Module III

SETS AND COMBINATIONS

Chapter 4

Set Theory

4.1 INTRODUCTION

Set theory, one of the most basic concept in Mathematics, was first formulated by German Mathematician Georg Cantor in later part of 19th century. Set theory and logic are closely related. The fundamental connectors like 'AND', 'OR' and 'NOT' in logic can be related to intersection, union and complement respectively in Set theory.

Set theory is widely used in other fields of Mathematics such as Real analysis, Measure theory, Number theory, Information theory and Artificial intelligence.

4.2 SET

A set is a collection of well defined objects, called *elements* or *members of the set.* These elements may be anything like numbers, letters of alphabets, points etc. Sets are denoted by capital letters and their elements by lower case letters. If an object x is an element of set A, we write it as $x \in A$ and read it as 'x belongs to A' otherwise $x \notin A$ (x does not belong to A). Following are some examples of sets :

1. Set of vowels in English.
2. Set of integers.
3. Set of days in a week.
4. Set of positive divisors of 20.

4.3 REPRESENTATION OF A SET

There are two ways of representing a set :

1. Tabular or Roster form
2. Set Builder or Rule method form

1. **Tabular or Roster form (Listing method) :** A set can be described by listing its members or elements being separated by commas and enclosed within braces :

   For example,

(i) $A = \{a, e, i, o, u\}$

(ii) $B = \{1, 3, 5, 7, 9, 11\}$

(iii) $C =$ {Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}

(iv) $D = \{1, 8, 27, 64, 125\}$

2. Set Builder or Rule method form : A set can be described by specifying property that elements of set have in common. Its general form is $\{x \mid P(x)\}$ where $P(x)$ indicates the property object x has '$\cdot$' is also used in place of '|' (vertical bar). It can be read as set of all object x such that x has the property $P(x)$.

For example :

(i) $A = \{x \mid x \text{ is a vowel in English alphabet}\}$

(ii) $B = \{x \mid x \text{ is an odd integer and } 1 \leq x \leq 11\}$

(iii) $C = \{x \mid x \text{ is days of a week}\}$

(iv) $D = \{x \mid x = n^3, \text{ where } n \text{ is a natural number less than equal to } 5\}$

4.4 TYPES OF SET

1. Finite set : A set with finite or countable number of elements is called *finite set.*

For example :

(i) $A = \{x \mid x \text{ is student in class V}\}$

(ii) $B = \{1, 2, 3, 4, 5\}$

(iii) $C = \{x \mid x \text{ is prime divisor of } 6\}$

(iv) $D = \{x \mid x \text{ is a vowel in English alphabet}\}$

2. Infinite set : A set with infinite number of elements is called *infinite set.*

For example :

(i) Z = Set of integers

(ii) R = Set of real numbers

(iii) A = Set of rational numbers between two real numbers

(iv) $B = \left\{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \ldots\ldots\ldots\right\}$

3. Null set : A set which contains no element at all is called a *null set.* It is denoted by ϕ or $\{\,\}$. It is also called *empty* or *void set.*

For example :

(i) $A = \{x \mid x^2 + 9 = 0, x \text{ is real}\}$

(ii) $B = \{x \mid x \text{ is odd and } x \text{ is even number}\}$

4. Singleton set : A set which has only one element is called singleton set.

For example :

$$A = \{a\}$$

5. Subset : Let A and B be two sets, if every element of A also belongs to B i.e., if every element of set A is also an element of set B, then A is called *subset of B* and it is denoted by $A \subseteq B$.

Symbolically, $A \subseteq B$ if $x \in A \Rightarrow x \in B$.

Note :

(a) Every set A is a subset of itself i.e., $A \subseteq A$.

(b) ϕ is subset of every set.

(c) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

(d) If a set contain n elements, then it has total 2^n subsets.

Example : Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6\}$ then all elements of set A are also elements of set B. Therefore, we can say $A \subseteq B$.

6. Superset : If A is subset of a set B, then B is called superset of A.

For example :

Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6\}$, then B is superset of A.

7. Proper subset : Any subset A is said to be proper subset of another set B, if there is atleast one element of B which does not belong to A, i.e., if $A \subseteq B$ but $A \neq B$. It is denoted by $A \subset B$.

For example :

(i) $A = \{1, 2, 3, 4, 5\}$

(ii) $B = \{1, 2, 3, 4, 5, 6\}$

(ii) $C = \{1, 2, 3, 4, 5, 6\}$

Here A and B are subsets of C but B is not proper subset of C.

8. Universal set : In many applications of sets, all the sets under consideration are considered as subsets of one particular set. This set is called *universal set* and is denoted by U.

For example :

$$A = \{x \mid x \text{ is an isosceles triangle}\}$$
$$B = \{x \mid x \text{ is an equilateral triangle}\}$$

then $U = \{x \mid x \text{ is a triangle}\}$ is universal set.

9. Equal sets : Two sets A and B are said to be equal if every element of A belongs to set B and symbolically : $A = B$ if $x \in A$ and $x \in B$.

Note : If $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$.

10. Comparable sets : Two sets A and B are said to be comparable if either of the following happens :

(a) $A \subset B$

(b) $B \subset C$

(c) $A = B$

11. Disjoint set : Let A and B be two sets if there is no common element between A and B, then they are said to be disjoint. [UPTU, B.Tech. 2007-08]

For example : $A = \{1, 2, 3\}$, $B = \{c, d, f\}$ are disjoint sets.

12. Power set : The collection of all subsets of a finite set A is called the *power set of A* and is denoted by $P(A)$.

Note : If A has n elements, then $P(A)$ has 2^n elements.

For example : $A = \{3, 4\}$

then $P(A) = \{\phi, \{3\}, \{4\}, \{3, 4\}\}$

No. of elements in $A = 2$

No. of elements in $P(A) = 2^2 = 4$

4.5 OPERATION ON SETS

(1) Union : Let A and B be two sets, then the union of sets A and B is a set that contain those elements that are either in A or B or in both. It is denoted by $A \cup B$ and is read as 'A union B'.

Symbolically : $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

For example : $A = \{1, 2, 3, 4\}$

$B = \{3, 4, 5, 6\}$

$A \cup B = \{1, 2, 3, 4, 5, 6\}$

(2) **Intersection :** Let A and B be two sets, then intersection of A and B is a set that contain those elements which are common to both A and B. It is denoted by $A \cap B$ and is read as 'A intersection B'.

Symbolically : $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

For example : $A = \{1, 2, 3, 4\}$

$B = \{2, 4, 5, 6\}$

then $A \cap B = \{2, 4\}$

(3) **Complement :** Let U be the universal set and A be any subset of U, then complement of A is a set containing elements of U which do not belong to A. It is denoted by A^C or A^1 or $\overline{A}$.

Symbolically : $A^C = \{x \mid x \in U \text{ and } x \notin A\}$

For example : $U = \{1, 2, 3, 4, 5, 6\}$

and $A = \{2, 3, 5\}$

then $A^C = \{1, 4, 6\}$

(4) **Difference of sets :** Let A and B be two sets. Then difference of A and B is a set of all those elements which belong to A but donot belong to B and is denoted by $A - B$.

Symbolically : $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

For example : Let $A = \{2, 3, 4, 5, 6, 7\}$

and $B = \{4, 5, 7\}$

then $A - B = \{2, 3, 6\}$

(5) **Symmetric difference of set :** Let A and B be two sets. Then symmetric difference of A and B is a set containing all the elements that belongs to A or B but not both. It is denoted by $A \oplus B$ or $A \Delta B$. [UPTU, B.Tech. 2007-08]

Also, $A \oplus B = (A \cup B) - (A \cap B)$

For example : Let $A = \{2, 3, 4, 6\}$

$B = \{1, 2, 5, 6\}$

then $A \oplus B = \{1, 3, 4, 5\}$

Remark : $A \oplus A = \phi$

4.6 CARDINALITY

The total number of elements in a finite set is called *cardinality* of that set. But for infinite set the concept of one-to-one correspondence between the element of two sets is taken into consideration. The sets like N, Z, Q and R seems to have infinitely many elements and we could say that their cardinality is "infinity". But later it was discovered that in fact not all the infinite sets have the same cardinality.

4.6.1 Countable Set

A set S is called *countable* if there exist an one-to-one function $\phi : S \to N$, i.e., ϕ is a function from set S to set a natural numbers N. If ϕ is onto also thus making ϕ bijective, then S is called *countably infinite*.

4.6.2 Uncountable Set

A set is uncountable if its cardinality is larger than that of natural numbers. A set S is uncountable if and only if any of the following conditions hold true :

1. There is no one-to-one function from S to set of natural number N.
2. The set S has cardinality strictly greater than cardinality of Natural numbers.

Example 4.1 : *Prove that union of two countably infinite sets is countably infinite.*

[UPTU, B.Tech. 2009-10]

Solution : Let us consider A_i be a countably infinite set and $A = \bigcup_{i=1}^{\infty} A_i$.

Now we will prove that A is countably infinite. Let the elements of A_i are given as below :

$$\begin{array}{ll} A_1 : & a_{11}, a_{12}, a_{13}, \ldots\ldots, a_{1n}, \ldots\ldots \\ A_2 : & a_{21}, a_{22}, a_{23}, \ldots\ldots, a_{2n}, \ldots\ldots \\ A_3 : & a_{31}, a_{32}, a_{33}, \ldots\ldots, a_{3n}, \ldots\ldots \\ \vdots & \\ A_n : & a_{n1}, a_{n2}, a_{n3}, \ldots\ldots, a_{nn}, \ldots\ldots \end{array}$$

Let us assumed that $A_i \cap A_j = \phi$ for $i \neq j$. Now we divide the elements of A into blocks such that k^{th} block contain k elements and a_{ij} will be in k^{th} block iff $i + j = k + 1$. Also in k^{th} block the second suffix j of a_{ij} varies from 1 to k. In this way A can be expressed in the following pattern.

$$A = (a_{11}, a_{21}, a_{12}, a_{31}, a_{22}, a_{13}, \ldots, a_{n1}, \ldots, a_{i1n}, \ldots) \quad \ldots (1)$$

The equation (1) show A can be expressed in a sequence of distinct numbers and hence $|A| = a$.

Now consider when $A_i \cap A_j \neq \phi$ for $i \neq j$. In this case A is equivalent to some subset of N i.e.,

$$|A| \leq |N|$$

or $$|A| \leq a$$

But $$A_1 \subset A$$

$\Rightarrow$ $$|A_1| \leq a \quad \ldots (2)$$

so that $$a = |A_1| \leq |A|$$

$\Rightarrow$ $$a \leq |A| \quad \ldots (3)$$

From equations (2) and (3)

$$|A| = a$$

Hence proved.

4.7 MULTISET

When we specify a set by its elements it doesnot make any difference how many times a particular element is repeated. However, sometimes the frequency with which a particular element appears in a set is important e.g., when we consider the marks of students obtained in some examination we like to count each figure as many times as the number of students who scored it. The mathematical concept in this situation is a multiset.

Definition : Multisets are sets where an element appear more than once.

e.g.,

$$A = \{1, 1, 1, 2, 2, 3\}$$
$$B = \{a, a, a, b, b, b, b, c, c\}$$

are multisets.

The multisets A and B can also be written as

$$A = \{3 \cdot 1, 2 \cdot 2, 1 \cdot 3\} \text{ and } B = \{3 \cdot a, 4 \cdot b, 2 \cdot c\}$$

The multiplicity of an element in a multiset is defined to be number of times an element appears in the multiset. In above examples, multiplicities of the elements 1, 2, 3 in multiset A are 3, 2, 1 respectively. The multiplicities of the elements a, b, c in the multiset B are 3, 4, 2 respectively. The cardinality of a multiset is defined as the cardinality of the set it corresponds to, assuming that the elements in the multiset are all distinct.

Let A and B be two multisets. Then, $A \cup B$, is the multiset where the multiplicity of an element is the maximum of its multiplicities in A and B. The intersection of A and B, $A \cap B$, is the multiset where the multiplicity of an element is the minimum of its multiplicities in A and B.

The difference of A and B, $A - B$, is the multiset where the multiplicity of an element is equal to the multiplicity of the element in A minus the multiplicity of the element in B if the difference is positive, and is equal to zero if the difference is zero and negative.

The sum of A and B, $A + B$, is the multiset where the multiplicity of an element is the sum of multiplicities of the elements in A and B.

Example 4.2 : *Let P and Q be two multisets $\{4 \cdot a, 3 \cdot b, 1 \cdot c\}$ and $\{3 \cdot a, 3 \cdot b, 2 \cdot d\}$ respectively. Find (i) $P \cup Q$ (ii) $P \cap Q$ (iii) $P - Q$ (iv) $P + Q$.*

Solution :

(i) $P \cup Q = \{4 \cdot a, 3 \cdot b, 2 \cdot d\}$

(ii) $P \cap Q = \{3 \cdot a, 3 \cdot b\}$

(iii) $P - Q = \{1 \cdot a, 1 \cdot c\}$

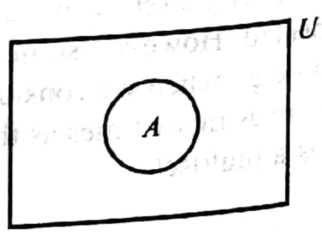
(iv) $Q - P = \{2 \cdot d\}$

(v) $P + Q = \{7 \cdot a, 6 \cdot b, 1 \cdot c, 2 \cdot d\}$

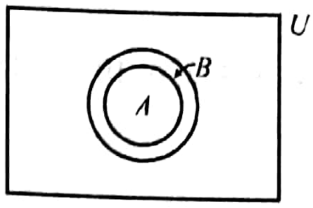
4.8 VENN DIAGRAM

A Venn diagram is a pictorial representation of sets which are used to show relationships between sets. The universal set is represented by interior of a rectangle and its subsets are represented by circular areas drawn within the rectangle.

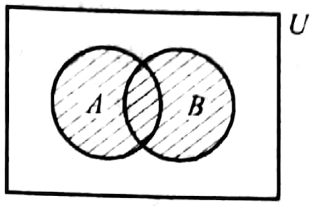
For example :

Set	Venn diagram
(i) $A \subset U$	

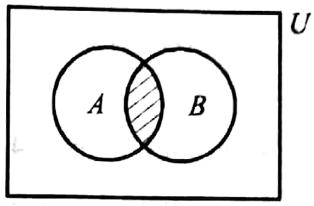
(ii) $A \subset B$



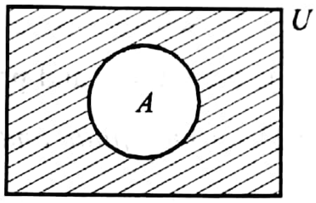
(iii) $A \cup B$



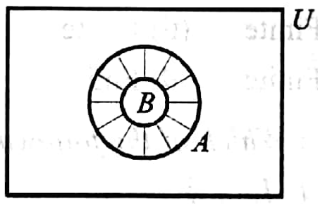
(iv) $A \cap B$



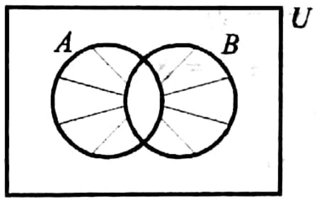
(v) A' or A^C



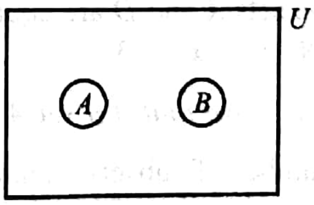
(vi) $A - B$



(vii) $A \oplus B$



(viii) If A and B are disjoint sets



Example 4.3 : *Describe each of following in roster form :*

(i) $A = \{x : x \text{ is an even prime}\}$

(ii) $B = \{x : x \text{ is a positive integral divisor of } 60\}$

(iii) $C = \{x \in R : x^2 - 1 = 0\}$

(iv) $D = \{x : x^2 - 2x + 1 = 0\}$

(v) $E = \{x : x \text{ is multiple of 3 or 5}\}$

Solution :

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(i) $A = \{2\}$

(ii) $B = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$

(iii) $C = \{1, -1\}$

(iv) $D = \{1\}$

(v) $E = \{3, 5, 6, 10, 15, ...\}$

Example 4.4 : *Describe the following in set builder form :*

(i) $A = \{-4, -3, -2, -1, 0, 1, 2, 3\}$

(ii) $B = \{1, 8, 27, 64\}$

(iii) $C = \{3, 6, 9, 12, 15, ...\}$

(iv) $D = \{2, 3, 5, 7, 11, 13, ...\}$

Solution :

(i) $A = \{x \mid x$ is an integer and $-4 \leq x \leq 3\}$

(ii) $B = \{x \mid x = n^3$, where $1 \leq n \leq 4$, n natural number$\}$

(iii) $C = \{x \mid x = 3n$, where n is natural number$\}$

(iv) $D = \{x \mid x$ is the integer and prime number$\}$

Example 4.5 : *Determine finite and infinite sets from example 4.3 and 4.4.*

Solution :

4.3. (i) Finite	**(ii)** Finite	**(iii)** Finite	**(iv)** Finite	**(v)** Infinite
4.4. (i) Finite	**(ii)** Finite	**(iii)** Infinite	**(iv)** Infinite	

Example 4.6 : *Which of the following sets are equal :*

(i) $A = \{1, 1, 1, ...\}$

(ii) $B = \{x \mid x^2 = 9\}$

(iii) $C = \{x \mid x^2 + 2 = 8\}$

(iv) $D = \{1\}$

(v) $E = \{3, -3\}$

Solution : The sets A and D are equal because 1 will be counted once in set A. Sets B and E are e[illegible] because $x^2 = 9 \Rightarrow x = \pm 3$.

