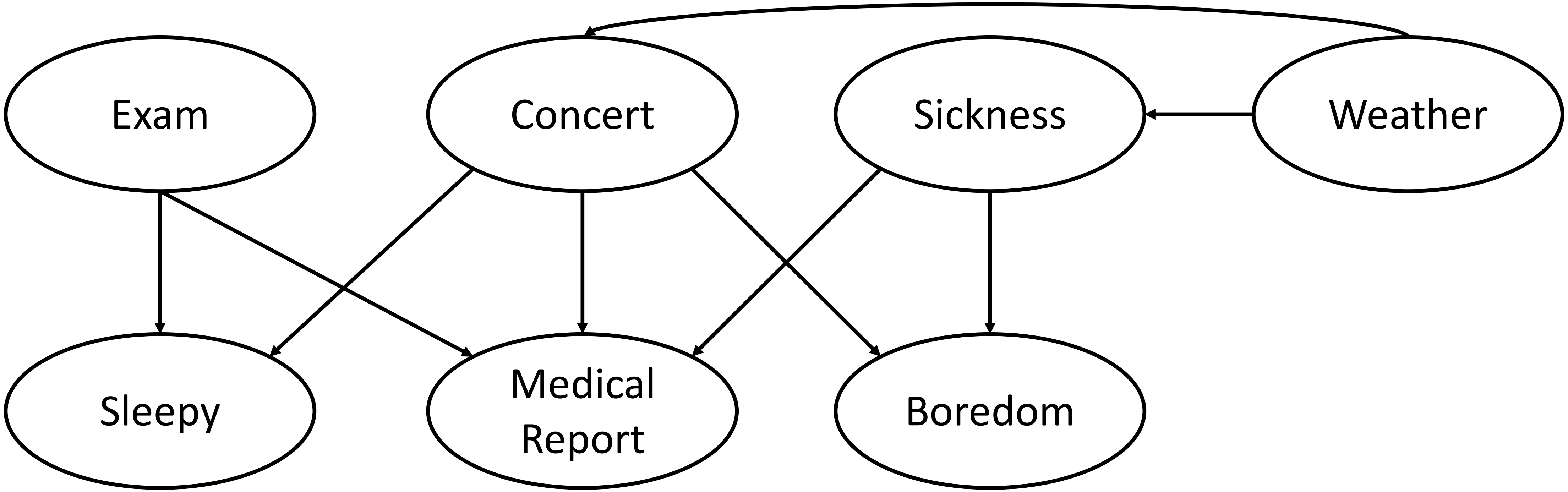
COMP 341 Intro to AI
Bayesian Networks - Representation



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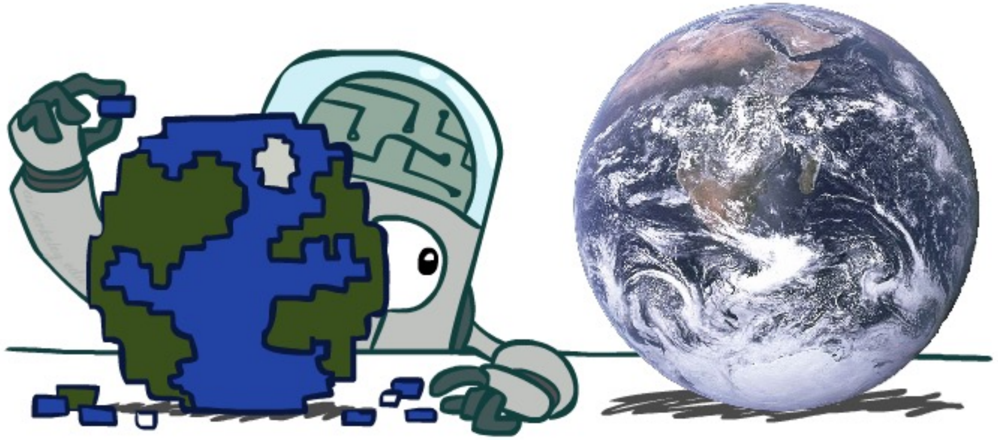
Probabilistic Models

- Models describe how (a portion of) the world works
- Models are always simplifications
  - May not account for every variable
  - May not account for all interactions between variables
  - "All models are wrong; but some are useful."
    – George E. P. Box



- What do agents do with probabilistic models?
  - Reason about unknown variables, given evidence
  - Example: explanation (diagnostic reasoning)
  - Example: prediction (causal reasoning)
  - Example: value of information

Probabilistic Models

- A probabilistic model is a joint distribution over a set of variables

$$P(X_1, X_2, \dots, X_n)$$

- Posterior probabilities are used to reason about the world, ask queries etc.

