Set Theory

Set:- A well defined collection of distinct objects is called a set.
The objects in a set can be numbers, people, letters, rivers, counteries etc.
The objects in a set are called elements or members of the set.

Ex:-

1. The set of numbers 2, 4, 6, 8, 10
2. The set of letters a, e, i, o, u.

Note -
Sets are usually denoted by capitalletters A, B, C ... or X, Y, Z ... and the elements are denoted by a, b, c, ... or x, y, z, ...

Standard notations for certain type of sets -

1. Natural Numbers $\mathbb{N} := \{1, 2, 3, \cdots\}$
2. Integers $\mathbb{Z} := \{\cdots -2, -1, 0, 1, 2, \ldots\}$
3. Rational Numbers $\mathbb{Q} = \{\frac{a}{b} \mid a, b \text{ are Integers}, b \neq 0\}$
4. Irrational Numbers $\mathbb{Q}^c$ $\mathbb{I} := \{$ a non rational number such as $\sqrt{2}, e, \pi, e^{\pi}, \ldots\}$
5. Real Numbers $\mathbb{R} := \{$ a rational or an irr. no$\}$
6. complex numbers $\mathbb{C} := \{a + ib \mid a, b \in \mathbb{R}, i = \sqrt{-1}\}$

Also $\mathbb{Z}^+ \rightarrow$ Set of positive integers
$\mathbb{Q}^* \rightarrow$ Set of non-zero rational numbers

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$2N = E \rightarrow$ Set of even Integers

$D_n \rightarrow$ set of divisors of a positive Integer n.

Cardinality or cardinal number of a Set :→

The number of elements in a Set A is called the cardinality or cardinal number of the set. It is denoted by $n(A)$ or $|A|$.

Equivalent Set :- A and B are said to be equivalent sets if $|A| = |B|$.

Ex:-

if $S = \{3, 5, 7, 9, 11\}$
Then $|S| = 5$.

Empty set / Null set :— A set having no element is called empty set. It is denoted by ϕ or $\{\}$.

eg → 1. The set of Integers b/w $\frac{9}{4}$ and $\frac{11}{4}$

2. The set of Integers less than 4 and greater than 8.

Finite and Infinite sets :- A set is called finite if it has n distinct elements. otherwise set is called Infinite.

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Subset :- if every element of the set A is also an element of the set B then A is called subset of B . denoted by $A \subseteq B$ which in read as A in contained in B.
i.e if $x \in A \Rightarrow x \in B$

Proper subset :- when $A \subseteq B$ and the set B has an element that is not in A. Then A is called proper subset of B and is denoted by $A \subset B$.

Equal Sets :- Two sets A and B are equal i.e $A = B$ iff $A \subseteq B$ and $B \subseteq A$

Comparability :- Two sets A and B are said to be comparable if $A \subset B$ or $B \subset A$ or $A = B$.

Power Set :→ Let X be a non-empty set. The set of all subsets of X is called the power set of the set X.
It is denoted by $\mathcal{P}(X)$.
Since $\phi \subseteq X$ and $X \subseteq X$, So $\phi, X \in \mathcal{P}(X)$.
Particularly, $\mathcal{P}(\phi) = \{\phi\}$.

Number of elements in Power Set :-

Let a set A has n elements. Then its Power set $\mathcal{P}(A)$ has 2^n elements

Proof :- For $0 \leq k \leq n$. there are ${}^nC_k$ ways to select a subset having k elements of the set A

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Therefore, in Total, Power Set $P(A)$ contains

$$^nC_0 + {}^nC_1 + {}^nC_2 + \cdots\cdots + {}^nC_{n-1} + {}^nC_n = 2^n$$

sets, as asserted.

Example:-1 Find the Power set of the Sets (i.) $\{a\}$, ii) $\{a, b\}$ (iii) $\{a, b, c\}$

Solⁿ:-

i.) let $A = \{a\}$
So $P(A) = \{\phi, \{a\}\}$

ii.) let $B = \{a, b\}$
So $P(B) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

iii.) let $C = \{a, b, c\}$
so $P(C) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}$.

Example:-2 Find the Power set of the sets
i.) $A = \{\phi, \{\phi\}\}$
ii.) $B = \{a, \{a\}\}$

Solⁿ:-

i.) Power set of A
$P(A) = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$

ii.) Power set of B
$P(B) = \{\phi, \{a\}, \{\{a\}\}, \{a, \{a\}\}\}$.

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Ex:-3 if $A = \{\phi, b\}$ Then find $A \cup P(A)$, $P(A)$ - Power set of A

Solⁿ:- since $P(A) = \{\phi, \{\phi\}, \{b\}, A\}$
$\therefore$ $A \cup P(A) = \{\phi, \{\phi\}, \{b\}, b, A\}$

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Operations on set :→

1. Union of Sets :- The union of two sets A and B is defined by
$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

2. Intersection of sets :- The Intersection of two sets A and B is defined by
$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

3. Difference of sets :- The difference of two sets A and B is defined by
$$A - B = \{x : x \in A ,\ x \notin B\}.$$

4. complement of a set :- The complement of a set A is the set of elements which do not belong to A denoted by A'. A^c
$$A^c = U - A$$
where U is the Universal Set.

5. Symmetric difference of two sets :-

Symmetric difference of two sets A and B is given by

$$A \oplus B = A \Delta B = (A-B) \cup (B-A)$$
or
$$A \vee B = \{x \in X \mid x \in (A \cup B) - (A \cap B)\}$$
where $X \neq \phi$, $A, B \in P(X)$.

Note –
$$A \Delta B = B \Delta A \qquad A \Delta A = \phi$$
$$A \Delta \phi = A ,\quad A \Delta X = A' ,\quad A \Delta A' = X$$

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Ques. Let $U = \{1, 2, \ldots, 8, 9\}$ be a universal set and the six sets given by
$A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7\}$,
$C = \{5, 6, 7, 8, 9\}$, $D = \{1, 3, 5, 7, 9\}$
$E = \{2, 4, 6, 8\}$, $F = \{1, 5, 9\}$ Then
Find the sets
(a.) $A \cup B$, $D \cup F$ (b.) B^c, D^c, E^c
(c.) $A - B$, $D - E$, $E - D$ (d.) $C \vee D$, $E \vee F$, $A \vee B$
(e.) $(A \vee C) - B$, $(B \vee C) - A$.

Solⁿ:-

(a.) $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$
$D \cup F = \{1, 3, 5, 7, 9\}$

(b.) $B^c = \{1, 2, 3, 8, 9\}$, $D^c = \{2, 4, 6, 8\}$
$E^c = \{1, 3, 5, 7, 9\}$

(c.) $A - B = \{1, 2, 3\}$
$D - E = \{1, 3, 5, 7, 9\}$
$E - D = \{2, 4, 6, 8\}$

(d.) $C \vee D = \{1, 3, 6, 8\}$
$E \vee F = \{1, 2, 4, 5, 6, 8, 9\}$
$A \vee B = \{1, 2, 3, 6, 7\}$

(e.) $(A \vee C) = \{1, 2, 3, 4, 6, 7, 8, 9\}$
So $(A \vee C) - B = \{1, 2, 3, 8, 9\}$
$B \vee C = \{4, 8, 9\}$
So $(B \vee C) - A = \{6, 7, 8, 9\}$.

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Laws of Algebra of Sets :— Let $A, B, C \in \mathcal{P}(X)$

1. Idempotent Laws : a.) $A \cup A = A$
   b.) $A \cap A = A$
2. Associative Laws : a.) $(A \cup B) \cup C = A \cup (B \cup C)$
   $(A \cap B) \cap C = A \cap (B \cap C)$
3. commutative Laws : a.) $A \cup B = B \cup A$
   b.) $A \cap B = B \cap A$
4. Distributive Laws : a.) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   b.) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5. Identity Laws :
   a.) $A \cup \phi = A$, $A \cup U = U$
   b.) $A \cap \phi = \phi$, $A \cap U = A$
6. Involution law :
   $(A')' = A$
7. complement laws : a.) $A \cup A' = U$, $A \cap A' = \phi$
   b.) $U' = \phi$, $\phi' = U$
8. De Morgan's law : a.) $(A \cap B)' = A' \cup B'$
   b.) $(A \cup B)' = A' \cap B'$
9. a.) $A - (B \cup C) = (A - B) \cap (A - C)$
   b.) $A - (B \cap C) = (A - B) \cup (A - C)$
10. Absorption law : a.) $A \cup (A \cap B) = A$
    b.) $A \cap (A \cup B) = A$
11. $A \cap (B - C) = (A \cap B) - (A \cap C)$
12. a.) $A - B = A \cap B'$
    b.) $B - A = B \cap A'$

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Note:- a.) $x \in A$ and $x \in B \Leftrightarrow x \in A \cap B$
$x \in A$ or $x \in B \Leftrightarrow x \in A \cup B$
$x \notin A$ and $x \notin B \Leftrightarrow x \notin A \cup B$
$x \notin A$ or $x \notin B \Leftrightarrow x \notin A \cap B$

Ques.1 Prove that if $A \subseteq B$ then $A \cup B = B$.

Soln:-

we know that $B \subseteq A \cup B$ —(1)

Now we have to show $A \cup B \subseteq B$

let $x \in A \cup B \Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in B$ as $A \subseteq B$

So $A \cup B \subseteq B$ —(2)

Hence (1) and (2) $\Rightarrow$

$A \cup B = B$

Ques.2 Prove that if $A \subseteq B$ then $A \cap B = A$.

Soln:-

we know that $A \cap B \subseteq A$ —(1)

Now we have to show $A \subseteq A \cap B$

let $x \in A \Rightarrow x \in B$ as $A \subseteq B$

$\Rightarrow x \in A \cap B$

i.e $A \subseteq A \cap B$ —(2)

Hence from (1) and (2),

$\boxed{A = A \cap B}$

Ques.3 Prove that $A - B = A \cap B^c$.

Let $x \in A - B \Leftrightarrow x \in A$ and $x \notin B$

$\Leftrightarrow x \in A$ and $x \in B^c$

$\Leftrightarrow x \in A \cap B^c$

Hence $A - B = A \cap B^c$.

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Ques. 4. Let A, B, C be three non-empty sets. Prove the following Identities:

1. $(A-C) \cap (C-B) = \phi$
2. $A - (B \cap C) = (A-B) \cup (A-C)$
3. $(A^c \cup B^c)^c \cup (A^c \cup B)^c = A$.

Soln:-

(1.) For, if $(A-C) \cap (C-B)$ is non-empty let $x \in (A-C) \cap (C-B)$

$\Rightarrow x \in (A-C)$ and $x \in (C-B)$

However, the two conditions can not hold simultaneously because

$x \in A - C \Rightarrow x \notin C$

and $x \in C - B \Rightarrow x \in C$

which is a contradiction.

Therefore, $(A-C) \cap (C-B) = \phi$.

(2.) we know that $X - Y = X \cap Y^c$

Therefore, we have

$A - (B \cap C) = A \cap (B \cap C)^c$

$= A \cap (B^c \cup C^c)$ (by De Morgan law)

$= (A \cap B^c) \cup (A \cap C^c)$ (by Distributive law)

$= (A-B) \cup (A-C)$

(3.) $(A^c \cup B^c)^c \cup (A^c \cup B)^c = (A \cap B) \cup (A \cap B^c)$ De Morgan's law & Involution law

$= A \cap (B \cup B^c)$ Distributive law

$= A \cap U$. by complement law

$= \underline{\underline{A}}$

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Multiset :→

Multisets are sets where an element can occur as a member more than once.

OR

A multiset (mset) is an unordered collection having multiple instances of the same object, where the multiplicity of an object may be zero, one, or more than one. We usually write a Multiset M as

$$M := [k_1 \cdot a_1, k_2 \cdot a_2, k_3 \cdot a_3, \ldots]$$

where $k_i \in \mathbb{Z}^+$

where the multiplicity k_i of an object a_i in the multiset M is the number of times it appears in the collection.

The set $\{a_1, a_2, a_3, \ldots\}$ is called the base set of the multiset M.

Example:- The two multisets A and B given by

$$A = [1, 1, 2, 2, 3, 4, 6, 6, 6]$$
$$B = [1, 2, 2, 2, 3, 3, 4, 4, 6].$$

can be written as

$$A = [2 \cdot 1, 2 \cdot 2, 3, 4, 3 \cdot 6]$$

and $B = [1, 3 \cdot 2, 2 \cdot 3, 2 \cdot 4, 6]$

Cardinality of Multiset :-

The cardinality of a multiset is the sum of the multiplicities of all its elements.

Cartesian Product of Sets :-

Ordered Pair :- An ordered pair, generally denoted by (a, b), is a pair of elements a and b of two sets, which is ordered in sense that $(a, b) \neq (b, a)$ when $a \neq b$ and a and b are known as first and second coordinates in (a, b).