Example 4.7 : *Prove that if a set A has n elements, then P(A) has 2^n elements.*

Solution : Number of subsets containing one element are ${}^nC_1$.

No. of subsets containing two elements are ${}^nC_2$.

No. of subsets containing three elements are ${}^nC_3$.

....

No. of subsets containing n elements are ${}^nC_n$.

Also ϕ is subset of every set.

$\therefore$ Total no. of subsets of $A = {}^nC_1 + {}^nC_2 + {}^nC_3 + ... + {}^nC_n + 1$

We know that ${}^nC_0 + {}^nC_1 + ... + {}^nC_n = 2^n$ and ${}^nC_0 = 1$

$\therefore \quad {}^nC_1 + {}^nC_2 + ... + {}^nC_n + 1 = 2^n$.

Hence total number of subsets of A are 2^n.

Therefore, $P(A)$ contain 2^n elements.

Example 4.8 : *Determine the power set of following :*

(i) $A = \{a, b, c, d\}$ *(ii)* $B = \{a, \{a\}\}$ [UPTU, B.Tech, 2006-07]

Solution :

(i) $A = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}, \{a, b, d\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\}$

(ii) $B = \{\phi, \{a\}, \{\{a\}\}, \{a, \{a\}\}\}$

Example 4.9 : *Determine the cardinality of following sets :*

(i) $A = \{1, 2, 3, 4\}$ **(ii)** $B = \{\phi\}$

(iii) $C = \phi$ **(iv)** $D = \{1, 1, 2, 2, 2, 3, 4, 4, 4, 4\}$

(v) $E = \{a, e, i, o, u\}$

Solution :

(i) $|A| = 4$

(ii) $|B| = 1$

(iii) $|C| = 0$

(iv) $|D| = 4$

(v) $|E| = 5$

Example 4.10 : *Let* $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{3, 4, 6, 7\}$, $B = \{1, 2, 5\}$, $C = \{6, 8, 9\}$, $D = \phi$.

Find the following :

(i) $A \cup B$ *(ii)* $B \cap C$ *(iii)* $B \cup D$ *(iv)* $C - D$

(v) $(C \cup D)^C$ *(vi)* $B \oplus D$ *(vii)* $A \oplus C$ *(viii)* A^C *(ix)* D^C

Solution :

(i) $\{1, 2, 3, 4, 5, 6, 7\}$ **(ii)** ϕ

(iii) $\{1, 2, 5\}$ **(iv)** $\{6, 8, 9\}$

(v) $\{1, 2, 3, 4, 5, 7\}$ **(vi)** $\{1, 2, 5\}$

(vii) $\{3, 4, 7, 8, 9\}$ **(viii)** $\{1, 2, 5, 8, 9\}$

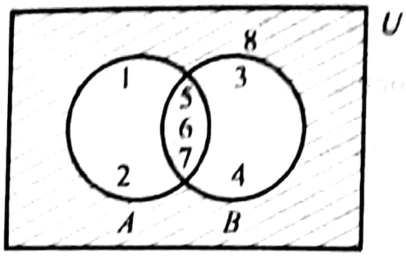
(ix) U

Example 4.11 : *Let* $A = \{1, 2, 5, 6, 7\}$, $B = \{3, 4, 5, 6, 7\}$ *and* $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$. *Draw Venn diagrams of the sets*

(i) $(A \cup B)'$ *(ii)* $A' \cup B'$

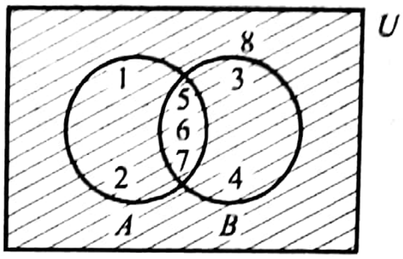
Solution : (i) $(A \cup B)'$

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$$
$$(A \cap B)' = \{8\}$$

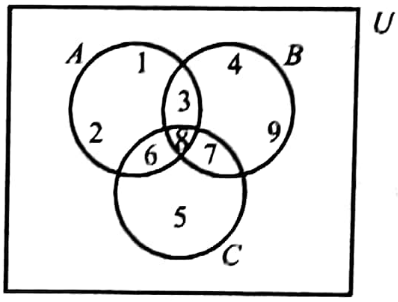


(ii) $A' \cup B'$

$$A' = \{3, 4, 8\}$$
$$B' = \{1, 2, 8\}$$
$$A' \cup B' = \{1, 2, 3, 4, 8\}$$



Example 4.12 : *Using Venn diagram shown below, find the following sets :*



(i) $A - (B \cap C)$ *(ii)* $(A \cup B) - C$

(iii) $B \oplus C$ *(iv)* $A - (B - C)$

(v) $A - (B \oplus C)$ *(vi)* $A \cap (B \oplus C)$

Solution : Here,

$$A = \{1, 2, 3, 6, 8\}$$
$$B = \{3, 4, 7, 8, 9\}$$
$$C = \{5, 6, 7, 8\}$$

(i) $$B \cap C = \{7, 8\}$$

$\therefore$ $$A - (B \cap C) = \{1, 2, 3, 6\}$$

(ii) $$A \cup B = \{1, 2, 3, 4, 6, 7, 8, 9\}$$
$$(A \cup B) - C = \{1, 2, 3, 4, 9\}$$

(iii) $$B \oplus C = (B \cup C) - (B \cap C)$$
$$B \cup C = \{3, 4, 5, 6, 7, 8, 9\}$$
$$B \cap C = \{7, 8\}$$

(iv) $$B - C = \{3, 4, 9\}$$
$$A - (B - C) = \{1, 2, 6, 8\}$$

(v) $B \oplus C = \{3, 4, 5, 6, 9\}$

$A - (B \oplus C) = \{1, 2, 8\}$

(vi) $B \oplus C = \{3, 4, 5, 6, 9\}$

$\therefore$ $A \cap (B \oplus C) = \{3, 6\}$

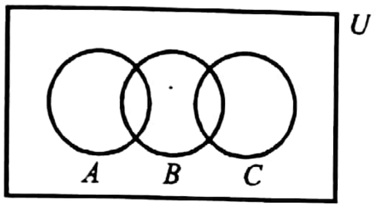
Example 4.13 : *Draw a Venn diagram of sets A, B, C where*

(i) A and B have elements in common, B and C have elements in common but A and C are disjoint.

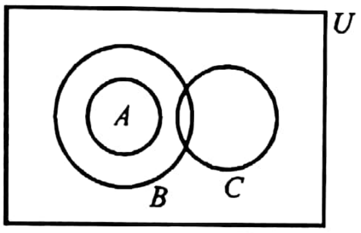
(ii) $A \subseteq B$, set A and C are disjoint but B and C have elements in common.

[UPTU, B.Tech. 2005-06]

Solution : (i)



(ii)



4.9 LAWS OF ALGEBRA OF SETS

Sets under operation of union, intersection and complement satisfy various laws or identities. Let A, B, C be any three sets and U be the universal set, then following are the laws of algebra of sets :

1. **Idempotent laws :**
   (a) $A \cup A = A$
   (b) $A \cap A = A$
2. **Commutative laws :**
   (a) $A \cup B = B \cup A$
   (b) $A \cap B = B \cap A$
3. **Associative laws :**
   (a) $A \cup (B \cup C) = (A \cup B) \cup C$
   (b) $A \cap (B \cap C) = (A \cap B) \cap C$
4. **Distributive laws :**
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5. **Identity laws :**
   (a) $A \cup \phi = A$
   (b) $A \cap U = A$

(c) $A \cup U = U$

(d) $A \cap \phi = \phi$

6. **Involution law :**

$(A^C)^C = A$

7. **Complement laws :**

(a) $A \cup A^C = U$

(b) $A \cap A^C = \phi$

(c) $U^C = \phi$

(d) $\phi^C = U$

8. **De Morgan's laws :**

(a) $(A \cup B)^C = A^C \cap B^C$

(b) $(A \cap B)^C = A^C \cup B^C$

9. **Absorption laws :**

(a) $A \cup (A \cap B) = A$

(b) $A \cap (A \cup B) = A$

Proof : 1(a). Let $x \in A \cup A$

$\Rightarrow$ $x \in A$ or $x \in A$

$\Rightarrow$ $x \in A$

$\Rightarrow$ $A \cup A \subseteq A$ .... (1)

Conversely, let

$x \in A$

$\Rightarrow$ $x \in A$ or $x \in A$

$\Rightarrow$ $x \in A \cup A$

$\Rightarrow$ $A \subseteq A \cup A$ .... (2)

From (1) and (2) $A \cup A = A$

Proof 2(a). Let, $x \in A \cup B$

$\Rightarrow$ $x \in A$ or $x \in B$

$\Rightarrow$ $x \in B$ or $x \in A$

$\Rightarrow$ $x \in B \cup A$

$\Rightarrow$ $A \cup B \subseteq B \cup A$ .... (1)

Conversely let,

$x \in B \cup A$

$\Rightarrow$ $x \in B$ or $x \in A$

$\Rightarrow$ $x \in A$ or $x \in B$

$\Rightarrow$ $x \in A \cup B$

$\Rightarrow$ $B \cup A \subseteq A \cup B$ .... (2)

From (1) and (2), $A \cup B = B \cup A$

Proof 3(a). Let $x \in A \cup (B \cup C)$

$\Rightarrow$ $x \in A$ or $(x \in B$ or $x \in C)$

$\Rightarrow$ $(x \in A$ or $x \in B)$ or $x \in C$

$\Rightarrow$ $x \in A \cup B$ or $x \in C$

$\Rightarrow$ $x \in (A \cup B) \cup C$

$\Rightarrow$ $A \cup (B \cup C) \subseteq (A \cup B) \cup C$ .... (1)

Conversely let, $x \in (A \cup B) \cup C$

$\Rightarrow$ $x \in (A \cup B)$ or $x \in C$

$\Rightarrow$ $(x \in A$ or $x \in B)$ or $x \in C$

$\Rightarrow$ $x \in A$ or $(x \in B$ or $x \in C)$

$\Rightarrow$ $x \in A$ or $x \in (B \cup C)$

$\Rightarrow$ $x \in A \cup (B \cup C)$

$\Rightarrow$ $(A \cup B) \cup C \subseteq A \cup (B \cup C)$ .... (2)

From (1) and (2), $A \cup (B \cup C) = (A \cup B) \cup C$

Proof 4(a). Let $x \in A \cup (B \cap C)$

$\Rightarrow$ $x \in A$ or $(x \in B$ and $x \in C)$

$\Rightarrow$ $(x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$

$\Rightarrow$ $x \in (A \cup B)$ and $x \in (A \cup C)$

$\Rightarrow$ $x \in (A \cup B) \cap (A \cup C)$

$\Rightarrow$ $A \cup (A \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Conversely let, $x \in (A \cup B) \cap (A \cup C)$

$\Rightarrow$ $x \in (A \cup B)$ and $x \in (A \cup C)$

$\Rightarrow$ $(x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$

$\Rightarrow$ $x \in A$ or $(x \in B$ and $x \in C)$

$\Rightarrow$ $x \in A$ or $x \in (B \cap C)$

$\Rightarrow$ $x \in A \cup (B \cap C)$

$\Rightarrow$ $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ .... (2)

From (1) and (2), $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof 5(a). Let, $x \in A \cup \phi$

$\Rightarrow$ $x \in A$ or $x \in \phi$

$\Rightarrow$ $x \in A$

$\Rightarrow$ $A \cup \phi \subseteq A$ .... (1)

Conversely, let $x \in A$

$\Rightarrow$ $x \in A$ or $x \in \phi$

$\Rightarrow$ $x \in A \cup \phi$

$\Rightarrow$ $A \subseteq A \cup \phi$ .... (2)

From (1) and (2), $A = A \cup \phi$

Proof 5(c). Let, $x \in U$

$\Rightarrow$ $x \in U$

$\Rightarrow$ $x \in U$ or $x \in A$

$\Rightarrow$ $U \subseteq A \cup U$ .... (1)

Now we know that every set is subset of universal set.

$\therefore$ $A \cup U \subseteq U$ .... (2)

From (1) and (2), $A \cup U = U$

Proof 6(a). Let $x \in (A^C)^C$

$\Rightarrow$ $x \notin A^C$

$\Rightarrow$ $x \in A$

$\Rightarrow$ $(A^C)^C \subseteq A$ .... (1)

Conversely let, $x \in A$

$\Rightarrow$ $x \notin A^C$

$\Rightarrow$ $x \in (A^C)^C$

$\Rightarrow$ $A \subseteq (A^C)^C$ .... (2)

From (1) and (2), $(A^C)^C = A$

Proof 7(a). Let, $x \in U$

$\Rightarrow$ $x \in A$ or $x \in A^C$

$\Rightarrow$ $x \in A \cup A^C$

$\Rightarrow$ $U \subseteq A \cup A^C$ .... (1)

Conversely, we know that every set is subset of universal set

$\therefore$ $A \cup A^C \subseteq U$ .... (2)

From (1) and 2), $A \cup A^C = U$

Proof 7(c). Let, $x \in U^C$

$\Rightarrow$ $x \notin U$

$\Rightarrow$ $x \in \phi \Rightarrow U^C \subseteq \phi$ .... (1)

Conversely, let, $x \notin \phi$

$\Rightarrow$ $x \notin U$

$\Rightarrow$ $x \in U^C \Rightarrow \phi \subseteq U^C$ .... (2)

From (1) and (2), $U^C = \phi$

Proof 8(a). Let, $x \in (A \cup B)^C$

$\Rightarrow$ $x \notin (A \cup B)$

$\Rightarrow$ $x \notin A$ and $x \notin B$

$\Rightarrow$ $x \in A^C$ and $x \in B^C$

$\Rightarrow$ $x \in A^C \cap B^C$

$\Rightarrow$ $(A \cup B)^C \subseteq A^C \cap B^C$ .... (1)

Conversely, let, $x \in A^C \cap B^C$

$\Rightarrow$ $x \in A^C$ and $x \in B^C$

$\Rightarrow$ $x \notin A$ and $x \notin B$

$\Rightarrow$ $x \notin A \cup B \Rightarrow x \in (A \cup B)^C$

$\Rightarrow$ $A^C \cap B^C \subseteq (A \cup B)^C$ .... (2)

From (1) and (2), $(A \cup B)^C = A^C \cap B^C$

Proof 9(a). Let $x \in A \cup (A \cap B)$

$\Rightarrow$ $x \in A$ or $x \in (A \cap B)$

$\Rightarrow$ $x \in A$ or $(x \in A$ and $x \in B)$

$\Rightarrow$ $(x \in A$ or $x \in A)$ and $(x \in A$ or $x \in B)$

$\Rightarrow$ $x \in A$

$\Rightarrow$ $A \cup (A \cap B) \subseteq A$ .... (1)

Conversely, let $x \in A \Rightarrow (x \in A$ or $x \in A)$ and $(x \in A$ or $x \in B)$

$\Rightarrow$ $(x \in A)$ or $(x \in A$ and $x \in B)$

$\Rightarrow$ $x \in A$ or $x \in A \cap B$

$\Rightarrow$ $x \in A \cup (A \cap B)$

$\Rightarrow$ $A \subseteq A \cup (A \cap B)$

From (1) and (2), $A = A \cup (A \cap B)$

Example 4.14 : *Prove the following :*

(a) $(A - B) \cap (B - A) = \phi$

(b) $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$

(c) $(A - B) = B^C - A^C$

(d) $A \cap (B - C) = (A \cap B) - (A \cap C)$

(e) $(A \cap B) - C = (A - C) \cap (B - C)$

(f) $A = (A \cap B) \cup (A - B)$

Solution :

(a) Let $x \in (A - B) \cap (B - A)$

$\Rightarrow$ $x \in (A - B)$ and $x \in (B - A)$

$\Rightarrow$ $(x \in A$ and $x \notin B)$ and $(x \in B$ and $x \notin A)$

$\Rightarrow$ $(x \in A$ and $x \notin A)$ and $(x \in B$ and $x \notin B)$

$\Rightarrow$ $x \in \phi$

$\Rightarrow$ $(A - B) \cap (B - A) \subseteq \phi$ .... (1)

Since ϕ is subset of every set

$\therefore$ $\phi \subseteq (A - B) \cap (B - A)$ .... (2)

From (1) and (2), $(A - B) \cap (B - A) = \phi$

(b) Let $x \in A \cap (B \oplus C)$

$\Leftrightarrow$ $x \in A$ and $x \in B \oplus C$

$x \in A$ and $x \in [(B \cup C) - (B \cap C)]$

$\Leftrightarrow$ $x \in A$ and $[x \in B \cup C$ and $x \notin B \cap C]$

$\Leftrightarrow$ $x \in A$ and $[(x \in B$ or $x \in C)$ and $(x \notin B$ or $x \notin C)]$

