1 Pen-and-Paper

1. Given an input vector $\mathbf{x} \in \mathbb{R}^{n\times 1}$, a linear layer with weights $W = [w_0, w_1, ..., w_{n-1}] \in \mathbb{R}^{n\times 1}$, and a forward calculation $\hat{y} = W^T x$, Smooth-L_1 loss is defined as:

$$\mathcal{L} = \begin{cases} 0.5(\hat{y}-y)^2/\beta & |\hat{y}-y| \leq \beta \\ |\hat{y}-y| - 0.5 * \beta & \text{else} \end{cases}$$

where β is a predefined positive threshold.

Your task is to compute the derivative of the loss with respect to each weight w_i, which is $\frac{\partial \mathcal{L}}{\partial w_i}$.

Hint: The derivative of piecewise functions can be computed separately for each case, and then recombined as a single function. You can express the piecewise function as a single formula by multiplying by an indicator function $\mathbb{1}(\text{condition})$, where $\mathbb{1}(\text{condition}) = 1$ if the condition holds, and 0 otherwise. For example, $x \cdot \mathbb{1}(c > 5) = x$ if $c > 5$, otherwise 0.

$$\mathcal{L} = \begin{cases} \frac{1}{2}(W^T x - y)^2/\beta, & |W^T x - y| \leq \beta \\ |W^T x - y| - \frac{1}{2}\beta, & \text{else} \end{cases}$$

$$\frac{\partial \mathcal{L}}{\partial W} = \begin{cases} a, & |W^T x - y| \leq \beta \\ b, & \text{else} \end{cases}$$

$$a := \frac{(W^T x - y)^2}{2\beta} \circ \frac{\partial}{\partial W} = \frac{1}{\beta} \cdot (W^T x - y) \cdot x$$

$$= \frac{(W^T x - y) \cdot x}{\beta}$$

$$b := \left(|W^T x - y| - \frac{\beta}{2}\right) \cdot \frac{\partial}{\partial W}$$

$$= \frac{\partial |W^T x - y|}{\partial W} + \cancel{\frac{\partial (-\beta)}{\partial W \cdot 2}}$$

$$x \cdot \operatorname{sign}(W^T x - y)$$

$$\Uparrow$$

$$= \begin{cases} x, & W^T x \geq y \\ -x, & W^T x < y \end{cases} = x \cdot (1 - 2\mathbb{1}(W^T x < y))$$

$$\frac{\partial \mathcal{L}}{\partial W} = (x \cdot (W^T x - y) \cdot 1/\beta) \cdot \mathbb{1}(|W^T x - y| \leq \beta) + (x \cdot \operatorname{sign}(W^T x - y)) \cdot \mathbb{1}(|W^T x - y| > \beta)$$

$$= (x \cdot (\hat{y} - y) \cdot \tfrac{1}{\beta}) \cdot \mathbb{1}(|\hat{y} - y| \leq \beta) + (x \cdot \operatorname{sign}(\hat{y} - y)) \cdot \mathbb{1}(|\hat{y} - y| \leq \beta)$$