Cartesian Product of two sets :-
Let A and B two non-empty sets. The cartesian product of A and B, denoted by $A \times B$, is the set of ordered pairs given by

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

For Example -
Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$
Then we have

$$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

Note :- $A \times B \neq B \times A$

More generally if A and B are finite sets given by $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$
Then the set $A \times B$ contains mn elements

$$|A \times B| = |A| \times |B| = mn$$

The Product set $A \times B$ in matrix form

$$\begin{array}{c|cccc}
 & b_1 & b_2 & \cdots & b_n \\
\hline
a_1 & (a_1, b_1) & (a_2, b_2) & \cdots & (a_1, b_n) \\
a_2 & (a_2, b_1) & (a_2, b_2) & \cdots & (a_2, b_n) \\
\vdots & & \vdots & & \\
a_m & (a_m, b_1) & (a_m, b_2) & \cdots & (a_m, b_n)
\end{array}$$

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Properties of cartesian Product

For the four sets A, B, C and D

1. $(A\cap B)\times(C\cap D) = (A\times C)\cap(B\times D)$
2. $(A-B)\times C = (A\times C)-(B\times C)$
3. $(A\cup B)\times C = (A\times C)\cup(B\times C)$
4. $A\times(B\cap C) = (A\times B)\cap(A\times C)$

Ques. Let A, B, C be three non empty sets, Prove that

1. $A\cap C = B\cap C$ and $A\cup C = B\cup C \Rightarrow A = B$
2. $A\times(B\cup C) = (A\times B)\cup(A\times C)$
3. $A\times(B\cap C) = (A\times B)\cap(A\times C)$

Proof:-

1.) we have,

$A = A\cap(A\cup C)$ by absorption
$= A\cap(B\cup C)$ by given condition
$= (A\cap B)\cup(A\cap C)$ by distributivity
$= (A\cap B)\cup(B\cap C)$ by given condition
$= (A\cap B)\cup(C\cap B)$ by commutativity
$= (A\cup C)\cap B$ by distributivity
$= (B\cup C)\cap B$ by given condition
$= (C\cup B)\cap B$ by commutativity
$= B$ by absorption

Hence

$A\cup C = B\cup C$ and $A\cap C = B\cap C \Rightarrow A = B$

2.) $A\times(B\cup C) = \{(x,y) \mid x\in A \text{ and } (y\in B\cup C)\}$
$= \{(x,y) \mid x\in A \text{ and } (y\in B \text{ or } y\in C)\}$
$= \{(x,y) \mid (x\in A \text{ and } y\in B) \text{ or } (x\in A \text{ and } y\in C)\}$
$= \{(x,y) \mid (x,y)\in A\times B \text{ or } (x,y)\in A\times C\}$
$= (A\times B)\cup(A\times C)$

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3.) $A \times (B \cap C) = \{(x,y) \mid x \in A \text{ and } y \in (B \cap C)\}$
$= \{(x,y) \mid x \in A \text{ and } (y \in B \text{ and } y \in C)\}$
$= \{(x,y) \mid (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C)\}$
$= \{(x,y) \mid (x,y) \in A \times B \text{ and } (x,y) \in A \times C\}$
$= (A \times B) \cap (A \times C)$

Ques. if $A = \{1,4\}$, $B = \{4,3\}$, $C = \{3,6\}$ verify that $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Soln:-
$B \cup C = \{4,3,6\}$
So $A \times (B \cup C) = \{1,4\} \times \{4,3,6\}$
$= \{(1,4), (1,3), (1,6), (4,4), (4,3), (4,6)\}$
$A \times B = \{1,4\} \times \{4,3\}$
$= \{(1,4), (1,3), (4,4), (4,3)\}$
$A \times C = \{1,4\} \times \{3,6\}$
$= \{(1,3), (1,6), (4,3), (4,6)\}$
So $(A \times B) \cup (A \times C)$
$= \{(1,4), (1,3), (1,6), (4,4), (4,3), (4,6)\}$
Hence
$A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Ques. if $A = \{1,3,5,7\}$, $B = \{2,3,5,8\}$
Then find $(A \cap B) \times (B - A)$
Soln:- $A \cap B = \{3,5\}$, $B - A = \{2,8\}$
So $(A \cap B) \times (B - A)$
$= \{(3,2), (5,2), (3,8), (5,8)\}$.

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Relation or Binary Relation :-

Let A and B be non-empty sets. Then any subset R of the cartesian product $A \times B$ is called a relation from A to B and is denoted by R.

$\therefore$ R is a relation from A to B $\Rightarrow R \subseteq A \times B$
and
$$R = \{(x, y) \mid x \in A, y \in B \text{ and } xRy\}$$
where
xRy denotes that x is Related to y by R
$x \not R y$ denotes that x is not related to y

Ex:- Let $A = \{1, 2, 5\}$ and $B = \{2, 4\}$ be two given sets. Find out the Relation from A to B defined by "is less than".

Solⁿ:-

Given $A = \{1, 2, 5\}$
$B = \{2, 4\}$

$\therefore A \times B = \{(1,2), (1,4), (2,2), (2,4), (5,2), (5,4)\}$
Then
$R = \{(1,2), (1,4), (2,4)\}$

Domain of R :- The set of first coordinates of every element of R
Domain $R = \{x \in A : (x, y) \in R \text{ for some } y \in B\}$

Range of R :- The set of second coordinates of every element of R.
Range $R = \{y \in B : (x, y) \in R \text{ for some } x \in A\}$

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Ex:- Let $A = \{2, 3, 5\}$ and $B = \{2, 4, 6, 10\}$
and $R = \{(2,2), (2,4), (2,6), (2,10), (3,6), (5,10)\}$

Then Dom $R = \{2, 3, 5\}$
Range $R = \{2, 4, 6, 10\}$

Total number of distinct Relation from A to B

$$= 2^{|A \times B|} = 2^{|A| \cdot |B|} = 2^{mn}$$

where $m = |A|$
$n = |B|$.

Relation on a Set :-

A Relation R from a Set A to the set A itself is called a Relation on A.
i.e $R \subseteq A \times A$
$$R = \{(x, y) : x, y \in A \text{ and } xRy\}$$

Ex:- Let $A = \{1, 2, 3\}$ find the Relation on A defined by $\leq$.

Soln:-

$$A \times A = \{1, 2, 3\} \times \{1, 2, 3\}$$
$$= \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$$

Then Relation $\leq$ on A is given by

$$R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}.$$

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(xii) Identity Relation :-

The Identity Relation I_A on a Set A is defined by

$$I_A = \{(x, y) : x, y \in A \text{ and } x = y\}$$
$$= \{(x, x) : x \in A\}$$

Ex:-

Let $A = \{a, b, c\}$

$$I_A = \{(a, a), (b, b), (c, c)\}$$

(xiii) Inverse Relation :-

Let R be any Relation from a Set A to B. The Inverse of R, R^{-1} is the Relation from B to A defined by

$$R^{-1} = \{(y, x) : (x, y) \in R\}$$

Consequently

$$xRy = yR^{-1}x$$

Ex:- Let $A = \{2, 3, 5\}$ and $B = \{6, 8, 10\}$ and Relation R is defined by "x divides y" then

$$R = \{(2,6), (2,8), (2,10), (3,6), (5,10)\}$$

and

$$R^{-1} = \{(6,2), (8,2), (10,2), (6,3), (10,5)\}$$

Operations on Relation :—

1.) Complement of a Relation:—
Let R be a Relation from set A to B. Then the complement of Relation R denoted by $\bar{R}$ or R' is a Relation from A to B such that
$$\bar{R} = \{(a,b) : (a,b) \notin R\}$$

Ex:— Let R be Relation from A to B where
$A = \{1, 2, 3\}$ $B = \{8, 9\}$
and $R = \{(1,8), (2,8), (1,9), (3,9)\}$
Find the complement of R.
We have
$$A \times B = \{(1,8), (1,9), (2,8), (2,9), (3,8), (3,9)\}$$
$$\bar{R} = \{(2,9), (3,8)\}, \cancel{(3,9)}$$

2.) Intersection of Relations:—
The Intersection of two Relations R and S is defined by
$$R \cap S = \{(x,y) : xRy \text{ and } xSy\}$$

3. Union of Relation:—

The Union of two Relations R and S is defined by
$$R \cup S = \{(x,y) : xRy \text{ or } xSy\}$$

Ex:— Let $R_1 = \{(1,1), (2,2), (3,3), (4,4), (3,4), (4,3)\}$
& $R_2 = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1)\}$

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Then
$R_1 \cup R_2 = \{(1,1), (2,2), (3,3), (4,4), (3,4), (4,3), (1,2), (2,1)\}$

$R_1 \cap R_2 = \{(1,1), (2,2), (3,3), (4,4)\}$

Properties of Relations :-

1.) **Reflexive Relation :-**
A relation R on a set A is Reflexive if aRa for every $a \in A$
i.e. $(a,a) \in R \ \forall \ a \in A$.

Ex:- if $R_1 = \{(1,1), (2,2), (2,3), (3,3)\}$ be a Relation on a set $A = \{1,2,3\}$ Then R_1 is Reflexive since aRa for every $a \in A$.

2.) **Irreflexive :-**
A relation R on a set A is Irreflexive if $a \not R a$ for all $a \in A$
i.e $(a,a) \notin R$ for all $a \in A$.

Ex:- The Relation $R_2 = \{(1,2), (1,3), (2,1), (2,3)\}$ on $A = \{1,2,3\}$ is irreflexive.

3. **Symmetric Relation :-**
A Relation R on a set A is Symmetric if $aRb \Rightarrow bRa$,
i.e whenever $(a,b) \in R \Rightarrow (b,a) \in R$. $\forall a,b \in A$

Ex:-
Sol: Relation $R_3 = \{(1,1), (1,2), (1,3), (2,2), (2,1), (3,1)\}$ on A is symmetric

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Asymmetric Relation:-

A Relation R on a Set A is asymmetric if $aRb \Rightarrow b \not R a$
i.e whenever $(a,b) \in R \Rightarrow (b,a) \notin R$ for $a \neq b$.

- The Relation $R_1 = \{(1,1), (1,2), (2,3), (3,1)\}$ on $A = \{1,2,3\}$ is an asymmetric Relation.

Antisymmetric Relation:-

A Relation R on a set A is antisymmetric if aRb and $bRa \Rightarrow a = b \ \forall a,b \in A$
i.e $(a,b) \in R$ and $(b,a) \in R \Rightarrow a = b \ \forall a,b \in A$

- The Relation $R = \{(1,2), (2,2), (2,3)\}$ on Set $A = \{1,2,3\}$ is an Antisymmetric Relation.

Transitive Relation:-

A Relation R on a Set A is transitive if aRb and $bRc \Rightarrow aRc$
i.e whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$.

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(#) Equivalence Relation:-

A Relation R on a Set A is called an equivalence Relation or RST relation if it is reflexive, Symmetric and Transitive i.e

R is an equivalence Relation on A if It has the following three Properties:

1. $(a, a) \in R$ for all $a \in A$ (Reflexive)
2. $(a,b) \in R \Rightarrow (b, a) \in R$ (Symmetric)
3. (a,b) and $(b,c) \in R \Rightarrow (a, c) \in R$ (Transitive)

Ex:- If R be a Relation in the set of Integer Z defined by.

$R = \{(x,y) : x \in Z,\ y \in Z,\ (x-y)$ is divisible by $6\}$

Then Prove that R is an equivalence Relation

Sol^n:- i.) Reflexive - Let $x \in Z$

Then $x - x = 0$ which is divisible by 6

Therefore $xRx \quad \forall x \in Z$

Hence R is Reflexive

ii.) Symmetric :- Let $x, y \in Z$

Then $xRy \Rightarrow x-y$ is divisible by 6

$\Rightarrow -(x-y)$ is divisible by 6

$\Rightarrow (y-x)$ is divisible by 6

$\Rightarrow yRx$

Hence R is Symmetric

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iii) <u>Transitive</u> :- Let $x, y, z \in \mathbb{Z}$

Then xRy and $yRz \Rightarrow$ ~~x is divisible by y~~
and y ~~is divisible by z~~

$\Rightarrow x - y$ is divisible by 6 and
$y - z$ is divisible by 6

$\Rightarrow [(x-y) + (y-z)]$ is divisible by 6

$\Rightarrow x - z$ is divisible by 6

$\Rightarrow xRz$

Hence, R is Transitive

Thus R is an equivalence Relation.

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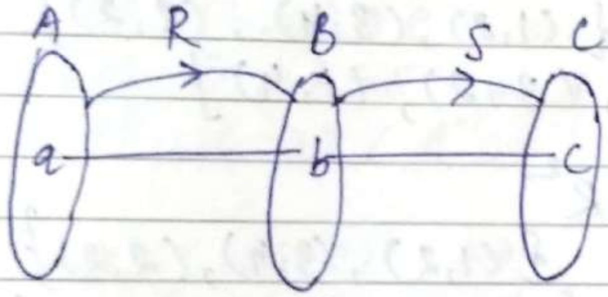
(#) Composite Relation :-

Let A, B and C be three non-empty sets. Suppose R be a relation from A to B and S be a relation from B to C. Then the composite relation of the R and S is a relation from A to C and is denoted by SoR, and defined as

$SoR = \{(a, c) : \exists$ an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S\}$



If $(a, b) \in R$
$(b, c) \in S$ Then $(a, c) \in SoR$

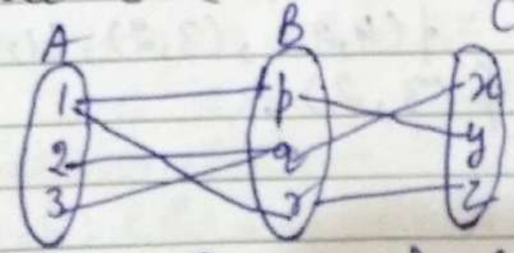
Note - $RoR = R^2$, $R^2 oR = R^3$

Ex! - Let $A = \{1, 2, 3\}$, $B = \{p, q, r\}$, $C = \{x, y, z\}$ and let

$R = \{(1, p), (1, r), (2, q), (3, q)\}$

and $S = \{(p, y), (q, x), (r, z)\}$ Then compute SoR.

Solⁿ:-



$SoR = \{(1, y), (1, z), (2, x), (3, x)\}$.