$\Leftrightarrow$ $[(x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)]$ and $[(x \in A$ and $x \notin B)$ or $(x \in A$ and $x \notin C)]$

$\Leftrightarrow$ $[x \in (A \cap B)$ or $x \in (A \cap C)]$ and $[x \notin (A \cap B)$ or $x \notin (A \cap C)]$

$\Leftrightarrow$ $x \in [(A \cap B) \cup (A \cap C)]$ and $[x \notin (A \cap B) \cap (A \cap C)]$

$\Leftrightarrow$ $x \in [(A \cap B) \cup (A \cap C)] - [(A \cap B) \cap (A \cap C)]$

$\Leftrightarrow$ $x \in (A \cap B) \oplus (A \cap C)$

$\Leftrightarrow$

Therefore, $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$

(c) Let $x \in (A - B)$

$\Leftrightarrow$ $x \in A$ and $x \notin B$

$\Leftrightarrow$ $x \notin A^C$ and $x \in B^C$

$\Leftrightarrow$ $x \in B^C$ and $x \notin A^C$

$\Leftrightarrow$ $x \in B^C - A^C$

Therefore, $A - B = B^C - A^C$

(d) Let $x \in A \cap (B - C)$

$\Leftrightarrow$ $x \in A$ and $x \in B - C$

$\Leftrightarrow$ $x \in A$ and $(x \in B$ and $x \notin C)$

$\Leftrightarrow$ $(x \in A$ and $x \in B)$ and $(x \in A$ and $x \notin C)$

$\Leftrightarrow$ $x \in (A \cap B)$ and $x \notin (A \cap C)$

$\Leftrightarrow$ $x \in (A \cap B) - (A \cap C)$

Therefore, $A \cap (B - C) = (A \cap B) - (A \cap C)$

(e) Let $x \in (A \cap B) - C$

$\Leftrightarrow$ $x \in (A \cap B)$ and $x \notin C$

$\Leftrightarrow$ $(x \in A$ and $x \in B)$ and $x \notin C$

$\Leftrightarrow$ $(x \in A$ and $x \notin C)$ and $(x \in B$ and $x \notin C)$

$\Leftrightarrow$ $x \in (A - C)$ and $x \in (B - C)$

$\Leftrightarrow$ $x \in (A - C) \cap (B - C)$

$\Leftrightarrow$ $(A \cap B) - C \subseteq (A - C) \cap (B - C)$ .... (1)

Conversely, let $x \in (A - C) \cap (B - C)$

$\Rightarrow$ $x \in (A - C)$ and $x \in (B - C)$

$\Rightarrow$ $(x \in A$ and $x \notin C)$ and $(x \in B$ and $x \notin C)$

$\Rightarrow$ $(x \in A$ and $x \in B)$ and $x \notin C$

$\Rightarrow$ $(x \in A \cap B)$ and $x \notin C$

$\Rightarrow$ $x \in (A \cap B) - C$

$\Rightarrow$ $(A - C) \cap (B - C) \subseteq (A \cap B) - C$ .... (2)

From (1) and (2), $(A \cap B) - C = (A - C) \cap (B - C)$

(f) Let, $x \in A$

$\Rightarrow$ $x \in A$ and $(x \in B$ or $x \notin B)$

$\Rightarrow$ $(x \in A$ and $x \in B)$ or $(x \in A$ and $x \notin B)$

$\Rightarrow$ $x \in (A \cap B)$ or $x \in (A - B)$

$\Rightarrow$ $x \in (A \cap B) \cup (A - B)$

$\Rightarrow$ $A \subseteq (A \cap B) \cup (A - B)$ .... (1)

Conversely, let $x \in (A \cap B) \cup (A - B)$

$\Rightarrow$ $(x \in A \cap B)$ or $(x \in A - B)$

$\Rightarrow$ $(x \in A$ and $x \in B)$ or $(x \in A$ and $x \notin B)$

$\Rightarrow$ $x \in A$ and $(x \in B$ or $x \notin B)$

$\Rightarrow$ $x \in A$

$\Rightarrow$ $(A \cap B) \cup (A - B) \subseteq A$ .... (2)

From (1) and (2), $A = (A \cap B) \cup (A - B)$

Example 4.15 : *Show that*

(a) $(A - B) - C = A - (B \cup C)$

(b) $(A - B) - C = (A - C) - B$

(c) $(A - B) - C = (A - C) - (B - C)$

where, A, B and C are arbitrary sets.

Solution :

(a) Let $x \in (A - B) - C$

$\Rightarrow$ $x \in (A - B)$ and $x \notin C$

$\Rightarrow$ $(x \in A$ and $x \notin B)$ and $x \notin C$

$\Rightarrow$ $x \in A$ and $(x \notin B$ and $x \notin C)$

$\Rightarrow$ $x \in A$ and $x \notin B \cup C$ [$\because$ If $x \notin B$ and $x \notin C \Rightarrow x \notin B \cup C$]

$\Rightarrow$ $x \in A - (B \cup C)$

$\Rightarrow$ $(A - B) - C \subseteq A - (B \cup C)$

Conversely, let $x \in A - (B \cup C)$

$\Rightarrow$ $x \in A$ and $x \notin (B \cup C)$

$\Rightarrow$ $x \in A$ and $(x \notin B$ and $x \notin C)$

$\Rightarrow$ $(x \in A$ and $x \notin B)$ and $x \notin C$

$\Rightarrow$ $x \in A - B$ and $x \notin C$

$\Rightarrow$ $x \in (A - B) - C$ .... (2)

$\Rightarrow$ $A - (B \cup C) \subseteq (A - B) - C$

From (1) and (2), $(A - B) - C = A - (B \cup C)$

(b) Let $x \in (A - B) - C$

$\Rightarrow$ $x \in (A - B)$ and $x \notin C$

$\Rightarrow \quad x \in A$ and $x \notin B$ and $x \notin C$

$\Rightarrow \quad x \in A$ and $x \notin C$ and $x \notin B$

$\Rightarrow \quad x \in (A - C)$ and $x \notin B$

$\Rightarrow \quad x \in (A - C) - B$

$\Rightarrow \quad (A - B) - C \subseteq (A - C) - B$

Conversely, let $\quad x \in (A - C) - B$

$\Rightarrow \quad x \in (A - C)$ and $x \notin B$

$\Rightarrow \quad x \in A$ and $x \notin C$ and $x \notin B$

$\Rightarrow \quad x \in A$ and $x \notin B$ and $x \notin C$

$\Rightarrow \quad x \in A - B$ and $x \notin C$

$\Rightarrow \quad x \in (A - B) - C$

$\Rightarrow \quad (A - C) - B \subseteq (A - B) - C$ .... (2)

From (1) and (2), $\quad (A - B) - C = (A - C) - B$

(c) Let $\quad x \in (A - B) - C$

$\Rightarrow \quad x \in (A - B)$ and $x \notin C$

$\Rightarrow \quad x \in A$ and $x \notin B$ and $x \notin C$

$\Rightarrow \quad (x \in A$ and $x \notin C)$ and $(x \notin B$ and $x \notin C)$

$\Rightarrow \quad x \in (A - C)$ and $x \notin (B - C)$

$\Rightarrow \quad x \in (A - C) - (B - C)$

$\Rightarrow \quad (A - B) - C \subseteq (A - C) - (B - C)$ .... (1)

Conversely, let $\quad x \in (A - C) - (B - C)$

$\Rightarrow \quad x \in (A - C)$ and $x \notin (B - C)$

$\Rightarrow \quad (x \in A$ and $x \notin C)$ and $(x \notin B$ and $x \notin C)$

$\Rightarrow \quad x \in A$ and $x \notin B$ and $x \notin C$

$\Rightarrow \quad x \in (A - B)$ and $x \notin C$

$\Rightarrow \quad x \in (A - B) - C$

$\Rightarrow \quad (A - C) - (B - C) \subseteq (A - B) - C$ .... (2)

From (1) and (2), $\quad (A - B) - C = (A - C) - (B - C)$

Example 4.16 : *If A and B are two subsets of universal set, then prove the following :*

(a) *$(A - B) = (B - A)$ iff $A = B$*

(b) *$(A - B) = A$ iff $A \cap B = \phi$*

[UPTU, *B*.Tech, 2008-09]

Solution :

(a) Let $\quad A = B$

Consider any element $\quad x \in A - B$

$\Rightarrow \quad x \in A$ and $x \notin B$

$\Rightarrow \quad x \in B$ and $x \notin A$

$\Rightarrow \quad x \in B - A$ $\quad [\because A = B]$

$\therefore$	$A - B \subseteq B - A$	
Conversely, if	$x \in B - A$	
$\Rightarrow$	$x \in B$ and $x \notin A$	
$\Rightarrow$	$x \in A$ and $x \notin B$	
$\Rightarrow$	$x \in A - B$	
	$B - A \subseteq A - B$	.... (2)

From (1) and (2), we have

If	$A = B \Rightarrow A - B = B - A$	
Now, let	$A - B = B - A$	
Let	$x \in A - B$ as $A - B = B - A$	
$\therefore$	$x \in B - A$	
Now,	$x \in A - B \Rightarrow x \in A$ and $x \notin B$	.... (1)
and	$x \in B - A \Rightarrow x \in B$ and $x \notin A$	.... (2)

(1) and (2), can hold true when $A = B$

(b) Let	$A - B = A$	
To show	$A \cap B = \phi$	
Let	$A \cap B \neq \phi$ and let $x \in A \cap B$ and $x \notin \phi$	
$\Rightarrow$	$x \in A$ and $x \in B$	
$\Rightarrow$	$x \in (A - B)$ and $x \in B$	$[\because A - B = A]$
$\Rightarrow$	$x \in A$ and $x \notin B$ and $x \in B$	
$\Rightarrow$	$x \in \phi$	

which is a contradiction.

$\therefore$	$A \cap B = \phi$	
Now conversely, let	$A \cap B = \phi$	
To show	$A - B = A$	
Let	$x \in A - B$	
$\Rightarrow$	$x \in A$ and $x \notin B$	
$\Rightarrow$	$x \in A$	[as $A \cap B = \phi$]
$\Rightarrow$	$A - B \subseteq A$	.... (1)
Conversely, let	$x \in A$	
$\Rightarrow$	$x \in A$ and $x \notin B$	[as $A \cap B = \phi$]
$\Rightarrow$	$x \in A - B$	
$\therefore$	$A \subseteq A - B$	.... (2)
From (1) and (2),	$A = A - B$	

Example 4.17 : *Show that for any two sets A and B, $A - (A \cap B) = A - B$.* [UPTU, *B*.Tech, 2006-07]

Solution : Let $x \in A - (A \cap B)$

$$x \in A \text{ and } x \notin (A \cap B)$$

$\Leftrightarrow$ $x \in A$ and $(x \notin A$ or $x \notin B)$

$\Leftrightarrow$ $(x \in A$ and $x \notin A)$ or $(x \in A$ or $x \notin B)$

$\Leftrightarrow$ $x \in \phi$ or $x \in (A - B)$

$\Leftrightarrow$ $x \in A - B$

$\Leftrightarrow$

Hence, $A - (A \cap B) = A - B$

Example 4.18 : *If A and B are sets then $(A \cup B) \cup (A \cap \sim B)$ and $A \cap (\sim A \cup B)$ are equal to?* [UPTU, *B*.Tech. 2008-09]

Solution :

$(A \cap B) \cup (A \cap \sim B) = A \cap (B \cup \sim B)$ (by distributive law)

$= A \cap U$ (by complement law)

$= A$ (by identity law)

Consider $A \cap (\sim A \cup B) = (A \cap \sim A) \cup (A \cap B)$ (by distributive law)

$= \phi \cup (A \cap B)$ (by complement law)

$= A \cap B$ (by identity law)

4.10 ORDERED PAIR

An ordered pair is a pair of objects formed by listing the two components in specified order. In the ordered pair (x, y), x is called *first component* and y is called *second component*.

4.11 CARTESIAN PRODUCT

Consider two sets A and B, then Cartesian product of A and B denoted by $A \times B$ is the set of all ordered pairs (x, y) with $x \in A$ and $y \in B$.

Symbolically, $A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$

Example 4.19 : *Let $A = \{1, 2, 3\}$. Find $A \times A$.*

Solution : $A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$.

Example 4.20 : *Let $A = \{a, b\}$ and $B = \{2, 3\}$. Prove that $A \times B \neq B \times A$.*

Solution :

$$A \times B = \{(a, 2), (a, 3), (b, 2), (b, 3)\}$$

$$B \times A = \{(2, a), (2, b), (3, a), (3, b)\}$$

Therefore, $A \times B \neq B \times A$

Example 4.21 : *If S and T have n elements in common. Show that $S \times T$ and $T \times S$ has n^2 elements in common.*

Solution : Let a set R contains n common elements of S and T.

$\therefore$ $R \subset S$ and $R \subset T$

Let, $(x, y) \in R \times R$

$\Leftrightarrow$ $x \in R$ and $y \in R$

$\Leftrightarrow$ $(x \in R$ and $y \in R)$ and $(x \in R$ and $y \in R)$

$\Leftrightarrow$ $(x \in S$ and $y \in T)$ and $(x \in T$ and $y \in S)$

$\Leftrightarrow$ $(x, y) \in (S \times T)$ and $(x, y) \in (T \times S)$

$\Leftrightarrow \quad (x, y) \cap (S \times T) \cap (T \times S)$

$\therefore \quad R \times R = (S \times T) \cap (T \times S)$

The left hand side contain n^2 elements. Since two sets are equal therefore, both have the same mber of elements. Hence, $S \times T$ and $T \times S$ has n^2 common elements.

11.1 Properties of Cartesian Product

Let A, B, C and D be four sets, then

1. $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$
2. $(A - B) \times C = (A \times C) - (B \times C)$
3. $(A \cup B) \times C = (A \times C) \cup (B \times C)$
4. $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Proof : 1. Let $(x, y) \in (A \cap B) \times (C \cap D)$

$\Rightarrow \quad x \in A \cap B$ and $y \in C \cap D$

$\Rightarrow \quad (x \in A$ and $x \in B)$ and $(y \in C$ and $y \in D)$

$\Rightarrow \quad (x \in A$ and $y \in C)$ and $(x \in B$ and $y \in D)$

$\Rightarrow \quad (x, y) \in (A \times C)$ and $(x, y) \in (B \times D)$

$\Rightarrow \quad (x, y) \in (A \times C) \cap (B \times D)$

$\Rightarrow \quad (A \cap B) \times (C \cap D) \subseteq (A \times C) \cap (B \times D)$ .... (1)

Conversely, let $(x, y) \in (A \times C) \cap (B \times D)$

$\Rightarrow \quad (x, y) \in (A \times C)$ and $(x, y) \in (B \times D)$

$\Rightarrow \quad (x \in A$ and $y \in C)$ and $(x \in B$ and $y \in D)$

$\Rightarrow \quad (x \in A$ and $x \in B)$ and $(y \in C$ and $y \in D)$

$\Rightarrow \quad (x \in (A \cap B))$ and $(y \in (C \cap D))$

$\Rightarrow \quad (x, y) \in (A \cap B) \times (C \cap D)$

$\Rightarrow \quad (A \times C) \cap (B \times D) \subseteq (A \cap B) \times (C \cap D)$ .... (2)

From (1) and (2), $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

. Let $(x, y) \in (A - B) \times C$

$\Rightarrow \quad x \in (A - B)$ and $y \in C$

$\Rightarrow \quad (x \in A$ and $x \notin B)$ and $y \in C$

$\Rightarrow \quad (x \in A$ and $y \in C)$ and $(x \notin B$ and $y \in C)$

$\Rightarrow \quad (x, y) \in (A \times C)$ and $(x, y) \notin (B \times C)$

$\Rightarrow \quad (x, y) \in (A \times C) - (B \times C)$

$\Rightarrow \quad (A - B) \times C \subseteq (A \times C) - (B \times C)$ .... (1)

Conversely, let $(x, y) \in (A \times C) - (B \times C)$

$\Rightarrow \quad (x, y) \in (A \times C)$ and $(x, y) \notin (B \times C)$

$\Rightarrow \quad (x \in A$ and $y \in C)$ and $(x \notin B$ and $y \in C)$

$\Rightarrow \quad (x \in A$ and $x \notin B)$ and $y \in C$

$\Rightarrow \quad x \in (A - B)$ and $y \in C$

$\Rightarrow \quad (x, y) \in (A - B) \times C$

$\Rightarrow \quad (A \times C) - (B \times C) \subseteq (A - B) \times C$ .... (2)

From (1) and (2), $\quad (A - B) \times C = (A \times C) - (B \times C)$

3. Let $\quad (x, y) \in (A \cup B) \times C$

$\Rightarrow \quad x \in (A \cup B)$ and $y \in C$

$\Rightarrow \quad (x \in A$ or $x \in B)$ and $y \in C$

$\Rightarrow \quad (x \in A$ and $y \in C)$ or $(x \in B$ and $y \in C)$

$\Rightarrow \quad (x, y) \in (A \times C)$ or $(x, y) \in (B \times C)$

$\Rightarrow \quad (x, y) \in (A \times C) \cup (B \times C)$

$\Rightarrow \quad (A \cup B) \times C \subseteq (A \times C) \cup (B \times C)$ .... (1)

