Chapter 2
Finite Element Equations for Heat Transfer

Abstract Solution of heat transfer problems is considered. Finite element equations are obtained using the Galerkin method. The conductivity matrix for a triangular finite element is calculated.

2.1 Problem Statement

Let us consider an isotropic body with temperature-dependent heat transfer. A basic equation of heat transfer has the following form [15]:

$$-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)+Q=\rho c\frac{\partial T}{\partial t}. \tag{2.1}$$

Here, q_x, q_y and q_z are components of heat flow through the unit area; $Q = Q(x,y,z,t)$ is the inner heat-generation rate per unit volume; ρ is material density; c is heat capacity; T is temperature and t is time. According to Fourier's law the components of heat flow can be expressed as follows:

$$\begin{aligned} q_x &= -k\frac{\partial T}{\partial x},\\ q_y &= -k\frac{\partial T}{\partial y},\\ q_z &= -k\frac{\partial T}{\partial z}, \end{aligned} \tag{2.2}$$

where k is the thermal-conductivity coefficient of the media. Substitution of Fourier's relations gives the following basic heat transfer equation:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+Q=\rho c\frac{\partial T}{\partial t}. \tag{2.3}$$

It is assumed that the *boundary conditions* can be of the following types:

1. Specified temperature

$$T_s = T_1(x,y,z,t) \text{ on } S_1 \ .$$

2. Specified heat flow

$$q_x n_x + q_y n_y + q_z n_z = -q_s \text{ on } S_2 \ .$$

3. Convection boundary conditions

$$q_x n_x + q_y n_y + q_z n_z = h(T_s - T_e) \text{ on } S_3 \ ,$$

4. Radiation

$$q_x n_x + q_y n_y + q_z n_z = \sigma \varepsilon T_s^4 - \alpha q_r \text{ on } S_4,$$

where h is the convection coefficient; T_s is an unknown surface temperature; T_e is a convective exchange temperature; σ is the Stefan–Boltzmann constant; ε is the surface emission coefficient; α is the surface absorption coefficient, and q_r is the incident radiant heat flow per unit surface area. For transient problems it is necessary to specify an initial temperature field for a body at the time $t = 0$:

$$T(x,y,z,0) = T_0(x,y,z). \tag{2.4}$$

2.2 Finite Element Discretization of Heat Transfer Equations

A domain V is divided into finite elements connected at nodes. We shall write all the relations for a finite element. Global equations for the domain can be assembled from finite element equations using connectivity information.

Shape functions N_i are used for interpolation of temperature inside a finite element:

$$\begin{aligned} T &= [N]\{T\}, \\ [N] &= \left[N_1 \ N_2 \ ...\right], \\ \{T\} &= \left\{T_1 \ T_2 \ ...\right\}. \end{aligned} \tag{2.5}$$

Differentiation of the temperature-interpolation equation gives the following interpolation relation for temperature gradients:

$$\begin{Bmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \\ \dfrac{\partial T}{\partial z} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} & ... \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} & ... \\ \dfrac{\partial N_1}{\partial z} & \dfrac{\partial N_2}{\partial z} & ... \end{bmatrix} \{T\} = [B]\{T\}. \tag{2.6}$$

Here, $\{T\}$ is a vector of temperatures at nodes, $[N]$ is a matrix of shape functions, and $[B]$ is a matrix for temperature-gradient interpolation.

Using the Galerkin method, we can rewrite the basic heat transfer equation in the following form:

$$\int_V \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} - Q + \rho c \frac{\partial T}{\partial t} \right) N_i dV = 0. \tag{2.7}$$

Applying the divergence theorem to the first three terms, we arrive at the relations:

$$\begin{aligned} &\int_V \rho c \frac{\partial T}{\partial t} N_i dV - \int_V \left[\frac{\partial N_i}{\partial x} \ \frac{\partial N_i}{\partial y} \ \frac{\partial N_i}{\partial z} \right] \{q\} dV \\ &= \int_V Q N_i dV - \int_S \{q\}^{\mathrm{T}} \{n\} N_i dS, \\ &\{q\}^{\mathrm{T}} = \left[q_x \ q_y \ q_z \right], \\ &\{n\}^{\mathrm{T}} = \left[n_x \ n_y \ n_z \right], \end{aligned} \tag{2.8}$$

