) = \frac{x-1}{x-3}$. Show that f is bijective & find f^{-1} .

Sol:- To show that f is one-one,

let $x_1, x_2 \in R - \{3\}$ so that

$$f(x_1) = f(x_2)$$

$$\Rightarrow \frac{x_1-1}{x_1-3} = \frac{x_2-1}{x_2-3}$$

$$\Rightarrow x_1x_2 - 3x_1 - x_2 + 3 = x_1x_2 - x_1 - 3x_2 + 3$$

$$\Rightarrow -2x_1 = -2x_2 \Rightarrow x_1 = x_2$$

$$\text{So } f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

Therefore f is one-one.

$$\text{Now } y = f(x)$$

$$\Rightarrow y = \frac{x-1}{x-3}$$

$$\Leftrightarrow x = \frac{3y-1}{y-1}$$

So if $y \in R - \{1\}$, if $x = \frac{3y-1}{y-1} \in R - \{3\}$ so $f(x) = y$
 $\therefore f$ is onto.

Therefore, $g: R - \{1\} \rightarrow R - \{3\}$ given by

$g(y) = \frac{3y-1}{y-1}$ is a well defined bijective

function such that

$$fog = I_{R-13} \text{ and } gof = I_{R-3}$$

Hence g is the inverse of f .

#

Growth of Functions : →

The Growth of functions is described using three Important Notations :

- 1.) Big - oh (O)
- 2.) Big - omega (Ω)
- 3.) Big - theta (Θ)

They Provide a special way to compare relative size of functions that is very useful in analysis of computer algorithm.

Suppose M is an algorithm and n is the size of the Input data. Clearly the complexity $f(n)$ of M increases as n increases. If we know how the function $f(n)$ grows, we can determine the efficiency of the algorithm and this done by comparing $f(n)$ with some standard functions.

$$1 < \log n < \sqrt{n} < n < n \log n < n^2 < n^3 \dots < 2^n < 3^n \dots$$

Imp
①

Big - Oh Notation : →

Let $f(n)$ and $g(n)$ be two Real Valued functions, where the argument x takes a real value or it is non-negative integer. Then $f(x)$ is of order g i.e $f(x) = O(g(x))$ if there are positive constants C and K such that

$$|f(x)| \leq C |g(x)| \text{ , for all } x > K$$

The two constants C and K are called witness. Equivalently, if $\lim_{x \rightarrow \infty} [f(x)/g(x)] < \infty$.

Big-omega and Big-theta Notations / Big- Ω & Big- Θ

Big- Ω →

Let $f(n)$ and $g(n)$ be two functions from the set of integers or the set of real numbers to the set of real numbers. Then $f(n) = \Omega(g(n))$ if there are positive constants C and K such that $|f(n)| \geq C|g(n)|$ whenever $n > K$

which is read as " $f(n)$ is big-omega of $g(n)$ "

Big- Θ →

Let $f(n)$ and $g(n)$ be two functions from the set of integers or the set of real numbers to the set of real numbers. Then $f(n)$ is $\Theta(g(n))$ if $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$.

When $f(n)$ is $\Theta(g(n))$, we say that " f is big-theta of $g(n)$ " and denoted by $f(n) = \Theta(g(n))$

Note:-

If $f(n)$ is $O(g(n))$, then $g(n)$ is an upper bound for $f(n)$ and whenever $f(n)$ is $\Omega(g(n))$, then $g(n)$ is a lower bound for $f(n)$.

The Big-O Notation compares the rate of growth of functions rather than their values.

So when $f(n)$ is $\Theta(g(n))$, $f(n)$ and $g(n)$ have the same rates of growth, but can be very different in their values.

#

Boolean Algebra : →

A non-empty set B with two binary operations $+$ and \cdot , a unary operation $'$, and two distinct elements 0 and 1 is called a Boolean Algebra, denoted by $(B, +, \cdot, ', 0, 1)$ iff the following properties are satisfied,
for any $a, b, c \in B$

1. commutative laws : →

(a) $a+b = b+a$ (b) $a \cdot b = b \cdot a$.

2. Distributive laws : →

(a) $a+b \cdot c = (a+b) \cdot (a+c)$

(b) $a \cdot (b+c) = a \cdot b + a \cdot c$

3. Identity laws : →

(a) $a+0 = a$ (b) $a \cdot 1 = a$

4. Complement laws : →

(a) $a + a' = 1$ (b) $a \cdot a' = 0$.

Some Basic Theorems

Let $(B, +, \cdot, ', 0, 1)$ be a Boolean Algebra and $a, b, c \in B$ then

1. Idempotent laws :

(a) $a+a = a$ (b) $a \cdot a = a$.

