

## Math 103, Fall 2024, Student Report # 2

### (Lectures 6 & 7, Oct. 28-Nov. 06, 2024)

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#### Part 1: Information on study habits (2 points)

Number of hours of study per day:

Week of Oct. 28	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Number of hours of study		2 h		2 h			
Week of Nov. 4	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Number of hours of study		2 h		5h			

- Amount of time (in hours or minutes) spent on studying the content of the lecture notes and reproducing the results derived in class:  $3h$ .....
- Amount of time (in hours or minutes) spent on studying the part(s) of the textbook that is related to the content of the lecture notes:  $4h$ .....
- Amount of time (in hours or minutes) spent on working on the homework assignments and solving other exercise problems:  $2h$ .....
- Amount of time (in hours or minutes) spent on preparing the summary of the lecture notes: 10.minutes..
- Amount of additional time (in hours or minutes) spent on preparing for upcoming exam:  $2h$ .....

#### Part 2: Summary lecture notes including Definition and Theorem Lists (8 points)

1. Prepare and attach to this form a 1-3 pages long hand-written summary of the notes you have taken during **Lectures 6 & 7**.
2. Prepare and attach to this form a (hand-written) list of all definitions given in Chapter 3 of Textbook 1.
3. Prepare and attach to this form a (hand-written) list of all theorems (including lemmas, propositions, corollaries) given in Chapter 3 of Textbook 1 without proof.

In the last page of your Theorem Lists include a hand-written copy of the following **honor code** and **sign your name**.

**“This is to certify that in preparing the content of Part 2 of this report I have not received help from anyone else. In particular, I have used the notes that I have taken while attending the lectures. Therefore, the work presented in this report belongs to me.”**

Note: You must upload the completed report form and its attachments to KU-Learn in the form of a **single PDF** before the due time. To avoid major grade deduction check the PDF of your report and make sure that its hand-written part is legible.

## Notes CH3) Theorem Types and Proof Methods

### 3.1 Existence theorems

Either one constructs an example of the object in question  $\rightarrow$  Proof by construction  
Shows that if it did not exist a known fact would be falsified  $\rightarrow$  Proof by contradiction

### 3.2 Uniqueness Theorems

Theorems stating the uniqueness of a mathematical object  $X$  whose existence has already been established.  
To prove a uniqueness theorem one takes an arbitrary mathematical object  $Y$  that satisfies the conditions defining  $X$ , and shows that  $X = Y$ .

### 3.3 Classification Theorems

Once the existence and non-uniqueness of a mathematical object are established, one may pose the question of its classification



### 3.4 Proving Implications

Majority of theorems in mathematics are implications. They involve a hypothesis  $a$  and a conclusion  $b$ , and the theorem states that  $c := (a \Rightarrow b)$  is a true statement.  $c$  means that if  $a$  is true, then so is  $b$ .

#### 3.4.1 Trivial Proof

The easiest way of proving an implication is to show that its hypothesis is false. The main problem with this method is that it cannot be applied to arbitrary implication.

#### 3.4.2 Direct Proof

Assume  $a$  is true and show that  $b$  is true. May be applied to any implication.

#### 3.4.3 Contrapositive Proof

$$(a \Rightarrow b) = (\neg a \vee b) = (b \Rightarrow a)$$

To establish  $a \Rightarrow b$  is true we can prove that  $\neg b \Rightarrow \neg a$  is true.

#### 3.4.4 Deductive Proof

$(a \Rightarrow c) \wedge (c \Rightarrow b) \Rightarrow (a \Rightarrow b)$  in other words, to prove  $a \Rightarrow b$  is true, one can use an intermediary  $c$  s.t.  $(a \Rightarrow c) \wedge (c \Rightarrow b)$  is true.