where $\{n\}$ is an outer normal to the surface of the body. After insertion of boundary conditions into the above equation, the discretized equations are as follows:

$$\begin{aligned} &\int_V \rho c \frac{\partial T}{\partial t} N_i dV - \int_V \left[\frac{\partial N_i}{\partial x} \ \frac{\partial N_i}{\partial y} \ \frac{\partial N_i}{\partial z} \right] \{q\} dV \\ &= \int_V Q N_i dV - \int_{S_1} \{q\}^{\mathrm{T}} \{n\} N_i dS \\ &+ \int_{S_2} q_{\mathrm{s}} N_i dS - \int_{S_3} h(T - T_{\mathrm{e}}) N_i dS - \int_{S_4} (\sigma \varepsilon T^4 - \alpha q_{\mathrm{r}}) N_i dS. \end{aligned} \tag{2.9}$$

It is worth noting that

$$\{q\} = -k[B]\{T\}. \tag{2.10}$$

The discretized finite element equations for heat transfer problems have the following form:

$$\begin{aligned} &[C]\{\dot{T}\} + ([K_c] + [K_h] + [K_r])\{T\} \\ &= \{R_T\} + \{R_Q\} + \{R_q\} + \{R_h\} + \{R_r\}, \end{aligned} \tag{2.11}$$

$$
\begin{aligned}
[C] &= \int_V \rho c [N]^{\mathrm{T}} [N] dV, \\
[K_c] &= \int_V k [B]^{\mathrm{T}} [B] dV, \\
[K_h] &= \int_{S_3} h [N]^{\mathrm{T}} [N] dS, \\
[K_r]\{T\} &= \int_{S_4} \sigma \varepsilon T^4 [N]^{\mathrm{T}} dS, \\
\{R_T\} &= -\int_{S_1} \{q\}^{\mathrm{T}} \{n\} [N]^{\mathrm{T}} dS, \\
\{R_Q\} &= \int_V Q [N]^{\mathrm{T}} dV, \\
\{R_q\} &= \int_{S_2} q_{\mathrm{s}} [N]^{\mathrm{T}} dS, \\
\{R_h\} &= \int_{S_3} h T_{\mathrm{e}} [N]^{\mathrm{T}} dS, \\
\{R_r\} &= \int_{S_4} \alpha q_{\mathrm{r}} [N]^{\mathrm{T}} dS.
\end{aligned}
\tag{2.12}
$$

Here, $\{\dot{T}\}$ is a nodal vector of temperature derivatives with respect to time.

2.3 Different Type Problems

Equations for different types of problems can be deducted from the above general equation:

Stationary linear problem

$$
([K_c] + [K_h])\{T\} = \{R_Q\} + \{R_q\} + \{R_h\}. \tag{2.13}
$$

Stationary nonlinear problem

$$
\begin{aligned}
&([K_c] + [K_h] + [K_r])\{T\} \\
&= \{R_Q(T)\} + \{R_q(T)\} + \{R_h(T)\} + \{R_r(T)\}.
\end{aligned}
\tag{2.14}
$$

Transient linear problem

$$[C]\{\dot{T}(t)\} + ([K_c] + [K_h(t)])\{T(t)\} \\ = \{R_Q(t)\} + \{R_q(t)\} + \{R_h(t)\}. \tag{2.15}$$

Transient nonlinear problem

$$[C(T)]\{\dot{T}\} + ([K_c(T)] + [K_h(T,t)] + [K_r(T)])\{T\} \\ = \{R_Q(T,t)\} + \{R_q(T,t)\} + \{R_h(T,t)\} + \{R_r(T,t)\}. \tag{2.16}$$

2.4 Triangular Element

Calculation of element conductivity matrix $[k_c]$ and heat flow vector $\{r_q\}$ is illustrated for a two-dimensional triangular element with three nodes. A simple triangular finite element is shown in Figure 2.1. The temperature distribution $T(x,y)$ inside the triangular element is described by linear interpolation of its nodal values:

$$T(x,y) = N_1(x,y)T_1 + N_2(x,y)T_2 + N_3(x,y)T_3, \\ N_i(x,y) = \alpha_i + \beta_i x + \gamma_i y. \tag{2.17}$$