2. Boundedness laws :

(a) $a+1 = 1$ (b) $a \cdot 0 = 0$

3. Absorption laws :

(a) $a + (a \cdot b) = a$ (b) $a \cdot (a+b) = a$

4. Associative laws :

(a) $(a+b)+c = a+(b+c)$

(b) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

5. Uniqueness of complement :-
 $a + x = 1$ and $a \cdot x = 0$, then $x = a'$

6. Involution law :-

$$(a')' = a$$

$$7. (a) 0' = 1 \quad (b) 1' = 0$$

8. De morgan's law :-

$$(a+b)' = a' \cdot b' \quad (b) (a \cdot b)' = a' + b'$$

Proofs :-

$$(1) (a) a + a = a$$

$$\begin{aligned} a &= a + 0 && \text{by Identity law} \\ &= a + a \cdot a' && \text{using Complement law} \\ &= (a+a) \cdot (a+a') && \text{using Distributive law} \\ &= (a+a) \cdot 1 && \text{using Complement law} \\ &= a+a && \text{Using Identity law.} \end{aligned}$$

$$(b) a \cdot a = a$$

$$\begin{aligned} a \cdot a &= a \cdot a + 0 && \text{by Identity law} \\ &= a \cdot a + a \cdot a' && \text{using complement law} \\ &= a \cdot (a+a') && \text{using distributive law} \\ &= a \cdot 1 && \text{using complement law} \\ &= a && \text{using Identity law.} \end{aligned}$$

$$(2) (a) a + 1 = 1$$

$$\begin{aligned} a + 1 &= a + (a+a') && \text{by complement law} \\ &= (a+a) + a' && \text{using Associative law} \\ &= a + a' && \text{using Idempotent law} \\ &= 1 && \text{using Identity law.} \end{aligned}$$

$$(b) a \cdot 0 = 0$$

$$\begin{aligned} \text{By } a \cdot 0 &= a \cdot (a \cdot a') = (a \cdot a) \cdot a' && \text{using complement} \\ &= a \cdot a' = 0 && \text{Associative, Idempotent laws} \end{aligned}$$

(3) (a) $a + a \cdot b = a$

$$\begin{aligned} a + (a \cdot b) &= a \cdot 1 + a \cdot b && \text{by Identity law} \\ &= a \cdot (1+b) && \text{using Distributive law} \\ &= a \cdot 1 && \text{using Boundedness law} \\ &= a && \text{using Identity law} \end{aligned}$$

(b) $a \cdot (a+b) = a$

$$\begin{aligned} a \cdot (a+b) &= a \cdot a + a \cdot b && \text{by distributive law} \\ &= a + a \cdot b && \text{by Idempotent law} \\ &= a && \text{from (a)} \end{aligned}$$

(c) $a \cdot (a+b) = a$

$$\begin{aligned} a \cdot (a+b) &= (a+0) \cdot (a+b) && \text{by Identity law} \\ &= a + 0 \cdot b && \text{by distributive law} \\ &= a + 0 && \text{by Boundedness law} \\ &= a && \text{by Identity law} \end{aligned}$$

Thm: In a Boolean Algebra B, the identity elements 0 and 1 are complementary to each other
OR

for $0, 1 \in B$, we have (i) $0' = 1$ (ii) $1' = 0$.

Proof:- (i) $0' = 1$

$$\begin{aligned} 0' &= 0' + 0 && [\text{by Identity law}] \\ &= 1 && [\begin{matrix} \text{by complement law} \\ a + a' = 1 \end{matrix}] \end{aligned}$$

(ii) $1' = 0$

$$\begin{aligned} 1' &= 1' \cdot 1 && [\text{by Identity law } a \cdot 1 = a] \\ &= 0 && [\text{by complement law } a \cdot a' = 0] \end{aligned}$$

"Faith is the bird that feels the light when the dawn is still dark." — Rabindranath Tagore

Theorem 5: For each $a \in B$, a' is unique.

Proof:-

Let if possible, a' and a'' be two complements of a .

Then $a + a' = a' + a = 1$ & $a \cdot a' = a' \cdot a = 0$
 $a + a'' = a'' + a = 1$ & $a \cdot a'' = a'' \cdot a = 0$

Now

$$\begin{aligned} a' &= a \cdot 1 && \text{(Identity law)} \\ &\equiv a' \cdot (a + a'') \\ &= a' \cdot a + a' \cdot a'' \\ &= 0 + a' \cdot a'' \\ &= a \cdot a'' + a' \cdot a' \\ &= (a + a') \cdot a'' \\ &= 1 \cdot a'' \\ &= a'' \end{aligned}$$

So $a' = a''$

Hence the complement a' of $a \in B$ is unique.

Theorem:- The elements 0 and 1 in a Boolean Algebra are unique.

Proof:- Suppose that there are two zero elements 0_1 and 0_2 . Then for each element $a \in B$ and $b \in B$, we have

$a + 0_1 = a$ and $b + 0_2 = b$ (Identity law)

Let $a = 0_2$ and $b = 0_1$,

Thus $0_2 + 0_1 = 0_2$ and $0_1 + 0_2 = 0_1$ — (1)

But $0_2 + 0_1 = 0_1 + 0_2$ (Commutative law)
 from (1)

$$\boxed{0_2 = 0_1}$$

thus the zero element is unique.