### 3.5 Proof By Contradiction

1. Find appropriate statement  $b$
2. Prove that  $\neg a \Rightarrow b$ , and
3. Prove that  $b$  is false.

OR

Prove that  $(\neg a \Rightarrow a \text{ is true} \Rightarrow a \text{ is true}) \Rightarrow (a \vee \neg a)$

In practice, one usually finds the appropriate  $b$  in the process of applying the method of direct proof to  $\neg a \Rightarrow b$ . In other words one assumes that  $\neg a$  is true and seeks among the logical consequences of this assumption for one that contradicts a known fact. If the assumption of  $\neg a$  leads to a false statement, then one has found an appropriate  $b$  and in the process has also achieved 2 and 3.

### 3.6 Proof by Cases

Basically proves all cases.

### 3.7 Proof by Induction

$$(a_1 \wedge (\forall m \in \{1, 2, \dots, N-1\}, (a_m \Rightarrow a_{m+1}))) \Rightarrow (\forall n \in \{1, 2, \dots, N\}, a_n).$$

1. Prove  $a_1$
2. Assume  $\exists m \in \mathbb{Z}^+$  s.t.  $a_m$  is true, and
3. prove  $a_{m+1}$

~~Induction Axiom:~~

$$(a_1 \wedge (\forall m \in \mathbb{Z}^+, (a_m \Rightarrow a_{m+1}))) \Rightarrow (\forall n \in \mathbb{Z}^+, a_n)$$



### 3.7.2 Complete Induction

1. Prove  $a_1$
2. Assume  $\exists m \in \mathbb{Z}^+$  such that for all  $l \in \{1, \dots, m\}$ ,  $a_l$  is true and
3. Prove  $a_{m+1}$

$$(a_1 \wedge (\forall m \in \mathbb{Z}^+ ((a_1 \wedge a_2 \wedge \dots \wedge a_m) \Rightarrow a_{m+1}))) \Rightarrow (\forall n \in \mathbb{Z}^+, a_n).$$

## Definitions:

Definition 3.1.1 Let  $m$  and  $n$  be integers. Then  $m$  is said to divide  $n$ , if there exists an integer  $k$  such that  $n = km$ , i.e., the following statement holds.  $\exists k \in \mathbb{Z}, n = km$ .

In this case,  $m$  is said to be a divisor of  $n$ , and  $n$  is called a multiple of  $m$ .  $m | n$

Definition 3.5.1 Let  $m$  and  $n$  be integers, then an integer that divides both  $m$  and  $n$  is said to be their common divisor. The largest of which is called the greatest common divisor. We denote it by " $\gcd(m, n)$ ".

Definition 3.5.2 Two integers  $m$  and  $n$  (not both zero) are said to be relatively prime if  $\gcd(m, n) = 1$ , i.e., the only positive integer that divides  $m$  and  $n$  is 1. We can express this as

$$\forall k \in \mathbb{Z}^+, (k | m \wedge k | n) \Rightarrow (k = 1) \quad \blacksquare$$

Definition 3.8.1 Let  $n \in \mathbb{Z}^+$  and  $j, k \in \mathbb{Z}$ . Then  $j$  is said to be congruent to  $k$  modulo  $n$ , if  $j$  and  $k$  have the same remainder.



Theorems (Lemmas, Propositions, Corollaries included) 1.8.8 mod A

Theorem 3.1.1 Every integer has a divisor  $\geq 2$ .

Theorem 3.2.1 The only natural number  $n$  satisfying the following condition is 0.  
 $\forall j \in \mathbb{N}, n+j=j$ .

Proposition 3.4.1 Let  $m, n, p$  be integers. If  $m$  divides  $n$  and  $n$  divides  $p$ , then  $m$  divides  $p$ , i.e.,  $(m|n \wedge n|p) \Rightarrow m|p$ .

Proposition 3.4.2 For every integer  $n$ , if  $n^2$  is even, then so is  $n$ .  
 $2|n^2 \Rightarrow 2|n$

Theorem 3.4.1 Let  $n \in \mathbb{Z}$  and  $c_1, c_2, \dots, c_n, a, b$  be statements. Then the following is a tautology:

$$t := ((a \Rightarrow c_1 \Rightarrow c_2 \Rightarrow \dots \Rightarrow c_n \Rightarrow b) \Rightarrow (a \Rightarrow b)).$$

Theorem 3.5.1 Let  $m$  and  $n$  be integers such that at least one of them is nonzero. Then  $m$  and  $n$  have a greatest common divisor that is greater than or equal to 1.