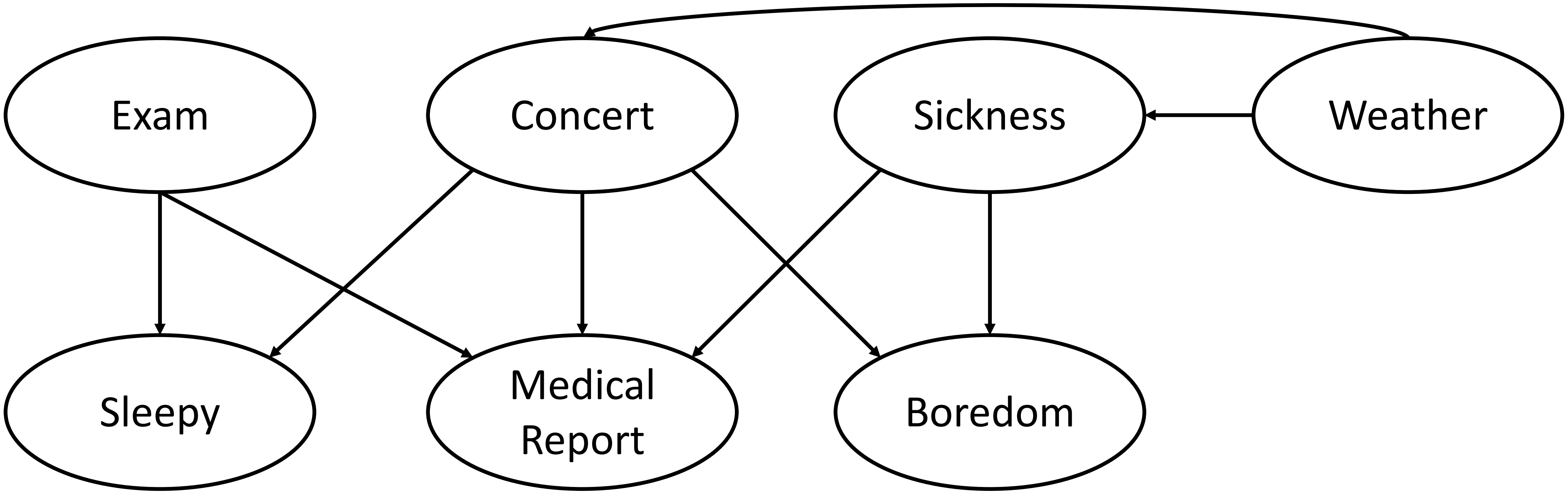
$$P(X_q |, x_{e1}, \dots, x_{ek})$$

Bayesian Networks

- Two problems with using full joint distribution tables as probabilistic models:
  - Too big to represent($O(d^n)$)
  - Hard to specify
  - Hard to learn anything empirically

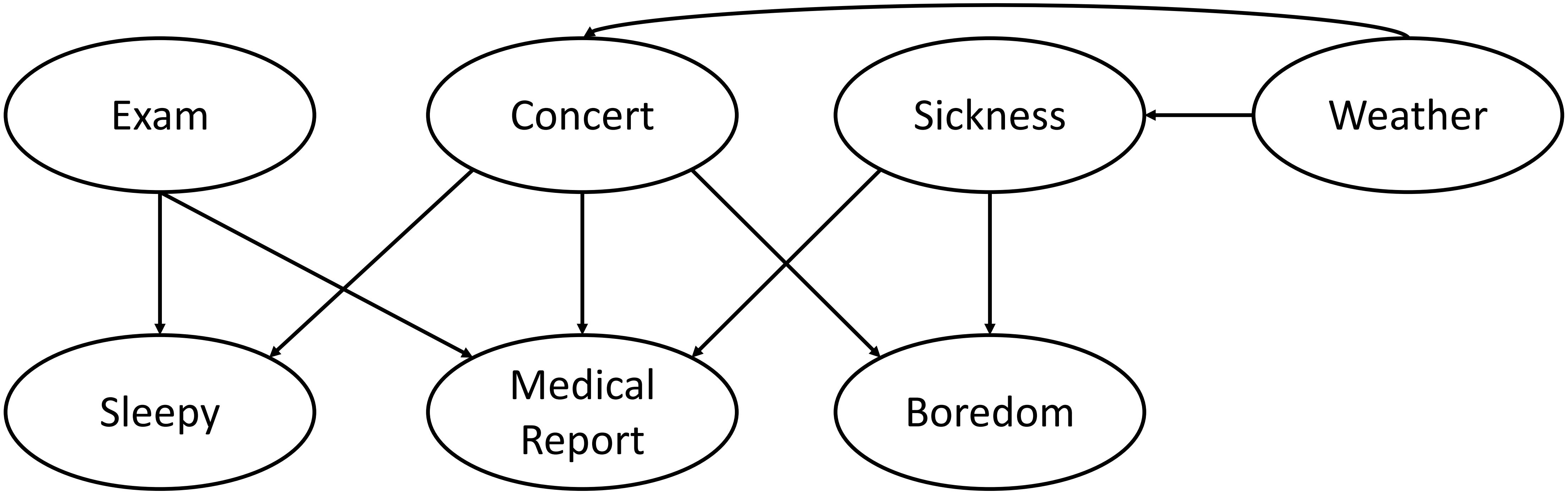
- Bayes' nets: a technique for describing complex joint distributions (models) using simple, local distributions (conditional probabilities)
  - A variety of graphical models
  - We describe how variables interact locally
  - Local interactions chain together to give global, indirect interactions

Bayesian Network Example



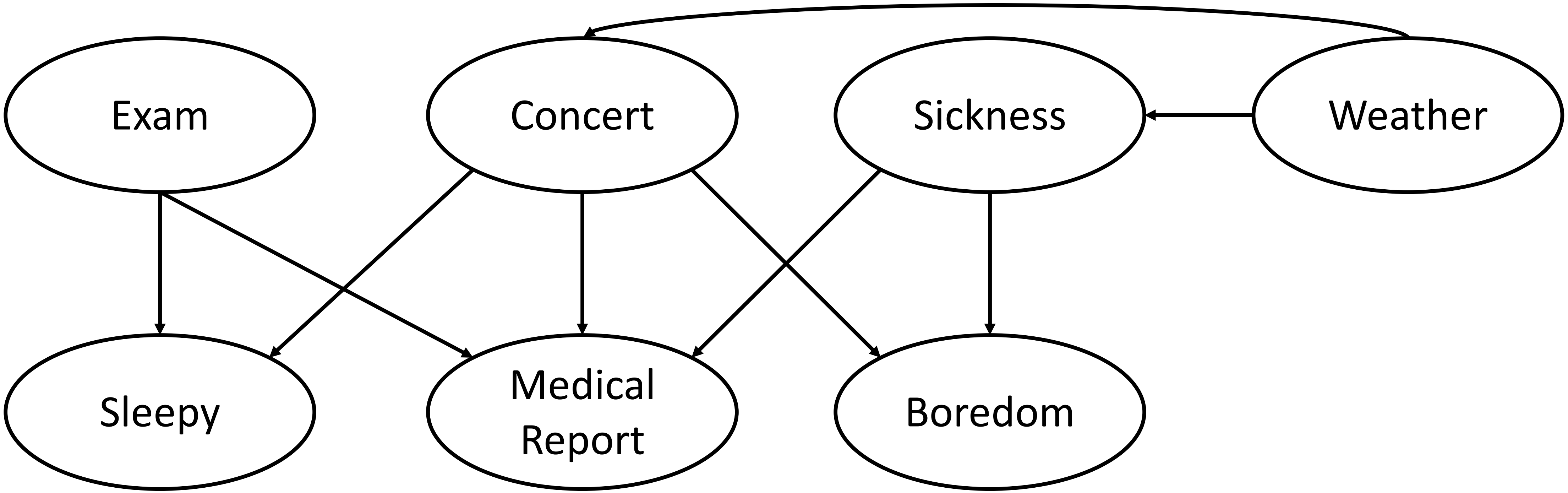
- If Mehmet is sick or there is an exam or there is a concert, he is likely to get a medical report
- If the weather is bad, Mehmet is likely to get sick. The weather also affects the concert
- Mehmet gets bored if he is sick or misses the concert
- Mehmet is sleepy the next day if he studies for an exam or goes to the concert