Similarly, let there are two identity elements 1_a and 1_b then for each $c \in B$ and $d \in B$ we have,

$$\text{Q. } c \cdot 1_a = c \text{ and } d \cdot 1_b = d \text{ (Identity law)}$$

$$\text{Let } c = 1_b \text{ and } d = 1_a$$

$$\text{Thus } 1_b \cdot 1_a = 1_b \text{ and } 1_a \cdot 1_b = 1_a \quad \text{--- (1)}$$

$$\text{But } 1_b \cdot 1_a = 1_a \cdot 1_b \quad (\text{commutative law})$$

from (1),

$$\boxed{1_b = 1_a}$$

Thus, the Identity element 1 is unique.

Theorem :- De-morgan's laws:

for any $a, b \in B, +, ., ', 0, 1$

$$(i) (a + b)' = a' \cdot b'$$

$$(ii) (a \cdot b)' = a' + b'.$$

Proof:-

Theorem :- 6: For each element a of Boolean Algebra B , $(a')' = a$.

Proof:- Let a' be the complement of $a \in B$
Then $a' \in B$

$$\Rightarrow a + a' = 1 \text{ and } a \cdot a' = 0 \quad \text{--- (1)}$$

$$\text{But } a + a' = a' + a \text{ and } a \cdot a' = a' \cdot a \quad (\text{comm. law})$$

$$\Rightarrow a' + a = 1 \text{ and } a' \cdot a = 0 \quad \text{from (1)} \quad \text{--- (2)}$$

egn (2) shows that complement of $a' \in B$ is $a \in B$
But complement of each element of Boolean algebra
is unique. Hence, $\boxed{(a')' = a}$

Theorem 1.8 De-Morgan's laws:

for any $a, b \in B$, we have

$$(i) (a+b)' = a' \cdot b'$$

$$(ii) (a \cdot b)' = a' + b'$$

Proof:

$$(i) \text{ To prove that } (a+b)' = a' \cdot b'$$

we have to show that the complement of $(a+b)$ is $a' \cdot b'$ and for that we have to show that

$$(a+b) + a' \cdot b' = 1 \text{ and } (a+b) \cdot (a' \cdot b') = 0$$

Now

$$(a+b) + a' \cdot b' = [(a+b) + a'] \cdot [(a+b) + b'] \quad (\text{by distributive law})$$

$$= [a' + (a+b)] \cdot [a + (b+b')] \quad \begin{matrix} \text{using commutation} \\ \text{of Association law} \end{matrix}$$

$$= [(a' + a) + b] \cdot [a + (b+b')] \quad "$$

$$= (1+b) \cdot (a+1) \quad \begin{matrix} \text{using complement law} \\ \text{using Boundedness law} \end{matrix}$$

$$= 1 \cdot 1$$

$$= 1 \quad \begin{matrix} \text{using Identity law.} \\ \text{using Identity law.} \end{matrix}$$

$$\text{and } (a+b) \cdot (a' \cdot b') = a \cdot (a' \cdot b') + b \cdot (a' \cdot b') \quad \begin{matrix} \text{by distributive law} \\ \text{using Association law} \end{matrix}$$

$$= (aa')b' + b(b'a') \quad \begin{matrix} \text{using Association law} \\ \text{using commutative law} \end{matrix}$$

$$= (aa')b' + (bb')a' \quad "$$

$$= 0 \cdot b' + 0 \cdot a' \quad \begin{matrix} \text{Complement law} \\ \text{using Boundedness law} \end{matrix}$$

$$= 0 + 0$$

$$= 0 \quad \begin{matrix} \text{Boundedness law} \\ \text{Idempotent law} \end{matrix}$$

Hence

$$(a+b)' = a' \cdot b'$$

(ii) To show that the complement of $(a \cdot b)$ is $a' + b'$
we have to show that

$$(a \cdot b) + (a' + b') = 1 \text{ and } (a \cdot b) \cdot (a' + b') = 0$$

$$\begin{aligned} \text{Now } (a \cdot b) + (a' + b') &= [a + (a' + b')] \cdot [b + (a' + b')] \\ &= [(a + a') + b'] \cdot [b + (b' + a')] \text{ by associative} \\ &\quad \& commutative law \\ &\equiv [(a + a') + b'] \cdot [(b + b') + a'] \text{ by -associative} \\ &\quad \text{law} \\ &= (1 + b') \cdot (1 + a') \text{ by complement law} \\ &= 1 \cdot 1 \text{ by Boundedness law} \\ &= 1 \text{ by Identity law.} \end{aligned}$$

$$\text{and } (a \cdot b) \cdot (a' + b') = (a \cdot b) \cdot a' + (a \cdot b) \cdot b' \\ \text{By distributive law.}$$

$$\begin{aligned} &= (b \cdot a) \cdot a' + (a \cdot b) \cdot b' \text{ by commutative law} \\ &= b(a \cdot a') + a \cdot (b \cdot b') \text{ by associative law} \\ &= b \cdot 0 + a \cdot 0 \text{ by complement law} \\ &= 0 + 0 \text{ by Boundedness law} \\ &= 0 \text{ by Idempotent law.} \end{aligned}$$

Hence $\boxed{(a \cdot b)' = a' + b'}$

Boolean SubAlgebra's -

Example-1 Let $B = \{1, 5, 7, 35\}$ be the set of integers and operations $+$ and \cdot are defined as

$$a+b = \text{lcm}(a, b)$$

$$\text{and } a \cdot b = \text{gcd}(a, b) \quad \forall a, b \in B$$

A unary operation ' $'$ ' on B is defined as

$$a' = 35/a \quad \forall a \in B$$

Show that $(B, +, \cdot, ')$ is a Boolean algebra.