Interpolation functions (usually called shape functions) $N_i(x,y)$ should satisfy the following conditions:

$$T(x_i,y_i) = T_i\,, \quad i = 1,\ 2,\ 3. \tag{2.18}$$

Solution of the above equation system provides expressions for the shape functions:

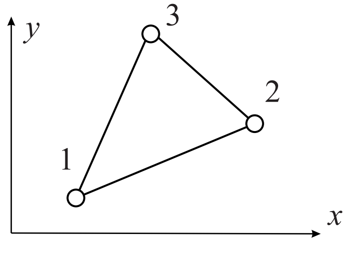


Fig. 2.1 Triangular finite element

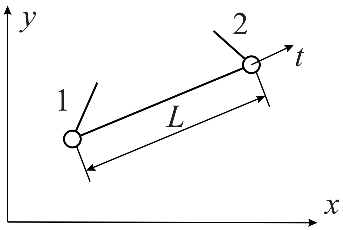


Fig. 2.2 Integration along an element side

$$
\begin{aligned}
N_i &= \frac{1}{2\Delta}(a_i + b_i x + c_i y),\\
a_i &= x_{i+1}y_{i+2} - x_{i+2}y_{i+1},\\
b_i &= y_{i+1} - y_{i+2},\\
c_i &= x_{i+2} - x_{i+1},\\
\Delta &= \frac{1}{2}(x_2y_3 + x_3y_1 + x_1y_2 - x_2y_1 - x_3y_2 - x_1y_3),
\end{aligned}
\tag{2.19}
$$

where Δ is the element area.

The conductivity matrix of the triangular element is determined by integration over element area A (assuming that the element has unit thickness),

$$[k_c] = \int_A k[B]^{\mathrm{T}}[B]dxdy. \tag{2.20}$$

The temperature differentiation matrix $[B]$ has expression

$$[B] = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial x} \\[2ex] \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_3}{\partial y} \end{bmatrix} = \frac{1}{2\Delta}\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}. \tag{2.21}$$

Since the temperature differentiation matrix does not depend on coordinates, integration of the conductivity matrix is simple;

$$[k_c] = \frac{k}{4\Delta}\begin{bmatrix} b_1^2 + c_1^2 & b_1b_2 + c_1c_2 & b_1b_3 + c_1c_3 \\ b_1b_2 + c_1c_2 & b_2^2 + c_2^2 & b_2b_3 + c_2c_3 \\ b_1b_3 + c_1c_3 & b_2b_3 + c_2c_3 & b_3^2 + c_3^2 \end{bmatrix}. \tag{2.22}$$

The heat-flow vector $\{r_q\}$ is evaluated by integration over the element side, as shown in Figure 2.2

$$\{r_q\} = -\int_L q_s [N]^T dL = -\int_0^1 q_s [N_1\ N_2]^T L dt. \tag{2.23}$$

Here, integration over an element side L is replaced by integration using variable t ranging from 0 to 1. Shape functions N_1 and N_2 on element side 1–2 can be expressed through t:

$$N_1 = 1 - t, \quad N_2 = t. \tag{2.24}$$

After integration with substituting integration limits, the heat-flow vector equals

$$\{r_q\} = -q_s \frac{L}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{2.25}$$

Element matrices and vectors are calculated for all elements in a mesh and assembled into the global equation system. After application of prescribed temperatures, solution of the global equation system produces temperatures at nodes.

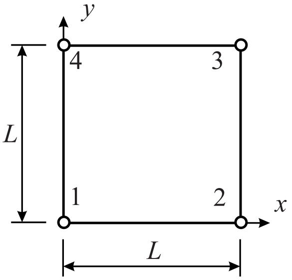
Problems

2.1. Calculate matrix $[k_h]$ describing convection boundary conditions

$$[k_h] = \int_L h[N]^T [N] dL$$

for a side of a triangular element (see Figure 2.2).

2.2. Obtain shape functions N_1, N_2, N_3 and N_4 for the square element shown below.



Assume that its size is $L = 1$ and that shape functions can be represented as $N_i = a_1(a_2 + x)(a_3 + y)$.

2.3. For the square element of the previous problem, estimate the heat-generation vector

$$\{r_Q\} = \int_V Q[N]^T dV.$$

Use the shape functions obtained in the previous problem.

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