Mathematics II

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You can find the last version of these course materials at

https://mbujosab.github.io/MatematicasII/

Contents

Part VI
Eigenvalues and eigenvectors

LECTURE 16: Eigenvalues and eigenvectors

Lecture 16

(**Lecture** 16) S-**1** Highlights of Lesson 16

Always squared matrices in this topic

Highlights of *Lesson 16*

- **Eigenvalues, eigenvectors** (prefix eigen is the German word for innate, distinct, self)
- $\left|\mathbf{A}-\lambda\mathbf{I}\right|=0$ *Characteristic equation*
- $\operatorname{tr}(\mathbf{A})$, $\det\mathbf{A}$ (demo in the next lesson)

F1

(**Lecture** 16) S-**2** Eigenvalues and eigenvectors

Consider the equation

$$\mathbf{A}\boldsymbol{x}=\lambda\boldsymbol{x} \qquad (\text{with} \quad \boldsymbol{x}\neq\mathbf{0})$$

- *Eigenvalue* is any λ such that there are solutions.
- Such *non-null* solutions $\boldsymbol{x}$ are called *eigenvectors*.

$$\boldsymbol{x}\neq\mathbf{0} \text{ such that } \mathbf{A}\boldsymbol{x} \text{ is multiple } \boldsymbol{x}$$

When λ is 0, What are the eigenvectors?

.

F2

(**Lecture** 16) S-**3** Example: projection matrix

- Orthogonal projection
- Which vectors are eigenvectors?
  What vectors are projected in the same starting direction?
- What are the eigenvalues of those eigenvectors?
- are there any other eigenvectors? with what eigenvalue?
- Two eigen-spaces

F3

(**Lecture** 16) S-4 **Another example: Interchange or swap matrix**

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- A vector that does not change after interchange?
- What is the eigenvalue?
- Is there an eigenvector corresponding to $\lambda_2 = -1$?

$$\mathbf{A}\boldsymbol{x}_2 = -\boldsymbol{x}_2$$

Note: $\operatorname{tr}\left(\mathbf{A}\right) = 0 = \lambda_1 + \lambda_2$; $\det \mathbf{A} = -1 = \lambda_1 \cdot \lambda_2$. F4

The trace of an n-by-n square matrix $\mathbf{A}$ is defined to be the sum of the elements on the main diagonal:

$$\operatorname{tr}\left(\mathbf{A}\right) = a_{11} + a_{22} + \cdots + a_{nn}.$$

(**Lecture** 16) S-5 **how to find eigenvalues and eigenvectors?**

How to solve

$$\mathbf{A}\boldsymbol{x} = \overbrace{\lambda}^{?}\,\overbrace{\boldsymbol{x}}^{?}\ ?$$

Here's the trick (simple idea). Bring the $\boldsymbol{x}$ s onto the same side ...

$$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{x} =$$

idea If $\boldsymbol{x} \neq \mathbf{0}$ what kind of matrix must be $(\mathbf{A} - \lambda\mathbf{I})$?
and then its determinant must be? $|\mathbf{A} - \lambda\mathbf{I}| =$ F5

(**Lecture** 16) S-6 **how to find eigenvalues and eigenvectors?**

1. Eigenvalues are λ's such that: $|\mathbf{A} - \lambda\mathbf{I}| =$

   (Characteristic polynomial $P_{\mathbf{A}}(\lambda)$)

2. How to compute $\boldsymbol{x}$ so that $\left(\mathbf{A} - \lambda\mathbf{I}\right)\boldsymbol{x} = \mathbf{0}$?

Eigenspace (Set of eigenvectors + $\mathbf{0}$):

$$\mathcal{E}_{\lambda}(\mathbf{A}) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x} \right\}$$

Spectrum: set $\{\lambda_1, \ldots \lambda_k\}$ of eigenvalues $\left(\text{roots of } P_{\mathbf{A}}(\lambda)\right)$ F6

(**Lecture** 16) S-7 Example (we must compute the eigevalues first!)

We are looking for a null determinant (Characteristic polynomial)

$$\mathbf{A} = \begin{bmatrix}3 & 1\\1 & 3\end{bmatrix}; \qquad \det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix}3-\lambda & 1\\1 & 3-\lambda\end{vmatrix} = (3-\lambda)^2 - 1 = 0$$

Note: $\operatorname{tr}\left(\mathbf{A}\right) = 6 = \lambda_1 + \lambda_2$; $\qquad \det \mathbf{A} = 8 = \lambda_1 \cdot \lambda_2$.

F7

Remember, if $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

See:
https://itempool.com/3b1b/c/ag9L0EqZFtK
http://en.wikipedia.org/wiki/Quadratic_equation

```
L = sympy.symbols('\lambda')
A = Matrix([[3,1],[1,3]])
D = Determinante(A-L*I(2),1).valor
display(D)

d = sympy.poly(D)
r = sympy.real_roots(d)
display(r)

ElimG(A-L*I(2), 1)
```

(**Lecture** 16) S-8 Example (...and then the eigenspaces)

And now we compute the null space $\mathcal{N}(\mathbf{A}-\lambda\mathbf{I})$ ...for each λ.

For $\lambda_1 = 4$

$$(\mathbf{A}-4\mathbf{I}) = \begin{bmatrix}3-4 & 1\\1 & 3-4\end{bmatrix} = \begin{bmatrix}-1 & 1\\1 & -1\end{bmatrix} \Rightarrow$$

For $\lambda_2 = 2$

$$(\mathbf{A}-2\mathbf{I}) = \begin{bmatrix}3-2 & 1\\1 & 3-2\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} \Rightarrow$$

Are they the only two eigenvectors?

$$\mathbf{A}\boldsymbol{x}_i = \lambda\boldsymbol{x}_i; \qquad \begin{bmatrix}3 & 1\\1 & 3\end{bmatrix}\boldsymbol{x}_i = \lambda\boldsymbol{x}_i.$$

F8

(**Lecture** 16) S-9 **Another example: 90º rotation matrix**

$$\mathbf{Q} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- How much do the eigenvalues add up to?
- What is the determinant?

Difficulties
$\lambda_1 + \lambda_2 = 0$ and $\lambda_1 \cdot \lambda_2 = 1$ $(+) \cdot (-) = (+)?$
What kind of vector can be parallel to itself after a 90º rotation?

$$\det(\mathbf{Q} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 =$$

F9

$$\mathbf{Q}\boldsymbol{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

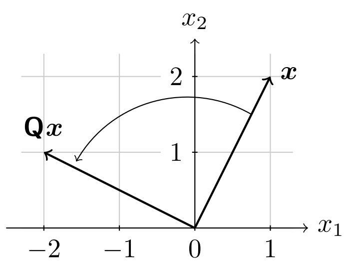

Figure 1: Rotation

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ i \end{pmatrix} = i \begin{pmatrix} i \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix} = -i \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Somehow these complex vectors keep their direction as they are rotated. Don ask me how!

Symmetric matrices have always *real* eigenvalues (and perpendicular eigenvectors —as we will see soon), but *anty-symmetric* (like $\mathbf{Q}$) matrices have pure imaginary eigenvalues.

(**Lecture** 16) S-**10** There are even worse examples

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

- Eigenvalues

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)(3-\lambda) = 0 \begin{cases} \lambda_1 = 3 \\ \lambda_2 = 3 \end{cases}$$

- Eigenvectors
  - for λ_1: $(\mathbf{A} - \lambda \mathbf{I})\boldsymbol{x} = \mathbf{0} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x}_1$; $\quad \boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
  - for λ_2:

$\lambda = 3$ is repeated twice, but $\dim \mathcal{E}_3(\mathbf{A}) = 1$

$$\mu(3) = 2 \neq 1 = \gamma(3)$$

F10

Summary:

1. The eigenvalues are those numbers λ that makes the matrix $(\mathbf{A} - \lambda \mathbf{I})$ singular. In other words, they are the roots of the Characteristic polynomial: $\det(\mathbf{A} - \lambda \mathbf{I})$.
2. Any n by n matrix has a caracteristic polynomial of degree n
3. A polynomial of degree n has n roots (perhaps some repeated roots).
4. The sum of eigenvalues of a matrix equals its trace
5. The product of eigenvalues of a matrix equals its determinant
6. The eigenvectors associated with λ are the non-zero vectors in $\mathcal{N}(\mathbf{A} - \lambda \mathbf{I})$.

The lecture ends here

Questions of the Lecture 16

(L-16) Question 1. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} -3 & 4 & -4 \\ -3 & 5 & -3 \\ -1 & 2 & 0 \end{bmatrix}$$

(a) The three eigenvalues of $\mathbf{A}$ are -1, 1 and 2; and two of its eigenvectors are

$$\boldsymbol{v} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}; \qquad \boldsymbol{w} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Check that both vectors are eigenvenctors of $\mathbf{A}$. What are the corresponding eigenvalues?

(b) Find a third linearly independent eigenvector.

(L-16) Question 2. Find the eigenvalues and eigenvectors of

(a)

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

(b)

$$\mathbf{B} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

(Strang, 2006, exercise 12 from section 5.1.)

(L-16) QUESTION 3. If $\mathbf{B}$ has eigenvalues 1, 2, 3, $\mathbf{C}$ has eigenvalues 4, 5, 6, and $\mathbf{D}$ has eigenvalues 7, 8, 9, what are the eigenvalues of the 6 by 6 matrix $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}$? where $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are upper triangular matrices.
(Strang, 2006, exercise 13 from section 5.1.)

(L-16) QUESTION 4. Find the eigenvalues and eigenvectors of
(a)

$$\mathbf{A} = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

(b)

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

(Strang, 2006, exercise 5 from section 5.1.)

(L-16) QUESTION 5. The eigenvalues of $\mathbf{A}$ equal the eigenvalues of $\mathbf{A}^\intercal$. This is because $\det(\mathbf{A}-\lambda\mathbf{I})$ equals $\det(\mathbf{A}^\intercal-\lambda\mathbf{I})$.
(a) That is true because ____________
(b) Show by an example that, nevertheless, the eigenvectors of $\mathbf{A}$ and $\mathbf{A}^\intercal$ are not the same.
(Strang, 2006, exercise 11 from section 5.1.)

(L-16) QUESTION 6. Consider the matrix $\mathbf{B}$ and its eigenvector $\boldsymbol{x}$ associated to the eigenvalue λ, that is $\mathbf{B}\boldsymbol{x} = \lambda\boldsymbol{x}$; and also consider the matrix $\mathbf{A} = (\mathbf{B} + \alpha\mathbf{I})$. Prove that $\boldsymbol{x}$ is also an eigenvector of $\mathbf{A}$ with eigenvalue $(\lambda + \alpha)$.

(L-16) QUESTION 7.

(a) Encuentre los autovalores y los auto-vectores de la matriz $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. Compruebe que la traza es igual a la suma de los autovalores, y que el determinante es igual a su producto.
(b) Si consideramos una nueva matriz, generada a partir de la anterior como

$$\mathbf{B} = (\mathbf{A} - 7\mathbf{I}) = \begin{bmatrix} -6 & -1 \\ 2 & -3 \end{bmatrix}.$$

¿Cuáles son los autovalores y auto-vectores de la nueva matriz, y como están relacionados con los de $\mathbf{A}$?
(Strang, 2006, exercise 1 and 3 from section 5.1.)