Solution:-

Here $B = \{1, 5, 7, 35\}$.

The composition table for the binary operations $+$ and \cdot are given as

$+$	1	5	7	35	\cdot	1	5	7	35
1	1	5	7	35	1	1	1	1	1
5	5	5	35	35	5	1	5	1	5
7	7	35	7	35	7	1	1	7	7
35	35	35	35	35	35	1	5	7	35

Closure law!

(1) It is clear from the table that $+$ and \cdot are binary operations on B .
i.e $a+b, a \cdot b \in B, \forall a, b \in B$

(2) Commutative law: Since all rows and columns in both the composition tables are identical
Therefore, the commutative law hold for $+$ & \cdot .
i.e (i) $a+b = b+a$ & (ii) $a \cdot b = b \cdot a \quad \forall a, b \in B$

(3) Distributive law: - $\forall a, b, c \in B$

$$a + (b \cdot c) = (a+b) \cdot (a+c)$$

$$a \cdot (b+c) = a \cdot b + a \cdot c$$

Hence the distributive law holds.

for $a = 7, b = 5, c = 1$.

$$a + (b \cdot c) = 7 + (5 \cdot 1) = 7 + 1 = 7$$

$$a \cdot (b+c) = 7 \cdot (5+1) = 7 \cdot 5 =$$

$$(a+b) \cdot (a+c) = (7+5) \cdot (7+1) = 35 \cdot 7 =$$

$$\Rightarrow a + (b \cdot c) = (a+b) \cdot (a+c)$$

$$\text{and } a \cdot (b+c) = 7 \cdot (5+1) = 7 \cdot 5 = 1$$

$$(a \cdot b) + (a \cdot c) = (7 \cdot 5) + (7 \cdot 1) = 1 + 1 = 1$$

$$\Rightarrow a \cdot (b+c) = (a \cdot b) + (a \cdot c).$$

(4) Identity law: - here 1 and 35 are least & greatest element respectively

<u>a</u>	<u>1</u>	<u>a+1</u>	<u>a</u>	<u>35</u>	<u>a·35</u>
1	1	1	1	35	1
5	1	5	5	35	5
7	1	7	7	35	7
35	1	35	35	35	35

Since $a+1 = a$, $\forall a \in B$

& $a \cdot 35 = a$, $\forall a \in B$.

Hence identity law holds. Here 1 is the zero element and 35 is the unit element.

(5) Complement law: -

$$\text{given } a' = 35/a \quad \forall a \in B$$

a	a'	$a+a'$	$a \cdot a'$	$\therefore a+a' = 35 \quad \forall a \in B$
1	35	35	1	$\& a \cdot a' = 1 \quad \forall a \in B$
5	7	35	1	
7	5	35	1	
35	1	35	1	

Hence complement law holds

$\therefore (B, +, \cdot, ')$ is a Boolean Algebra.

~~Ans~~

Example: 2 Show that the set $B = \{1, 2, 3, 6\} = D_6$ is a Boolean algebra with respect to binary operations $+$ and \cdot and unary operation $'$ respectively given by

$$a+b = \text{lcm}\{a, b\}$$

$$a \cdot b = \text{hcf}\{a, b\}$$

$$a' = 6/a \quad \forall a, b \in B.$$

Solutn—

Here $B = \{1, 2, 3, 6\}$

The composition table for the binary operation $+$ and \cdot are as follows

$+$	1	2	3	6	\cdot	1	2	3	6
1	1	2	3	6	1	1	1	1	1
2	2	2	6	6	2	1	2	1	2
3	3	6	3	6	3	1	1	3	3
6	6	6	6	6	6	1	2	3	6

(1) closure law:—Since all the entries are from the set B in both tables i.e. $a+b, a \cdot b \in B$, $\forall a, b \in B$.

Therefore, closure law holds.

(2) Commutative law :- Since all the corresponding rows and columns are identical in both composition tables
i.e. $a+b = b+a$ & $a \cdot b = b \cdot a \quad \forall a, b \in B$.

\Rightarrow commutative law holds.

(3) Distributive law :- $\forall a, b \in B$

$$a + (b \cdot c) = (a + b) \cdot (a + c)$$

$$\text{&} \quad a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

Hence distributive law holds.

for $a = 3, b = 2$ & $c = 1$, we have

$$a + (b \cdot c) = 3 + (2 \cdot 1) = 3 + 1 = 3$$

$$(a + b) \cdot (a + c) = (3 + 2) \cdot (3 + 1) = 5 \cdot 3 = 3$$

$$\Rightarrow a + (b \cdot c) = (a + b) \cdot (a + c)$$

$$\text{And } a \cdot (b + c) = 3 \cdot (2 + 1) = 3 \cdot 2 = 1$$

$$(a \cdot b) + (a \cdot c) = 3 \cdot 2 + 3 \cdot 1 = 1 + 1 = 1$$

$$\Rightarrow a \cdot (b + c) = (a \cdot b) + (a \cdot c).$$

(4) Identity law :- Here 1 and 6 are zero and identity elements respectively.