Lemma 3.5.1 Every rational can be expressed as the ratio of two relatively prime integers, i.e.,  
 $\forall r \in \mathbb{Q}, \exists p, q \in \mathbb{Z}, (q \neq 0) \wedge (r = \frac{p}{q}) \wedge (\text{g.c.d.}(p, q) = 1).$

Theorem 3.5.2  $\sqrt{2}$  is not a rational number.

$$(p_1 \leq p_2 \leq \dots \leq p_n) \Leftrightarrow (p_1 \leq p_2 \leq \dots \leq p_n) = \text{true}$$
$$(p_1 \leq p_2 \leq \dots \leq p_n) \Leftrightarrow (p_1 \leq p_2 \leq \dots \leq p_n) = \text{true}$$



### Axiom 3.7.1 (Induction Axiom)

Let  $S$  be a collection of positive integers (a subset of  $\mathbb{Z}^+$ ). If  $1 \in S$  and  $\forall m \in \mathbb{Z}^+, ((m \in S) \Rightarrow ((m+1) \in S))$ , then  $S$  includes all positive integers, i.e.  $S = \mathbb{Z}^+$ .

### Theorem 3.7.1 (Principle of Mathematical Induction)

Let  $a_n$  be a statement for all  $n \in \mathbb{Z}^+$ . Suppose that  $a_1$  and " $\forall m \in \mathbb{Z}^+, (a_m \Rightarrow a_{m+1})$ " are true, then  $a_n$  is true for all  $n \in \mathbb{Z}^+$ .

Proposition 3.7.1 Every positive integer can be written as the sum of distinct powers of 2, i.e.,

$\forall n \in \mathbb{Z}^+, \exists l \in \mathbb{Z}^+, \exists p_1, p_2, \dots, p_l \in \mathbb{N}$ , s.t.  $p_1 < p_2 < \dots < p_l$ , and

$$n = 2^{p_1} + 2^{p_2} + \dots + 2^{p_l} = \sum_{k=1}^l 2^{p_k}$$

### Theorem 3.7.2 (Principle of Complete Induction)

Let  $a_n$  be a statement for all  $n \in \mathbb{Z}^+$ . Suppose that  $a_1$  and  $(\forall m \in \mathbb{Z}^+, ((a_1 \wedge a_2 \wedge \dots \wedge a_m) \Rightarrow a_{m+1}))$  are true. Then  $a_n$  is true for all  $n \in \mathbb{Z}^+$ .

### Theorem (Division Algorithm)

Let  $n \in \mathbb{Z}^+$  and  $i \in \mathbb{Z}$ . Then  $\exists! m \in \mathbb{Z}$  and  $\exists! r \in \{0, 1, \dots, n-1\}$  s.t.  $i = mn + r$ .  $r$  is called the remainder of the division.

### Theorem 3.8.1 ( $(a \equiv b \pmod{n}) \wedge (b \equiv c \pmod{n}) \Rightarrow a \equiv c \pmod{n}$ )

Let  $n \in \mathbb{Z}^+$  and  $j, k \in \mathbb{Z}$ . Then  $j$  is congruent to  $k$  modulo  $n$  if and only if  $n$  divides  $j-k$ .

### Theorem 3.8.2

Let  $\forall n \in \mathbb{Z}^+, a_n$  be a statement. Then for every integer  $k$  greater than 1, the following two statements are logically equivalent.

$$b_k := (a_1 \Leftrightarrow a_2 \Leftrightarrow \dots \Leftrightarrow a_k),$$

$$c_k := (a_1 \Rightarrow a_2 \Rightarrow \dots \Rightarrow a_k \Rightarrow a_1)$$



This is to certify that in preparing the content of Part 2 of this report I have not received help from anyone else. In particular, I have used the notes I have taken ~~during~~ ~~while~~ while attending the lectures. Therefore, the work presented in this report belongs to me.

Mahmet Murat Budak  
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