Conversely, let $\quad (x, y) \in (A \times C) \cup (B \times C)$

$\Rightarrow \quad (x, y) \in (A \times C)$ or $(x, y) \in (B \times C)$

$\Rightarrow \quad (x \in A$ and $y \in C)$ or $(x \in B$ and $y \in C)$

$\Rightarrow \quad (x \in A$ or $x \in B)$ and $(y \in C)$

$\Rightarrow \quad (x \in (A \cup B))$ and $(y \in C)$

$\Rightarrow \quad (x, y) \in (A \cup B) \times C$

$\Rightarrow \quad (A \times C) \cup (B \times C) \subseteq (A \cup B) \times C$ .... (2)

From (1) and (2), $\quad (A \cup B) \times C = (A \times C) \cup (B \times C)$

4. Let $\quad (x, y) \in A \times (B \cap C)$

$\Rightarrow \quad x \in A$ and $y \in (B \cap C)$

$\Rightarrow \quad x \in A$ and $(y \in B$ and $y \in C)$

$\Rightarrow \quad (x \in A$ and $y \in B)$ and $(x \in A$ and $y \in C)$

$\Rightarrow \quad (x, y) \in (A \times B)$ and $(x, y) \in (A \times C)$

$\Rightarrow \quad (x, y) \in (A \times B) \cap (A \times C)$

$\Rightarrow \quad A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$ .... (1)

Conversely, let $\quad (x, y) \in (A \times B) \cap (A \times C)$

$\Rightarrow \quad (x, y) \in (A \times B)$ and $(x, y) \in (A \times C)$

$\Rightarrow \quad (x \in A$ and $y \in B)$ and $(x \in A$ and $y \in C)$

$\Rightarrow \quad x \in A$ and $(y \in B$ and $y \in C)$

$\Rightarrow \quad x \in A$ and $y \in (B \cap C)$

$\Rightarrow \quad (x, y) \in A \times (B \cap C)$

$\Rightarrow \quad (A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$ .... (2)

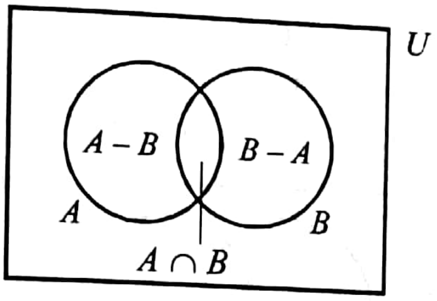
From (1) and (2), $\quad A \times (B \cap C) = (A \times B) \cap (A \times C)$

4.12 PRINCIPLE OF INCLUSION EXCLUSION : (COUNTING PRINCIPLE)

Statement : Let A and B be two non-disjoint sets, then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Proof : From the Venn diagram, we have,



$$n(A) = n(A - B) + n(A \cap B) \quad \text{.... (1)}$$

$$n(B) = n(B - A) + n(A \cap B) \quad \text{.... (2)}$$

Adding (1) and (2) we get,

$$n(A) + n(B) = n(A - B) + n(B - A) + 2n(A \cap B) \quad \text{.... (3)}$$

Also from Venn diagram,

$$n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B) \quad \text{.... (4)}$$

Using equation (4) in (3),

$$n(A) + n(B) = n(A \cup B) + n(A \cap B)$$

$$\therefore \quad n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Hence proved.

Cor : If A, B and C are three finite sets, then

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

Example 4.22 : *40 lecturers interviewed for a job. 25 were mathematicians. 28 were physicists and 7 were neither. How many lecturers were both mathematicians and physicists?*

Solution : $n(M) = 25, \quad n(P) = 28$

$$n(M \cup P) = 40 - 7 = 33$$

By principle of inclusion exclusion :

$$n(M \cup P) = n(M) + n(P) - n(M \cap P)$$

$$\therefore \quad 35 = 25 + 28 - n(M \cap P)$$

$$\therefore \quad n(M \cap P) = 25 + 28 - 33 = 20$$

Example 4.23 : *In a survey of 600 television viewers given the following information. 385 watch cricket matches, 295 watch hockey matches, 215 watch football matches. 145 watch cricket and football matches. 170 watch cricket and hockey matches and 150 watch hockey and football matches and 150 do not watch any of three.*

(i) *How many people watch all three kind of matches?*

(ii) *How many people watch exactly one sport?*

Solution : Here

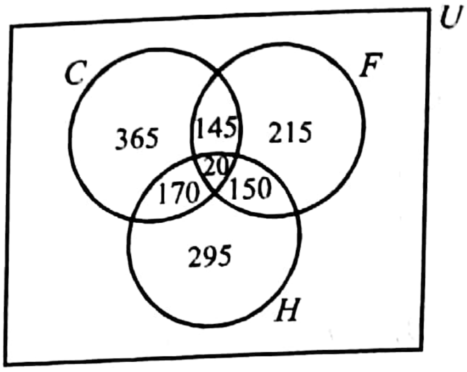
$$n(C) = 385$$

$$n(H) = 295$$

$$n(F) = 215$$

$$n(C \cap F) = 145$$

$$n\,(C \cap H) = 170$$
$$n\,(H \cap F) = 150$$
$$n\,(C \cup F \cup H) = 150 - 150 = 450$$



(i) By principle of inclusion exclusion,

$$n(C \cup F \cup H) = n(C) + n(F) + n(H) - n(C \cap F) - n(C \cap H) - n(F \cap H) + n(C \cap H \cap F$$

$$n(C \cap H \cap F) = 450 - 385 - 295 - 215 + 145 + 170 + 150 = 20$$

(ii) $n(\text{exactly one game}) = n(\text{only cricket}) + n(\text{only hockey}) + n(\text{only football})$

$$= n(C) - n(C \cap H) - n(C \cap F) + n(C \cap H \cap F) + n(H) - n(C \cap H) - n(H\ \ F) + n(C \cap H \cap F) + n(F) - n(F \cap C) - n(F \cap H) + n(C \cap H \cap F)$$

$$= 295 - 170 - 150 + 20 + 385 - 145 - 170 + 20 + 215 - 145 - 150 + 20 = 25$$

Example 4.24 : *Among first 500 positive integers :*

(a) *Determine the integer which are not divisible by 2, nor by 3 nor by 5.*

(b) *Determine the integer which are exactly divisible by one of them.*

Solution : Let A be the set of integers divisble by 2.

Let B be the set of integers divisible by 3.

Let C be the set of integers divisible by 5.

$$n(A) = \frac{500}{2} = 250 \qquad n(B) = \frac{500}{3} = 166 \qquad n(C) = \frac{500}{5} = 100$$

$$n(A \cap B) = \frac{500}{2 \times 3} = 83 \qquad n(A \cap C) = \frac{500}{2 \times 5} = 50$$

$$n(A \cap C) = \frac{500}{3 \times 5} = 33 \qquad n(A \cap B \cap C) = \frac{500}{2 \times 3 \times 5} = 16$$

(a) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

$$= 250 + 166 + 100 - 83 - 33 - 50 + 16 = 366$$

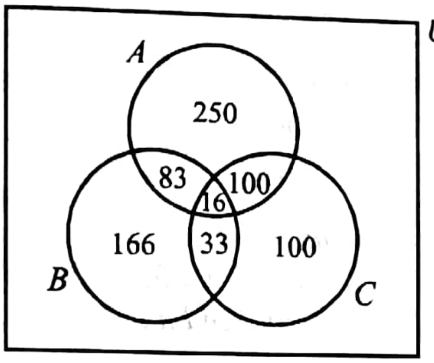
$$n(A' \cap B' \cap C') = 500 - n(A \cup B \cup C)$$
$$= 500 - 366 = 134$$

(b) n (divisible by exactly one) $= n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$

$$+ n(B) - n(A \cap B) - n(B \cap C) + n(A \cap B \cap C)$$

$$+ n(C) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

$$= 250 - 83 - 50 + 16 + 166 - 83 - 33 + 16 + 100 - 50 - 33 + 16 = 232$$



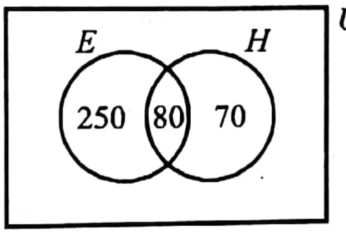
Example 4.25 : *In a group of 400 people 250 can speak English only and 70 can speak Hindi only*

(a) How many can speak English?

(b) How many can speak Hindi?

(c) How many can speak both English and Hindi?

Solution : Here, n (English only) = 250, n (Hindi only) = 70



(a) $n(E) = 400 - n$ (Hindi only)

$= 400 - 70 = 330$

(b) $n(H) = 400 - n$ (English only)

$= 400 - 250 = 150$

(c) $n(A \cap E) = 400 - n$ (English only) $- n$ (Hindi only)

$= 400 - 250 - 70 = 80$

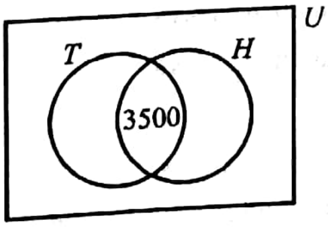
Example 4.26 : *In a population of 25000, 13000 read "TOI", 10500 read Hindustan Times and 3500 read both papers. Find the percentage of population that neither of these paper?*

Solution :

$$n(T) = 13000$$

$$n(H) = 10500$$

$$n(T \cap H) = 3500$$



By principle of inclusion exclusion,

$$n(T \cup H) = n(T) + n(H) - n(T \cap H)$$
$$= 13000 + 10500 - 3500 = 20000$$

Number people which read neither of these papers

$$= 25000 - 20000 = 5000$$

Percentage of population that reads neither of these papers

$$= \frac{5000}{25000} \times 100 = 20\%$$

Example 4.27 : *75 Children went to a circus, where they can attend a magic show, a comedy and an animal show. 20 of them attended all three shows and 55 attended atleast two of 3 shows. Each show costs Rs. 5 and total money collected is Rs. 700. Find number of children who did not attend any of 3 shows.*

Solution :

Number of children = 75

Total money collected = Rs. 700

Cost of one show = Rs. 5

$\therefore$ Total shows attended $= \frac{700}{5} = 140$

Number of children who attended all three shows $= n(M \cap C \cap A) = 20$

$\therefore$ Children who attended 2 shows = Children who attended either 2 shows or 3 shows = 55

$\therefore$ Children who attended 2 shows = 55 – 20 = 35

Now children who attended exactly one show

$$= 140 - \underbrace{(35 \times 2)}_{\text{Two shows attended}} - \underbrace{(20 \times 3)}_{\text{Three shows attended}}$$

$$= 140 - 70 - 60 = 10$$

Therefore, children who attended no shows

$$= 75 - (\underbrace{10}_{\text{1 show}} + \underbrace{35}_{\text{2 shows}} + \underbrace{20}_{\text{3 shows}})$$

$$= 75 - 65 = 10$$

4.13 FIRST COUNTING PRINCIPLE : (PRODUCT RULE)

If an event can occur in r different steps and

Step 1 can occur in n_1 ways

Step 2 can occur in n_2 ways

....

Step r can occur in n_r ways.

The no. of possible events that can occur $= n_1 \cdot n_2 \cdot n_3 \, ... \, n_r$.

Example 4.28 : *A child has 4 hats, 3 pair of gloves and five pair of socks. Determine different possible triplets that can occur?*

Solution : A hat can be selected in 4 ways.

A pair of gloves be selected in 3 ways.

A pair of socks can be selected in 5 ways.

$\therefore$ Total number of possible ways = $4 \times 3 \times 5 = 60$ ways

Example 4.29 : *A person has to arrange 5 books on a shelf. In how many ways can he do so?*

Solution : The first book can be arranged in 5 ways.

The second book can be arranged in 4 ways.

The third book can be arranged in 3 ways.

The fourth book can be arranged in 2 ways.

The fifth book can be arranged in 1 ways.

$\therefore$ Total number of ways = $5 \times 4 \times 3 \times 2 \times 1 = 120$ ways

Example 4.30 : *How many different 2-digit numbers can be made from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, 0 when repetition is allowed and when repetition is not allowed?*

Solution : When repetition allowed.

The ten place can be filled by 9 ways

and the units place can be filled by 10 ways

$\therefore$ Total number of 2 digit number = $10 \times 9 = 90$

When repetition is not allowed.

The tens place can be filled by 9 ways

and unit place can be filled by 9 ways

$\therefore$ Total number of 2-digit number = 81.

Example 4.31 : *A person committee having members Ankit, Ajit, Sonu, Monu and Nonu is to select a president, vice-president and secretary.*

(a) *How many selections exclude Nonu?*

(b) *How many selections include Sonu and Monu?*

(c) *How many selections exclude Sonu and Monu?*

(d) *How many selections are there in which Ankit is President?*

Solution : (a) After including Nonu, we have to select 3 persons from the remaining four.

President can be selected in 4 ways.

Vice President can be selected in 3 ways.

Secretary can be selected in 2 ways.

$\therefore$ Total number of ways = $4 \times 3 \times 2 = 24$

(b) We have 3 ways to assign any post to Sonu. After selecting Sonu, there are 2 ways to assign any post to Monu. After selecting Sonu and Monu, we can assign the remaining post of any of the three persons.

$\therefore$ Total number of selections $= 3 \times 2 \times 3 = 18$

(c) After excluding Sonu and Monu, we have to select three persons from the remaining three.

$\therefore$ President can be selected in 3 ways.

Vice President can be selected in 2 ways.

Secretary can be selected in 1 ways.

Total number of selections $= 3 \times 2 \times 1 = 6$

(d) When Ankit is selected as president, then we have to select Vice-president and secretary from remaining four.

$\therefore$ Vice president can be selected in 4 ways.

and Secretary can be selected in 3 ways.

Total number of selections $= 4 \times 3 = 12$

Example 4.32 : *Find the number of ways in which we can post 5 letters in 10 letter boxes?*

Solution : One letter can be posted in 10 ways

Second letter can be posted in 10 ways

Third letter can be posted in 10 ways

Fourth letter can be posted in 10 ways

Fifth letter can be posted in 10 ways

We can post five letters in = 10^5 ways

Example 4.33 : *Three travellers arrive at a town when there are four hotels. In how many ways can they take up their quarters each at different hotel?*

Solution : 1st traveller can choose from 4 hotels.

2nd traveller can choose from 3 hotels.

3rd traveller can choose from 2 hotels.

Total number of ways = $4 \times 3 \times 2 = 24$ ways

Example 4.34 : *How many ways are there to select 2 cards from a deck of 52 such that*

(a) 1^{st} card is diamond and 2^{nd} is a club?

(b) 1^{st} card is queen and 2^{nd} is not a king?

(c) 1^{st} card is club and 2^{nd} is queen?

Solution :

(a) 1^{st} card can be selected in 13 ways [$\because$ 13 diamond cards]

2^{nd} card can be selected in 13 ways [$\because$ 13 club cards]

$\therefore$ Total number of ways = $13 \times 13 = 169$

(b) 1^{st} card can be selected in 4 ways [$\because$ 4 queen cards]

2^{nd} card can be selected in 47 ways [$\because$ $52 - 4 - 1 = 47$ cards are not king]

$\therefore$ Total number ways $= 4 \times 47 = 188$

(c) If 1^{st} card is queen of club

1^{st} card can be selected in 1 way

2^{nd} card can be selected in 3 ways

$\therefore$ Total number of ways $= 3 \times 1 = 3$ .... (1)

If 1^{st} card is not queen of club

1^{st} card can be selected in 12 ways

2^{nd} card can be selected 8 in 4 ways

$\therefore$ Total number of ways $= 12 \times 4 = 48$ ways .... (2)

Therefore from (1) and (2)

Total number of selections are $48 + 3 = 51$ ways.

4.14 SECOND COUNTING PRINCIPLE : (SUM RULE)

Consider that $\{D_1, D_2, D_3, ..., D_r\}$ is a pairwise disjoint family of sets and set D_j has n_j elements. Then the number of possible selection of elements from the sets D_1 or D_2 or D_3 or .... or D_r is $n_1 + n_2 + ... + n_r$ or consider an event A_1 can occur in n_1 ways and another event A_2 can occur in n_2 ways and A_1 and A_2 are mutually exclusive, then A_1 or A_2 can occur in $(n_1 + n_2)$ ways. It is applicable for any number of events.

Example 4.35 : *A five person committee having members Ankit, Arjit, Sonu, Monu and Nonu is a select a president, vice-president and secretary.*

(a) *In how many ways can this occur if either Sonu or Monu must be president?*

(b) *How many selections are there in which either Nonu is secretary or he is excluded?*

(c) *How many selections exclude Ankit or Arjit?*

Solution :

(a) If Sonu is president then Vice President can be selected at 4 ways and secretary in 3 ways.

$\therefore$ Total number of ways to select the remaining is $= 4 \times 3 = 12$

Similarly, if Monu is president then remaining persons can be selected in 12 ways.

As these are mutually exclusive events, total number of ways if either Sonu or Monu must be president is $= 12 + 12 = 24$.