(L-16) QUESTION 8. Suponga que λ es un auto-valor de $\mathbf{A}$, y que $\boldsymbol{x}$ es un auto-vector tal que $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$.
(a) Demuestre que ese mismo $\boldsymbol{x}$ es un auto-vector de $\mathbf{B} = \mathbf{A} - 7\mathbf{I}$, y encuentre el correspondiente auto-valor de $\mathbf{B}$.
(b) Suponga que $\lambda \neq 0$ (y que $\mathbf{A}$ es invertible), demuestre que $\boldsymbol{x}$ también es un auto-vector de $\mathbf{A}^{-1}$, y encuentre el correspondiente auto-valor. ¿Qué relación tiene con λ?
(Strang, 2006, exercise 7 from section 5.1.)

(L-16) QUESTION 9. Suponga que $\mathbf{A}$ es una matriz de dimensiones $n \times n$, y que $\mathbf{A}^2 = \mathbf{A}$. ¿Qué posibles valores pueden tomar los autovalores de $\mathbf{A}$?

(L-16) QUESTION 10. Suponga la matriz $\underset{3\times 3}{\mathbf{A}}$ con autovalores 1, 2 y 3. Si $\boldsymbol{v}_1$ es un auto-vector asociado al auto-valor 1, $\boldsymbol{v}_2$ al auto-valor 2 y $\boldsymbol{v}_3$ al auto-valor 3; entonces ¿cuanto es $\mathbf{A}(\boldsymbol{v}_1 + \boldsymbol{v}_2 - \boldsymbol{v}_3)$?

(L-16) QUESTION 11. Proporcione un ejemplo que muestre que los auto-valores pueden cambiar cuando un múltiplo de una columna se resta de otra. ¿Por qué los pasos de eliminación no modifican los autovalores nulos?
(Strang, 2006, exercise 6 from section 5.1.)

(L-16) QUESTION 12. El polinomio característico de una matriz $\mathbf{A}$ se puede factorizar como

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda).$$

Demuestre, partiendo de esta factorización, que el determinante de $\mathbf{A}$ es igual al producto de sus valores propios (autovalores). Para ello haga una elección inteligente del valor de λ.
(Strang, 2006, exercise 8 from section 5.1.)

(L-16) QUESTION 13. Calcule los valores característicos (autovalores o valores propios) y los vectores característicos de $\mathbf{A}$ y $\mathbf{A}^2$:

$$\mathbf{A} = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \qquad \text{y} \qquad \mathbf{A}^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}$$

$\mathbf{A}^2$ tiene los mismos ________ que $\mathbf{A}$. Cuando los autovalores de $\mathbf{A}$ son λ_1 y λ_2, los autovalores de $\mathbf{A}^2$ son ________.
(Strang, 2006, exercise 22 from section 5.1.)

(L-16) QUESTION 14. Suponga que los valores característicos de $\underset{3\times 3}{\mathbf{A}}$ son 1, 2 y 4, ¿cuál es la traza de $\mathbf{A}^2$? ¿Cuál es el determinante de $\left(\mathbf{A}^{-1}\right)^{\intercal}$?
(Strang, 2006, exercise 10 from section 5.2.)

(L-16) QUESTION 15. The equation $(\mathbf{A}^2 - 4\mathbf{I})\boldsymbol{x} = \boldsymbol{b}$ has no solution for some right-hand side $\boldsymbol{b}$. Give as much information as possible about the eigenvalues of the matrix $\mathbf{A}$ (the matrix $\mathbf{A}$ is diagonalizable). *MIT Course 18.06 Quiz 3. Spring, 2009*

(L-16) QUESTION 16. You are given the matrix

$$\mathbf{A} = \begin{bmatrix} 0.5 & 0.2 & 0.2 \\ 0.1 & 0.5 & 0.5 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$$

One of the eigenvalues is $\lambda = 1$. What are the eigenvalues of $\mathbf{A}$? [Hint: Very little calculation required! You should be able to see another eigenvalue by inspection of the form of $\mathbf{A}$, and the third by an easy calculation. You shouldn't need to compute $\det(\mathbf{A} - \lambda\mathbf{I})$ unless you really want to do it the hard way.] *MIT Course 18.06 Quiz 3. Spring, 2009*

End of Questions of the Lecture 16

LECTURE 17: Diagonalization in triangular blocks by similarity

Lecture 17

(**Lecture** 17) S-1 Highlights of Lesson 17

Highlights of *Lesson 17*

- Similar matrices: $\mathbf{C} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$
- Triangular block diagonalizing a matrix

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] \xrightarrow[esp(\boldsymbol{\tau}_p^{-1}\cdots\boldsymbol{\tau}_1^{-1})]{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_p} \left[\frac{\mathbf{C}}{\mathbf{S}}\right] \qquad \text{where} \quad \mathbf{S} = \mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_p}.$$

- Diagonalizable matrices: when $\mathbf{C}$ is diagonal.

F12

(**Lecture** 17) S-2 Similar matrices

Similarity
$\mathbf{A}$ and $\mathbf{C}$ are *similar* if there is an invertible $\mathbf{S}$ such that

$$\mathbf{C} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$$

If $\mathbf{A}$ and $\mathbf{C}$ are similar (*see demos in the book*):

- The same determinant: $\det \mathbf{A} = \det \mathbf{C}$
- The same caracteristic polinomial: $|\mathbf{A} - \lambda\mathbf{I}| = |\mathbf{C} - \lambda\mathbf{I}|$
- The same eigenvalues (same *algebraic* and *geometric* multiplicities).
- The same trace.

***Mirror* inverse transf.:** $\left(\mathbf{I}_{(\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k)}\right)^{-1} = {}_{esp(\boldsymbol{\tau}_k^{-1}\cdots\boldsymbol{\tau}_1^{-1})}\mathbf{I}$

$$\mathbf{I} = {}_{\underset{[(-\alpha)\mathbf{j}+\mathbf{i}]}{\boldsymbol{\tau}}}\mathbf{I}_{\underset{[(\alpha)\mathbf{i}+\mathbf{j}]}{\boldsymbol{\tau}}} = {}_{\underset{[(\frac{1}{\alpha})\mathbf{j}]}{\boldsymbol{\tau}}}\mathbf{I}_{\underset{[(\alpha)\mathbf{j}]}{\boldsymbol{\tau}}} \quad \Rightarrow \quad \mathbf{A} \text{ similar to } {}_{esp(\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k)^{-1}}\mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}$$

F13

(**Lecture** 17) S-3 **Block diagonalizing a matrix (toothed matrix)**

$$\text{Consider} \qquad \mathbf{A} = \left[\begin{array}{c||c} \mathbf{C} & \\ \hline * & \mathbf{L} \end{array}\right] \in \mathbb{C}^{n\times n} \qquad \text{where}$$

$\mathbf{C}$ (of order m) is *singular* and $\mathbf{L}$ is *full rank lower triangular*; then there exists an invertible $\mathbf{R}$ such that

$$\mathbf{R}^{-1}\mathbf{A}\mathbf{R} \quad = \left[\begin{array}{c|c||cccc} \underset{m\times(m-1)}{*} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & & & & \\ \hline & d_{m+1} & \beta_{m+1} & & & \\ & d_{m+2} & * & \beta_{m+2} & & \\ * & \vdots & * & * & \ddots & \\ & d_n & * & * & \cdots & \beta_n \end{array}\right]$$

$$\left(\cdots_{[(-\alpha_j)\overset{\tau}{} m+j]}\cdots\right)\mathbf{A}\left(\cdots_{[(\alpha_j)\overset{\tau}{} j+m]}\cdots\right); \qquad \boldsymbol{j} = 1, \ldots, m-1.$$

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(**Lecture** 17) S-4 **Block diagonalizing a matrix (toothed matrix)**

$$\text{Consider} \qquad \mathbf{A} = \left[\begin{array}{c||c} \mathbf{C} & \\ \hline * & \mathbf{L} \end{array}\right] \in \mathbb{C}^{n\times n} \qquad \text{where}$$

$\mathbf{C}$ (of order m) is singular and $\mathbf{L}$ is full rank lower triangular, then there exists $\mathbf{S} = \mathbf{RP}$ (invertible) such that

$$\mathbf{P}^{-1}\mathbf{R}^{-1}\mathbf{A}\mathbf{R}\mathbf{P} = \left[\begin{array}{c|c||cccc} \underset{m\times(m-1)}{*} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & & & & \\ \hline & 0 & \beta_{m+1} & & & \\ & 0 & * & \beta_{m+2} & & \\ * & \vdots & * & * & \ddots & \\ & 0 & * & * & \cdots & \beta_n \end{array}\right]$$

$$\left(\cdots_{[(-\alpha_j)\overset{\tau}{} m+j]}\cdots\right)\mathbf{R}^{-1}\mathbf{A}\mathbf{R}\left(\cdots_{[(\alpha_j)\overset{\tau}{} j+m]}\cdots\right); \qquad \boldsymbol{j} = m+1, \ldots, n.$$

F15

(**Lecture** 17) S-5 A very simple example

Example 1. Consider $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ with eigenvalues 0, 1 and 1.

$$\begin{bmatrix} \mathbf{A} \\ \hline \mathbf{I} \end{bmatrix} \xrightarrow[0\mathbf{I}]{(-)} \left[\begin{array}{ccc} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(1)\mathbf{1}+\mathbf{2}] \\ [(2)\mathbf{3}+\mathbf{2}] \\ [\mathbf{2}\rightleftharpoons\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow[\substack{[\mathbf{2}\rightleftharpoons\mathbf{3}] \\ [(-2)\mathbf{2}+\mathbf{3}] \\ [(-1)\mathbf{2}+\mathbf{1}]}]{\tau} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow[0\mathbf{I}]{(+)} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] = \begin{bmatrix} \mathbf{C} \\ \hline \mathbf{S} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{diagonal}}$$

F16

(**Lecture** 17) S-6 A not so simple example

Example 2. Consider $\mathbf{A} = \begin{bmatrix} -2 & 0 & 3 \\ 3 & -2 & -9 \\ -1 & 2 & 6 \end{bmatrix}$ with eigenvalues *1*, *1* and *0*.

$$\xrightarrow[1\mathbf{I}]{(-)} \left[\begin{array}{ccc} -3 & 0 & 3 \\ 3 & -3 & -9 \\ -1 & 2 & 5 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(1)\mathbf{1}+\mathbf{3}] \\ [(-2)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} -3 & 0 & 0 \\ 3 & -3 & 0 \\ -1 & 2 & 0 \\ \hline 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(2)\mathbf{3}+\mathbf{2}] \\ [(-1)\mathbf{3}+\mathbf{1}]}]{\tau} \left[\begin{array}{ccc} -2 & -2 & 0 \\ 1 & 1 & 0 \\ -1 & 2 & 0 \\ \hline 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[1\mathbf{I}]{(+)} \left[\begin{array}{ccc} -1 & -2 & 0 \\ 1 & 2 & 0 \\ -1 & 2 & 1 \\ \hline 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[1\mathbf{I}]{(-)} \left[\begin{array}{ccc} -2 & -2 & 0 \\ 1 & 1 & 0 \\ \hline -1 & 2 & 0 \\ \hline 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc} -2 & 0 & 0 \\ 1 & 0 & 0 \\ \hline -1 & 3 & 0 \\ \hline 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{2}+\mathbf{1}]}]{\tau} \left[\begin{array}{ccc} -1 & 0 & 0 \\ 1 & 0 & 0 \\ \hline -1 & 3 & 0 \\ \hline 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[1\mathbf{I}]{(+)} \left[\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 3 & 1 \\ \hline 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[0\mathbf{I}]{(-)} \left[\begin{array}{ccc} 0 & 0 & 0 \\ \hline 1 & 1 & 0 \\ -1 & 3 & 1 \\ \hline 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-1)\mathbf{2}+\mathbf{1}] \\ [(4)\mathbf{3}+\mathbf{1}]}} \left[\begin{array}{ccc} 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 3 & 1 \\ \hline 6 & -1 & 1 \\ -9 & 1 & -2 \\ 4 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-4)\mathbf{1}+\mathbf{3}] \\ [(1)\mathbf{1}+\mathbf{2}]}]{\tau} \left[\begin{array}{ccc} 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 3 & 1 \\ \hline 6 & -1 & 1 \\ -9 & 1 & -2 \\ 4 & 0 & 1 \end{array}\right] \xrightarrow[0\mathbf{I}]{(+)} \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \\ \hline 6 & -1 & 1 \\ -9 & 1 & -2 \\ 4 & 0 & 1 \end{array}\right]$$