(Least) (Greatest)

a	1	$a+1$	a	6	$a \cdot 6$
1	1	1	1	6	1
2	1	2	2	6	2
3	1	3	3	6	3
6	1	6	6	6	6

$$\text{Since } a+1 = a$$

$$\text{&} \quad a \cdot 6 = a \quad \forall a \in B$$

Hence the Identity law holds

"Faith is the bird that feels the light when the dawn is still dark." — Rabindranath Tagore

Date: _____

(S.) complement law :-
we have $a' = 6/a \quad \forall a \in B$

a	a'	$a+a'$	$a \cdot a'$
1	6	6	1
2	3	6	1
3	2	6	1
6	1	6	1

Since $a+a' = 6$
and $a \cdot a' = 1 \quad \forall a \in B$

\Rightarrow complement law holds

Hence $(B = \{1, 2, 3, 6\}, +, \cdot, ', 1)$ is a Boolean Algebra.

Ex:-3: Show that the set $B = \{1, 2, 4, 8\}$ is not a Boolean algebra with respect to the binary operations $+$, \cdot and unary operation ' $'$ respectively given by

$$a+b = \text{lcm}\{a, b\}$$

$$a \cdot b = \text{hcf}\{a, b\}$$

$$\text{and } a' = 8/a$$

Solution:- Here $B = \{1, 2, 4, 8\}$.

The composition tables for the binary operations $+$ and \cdot are as

$+$	1	2	4	8	\cdot	1	2	4	8
1	1	2	4	8	1	1	1	1	1
2	2	2	4	8	2	1	2	2	2
4	4	4	4	8	4	1	2	4	4
8	8	8	8	8	8	1	2	4	8

(1) closure law:- since all the entries in both composition tables are coming from B
 i.e. $a+b, a \cdot b \in B$ $\forall a, b \in B$
 hence closure law holds.

(2) commutative law:- Since all corresponding rows & columns are identical in both the tables
 Hence commutative law holds
 i.e. $a+b = b+a$ & $a \cdot b = b \cdot a, \forall a, b \in B$

(3) Distributive law:- for $a=4, b=2, c=1$
 $a+(b \cdot c) = 4 + (2 \cdot 1) = 4 + 1 = 4$
 $(a+b) \cdot (a+c) = (4+2) \cdot (4+1) = 4 \cdot 4 = 4$
 for $a=8, b=4, c=2$.
 $a+(b \cdot c) = 8 + (4 \cdot 2) = 8 + 2 = 8$
 $(a+b) \cdot (a+c) = (8+4) \cdot (8+2) = 8 \cdot 8 = 8$
 for $a=8, b=2, c=1$
 $8 + (2 \cdot 1) = 8 + 1 = 8$
 $(8+2) \cdot (8+1) = 8 \cdot 8 = 8$
 $\therefore a+(b \cdot c) = (a+b) \cdot (a+c) \quad \forall a, b, c \in B$
 Similarly $a \cdot (b+c) = (a \cdot b) + (a \cdot c) \quad \forall a, b, c \in B$
 hence Distributive law holds.

(4.) Identity law:- here 1 and 8 are zero (least) and unit (greatest) element

<u>a</u>	<u>1</u>	<u>$a+1$</u>	<u>a</u>	<u>8</u>	<u>$a \cdot 8$</u>
1	1	1	1	8	1
2	1	2	2	8	2
4	1	4	4	8	4
8	1	8	8	8	8

Since $a + 1 = a$
 $\& a \cdot 0 = a$ $\forall a \in B$

Hence Identity law holds.

(8.) Commutative law :-

Given $a' = 8/a$ ~~for all a~~

a	a'	$a+a'$	$a \cdot a'$	
1	8	8	1	$\therefore 2+4=4 \neq 8$
2	4	8	1	$\& 4+2=4 \neq 8$
4	2	8	1	and $2 \cdot 4=2 \neq 1$
8	1	8	1	$4 \cdot 2=2 \neq 1$

Since \nexists any $a \in B$ s.t

$$2+2=8 \& 2 \cdot 2=1$$

Similarly \nexists any $b \in B$ s.t

$$4+4=8 \text{ and } 4 \cdot 4=1$$

i.e The complements of 2 and 4 does not exist

Therefore, The complement law does not hold

Hence $(B = \{1, 2, 4, 8\}, +, \cdot, ', 0)$ is not a Boolean Algebra.

#

Boolean SubAlgebra : →

A non-empty subset A of a Boolean algebra $(B, +, \cdot, ', 0, 1)$ is called a Boolean Subalgebra if A is a Boolean Algebra under the same set of binary and unary operations i.e. $a, b \in A \Rightarrow a+b, a \cdot b, a' \in A$.