Bayesian Network Example



- What is the reason that Mehmet had a Medical Report?
- The weather was bad
- There was an exam
- Mehmet was sleepy
- Mehmet was not bored
- TODO: Tell Mehmet he will not be getting a Make Up!

Bayesian Network Example

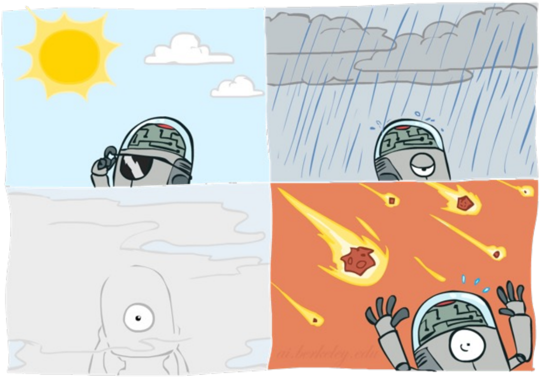


$$P(Exam, Concert, Sickness, Weather, Sleepy, MedicalReport, Boredom) =?$$

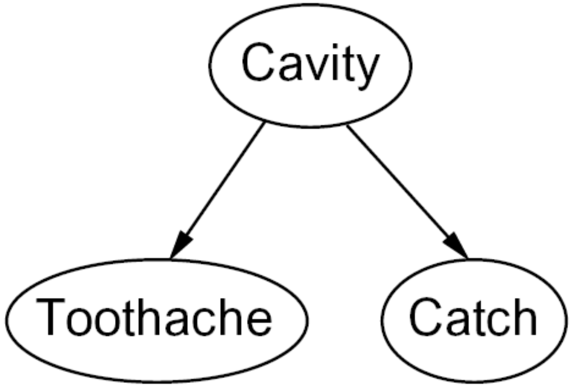
- Answer queries, e.g., about Mehmet getting a medical report
- Use the full joint distribution
- Simplify by using the chain rule and conditional independence!

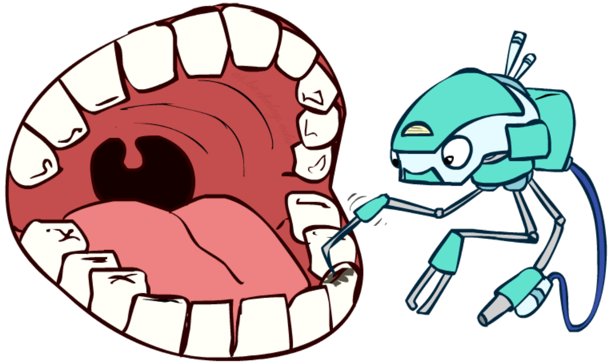
Bayesian Network Semantics

- Nodes: variables (with domains)
  - Can be assigned (observed) or unassigned (unobserved)



- Arcs: interactions
  - Directional
  - Indicate "direct influence" between variables
  - Formally: encode conditional independence



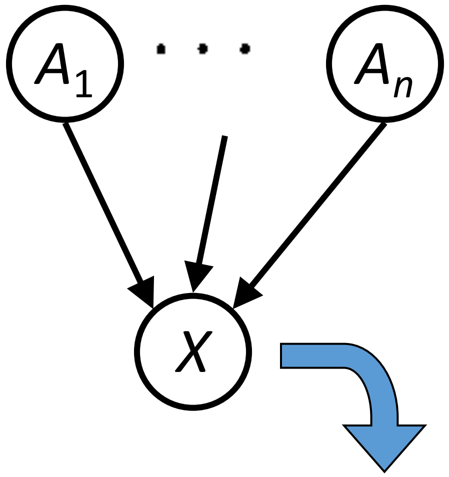


Bayesian Network Semantics

- Represented as a directed acyclic graphs
- A set of nodes, one per variable X
- A conditional distribution for each node
  - A conditional distribution for each node given its parents

$$P(X_i|Parents(X_i))$$

  - E.g. as a conditional probability table (CPT)
  - Description of a noisy "causal" process



$$P(X|A_1 \ldots A_n)$$

Bayes net = Graph Topology (graph) + CPTs

Probabilities in Bayesian Networks

- Bayesian Networks encode joint distributions implicitly
- … as a product of conditional distributions
- To get the probability of a full assignment, multiply all the conditionals:

$$P(X_1, X_2, \ldots, X_n) = \prod_{i=1}^{n} P\big(X_i | Parents(X_i)\big)$$

- E.g.