F17

(**Lecture** 17) S-7 **Every matrix is similar to a toothed matrix**

For every $\mathbf{A}$ there exists $\mathbf{S}$ such that

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{C} \qquad \Rightarrow \qquad \mathbf{A}\mathbf{S} = \mathbf{S}\mathbf{C}$$

where $\mathbf{C}$, toothed, has the eigenvalues on the diagonal

Example 3.

$$\begin{bmatrix} 6 & -1 & 1 \\ -9 & 1 & -2 \\ 4 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -2 & 0 & 3 \\ 3 & -2 & -9 \\ -1 & 2 & 6 \end{bmatrix} \begin{bmatrix} 6 & -1 & 1 \\ -9 & 1 & -2 \\ 4 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}}_{\text{toothed}}$$

Consequences

- $\sum \lambda_i = \operatorname{tr}(\mathbf{A})$ and $\prod \lambda_i = \det \mathbf{A}$
- $\mathbf{A}\mathbf{S}_{|j} = \mathbf{S}\mathbf{C}_{|j} \qquad \Rightarrow \qquad$ for j such that $\mathbf{C}_{|j} = \lambda_i \mathbf{I}_{|j}$:

$$\mathbf{A}(\mathbf{S}_{|j}) = \lambda_i(\mathbf{S}_{|j}) \qquad \Rightarrow \qquad \mathbf{S}_{|j} \text{ is an eigenvector.}$$

F18

(**Lecture** 17) S-8 **Back to the simple, "toothless" example**

Consider $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ with eigenvalues 0, 1 and 1.

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] \xrightarrow[0\mathbf{I}]{(-)} \left[\begin{array}{ccc} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(1)\mathbf{1}+\mathbf{2}] \\ [(2)\mathbf{3}+\mathbf{2}] \\ [\mathbf{2}\rightleftharpoons\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow[\substack{\tau \\ [\mathbf{2}\rightleftharpoons\mathbf{3}] \\ [(-2)\mathbf{2}+\mathbf{3}] \\ [(-1)\mathbf{2}+\mathbf{1}]}]{} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow[0\mathbf{I}]{(+)} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] = \left[\frac{\mathbf{C}}{\mathbf{S}}\right]$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}(\mathbf{S}_{|j}) = \lambda_i(\mathbf{S}_{|j}) \qquad \Rightarrow \qquad \mathbf{S}_{|j} \text{ is an eigenvector.}$$

F19

(**Lecture** 17) S-9 **Diagonalizable matrices**

- A matrix is diagonalizable **if and only if** *algebraic* and *geometric* multiplicities are equal for each eigenvalue
- If there are no repeated eigenvalues, there are no "teeth" either
- When there are no repeated eigenvalues $\underset{n\times n}{\mathbf{A}}$ is diagonalizable (is sure to have n independent eigenvectors)

F20

(**Lecture** 17) S-**10** Diagonalizing a matrix

- Find the spectrum: $\{\lambda_1, \lambda_2, \ldots\}$
- Find the *algebraic multiplicity* of each eigenvalue: $\mu(\lambda_i)$

then choose one of these alternatives:

1. teething the matrix (implemented in NAcAL)
2. ...or for every λ_i
   - find the eigenspace

$$\mathcal{E}_{\lambda_i}(\mathbf{A}) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \middle| \mathbf{A}\boldsymbol{x} = \lambda_i \boldsymbol{x} \right\} = \mathcal{N}(\mathbf{A} - \lambda_i \mathbf{I}).$$

   - check $\mu(\lambda_i) = \dim \mathcal{E}_{\lambda_i}(\mathbf{A})$ (algebraic and geometric multiplicities are equal)

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix}; \quad \mathbf{S} = \Big[\text{Basis for } \mathcal{E}_{\lambda_1}(\mathbf{A}) \mathbin{+\!\!+} \cdots \mathbin{+\!\!+} \text{Basis for } \mathcal{E}_{\lambda_k}(\mathbf{A}) \Big]$$

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D} \quad \Leftrightarrow \quad \mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$$

F21

(**Lecture** 17) S-**11** Matrix powers

If $\mathbf{A}\boldsymbol{x} = \lambda \boldsymbol{x}$ then $\mathbf{A}^2\boldsymbol{x} = \mathbf{A}\mathbf{A}\boldsymbol{x} = \mathbf{A}(\lambda\boldsymbol{x}) = \lambda\mathbf{A}\boldsymbol{x} =$

- What can I say about the eigenvectors?

- What is the relationship between the eigenvalues of $\mathbf{A}$ and those of $\mathbf{A}^2$

In a matrix form (if $\mathbf{A}$ is diagonalizable, $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$):

$$\mathbf{A}^2 = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}\mathbf{S}\mathbf{D}\mathbf{S}^{-1} = \mathbf{S}\mathbf{D}^2\mathbf{S}^{-1}$$

In general, for, $n \in \mathbb{Z}$, $n \geq 0\ldots$ $\quad \mathbf{A}^n =$
what about $\mathbf{A}$ both diagonalizable and invertible?

F22

The lecture ends here

Questions of the Lecture 17

(L-17) Question 1. Factor these two matrices into $\mathbf{S}\mathbf{D}\mathbf{S}^{-1}$;

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

(Strang, 2006, exercise 15 from section 5.2.)

(L-17) Question 2. Which of these matrices cannot be diagonalized?

(a)

$$\mathbf{A}_1 = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$$

(b)

$$\mathbf{A}_2 = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}$$

(c)

$$\mathbf{A}_3 = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

(Strang, 2006, exercise 5 from section 5.2.)

(L-17) QUESTION 3. If $\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ find $\mathbf{A}^{100}$ by diagonalizing $\mathbf{A}$.
(Strang, 2006, exercise 7 from section 5.2.)

(L-17) QUESTION 4. If the eigenvalues of $\underset{3\times 3}{\mathbf{A}}$ are 1, 1 and 2, which of the following are certain to be true? Give a reason if true or a counterexample if false:

(a) $\mathbf{A}$ is invertible.
(b) $\mathbf{A}$ is diagonalizable.
(c) $\mathbf{A}$ is not diagonalizable

(Strang, 2006, exercise 11 from section 5.2.)

(L-17) QUESTION 5. Let $\mathbf{A}$ be the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

(a) (1^{pts}) Determine if $\mathbf{A}$ is diagonalizable, and if so, diagonalize it.
(b) (0.5^{pts}) Compute $\left(\mathbf{A}^6\right)\boldsymbol{v}$, where $\boldsymbol{v} = \left(0, \quad 0, \quad 0, \quad 1,\right)$.
(c) (0.5^{pts}) Using the the eigenvalues found in part (a) justify that $\mathbf{A}$ is invertible.
(d) (0.5^{pts}) What is the relation between the eigenvalues of $\mathbf{A}$ and the eigenvalues of $\mathbf{A}^{-1}$?

(L-17) QUESTION 6. Si $\mathbf{A} = \mathbf{SDS}^{-1}$; entonces $\mathbf{A}^3 = (\quad)(\quad)(\quad)$ y $\mathbf{A}^{-1} = (\quad)(\quad)(\quad)$.
(Strang, 2006, exercise 16 from section 5.2.)

(L-17) QUESTION 7. Considere la matriz

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

(a) Encuentre los autovalores de $\mathbf{A}$
(b) Encuentre los auto-vectores de $\mathbf{A}$
(c) Diagonalice $\mathbf{A}$: escríbalo como $\mathbf{A} = \mathbf{SDS}^{-1}$.

(L-17) QUESTION 8. ¿Falso o verdadero? Si los autovalores de $\mathbf{A}$ son 2, 2 y 3 entonces sabemos que la matriz es

(a) Invertible
(b) Diagonalizable
(c) No diagonalizable.

(L-17) QUESTION 9. Sean las matrices

$$\mathbf{A}_1 = \begin{bmatrix} 8 & \\ & 2 \end{bmatrix}; \qquad \mathbf{A}_2 = \begin{bmatrix} 9 & 4 \\ & 1 \end{bmatrix}; \qquad \mathbf{A}_3 = \begin{bmatrix} 10 & 5 \\ -5 & \end{bmatrix}$$

(a) Complete dichas matrices de modo que en los tres casos $\det \mathbf{A}_i = 25$. Así, la traza es en todos los casos igual a 10, y por tanto para las tres matrices el único auto-valor $\lambda = 5$ está repetido dos veces ($\lambda^2 = 25$ y $\lambda + \lambda = 10$ implica $\lambda = 5$).
(b) Encuentre un vector característico con $\mathbf{A}\boldsymbol{x} = 5\boldsymbol{x}$. Estas tres matrices no son diagonalizable porque no hay un segundo auto-vector linealmente independiente del primero.

(Strang, 2006, exercise 27 from section 5.2.)

(L-17) Question 10. Factorice las siguientes matrices en $\mathbf{S}\ \mathbf{D}\ \mathbf{S}^{-1}$

(a) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$

(Strang, 2006, exercise 1 from section 5.2.)

(L-17) Question 11. Encuentre la matriz $\mathbf{A}$ cuyos autovalores son 1 y 4, cuyos autovectores son $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ y $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ respectivamente.
(Strang, 2006, exercise 2 from section 5.2.)

(L-17) Question 12. Si los elementos diagonales de una matriz triangular superior de orden 3×3 son 1, 2 y 7, ¿puede saber si la matriz es diagonalizable? ¿Quién es $\mathbf{D}$?
(Strang, 2006, exercise 4 from section 5.2.)

(L-17) Question 13.

(a) Encuentre los autovalores y auto-vectores de la matriz $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

(b) Explique por qué (o por qué no) la matriz $\mathbf{A}$ es diagonalizable.

(L-17) Question 14. Sea $\mathbf{A}$ una matriz 3×3. Asuma que sus autovalores son 1 y 0, que una base de los autovectores asociados a $\lambda = 1$ son $[1, 0, 1]$ y $[0, 0, 1]$; mientras que los asociados a $\lambda = 0$ son paralelos a $[1, 1, 2]$.

(a) ¿Es $\mathbf{A}$ diagonalizable? En caso afirmativo escriba la matriz diagonal $\mathbf{D}$ y la matriz $\mathbf{S}$ tales que $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$.

(b) Encuentre $\mathbf{A}$.

(L-17) Question 15. Let $\mathbf{A}$ be a 2×2 matrix such that $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ is an eigenvector for $\mathbf{A}$ with eigenvalue 2, and $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is another eigenvector for $\mathbf{A}$ with eigenvalue -2. If $\boldsymbol{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, compute $\left(\mathbf{A}^3\right)\boldsymbol{v}$.