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Ex:- Let $R = \{(1,2), (3,4), (2,2)\}$
and $S = \{(4,2), (2,5), (3,1), (1,3)\}$
Find SoR, RoS, RoR, $Ro(SoR)$, $(RoS)oR$

Solⁿ:-

i.) SoR :-

$R = \{(1,2)\ (3,4), (2,2)\}$

$S = \{(4,2), (2,5), (3,1), (1,3)\}$

$\therefore SoR = \{(1,5), (3,2), (2,5)\}$

ii.) RoS :-

$S = \{(4,2), (2,5), (3,1), (1,3)\}$

$R = \{(1,2), (3,4), (2,2)\}$

$\therefore RoS = \{(4,2), (3,2), (1,4)\}$

iii.) $RoR = R^2$

$R = \{(1,2), (3,4), (2,2)\}$

$R = \{(1,2), (3,4), (2,2)\}$

$RoR = R^2 = \{(1,2), (2,2)\}$

iv.) $Ro(SoR) \rightarrow$

$SoR = \{(1,5), (3,2), (2,5)\}$

$R = \{(1,2), (3,4), (2,2)\}$

$Ro(SoR) = \{(3,2)\}$

v.) $(RoS)oR \rightarrow$

$R = \{(1,2), (3,4), (2,2)\}$

$RoS = \{(4,2), (3,2), (1,4)\}$

$(RoS)oR = \{(3,2)\}$.

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(11) Reversal law in composite relation →

Theorem:- Let R be a relation from the set A to the set B and S be a relation from the set B to set C, then

$$(SoR)^{-1} = R^{-1}oS^{-1}$$

Proof:-

let $(c,a) \in (SoR)^{-1}$

$\Leftrightarrow (a,c) \in (SoR) \quad \forall a \in A, c \in C$

$\Leftrightarrow \exists$ an element b s.t

$(a,b) \in R$ and $(b,c) \in S$

$\Leftrightarrow (b,a) \in R^{-1}$ and $(c,b) \in S^{-1}$

$\Leftrightarrow (c,b) \in S^{-1}$ and $(b,a) \in R^{-1}$

$\Leftrightarrow (c,a) \in R^{-1}oS^{-1}$.

$\therefore (c,a) \in (SoR)^{-1} \Leftrightarrow (c,a) \in R^{-1}oS^{-1}$

Thus $(SoR)^{-1} = R^{-1}oS^{-1}$.

(11) Order of Relation :→

1.) Partial order Relation
2.) Total order Relation.

(1.) Partial order Relation →

A relation R on a set A is called partial order relation if R is

1.) Reflexive :- $(a,a) \in R, \ \forall a \in A$

i.e $aRa \quad \forall a \in A$

2.) Antisymmetric :-

$(a,b) \in R$ and $(b,a) \in R \Rightarrow a=b, \forall a,b \in A$

i.e aRb & $bRa \Rightarrow a=b$

3.) Transitive :- $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$

i.e aRb & $bRc \Rightarrow aRc$

POSET or Partial ordered Set

The set A together with a partially order relation R on the set A i.e (A, R) is called Partial order set or POSET.

Ex:- Show that the Relation $R = \{(x, y) : x \geq y\}$ where $x, y \in I^+$ the set of Positive Integers, is a partial order Relation.

Soln:-

Let $x, y, z \in I^+$
and given $R = \{(x, y) : x \geq y\}$.

i.) Reflexive :- Let $x \in I^+$
since $x \geq x \Rightarrow (x, x) \in R, \forall x \in I^+$
$\Rightarrow xRx, \forall x \in I^+$
$\Rightarrow$ R is Reflexive

ii.) Anti-Symmetric :- Let $x, y \in I^+$
Now $(x, y) \in R$ and $(y, x) \in R$
$\Rightarrow x \geq y$ and $y \geq x$
$\Rightarrow x = y$
$\Rightarrow (x, y) \in R$ and $(y, x) \in R \Rightarrow x = y$
$\therefore$ R is antisymmetric

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3.) Transitive :- Let $x, y, z \in I^+$
Let $(x, y) \in R$ and $(y, z) \in R$
$\Rightarrow x \geq y$ and $y \geq z$
$\Rightarrow x \geq z$
$\Rightarrow (x, z) \in R$
$\therefore (x, y) \in R$ & $(y, z) \in R$
$\Rightarrow (x, z) \in R$.

$\therefore$ R is Transitive
Hence R is a Partial order Relatn on I^+
Thus (I^+, R) is a POSET.

Total order Relation :-

Consider the Relation R on the set A.
if $\forall a, b \in A$, we have
either $(a, b) \in R$
or $(b, a) \in R$
or $a = b$
Then R is called total order relation on A.

Ex:- Show that the Relation '<' (less than) defined on N, the set of +ve Integers is neither an equivalence relation nor partially ordered relation but is a Total order relation.

Solⁿ:-
Let $a, b \in N$
$R = \{(a, b) : a < b\}$
1.) Reflexive :- let $a \in N$
Since $a \not< a$
$\Rightarrow (a, a) \notin R \Rightarrow a \not R a$
$\Rightarrow$ R is not reflexive

∴ The Relation '<' is not Reflexive & Hence it is neither an equivalence relation nor the partial order relation.
But

as $\forall\, a, b \in N$
we have

either $a < b$ $(a,b) \in R$
or $b < a$ $(b,a) \in R$
or $a = b$

So, the Relation is a total ordered relation.

composition of relations SoR by using matrix of Relations. →

Matrix Representation of Relation :—

Let R be the Relation from set A to set B where

$$A = \{a_1, a_2, \ldots, a_m\}$$
$$B = \{b_1, b_2, \ldots, b_n\}$$

be finite sets having m and n elements resp. Then R can be represented by mn matrix and defined as

$$M_R = [m_{ij}]_{m\times n}$$
$$= \begin{cases} 1, & \text{if } aRb \text{ i.e } (a,b) \in R \\ 0, & \text{if } a\not R b \text{ i.e } (a,b) \notin R \end{cases}$$

The matrix M_R is called matrix of Relation R.

Ex:- 1 Let R be the Relation from set $A = \{1, 3, 4\}$ to itself and defined by

$$R = \{(1,1), (1,3), (3,3), (4,4)\}$$

then find relation matrix.

Sol^n:-

$$M_R = \begin{array}{c} \\ 1 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Ex:- 2.

Let $A = \{1, 2, 3, 4, 8\}$, $B = \{1, 4, 6, 9\}$

Let aRb iff $a|b$ i.e a divides b

Find the Relation matrix M_R.

Sol^n:-

Here $R = \{(1,1), (1,4), (1,6), (1,9), (2,4), (2,6), (3,6), (3,9), (4,4)\}$

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 8 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 4 & 6 & 9 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Ex:- Let $P = \{2, 3, 4, 5\}$. Consider the Relation R and S defined by

$$R = \{(2,2), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5), (5,3)\}$$

$$S = \{(2,3), (2,5), (3,4), (3,5), (4,2), (4,5), (5,2), (5,5)\}.$$

Find the matrices of the above relations. Use matrix to find the composition relation of R and S.

Sol^n:- The matrices of the relation R and S are

$$M_R = \begin{array}{cc} & \begin{array}{cccc} 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

and

$$M_S = \begin{array}{cc} & \begin{array}{cccc} 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The composite Relation of R and S is SOR

So Multiplying the matrix M_S with M_R to obtain the matrix $M_S \times M_R$

$$M_S \times M_R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{array}{cc} & \begin{array}{cccc} 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 1 & 1 \end{bmatrix} \end{array}$$

The non-zero enteries in the matrix $M_S \times M_R$ tell the elements related in composition relation SOR of R and S.

Hence

SOR = {(2,3), (2,4), (2,5), (3,3), (3,5), (4,2), (4,3), (4,4), (4,5), (5,2), (5,3), (5,4), (5,5)}

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Total no. of Relations :—

Let A be a set with n elements i.e $|A| = n$
Then

1.) Total no. of Relations = 2^{n^2}

2.) Total no. of Reflexive relation = $2^{n(n-1)}$ or 2^{n^2-n}

3.) Total no. of Irreflexive relation = 2^{n^2-n}

4.) Total no. of Symmetric relation = $2^{\frac{n(n+1)}{2}}$

5.) Total no. of Asymmetric relation = $3^{\frac{n(n-1)}{2}}$

6.) Total no. of Antisymmetric relation = $2^n \cdot 3^{\frac{n(n-1)}{2}}$

7.) Total no. of both Reflexive and Symmetric = $2^{\frac{1}{2}(n^2-n)}$

Digraph / Directed graph of Relation R →

If A is finite set and R is a Relation on A, then we can also represent R pictorially as

1. Draw a small circle for each element of A and label the circle with corresponding elements of A. These circles are called vertices or nodes.
2. Draw an arrow, called an edge, from vertex a_i to $a_j \Leftrightarrow a_i R a_j$.

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The Resulting pictorial Representation of R is called a Directed graph or Digraph of R.

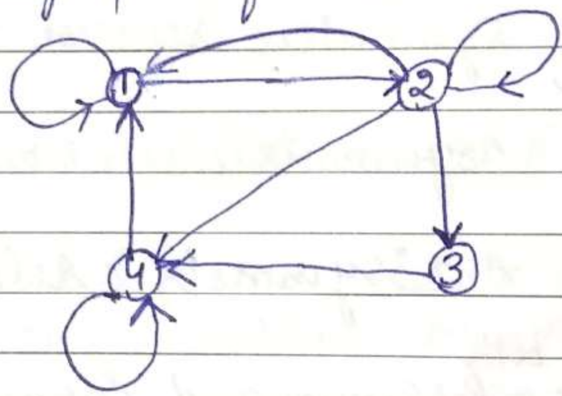
Ex:- Let $A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,2), (2,1), (2,2), (2,3), (2,4), (3,4), (4,1), (4,4)\}$

Construct the diagraph of R.

Soln:-

The digraph of R is



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Closure of Relations :→

Let R be a Relation on a Set A.
R may or may not have some Property P, Such as reflexivity, symmetry or transitivity. If we add some pairs then we have the desired Property. The Smallest relation S on A that contains R and Possess the desired Property P is called closure of R with respect to that Property.

1. Reflexive closure :—

Let R be a relation on a set A
The Reflexive closure $R^{(r)}$ of a Relation R is the smallest reflexive relation that contains R as a Subset. and is given by

$$\boxed{R^{(r)} = R \cup \Delta_A}$$

where $\Delta_A = \{(a,a) : a \in A\}$ is the diagonal Relation on A.

Ex :—

Let $A = \{1,2,3,4\}$ and
$R = \{(1,1), (2,2), (1,2), (2,1)\}$
Then find the Reflexive closure of R.

Sol^n :—

Reflexive closure of R

$R^{(r)} = R \cup \Delta_A$

$\Delta_A = \{(1,1), (2,2), (3,3), (4,4)\}$

So $R^{(r)} = \{(1,1),(2,2),(1,2),(2,1)\} \cup \{(1,1),(2,2),(3,3),(4,4)\}$
$= \{(1,1),(2,2),(3,3),(4,4),(1,2),(2,1)\}$

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2. Symmetric closure :-

Let R be a Relation on a set A, which is not Symmetric. The symmetric closure $R^{(s)}$ is the Smallest symmetric relation that contains R as a Subset. and is given by

$$\boxed{R^{(s)} = R \cup R^{-1}}$$

where $R^{-1} = \{(y,x) : (x,y) \in R\}$ be the Inverse of R.

Ex:-

if $R = \{(1,2), (2,1), (2,2), (3,1), (4,3)\}$ be a Relation on $S = \{1,2,3,4\}$. Find the symmetric closure of R.

Sol^n:-

The symmetric closure of R can be found by taking the union of R and R^{-1}.

Now $R^{-1} = \{(2,1), (1,2), (2,2), (1,3), (3,4)\}$

So $R^{(s)} = R \cup R^{-1}$ $\{(2,1),(1,2),(2,2),(3,1),(1,3),(3,4),(4,3)\}$

$= \{(1,2), (2,1), (2,2), (3,1), (4,3)\} \cup \{(2,1), (1,2), (2,2), (1,3), (3,4)\}$

3. Transitive closure :-

The Relation obtained by adding the least number of ordered pairs to ensure Transitivity is called the Transitive closure of the Relat^n R. and is given by

$$\boxed{R^{*} = R \cup R^{2} \cup \cdots R^{n}}$$

where $n = |A|$ & R be a Relat^n on A.

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Ex:- Let $A = \{1, 2, 3, 4\}$
and $R = \{(1,2), (2,3), (3,4)\}$
be a Relation on A. Find transitive closure R^*
using composition of R.

Soln:- given $R = \{(1,2), (2,3), (3,4)\}$

→ Now $R^2 = R \circ R$

$R = \{(1,2), (2,3), (3,4)\}$

$R = \{(1,2), (2,3), (3,4)\}$

So $R^2 = \{(1,3), (2,4)\}$

→ $R^3 = R^2 \circ R$

$R = \{(1,2), (2,3), (3,4)\}$

$R^2 = \{(1,3), (2,4)\}$

So $R^3 = \{(1,4)\}$

→ $R^4 = R^3 \circ R$

~~R^3~~ $R = \{(1,2), (2,3), (3,4)\}$

$R^3 = \{(1,4)\}$

So $R^4 = \phi$

Hence the transitive closure of R

$R^* = R \cup R^2 \cup R^3 \cup R^4$

$= \{(1,2), (2,3), (3,4)\} \cup \{(1,3), (2,4)\} \cup \{(1,4)\} \cup \phi$

$= \{(1,2), (2,3), (3,4), (1,3), (2,4), (1,4)\}$

Warshall's Algorithm :→

An efficient method for computing the Transitive closure of a Relation is known as Warshall's algorithm named after Stephen Warshall.