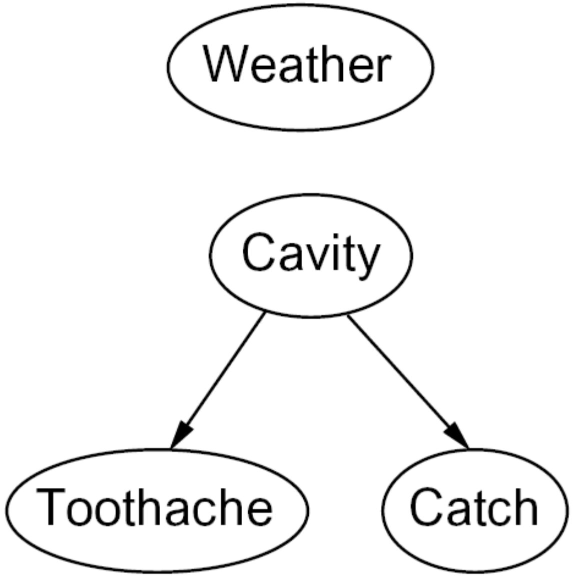
P(+cavity, +catch, -toothache, sunny)

= *P(+cavity)* x

P(-toothache|+cavity) x

P(+catch|+cavity) x

P(sunny)



Example: Coin Flips

X_1 X_2 $\cdots$ X_n

$P(X_1)$

h	0.5
t	0.5

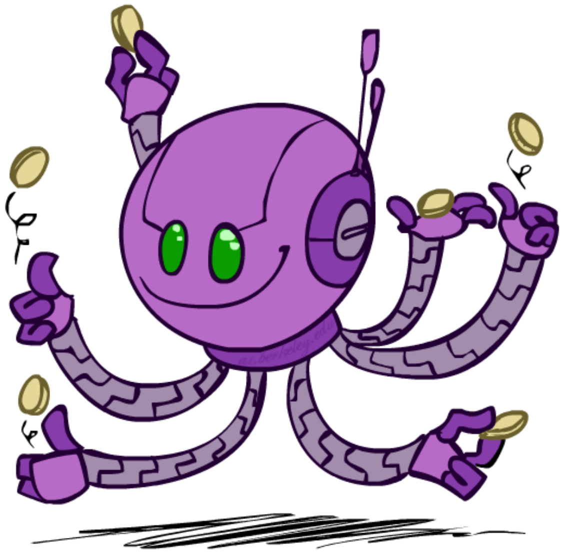
$P(X_2)$

h	0.5
t	0.5

$\cdots$

$P(X_n)$

h	0.5
t	0.5

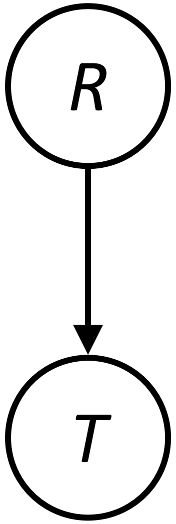


$P(h,t,t,h,t,h) = ??$

Only distributions whose variables are absolutely independent can be represented by a Bayes' net with no arcs.

Example: Traffic

- Rain affects traffic



$P(R)$

+r	1/4
-r	3/4

$P(T|R)$

+r	+t	3/4
	-t	1/4
-r	+t	1/2
	-t	1/2

$P(T,R) = P(T|R)\ P(R)$

$P(-t,+r) = ??$

Example: House Alarm

- I'm at work, neighbor John calls to say my alarm is ringing, but neighbor Mary doesn't call. Sometimes it's set off by minor earthquakes. Is there a burglar?

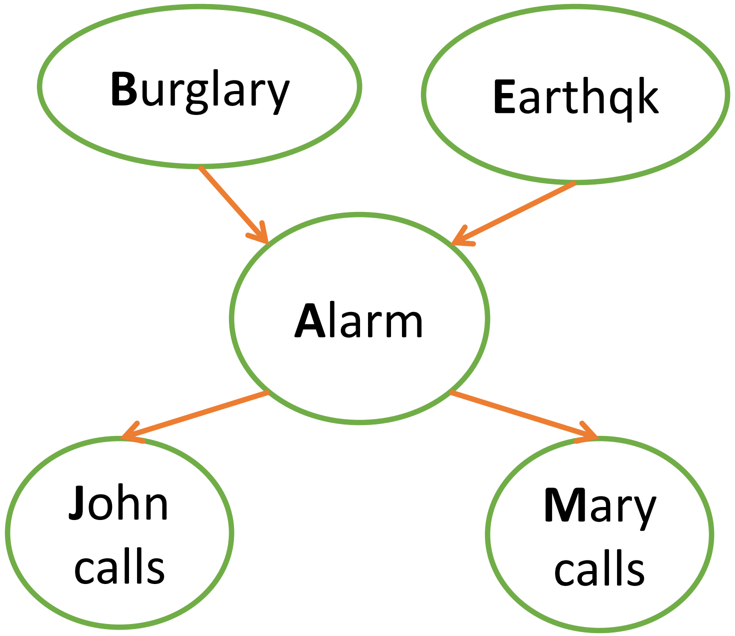
- Variables:
  - *Burglar*, *Earthquake*, *Alarm*, *JohnCalls*, *MaryCalls*

- A burglar can set the alarm off
- An earthquake can set the alarm off
- The alarm can cause Mary to call
- The alarm can cause John to call

- What does the Bayes Net look like?

Example: House Alarm

B	P(B)
+b	0.001
-b	0.999



E	P(E)
+e	0.002
-e	0.998

B	E	A	P(A|B,E)
+b	+e	+a	0.95
+b	+e	-a	0.05
+b	-e	+a	0.94
+b	-e	-a	0.06
-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

A	J	P(J|A)
+a	+j	0.9
+a	-j	0.1
-a	+j	0.05
-a	-j	0.95

A	M	P(M|A)
+a	+m	0.7
+a	-m	0.3
-a	+m	0.01
-a	-m	0.99

Constructing Bayesian networks

- 1. Choose an ordering of variables $X_1, X_2, \dots, X_n$
- 2. For i = 1 to n
  - add X_i to the network
  - select parents *from* $X_1, X_2, \dots, X_{i-1}$ such that:

    $P\left(X_i | Parents(X_i)\right) = P(X_i | X_1, X_2, \dots, X_{i-1})$
  - This choice of parents guarantees:

$$P(X_1, X_2, \dots, X_n) = \prod_{i=1}^{n} P(X_i | X_1, X_2, \dots, X_{i-1}) \text{ (chain rule)}$$

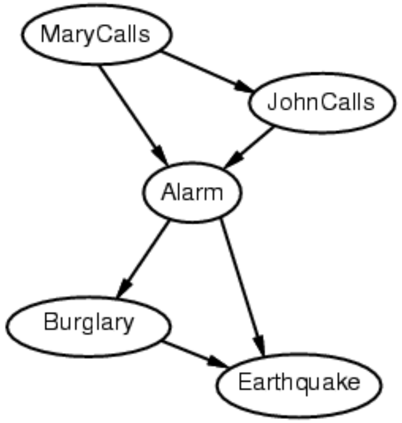
$$= \prod_{i=1}^{n} P\left(X_i | Parents(X_i)\right) \text{ (by construction)}$$