End of Questions of the Lecture 17

LECTURE 18: Symmetric matrices and orthogonal diagonalization

Lecture 18

(**Lecture** 18) S-**1** Highlights of Lesson 18

Highlights of *Lesson 18*

- Symetric matrices $\mathbf{A} = \mathbf{A}^{\mathsf{T}}$
  - Eigenvalues and eigenvectors
- Introd. positive Definiteness matrices

F23

(**Lecture** 18) S-**2** Symmetric matrices $\mathbf{A} = \mathbf{A}^{\mathsf{T}}$

what's special about $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ when $\underset{n\times n}{\mathbf{A}}$ is symmetric?

1. A symmetric matrix has only REAL EIGENVALUES
2. n EIGENVECTORS *can be choosen* ORTHOGONAL

(always diagonalizable)

The usual diagonalizable case:

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D} \quad \longleftrightarrow \quad \mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$$

Symmetric case:
I can choose perpendicular unit eigenvectors (ortho*normal* columns of $\mathbf{S} = \mathbf{Q}$)

$$(\text{if } \mathbf{A} = \mathbf{A}^{\mathsf{T}}) \qquad \mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}^{-1} = \mathbf{Q}\mathbf{D}\mathbf{Q}^{\mathsf{T}} \qquad \text{Spectral theorem}$$

Orthogonally diagonalizable.

F24

(**Lecture** 18) S-**3** Eigenspaces are orthogonal for symmetric matrices

Eigenvectors (corresponding to different eigenvalues) of a symmetric matrix are orthogonal.

Proof. Consider $\mathbf{A}\boldsymbol{x} = \lambda_1\boldsymbol{x}$ and $\mathbf{A}\boldsymbol{y} = \lambda_2\boldsymbol{y}$ (with $\lambda_1 \neq \lambda_2$). then

$$\lambda_1\boldsymbol{x}\cdot\boldsymbol{y} \;=\; \mathbf{A}\boldsymbol{x}\cdot\boldsymbol{y} \;=\; \boldsymbol{x}(\mathbf{A}^{\mathsf{T}})\boldsymbol{y} \;=\; \boldsymbol{x}\mathbf{A}\boldsymbol{y} \;=\; (\boldsymbol{x}\cdot\boldsymbol{y})\lambda_2.$$

Since $\lambda_1 \neq \lambda_2$ then:

$$\lambda_1(\boldsymbol{x}\cdot\boldsymbol{y}) - \lambda_2(\boldsymbol{x}\cdot\boldsymbol{y}) = 0 \quad \Longrightarrow \quad (\lambda_1 - \lambda_2)\boldsymbol{x}\cdot\boldsymbol{y} = 0 \quad \Longrightarrow \quad \boldsymbol{x}\cdot\boldsymbol{y} = 0.$$

□

F25

(**Lecture** 18) S-4 Quadratic forms

Quadratic form:

$$\boldsymbol{x}\mathbf{A}\boldsymbol{x}; \quad \text{with} \quad \mathbf{A}^\intercal = \mathbf{A}$$

Since $\mathbf{A} = \mathbf{QDQ}^\intercal$ (with $\mathbf{Q}^\intercal\mathbf{Q} = \mathbf{QQ}^\intercal = \mathbf{I}$), then

$$\boldsymbol{x}\mathbf{A}\boldsymbol{x} = \boldsymbol{x}\mathbf{QDQ}^\intercal\boldsymbol{x} = (\mathbf{Q}^\intercal\boldsymbol{x})\mathbf{D}(\mathbf{Q}^\intercal\boldsymbol{x}) \text{ (weighted sum of squares)}$$

Positive definite quadratic form:

$$\boldsymbol{x}\mathbf{A}\boldsymbol{x} > 0 \quad \forall \boldsymbol{x} \neq \mathbf{0} \qquad \Longleftrightarrow \qquad \lambda_i > 0, \quad i = 1:n.$$

then we also say $\mathbf{A}$ is positive definite.

F26

(**Lecture** 18) S-5 Positive definite matrices

Meaning:

$$\boldsymbol{x}\mathbf{A}\boldsymbol{x} > 0 \qquad (\text{except for } \boldsymbol{x} = \mathbf{0})$$

Some properties

Consider a positive definite symmetric $\mathbf{A}$: What about $\mathbf{A}^{-1}$?

$\mathbf{A} = \mathbf{QDQ}^{-1} = \mathbf{QDQ}^\intercal$

Consider two positive definite symmetric matrices $\mathbf{A}$, $\mathbf{B}$: What about $\mathbf{A} + \mathbf{B}$?

the answer must be...

F27

(**Lecture** 18) S-6 The matrix product $\mathbf{A}^\intercal\mathbf{A}$

Consider the rectangular matrix $\underset{m\times n}{\mathbf{A}}$. Is $\mathbf{A}^\intercal\mathbf{A}$ positive definite?

$$\boldsymbol{x}(\mathbf{A}^\intercal\mathbf{A})\boldsymbol{x} =$$

It can only be 0 when $\mathbf{A}\boldsymbol{x}$ is $\mathbf{0}$

How can we guarantee that $\mathbf{A}\boldsymbol{x} \neq \mathbf{0}$ when $\boldsymbol{x} \neq \mathbf{0}$?

F28

(**Lecture** 18) S-7 Symmetric matrices: signs of eigenvalues

are all λ_i positive? are they negative?

Computing eigenvalues of $\underset{5\times 5}{\mathbf{A}}$ is impossible in general! (5th degree polynomial)

Good news: The signs of the pivots of echelon form are the same as the signs of the eigenvalues λ_i (if we do not change the sign of the determinant with *Type II* elementary transformations)

$$\text{num. of positive pivots} = \text{num. of positive eigenvalues}$$

F29

(**Lecture** 18) S-8 Positive definite symmetric matrices

- All eigenvalues are:
- All pivots are:

$$\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$$

Pivots:

What is the sign of each eigenvalue?

$$\lambda^2 - 8\lambda + 11 = 0 \rightarrow \qquad \lambda = 4 \pm \sqrt{5} > 0$$

F30

Summary (for symmetric matrices):

1. Symmetric matrices have *real eigenvalues* and *perpendicular eigenvectors* can be choosen
2. $\mathbf{A} = \mathbf{QDQ^T}$ where $\mathbf{Q}$ is orthogonal
3. $\mathbf{A}$ is symmetric if and only if it is *orthogonally* diagonalizable
4. The signs of the pivots in the echelon form are same as the signs of the eigenvalues λ_i (only if we do not change the sign of the determinant with *Type II* elementary transformations)

The lecture ends here

Questions of the Lecture 18

(L-18) Question 1. Write $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ in the form $\mathbf{QDQ^T}$ of the spectral theorem:

(a) $\mathbf{A} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\mathbf{C} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

(Strang, 2006, exercise 11 from section 5.5.)

(L-18) Question 2. Find the eigenvalues and the unit eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(Strang, 2003, exercise 3 from section 6.4.)

(L-18) Question 3. Find an orthonormal $\mathbf{Q}$ that diagonalizes this symmetric matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$$

(Strang, 2003, exercise 5 from section 6.4.)

(L-18) Question 4. Suppose $\mathbf{A}$ is a symmetric 3 by 3 matrix with eigenvalues 0, 1,2.

(a) What properties can be guaranteed for the corresponding unit eigenvectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$
(b) In terms of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, describe the nullspace, left nullspace, row space, and column space of $\mathbf{A}$.
(c) Find a vector $\boldsymbol{x}$ that satisfies $\mathbf{A}\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$. Is $\boldsymbol{x}$ unique?
(d) Under what conditions on $\boldsymbol{b}$ does $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ have a solution?
(e) If $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are the columns of $\mathbf{S}$, what are $\mathbf{S}^{-1}$ and $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$.

(Strang, 2006, exercise 13 from section 5.5.)

(L-18) Question 5. Escriba un hecho destacado sobre los valores característicos de cada uno de estos tipos de matrices:

(a) Una matriz simétrica real.
(b) Una matriz diagonalizable tal que $\mathbf{A}^n \to \mathbf{0}$ cuando $n \to \infty$.
(c) Una matriz no diagonalizable
(d) Una matriz singular

(Strang, 2006, exercise 16 from section 5.5.)

(L-18) Question 6. Sean

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}; \qquad \mathbf{B} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

(a) Encuentre los valores característicos de $\mathbf{A}$ (recuerde que $i^2 = -1$).
(b) Encuentre los valores característicos de $\mathbf{B}$ (en este caso quizá le resulte más sencillo encontrar primero los autovectores, y deducir entonces los autovalores).

(Strang, 2006, exercise 14 from section 5.5.)

(L-18) Question 7. Si $\mathbf{A}^3 = \mathbf{0}$ entonces los autovalores de $\mathbf{A}$ deben ser ________. De un ejemplo tal que $\mathbf{A} \neq \mathbf{0}$. Ahora bien, si $\mathbf{A}$ es además simétrica, demuestre que entonces $\mathbf{A}^3$ es necesariamente $\mathbf{0}$.

(L-18) Question 8. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} a & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

(a) Prove that $\mathbf{A}$ is not diagonalizable when $a = 3$.
(b) Is $\mathbf{A}$ diagonalizable when $a = 2$? (explain). If it is diagonalizable, find an eigenvalue diagonal matrix $\mathbf{D}$ and an eigenvector matrix $\mathbf{S}$ such as $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$.
(c) Is $\mathbf{A}^\intercal\mathbf{A}$ diagonalizable for any value a? Is it possible to find a full set of orthonormal eigenvectors of $\mathbf{A}^\intercal\mathbf{A}$?
(d) Find all posible values a such as $\mathbf{A}$ is invertible and diagonalizable.

(L-18) Question 9. Sea la matriz

$$\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};$$

(a) Exprese $\mathbf{B}$ en la forma $\mathbf{B} = \mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}^\intercal$ del teorema espectral.
(b) ¿Es $\mathbf{B}$ diagonalizable? Si no lo es, diga las razones; y en caso contrario genere una matriz $\mathbf{S}$ que diagonalice a $\mathbf{B}$.

(L-18) Question 10. Suppose the vectors $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$ form an orthonormal basis for $\mathbb{R}^3$ and the matrix $\mathbf{A}$ satisfies $\mathbf{A}\boldsymbol{q}_1 = (1, 0, 0,)$, $\mathbf{A}\boldsymbol{q}_2 = (0, 1, 0,)$, and $\mathbf{A}\boldsymbol{q}_3 = (0, 0, 1,)$.

(a) (0.5^{pts}) Write the matrix $\mathbf{A}$ explicitly in terms of the vectors $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$.
(b) (1^{pts}) Write down all possibilities for $\det \mathbf{A}$.
(c) Which of the following statementes are correct: The eigenvalues of $\mathbf{A}$ must...

- be real numbers.
- be positive real numbers.
- be imaginary numbers.
- have absolute value $|\lambda| = 1$.

End of Questions of the Lecture 18

LECTURE 19: Diagonalization by congruence. Definite Matrices

Lecture 19

(**Lecture** 19) S-1 Highlights of Lesson 19

Highlights of *Lesson 19*

- Positive and Negative (semi)definite matrices
- Completing the squares
- Diagonalization by congruence

F32

(**Lecture** 19) S-2 Quadratic forms

- Positive definite: $\forall \boldsymbol{x} \neq \mathbf{0} \;\Rightarrow\; \boldsymbol{x}\mathbf{A}\boldsymbol{x} > 0.$
- Positive semi-definite: $\forall \boldsymbol{x} \neq \mathbf{0} \;\Rightarrow\; \boldsymbol{x}\mathbf{A}\boldsymbol{x} \geq 0.$
- Negative definite: $\forall \boldsymbol{x} \neq \mathbf{0} \;\Rightarrow\; \boldsymbol{x}\mathbf{A}\boldsymbol{x} < 0.$
- Negative semi-definite: $\forall \boldsymbol{x} \neq \mathbf{0} \;\Rightarrow\; \boldsymbol{x}\mathbf{A}\boldsymbol{x} \leq 0.$
- Indefinite: neither positive semi-definite, nor negative semi-definite.