Ex:- consider the set of all the divisors of 30

i.e. $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$ and let $+ \& \cdot \& '$ be defined as:

$$a+b = \text{lcm}(a, b)$$

$$a \cdot b = \text{gcd}(a, b) \text{ and } a' = 30/a.$$

Then, D_{30} is a Boolean Algebra

and $A_1 = \{1, 5, 6, 30\}$, $A_2 = \{1, 3, 10, 30\}$

& $A_3 = \{1, 5, 6, 30\}$ are Boolean SubAlgebras of D_{30} .

Sol:- $A_1 = \{1, 5, 6, 30\}$.

The composition table for binary operatⁿ $+$ & \cdot .
 & unary operation ' are as follows

$+$	1	5	6	30	.	1	5	6	30
1	1	5	6	30	1	1	1	1	1
5	5	5	30	30	5	1	5	1	5
6	6	30	6	30	6	1	1	6	6
30	30	30	30	30	30	1	5	6	30

a a' Since $a+b, a \cdot b, a' \in A$,

if $a, b \in A$,

Hence A_1 is a Subalgebra

of B .

30 1

Date _____

$$A_2 = \{1, 3, 10, 30\}$$

$+$	1	3	10	30	.	1	3	10	30	<u>a</u>	<u>a'</u>
1	1	3	10	30	.	1	1	1	1	1	30
3	3	3	30	30	.	3	1	3	13	3	10
10	10	30	10	30	.	10	1	1	10	10	3
30	30	30	30	30	.	30	1	3	10	30	1

$$\because a+b, a \cdot b \notin A_2 \quad \forall a, b \in A_2$$

Hence A_2 is a subalgebra of B .

$$A_3 = \{1, 5, 6, 30\}$$

$+$	1	5	6	30	.	1	5	6	30	<u>a</u>	<u>a'</u>
1	1	5	6	30	.	1	1	1	1	1	30
5	5	5	30	30	.	5	1	5	15	5	6
6	6	30	6	30	.	6	1	1	6	6	5
30	30	30	30	30	.	30	1	5	6	30	1

Since $a+b, a \cdot b, a' \in A_3, \forall a, b \in A_3$
Hence A_3 is a subalgebra of B .

(#)

Boolean Function or Boolean Polynomial :-
 An expression obtained by the application of binary operations (+, ·) and unary operation ('') on finite no. of elements of Boolean Algebra ($B, +, \cdot, '$) is called a Boolean function or Boolean ~~Algebra~~ Polynomial.

Example :-

$$(1) f(n) = x + x'$$

$$(2) f(n, y) = n \cdot y' + x' \cdot y$$

$$(3) f(n, y, z) = x \cdot y \cdot z + x \cdot y' \cdot z + x' \cdot y \cdot z$$

(*) Minterm :-

A minterm of n variables x_1, x_2, \dots, x_n is a product of n literals in which each variable appears exactly once in either true or complemented form (i.e x_i or x_i'), but not both.

Ceg -

The minterms of two variables x and y are $xy, x'y, xy', x'y'$.

The minterms of three variables x, y, z are $xyz, x'yz, xy'z, xyz', x'y'z, xy'z', x'yz', x'yz'$.

Note :- (1) The No. of minterms in n -variables are 2^n .
 (2) The output value of a minterm is 1.



Maxterm : \rightarrow

A maxterm of n -variables x_1, x_2, \dots, x_n is sum of n - literals in which each variable appears exactly once in either true or complemented form, but not both.

e.g. \rightarrow

The maxterms of two variables x and y are $x+y, x'+y, x+y', x'+y'$.

The maxterms of three variables x, y, z are $x+y+z, x'+y+z, x+y'+z, x+y+z', x'+y'+z, x'+y+z', x+y'+z'$.

Note:-

- (1) The No. of maxterms in n -variables are 2^n .
- (2) The output of maxterm is always 1.



Disjunctive Normal form (DNF) or Sum of Product (SOP) : \rightarrow

A Boolean function of which is written as a sum of minterms is called disjunctive normal form or DNF form or SOP or Canonical form.

Example :-

$$(1) f(x, y) = xy + x'y$$

$$(2) f(x, y, z) = x'y'z + x'y'z' + xy'z' + xyz$$

\Rightarrow Complete Disjunctive Normal form : \rightarrow

A Boolean function of which is written as a sum of all 2^n minterms of n -variables is called complete disjunctive normal form.

$$\text{Log} - f(x, y) = xy + x'y' + x'y + x'y'.$$

Example: Express the following function into DN form

$$f(x, y) = x + x'y$$

Sol:- No. of minterms in 2 variables = $2^2 = 4$

x	y	x'	$x' \cdot y$	$x + x' \cdot y$	
1	1	0	0	1	
1	0	0	0	1	$1 \rightarrow x \text{ or } y$
0	1	1	1	1	$0 \rightarrow x' \text{ or } y'$
0	0	1	0	0	

Truth Table
Method:
Minterms present - $xy, x'y', x'y$
Therefore the required DN form or SOP is

$$f(x, y) = m_1 + m_2 + m_3 \\ = xy + x'y' + x'y.$$

By Algebraic method:

$$f(x, y) = x + x'y$$

$$f(x, y) = x + x'y \\ = x \cdot 1 + x'y \quad (\because a \cdot 1 = a \text{ identity law}) \\ = x \cdot (y + y') + x'y \quad (\because a + a' = 1) \\ = xy + x \cdot y' + x'y \quad [\text{Distributive law } a \cdot (b + c) = a \cdot b + a \cdot c]$$

This is the Required DN form.