  - Ordering matters!
  - Recall that a causal structure is often simpler (rule of thumb!)
  - This is just a way to build valid networks and is not too useful in practice (unless you have a good ordering and prune after the construction)

Example

- Suppose we choose the ordering M, J, A, B, E

$\mathbf{P}(J \mid M) = \mathbf{P}(J)$?	**No**
$\mathbf{P}(A \mid J, M) = \mathbf{P}(A \mid J)$? $\mathbf{P}(A \mid J, M) = \mathbf{P}(A)$?	**No**
$\mathbf{P}(B \mid A, J, M) = \mathbf{P}(B \mid A)$?	**Yes**
$\mathbf{P}(B \mid A, J, M) = \mathbf{P}(B)$?	**No**
$\mathbf{P}(E \mid B, A, J, M) = \mathbf{P}(E \mid A)$?	**No**
$\mathbf{P}(E \mid B, A, J, M) = \mathbf{P}(E \mid A, B)$?	**Yes**



Size of a Bayes Net

- Joint dist. table size for *n* variables with largest domain size *d*:

  $O(d^n)$

- Size of a Bayes Net. with *n* nodes where each variable has no more than *k* parents:

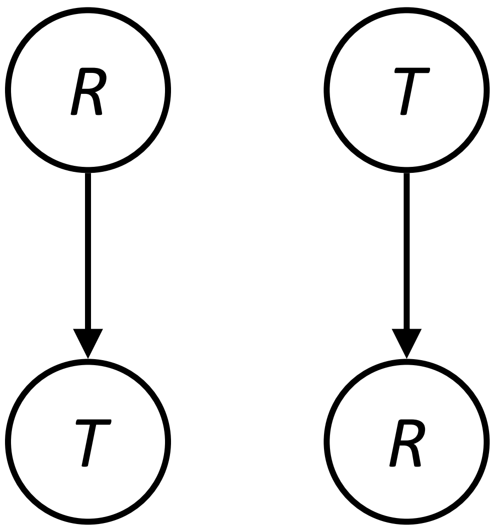
  $O(n \cdot d^{k+1})$

- If $k \ll n$, then a huge difference! (Exponential to linear)

- For burglary net, 1 + 1 + 4 + 2 + 2 = 10 numbers (vs. 2^5-1 = 31)

Causality

- Do BNs always represent causality?
  - No!
- Both are valid
- Topology encodes *conditional independence*, not causality
- Same joint distribution can be encoded by many different BNs



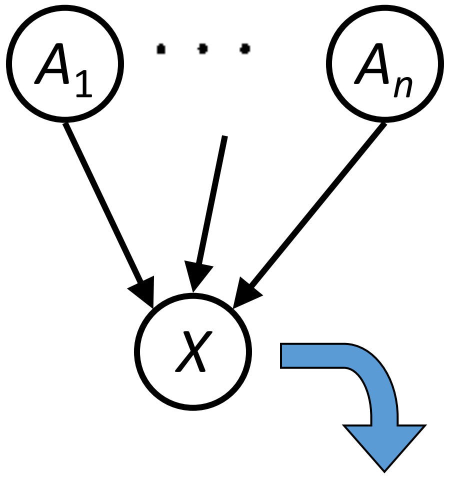
- When Bayes' nets reflect the true causal patterns:
  - Often simpler (nodes have fewer parents)
  - Often easier to think about
  - Often easier to elicit from experts

Bayesian Network Recap

- Represented as directed acyclic graphs
- A set of nodes, one per variable X
- Implicitly encode the joint probability distribution as a product of local conditional distributions

$$P(X_1, X_2, \dots, X_n) = \prod_{i=1}^{n} P(X_i | Parents(X_i))$$

- Connections do not have to represent causality! (But better if they do)

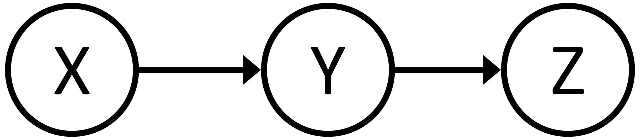


$$P(X|A_1 \dots A_n)$$

Independence in a BN

$$X \perp\!\!\!\perp Y | Z$$
$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$
$$\forall x, y, z : P(x|z, y) = P(x|z)$$

- Are two nodes independent given certain evidence?



- Question: are X and Z necessarily independent?
  - No!
  - X can influence Z, Z can influence X (via Y)
  - E.g. low pressure –> rain –> traffic
  - They *could* be independent: how?

Counter Example:

P(+y | +x) = 1, P(-y | - x) = 1

P(+z | +y) = 1, P(-z | -y) = 1

P(X,Y,Z) = P(X)

P(X,Z) = P(X) ≠ P(X)P(Z) (in general)

$P(X,Y,Z) = P(X)P(Y|X)P(Z|Y)$ (from the BN)

$P(X,Z) = \sum_i P(X)P(Y = y_i|X)P(Z|Y = y_i)$ (marginalize)

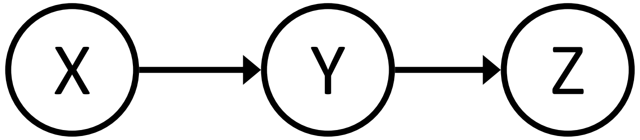
$P(X,Z) = P(X)\sum_i P(Y = y_i|X)P(Z|Y = y_i) \stackrel{?}{=} P(X)P(Z)$ (defn. independence)

$P(Z) \stackrel{?}{=} \sum_i P(Y = y_i|X)P(Z|Y = y_i)$ -> Not in general!