F33

Example 4. What number do I have to put there for the matrix $\mathbf{A}$ to be singular?

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & \end{bmatrix}$$

- Eigenvalues:
- Leading principal minors:
- For the following quadratic form

$$q_{\mathbf{A}}(\boldsymbol{x}) = \boldsymbol{x}\mathbf{A}\boldsymbol{x} = (x, \quad y,) \begin{bmatrix} 2 & 6 \\ 6 & \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 12xy + \quad y^2$$

Is there a $\boldsymbol{x} \neq \mathbf{0}$ such that $\boldsymbol{x}\mathbf{A}\boldsymbol{x} = 0$?

Example 5.

$$\text{If } \mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix} \quad \text{then} \quad (x, \quad y,) \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 12xy + \quad y^2$$

- Are there numbers x and y that make $\boldsymbol{x}\mathbf{A}\boldsymbol{x}$ negative?
- Does the function go through the origin?
- When $y = 0$ and $x = 1$, is it possitive? (and when $x = -1$?)

- When $x = 0$ and $y = 1$, is it possitive? (and when $y = -1$?)
- Is it always positive?

$(0, 0,)$ **saddle point**: minimum in some directions, maximum in others.

$$\lambda_1 = -2, \quad \begin{pmatrix} -6 \\ 4 \end{pmatrix}; \qquad \lambda_1 = 11, \quad \begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

Example 6.

$$\text{If } \mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \text{ then } \begin{pmatrix} x, & y, \end{pmatrix} \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 12xy + 20y^2$$

Positive definite.

Does it pass the tests?

- Are the leading principal minors positive?
- Are the eigenvalues positive?

$$q_{\mathbf{A}}(\boldsymbol{x}) = \boldsymbol{x}\mathbf{A}\boldsymbol{x} > 0 \qquad \text{for all } \boldsymbol{x} \neq \mathbf{0}$$

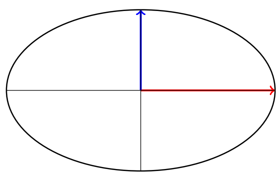
(**Lecture** 19) S-3 **Completing the squares**

If we could express $q(\boldsymbol{x})$ as a sum of squares, we would know whether $q(\boldsymbol{x})$ is positive definite.

Let's complete the square!

- $q(x, y) = 2x^2 + 12xy + 20y^2 \ = \ 2(x + \quad y)^2 +$
- $q(x, y) = 2x^2 + 12xy + 7y^2$
- $q(x, y) = 2x^2 + 12xy + 18y^2$
- $q(x, y) = 2x^2 + 12xy + 200y^2$ (graph)

If positive definite: $q(x, y) = a$; $\quad a > 0$: ellipse



is $q(x, y, z, w, t) = 2t^2 - 2tx - 2tz + w^2 - 2wy + 2x^2 - 2xy + 2y^2 + z^2$ positive definite? ☹☹☹☹☹!!!???