Step 1:- Let n be the number of elements in a given set A.
To find transitive closure of Relation R on A, maximum n warshall sets can be find $W_0, W_1, \ldots, W_n$ where $W_0 = M_R$ and W_n is the required matrix to find transitive closure.

Step 2:- Procedure to compute W_K from W_{K-1}.

1. copy 1 entries from W_{K-1} in W_K with same position.
2. In K^{th} row and K^{th} coloum of W_{K-1},
   See K^{th} coloum and check the position of 1 and put below p_i.
   See K^{th} row and check the position of 1 and put below q_i.
3. mark entries in W_K as 1 for (p_i, q_i) of W_K if there are not already 1.

Step 3:- stop when W_n is obtained and it is the required transitive closure

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Ex:- 1. Let $A = \{4, 6, 8, 10\}$
& $R = \{(4,4), (4,10), (6,6), (6,8), (8,10)\}$
is a Relation on set A. Determine the transitive closure of R using warshall's algorithm.

Soln - Given $A = \{4, 6, 8, 10\}$
and $R = \{(4,4), (4,10), (6,6), (6,8), (8,10)\}$

$$M_R = \begin{array}{c|c} & \begin{array}{cccc} 4 & 6 & 8 & 10 \end{array} \\ \begin{array}{c} 4 \\ 6 \\ 8 \\ 10 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Now we have $|A| = n = 4$
Thus we have to find warshall's sets W_0, W_1, W_2, W_3, W_4.
Here

$$W_0 = M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1. To compute W_1 - Transfer all 1's from W_0 to W_1
The locations of non-zero entries in C_1 :- $P_i = 1$
The locations of non-zero entries in R_1 :- $q_i = 1, 4$

$$W_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (1,1), (1,4)$$

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2. To compute W_2 – Transfer all 1's from W_1 to W_2

Positions of the non zero entries in C_2 : $p_i = 2$
Positions of non zero entries in R_2 : $q_i = 2, 3$

$$W_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (2,2), (2,3)$$

3. To compute W_3 :- Transfer all 1's from W_2 to W_3

Positions of non zero entries in C_3 : $p_i = 2$
Positions of non zero entries in R_3 : $q_i =$

$$W_3 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (2,4)$$

4. To compute W_4 :- Transfer all 1's from W_3 to W_4
Positions of non-zero entries in C_4 : $p_i = 1, 2, 3$
Positions of non-zero entries in R_4 : q_i —
No New pairs formes

So $W_4 = W_3$

$$= \begin{array}{c} \\ 4 \\ 6 \\ 8 \\ 10 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 6 & 8 & 10 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\therefore \quad R^* = \{ (4,4), (4,10), (6,6), (6,8), (6,10), (8,10) \}$

Ex:- 2. Let $A = \{1, 2, 3, 4\}$ & $R = \{(1, 2), (2, 3), (3, 4)\}$ be a relation on A. Find transitive closure R^* by warshall's Algorithm.

Soln -

Here $|A| = 4$

So we have to find warshall's sets W_0, W_1, W_2, W_3, W_4.

$R = \{(1, 2), (2, 3), (3, 4)\}$

$$\therefore M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array}_{4\times 4}$$

Now $W_0 = M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

1.) $W_1 \rightarrow$

$$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

C_1	R_1
p_i	q_i
ϕ	2

No new pairs formed

2.) $W_2 \rightarrow$

$$W_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

C_2	R_2
p_i	q_i
1	3

(1, 3)

3.) $W_3 \rightarrow$

$$W_3 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

C_3	R_3
p_i	q_i
1	4
2	

$(1, 4), (2, 4)$

4.) $W_4 \rightarrow$

$$W_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

C_4	R_4
p_i	q_i
1	ϕ
2	
3	

No new pairs formed

Hence the Required Transitive closure is

$$R^* = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$$

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Equivalence classes: →

Imp

Suppose R is an equivalence Relation on a Set A. For each x in A, the Equivalence class of x denoted by $[x]$ is the set of all elements y in A such that x is Related to y by R.

i.e $[x] = \{ y \in A : (x, y) \in R \}$

Ex:- Let $A = \{1, 2, 3, 4\}$ and
$R = \{(1,2), (2,1), (1,1), (2,2), (3,3), (4,4)\}$
is equivalence relation. Determine Equivalence classes.

Soln:- R is an equivalence Relation on A
So $[1] = \{1, 2\}$
$[2] = \{1, 2\}$
$[3] = \{3\}$
$[4] = \{4\}$

Quotient Set: →

The collection of all equivalence classes of elements of set A under an equivalence relation R is called Quotient Set, denoted by $A|R$

i.e $A|R = \{[x] : x \in A\}$

Ex:- In above example,

$A|R = \{[1], [2], [3], [4]\}$
or $A|R = \{\{1,2\}, \{3\}, \{4\}\}$

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Partions of a Set : →

A partition of a Set A is a Set of non-empty subsets of A denoted by $\{A_1, A_2, \dots, A_k\}$ such That

i.) $A_1 \cup A_2 \cup \dots \cup A_k = A$
ii.) $A_i \cap A_j = \phi$ for $i \neq j$

Ex! - Let $A = \{1, 2, 3, 4\}$
$R = \{(1,2), (2,1), (1,1), (2,2), (3,3), (4,4)\}$
Then

$[1] = \{1, 2\}, [2] = \{1, 2\}$
$[3] = \{3\}, [4] = \{4\}$

So $P_1 = \{1, 2\}$
$P_2 = \{3\}$
$P_4 = \{4\}$
$P_1 \cup P_2 \cup P_3 = A$
& $P_i \cap P_j = \phi$ for $i \neq j$.

POSETS & Lattices

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Partially ordered Sets (POSETs) :→

A Relation R on a Set S is called a Partial order if it is Reflexive, antisymmetric and transitive i.e for $a, b, c \in S$

1.) aRa, $\forall a \in S$ (Reflexive)
2.) aRb and $bRa \Rightarrow a = b$ (Antisymmetric)
3.) aRb and $bRc \Rightarrow aRc$ (Transitive)

Then "A set S together with a Partial order Relation R is called a Partially ordered set or Poset". It is denoted by (S, R).

* The Relation R is often denoted by $\preceq$ (different from the usual less than or equal to $\leq$) Hence a Poset is denoted by $(S, \preceq)$

* In a Poset $(S, \preceq)$
$a \preceq b$ means 'a precedes b' and 'b succeed a'.
$a \prec b$ means 'a strictly precedes b' and 'b strictly succeed a'.

Important Posets :—

1.) The set of Integers $\mathbb{Z}$ with relation $\geq$ i.e $(\mathbb{Z}, \geq)$ is a Poset.

Since i.) $a \geq a$, $\forall a \in \mathbb{Z}$ (Reflexive)
ii.) $a \geq b$ and $b \geq a \Rightarrow a = b$ (Antisymmetric)
iii.) $a \geq b$ and $b \geq c \Rightarrow a \geq c$ (Transitive)
Hence $(\mathbb{Z}, \geq)$ is a Poset.

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2.) The Power Set $P(S)$ of a set S together with the inclusion Relation $\subseteq$ i.e $(P(S), \subseteq)$ is a Poset.

Since for A, B and $C \in P(S)$

i.) $A \subseteq A$, $\forall A \in P(S)$

$\Rightarrow$ $\subseteq$ is reflexive

ii.) $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$

$\Rightarrow$ $\subseteq$ is Antisymmetric

iii.) $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$

$\Rightarrow$ $\subseteq$ is Transitive

Hence $(P(S), \subseteq)$ is a Poset.

3.) The Set of all Positive Integers $\mathbb{Z}^+$ with divisibility "/" relation i.e $(\mathbb{Z}^+, /)$ is a Poset.

Since, for $a, b, c \in \mathbb{Z}^+$

i.) a/a, $\forall a \in \mathbb{Z}^+$ Reflexive

ii.) a/b and $b/a \Rightarrow a = b$ Antisymmetric

iii.) a/b and $b/c \Rightarrow a/c$ Transitive

Thus / is a Partial order Relation on $\mathbb{Z}^+$ and $(\mathbb{Z}^+, /)$ is a Poset.

Note:- i.) $(\mathbb{Z}, /)$ is not Poset, as $a/(-a)$ and $(-a)/a$ but $a \neq -a$

So the relation is not antisymmetric on $\mathbb{Z}$

ii.) $(\mathbb{Z}^+, <)$ is not Poset as the Relation is not Reflexive on $\mathbb{Z}^+$ for $a \nless a$ for any $a \in \mathbb{Z}^+$.

(iv) Comparability :—

The elements a and b of a Poset $(S, \preceq)$ are comparable if either $a \preceq b$ or $b \preceq a$.

if $(S, \preceq)$ is a Poset and every two elements of S are Comparable. Then S is called a Totally ordered Set. It is also called a chain.

e.g → $(\mathbb{Z}, \leq)$ is a Totally ordered Set.
but $(\mathbb{Z}^+, |)$ is not Totally ordered set.

Product order → For $s, s' \in S$ and $t, t' \in T$ the Product set defined by
$(s,t) \preceq (s', t')$ if $s \preceq_1 s'$ and $t \preceq_2 t'$ in S and T resp., where (s,t) and $(s', t') \in S \times T$. is called the Product order.

Lexicographic order →
suppose $(S, \preceq_1)$ and $(T, \preceq_2)$ are Posets. The Cartesian Product $S \times T$ defined by $(s,t) \preceq (s', t')$ either if $s \preceq_1 s'$ or if both $s = s'$ and $t \preceq_2 t'$.

(v) Immediate Successor and Predecessor :—
Let $(X, \prec)$ be a Poset and suppose $x, y \in X$. Then y is said to be immediate successor of x if $x \prec y$ and there is no element z in X such that $x < z < y$. we also say y covers x or x is an immediate Predecessor of y.

e.g →
In Posel $(P, \subseteq)$, $P = \{1, 2, 3, 4, 5\}$ the two elements $A = \{2, 4\}$ & $B = \{2, 4, 5\}$ are s.t A is immediate Predecessor of B

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(#) Hasse Diagram :→

A Graphical representation of a Partial order Relation in which all arrow heads are understood to be pointing upward is known as the Hasse Diagram.

Procedure for Drawing Hasse Diagram :—

Step-1 Draw Digraph of the Relation

Step-2 Remove self loops

Step-3 Remove all Transitive edges

Step-4 Arrange all edges pointing upward and remove arrows from edges.

Step-5 Replace circle by dots or vertex.

Note:- Let $\leq$ be a Partial Relation on a Set X. if $x, y \in X$ and y is an immediate Successor of x, Then y is replaced at a higher level than x and x and y are joined by a Straight line.

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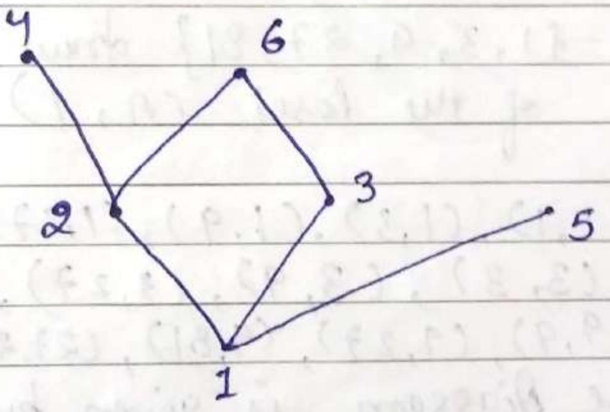
Ex:-1. Let $X = \{1, 2, 3, 4, 5, 6\}$ and $|$ is a partial order Relation on X. Draw the Hasse Diagram of $(X, |)$.

Soln:- $R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,4),$
~~Hence 2, 3, 5~~ $(2,6), (3,3), (3,6), (4,4), (5,5)$
$(6,6)\}$

Here 2, 3, 5 are immediate successors of 1 and hence they are placed at Higher level than 1 and connected with 1.
Similarly, 6 is an immediate successor of 2, 3
4 is an immediate successor of 2



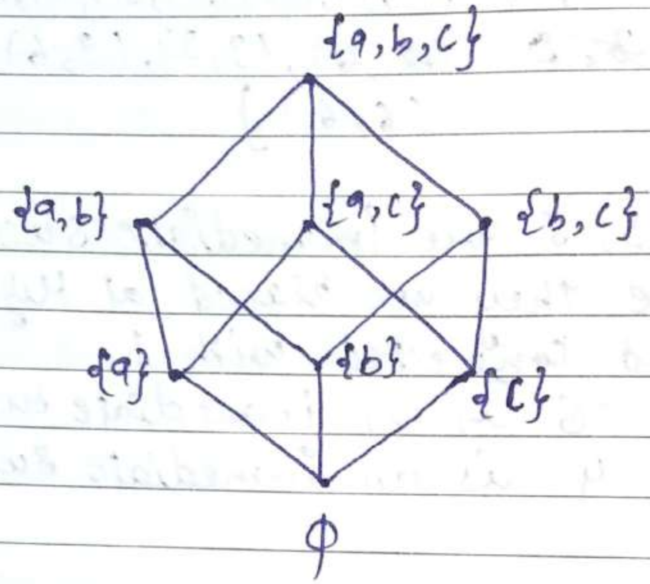
Ex-2. Draw the Hasse Diagram for the Poset $(P(S), \subseteq)$, where $S = \{a, b, c\}$.