(b) If Nonu is secretary, total number of ways $= 12$

If Nonu is excluded, total number of selections $= 24$

Total number of ways in which Nonu is secretary or is excluded $= 24 + 12 = 36$

(c) Number of selections when Ankit is excluded $= 24$

Number of selections when Arjit is excluded $= 24$

Total number of selections which exclude Ankit or Arjit $= 24 + 24 = 48$

Example 4.36 : *A computer password consists of an alphabet followed by 3 or 4 digit. Find (a) the total number of passwords than can be created (b) the number of passwords in which no digit repeats.*

Solution :

(a) Total number of password

= Number of passwords having 4 characters + Number of passwords having 5 characters

$= 26 \cdot 10 \cdot 10 \cdot 10 + 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 286000$

(b) When no digit repeats then number of passwords (with 4 characters)

$= 26.10.9.8 = 18720$

Number of 5 character password is $= 26.10.9.8.7 = 131040$

Total number of passwords = 149760.

4.15 PERMUTATION

A permutation is an arrangement of a number of objects in some definite order taken some or all at a time. The total number of permutations of n distinct objects taken r at a time is denoted by ${}^nP_r$ or by $P(n, r)$ where $1 \leq r \leq n$.

Example 4.37 : *Prove that the number of permutations of n things taken all at a time is n!*

Solution : Total number of arrangements $= {}^nP_n$

$${}^nP_n = \frac{n!}{(n-n)!} = n!$$

Example 4.38 : *Determine the value of n when* $6\ {}^nP_3 = 3\ {}^{n+1}P_3$.

Solution :

$$6 \times \frac{n!}{(n-3)!} = 3 \times \frac{(n+1)!}{(n-2)!}$$

$$6 \times \frac{n!}{(n-3)!} = \frac{3(n+1)\,n!}{(n-2)\,(n-3)!}$$

$$\Rightarrow \quad 2(n-2) = n+1$$

$$n = 5$$

Example 4.39 : *How many 6-digit numbers can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, if no digits is repeated?*

Solution : Total number of 6-digit number $= {}^8P_6 = \frac{8!}{2!} = 22560.$

Example 4.40 : *There are 10 persons called for an interview. Each one is capable to be selected for the job. How many permutations are there to select 4 from the 10.*

Solution : Total number of permutations to select 4 person is $= {}^{10}P_4 = \frac{10!}{6!} = 5040.$

4.15.1 Permutations with Restrictions

Example 4.41 : *How many 6 digit number can be formed by using the digits 0, 1, 2, 3, 4, 5, 6, 7 and 8 if every number start with '30' with no digit repeated.*

Solution : All numbers begin with '30'. So we have to choose 4 digits from 7 digits.

Total number of numbers = $^7P_4 = 840$.

Example 4.42 : *In how many ways 5 different microprocessor books and 4 different digital electronic books be arranged in a shelf so that all the four digital electronic books are together?*

Solution : Consider the 4 digital books as one unit.

Thus, we have 6 units that can be arranged in 6! ways. For each of the arrangement, 4 electronic books can be arranged in 4! ways.

$\therefore$ Total number of ways $= 6! \times 4! = 17280$

Example 4.43 : *How many permutations can be made out of the letters of word 'COMPUTER'? How many of these*

(i) begin with C?

(ii) end with R?

(iii) begin with C and end with R?

(iv) C and R occupy the end places?

Solution :

(i) The first position can be filled with only one way i.e., C and the remaining 7 letters can be arranged in 7! ways.

$\therefore$ Total number of permutations $= 1 \times 7! = 5040$

(ii) Total number of permutations $= 1 \times 7! = 5040$

(iii) 1st and last positions are fixed. 6 letters in between can be arranged in 6! ways. Total number of permutations

$$= 1 \times 6! \times 1 = 720$$

(iv) C & R occupy end positions in 2! ways

Total number of permutations $= 2! \times 6! = 1440$

4.15.2 Permutations when all the Objects are not Distinct

The number of permutations of n objects of which n_1 objects are of one kind and n_2 objects are of other kind when all are taken at a time is $\dfrac{n!}{n_1!\, n_2!}$

Example 4.44 : *Determine the number of permutations that can be made out of letters of word 'PROGRAMMING'.*

Solution : Total number of letters = 11

Letters R, M and G are repeating twice

Total number of permutations $= \dfrac{11!}{2!\ 2!\ 2!} = 4989600$

Example 4.45 : *There are 4 blue, 3 red and 2 black pens in the box. These are drawn one by one. Determine all the different permutations.*

Solution : Total number of pens $= 9$

$$\text{Total number of permutations} = \frac{9!}{4!\ 3!\ 2!} = 1080$$

Example 4.46 : *How many different variable names can be formed by using the letters a, a, a, b, b, b, c, c, c.*

Solution : $$\text{Total number of names} = \frac{10!}{3!\ 4!\ 3!} = 4200$$

4.15.3 Permutations with Repeated Objects

The number of different permutations of n distinct objects taken r at a time when every object is allowed to repeat any number of times is given by n^r.

Example 4.47 : *How many 4 digit numbers can be formed by using the digits 2, 4, 6, 8 when repetition of digits is allowed.*

Solution : Number of ways of filling unit's place = 4.

Number of ways of filling ten's place = 4

Number of ways of filling hundred's place = 4

Number of ways of filling thousand's place = 4

Total number of ways $= 4 \times 4 \times 4 \times 4 = 256$

Example 4.48 : *How many 2 digit even numbers can be formed by using the digits 1, 3, 4, 6, 8 when repetition of digits is allowed?*

Solution : Number of ways of filling unit's place = 3

Number of ways of filling ten's place = 5

$\therefore$ Total number of ways $= 3 \times 5 = 15$

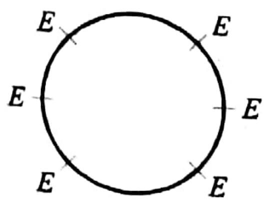
4.15.4 Circular Permutations

The total number of circular permutations of n different objects in $(n-1)!$.

Example 4.49 : *Determine the number of ways in which 5 software engineers and 6 electronics engineers can be seated at a round table so that no two software engineers can sit together?*

Solution : There are 6 electronics engineers that can be arranged round a table in $(6-1)!$ ways. There are 5 software engineer and they are not to sit together so we have six places for software engineers and can be placed in 6! ways.

Total number of ways to arrange the engineers on a round table is $= (6-1)! \times 6! = 86400$.



where E denotes electronic engineer.

4.16 COMBINATION

A combination is a selection of some or all objects from a set of given objects where order of objects does not matter. The number of combinations of n objects taken r at a times is ${}^nC_r$.

Example 4.50 : *Determine the value of n if* ${}^nC_{n-2} = 10$.

Solution :
$$\frac{n!}{(n-2)!\,2!} = 10$$
$$\Rightarrow \quad n(n-1) = 20$$
$$n^2 - n - 20 = 0$$
$$n = -4, 5 \quad \therefore \; n = 5$$

Example 4.51 : *How many ways can we select a software development group of 1 project leader, 5 programmers and 6 data entry operators from a group of 5 project leaders, 20 programmers and 25 data entry operators?*

Solution : Project leader can be selected in ${}^5C_1$ ways

5 Programmers can be selected in ${}^{20}C_5$ ways

6 data entry operators can be selected in ${}^{25}C_6$ ways

Total number of selections = ${}^5C_1 \times {}^{20}C_5 \times {}^{25}C_6$.

Example 4.52 : *In how many ways can a selection of 5 books out of 12 books be made.*

(a) *When one specified book is always included?*

(b) *When one specified book is always excluded?*

Solution : (a) Since one book is always included.

$\therefore$ 4 books are to be selected from 11 books.

Total number of ways ${}^{11}C_4 = 330$.

(b) Since one book is always excluded therefore 5 books are to be selected from 11 books.

Total number of ways = ${}^{11}C_5 = 462$

Example 4.53 : *Prove that* ${}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$

Solution :
$$\text{RHS} = \frac{(n-1)!}{r!\,(n-r-1)!} + \frac{(n-1)!}{(r-1)!\,(n-r)!}$$
$$= \frac{(n-1)!}{(r-1)!\,(n-r-1)!}\left[\frac{1}{r} + \frac{1}{n-r}\right] = \frac{n(n-1)!}{r(r-1)!\,(n-r)\,(n-r-1)!}$$
$$= \frac{n!}{r!\,(n-r)!} = {}^nC_r$$

Example 4.54 : *A collection of 100 pens contain 8 deflective ones*

(a) *In how many ways can a sample of 10 pens be selected?*

(b) *In how many ways can a sample of 10 pens be selected which contain 7 good and 3 deflective.*

(c) *In how many ways can the sample contains 10 pens such that either the sample contain 7 goods ones and 3 deflective or 5 good and 5 defective?*

Solution :

(a) Total number of ways = ${}^{100}C_{10} = \dfrac{100!}{10!\ 90!}$

(b) 7 good pens can be selected in ${}^{92}C_7$ ways

3 defective pens can be selected in ${}^{8}C_3$ ways

$\therefore$ Total number of ways = ${}^{92}C_7 \cdot {}^{8}C_3$.

(c) Total number of ways of selecting a sample containing 7 good and 3 defective pens = ${}^{92}C_7 \cdot {}^{8}C_3$.

Also, total number of ways of selecting a sample containing 5 good and 5 defective pens = ${}^{95}C_5 \cdot {}^{8}C_5$.

$\therefore$ Total number of ways that a sample contains either 7 good and 3 defective or 5 good and 5 defective = ${}^{92}C_7\ {}^{8}C_3 + {}^{92}C_5\ {}^{8}C_5$

Example 4.55 : *From 10 programmers in how many ways can 5 be selected when*

(a) A particular programmer is included every time

(b) A particular programmer is not included at all

Solution : (a) ${}^{9}C_4 = 126$ **(b)** ${}^{9}C_5 = 126$

Example 4.56 : *Determine the number of diagonals that can be drawn by joining the nodes of octagon?*

Solution : The number of lines that can be formed by joining 2 out of 8 points is = ${}^{8}C_2 = 28$.

Out of 28 lines, the 8 are sides of octagon.

$\therefore$ Number fo diagonals = 28 – 8 = 20.

Example 4.57 : *How many lines can be drawn through 10 points on a circle.*

Solution : Number of lines drawn by taking 2 points at a time = ${}^{10}C_2$.

Example 4.58 : *Determine the number of Δs that are formed by selecting points from a set of 15 points out of which 8 are collinear.*

Solution : When we take all 15 points, the number of Δs is ${}^{15}C_3$.

As 8 points lie on a line, they do not form any Δ.

Thus, ${}^{8}C_3$ Δs are lost.

Total number of $\Delta s = {}^{15}C_3 - {}^{8}C_3 = 854$

4.16.1 Combination with Repetition

If repetition is allowed the number of unordered samples of size r from an element set is ${}^{n+r-1}C_r$.

Example 4.59 : *Consider {a, b, c, d}. In how many ways can we select 2 of these letters when repetition is allowed?*

Solution : $n = 4,\ r = 2$

$\therefore$ Total number of way $= {}^{4+2-1}C_2 = {}^{5}C_2 = 10$

Example 4.60 : *In how many ways can 12 balloons be distributed at a Birthday party among 10 chidlren?*

Solution : $n = 10,\ r = 12$

Total number of ways = $^{10+12-1}C_{12} = {}^{21}C_{12} = 293939$

If we ensure that each child get a balloon then distribute each balloon to a child. Now 2 balloons are left. $\therefore$ it can be distributed in $^{10+2-1}C_2 = {}^{11}C_2 = 55$ ways.

Example 4.61 : *Find the number of unordered samples of size five (repetition allowed) from {a, b, c, d, e, f}*

(a) No further restriction

(b) a occurs atleast twice

(c) a occurs exactly twice

Solution : (a) $n = 6, r = 5$

$\therefore$ Total number of samples = $^{6+5-1}C_5 = {}^{10}C_5 = 252$

(b) Since a occur atleast twice. $\therefore$ We have to find sample of 3 from 6 elements.

$$\therefore \quad n = 6, \quad r = 3$$

$$\therefore \quad \text{Total number of ways} = {}^{n+r-1}C_r = {}^8C_3 = 56$$

(c) Since a occurs exactly twice. $\therefore$ we have to find sample of 3 from 5

$$n = 5, \quad r = 3$$

$$\therefore \quad \text{Total number of ways} = {}^{5+3-1}C_3 = {}^7C_3 = 35$$

Example 4.62 : *How many solution does the equation*

$x + y + z = 17$ have where x, y, z are non-ve integers?

Solution : Each solution of given equation is selecting 17 items from 3 numbers (x, y, z)

i.e., selecting 17 items from set (x, y, z)

$\therefore$ Required number of solution = $^{17+3-1}C_{17} = {}^{19}C_{17} = {}^{19}C_2 = 171$

Example 4.63 : *How many solutions are there of $x + y + z = 17$ subject to constraints $x \geq 1, y \geq 2, z \geq 3$.*

Solution : $x = 1 + u, \; y = 2 + v, \; z = 3 + w$

$$\therefore \quad u + v + w = 11$$

$\therefore$ we have to select 11 elements from 3.

$\therefore$ Required number of solution = $^{11+3-1}C_{11} = {}^{13}C_{11} = 78$

Example 4.64 : *How many different numbers can be formed by using six out of 9 digits 1, 2, 3, ... 9?*

Solution : $^9P_6$

Example 4.65 : *How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1 so that odd digits always occupy the odd places?*

Solution : There are total 7 places out of which 4 are odd and 3 are even. The 4 odd places can be occupied by 4 odd digits 1, 3, 3, 1 in $\frac{4!}{2!\,2!}$ ways.

The remaining 3-even places can be occupied in $\frac{3!}{2!}$ ways.

Total number of permutation are $\frac{4!}{2!\,2!} \times \frac{3!}{2!} = 18$

Example 4.66 : *How many words can be made by using the letters of 'ENGLAND' word such that*

(a) words begin and end with vowel

(b) all the vowels are together

Solution :

(a) We have 2 vowels and 5 consonants of which 2 are alike. The first place can be filled in 2 ways and last in 1 ways. The rest 5 places can be filled by 5 consonants in which 2 are alike

$$= \frac{2!\ 5!}{2!} = 120$$

(b) Let (E, A) is single letter than (E, A), N, G, L, N, D can be arranged in $\frac{6!}{2!}$ ways. E and A can be arranged in 2! ways.

Total number of words $= \frac{6!}{2!} \times 2! = 6! = 720$

Example 4.67 : *In how many ways can 7 boys and 5 girls be seated together in a row so that no two girls are together?*

Solution : 7 boys can take seats in 7! ways and girls can take their seat at 8 places.

So total number of ways ${}^8P_5$.

Total number of ways $= 7!\,{}^8P_5 = 33868800$.

Example 4.68 : *Out of 7 consonants and 4 vowels, how many words can be made each containing 3 consonants and 2 vowels.*

Solution : 3 consonants an be selected in ${}^7C_3$ ways

2 vowels can be selected in ${}^4C_2$ ways

Alphabets in five places can be arranged in 5! ways.

$\therefore$ Total number of words $= {}^nC_3\,{}^4C_2\,5! = \frac{7!}{3!\ 4!}\ \frac{4!}{2!\ 2!}\ 5!$

$$= 7!\,\frac{5.4}{2\cdot 2}25200$$

Example 4.69 : *In a shipment, there are 40 floppy discs of which 5 are defective. Determine*

(a) In how many ways we can select 5 floppy discs?

(b) In how many ways we can select 5 non-defective discs?

(c) In how many ways we can select 5 discs containing exactly 3 defective floppy discs?

(d) In how many ways we can select 5 discs containing atleast 1 defective floppy discs?