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```
x, y, z, w, t  = sympy.symbols('x y z w t')
B = Matrix([[-1,1,0,0,0],[1,0,0,0,-1],[0,-1,0,1,0],[0,0,1,0,-1]])
A = ~B*B
v = Vector([x,y,z,w,t])
q = sympy.factor(v*A*v)
D = DiagonalizaC(A)
dispElimFyC(A,D.pasos)
q
```

(**Lecture** 19) S-4 Congruent matrices

$\mathbf{A}$ and $\mathbf{C}$ are congruent if there exists an invertible $\mathbf{B}$ such that $\boxed{\mathbf{C} = \mathbf{B}^{\mathsf{T}}\mathbf{A}\mathbf{B}}$

Diagonalization by congruence
For each $\mathbf{A}$ (symmetric) exists $\mathbf{B} = \mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}$ *(invertible)* such that

$$\mathbf{D} = \mathbf{B}^{\mathsf{T}}\mathbf{A}\mathbf{B} \quad \text{is diagonal} \qquad (\mathbf{B}^{\mathsf{T}} = {}_{\boldsymbol{\tau}_k\cdots\boldsymbol{\tau}_1}\mathbf{I})$$

Spectral Theorem: ¡Diagonalization by similarity and congruence!

$$\mathbf{D} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \mathbf{Q}^{\mathsf{T}}\mathbf{A}\mathbf{Q}.$$

Hence, every quadratic form can be written as a sum of squares

$$\boldsymbol{x}\mathbf{A}\boldsymbol{x} \;=\; \boldsymbol{x}\left(\mathbf{B}^{-1}\right)^{\mathsf{T}}\mathbf{D}\mathbf{B}^{-1}\boldsymbol{x} \;=\; \boldsymbol{y}\mathbf{D}\boldsymbol{y}; \qquad \text{where} \qquad \boldsymbol{y} = \mathbf{B}^{-1}\boldsymbol{x}.$$

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(**Lecture** 19) S-5 Completing the squares

$$2x^2 + 12xy + 20y^2$$

$$\begin{bmatrix}2&6\\6&20\end{bmatrix} \xrightarrow{\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}} \begin{bmatrix}2&0\\6&2\end{bmatrix} \xrightarrow[\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}]{} \begin{bmatrix}2&0\\0&2\end{bmatrix};$$

therefore, we get:

$$\begin{bmatrix}2&0\\0&2\end{bmatrix} = \mathbf{D} = \mathbf{E}^{\mathsf{T}}\mathbf{A}\mathbf{E} = \begin{bmatrix}1&0\\-3&1\end{bmatrix}\begin{bmatrix}2&6\\6&20\end{bmatrix}\begin{bmatrix}1&-3\\0&1\end{bmatrix}$$

hence $\mathbf{A} = (\mathbf{E}^{\mathsf{T}})^{-1}\mathbf{D}\mathbf{E}^{-1}$ so

$$\boldsymbol{x}\mathbf{A}\boldsymbol{x} = (x, \quad y,)\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}1&3\\0&1\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \left(\boldsymbol{x}\,(\mathbf{E}^{-1})^{\mathsf{T}}\right)\mathbf{D}\left(\mathbf{E}^{-1}\boldsymbol{x}\right)$$

$$= \Big((x+3y), \quad y,\Big)\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{pmatrix}(x+3y)\\y\end{pmatrix} = 2(x+3y)^2 + 2y^2$$

F39

(**Lecture** 19) $\boxed{\text{S-6}}$ example 3 by 3

$$\text{Is } \mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \text{ positive definite?}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow[{[(\frac{1}{2})\mathbf{1}+\mathbf{2}]}^{\tau}]{{[(\frac{1}{2})\mathbf{1}+\mathbf{2}]}^{\tau}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow[{[(\frac{2}{3})\mathbf{2}+\mathbf{3}]}^{\tau}]{{[(\frac{2}{3})\mathbf{2}+\mathbf{3}]}^{\tau}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

$$\boldsymbol{x}\mathbf{A}\boldsymbol{x} = 2x^2 + 2y^2 + 2z^2 - 2xy - 2yz > 0$$

$\boldsymbol{x}\mathbf{A}\boldsymbol{x} = 1$: (ellipsoid) axes are eigenvectors $\mathbf{A} = \mathbf{Q}^{\mathsf{T}}\boldsymbol{\lambda}\mathbf{Q}$ $\boxed{\text{F40}}$

Si $\mathbf{B} = \mathbf{I}_{{[(\frac{1}{2})\mathbf{1}+\mathbf{2}]}^{\tau}{[(\frac{2}{3})\mathbf{2}+\mathbf{3}]}^{\tau}}$, es decir, si

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{{[(\frac{1}{2})\mathbf{1}+\mathbf{2}]}^{\tau} \\ {[(\frac{2}{3})\mathbf{2}+\mathbf{3}]}}} \begin{bmatrix} 1 & -1/2 & 1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{B} \qquad \Rightarrow \qquad \mathbf{B}^{-1} = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix},$$

tenemos que la forma cuadrática $\boldsymbol{x}\mathbf{A}\boldsymbol{x}$ se puede expresar como:

$$\boldsymbol{x}\mathbf{A}\boldsymbol{x} = (x, \quad y, \quad z,) \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & 3/2 & \\ & & 4/3 \end{bmatrix} \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad = \boldsymbol{x}\left(\mathbf{B}^{-1}\right)^{\mathsf{T}}\mathbf{D}\mathbf{B}^{-1}\boldsymbol{x}$$

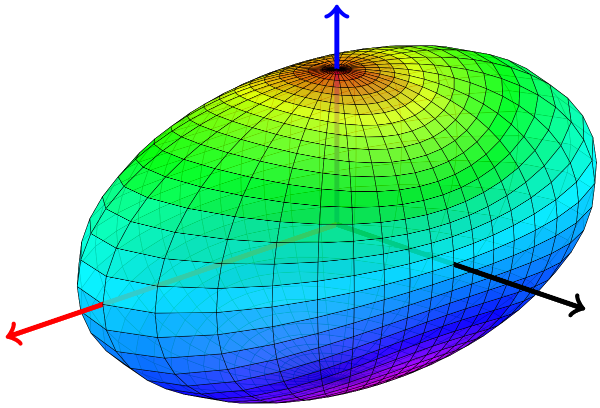
$$= \Big((x-1/2y), \quad (y-2/3z), \quad z,\Big) \begin{bmatrix} 2 & & \\ & 3/2 & \\ & & 4/3 \end{bmatrix} \begin{pmatrix} (x-1/2y) \\ (y-2/3z) \\ z \end{pmatrix} \qquad = \Big(\mathbf{B}^{-1}\boldsymbol{x}\Big) \cdot \mathbf{D} \cdot \Big(\mathbf{B}^{-1}\boldsymbol{x}\Big)$$

$$= 2\,(x-1/2y)^2 \; + \; 3/2\,(y-2/3z)^2 \; + \; 4/3z^2 \qquad = \sum_{j|}\mathbf{D}_{|j} \cdot (y_j)^2 \quad (\text{con} \;\; \boldsymbol{y} = \mathbf{B}^{-1}\boldsymbol{x});$$

es una suma de tres términos al cuadrado, cada uno de ellos multiplicado por uno de los pivotes de $\mathbf{D}$; como todos los pivotes son positivos, la forma cuadrática es evidentemente definida positiva.

(**Lecture** 19) S-**7** **Positive definite matrices and ellipsoids: example 3 by 3**

- The region ($\boldsymbol{x}\mathbf{A}\boldsymbol{x} = a$) is an (ellipsoid).
- The eigenvectors of $\mathbf{Q}$ are in the direction of the three principal axes.
- Lengths of axes determined by the eigenvalues



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(**Lecture** 19) S-**8** **Another example 3 by 3**

$$\text{Is } \mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ positive definite?}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow[{[(1)\overset{\tau}{\mathbf{3}}+\mathbf{1}]}]{[(1)\overset{\tau}{\mathbf{3}}+\mathbf{1}]} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow[{[(-\frac{1}{2})\overset{\tau}{\mathbf{1}}+\mathbf{3}]}]{[(-\frac{1}{2})\overset{\tau}{\mathbf{1}}+\mathbf{3}]} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \xrightarrow[{[\mathbf{2}\overset{\tau}{\rightleftharpoons}\mathbf{3}]}]{[\mathbf{2}\overset{\tau}{\rightleftharpoons}\mathbf{3}]} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Indefinite matrix

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(**Lecture** 19) S-**9** **"Classification" of quadratic**

$$\boxed{\boldsymbol{x}\mathbf{A}\boldsymbol{x} \lesseqqgtr 0; \quad \text{for all } \boldsymbol{x} \neq \mathbf{0}}$$

Methods
Check the signs of

1. Elem. diag.: $\mathbf{D} = \mathbf{B}^{\intercal}\mathbf{A}\mathbf{B}$ (*Diagonalization by congruence*) ☺
2. Computing eigenvalues: (Roots of a polynomial) ☹
3. Leading principal minors: (Sylvester's criterion) ☹

Law of inertia
the number of positive, negative and zero entries of the diagonal of $\mathbf{D}$ is an invariant of $\mathbf{A}$, i.e. it does not depend on $\mathbf{B}$

(Orthogonal diagonalization $\mathbf{D} = \mathbf{Q}^{\intercal}\mathbf{A}\mathbf{Q}$ is a special case)

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El criterio de los menores de Sylvester indica que si todos los menores principales son positivos, entonces la matriz es definida positiva. Pero este criterio no da pistas sobre el signo de los autovalores cuando alguno de los menores es cero. Así ocurre en el último ejemplo, donde uno de los autovalores es cero (con multiplicidad 1); como la traza es cero pero el rango no es nulo, los otros dos autovalores deben ser no nulos y de signos opuestos. Es decir, la matriz es indefinida, sin embargo ¡los tres menores principales son cero!

Existe una verificación por menores para matrices semidefinidas positivas, pero es más complicada (y si la matriz es negativa, hay una complicación más con la alternancia de los signos de los menores). Por ello, es mucho más práctico completar el cuadrado mediante la diagonalización por congruencia: $\mathbf{D} = \mathbf{B}^{\intercal}\mathbf{A}\mathbf{B}$.

The lecture ends here

Questions of the Lecture 19

(L-19) QUESTION 1. Decide for or against the positive definiteness of these matrices, and write out the corresponding quadratic form $f = \boldsymbol{x}\mathbf{A}\boldsymbol{x}$:

(a) $\begin{bmatrix}1 & 3\\3 & 5\end{bmatrix}$

(b) $\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$

(c) $\begin{bmatrix}2 & 3\\3 & 5\end{bmatrix}$

(d) $\begin{bmatrix}-1 & 2\\2 & -8\end{bmatrix}$

(e) The determinant in (b) is zero; along what line is $f(x, y) = 0$?

(Strang, 2006, exercise 2 from section 6.1.)

(L-19) QUESTION 2. What is the quadratic $f = ax^2 + 2bxy + cy^2$ for each of these matrices? Complete the square to write f as a sum of one or two squares $d_1(\quad)^2 + d_2(\quad)^2$.

(a) $\mathbf{A} = \begin{bmatrix}1 & 2\\2 & 9\end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix}1 & 3\\3 & 9\end{bmatrix}$

(Strang, 2006, exercise 15 from section 6.1.)

(L-19) QUESTION 3. Which one of the following matrices has two positive eigenvalues? Test $a > 0$ and $ac > b^2$, don't compute the eigenvalues. $\boldsymbol{x}\mathbf{A}\boldsymbol{x} < 0$.

(a) $\mathbf{A} = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix}$
(b) $\mathbf{B} = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix}$
(c) $\mathbf{C} = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix}$
(d) $\mathbf{D} = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}$
(Strang, 2006, exercise 14 from section 6.1.)

(L-19) Question 4. Show that $f(x, y) = x^2 + 4xy + 3y^2$ does not have a minimum at $(0, 0)$ even though it has positive coefficients. Write $f(x, y)$ as a difference of squares and find a point (x, y) where $f(x, y)$ is negative.
(Strang, 2006, exercise 16 from section 6.1.)

(L-19) Question 5. Show from the eigenvalues that if $\mathbf{A}$ is positive definite, so is $\mathbf{A}^2$ and so is$\mathbf{A}^{-1}$.
(Strang, 2006, exercise 4 from section 6.2.)

(L-19) Question 6. Consider the following quadratic forms

$$\begin{aligned} q_1(x, y, z) &= x^2 + 4y^2 + 5z^2 - 4xy. \\ q_2(x, y, z) &= -x^2 + 4y^2 + z^2 + 2xy - 2axz. \end{aligned}$$

(a) Show that $q_1(x, y, z)$ is positive semi-definite.
(b) Find, if it is possible, any value of a such as $q_2(x, y, z)$ is negative definite.

(L-19) Question 7. Decide for or against the positive definiteness of
(a) $\mathbf{A} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$
(b) $\mathbf{B} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$
(c) $\mathbf{C} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}^2$
(Strang, 2006, exercise 2 from section 6.2.)

(L-19) Question 8. Consider the following quadratic form

$$q(x, y, z) = x^2 + 6xy + y^2 + az^2;$$

Decide for which values a the quadratic form is positive definite, negative definite, semidefinite, or indefinite.

(L-19) Question 9. Si $\mathbf{A} = \left[\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right]$ es definida positiva, pruebe que $\mathbf{A}^{-1}$ es definida positiva.
(Strang, 2006, exercise 8 from section 6.1.)

(L-19) Question 10. Si una matriz simétrica de 2 por 2 satisface $a > 0$, y $ac > b^2$, demuestre que sus autovalores son reales y positivos (definida positiva). Emplee la ecuación característica y el hecho de que el producto de los autovalores es igual al determinante.
(Strang, 2006, exercise 3 from section 6.1.)

(L-19) Question 11. Decida si las siguientes matrices son definidas positivas, definidas negativas, semi-definidas, o indefinidas.
(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 6 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 3 \end{bmatrix}$
(c) $\mathbf{C} = -\mathbf{B}$
(d) $\mathbf{D} = \mathbf{A}^{-1}$

(L-19) QUESTION 12. Una matriz definida positiva no puede tener un cero (o incluso peor; un número negativo) en su diagonal principal. Demuestre que esta matriz no cumple $\boldsymbol{x}\mathbf{A}\boldsymbol{x} > 0$, para todo $\boldsymbol{x} \neq \mathbf{0}$:

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{no es positiva cuando} \quad \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} = (\qquad)$$

(Strang, 2006, exercise 21 from section 6.2.)

(L-19) QUESTION 13. Demuestre que si $\mathbf{A}$ y $\mathbf{B}$ son definidas positivas entonces $\mathbf{A}+\mathbf{B}$ también es definida positiva. Para esta demostración los pivotes y los valores característicos no son convenientes. Es mejor emplear $\boldsymbol{x}(\mathbf{A}+\mathbf{B})\boldsymbol{x} > 0$ (Strang, 2006, exercise 5 from section 6.2.)

(L-19) QUESTION 14. Find the $\dot{\mathbf{L}}\mathbf{D}\dot{\mathbf{L}}^{\intercal}$ factorization for the following symmetric matrices.
(a)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

(L-19) QUESTION 15. La forma cuadrática $f(x, y) = 3(x + 2y)^2 + 4y^2$ es definida positiva. Encuentre la matriz $\mathbf{A}$, factorícela en $\mathbf{L}\mathbf{D}\mathbf{L}^{\intercal}$, y relacione los elementos en $\mathbf{D}$ y $\mathbf{L}$ con 3, 2 y 4 en f.
(Strang, 2006, exercise 9 from section 6.1.)

(L-19) QUESTION 16. Consider the following matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & a & a \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) (0.5^{pts}) Compute the eigenvalues of $\mathbf{A}$.
(b) (0.5^{pts}) Prove that when $a = 2$ the matrix $\mathbf{A}$ is not diagonalisable.
(c) (1^{pts}) For matrix $\mathbf{B}$, find a diagonal matrix $\mathbf{D}$ and an orthonormal matrix $\mathbf{P}$ such as $\mathbf{B} = \mathbf{P}\mathbf{D}\mathbf{P}^{\intercal}$.
(d) (0.5^{pts}) Find the quadratic form $f(x, y, z)$ associated to $\mathbf{B}$, and prove it is positive defined.

Versión de un ejercicio proporcionado por Mercedes Vazquez