Date _____

Ques.

Express the Boolean function

 $f(x, y, z) = x + y'z$ in DNF form or
Sum of minterms

Soln. -
$$\begin{aligned} f(x, y, z) &= x + y'z \quad (\because a + a' = 1) \\ &= x \cdot (y + y') + y'z \\ &= xy + xy' + y'z \\ &= xy(z + z') + xy'(z + z') + y'z(x + x') \\ &= xyz + xyz' + xy'z + xy'z' + \cancel{xy'z + xy'z'} \quad (\because a + a = a) \\ &= xyz + xyz' + xy'z + xy'z' + x'y'z \end{aligned}$$

this is the required DNF form.

Q.2. find the sum of product of the Boolean
 $f^n f(x, y, z) = (x + y)z'$.

Soln:-

$$\begin{aligned} f(x, y, z) &= (x + y)z' \\ &= xz' + yz' \\ &= x(y + y')z + (x + x')yz' \\ &= xyz + xyz' + xy'z + x'y'z \end{aligned}$$

using truth table,

x	y	z	$(x+y)$	z'	$f = (x+y) \cdot z'$	for minterm - 1
1	1	1	1	0	0	& maxterm - 0
1	1	0	1	1	1	DNF form
1	0	1	1	0	0	$f(x, y, z) = xyz' + xy'z' + x'y'z$
1	0	0	1	1	1	
0	1	1	1	0	0	CNF form
0	1	0	1	1	1	$f(x, y, z) = (x+y+z)(x+y'+z)$
0	0	1	0	0	0	$(x'+y+z)(x'+y'+z)(x'+y'+z')$
0	0	0	1	0	0	

Ques.

Obtain the DNF associated with the Boolean expressions given by

$f(x, y, z) = (yz + xz')(xy' + z')'$, by using truth table, and also by using equivalences.

Proof:- the Given Boolean expression is

$$f(x, y, z) = (yz + xz')(xy' + z')'$$

x	y	z	z'	y'	xy'	$xy' + z'$	$(xy' + z')'$	yz	xz	$y + xz$	$(xy' + z')'(yz + xz')$
1	1	1	0	0	0	0	1	1	0	1	1
1	1	0	1	0	0	1	0	0	1	1	0
1	0	1	0	1	1	1	0	0	0	0	0
1	0	0	1	1	1	1	0	0	1	1	0
0	1	1	0	0	0	0	1	1	0	1	1
0	1	0	1	0	0	1	0	0	0	0	0
0	0	1	0	1	0	0	1	0	0	0	0
0	0	0	1	1	0	1	0	0	0	0	0

for minterms - 1

The Required DNF form is

$$f(x, y, z) = xyz + x'y'z$$

using Algebraic method or equivalences

$$\begin{aligned}
 f(x, y, z) &= (yz + xz')(xy' + z')' \\
 &= (yz + xz')[((xy')'(z')')] \quad \text{By De-morgan's law} \\
 &= (yz + xz')[((x' + y) \cdot z)] \quad ' + (a')' = a \\
 &= (yz + xz')(x'z + yz) \quad \text{Distributive law} \\
 &= (yz + xz')(yz + x'z) \quad \text{commutative law} \\
 &= yz + (xz')(x'z) \quad \text{Distributive law} \\
 &= yz + (xx')(zz') \quad \text{Associative law}
 \end{aligned}$$

"I sincerely believe that Indians have the ability to compete with the best in the world." —Dhirubhai Ambani

$$\begin{aligned}
 &= yz + 0 \cdot 0 \quad (\because a \cdot a' = 0) \\
 &= 1 \cdot yz \\
 &= (x + x')yz \\
 &= xyz + x'y'yz
 \end{aligned}$$

Hence

$f(x, y, z) = xyz + x'y'yz$ is in the DNF form.

$$(xy' + z')(yz + xz')$$

Ques. Obtain the DNF associated with the Boolean expressions given by
 $g(x_1, x_2, x_3) = x_1 x_2' + x_3$, by using truth table, and also by using equivalences.

Sol:- using Truth Table →

The Given Boolean expression

$$g(x_1, x_2, x_3) = x_1 x_2' + x_3$$

x_1	x_2	x_3	x_2'	$x_1 x_2'$	$x_1 x_2' + x_3$
1	1	1	0	0	1
1	1	0	0	0	0
1	0	1	1	1	1
1	0	0	1	1	1
0	1	1	0	0	1
0	1	0	0	0	0
0	0	1	1	0	1
0	0	0	1	0	0