Solution :

(a) ${}^{40}C_5$

(b) ${}^{35}C_5 = 324632$

(c) ${}^5C_3 \times {}^{35}C_2 = 5950$

(d) ${}^{40}C_5 - {}^{35}C_4 = 611625$

EXERCISE

1. Write the following sets in Roster form :
   (a) $A = \{x : x \text{ precedes d in alphabets}\}$
   (b) $B = \{x : x \text{ is a letter in 'Prayer'}\}$
   (c) $C = \{x : x^2 - 3x + 2 = 0\}$
   (d) $D = \{x : x \text{ is even and } 0 < x < 10\}$
2. Write the following sets in set builder form :
   (a) $A = \{5, 10, 15, 20, 25, ...\}$
   (b) $B = \{a, b, c, d, e, ...\}$
   (c) $C = \{3, 5, 6, 9, 10, 12, 15, ...\}$
   (d) $D = \{10, 100, 1000, 10000, 100000\}$
3. Which of the following sets are equal?
   $A = \{x : x^2 - 2x + 1 = 0\}$ $\qquad$ $B = \{1, 2\}$
   $C = \{1\}$ $\qquad$ $D = \{x : x^2 - 4x + 3 = 0\}$
   $E = \{1, 3, 1\}$ $\qquad$ $F = \{x : x \text{ is odd}, 0 < x < 5\}$
4. Let $A = \{\{1, 1\}, \{1, 2\}\}$. Determine whether each of the following is true or false. Justify your answer :
   (a) $\phi \in P(A)$ $\qquad$ (b) $\phi \subseteq P(A)$
   (c) $\{1\} \in P(A)$ $\qquad$ (d) $\{\phi\} \in P(A)$
   (e) $\{1\} \subseteq P(A)$ $\qquad$ (f) $\{\{1\}\} \in P(A)$
   (g) $\{\{1\}, \{1, 2\}\} \in P(A)$ $\qquad$ (h) $\{1, \{1\}, \{1, 2\}\} \in P(A)$
   (i) $\{1, 2\} \in P(A)$ $\qquad$ (j) $\{1, \{1\}\} \subseteq P(A)$
5. What is cardinality of following sets :
   (a) ϕ $\qquad$ (b) $\{a\}$
   (c) $\{\{a\}\}$ $\qquad$ (d) $\{\phi, \{\phi\}\}$
   (e) $P\{a, b, \{a, b\}\}$ $\qquad$ (f) $\{\phi\}$
   (g) $\{a, \{a\}\}$ $\qquad$ (h) $\{a, \{a\}, \{a, \{a\}\}\}$
6. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
   $A = \{1, 3, 4, 9\}$ $\qquad$ $B = \{1, 2, 3, 4\}$
   $C = \{2, 4, 6, 8\}$ $\qquad$ $D = \{1, 3, 5, 7\}$
   Find each of following :
   (a) $(A \cup B) \cap C$ $\qquad$ (b) $A \cup (B - C)$
   (c) $A \cup (B \cap C$ $\qquad$ (d) $C^C \cup D^C$
   (e) $A^C \cap D^C$ $\qquad$ (f) $(C \cup D)^C$
   (g) $(A \cup B) - C$ $\qquad$ (h) $(B - C) - D$
   (i) $(A \cup B) - (C \cup D)$ $\qquad$ (j) $(A - B) \cup (C - D)$
   (k) $A \oplus (B \cap D)$ $\qquad$ (l) $A \cup (B \oplus C)$

7. Find the set A and B if $A - B = \{1, 5, 7, 8\}$, $B - A = \{2, 10\}$ and $A \cap B = \{3, 6, 9\}$.
8. Prove following analytically :
   (a) $A - B = A \cap B^C$
   (b) $A - (A \cap B) = A - B$
   (c) $(A \cap B) \cup (A \cap B^C) = A$
   (d) $(A \cap B) \cup (B - A) = A$
   (e) $(A - B)C = A^C \cup B$
   (f) $A \cap (B - A) = \phi$
   (g) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
   (h) $A \cup B = (A \cap B) \cup (A \cap B^C) \cup (A \cap B)$
9. Prove that $A \cup B = A$ iff $B \subseteq A$.
10. Prove that $A \cap B = A$ iff $A \subseteq B$.
11. Draw Venn diagram of following :
    (a) $A^C \cap B^C$
    (b) $A \cap B^C \cap C^C$
    (c) $A^C \cap (B \cup C)$
    (d) $(A - B) \cap (A - C)$
12. Prove that $A \times B = B \times A$ iff $A = B$.
13. For sets A, B and C prove that

$$(A \cap B) \times C = (A \times C) \cap (B \times C)$$

14. In a survey of 60 people, it was found that 25 read India today, 26 read Forbes and 26 read Business today. 9 read both India today and Business today. 11 read India today and Forbes and 8 read both Forbes and Business today. 3 read all three magazines.
15. Among integers 1 to 1000.
    (a) How many are not divisible by 3 nor by 5 nor by 7?
    (b) How many are not divisible by 5 or 7 but divisible by 3?
16. In a college, 60% of student took Mathematics, 50% of them took Physics and 70% took Chemistry. 20% studied Mathematics and Physics, 40% studied Physics and Chemistry and 30% took Mathematics and Chemistry. How many percent of student took all three subjects?
17. In a group fo 26 people, 8 take tea but not coffee and 16 take tea. How many take coffee but not tea?
18. The members of a group of 400 peoples speak either Hindi or English or both. If 270 speak Hindi only and 50 speak both Hindi and English. How many of them speak English only?
19. In a town with population 5000, 3200 peoples are egg-eater, 2500 are meat eater and 1500 eat both egg and meat. How many are pure vegetarian?
20. Illustrate the Distributive law

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

using a Venn diagram.

21. A room has 6 doors. In how many ways can a man enter the room through one door and come out through a different door?
22. How many words (with or without meaning) of three distinct letters of English alphabet are there?
23. A gentleman has 6 friends to invite. In how many ways can he send invitation cards to them, if he has three servants to carry the card?

24. How many three letter words can be formed using a, b, c, d, e if
    (i) Repetition is not allowed
    (ii) Repetition is allowed
25. In how many ways 5 letters can be posted in 4 letter boxes?
26. If ${}^9P_5 + 5 \cdot {}^9P_4 = {}^{10}P_r$. Find r?
27. If ${}^{22}P_{r+1} : {}^{20}P_{r+2} = 11 : 52$. Find r?
28. In how many ways six persons can stand in a queue?
29. How many three digit numbers are there, with distinct digits, with each digit odd?
30. In how many ways can 6 boys and 5 girls be arranged in a group photograph if the girls are to sit on chairs in a row and boys are to stand in row behind them?
31. How many words, with or without meaning, can be formed by using the letters of word 'TRIANGLE'.
32. In how many ways can 5 children be arranged in row such that
    (i) two of them, Ram and Shyam, are always together?
    (ii) two of them, Ram and Shyam, are never together?
33. Find the number of ways in which five boys and five girls be seated in a row so that
    (i) No two girls may sit together
    (ii) All the girls sit together and all the boys sit together
    (iii) All the girls are never together
34. Find the number of arrangements of the letters of the word 'INDEPENDENCE'. In how many of these arrangements
    (i) do the words start with?
    (ii) do all the vowels always occur together?
    (iii) do all the vowels never occur together?
    (iv) do the words begin with I and end with P?
35. If ${}^{n+2}C_8 : {}^{n-2}P_4 = 57 : 16$. Find n.
36. A question paper has 2 parts, part A and part B, each containing 10 questions. If the student has to choose 8 from part A and 5 from part B, in how many ways he can choose the questions?
37. A person wishes to make up as many different parties as he can out of 20 friends such that each party consists of same number of persons. How many friends should he invite?
38. How many diagonals are there in a polygon with n sides?
39. How many chords can be drawn through 20 points on a circle?
40. A group consists of 4 girls and 7 boys. In how many ways can a team of 5 members be selected if the team has
    (i) no girls
    (ii) atleast one boy and one girl
    (iii) atleast 3 girls

ANSWERS

1. **(a)** $A = \{a, b, c\}$
   (b) $B = \{p, r, a, y, e\}$
   (c) $C = \{1, 2\}$
   (d) $D = \{2, 4, 6, 8\}$
2. **(a)** $A = \{x : x$ is multiple of 5 and $x \in N\}$
   (b) $B = \{x : x$ is alphabet of English$\}$
   (c) $C = \{x : x$ is multiple of 3 or 5 and $x \in N\}$
   (d) $D = \{x : x = 10^n$ and $1 \leq n \leq 5\}$
3. $A = C, D = E = F$
4. **(a)** True **(b)** True
   (c) True **(d)** True
   (e) False **(f)** True
   (g) True **(h)** True
   (i) True **(j)** False
5. **(a)** 0 **(b)** 1
   (c) 1 **(d)** 2
   (e) 8 **(f)** 1
   (g) 2 **(h)** 3
6. **(a)** $\{2, 4\}$ **(b)** $\{1, 3, 4, 9\}$
   (c) $\{1, 2, 3, 4, 9\}$ **(d)** $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
   (e) $\{2, 6, 8\}$ **(f)** $\{9\}$
   (g) $\{1, 3, 9\}$ **(h)** ϕ
   (i) $\{9\}$ **(j)** $\{2, 4, 6, 8, 9\}$
   (k) $\{4, 9\}$ **(l)** $\{1, 3, 4, 6, 8, 9\}$
7. $A = \{1, 3, 5, 6, 7, 8, 9\}$
   $B = \{2, 3, 6, 9, 10\}$

14. **(a)** 52 **(b)** 30
15. **(a)** 457 **(b)** 229
16. 10%
17. 10
18. 80
19. 800
21. 30
22. 15600
23. 729
24. **(i)** 60 **(ii)** 125

25. 1024
26. $r = 5$
27. $r = 7$
28. 720
29. 60
30. 86400
31. 8!
32. **(i)** 48 **(ii)** 72
33. **(i)** $5! \times 6!$ **(ii)** $2(5!)^2$ **(iii)** $10! - 5! \times 6!$
34. 1663200
    (i) 138600 **(ii)** 16800
    (iii) 1646400 **(iv)** 12600
35. $n = 19$
36. 11340
37. 10
38. $n(n-3)/2$
39. 190
40. **(i)** 21 **(ii)** 441 **(iii)** 91

Chapter 5

Rate of Growth

5.1 GROWTH OF FUNCTIONS

We need to approximate the number of steps required to execute any algorithm because of the difficulty involved in expression or difficulty in evaluating an expression. We compare one function with another function to know their rate of growths.

If f and g are two functions we can give the statements like 'f has same growth rate as g' or 'f has higher growth rate than g'.

In this section we will introduce the big oh, big omega, big theta and little oh notations to compare growth of functions.

5.2 BIG Oh OR BIG-O

Big-O notation is used extensively to estimate the number of operations an algorithm uses as its input grows. Also we can compare two algorithms to determine which is more efficient at the size of input grows using Big-O notations.

Definition : Let f and g be functions from the set of integers or set of real numbers to the set of real numbers. We say that $f(x)$ is $O(g(x))$ if there are positive constants C and k such that

$$|f(x)| \leq C|g(x)| \text{ whenever } x \geq k \quad \text{.... (1)}$$

To establish that $f(x)$ is Big-O of $g(x)$ we need one pair of C and k satisfying equation (1).

Remark : C and k are not unique i.e., there are infinitely many pairs of C and k to show $f(x)$ is Big-O of $g(x)$.

For example : Let $|3x^3 + 2x + 7| \leq 12|x^3|$ whenever $x \geq 1$

Here $C = 12$ and $k = 1$

$\therefore \quad 3x^2 + 2x + 7$ is $O(x^3)$.

Remarks :

(i) Big-O notation gives an upper bound on the growth rate of the function.

(ii) g is an asymptotic upper bound of f.

5.3 ORDER OF POWER FUNCTION

Theorem : For any positive integers r and s, if $r < s$ then x^r is $O(x^s)$.

Proof : Let $\qquad x \geq 1$

$$x^2 \geq x$$
$$\Rightarrow \quad x^3 \geq x^2$$
$$\Rightarrow \quad x^{r+1} \geq x^r$$
$$x^s \geq x^{s-1}$$

In general, $1 \leq x \leq x^2 \ldots\ldots x^r \leq x^{r+1} \ldots\ldots \leq x^{s-1} \leq x^s$

Therefore for any rational number r and s if $r < s$ then x^r is $O(x^s)$.

For examples : **1.** x^2 is $O(x^3)$

2. $7x^4$ is $O(x^6)$

In (2), 7 is a constant and growth of function is not affected by constant factor.

5.4 ORDER OF POLYNOMIAL FUNCTION

Theorem : Let $f(x) = a_n x^n + a_{n-1} x^{n-1} \ldots + a_1 x + a_0$

where $a_0, a_1, a_2, \ldots\ldots a_{n-1}, a_n$ are real numbers and $a_n \neq 0$ then $f(x)$ is Big O of x^n or $f(x)$ is $O(x^n)$.

Proof : Using triangular inequality is $x \geq 1$ we have,

$$\begin{aligned}
|f(x)| &= |a_n x^n + a_{n-1} x^{n-1} \ldots\ldots + a_1 x + a_0| \\
&\leq |a_n x^n| + |a_{n-1} x^{n-1}| + \ldots + |a_1 x| + |a_0| \\
&= |a_n||x^n| + ||x^{n-1}| \ldots\ldots + |a_1||x| + |a_0| \\
&\leq |a_n||x^n| + |a_{n-1}||x^n| + \ldots + |a_1||x^n| + |a_0||x^n| \\
&= (|a_n| + |a_{n-1}| + \ldots\ldots + |a_1| + |a_0|)|x^n| \\
|f(x)| &\leq C|x^n| \text{ whenever } x \geq 1
\end{aligned}$$

where $C = |a|_n + |a_{n-1}| \ldots\ldots |a_1| + |a_0|$

Hence $f(x)$ is $O(x^n)$.

For example : **(i)** $f(x) = 2x^4 + 4x^3 + 5$ is $O(x^4)$

(ii) $f(x) = 5x^3 + 3x^2 + 1$ is $O(x^3)$

5.5 O-ARITHMETIC

Let f and g be functions and k be a constant then

1. $O(kf) = O(f)$
2. $O(fg) = O(f)\, O(g)$
3. $O(f/g) = O(f)/O(g)$
4. $O(f+g) = \text{Max}\,[O(f), O(g)]$

5.6 BIG OMEGA AND BIG THETA NOTATION

[UPTU, B.Tech. 2009-10]

When $f(x)$ is Big-O of $g(x)$ then $g(x)$ is upper bound of $f(x)$ for large values of x. However Big-O notation does not provide lower bound of $f(x)$ for large x. To get this we use Big-Omega (Big-Ω) Notation. When we want to get both lower and upper bound of $f(x)$ relative to $g(x)$, we use Big-θ Notation.

Definition : Let f and g be functions from the set of integers or the set of real number to the set of real numbers. We say that $f(x)$ is $\Omega\,(g(x))$ if there are positive constants C and k such that

$$|f(x)| \geq C|g(x)| \text{ whenever } x \geq k$$

Result : $f(x)$ is $\Omega(g(x))$ if and only if $g(x)$ is $O(f(x))$ i.e., Big Omega is reverse of Big-O.

Remarks : (i) Big W gives a lower bound on the growth rate of the function.

(ii) g is asymptotic lower bound of f.

For example : Let $|3x^3 + 4x^2 + 6| \geq 6|x^3|, \ x \geq 1$

Here, $C = 6$ and $k = 1$

$\therefore$ $3x^3 + 2x + 7$ is $\Omega(x^3)$.

Definition : Let f and g be functions from the set of integer or the set of real number to the set of real number. We say $f(x)$ is θ $(g(x))$ if $f(x)$ is $O(g(x))$ and $g(x)$ is $\Omega(g(x))$.

$f(x)$ is $\theta(g(x))$ is also read as $f(x)$ is of order $g(x)$.

Result : $f(x)$ is $\theta(g(x))$ if $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$.

Remarks : (i) Big θ gives both lower and upper bound on the growth rate of function.

(ii) g is an asymptotic tight bound of f.

For example : Let $f(x) = 5x^2$ and $g(x) = x^2$

then $|5x^2| \leq 5|x^2|$ where $x \geq 1$

$C = 5$ and $k = 1$

$\Rightarrow$ $|f(x)| \leq C \ |g(x)|, x \geq k$

$\therefore$ $f(x)$ is $O(g(x))$ .... (1)

Now, $|5x^2| \geq |x^2|, \ x \geq 1$

Here, $C = 1$ and $k = 1$

$|f(x)| \geq C|g(x)|, \ x \geq k$

$\therefore$ $g(x)$ is $O(f(x))$ .... (2)

From (1) and (2) we can say that $f(x)$ is $\theta(g(x))$.

5.7 LITTLE-o

Let f and g be functions from set of integer or the set of real number to set of real number. We say that $f((x)$ is $o(g(x))$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$$

It means that $g(x)$ grows much faster than $f(x)$.

Also we can write that

$$|f(x)| < C \ |g(x)| \text{ whenever } x \geq k$$

little-o bound is a bit tighter than Big O bound since we use '<' sign in little-o for comparison instead of '≤' used in Big-O.

For example : If $f(x) = 4x$ and $g(x) = x^2$ then

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{4x}{x^2} = 0$$

$\therefore$ $f(x)$ is little-o of $g(x)$ or $f(x)$ is $o(g(x))$.

Example 5.1 : *Use the definition of $f(x)$ is $O(g(x))$ to show that $3x^3 + 20x^2 + 5$ is $O(x^3)$.*

Solution : Let $f(x) = 3x^3 + 20x^2 + 5$ and $g(x) = x^3$.

We know that

for $x \geq 1$

$$x^3 \geq x^2 \text{ and } x^3 \geq 1$$

$$\therefore \quad |3x^3 + 20x^2 + 5| \leq |3x^3| + |20x^2| + |5|$$

$$\leq 3|x^3| + 20|x^3| + 5|x^3|$$

$$= 28|x^3|$$

Here $C = 28$ and $k = 1$

$\therefore$ By definition $\quad |f(x)| \leq C\,|g(x)|$ whenever $x \geq k$

where $C = 28$, $k = 1$

$\therefore \quad 3x^3 + 20x^2 + 5$ is $O(x^3)$.

Example 5.2 : *Show that $x^2 + 5x + 11$ is $O(x^3)$ but x^3 is not $O(x^2 + 5x + 11)$.*

Solution : We will first show that $x^2 + 5x + 11$ is $O(x^3)$.