Causal Chains

$$X \perp\!\!\!\perp Y | Z$$
$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$
$$\forall x, y, z : P(x|z, y) = P(x|z)$$

- This configuration is a *causal chain*



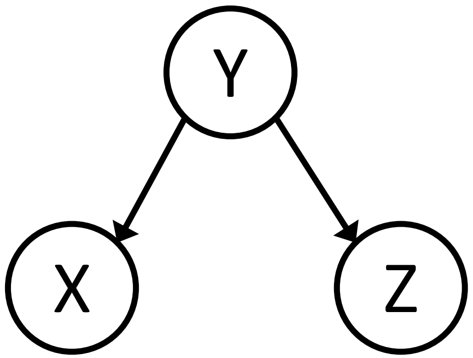
- Is X independent of Z given Y?
  - Yes!

$$P(X, Y, Z) = P(X)P(Y|X)P(Z|Y)$$
$$P(Z|X, Y) = \frac{P(X,Y,Z)}{P(X,Y)} = \frac{P(X)P(Y|X)P(Z|Y)}{P(X)P(Y|X)} = P(Z|Y)$$

- Evidence along the chain “blocks” influence

Common Cause

$$X \perp\!\!\!\perp Y | Z$$
$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$
$$\forall x, y, z : P(x|z, y) = P(x|z)$$



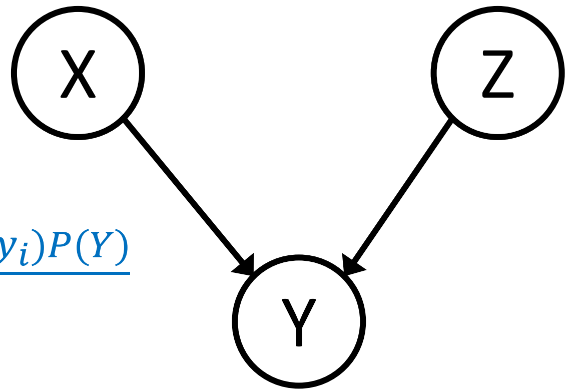
- Are X and Z independent?
  - No!
- Are X and Z independent given Y?
  - Yes!

$$P(Z|X,Y) = \frac{P(X,Y,Z)}{P(X,Y)} = \frac{P(Y)P(X|Y)P(Z|Y)}{P(X,Y)} = \frac{P(X,Y)P(Z|Y)}{P(X,Y)} = P(Z|Y)$$

- Observing the cause "blocks" influence between effects

Common Effect

$$X \perp\!\!\!\perp Y | Z$$
$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$
$$\forall x, y, z : P(x|z, y) = P(x|z)$$



- Are X and Z independent?
  - Yes!

$$P(X,Z) = \sum_i P(X, Y = y_i, Z) = \sum_i P(X)P(Z)P(Y = y_i | X, Z) = \sum_i \frac{P(X)P(Z)P(X,Z|Y=y_i)P(Y)}{P(X,Z)}$$
$$= \frac{P(X)P(Z)}{P(X,Z)} \sum_i P(X, Z | Y = y_i)P(Y) = \frac{P(X)P(Z)}{P(X,Z)} P(X,Z) = P(X)P(Z)$$

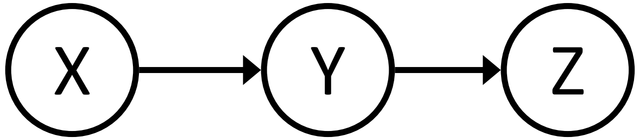
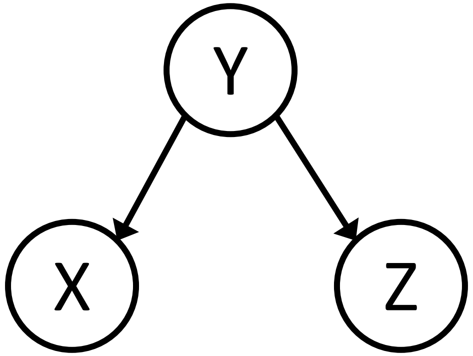
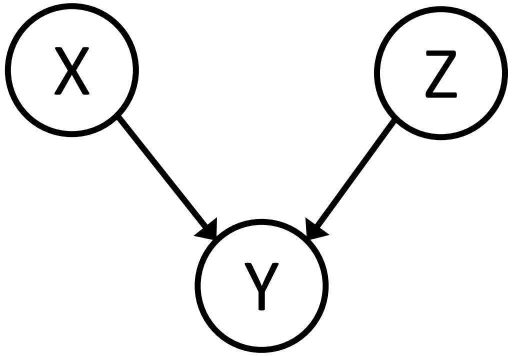
- Are X and Z independent given Y?
  - No!
  - Seeing Y puts X and Z in competition as the explanation

$$P(Z|X,Y) = \frac{P(X,Y,Z)}{P(X,Y)} = \frac{P(X)P(Z)P(Y|Z,X)}{P(X)P(Y|X)} = \frac{P(Z)P(Y|Z,X)}{P(Y|X)} \neq P(Z|Y) \text{ (inequality holds in general)}$$

- Seeing evidence "activates" influence

3-Node Cond. Indep. Recap

$$X \perp\!\!\!\perp Y | Z$$
$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$
$$\forall x, y, z : P(x|z, y) = P(x|z)$$

	Causal Chain	Common Cause	Common Effect
			
$X \perp\!\!\!\perp Z$?			
$X \perp\!\!\!\perp Z \mid Y$?			

General Case

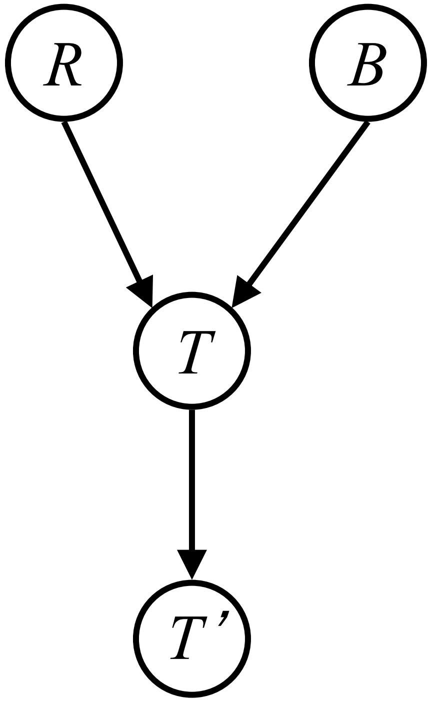
- Any complex example can be analyzed using these three canonical cases
- General question: In a given BN, are two variables independent (given evidence)?
- Solution: Analyze the graph