Soln:-

Here $P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\},$
$\{c, a\}, \{a, b, c\}\}$

Since ϕ is the subset of all sets, So it is the lowest point of the Diagram. Now $\{a\}$, $\{b\}$, $\{c\}$ are immediate successor of ϕ. They are placed at Higher level than ϕ and connected with ϕ. $\{a, b\}$ is an immediate successor of $\{a\}$ and $\{b\}$ so $\{a, b\}$ is placed at higher level

than $\{a\}$ and $\{b\}$ and connected with $\{a\}, \{b\}$
Similarly, the other Points are drawn.
So the Hasse Diagram of the Poset $(P(S), \subseteq)$ is

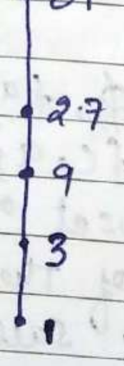


Ex:- 3. Let $A = \{1, 3, 9, 27, 81\}$ draw the Hasse Diagram of the Poset $(A, /)$

Sol^n:-

$$R = \{(1,1), (1,3), (1,9), (1,27), (1,81), (3,3), (3,9), (3,27), (3,81), (9,9), (9,27), (9,81), (27,27), (27,81), (81,81)\}$$

The Hasse Diagram is given by



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Ex:- 4. consider the Set $A = \{4, 5, 6, 7\}$. Let R be the Relation $\leq$ on A. Draw the Hasse Diagram of R.

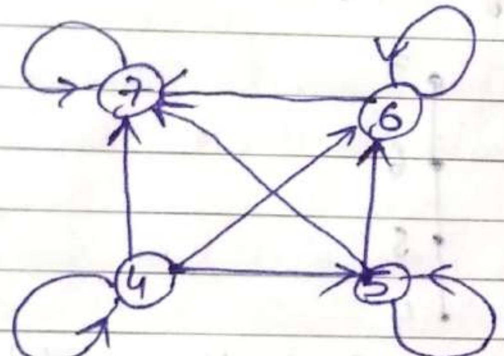
Ans -

Given $A = \{4, 5, 6, 7\}$, $R \rightarrow \leq$

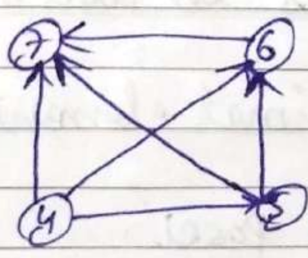
Then

$$R = \{(4,4), (5,5), (6,6), (7,7), (4,5), (4,6), (4,7), (5,6), (5,7), (6,7)\}$$

Step 1 → Directed graph of R →



Step 2:- Remove Self loops.



Step 3:- Remove transitive edges.

The edges given by the Transitive Property

$$\{(4,5), (4,6), (5,6), (5,7), (6,7), (4,7)\}$$

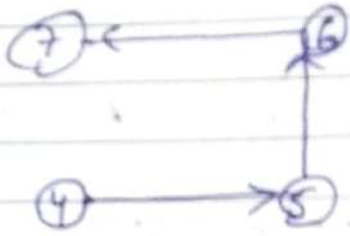
$(4,5) \in R$ & $(5,6) \in R \Rightarrow (4,6) \in R$

$(5,6) \in R$ & $(6,7) \in R \Rightarrow (5,7) \in R$

$(4,5) \in R$ & $(5,7) \in R \Rightarrow (4,7) \in R$

The ordered pairs implied by Transitivity
(4, 6), (4, 7), (5, 7)

$\therefore$



<u>Step 4</u>:- Remove arrows from edges and replace circle by dot and making all edges pointing upward.

This is the Required Hasse diagram of $(A, \leq)$

<u>Special Elements in Posets :</u> $\longrightarrow$

\# <u>Maximal and minimal elements :-</u>

Let $(P, \preceq)$ be a poset.
An element a in the poset is called a maximal element of P if $a < x$ for no x in P.
i.e no element of P strictly succeeds a

An element b in P is called Minimal element of P if $x < b$ for no x in P,
i.e no element of P strictly Precedes b.

Greatest and least element :—

* An element $a \in P$ is Greatest element of P if $x \le a$ for all $x \in P$ i.e every element in P preceds a.
* An element $b \in P$ is called the least element of P if $b \le x$ for all $x \in P$ i.e every element in P succeeds b.

Note:- ① The Greatest and least elements are unique if they exist.

② The greatest element if it exists, will be comparable with all elements of the Poset.

③ Maximal and minimal elements are easy to spot using a Hasse Diagram. They are just the top and bottom elements in the Diagram.

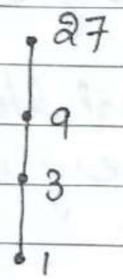
④ A Poset may have more than one maximal and minimal elements.

Ex:- Find the least and greatest element in the Poset $(\mathbb{Z}^+, |)$, if they exist.

Soln:-
Since $1 | n$ for all $n \in \mathbb{Z}^+$
So 1 is the least element of $(\mathbb{Z}^+, |)$
Since there is no integer which is divisible by all positive integers
So there is no greatest element.

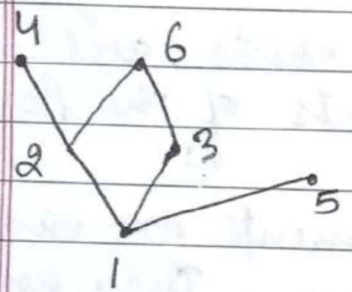
Ex:- Determine the greatest element, least element, maximal element and minimal elements for the Posets given by following Hasse Diagram.

(a)



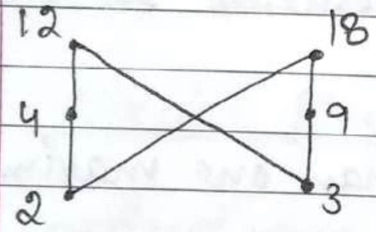
The least element of the Poset is 1 and is the only minimal element. The greatest element of the Poset is 27 and is the only Maximal element

(b)



The least element is 1 and is the only minimal element. There is no greatest element and 4, 6, and 5 are maximal element

(c)

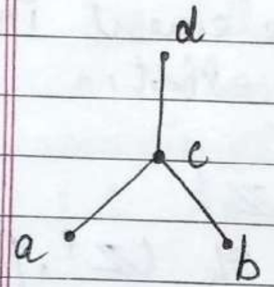


The Poset given by Hasse Diagram neither has least element nor has greatest element

minimal elements = 2, 3

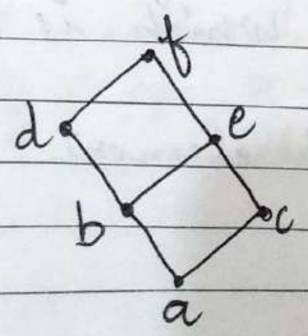
maximal elements = 12, 18

(d)



There is no least element and a and b are minimal elements. d is the greatest element and is the only maximal element.

(e)

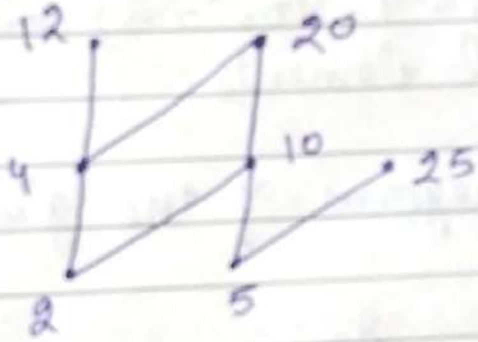


The greatest element is f and is the only Maximal element. The least element is a and is the only minimal element.

The greatest element

<u>Ques. 1</u> Find the maximal and minimal elements of the poset $(P, \leq)$ where $P = \{2, 4, 5, 10, 12, 20, 25\}$, and the partial order $\leq$ is 'a divides b'

<u>Solⁿ</u> – The Hasse Diagram for the given Poset $(P, \leq)$ is as follows:



maximal elements = 12, 20, 25
minimal elements = 2, 5

<u>Ques. 2</u> In each of the following cases, give an example of a poset $(P, \leq)$ such that

i.) P has precisely three maximal elements and 2 minimal elements.

ii.) P has minimal elements, but no maximal element

iii.) P has neither a minimal nor maximal elements

<u>Solⁿ</u>. –

i.) The Poset $P = \{2, 4, 5, 10, 12, 20, 25\}$ with relation divisibility '/' is poset which has 3 maximal elements = 12, 20, 25
and 2 minimal elements = 2, 5

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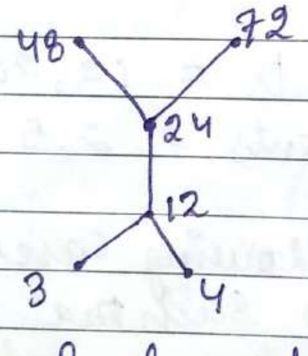
ii.) The Poset $(\mathbb{Z}^+, \leq)$ has a minimal element 1 but no maximal elements.
whereas The Poset $(\mathbb{Z}^-, \leq)$ has a maximal element -1 and no minimal elements

iii.) The Poset $(\mathbb{Z}, \geq)$ neither has a maximal element nor minimal element.

Ques 3 Draw the Hasse Diagram of the Poset $A = \{3, 4, 12, 24, 48, 72\}$ and find maximal and minimal elements.

Soln:-

The Hasse Diagram of the given Poset is as follows



maximal elements = 48, 72
minimal elements = 3, 4

There does not exist greatest and least elements

Upper and lower bound :—

Let B be a subset of a Poset $(A, \leq)$
an element $u \in A$ is called an upper bound of B
of B if u succeeds every elements of B
i.e $x \leq u$ for all $x \in B$.

An element $l \in A$ is called lower bound of B
of B if l Precedes every element of B
i.e $l \leq x$ for all $x \in B$.

Supremum = least upper bound (l.u.b)
Infimum = greatest lower bound (g.l.b).

Note:— ① The Greatest element is always the Supremum but the converse is not true.
$a = \sup(B)$ is greatest element iff $a \in B$

② least element is always the Infimum but the converse is not true.
$b = \inf(B)$ is the least element iff $b \in B$.

Ex:— suppose the set $P = \{1, 2, 3, 4, 5, 6, 7, 8\}$ is partially ordered by the Relation "divides". Find the lower and upper bounds of $A = \{1, 2\}$ and $B = \{3, 4, 5\}$ of the Poset $(P, \leq)$. Also find the $\sup(A)$, $\inf(A)$, $\sup(B)$ and $\inf(B)$.

Soln:— upper bounds of $A = \{u \in P : x \leq u, \forall x \in A\}$
$= \{2, 4, 6, 8\}$

So $\sup(A) = 2$

lower bounds of $A = \{l \in P : l \leq x, \forall x \in A\}$
$= \{1\}$

So $\mathrm{Inf}(A) = 1$

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upper bounds of $B = \{u \in P : x \leq u, \forall x \in B\}$
$= \phi$

So $\sup(B)$ does not exist

lower bounds of $A = \{l \in P : l \leq x, \forall x \in B\}$
$= \{1\}$

So $\operatorname{Inf}(B) = 1$

Ex:- 2. consider the poset $A = (\{1, 2, 3, 4, 6, 9, 12, 18, 36\}, /)$ find the greatest lower bound and the least upper bound of $A = \{6, 18\}$ and $B = \{4, 6, 9\}$.

Soln:-

lower bounds of $A = \{1, 2, 3, 6\}$

So $\operatorname{Inf}(A) = 6$

upper bounds of $A = \{18, 36\}$

So $\sup(A) = 18$

Now lower bounds of $B = \{1\}$

So $\operatorname{Inf}(B) = 1$

upper bounds of $B = \{36\}$

So $\sup(B) = 36$.

Well-ordered Set :→

A set with an ordering relation is well-ordered if every non-empty subset of the set has a least element.

e.g → The set of natural numbers is well ordered, But the set of Integers Z is not well ordered since the set of Negative integers, which is a subset of Z, has no least element.

Lattice :→

A Poset $(P, \leq)$ is called a lattice if every 2-element subset of P has both a least upper bound and a greatest lower bound

imp i.e if lub $\{x, y\}$ and glb $\{x, y\}$ exist for every x and y in P. It is denoted by $(L, \leq, \vee, \wedge)$ where

$$x \vee y = \text{lub}\{x, y\} \quad (\text{read as } x \text{ join } y)$$
$$x \wedge y = \text{glb}\{x, y\} \quad (\text{read as } x \text{ meet } y).$$

Examples :→ ① The Power set $(P(S), \subseteq)$ is a lattice

Let $A, B \subseteq S$ Then

$\sup\{A, B\} = A \vee B = A \cup B$

$\inf\{A, B\} = A \wedge B = A \cap B$.

As, $A \subseteq A \cup B$ and $B \subseteq A \cup B$

and if $A \subseteq C$ and $B \subseteq C$ Then

$A \cup B \subseteq C$.

So $A \cup B$ is the least subset of S which contains both A & B.

Also $A \cap B \subseteq A$ and $A \cap B \subseteq B$

and if $C \subseteq A$ & $C \subseteq B$ Then

$C \subseteq A \cap B$. So $A \cap B$ is the greatest subset of S which contained both A & B

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e.g In the Poset $(P\{a,b,c\}, \subseteq)$

$$\{a\} \vee \{b\} = \{a,b\}$$
$$\{a,b\} \vee \{b,c\} = \{a,b,c\}$$
$$\{a,b\} \wedge \{b,c\} = \{b\}$$
$$\{a\} \wedge \{b\} = \phi.$$

Therefore $(P(S), \subseteq)$ is a lattice

2. The Poset $(Z^+, /)$ is a lattice because any upper bound of $\{a,b\}$ where $a, b \in Z^+$, is Nothing but an element which is divisible by both a and b i.e common multiple of a and b which is an upper bound of a and b. The least upper bound is thus the least common multiple (lcm) IIIy, the greatest lower bound is the greatest common divisor of (gcd) of a and b.

$$\therefore \quad \left.\begin{aligned} lub\{a,b\} &= a \vee b = lcm(a,b) \\ glb\{a,b\} &= a \wedge b = gcd(a,b) \end{aligned}\right\}$$

e.g → $lub\{6,4\} = 6 \vee 4 = lcm(6,4) = 12$
$glb\{6,4\} = 6 \wedge 4 = gcd(6,4) = 2$

3. $(Z, \leq)$ is a lattice where
$$m \vee n = max\{m,n\}$$
$$m \wedge n = min\{m,n\}$$

(b) A chain $(P, \leq)$ is a lattice where $a \vee b = b$ and $a \wedge b = a$ if $a \leq b$ and $a \vee b = a$ and $a \wedge b = b$ if $b \leq a$.