Let $f(x) = x^2 + 5x + 11$ and $g(x) = x^3$

We know that

for $x \geq 1$

$$x^3 \geq x^2 \text{ and } x^3 \geq x \text{ and } x^3 \geq 1$$

Now, $\quad |x^2 + 5x + 11| \leq |x^2| + |5x| + |11| \quad$ [By triangle inequality]

$$\leq |x^3| + 5|x^3| + 11|x^3|$$

$$= 17|x^3|$$

$\therefore$ By definition of Big O

$$|f(x)| \leq C\,|g(x)| \text{ whenever } x \geq k$$

where $C = 17$ and $k = 1$

$\therefore \quad x^2 + 5x + 11$ is $O(x^3)$

then $\quad |x^3| \leq C|x^2 + 5x + 11|$ whenever $x \geq k$ .... (1)

By triangle inequality

$$|x^2 + 5x + 11| \leq |x^2| + |5x| + |11|$$

if $\quad x \geq 1 \Rightarrow x^2 \geq x$ and $x^2 \geq 1$

$\therefore \quad |x^2 + 5x + 11| \leq |x^2| + 5|x^2| + 11|x^2|$ .... (2)

$$= 17|x^2|$$

From (1) and (2) $\quad$ [$\because k = 1$]

$$|x^3| \leq 17C|x^2| \text{ whenever } x \geq 1$$

Since x is a positive number and $C > 0$

$$x^3 \leq 17Cx^2$$

$\Rightarrow \quad x \leq 17C \quad$ [$\because x \geq 1$]

which is a contradiction for sufficiently or arbitrarily large x.

Therefore x^3 is not $O(x^2 + 5x + 11)$.

Example 5.3 : *Use definition of O notation to prove that $1^2 + 2^2 + ... + n^2$ is $O(n^3)$.*

Solution : By formula for sum

$$1^2 + 2^2 + ... n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(2n^2 + 3n + 1)}{6}$$

$$= \frac{2n^3}{6} + \frac{3n^2}{6} + \frac{n}{6}$$

$$= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

Now,

$$|f(n)| = |1^2 + 2^2 + ... n^2|$$

$$= \left|\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right|$$

$$\leq \left|\frac{n^3}{3}\right| + \left|\frac{n^2}{2}\right| + \left|\frac{n}{6}\right| \quad \text{[By triangular inequality]}$$

$$= \frac{1}{3}|n^3| + \frac{1}{2}|n^2| + \frac{1}{6}|n| \quad .... (1)$$

Now, let

$$n \geq 1$$

$$n^2 \geq n$$

$$n^3 \geq n^2 \qquad \therefore\ n^3 \geq n^2 \geq n \geq 1$$

Substituting above inequalities in (1) we get,

$$|f(n)| \leq \frac{1}{3}|n^3| + \frac{1}{2}|n^3| + \frac{1}{6}|n^3|$$

$$= |n^3|$$

$\therefore$ for $C = 1$ and $k = 1$

$$|f(n)| \leq C\,|g(n)| \text{ whenever } n \geq k$$

$\therefore$ $1^2 + 2^2 + ... + n^2$ is $O(n^3)$

Alternate method : Each square of integer n is greater than square of every positive integer that precedes it.

$$\therefore \quad 1^2 + 2^2 + ... + n^2 = \underbrace{n^2 + n^2 + ... + n^2}_{n \text{ times}}$$

$$= n \times n^2$$

$$= n^3$$

$$\therefore \quad |f(n)| = |1^2 + 2^2 + ... + n^2|$$

$$\leq |n^3|$$

for $C = 1$ and $k = 1$

$$|f(n)| \leq C\,|g(n)| \text{ whenever } x \geq k$$

$\therefore$ $1^2 + 2^2 + \ldots n^2$ is $O(n^3)$

Example 5.4 : *Show that $x \log x$ is $o(x^2)$.*

Solution : Let

$$f(x) = x \log x$$
$$|f(x)| = |x \log x|$$
$$< |x \cdot x| = |x^2| \quad \text{where } x \geq 1 \qquad [\because \log x < x]$$

for $C = 1$ and $k = 1$

$$|f(x)| < C\,|g(x)| \text{ whenever } x \geq k$$

$\therefore$ $x \log x$ is $o(x^2)$

Example 5.5 : *Show that 2^x is $O(3^x)$ but 3^x is not $O(2^x)$.*

Solution : Let $f(x) = 2^x$ and $g(x) = 3^x$.

Since $2^x \leq 3^x \ \forall\ x \geq 0$

$\therefore$
$$|f(x)| = |2^x| \leq |3^x| \ \forall\ x \geq 0$$

for $C = 1$ and $k = 0$

$$|f(x)| \leq C\,|g(x)| \text{ whenever } x \geq k$$

$\therefore$ $2x$ is $O(3^x)$

Conversely let 3^x is $O(2^x)$ therefore $\exists$ C and k

such that $|3^x| \leq C|2^x|$ whenever $x \geq k$

x is positive integer [as we have choosen k such that it will make x positive]

$$3^x \leq C\,2^x$$

$$\Rightarrow \quad x \log 3 \leq \log (C) + x \log 2$$

$$\Rightarrow \quad x \leq \frac{\log C}{\log 3/2}$$

which is contradiction as x is ≥ 0 and is arbitrarily large.

$\therefore$ 3^x is not $O(2^x)$.

Example 5.6 : *Show that $x^2 + 100$ is $\Omega(x^2)$.*

Solution : Let

$$f(x) = x^2 + 100$$

and
$$g(x) = x^2$$

Now
$$|f(x)| = |x^2 + 100| \geq |x^2| \quad \text{for } x \geq 1$$

$\therefore$ for $C = 1$ and $k = 1$

$$|f(x)| \geq C\,|g(x)| \text{ whenver } x \geq k$$

$\therefore$ $x^2 + 100$ is $\Omega(x^2)$.

Example 5.7 : *Use the definition of $f(x)$ is $\Omega(g(x))$ to show $x^4 + 9x^3 + 4x + 7$ is $\Omega(x^4)$.*

Solution : Let

$$f(x) = x^4 + 9x^3 + 4x + 7$$
$$g(x) = x^4$$

let $\quad |f(x)| = |x^4 + 9x^3 + 4x + 7| \geq |x^4| \quad \forall\, x \geq 0$

$\therefore$ for $C = 1$ and $k = 0$

$$|f(x)| \geq C\,|g(x)| \text{ whenever } x \geq k$$

Example 5.8 : *Show that $2x^2 + x + 11$ is $\theta(x^2)$.*

Solution : Let

$$f(x) = 2x^2 + x + 11$$

$$g(x) = x^2$$

We know that for $\quad x \geq 1$

$\Rightarrow \quad x^2 \geq 1$ and $x^2 \geq x \geq 1$

$\therefore$

$$|f(x)| = |2x^2 + x + 11|$$
$$\leq 2|x^2| + |x^2| + 11|x^2|$$
$$= 14|x^2| \text{ whenever } x \geq 1$$

$\therefore$ for $C = 14$ and $k = 1$

$|f(x)| \leq C|g(x)|$ whenever $x \geq k$

$\therefore \quad f(x)$ is $O(g(x))$

$\Rightarrow \quad (2x^2 + x + 11)$ is $O(x^2)$ .... (1)

Now we have to show $2x^2 + x + 11$ is $\Omega(x^2)$

Let $\quad |f(x)| = |2x^2 + x + 11|$

$$\geq |2x^2| = 2|x^2| \text{ for } x \geq 1$$

$\therefore \quad$ for $C = 2$ and $k = 1$

$|f(x)| \geq C\,|g(x)|$ whenever $x \geq k$

$\therefore \quad 2x^2 + x + 11$ is $\Omega(x^2)$ .... (2)

From (1) and (2) we can say that $2x^2 + x + 11$ is $\theta(x^2)$.

Example 5.9 : *Show that $2x^3 + x^2 \log x$ is $\theta(x^3)$.*

Solution : Let, $\quad f(x) = 2x^3 + x^2 \log x$

and $\quad g(x) = x^3$

We know that $\log x < x$ for $x \geq 1$

$\therefore \quad x^2 \log x < x^3$ for $x \geq 1$

$\Rightarrow \quad 2x^3 + x^2 \log x \leq 3x^3$ for $x \geq 1$

$\therefore \quad |f(x)| = |2x^3 + x^2 \log x| \leq 3|x^3|$

So for $C = 3$ and $k = 1$

$|f(x)| \leq C\,|g(x)|$ whenever $x \geq k$

$\therefore \quad f(x)$ is $O(g(x))$ .... (1)

Conversely, $\quad |f(x)| = |2x^3 + x^2 \log x|$

$$\geq |2x^3| \quad \text{(since } x \text{ is positive)}$$
$$= 2|x^3|$$

$\therefore$ for $C = 2$ and $k = 1$

$$|f(x)| \geq C\,|g(x)| \text{ whenever } x \geq k$$

$\therefore \quad 2x^3 + x^2 \log x$ is $\Omega(x^3)$

$\therefore$ from (1) and (2), we get,

$$2x^3 + x^2 \log x \text{ is } \theta(x^3)$$

Example 5.10 : *Let $f(x) = a_n x^n + a_{n-1} x^{n-1} \ldots + a_1 x + a_0$ where $a_0, a_1, \ldots, a_n$ are real numbers with $a_n \neq 0$ then prove that $f(x)$ is $\theta(x^n)$.*

Solution : Let $\quad f(x) = a_n x^n + a_{n-1} x^{n-1} \ldots + a_1 x + a_0$

and $\quad g(x) = x^n$

Consider,

$$|f(x)| = |a_n x^n + a_{n-1} x^{n-1} \ldots + a_1 x + a_0|$$
$$\leq a_n |x^n| + a_{n-1} |x^{n-1}| + \ldots + a_1 |x| + a_0$$

[Using triangle inequality] .... (1)

Now, $\quad x \geq 1$

$\Rightarrow \quad x^2 \geq x$

$\Rightarrow \quad x^3 \geq x^2$

$\vdots$

$\Rightarrow \quad x^n \geq x^{n-1}$

$\Rightarrow \quad x^n \geq x^{n-1} \geq x^{n-2} \geq \ldots \geq x > 1$

Substituting above inequalities in (1)

$$|f(x)| \leq |a_n|\,|x^n| + |a_{n-1}|\,|x^n| + \ldots + |a_1|\,|x^n| + |a_0|\,|x^n|$$
$$= (|a_n| + |a_{n-1}| + \ldots |a_1| + |a_0|)\,|x^n|$$

Therefore for $C = |a_n| + |a_{n-1}| + \ldots + |a_1| + |a_0|$ and $k = 1$

$$|f(x)| \leq C\,|g(x)| \text{ whenever } x \geq k$$

$\therefore \quad f(x)$ is $O(g(x))$ .... (2)

Conversely consider,

$$|f(x)| = |a_n x^n + a_{n-1} x^{n-1} \ldots + a_1 x + a_0|$$
$$\geq |a_n x^n|$$
$$= a_n |x^n|$$

$\therefore$ for $C = \text{an}$ and $k = 1$

$$|f(x)| \geq C\,|g(x)| \text{ whenver } x \geq k$$

$\therefore \quad f(x)$ is $\Omega(g(x))$ .... (3)

From (2) and (3)

$$f(x) \text{ is } \theta(x^n)$$

i.e., $\quad f(x)$ is of order x^n.

Example 5.10 : *Show that x log x is $o(x^2)$.*

Solution : To show that $x \log x$ is $o(x^2)$ we have to show

$$\log_{x \to \infty} \frac{f(x)}{g(x)} = 0$$

let $f(x) = x \log x$ and $g(x) = x^2$

Consider, $$\lim_{x \to \infty} \frac{x \log x}{x^2} = \lim_{x \to \infty} \frac{\log x}{x}$$

Using L'Hospital Rule $$= \lim_{x \to \infty} \frac{1/x}{1} = 0$$

$\therefore$ $x \log x$ is $o(x^2)$.

Example 5.11 : *Show that x^2 is $o(2^x)$.*

Solution : Let $f(x) = x^2$ and $g(x) = 2^x$

Consider, $$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{x^2}{2^x}$$

Using L'Hospital Rule

$$\lim_{x \to \infty} \frac{2x}{2^x \log 2} = \lim_{x \to \infty} \frac{2}{2^x (\log 2)^2} = 0$$

$\therefore$ x^2 is $o(2^x)$.

EXERCISE

1. Show that $x^2(5x + 2)$ is $O(x^3)$.
2. Show that x^3 is $O(x^4)$ but x^4 is not $O(x)$.
3. Use definition of O notation to prove that $1 + 2 + ... + n$ is $O(n^2)$.
4. Give a Big O estimate as possible of $(x^2 + 8)(x + 2)$.
5. Show that if $f(x)$ is $O(x)$ then $f(x)$ is $O(x^2)$.
6. Show that $2x^2 + 15$ is $\Omega(x^2)$.
7. If $f(x) = 2x^4 - 5x^2$ and $g(x) = x^4$. Prove that f and g are of same order.
8. Show that $3x^5 + x(\log x)^4$ is $\theta(x^5)$.

ANSWERS

4. x^3

Chapter 6

Pigeonhole Principle

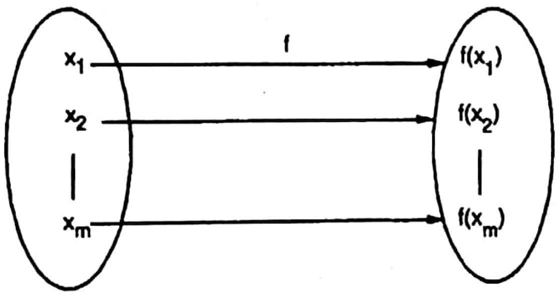
INTRODUCTION

According to Pigeonhole principle.

If n pigeons are assigned to m pigeonholes and $m < n$, then there is atleast one pigeonhole that ntains two or more pigeons.

In set theoretic notation it can be stated as follows : Let $f : X \rightarrow Y$ where X and Y are finite sets, iere cardinality of $X = m$ and cardinality of $Y = n$, $m > n$. Then there exist atleast two distinct ments x_1 and x_2 in X such that $f(x_1) = f(x_2)$.

Proof : Let $X = \{x_1, x_2, \ldots, x_m\}$. Suppose f is injective. Then $f(x_1), f(x_2), \ldots, f(x_m)$ are distinct ment in Y. So $m \leq n$ as f is one to one (injective).



But this a contradiction to our assumption $m > n$. Therefore f is not injective and there must be least two distinct elements x_1 and $x_2 \in X$ such that $f(x_1) = f(x_2)$. Hence proved.

Note : This principle is also called **Dirichelet's Drawer Principle.**

xample 6.1 : *Find the minimum of elements to be selected from the set $A = \{3, 4, 5, 6, 7, 8, 9\}$ such hat sum of two is 12.*

olution : The possible sets or cases of selecting 2 elements from set A such that sum of them is 12 will e as follows :

$$\{3, 9\}, \{4, 8\}, \{5, 7\}, \{6, 6\}$$

Now we can observe that each element of A belongs to one and only one of these sets.

If we select any 2 numbers from A then by Pigeonhole Principle two of them will belong to same set giving sum equal to twelve.

Thus minimum number of digits to be selected from set A should be greater than number of sets. So minimum number of digits selected is 5.

Example 6.2 : *Show that atleast 2 people must have their birthday is same month if there are 13 people under consideration.*

Solution : Let us consider months as pigeonholes. Then there are 12 pigeonholes. Each person will be assigned with one month.

Now number of people will be considered as pigeons that is 13 in number.

So number of pigeons is greater than number of pigeonholes. So by pigeonhole principle atleast 2 people must have their birthday in same month.

Example 6.3 : *Let there are some green balls, some blue balls and some black balls in a box. What is minimum number of balls that one should take out from the box to be sure of getting balls of same colour.*

Solution : Let number of colours = Number of pigeonholes = 3.

Now let number of balls to be taken out = Number of pigeons.

To get balls of same colours i.e., to get atleast two pigeons into same pigeonholes, the number of pigeons should be greater than number of pigeonholes by Pigeonhole Principle i.e., Number of balls taken out > Number of colours.

$\Rightarrow$ Number of balls taken out > 3.

Therefore, minimum four balls should be taken out from box to be sure of getting balls of same colour.

Example 6.4 : *Show that if any five positive integer are chosen, two of them will have same remainder when divided by four.*

Solution : When a number is divided by four we will get following remainders :

$$0, 1, 2, 3$$

Now we have 5 integer.

Number of pigeons = Number of integers = 5

Number of pigeonhole = (0, 1, 2, 3) = 4

Therefore number of pigeons is greater than number of pigeonholes. Therefore, by Pigeonhole Principle, there is atleast one pigeonhole that contain 2 or more pigeons. Therefore, two integers have same remainder.

Example 6.5 : *Let n players participate in a tournament. Each of n player plays against every other player. Also each player wins atleast once. Show that there are atleast 2 players having the same number of wins.*

Solution : Each player plays against each other so each player plays $(n - 1)$ times with other players.

Now he will win atleast once. So minimum number of wins is one and maximum number of wins is $(n - 1)$.

So number of wins are 1, 2, 3, 4, ... $(n - 2)$, $(n - 1)$.