Example

$R \perp\!\!\!\perp B$ *Yes*

$R \perp\!\!\!\perp B | T$ *No*

$R \perp\!\!\!\perp B | T'$ *No*



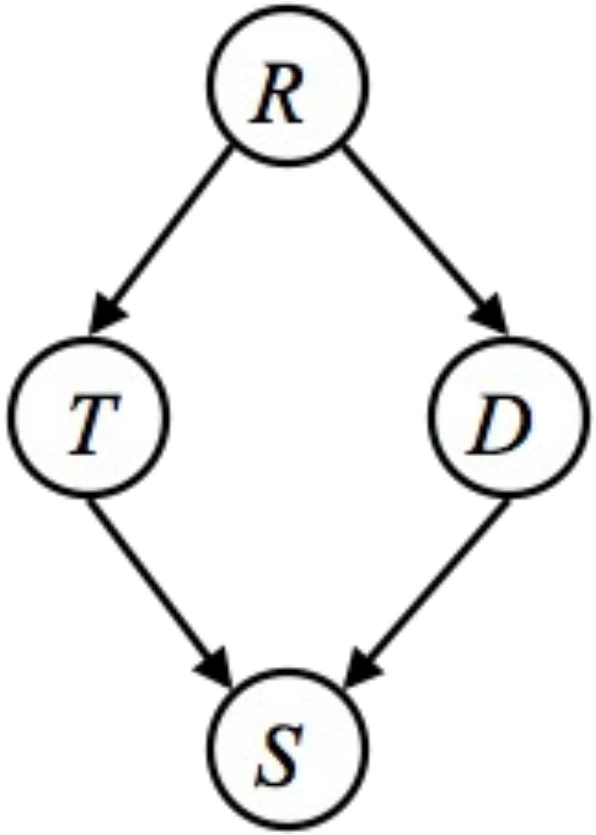
Example

R: Rain
T: Traffic
D: Roof drip
S: Sadness

$T \perp\!\!\!\perp D$ No, common cause

$T \perp\!\!\!\perp D | R$ Yes

$T \perp\!\!\!\perp D | R, S$ No, common effect



Example

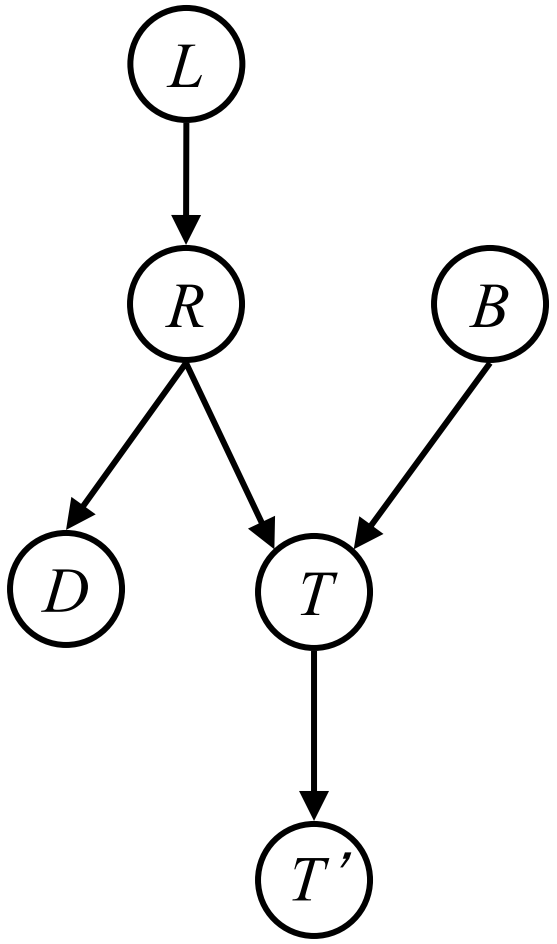
$L \perp\!\!\!\perp T' | T$ *Yes*

$L \perp\!\!\!\perp B$ *Yes*

$L \perp\!\!\!\perp B | T$ *No*

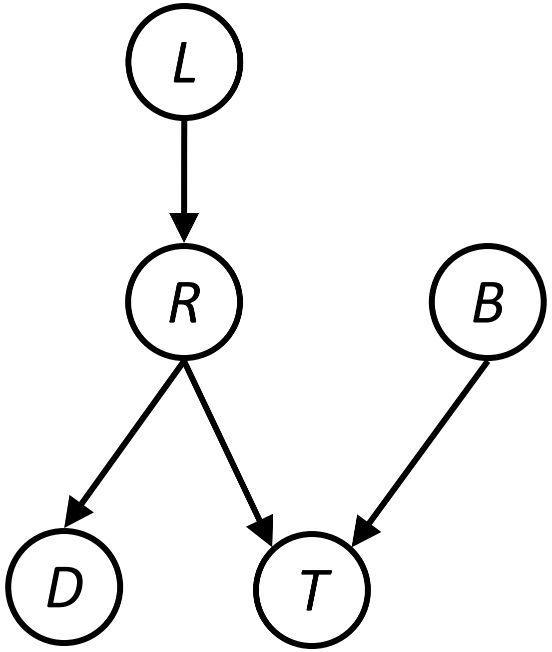
$L \perp\!\!\!\perp B | T'$ *No*

$L \perp\!\!\!\perp B | T, R$ *Yes*



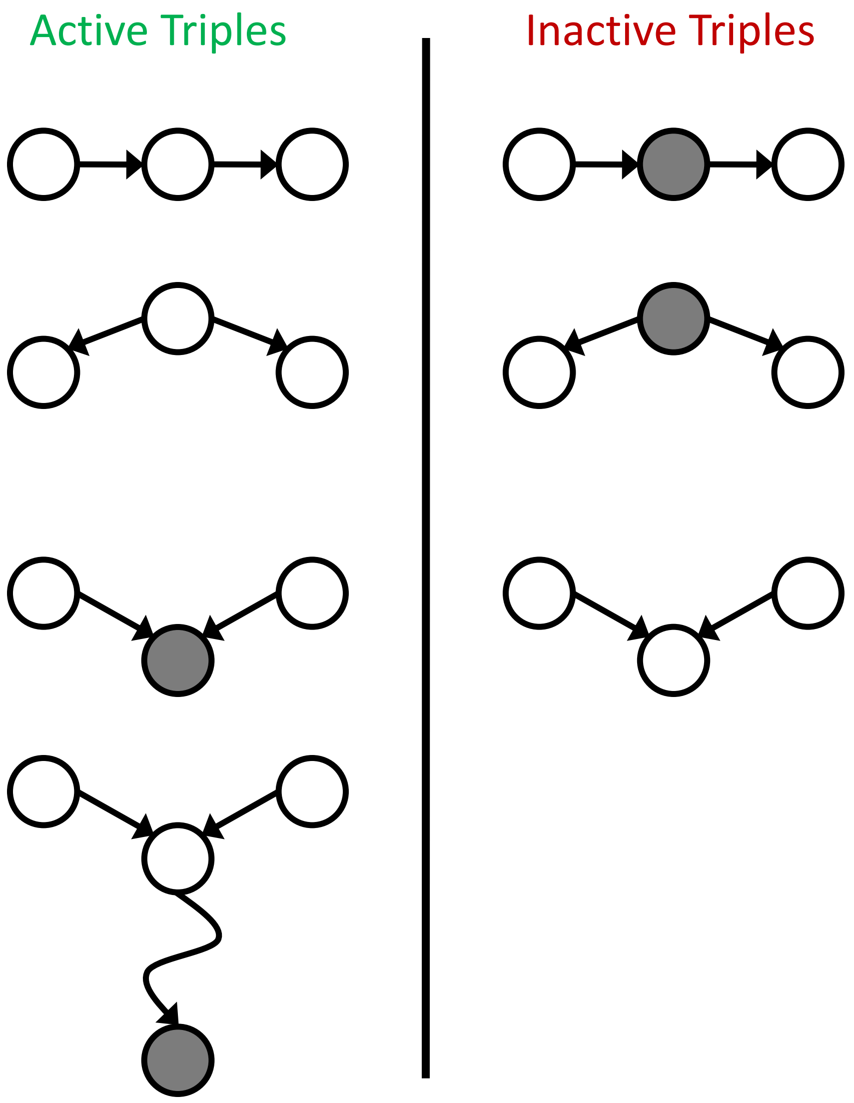
Reachability

- Recipe: shade evidence nodes, look for paths in the resulting graph
- Attempt 1: if two nodes are connected by an undirected path not blocked by a shaded node, they are conditionally independent
- Almost works, but not quite
  - Where does it break?
  - Answer: the v-structure at T doesn't count as a link in a path unless "active"



Active / Inactive Paths

- Question: Are X and Y conditionally independent given evidence variables {Z}?
  - Yes, if X and Y "d-separated" by Z
  - Consider all (undirected) paths from X to Y
  - No active paths = independence!
- A path is active if each triple is active:
  - Causal chain $A \rightarrow B \rightarrow C$ where B is unobserved (either direction)
  - Common cause $A \leftarrow B \rightarrow C$ where B is unobserved
  - Common effect (aka v-structure)
    $A \rightarrow B \leftarrow C$ where B *or one of its descendants* is observed
- All it takes to block a path is a single inactive segment



D-Separation

- Query: $X_i \perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$?
- Check all (undirected!) paths between X_i and X_j
  - If one or more active, then independence not guaranteed

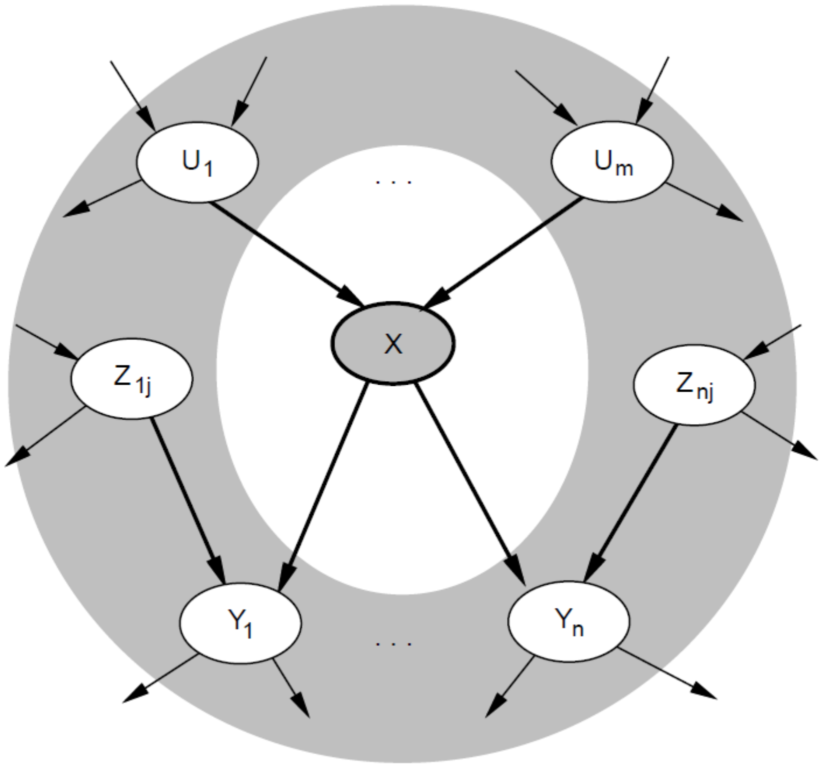
$$X_i \not\perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$

  - Otherwise (i.e. if all paths are inactive), then independence is guaranteed

$$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$

Markov Blanket

- Each node is conditionally independent of all others given its *Markov Blanket*
- Markov Blanket: parents + children + children's parents



Summary

- Bayesian networks provide a natural representation for conditional independence
- Topology + CPTs = compact representation of joint distribution
- Guaranteed independencies of distributions can be deduced from BN graph structure
- Generally easy for domain experts to construct