Note:- Not all posets are lattice.
e.g → The Poset $(\{2,3,6,12,24,36\}, /)$ is not lattice since $glb(2,3)$ and $lub(24,36)$ do not exist in S, hence $(S, /)$ is not a lattice.

Ques. Determine whether the posets represented by each of the Hasse Diagram are lattices.

(a)

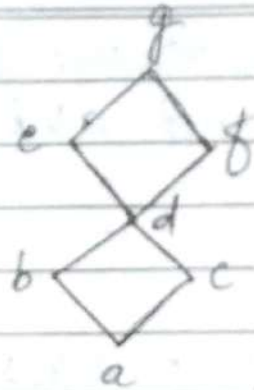
Sol^n:- The closure table for lub$(\vee)$ and glb$(\wedge)$ are given below

$\vee$	a	b	c	d
a	a	b	c	d
b	b	b	c	d
c	c	c	c	d
d	d	d	d	d

$\wedge$	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	b	c	c
d	a	b	c	d

Since each subset of two elements has least upper bound and a greatest lower bound. So the given Hasse Diagram represents a lattice.

(b).



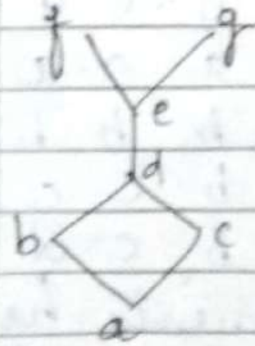
The closure table for lub ($\vee$) and glb ($\wedge$) are as follows:

$\vee$	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	d	d	e	f	g
c	c	d	c	d	e	f	g
d	d	d	d	d	e	f	g
e	e	e	e	e	e	f	g
f	f	f	f	f	g	f	g
g	g	g	g	g	g	g	g

$\wedge$	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	b	b	b	b
c	a	a	c	c	c	c	c
d	a	b	c	d	d	d	d
e	a	b	c	d	e	d	e
f	a	b	c	d	d	f	f
g	a	b	c	d	e	f	g

Since each subset of two elements has least upper bound & a greatest lower bound. So this in a lattice.

(c).



lub ($\vee$) table:

$\vee$	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	d	d	e	f	g
c	c	d	c	d	e	f	g
d	d	d	d	d	e	f	g
e	e	e	e	e	e	f	g
f	f	f	f	f	f	f	—
g	g	g	g	g	g	—	g

glb ($\wedge$) table:

$\wedge$	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	b	b	b	b
c	a	a	c	c	c	c	c
d	a	b	c	d	d	d	d
e	a	b	c	d	e	e	e
f	a	b	c	d	e	f	e
g	a	b	c	d	e	e	g

Since lub(f, g) does not exist in Poset
Hence this is not a lattice.

Properties of Lattices :→

Theorem 1: Let L be a lattice, then for every a and b in L.

(a) $a \vee b = b$ if and only if $a \le b$.
(b) $a \wedge b = a$ if and only if $a \le b$
(c) $a \wedge b = a$ if and only if $a \vee b = b$.

Proof:- (a) Let $a \vee b = b$
Since $a \le a \vee b = b$
$\Rightarrow a \le b$.

Conversely, let $a \le b$
Since $a \le b$ and $b \le b$
$\Rightarrow$ b is an upper bound of a and b
by def. of lub, $a \vee b \le b$ —(1)
Also, $a \vee b$ is an upper bound
$\Rightarrow b \le a \vee b$ —(2)

So, from (1) and (2), we have
$\boxed{a \vee b = b.}$ Hence Proved.

(b) Let $a \wedge b = a$
Since $a \wedge b$ is a lower bound of a and b
$\therefore$ $a \wedge b \le b$
$\Rightarrow$ $a \le b$
Conversely, if $a \le b$ then since
$a \le a$ and $a \le b$.
$\Rightarrow$ a is a lower bound of a and b.
by def. of glb, $a \le a \wedge b$ —(1)
also, $a \wedge b$ is a lower bound of a and b
$\therefore$ $a \wedge b \le a$ —(2)
So, from (1) and (2), we have
$\boxed{a \wedge b = a}$

(c) from (a) & (b) we have
$a \wedge b = a$ iff $a \le b$ and $a \le b$ iff $a \vee b = b$
$\Rightarrow$ $a \wedge b = a$ iff $a \vee b = b$.

Theorem 2: if L be any Lattice, then for any $a, b, c \in L$,
1. (a) $a \vee a = a$ (b) $a \wedge a = a$ Idempotency
2. (a) $a \vee b = b \vee a$ (b) $a \wedge b = b \wedge a$ Commutativity
3. (a) $(a \vee b) \vee c = a \vee (b \vee c)$
(b) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ Associativity
4. (a) $a \vee (a \wedge b) = a$ (b) $a \wedge (a \vee b) = a$
Absorption.

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Proof:- 1. (a) $a \vee a = \sup\{a, a\} = \text{lub}\{a, a\}$
$= \text{lub}\{a\} = a$.

(b) $a \wedge a = \inf\{a, a\} = \text{glb}\{a, a\} = \text{glb}\{a\} = a$

2. (a) $a \vee b = \text{lub}\{a, b\} = \text{lub}\{b, a\} = b \vee a$

(b) $a \wedge b = \text{glb}\{a, b\} = \text{glb}\{b, a\} = b \wedge a$.

3. (a) we have $(a \vee b) \vee c = \text{lub}\{a \vee b, c\}$.

$\Rightarrow a \vee b \leq (a \vee b) \vee c$ & $c \leq (a \vee b) \vee c$

also $a \leq (a \vee b)$ & $b \leq (a \vee b)$

So by transitivity

$a \leq (a \vee b) \vee c$, $b \leq (a \vee b) \vee c$

thus $(a \vee b) \vee c$ is an upper bound of a and b.

also $b \leq (a \vee b) \vee c$ & $c \leq (a \vee b) \vee c$

$\Rightarrow (a \vee b) \vee c$ is upper bound of b and c

by def. of lub of b & c i.e $b \vee c$,

$b \vee c \leq (a \vee b) \vee c$ $\Rightarrow$ $(a \vee b) \vee c$ is an upper bound of $a, b \vee c$

& $a \leq (a \vee b) \vee c$

$\Rightarrow a \vee (b \vee c) \leq (a \vee b) \vee c$ by def. of $\vee$ —— (1)

Again, $a \vee (b \vee c) = \text{lub}\{a, b \vee c\}$.

$\Rightarrow a \leq a \vee (b \vee c)$ & $b \vee c \leq a \vee (b \vee c)$

also $b \leq b \vee c$ & $c \leq b \vee c$

$\Rightarrow b \leq a \vee (b \vee c)$ & $c \leq a \vee (b \vee c)$ by transitivity

$\Rightarrow a \vee (b \vee c)$ is an upper bound of b & c.

also $a \leq a \vee (b \vee c)$ & $b \leq a \vee (b \vee c)$

$\Rightarrow a \vee (b \vee c)$ is an upper bound of a & b.

by def of $a \vee b$, $a \vee b \leq a \vee (b \vee c)$

& $c \leq a \vee (b \vee c)$

$\Rightarrow a \vee (b \vee c)$ is an upper bound of $a \vee b$ and c

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$\Rightarrow (a \vee b) \vee c \leq a \vee (b \vee c)$ — (2) by def of lub

Applying Antisymmetry on (1) & (2), we have,

$(a \vee b) \vee c = a \vee (b \vee c)$. Hence proved

Theorem 3:- Let $(L, \leq)$ be a lattice, then for any $a, b, c, d \in L$.

1. $a \leq b \Rightarrow a \vee c \leq b \vee c$
2. $a \leq b \Rightarrow a \wedge c \leq b \wedge c$
3. $a \leq b$ and $c \leq d \Rightarrow a \vee c \leq b \vee d$.
4. $a \leq b$ and $c \leq d \Rightarrow a \wedge c \leq b \wedge d$.

Proof:-

1. We know,

$a \leq b \Leftrightarrow a \vee b = b$ —(1) (from Th^m 1)

Now, we get

$(a \vee c) \vee (b \vee c) = (a \vee c) \vee (c \vee b)$ (commutative law)
$= a \vee (c \vee c) \vee b$ (Associative law)
$= a \vee (c \vee b)$ ($\because c \vee c = c$)
$= a \vee (b \vee c)$ (commutative law)
$= (a \vee b) \vee c$ (Associative law)
$= b \vee c$ from (1)

Thus, $a \vee c \leq b \vee c$

2. We know,

$a \leq b \Leftrightarrow a \wedge b = a$ —(2) (from Th^m 1)

Now let $a \leq b$, we have to prove that

$a \wedge c \leq b \wedge c$

So consider, $(a \wedge c) \wedge (b \wedge c)$
$= (a \wedge c) \wedge (c \wedge b)$ (commutative law)
$= (a \wedge (c \wedge c) \wedge b$ (Associative law)
$= a \wedge (c \wedge b)$ ($\because c \wedge c = c$)
$= a \wedge (b \wedge c)$ (commutative law)
$= (a \wedge b) \wedge c$ (Associative law)
$= a \wedge c$ from (2)

$\Rightarrow a \wedge c \leq b \wedge c$.

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3. Let $a \le b$ and $c \le d$ —(1)

we know

$a \le b \Leftrightarrow a \vee b = b$ } (from Th^m 1)

& $c \le d \Leftrightarrow c \vee d = d$ } —(2)

Now,

$(a \vee c) \vee (b \vee d) = a \vee (c \vee b) \vee d$ (Associative law)

$= a \vee (b \vee c) \vee d$ (commutative law)

$= (a \vee b) \vee (c \vee d)$ (Associative law)

$= b \vee d$ from (1) & (2)

$\Rightarrow a \vee c \le b \vee d$ (from Th^m 1. i.e $a \le b \Leftrightarrow a \vee b = b$)

4. Let $a \le b$ and $c \le d$ —(1)

we know,

$a \le b \Leftrightarrow a \wedge b = a$ }

$c \le d \Leftrightarrow c \wedge d = c$ } —(2)

we have to prove that $a \wedge c \le b \wedge d$

So consider

$(a \wedge c) \wedge (b \wedge d) = a \wedge (c \wedge b) \wedge d$ (Associative law)

$= a \wedge (b \wedge c) \wedge d$ (commutative law)

$= (a \wedge b) \wedge (c \wedge d)$ (Associative law)

$= a \wedge c$ from (1) & (2)

$\Rightarrow a \wedge c \le b \wedge d$.

Theorem 4:— Let $(L, \le, \vee, \wedge)$ be a lattice and $a, b, c \in L$ then the distributive Inequalities are as follows:

(1.) $a \wedge (b \vee c) \ge (a \wedge b) \vee (a \wedge c)$

(2.) $a \vee (b \wedge c) \le (a \vee b) \wedge (a \vee c)$

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Proof:- (1.) Since $a \wedge b$ is greatest lower bound of a & b

$\Rightarrow$ $a \wedge b \leq a$ and $a \wedge b \leq b$.

also, $b \vee c$ is least upper bound of b & c

$\Rightarrow$ $b \leq b \vee c$ and $c \leq b \vee c$

So $a \wedge b \leq b$ and $b \leq b \vee c$

$\Rightarrow$ $a \wedge b \leq b \vee c$ (by transitivity)

& $a \wedge b \leq a$

$\Rightarrow$ $a \wedge b$ is lower bound of $\{a, b \vee c\}$

$\Rightarrow$ $a \wedge b \leq a \wedge (b \vee c)$ —— (1)

Again $a \wedge c \leq a$ & $a \wedge c \leq c$.

also $c \leq b \vee c$

$\Rightarrow$ $a \wedge c \leq b \vee c$ by transitivity

& $a \wedge c \leq a$

$\Rightarrow$ $a \wedge c$ is lower bound of $\{a, b \vee c\}$

$\Rightarrow$ $a \wedge c \leq a \wedge (b \vee c)$ —— (2)

(1) and (2) shows that

$a \wedge (b \vee c)$ is an upper bound of $\{a \wedge b, a \wedge c\}$

$\Rightarrow$ $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$ by def. of lub($\vee$)

Hence

$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$.

(2.) $a \leq a \vee b$ & $b \leq a \vee b$

and $b \wedge c \leq b$ & $b \wedge c \leq c$

So $b \wedge c \leq b$ & $b \leq a \vee b$

$\Rightarrow$ $b \wedge c \leq a \vee b$ (by transitivity)

& $a \leq a \vee b$

$\Rightarrow$ $a \vee b$ is upper bound of $\{a, b \wedge c\}$

$\Rightarrow$ $a \vee (b \wedge c) \leq a \vee b$ —— (1) def. of $\vee$

Again, $a \leq a \vee c$

$b \wedge c \leq c \leq a \vee c \Rightarrow b \wedge c \leq a \vee c$

$\Rightarrow$ $a \vee c$ is upper bound of $\{a, b \wedge c\}$

$\Rightarrow$ $a \vee (b \wedge c) \leq a \vee c$ —— (2) by def. of $\vee$

(1) & (2) shows that

$a \vee (b \wedge c)$ is lower bound of $\{a \vee b, a \vee c\}$

$\Rightarrow$ $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$.