(L-19) QUESTION 17. Given the matrix $\mathbf{A} = \left(\begin{smallmatrix} a & 3/5 \\ b & 4/5 \end{smallmatrix}\right)$, compute the values (if they exist) of a and b such as

(a) (0.5^{pts}) $\mathbf{A}$ is ortho-normal.
(b) (0.5^{pts}) Columns of $\mathbf{A}$ are linearly independent.
(c) (0.5^{pts}) $\lambda = 0$ is an eigenvalue of $\mathbf{A}$.
(d) (0.5^{pts}) $\mathbf{A}$ is a symmetric definite negative matrix.

(L-19) QUESTION 18.

(a) Consider the quadratic form $q(x, y, z) = x^2 + 2xy + ay^2 + 8z^2$ and find its corresponding symmetric matrix $\mathbf{Q}$; determine if $\mathbf{Q}$ is positive-definite, positive-semidefinite, negative-definite, negative-semidefinite or indefinite when the parameter a is equal to one ($a = 1$).

(b) If $a \neq 1$, determine whether the matrix is positive-definite, positive-semidefinite, negative-definite, negative-semidefinite or indefinite.

End of Questions of the Lecture 19

OPTIONAL LECTURE II: Review execises

Questions of the Optional Lecture 2

(L-Opt-2) Question 1. Consider the following matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(a) (0.5^{pts}) Prove $\mathbf{A}$ is invertible if and only if $a \neq 0$.
(b) (0.5^{pts}) Is $\mathbf{A}$ positive definite when $a = 1$? Explain your answer.
(c) (1^{pts}) Compute $\mathbf{A}^{-1}$ when $a = 2$.
(d) (0.5^{pts}) How many variables can be chosen as pivot (or exogenous) variables in the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{o}$ when $a = 0$? Which ones?

(L-Opt-2) Question 2. True or false (to receive full credit you must explain your answers in a clear and concise way)
(a) If $\mathbf{A}$ is symetric, then so it is $\mathbf{A}^2$.
(b) If $\mathbf{A}^2 = \mathbf{A}$ then $(\mathbf{I} - \mathbf{A})^2 = (\mathbf{I} - \mathbf{A})$ where $\mathbf{I}$ is the identity matrix.
(c) If $\lambda = 0$ is an eigenvalue of the squared matrix $\mathbf{A}$, then the linear system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ is is always solvable and has only one solution.
(d) If $\lambda = 0$ is an eigenvalue of the squared matrix $\mathbf{A}$, then the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ could be unsolvable.
(e) If a matrix is orthogonal (perpendicular columns of norm one), then so it is the inverse of that matrix.
(f) If 1 is the only eigenvalue of a 2×2 matrix $\mathbf{A}$, then $\mathbf{A}$ must be the identity matrix $\mathbf{I}$.

(L-Opt-2) Question 3. complete los blancos, o responda **V**erdadero/**F**also.
(a) Cualquier sistema generador de un espacio vectorial contiene una base del espacio (V/F)

______________.

(b) Que los vectores $\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \ldots,\ \boldsymbol{v}_n$ sean linealmente independientes significa que

____.

(c) El conjunto que sólo contiene el vector $\mathbf{0}$ es un conjunto linealmente independiente. (V/F)

______________.

(d) Una matriz cuadrada de orden n por n es diagonalizable cuando:

____.

(e) Si $\boldsymbol{u} = (1, 2, -1, 1)$, entonces $||\boldsymbol{u}|| =$ ____________.
(f) Si $\boldsymbol{u} = (1, 2, -1, 1)$ y $\boldsymbol{v} = (-2, 1, 0, 0)$, entonces $\boldsymbol{u} \cdot \boldsymbol{v} =$ ____________.

(L-Opt-2) Question 4. En las preguntas siguientes $\mathbf{A}$ y $\mathbf{B}$ son matrices $n \times n$. Indique si las siguientes afirmaciones son verdaderas o falsas (incluya una breve explicación, o un contra ejemplo que justifique su respuesta):
(a) Si $\mathbf{A}$ no es cero entonces $\det(\mathbf{A}) \neq 0$
(b) Si $\det(\mathbf{AB}) \neq 0$ entonces $\mathbf{A}$ es invertible.
(c) Si intercambio las dos primeras filas de $\mathbf{A}$ sus autovalores cambian.
(d) Si $\mathbf{A}$ es real y simétrica, entonces sus autovalores son reales **(aquí no es necesaria una justificación)**.
(e) Si la forma reducida de echelon de $(\mathbf{A} - 5\mathbf{I})$ es la matriz identidad, entonces 5 no es un autovalor de $\mathbf{A}$.
(f) Sea $\boldsymbol{b}$ un vector columna de $\mathbb{R}^n$. Si el sistema $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ no tiene solución, entonces $\det(\mathbf{A}) \neq 0$
(g) Sea $\mathbf{C}$ de orden 3×5. El rango de $\mathbf{C}$ puede ser 4.
(h) Sea $\mathbf{C}$ de orden $n \times m$, y $\boldsymbol{b}$ un vector columna de $\mathbb{R}^n$. Si $\mathbf{C}\boldsymbol{x} = \boldsymbol{b}$ no tiene solución, entonces $\operatorname{rg}(\mathbf{C}) < n$.
(i) Toda matriz diagonalizable es invertible.
(j) Si $\mathbf{A}$ es invertible, entonces su forma reducida de echelon es la matriz identidad.

(L-Opt-2) Question 5. Sean

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & 5 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 0 & 4 \\ 0 & 0 & 6 \end{bmatrix}; \quad \mathbf{D} = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Los autovalores de $\mathbf{B}$ son 0 y 2. Use esta información para responder a las siguientes cuestiones. Para cada matriz debe dar una explicación. Puede haber más de una matriz que cumpla la condición:

(a) ¿Qué matrices son invertibles?
(b) ¿Qué matrices tienen un autovalor repetido?
(c) ¿Qué matrices tienen rango menor a tres?
(d) ¿Qué matrices son diagonalizables?
(e) ¿Para qué matrices diagonalizables podemos encontrar tres autovectores ortogonales entre si?

(L-Opt-2) Question 6. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

(a) Find the eigenvalues and eigenvectors of $\mathbf{A}$.
(b) Is $\mathbf{A}$ diagonalizable?
(c) Is it possible to find a matrix $\mathbf{P}$ such as $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{\intercal}$, where $\mathbf{D}$ is diagonal?
(d) Find $|\mathbf{A}^{-1}|$.

(L-Opt-2) Question 7. Consider a 3 by 3 matrix $\mathbf{A}$ with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = -1$; and let $\boldsymbol{v}_1 = (1, 0, 1)^{\intercal}$ and $\boldsymbol{v}_2 = (1, 1, 1)^{\intercal}$ be the corresponding eigenvectors to λ_1 and λ_2.

(a) Is $\mathbf{A}$ diagonalizable?
(b) Is $\boldsymbol{v}_3 = (-1, 0, -1)^{\intercal}$ an eigenvector associated to the eigenvalue $\lambda_3 = -1$?
(c) Compute $\mathbf{A}(\boldsymbol{v}_1 - \boldsymbol{v}_2)$.

(L-Opt-2) Question 8.

(a) (0.5^{pts}) Find an homogeneous system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ such as its solutions set is

$$\left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \exists \alpha, \beta, \gamma \in \mathbb{R} \quad \text{such that} \quad \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \right\}$$

(b) (0.5^{pts}) If the characteristic polynomial of a matrix $\mathbf{A}$ is $p(\lambda) = \lambda^5 + 3\lambda^4 - 24\lambda^3 + 28\lambda^2 - 3\lambda + 10$, find the rank of $\mathbf{A}$.

(L-Opt-2) Question 9. Suponga una matriz cuadrada e *invertible* $\underset{n\times n}{\mathbf{A}}$.

(a) ¿Cuáles son sus espacios columna $\mathcal{C}\left(\mathbf{A}\right)$ y espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$? (no responda con la definición, diga qué conjunto de vectores compone cada espacio).
(b) Suponga que $\mathbf{A}$ puede ser factorizada en $\mathbf{A} = \mathbf{L}\mathbf{U}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 7 & 3 & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{12} & u_{13} \\ 0 & 0 & u_{13} \end{bmatrix}$$

Describa el primer paso de eliminación en la reducción de $\mathbf{A}$ a $\mathbf{U}$. ¿porqué sabe que $\mathbf{U}$ es también una matriz invertible? ¿Cuanto vale el determinante de $\mathbf{A}$?
(c) Encuentre una matriz particular de dimensiones 3×3 e invertible $\mathbf{A}$ que no pueda ser factorizada en la forma $\mathbf{L}\mathbf{U}$ (sin permutar previamente las filas). ¿Qué factorización es todavía posible en su ejemplo? (no es necesario que realice la factorización). ¿Cómo sabe que su matriz $\mathbf{A}$ es invertible?

End of Questions of the Optional Lecture 2

References

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Solutions

(L-16) Question 1(a)

$$\begin{bmatrix} -3 & 4 & -4 \\ -3 & 5 & -3 \\ -1 & 2 & 0 \end{bmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

Therefore, the corresponding eigenvalue to $\boldsymbol{v}$ is -1.

$$\begin{bmatrix} -3 & 4 & -4 \\ -3 & 5 & -3 \\ -1 & 2 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Therefore, the correspondant eigenvalue to $\boldsymbol{w}$ is 1.

□

(L-16) Question 1(b) We need to find a vector in the null space of $(\mathbf{A} - 2\mathbf{I})$, in other words, we need to find a solution to

$$\begin{bmatrix} -5 & 4 & -4 \\ -3 & 3 & -3 \\ -1 & 2 & -2 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We can find a solution by gaussian elimination, but as the third column equals minus the second one, it is easy to realize that one solution is:

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

□

(L-16) Question 2(a) The eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

are the solutions to the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = (3-\lambda)(-3-\lambda) - 16 = 0; \quad \Rightarrow \quad \lambda_1 = 5;\ \lambda_2 = -5.$$

We can also use

$$\begin{cases} \lambda_1 \cdot \lambda_2 = & \det \mathbf{A} = -25 \\ \lambda_1 + \lambda_2 = & \operatorname{tr} \mathbf{A} = 0 \end{cases}$$

with the same result.

For $\lambda_1 = 5$, the null space of

$$(\mathbf{A} - \lambda_1\mathbf{I}) = \begin{bmatrix} 3-5 & 4 \\ 4 & -3-5 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix}$$

consists of all multiples of $\boldsymbol{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Hence, the eigenvectors are the non-null multiples of $\boldsymbol{x}_1$

For $\lambda_2 = -5$, the null space of

$$(\mathbf{A} - \lambda_2\mathbf{I}) = \begin{bmatrix} 3+5 & 4 \\ 4 & -3+5 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}$$

consists of all multiples of $\boldsymbol{x}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Hence, the eigenvectors are the non-null multiples of $\boldsymbol{x}_2$

□

(L-16) Question 2(b) The eigenvalues of

$$\mathbf{B} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

are the solutions to the characteristic equation

$$\det(\mathbf{B}-\lambda\mathbf{I}) = \begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} = (a-\lambda)^2 - b^2 = \lambda^2 - 2a\lambda + (a^2-b^2) = 0.$$

Therefore

$$\lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2-b^2)}}{2} = a \pm b.$$

For $\lambda_1 = a+b$, the null space of

$$(\mathbf{B}-\lambda_1\mathbf{I}) = \begin{bmatrix} -b & b \\ b & -b \end{bmatrix}$$

consists of all multiples of $\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Hence, the eigenvectors are the non-null multiples of $\boldsymbol{x}_1$.

For $\lambda_2 = a-b$, the null space of

$$(\mathbf{B}-\lambda_2\mathbf{I}) = \begin{bmatrix} b & b \\ b & b \end{bmatrix}$$

consists of all multiples of $\boldsymbol{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Hence, the eigenvectors are the non-null multiples of $\boldsymbol{x}_2$.

□

(L-16) Question 3. The numbers on the diagonal of $\mathbf{A}$: 1, 2, 3, 7, 8, and 9.

□

(L-16) Question 4(a) Since the matrix is triangular, the numbers on the diagonal are the eigenvectors of this matrix For $\lambda_1 = 3$ we need to compute a basis for the nullspace of $\big(\mathbf{A}-3\mathbf{I}\big)$.

$$\left[\begin{array}{c} \mathbf{A}-3\mathbf{I} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{ccc} 0 & 4 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & -3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[\left[\left(\frac{-1}{2}\right)\mathbf{2}+\mathbf{3}\right]]{\boldsymbol{\tau}} \left[\begin{array}{ccc} 0 & 4 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & -3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{array}\right]. \qquad \text{Eigenvector: the non-null multiples of } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

For $\lambda_2 = 1$ we need to compute a basis for the nullspace of $\big(\mathbf{A}-\mathbf{I}\big)$.

$$\left[\begin{array}{c} \mathbf{A}-\mathbf{I} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{ccc} 2 & 4 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-2)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \\ \hline 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]. \qquad \text{Eigenvector: the non-null multiples of } \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

For $\lambda_3 = 0$ we need to compute a basis for the nullspace of $\big(\mathbf{A}-0\mathbf{I}\big)$.

$$\left[\begin{array}{c} \mathbf{A}-0\mathbf{I} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{ccc} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(3)\mathbf{2}] \\ [(-4)\mathbf{1}+\mathbf{2}] \\ [(3)\mathbf{3}] \\ [(-2)\mathbf{1}+\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc} 3 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \\ \hline 1 & -4 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{array}\right] \xrightarrow[{[(-2)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \\ \hline 1 & -4 & 6 \\ 0 & 3 & -6 \\ 0 & 0 & 3 \end{array}\right] = \left[\begin{array}{c} \mathbf{L} \\ \hline \mathbf{E} \end{array}\right]$$

The eigenvectors are non-null multiples of $\begin{pmatrix} 6 \\ -6 \\ 3 \end{pmatrix}$.

In addition $\lambda_1 + \lambda_2 + \lambda_3 = 3+1+0 = 4 = \operatorname{tr}\big(\mathbf{A}\big)$ and $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 3 \cdot 1 \cdot 0 = 0 = |\mathbf{A}|$.