Let number of pigeonholes = Number of wins = $(n - 1)$ [i.e., 1, 2, 3, ... $(n - 2)$, $(n - 1)$]

And number of pigeons = Number of player = n and $n > n - 1$.

So according to Pigeonhole Principle, atleast two pigeons will be in same pigeonhole. So atleast two players will have same number of wins.

Example 6.6 : *Show that if any 14 numbers from 1 to 25 are chosen, then one of them is multiple of another.*

Solution : We can express any positive integer x as follows :

$$x = 2^p r$$

where r is odd, $p \geq 0$ [e.g., $12 = 2^2 3$, $15 = 2^0.15$ ....]

Now if $x \in \{1, 2, 3, ..., 25\}$ then as r is odd part of x therefore, $r \in \{1, 3, 5, ..., 25\}$

Let X be a set of 14 chosen integers.

Define a mapping f such that, $f: X \to \{1, 3, ..., 25\}$

such that $f(x) = r =$ odd part of x.

Now number of element in $x = 14$ (Number of pigeons)

and number of element in $\{1, 3, ..., 25\} = 13$ (Number of pigeonholes)

Since number of element in $X \neq$ number of element in $\{1, 3, ..., 25\}$

$\therefore$ f is not one-one or by set theoretic form of pigeonhole principle.

$\Rightarrow$ Therefore exist 2 integers x_1 and x_2 such that $f(x_1) = f(x_2)$ i.e., their odd part is same.

Let, $f(x_1) = r = f(x_2)$

$\therefore$ $x_1 = 2^p r$ and $x_2 = 2^q r$ for integer p and q

Case I. If $p \geq q$

$$x_1 = 2^p r \quad x_2 = 2^q r$$

$\Rightarrow$
$$x_1 = 2^p \times \frac{x_2}{2^q} = 2^{p-q} x_2$$

$\therefore$ x_1 is multiple of x_2.

Case II. If $p < q$

$$x_2 = 2^q \times \frac{x_1}{2^p} = 2^{q-p} x_1$$

$\Rightarrow$
$$x_2 = 2^{q-p} x_1$$

So x_2 is multiple of x_1.

Thus either x_1 is multiple of x_2 or x_2 is multiple of x_1 i.e., one of them is multiple of other.

Example 6.7 : *Consider any 5 points in the interior of an equilateral triangle of side 1m. Show that there exist 2 points within a distance of atmost 1/2m.*

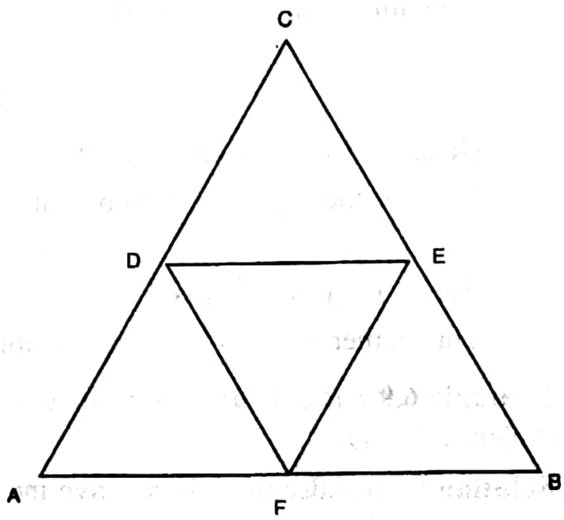
Solution : We will divide the given equilateral triangle into four equal parts such that length of each side of new triangle is 1/2m.

Now place the given 5 points in these 4 equal triangles.

Now number of pigeons = Number of points = 5 = n

Number of pigeonholes = Number of triangles = 4 = m

Since $m < n$. Therefore, by Pigeonhole principle atleast 2 of given points lie within a same triangle. The distance between two points can never exceed the side of triangle which is 1/2m. So there exist 2 points in a distance of atmost 1/2m.



Example 6.8 : *Show that any set of 7 distinct integers contain two integers x and y such that either $x + y$ or $x - y$ is divisible by 10.*

Solution : Let us consider set of 7 distinct integers $X = \{x_1, x_2, \ldots, x_7\}$. Let r_i be the remainder when any element $x_i \in X$ is divided by 10.

Consider following partition of X :

$$X_1 = \{x_i : x_i \in X \text{ and remainder } r_i = 0\}$$
$$X_2 = \{x_i : x_i \in X \text{ and remainder } r_i = 5\}$$
$$X_3 = \{x_i : x_i \in X \text{ and remainder } r_i = 1 \text{ or } 9\}$$
$$X_4 = \{x_i : x_i \in X \text{ and remainder } r_i = 2 \text{ or } 8\}$$
$$X_5 = \{x_i : x_i \in X \text{ and remainder } r_i = 3 \text{ or } 7\}$$
$$X_6 = \{x_i : x_i \in X \text{ and remainder } r_i = 4 \text{ or } 6\}$$

So if we divide any element of X by 10 then its remainder must belong to one of X_i $(i = 1, 2, 3, \ldots 6)$.

Now pigeons = 7 integers which are elements of X.

Pigeonholes = $X_1, X_2, X_3, X_4, X_5, X_6$.

Therefore, number of pigeonholes < Number of pigeons.

So, by Pigeonhole Principle, some X_i must contain atleast two integers $x = x_i$ and $y = x_j$ from X.

Case I. If both x and y are in X_1 or X_2 : If $x \in X_1$ and $y \in X_1$, then $x + y$ and $x - y$ have 0 remainder when divided by 10. So both $x + y$ and $x - y$ are divisible by 10 and if x and $y \in X_2$, then $x + y$ and $x - y$ have 0 remainder when divided by 10. So again both $x + y$ and $x - y$ are divisible by 10.

Case II. If x and y are in X_3 or X_4 or X_5 or X_6 :

If x and y is one of the last four sets then by remainder theorem

$$x = 10x^* + r_i \text{ and } y = 10y^* + r_j$$

where r_i and r_j are in one of the set X_3, X_4, X_5 or X_6.

Now,
$$x - y = 10(x^*) + r_i - 10y^* - r_j$$
$$= 10(x^* - y^*) + (r_i - r_j) \quad \ldots (1)$$

and
$$x + y = 10x^* + r_i + 10y^* + r_j :$$
$$= 10(x^* + y^*) + r_i + r_j \quad \ldots (2)$$

Now if $r_i \neq r_j$ then $r_i + r_j = 10$

Substituting in equation (2)

$$x + y = 10(x^* + y^*) + 10$$
$$= 10(x^* + y^* + 1)$$

Hence $x + y$ is divisible by 10.

If $r_i = r_j$ then $r_i - r_j = 0$. Substituting in equation (1)

$$x - y = 10(x^* - y^*) + 0 = 10(x^* - y^*)$$

Hence $x - y$ is divisible by 10.

Thus either $x + y$ or $x - y$ is divisible by 10 where x and y are in X.

Example 6.9 : *If m is an odd positive integer prove that there exists a positive integer n such that m divides $(2^n - 1)$.*

Solution : Consider $(m + 1)$ positive integers $2^1 - 1, 2^2 - 1, 2^3 - 1, \ldots, 2^m - 1$ and $2^{m+1} - 1$.

When these are divided by m two of the numbers will give the same remainder because by Pigeonhole principle there are $(m + 1)$ pigeons and m remainders are pigeonholes.

Let $2^r - 1$ and $2^s - 1$ be the two numbers which give the same remainder r_i upon division by m.

i.e.,
$$2^r - 1 = q_1 m + r_i \text{ and } 2^s - 1 = q_2 m + r_i$$

$\therefore$
$$2^r - 2^s = (q_1 - q_2)m$$

But
$$2^r - 2^s = 2^s(2^{r-s} - 1)$$

$\therefore$
$$(q_1 - q_2)m = 2^s(2^{r-s} - 1)$$

But m is odd and hence cannot be a factor of 2^s.

$\therefore$ m divides $(2^{r-s} - 1)$.

Taking $n = r - s$ we get, m divides $(2^n - 1)$.

Hence the result is proved.

Example 6.10 : *During a four-week vacation, a school student will attend atleast one computer class each day but he won't attend more than 40 classes in all during the vacation. Prove that no matter how he distributes his classes during the four weeks, there is a consecutive span of days during which he will attend exactly 15 classes.*

Solution : There are total 28 days in four weeks vacation. Let the student attend a_1 classes on day 1, a_2 classes on day 2, ..., a_i classes on day 28.

Then $b_i = a_1 + a_2 + ... + a_{28}$ be the total number of classes he will attend from day 1 to day i where $i = 1, 2, ..., 28$.

Clearly, $\quad 1 \leq b_1 < b_2 < .... < b_{28} \leq 40$

and $\quad b_1 + 15 < b_2 + 15 < .... < b_{28} + 15 \leq 55$

Thus there are 56 distinct numbers (Pigeons) $b_1, b_2, ..., b_{28}$ and $b_1 + 15, b_2 + 15, ..., b_{28} + 15$.

These can take only 55 (pigeonholes) different values. Then, by Pigeonholes principle, atleast two of the 56 numbers are equal.

Since $b_j > b_i$ if $j > i$, the only way for two number, to be equal is $b_j = b_i + 15$ for some i and j, where $j > i$.

$\therefore$
$$b_j - b_i = 15$$

i.e.,
$$a_{i+1} + a_{i+2} + ... + a_j = 15$$

i.e., from the start of day $(i + 1)$ to the end of day j, the student will attend exactly 15 classes.

6.2 GENERALISATION OF PIGEONHOLE PRINCIPLE (EXTENDED PIGEONHOLE PRINCIPLE)

If n pigeons are accommodated in m pigeonholes and $n > m$, then one of the pigeonholes must contain atleast $\left\lfloor \frac{(n-1)}{m} \right\rfloor + 1$ pigeons, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x which is a real number.

Proof : If possible, let each Pigeonhole contain atmost $\left\lfloor \frac{(n-1)}{m} \right\rfloor$ pigeons.

Then the maximum number of pigeons in all pigeonholes

$$= m\left\lfloor\frac{(n-1)}{m}\right\rfloor \leq m \cdot \frac{(n-1)}{m} \qquad \left[\because \left\lfloor\frac{(n-1)}{m}\right\rfloor \leq \frac{(n-1)}{m}\right]$$

i.e., the maximum number of pigeons in all pigeonholes $\leq (n-1)$

This is contradiction to the assumption that there are n pigeons. Hence one of the pigeonholes must contain atleast $\left\lfloor\frac{(n-1)}{m}\right\rfloor + 1$ pigeons.

Example 6.11 : *Prove that in any group of six people, atleast three must be mutual friends and least three must be mutual strangers.*

Solution : Consider x be any person at the party. Let X_1 be the set of persons who are friends to x. Let X_2 be the set of persons who are strangers to x.

Now there are six persons and excluding x there are five persons.

Consider remaining five persons as pigeons and X_1 and X_2 are pigeonholes. By extended pigeonhole principle either X_1 or X_2 contain atleast $\left\lfloor\frac{5-1}{2}\right\rfloor + 1 = 3$ persons.

Let us consider X_1 has 3 persons. If two of these 3 are friends then together with x they form a set of mutual friends. On other hand if no two of three are friend to each other then these three will be set of three mutual strangers.

If in set X_2 two are stranger to each other then these together with x form a set of mutual strangers.

If no two of three are strangers to each other then they form a set of 3 mutual friends.

Example 6.12 : *Show that if 9 colours are used to paint 100 cars, atleast 12 cars will be of the same colour.*

Solution : Let 9 colours be the pigeonholes and 100 cars be the pigeons.

$$\therefore \qquad n = 100 \text{ and } m = 9 \text{ and } n > m$$

Using generalised Pigeonhole Principle,

$$\left\lfloor\left(\frac{(n-1)}{m}\right)\right\rfloor + 1 = \left\lfloor\left(\frac{100-1}{9}\right)\right\rfloor + 1 = 12$$

Thus 12 cars will be of same colour.

Example 6.13 : *How many people must you have to guarantee that atleast 9 of them will have birthdays in the same day of the week.*

Solution : Let the days of week be the pigeonholes and people be the pigeons.

$$\therefore \qquad n = ? \text{ and } m = 7 \text{ and } n > m$$

$\therefore$ By Generalised Pigeonhole Principle :

$$\left\lfloor\frac{(n-1)}{m}\right\rfloor + 1 = 9$$

$$\Rightarrow \qquad \frac{(n-1)}{7} = 8 \quad \Rightarrow \quad (n-1) = 56 \quad \Rightarrow \quad n = 57$$

Thus, there must be 57 people to guarantee that atleast 9 of them will have birthdays on the same day of week.

Example 6.14 : *7 members of a family have total Rs. 2886 in their pockets. Show that atleast one of them must have atleast Rs. 413 in his pocket.*

Solution : Let the members be pigeonholes and rupees be the pigeons

$$n = 2886 \text{ and } m = 7$$

$\therefore$ Using the Extended Pigeonhole Principle,

$$\left\lfloor \frac{(n-1)}{m} \right\rfloor + 1 = \left\lfloor \frac{(2886-1)}{7} \right\rfloor + 1 = \text{Rs. } 413$$

must have in one member's pocket.

Example 6.15 : *Show that there must be atleast 90 ways to choose 6 members from 1 to 15 so that all the choices have the same sum.*

Solution : 6 numbers can be chosen from 1 to 15 in ${}^{15}C_6 = 5005$ ways. Let these be the pigeons.

Now the sum varies from $(1 + 2 + 3 + 4 + 5 + 6) = 21$ to $(10 + 11 + 12 + 13 + 14 + 15) = 75$.

Thus the number of sums that can be generated $= 75 - 21 + 1 = 55$.

Let these be the pigeonholes.

By Extended Pigeonhole Principle :

$$n = 5005, \quad m = 55$$

then
$$\left\lfloor \frac{(n-1)}{m} \right\rfloor + 1 = \left\lfloor \frac{(5005-1)}{55} \right\rfloor + 1 = 91 \text{ ways}$$

Thus there must be atleast 90 ways to choose 6 numbers from 1 to 15, so that all the choices have the same sum.

Example 6.16 : *What is the minimum number of students required in a class to be sure that atleast 5 will receive the same grade if there are four possible grades A, B, C, D?*

Solution : Let the four grades be the pigeonholes and let number of students be the pigeons. By Extended Pigeonhole Principle,

$$n = ? \quad m = 4$$

$$\left\lfloor \frac{(n-1)}{m} \right\rfloor \quad 1 = 5$$

$$\Rightarrow \qquad \frac{n-1}{4} = 4 \quad \Rightarrow \quad n = 17$$

Thus, there will be 17 minimum number of students required, so that atleast 5 will receive same grade.

Example 6.17 : *Show that if any 20 people are selected, then we may choose a subset of 3 so that all were born on same day of week.*

Solution : Let number of people be the pigeons and days of week be the Pigeonholes.

By Extended Pigeonhole Principles,

$$n = 20, \quad m = 7$$

$$\therefore \quad \left\lfloor \frac{(n-1)}{4} \right\rfloor + 1 = \left\lfloor \frac{(20-1)}{7} \right\rfloor + 1 = 2 + 1 = 3$$

We can choose subset of 3 people which have been born on same day of week.

Example 6.18 : *Ten people came forward to volunteer for a 3 person committee. Every possible committee of three that can be formed from these ten names is written on a slip of paper. One slip of each possible committee and the slips are put in 10 hats. Show that atleast one hat contains 12 or more slips of paper.*

Solution : A committee of 3 can be chosen from 10 person in ${}^{10}C_3 = 120$ ways. Let 120 slips on which these committees are written be the pigeons.

These slips are put in 10 hats. Let these be pigeonholes.

$\therefore$ By Extended Pigeonhole Principle,

$$n = 120, \quad m = 10$$

$\therefore$ One hat must contain atleast $\left\lfloor \frac{(n-1)}{m} \right\rfloor + 1 = \left\lfloor \frac{(120-1)}{10} \right\rfloor + 1$

$= 11 + 1 = 12$ or more slips of paper

EXERCISE

1. Let we select 367 students from a campus. Show that atleast two of them must have the same birthday.
2. If we select any group of 1000 students from a college, show that atleast three of them must have the same birthday.
3. If 10 points are selected inside an equilateral triangle of unit side, then atleast two of them are not more than 1/3 of a unit apart.
4. Show that in any set of element integers there are two whose difference is divisible by 10.
5. Show that if seven numbers from 1 to 12 are chosen then two of them add upto 13.
6. A student must take five classes from three areas of study. Numerous classes are offered in each discipline but the student cannot take more than two classes in any given area. Using Pigeonhole Principle show that the student will take atleast two classes in one area.
7. Show that if eleven numbers are chosen from the set {1, 2, ..., 20} then one of them will be a multiple of another.
8. Show that if 10 colour are used to paint houses then atleast 11 houses have the same colour.
9. How many students each of whom comes from one of 50 districts must be enrolled in an engineering college to guarantee that there are atleast 100 students who come from the same district?
10. If there are 38 different time periods during which classes at an engineering college can be scheduled. If there are 677 different classes how many different rooms will be needed?
11. Find the minimum number of persons in a joint family to be sure that 4 of them born in the same month.
12. If 30 story books contain a total of 61327 pages then show that one of the books must have atleast 2045 pages.