Hence proved

Remark:— These inequalities become equalities in many lattices (Including the Power set lattice) where the operation of union and inter-section are distributive over each other. But they do not hold in general.

Principle of duality:→

Principle of duality for lattices states that if we interchange $\vee$ with $\wedge$ and $\leq$ with $\geq$ in a true statement about lattices, we get another true statement.

Lattice Homomorphism (Homomorphism):—

Let L and M be two lattices then a mapping $f: L \to M$ is called lattice Homo. if

$f(x \vee y) = f(x) \vee f(y)$ (join Homomorphism)

and $f(x \wedge y) = f(x) \wedge f(y)$ (meet Homomorphism)

$\forall\, x, y \in L$

→ order-Homomorphism:— $f: L \to M$ is called order-Homomorphism if $x \leq y \Rightarrow f(x) \leq f(y)$ i.e it preserves the partial order, hold $\forall\, x, y \in L$

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→ If f is Homomorphism from L to M then f(L) is called Homomorphic Image of L.

→ Every Lattice Homomorphism is an order-Homomorphism, But converse is not True.

Isomorphism :— A Homomorphism $f : L \to M$ is called an Isomorphism if It is bijective i.e one-one and onto.

Automorphism :— An Isomorphism $f : L \to L$ is called an Automorphism.

Lattice as Algebraic Structure : →

A lattice is an algebraic Structure $(L, \vee, \wedge)$ with two binary operations $\vee$ and $\wedge$ which possesses the idempotent, commutative, Associative and Absorption Properties.

Sublattice : →

A non-empty subset L' of a lattice L is called a Sublattice of L if $a, b \in L' \Rightarrow a \vee b, a \wedge b \in L'$

i.e the Algebraic Structure $(L', \vee, \wedge)$ is a Sublattice of $(L, \vee, \wedge)$ iff L' is closed under both operation $\vee$ and $\wedge$.

→ a Sublattice itself is a lattice

→ Every Singleton of a lattice L is a sublattice of L.

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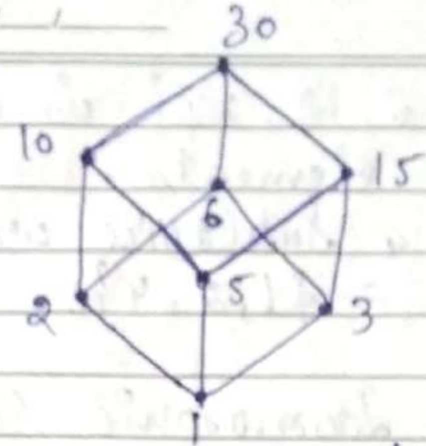
Example:-1 Let S be a set and $L = (P(S), \vee, \wedge)$ then only those sub collections of $P(S)$ are Sublattices of L which are closed under $\wedge$ and $\vee$. If $S = \{a, b, c, d\}$.
Then $\{\phi, \{a\}, \{a, c\}, \{c\}, \{a, b, c\}\}$ would be Sublattice but $\{\phi, \{a\}, \{b\}, \{c\}, \{b, c\}\}$ would not form a Sublattice as $\{a\} \cup \{c\} = \{a, c\}$ is not there.

\# union of two Sublattices may not be Sublattice.

Counter Example:-
consider the lattice $L = \{1, 2, 3, 4, 6, 12\} = D_{12}$ of factors of 12 under divisibility. Then $S = \{1, 2\}$ and $T = \{1, 3\}$ are Sublattices of L as $1 \vee 2 = lcm(1, 2) = 2 \in S$
& $1 \wedge 2 = gcd(1, 2) = 1 \in S$
also $1 \vee 3 = lcm(1, 3) = 3 \in T$
& $1 \wedge 3 = gcd(1, 3) = 1 \in T$.
But the union of S and T i.e $S \cup T = \{1, 2, 3\}$ is not a Sublattice as $2, 3 \in S \cup T$ but $2 \vee 3 = lcm(2, 3) = 6 \notin S \cup T$

\# Let D_n denote the set ~~the~~ of all positive divisors of n. Then D_m is a Sublattice of D_n if $m | n$.

Ex:- if $L = D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$. having the Hasse Diagram

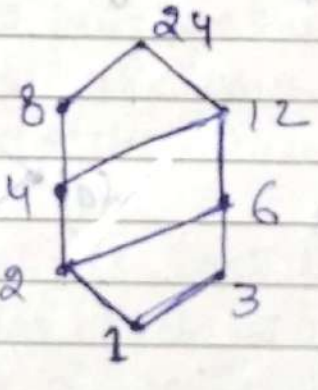


then $D_6 = \{1, 2, 3, 6\}$, $D_{10} = \{1, 2, 5, 10\}$,
$D_{15} = \{1, 3, 5, 15\}$ are sublattices of D_{30}
$\{5, 10, 15, 30\}$ is also sublattice of D_{30}

Ex:- Draw the Hasse Diagram of D_{24}, and find its sublattices with 4, 5, 6 and 7 elements. Does D_{24} contain a sublattice with 3 elements.

Solⁿ:-

$D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$.



Hasse Diagram of D_{24}.

Since $6 | 24 \Rightarrow D_6$ is a sublattice of D_{24}
So $D_6 = \{1, 2, 3, 6\}$ is a sublattice of D_{24} with 4 elements.

Since $12 | 24 \Rightarrow D_{12}$ is a sublattice of D_{24}
So $D_{12} = \{1, 2, 3, 4, 6, 12\}$ is sublattice with 6 elements

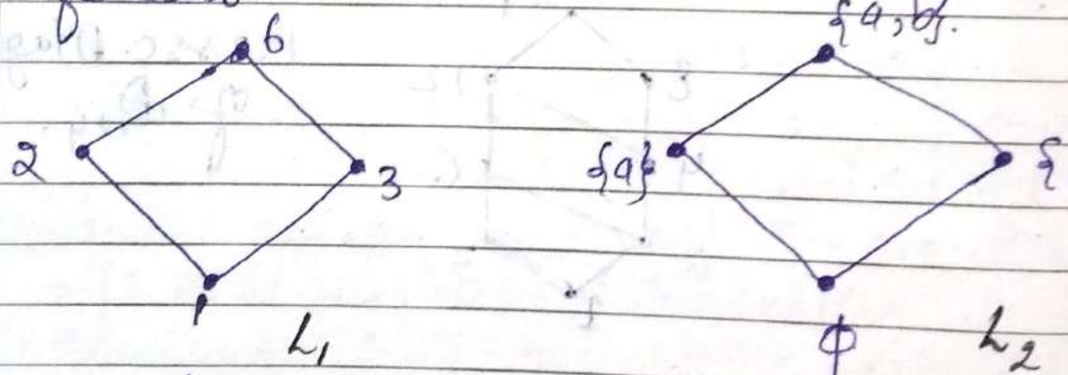
$S = \{1, 2, 3, 6, 12\}$ is a sublattice of D_{24} with 5 elements.

$T = \{1, 2, 3, 4, 6, 12, 24\}$ is a Sublattice of D_{24} with 7 elements.
also D_{24} has a Sublattice with 3 elements which is $D_4 = \{1, 2, 4\}$.

Example Based on Isomorphic lattice

Ex:- Let L_1 be the lattice $D_6 = \{1, 2, 3, 6\}$ and L_2 be another lattice $(P(S), \subseteq)$ where $S = \{a, b\}$. Then these two lattices are Isomorphic.

Soln:-
Here $L_1 = \{1, 2, 3, 6\}$
$L_2 = \{\phi, \{a\}, \{b\}, \{a, b\}\}$
The Hasse Diagram of L_1 and L_2 are as follows



Let $f : L_1 \to L_2$ be defined by

$$f(1) = \phi,\ f(2) = \{a\},\ f(3) = \{b\},\ f(6) = \{a, b\}$$

Then f is one-one & onto fn
Now we have to show that

$$f(x \vee y) = f(x) \vee f(y)$$

$$\& \quad f(x \wedge y) = f(x) \wedge f(y) \quad \forall x, y \in L_1$$

Date __/__/__ $\because 1 \vee 1 = 1$. Page No.: ____

$f(1 \vee 1) = f(1) = \phi$ $\Rightarrow f(1 \vee 1) = f(1) \vee f(1)$
$f(1) \vee f(1) = \phi \vee \phi = \phi$
$f(1 \vee 2) = f(2) = \{a\}$ $\Rightarrow f(1 \vee 2) = f(1) \vee f(2)$
$f(1) \vee f(2) = \phi \vee \{a\} = \{a\}$
$f(1 \vee 3) = f(3) = \{b\}$ $\Rightarrow f(1 \vee 3) = f(1) \vee f(3)$
$f(1) \vee f(3) = \phi \vee \{b\} = \{b\}$
$f(1 \vee 6) = f(6) = \{a, b\}$ $\Rightarrow f(1 \vee 6) = f(1) \vee f(6)$
$f(1) \vee f(6) = \phi \vee \{a, b\} = \{a, b\}$
Similarly we can do for $\wedge$

Now, $f(2 \vee 3) = f(6) = \{a, b\}$ $\Rightarrow f(2 \vee 3) = f(2) \vee f(3)$
$f(2) \vee f(3) = \{a\} \vee \{b\} = \{a, b\}$
& $f(2 \wedge 3) = f(1) = \phi$ $\Rightarrow f(2 \wedge 3) = f(2) \wedge f(3)$
$f(2) \wedge f(3) = \{a\} \wedge \{b\} = \phi$.

we can also prove for the pairs $(2, 6)$ & $(3, 6)$ under $\vee$ & $\wedge$.

Since the mapping is one-one & onto & preserves closure property under $\vee$ & $\wedge$

Hence the lattices are isomorphic.

* Isomorphic lattices look identical except for the label of nodes. Each is a mirror image of the other with the elements simply relabled.

* If S and T are any two finite sets with n elements, Then lattices $(P(S), \subseteq)$ and $(P(T), \subseteq)$ are isomorphic.

e.g. — if $S = \{a, b, c\}$ and $T = \{1, 2, 3\}$.
Then

$L_1 = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}$
$L_2 = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$

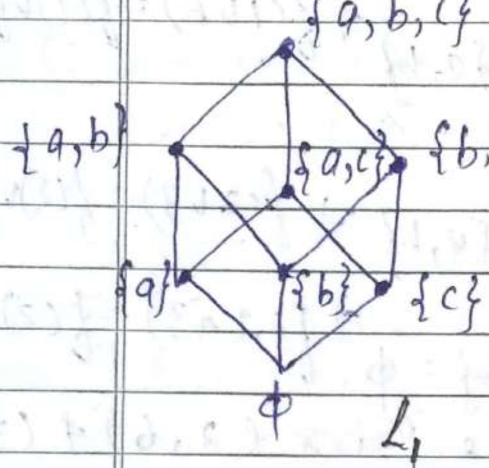
Let $f: L_1 \to L_2$ is an isomorphism defined by

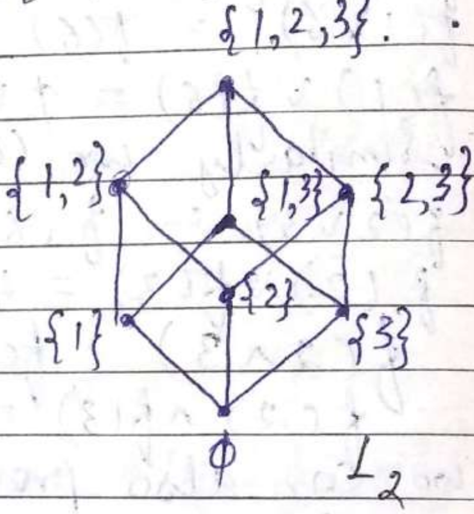
$f(\{1\}) = \{a\}$, $f(\{2\}) = \{b\}$, $f(\{3\}) = \{c\}$

$f(\{1,2\}) = \{a,b\}$, $f(\{1,3\}) = \{a,c\}$,

$f(\{2,3\}) = \{b,c\}$

$f(\{1,2,3\}) = \{a,b,c\}$.



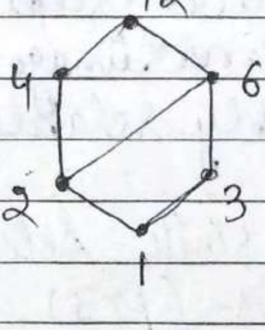


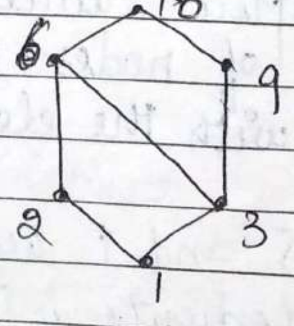
Ex:- Show that D_{12} and D_{18} are isomorphic lattices

Here $D_{12} = \{1, 2, 3, 4, 6, 12\}$

$D_{18} = \{1, 2, 3, 6, 9, 18\}$

The Hasse Diagram of D_{12} & D_{18} are as





we define a mapping $f: D_{12} \to D_{18}$ as

$f(1) = 1$, $f(2) = 3$, $f(3) = 2$

$f(4) = 9$, $f(6) = 6$, $f(12) = 18$

which is one-one and onto mapping

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Types of Lattices :→

Complete Lattice :- A lattice is called complete if each of its non-empty subsets has a least upper bound and a greatest lower bound.

Every finite lattice is complete b/c Every subset here is finite.

Every complete lattice must have a least element and a greatest element.
The least and the greatest elements of lattice are called bounds (units, universal bounds) of a lattice and are denoted by 0 and 1 respectively.