DNF →

$$\begin{aligned}
 g(x_1, x_2, x_3) &= x_1 x_2 x_3 + x_1 x_2' x_3 + x_1 x_2 x_3' + x_1' x_2 x_3 \\
 &\quad + x_1' x_2' x_3
 \end{aligned}$$

using equivalence laws \rightarrow

$$g(x_1, x_2, x_3) = x_1 x_2' + x_3$$

$$= x_1 x_2' (x_3 + x_3') + (x_1 + x_1') (x_2 + x_2') x_3$$

$$= x_1 x_2' x_3 + x_1 x_2' x_3' + (x_1 + x_1') (x_2 x_3 + x_2' x_3)$$

$$= \underline{x_1 x_2' x_3} + x_1 x_2' x_3' + x_1 x_2 x_3 + x_1' x_2 x_3 +$$

$$\underline{x_1 x_2' x_3 + x_1' x_2' x_3} \quad (\because x+x=0)$$

$$= x_1 x_2 x_3 + x_1 x_2' x_3 + x_1' x_2 x_3 + x_1 x_2' x_3' + x_1' x_2' x_3$$

Conjunctive Normal form (CNF)

OR Canonical Product of Sums (POS)

When a Boolean function f is written as a product of maxterm, it is referred as a Conjunctive Normal form or POS.

Examples:-

$$(1) f(x, y) = (x+y)(x'+y)$$

$$(2) f(x, y, z) = (x+y+z)(x+y'+z)(x'+y+z) \\ (x'+y'+z)$$

Complete Conjunctive Normal form: \rightarrow

A Boolean function when expressed as a product of all 2^n maxterms of n -variables is called the complete conjunctive Normal form.

Ex:- $f(x, y) = (x+y)(x'+y)(x+y')(x'+y')$,
is in complete conjunctive normal form.

Ques 1: Express the Boolean fn $f(a, b, c) = ab + a'c$ as a product of maxterms.

Soln:-

$$\begin{aligned}
 f(a, b, c) &= ab + a'c \\
 &= (ab + a')(ab + c) \quad [\because x + yz = (x+y)(x+z)] \\
 &= (a + a')(b + a')(a + c) \cancel{(b + c)} \\
 &= (b + a') + (a + c)(b + c) \quad [\because a + a' = 1] \\
 &= (a' + b + c \cdot c') + (a + c + b \cdot b') \cancel{(b + c + a \cdot a')} \\
 &= (a' + b + c)(a' + b + c') + (a + c + b)(a + c + b') \\
 &\quad (b + c + a) \cdot (b + c + a') \\
 &\text{By distributive law} \\
 &= (a + b + c)(a' + b + c)(a + b' + c)(a' + b + c')
 \end{aligned}$$

Ques 2: Obtain the CNF associated with the Boolean expressions given by $f(x, y, z) = (x + y) \cdot z$ by using truth table & Also by using equivalence Boolean fn

$$f(x, y, z) = (x + y) \cdot z.$$

x	y	z	$x + y$	$f = (x + y) \cdot z$
1	1	1	1	1
1	1	0	1	0
1	0	1	1	1
1	0	0	1	0
0 - x or y or z	0	1	1	1
1 - x or y or z	0	1	0	0
2)	0	0	1	0
0	0	0	0	0

Since the maxterms are represented by 0 in Boolean fn
 $f(x, y, z) = (x + y) \cdot z$
 Therefore. The CN form is

$$\begin{aligned}
 f(x, y, z) &= (x' + y' + z)(x' + y + z)(x + y' + z)(x + y + z) \\
 &\quad (x + y' + z)
 \end{aligned}$$

Ques 3: obtain the CNF associated with the Boolean expressions given by $g(x_1, x_2, x_3) = (x_1 x_3 + x_1 x_3')(x_1 x_2 + x_3)'$, by using Truth table and also by using equivalences.

Sol:-

Given Boolean fⁿ $g(x_1, x_2, x_3) = (x_1 x_3 + x_1 x_3')(x_1 x_2 + x_3)'$

x_1	x_2	x_3	$x_3' + x_2'$	x_2'	x_3'	$x_1 x_3'$	$x_1 x_2'$	$x_1 x_3$	$x_1 x_3 + x_1 x_3'$	$x_1 x_2 + x_3$
1	1	1	0	0	0	0	0	1	1	1
1	1	0	0	1	1	1	1	0	1	1
1	0	1	1	0	0	0	0	1	1	1
1	0	0	1	1	1	1	1	0	1	1
0	1	1	0	0	0	0	0	0	0	1
0	1	0	0	1	0	0	0	0	0	0
0	0	1	1	0	0	0	0	0	0	1
0	0	1	1	1	0	0	0	0	0	0

$$(x_1 x_2' + x_3)' (x_1 x_3 + x_1 x_3') (x_1 x_2' + x_3)'$$

0	0
0	0
0	0
0	0
0	0
1	0
0	0
1	0

for maxterm - 0

Hence The CNF of g is as follows

$$\begin{aligned} g(x_1, x_2, x_3) &= (x_1 + x_2 + x_3) (x_1' + x_2 + x_3') (x_1 + x_2' + x_3) \\ &\quad (x_1 + x_2' + x_3') (x_1' + x_2 + x_3) (x_1' + x_2 + x_3') \\ &\quad (x_1' + x_2' + x_3) (x_1' + x_2' + x_3') \end{aligned}$$