Bounded Lattice :— A lattice which has both elements 0 and 1 is called a Bounded lattice and is often denoted by $(L, \vee, \wedge, 0, 1)$.
where 0 = least element of lattice
1 = greatest element of lattice.

- Every finite lattice is Bounded.
Let $(L, \vee, \wedge)$ be a finite lattice with $L_n = \{a_1, a_2, \ldots, a_n\}$
Then
least element of $L = \bigwedge_{i=1}^{n} a_i = a_1 \wedge a_2 \wedge \ldots \wedge a_n = 0$

and greatest element of L
$= \bigvee_{i=1}^{n} a_i = a_1 \vee a_2 \vee \cdots \vee a_n = 1$
& both of them exist in L
So L is a Bounded Lattice.

Ex: (1) $(P(S), \cup, \cap)$ is bounded lattice
with $\phi = 0$ & $S = 1$.
Let $P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$.
Where $S = \{a, b, c\}$.
Here the greatest element $= S = \{a, b, c\} = 1$
& the least element $= \phi = 0$

(2.) $(Z^+, |)$ is not bounded lattice.
$N = Z^+ = \{1, 2, 3, 4, \cdots\}$
It has least element 1
but $\nexists$ any greatest element.
So $(Z^+, |)$ is not bounded lattice.

(3.) $(Z, \leq)$ is not bounded lattice.
Since It has neither greatest element nor least element.

\# In a Bounded lattice, the bounds 0 and 1 clearly satisfy following conditions:

$0 \wedge a = 0 = a \wedge 0, \quad \forall a \in L$
$0 \vee a = a = a \vee 0, \quad \forall a \in L$
$1 \wedge a = a = a \wedge 1, \quad \forall a \in L$
$1 \vee a = 1 = a \vee 1, \quad \forall a \in L$

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Complement of an element:

In a bounded lattice $(L, \vee, \wedge, 0, 1)$, an element $b \in L$, is called a complement of an element $a \in L$ if

$$a \wedge b = 0 \quad \text{and} \quad a \vee b = 1.$$

where 0 and 1 are least and greatest elements.

Note:-

1. if b is complement of a then a is also complement of b.
2. Any element $a \in L$ may or may not have a complement.
3. An element $a \in L$ may have more than one complement in L.
4. In any bounded lattice, the bound 0 & 1 are unique complements of each other b/c
   $$0 \vee 1 = 1 \quad \& \quad 0 \wedge 1 = 0.$$

Complemented lattice: →

A lattice $(L, \vee, \wedge)$ is called complemented lattice if

(i) L is bounded.

(ii) every element $a \in L$ has a complement

Ex:- Let $P = \{1, 2, 3, 5, 30\}$ with Relation R defined by 'a divides b' be a poset then $(P, |)$ is bounded and complemented lattice.

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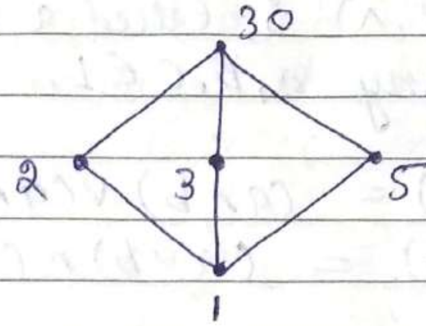
Solⁿ: Given that $P = \{1, 2, 3, 5, 30\}$.
Then the Hasse Diagram of $(P, |)$ is as follows:



This poset is a lattice as for every pair (a, b). The Sup$\{a, b\}$ & Inf$\{a, b\}$ exist in P
And this lattice is Bounded
as It has greatest element $= 30$
and least element $= 1$
Now, since $2 \vee 3 = 30$ $\quad 2 \vee 5 = 30$
& $2 \wedge 3 = 1$ & $2 \wedge 5 = 1$
$\Rightarrow$ 2 has two ~~elements~~ complements 3 & 5.
& 3 has also two complements 2 & 5
5 has two complements 2 & 3.
1 has one complement 30
30 has one complement 1
As $1 \vee 30 = 30$ & $1 \wedge 30 = 1$.
Hence this lattice is complemented.

(#) distributive lattice :→

A Lattice $(L, \vee, \wedge)$ is called a distributive lattice if for any $a, b, c \in L$.

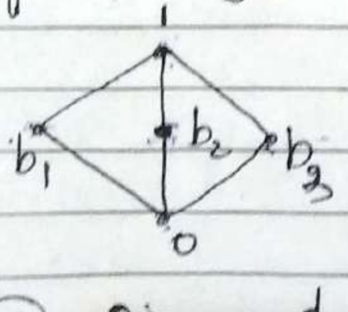
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

Notes:-

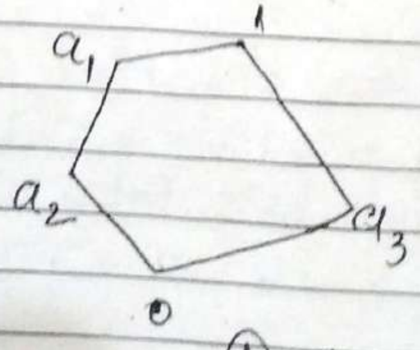
Since both the inequalities are equivalent to one another, hence to check whether the lattice is distributed or not, it is sufficient to verify one of them.

Ex:- 1 $L = (P(S), \cup, \cap)$ is distributive lattice
$P(S)$ is the set of all subsets of S
Let A, B and $C \in P(S)$ then
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
for elements A, B, C in $P(S)$.
$\Rightarrow$ L is distributive lattice.

Ex:- 2. check whether the lattices given in the following diagrams are distributive.



(a) Diamond



(b) Pentagon

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Sol^n:- (a) $b_1 \wedge (b_2 \vee b_3) = b_1 \wedge 1$ $(\because b_2 \vee b_3 = 1)$
$= b_1$
but $(b_1 \wedge b_2) \vee (b_1 \wedge b_3) = 0 \vee 0 = 0$
$\Rightarrow$ $b_1 \wedge (b_2 \vee b_3) \neq (b_1 \wedge b_2) \vee (b_1 \wedge b_3)$
Hence this lattice is not Distributive.

(b) $a_1 \wedge (a_2 \vee a_3) = a_1 \wedge 1 = a_1$ $(\because a_2 \vee a_3 = 1)$
but $(a_1 \wedge a_2) \vee (a_1 \wedge a_3) = a_2 \vee 0 = a_2$ $(\because a_1 \wedge a_2 = a_2$ & $a_1 \wedge a_3 = 0)$
$\therefore$ $a_1 \wedge (a_2 \vee a_3) \neq (a_1 \wedge a_2) \vee (a_1 \wedge a_3)$
Hence the given lattice (pentagon) is not Distributive.

Theorem:- A Lattice $(L, \vee, \wedge)$ is Distributive iff it does not contain the five element pentagonal, or, the Diamond lattice as one of its sublattices. (or an isomorphic copy thereof)

Th^m:- In a Distributive lattice, if an element has a complement then this complement is unique.
or
A lattice L is called Distributive if every element of L has atmost one complement.

Proof:- Suppose an element $a \in L$ has two complements b and c then
$a \vee b = 1$ and $a \wedge b = 0$
& $a \vee c = 1$ and $a \wedge c = 0$
we have $b = b \wedge 1$
$= b \wedge (a \vee c)$
$= (b \wedge a) \vee (b \wedge c)$ (by distributive law)

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$= (a \wedge b) \vee (b \wedge c)$

$= 0 \vee (b \wedge c)$ $\quad (\because a \wedge b = 0)$

$= (a \wedge c) \vee (b \wedge c)$

$= (a \vee b) \wedge c$

$= 1 \wedge c$

$= c$

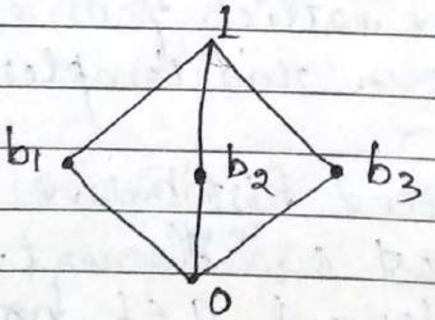
$\Rightarrow b = c$

Hence, In a distributive lattice, Every element has a unique complement if it exist.

Modular Lattice :→

A Lattice $(L, \wedge, \vee)$ is said to be modular if $a \vee (b \wedge c) = (a \vee b) \wedge c$ whenever $a \leq c$ for all $a, b, c \in L$.

Ex:→ The Lattice given by the following diagram is modular



~~Diag~~ Diamond Lattice.

Sol^n:– This lattice is not Distributive.
Since the lattice is symmetric w.r.t b_1, b_2, b_3
So the only situations for $a \leq c$ (condition of modularity) are $b_1 \leq 1$ and $0 < b_1$

$(a \vee b) \vee c = (a \vee (b \vee c)$

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Thus taking $a = b_1$ and $c = 1$.

$a \vee (b \wedge c) = b_1 \vee (b \wedge 1) = b_1 \vee b$ $(\because b \wedge 1 = b)$

and $(a \vee b) \wedge c = (b_1 \vee b) \wedge 1 = b_1 \vee b$.

whatever b may be.

Similarly for $a = 0$ and $c = b_1$

$a \vee (b \wedge c) = 0 \vee (b \wedge b_1) = b \wedge b_1$

$(a \vee b) \wedge c = (0 \vee b) \wedge b_1 = b \wedge b_1$

whatever b may be.

$\Rightarrow$ modularity holds

Hence the Diamond lattice is Modular.

Imp

Theorem :- Every Distributive lattice is modular.

proof :-

Let $(L, \leq)$ be a distributive lattice and $a, b, c \in L$ be such that $a \leq c$.

Thus if $a \leq c \Rightarrow a \vee c = c$ — (1)

Now

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ $\because L$ is Distributive

$= (a \vee b) \wedge c$ from (1)

$\Rightarrow$ $a \vee (b \wedge c) = (a \vee b) \wedge c$ whenever $a \leq c$

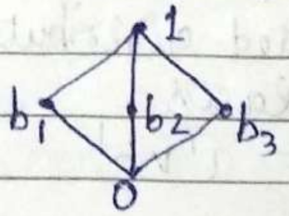
Hence, every distributive lattice is modular.

Note: The Converse of above Th^m is not True

i.e A modular lattice may not be distributive

e.g —

~~The modular~~ The Diamond lattice



is modular but NOT Distributive.

Ex:- Prove that a lattice $(L, \leq)$ is modular iff $(a\wedge b)\vee(a\wedge c) = a\wedge(b\vee(a\wedge c))$ $\forall\ a, b, c \in L$.

Sol^n:- Suppose $(L, \leq)$ is modular then

$(a\wedge b)\vee(a\wedge c) = (a\wedge c)\vee(a\wedge b)$ Comm. law

$= (a\wedge c)\vee(b\wedge a)$ "

$= ((a\wedge c)\vee b)\wedge a$ By modularity $\because a\wedge c \leq a$

$= a\wedge(b\vee(a\wedge c))$ by commutative law.

Conversely, suppose that $(a\wedge b)\vee(a\wedge c) = a\wedge(b\vee(a\wedge c))$, for all $a, b, c \in L$.

Let $a, b, c \in L$ be such that $a \leq c$

Then $a\wedge c = a$

Now $(c\wedge b)\vee(c\wedge a) = (a\wedge c)\vee(b\wedge c)$ by comm. law

$= c\wedge(b\vee(a\wedge c))$

Hence, $(c\wedge b)\vee a = c\wedge(b\vee a)$

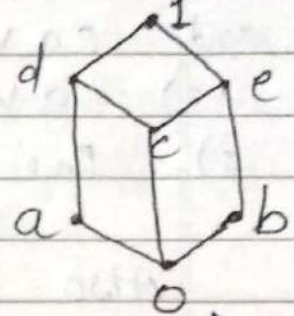
i.e. $a\vee(b\wedge c) = (a\vee b)\wedge c$

Ex:- The pentagonal lattice is not modular.

Th^m:- A lattice $(L, \wedge, \vee)$ is modular iff it does not have a sublattice isomorphic with the pentagonal lattice.

Th^m:- If $(L, \vee, \wedge)$ is a complemented distributive lattice, then DeMorgan's laws $(a\vee b)' = a'\wedge b'$ & $(a\wedge b)' = a'\vee b'$ holds for all $a, b \in L$.

Ex:- Show that the poset given in the following Hasse Diagram is a lattice. It is distributive and complemented?



Solution:- Here 0 is the least element & 1 is the greatest element.
Now to check whether the given Hasse Diagram represent lattice or not we form the tables for lub (V) and glb (^) of pairs of elements of the set S = {0, a, b, c, d, e, 1}.

lub(V)	0	a	b	c	d	e	1
0	0	a	b	c	d	e	1
a	a	a	1	d	d	1	1
b	b	1	b	e	1	e	1
c	c	d	e	c	d	e	1
d	d	d	1	d	d	1	1
e	e	1	c	e	1	e	1
1	1	1	1	1	1	1	1

glb(^)	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	0	0	a	0	a
b	0	0	b	0	0	b	b
c	0	0	0	c	c	c	c
d	0	a	0	c	d	c	d
e	0	0	b	c	c	e	e
1	0	a	b	c	d	e	1

Since lub and glb of every pair of elements, Hence the Poset is a lattice.

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The lattice is not Distributive as

$a \vee (b \wedge c) = a \vee 0 = a$

and $(a \vee b) \wedge (a \vee c) = 1 \wedge d = d.$

$\Rightarrow a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$

$\Rightarrow$ The lattice is not distributive.

Also The lattice is not complemented as c has no complement as $c \vee x \neq 1$ for any $x \neq 1$, $x \in S$.