```
A =  Matrix([[3,4,2],[0,1,2],[0,0,0]])
espectro = {3,1,0}
```

```
for l in espectro:
    display(l)
    L   = Elim(  (A-l*I(3)).apila(I(3),1) ,1)
    cL0 = tuple([c+1 for c,i in enumerate( (1,2,3)|L ) if i.es_nulo()])
    display(((4,5,6)|L|cL0).sis())
```

□

(L-16) Question 4(b) The characteristic equation is

$$\begin{vmatrix}-\lambda & 0 & 2\\ 0 & 2-\lambda & 0\\ 2 & 0 & -\lambda\end{vmatrix} = (\lambda^2)(2-\lambda) - 4(2-\lambda) = 0$$

It is clear that $\lambda_1 = 2$ is one eigenvalue.

Dividing the characteristic equation by $(2-\lambda)$ we get $\lambda^2 = 4$; therefore $\lambda_2 = 2$ and $\lambda_3 = -2$

For $\lambda = 2$ we need to compute a basis for the nullspace of

$$\left(\mathbf{A} - 2\mathbf{I}\right) = \begin{bmatrix}-2 & 0 & 2\\ 0 & 0 & 0\\ 2 & 0 & -2\end{bmatrix}$$

We get two independent eigenvectors for $\lambda = 2$ (a basis for the null space of $\left(\mathbf{A} - 2\mathbf{I}\right)$):

$$\text{Eigenvectors: all non-null linear combinations of } \begin{pmatrix}0\\1\\0\end{pmatrix};\begin{pmatrix}1\\0\\1\end{pmatrix}$$

For $\lambda_3 = -2$ we need to compute a basis for the nullspace of

$$\left(\mathbf{A} + 2\mathbf{I}\right) = \begin{bmatrix}2 & 0 & 2\\ 0 & 4 & 0\\ 2 & 0 & 2\end{bmatrix}$$

$$\text{The corresponding eigenvectors are all non-null multiples of } \begin{pmatrix}-1\\0\\1\end{pmatrix}$$

In addition $\lambda_1 + \lambda_2 + \lambda_3 = 2 + 2 - 2 = 2 = \operatorname{tr}\left(\mathbf{A}\right)$ and $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 2 \cdot 2 \cdot (-2) = -8 = |\mathbf{A}|$.

```
B = Matrix([[0,0,2],[0,2,0],[2,0,0]])
l = sympy.symbols('\lambda')
d = Determinante( (B-l*I(3)) ,1)
p = sympy.poly(d.valor)
e = sympy.real_roots(p)
display(Sistema([d, e]))
for i in set(e):
    caso = "\lambda = %d\n" % i
    display(Math(caso))
    L   = Elim(  (B-i*I(3)).apila(I(3),1),1 )
    cL0 = tuple([c+1 for c,v in enumerate((1,2,3)|L) if v.es_nulo()])
    display( ((4,5,6)|L|cL0).sis() )
```

□

(L-16) Question 5(a) when a matrix is transposed, the determinant doesn't change ($\det \mathbf{B} = \det \mathbf{B}^\intercal$); therefore

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det(\mathbf{A} - \lambda\mathbf{I})^\intercal = \det(\mathbf{A}^\intercal - \lambda\mathbf{I})$$

□

(L-16) Question 5(b) The eigenvalues of

$$\mathbf{A} = \begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}$$

are the solutions to the characteristic equation

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}1-\lambda & 2\\ 0 & 3-\lambda\end{vmatrix}=(1-\lambda)(3-\lambda)=0;$$

in this case, the eigenvalues are equal to the numbers on the diagonal (since this matrix is triangular) .

For $\lambda_1=1$, the null space of

$$(\mathbf{A}-\lambda_1\mathbf{I})=\begin{bmatrix}1-1 & 2\\ 0 & 3-1\end{bmatrix}=\begin{bmatrix}0 & 2\\ 0 & 2\end{bmatrix}$$

consists of all multiples of the eigenvector $\boldsymbol{x}_1=\begin{pmatrix}1\\0\end{pmatrix}$.

For $\lambda_2=3$, the null space of

$$(\mathbf{A}-\lambda_2\mathbf{I})=\begin{bmatrix}1-3 & 2\\ 0 & 3-3\end{bmatrix}=\begin{bmatrix}-2 & 2\\ 0 & 0\end{bmatrix}$$

consists of all multiples of the eigenvector $\boldsymbol{x}_2=\begin{pmatrix}1\\1\end{pmatrix}$.

Nevertheless, for the transposed matrix

$$\mathbf{A}^\intercal=\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix},$$

with the same eigevalues (same diagonal), we get

For $\lambda_1=1$, the null space of

$$(\mathbf{A}^\intercal-\lambda\mathbf{I})=\begin{bmatrix}0 & 0\\ 2 & 2\end{bmatrix}$$

consists of all multiples of the eigenvector $\boldsymbol{x}_1=\begin{pmatrix}-1\\1\end{pmatrix}$.

For $\lambda_2=3$, the null space of

$$(\mathbf{A}^\intercal-\lambda\mathbf{I})=\begin{bmatrix}-2 & 0\\ 2 & 0\end{bmatrix}$$

consists of all multiples of the eigenvector $\boldsymbol{x}_2=\begin{pmatrix}0\\1\end{pmatrix}$.

Therefore, the eigenvectors of $\mathbf{A}$ and $\mathbf{A}^\intercal$ are not the same.

$\square$

(L-16) Question 6.

$$\begin{aligned}\mathbf{A}\boldsymbol{x}=&\big(\mathbf{B}+\alpha\mathbf{I}\big)\boldsymbol{x}\\ =&\mathbf{B}\boldsymbol{x}+\alpha\mathbf{I}\boldsymbol{x}\\ =&\lambda\boldsymbol{x}+\alpha\boldsymbol{x}\\ =&(\lambda+\alpha)\boldsymbol{x};\end{aligned}$$

therefore, $\boldsymbol{x}$ is an eigenvector of $\mathbf{A}$ associated to the eigenvalue $(\lambda+\alpha)$.

$\square$

(L-16) Question 7(a) Primero calculemos los autovalores:

$$\begin{vmatrix}1-\lambda & -1\\ 2 & 4-\lambda\end{vmatrix}=(1-\lambda)(4-\lambda)+2=\lambda^2-5\lambda+6=0;\quad\Rightarrow\quad\begin{cases}\lambda_1 &=2\\ \lambda_2 &=3\end{cases}$$

Y ahora los auto-vectores.

Para $\lambda_1=2$ buscamos una base del espacio nulo de la matriz

$$\begin{bmatrix}1-2 & -1\\ 2 & 4-2\end{bmatrix}=\begin{bmatrix}-1 & -1\\ 2 & 2\end{bmatrix}$$

Puesto que las dos columnas son iguales, los autovectores son los múltiplos no nulos de $\boldsymbol{x}_1=\begin{pmatrix}-1\\1\end{pmatrix}$.

Para $\lambda_2 = 3$ buscamos una base del espacio nulo de la matriz

$$\begin{bmatrix} 1-3 & -1 \\ 2 & 4-3 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}$$

Puesto que la primera columna es el doble de la segunda, los correspondientes autovectores son los múltiplos no nulos de $\boldsymbol{x}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Por último, el determinante de $\mathbf{A}$ es 6 que es igual a 2×3 y la traza es 5 que es igual a $2 + 3$.

□

(L-16) Question 7(b) Calculemos los autovalores:

$$\begin{vmatrix} -6-\lambda & -1 \\ 2 & -3-\lambda \end{vmatrix} = (-6-\lambda)(-3-\lambda) + 2 = \lambda^2 + 9\lambda + 20 = 0; \Rightarrow \begin{cases} \lambda_1 &= -5 \\ \lambda_2 &= -4 \end{cases}.$$

Y ahora los auto-vectores.

Para $\lambda_1 = -5$ buscamos una base del espacio nulo de la matriz

$$\begin{bmatrix} -6-(-5) & -1 \\ 2 & -3-(-5) \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}.$$

Al igual que en el apartado anterior, los correspondientes autovectores son los múltiplos no nulos de $\boldsymbol{x}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Para $\lambda_2 = -4$ buscamos una base del espacio nulo de la matriz

$$\begin{bmatrix} -6-(-4) & -1 \\ 2 & -3-(-4) \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}$$

De manera idéntica al apartado (a), los correspondientes autovectores son los múltiplos no nulos de $\boldsymbol{x}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Así pues, tras restar 7 veces la matriz identidad ($\mathbf{B} = \mathbf{A} - 7\mathbf{I}$), los autovalores son los de la matriz original menos 7; es decir, $\lambda_1 = 2 - 7 = -5$ y $\lambda_2 = 3 - 7 = -4$; y los auto-vectores son idénticos a los de la matriz original $\mathbf{A}$.

□

(L-16) Question 8(a) Sea un vector $\boldsymbol{x}$ tal que verifica $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$; entonces

$$\mathbf{B}\boldsymbol{x} = (\mathbf{A} - 7\mathbf{I})\boldsymbol{x} = \mathbf{A}\boldsymbol{x} - 7\boldsymbol{x} = (\lambda - 7)\boldsymbol{x}$$

por tanto $\boldsymbol{x}$ también es auto-vector de $\mathbf{B}$ con un auto-valor asociado igual a $(\lambda - 7)$.

□

(L-16) Question 8(b)

$$\begin{aligned} \mathbf{A}\boldsymbol{x} =& \lambda\boldsymbol{x} \\ \boldsymbol{x} =& \lambda(\mathbf{A}^{-1})\boldsymbol{x} \\ \frac{1}{\lambda}\boldsymbol{x} =& (\mathbf{A}^{-1})\boldsymbol{x} \end{aligned}$$

La última igualdad $(\mathbf{A}^{-1})\boldsymbol{x} = (1/\lambda)\boldsymbol{x}$ implica que $\boldsymbol{x}$ es también auto-vector de $\mathbf{A}^{-1}$ con un auto-vector asociado igual a $1/\lambda$ para el caso de $\mathbf{A}^{-1}$.

□

(L-16) Question 9. Si λ es un auto-valor de $\mathbf{A}$, entonces $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$. Entonces

$$\mathbf{A}^2\boldsymbol{x} = \mathbf{A}\mathbf{A}\boldsymbol{x} = \mathbf{A}\lambda\boldsymbol{x} = \lambda\mathbf{A}\boldsymbol{x} = \lambda^2\boldsymbol{x}.$$

Pero, puesto que $\mathbf{A}^2 = \mathbf{A}$ entonces

$$\mathbf{A}^2\boldsymbol{x} = \mathbf{A}\boldsymbol{x} = \lambda^2\boldsymbol{x} = \lambda\boldsymbol{x},$$

por tanto
$$\lambda^2 = \lambda.$$
Los dos únicos valores posibles son, o bien 0, o bien 1.

$\square$

(L-16) Question 10.
$$\mathbf{A}(\boldsymbol{v}_1 + \boldsymbol{v}_2 - \boldsymbol{v}_3) = \mathbf{A}\boldsymbol{v}_1 + \mathbf{A}\boldsymbol{v}_2 - \mathbf{A}\boldsymbol{v}_3 = \boldsymbol{v}_1 + 2\boldsymbol{v}_2 - 3\boldsymbol{v}_3.$$

$\square$

(L-16) Question 11. La ecuación característica de la matriz
$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
es $(1-\lambda)^2 - 1 = 0$. Por tanto la matriz tiene autovalores $\lambda = 0$ y $\lambda = 2$.

Sin embargo, la ecuación característica de su forma de escalonada es
$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$
es $(1-\lambda)(-\lambda) = 0$. Por tanto los nuevos autovalores son $\lambda = 0$ y $\lambda = 1$.

Los autovalores nulos no cambian. Hay tantos cómo el número de columnas menos el rango de la matriz (es decir, tantos como numero de columnas libres); y ni el número de columnas ni el rango de la matriz cambia al aplicar transformaciones elementales. Por tanto el número de autovalores nulos se mantiene tras aplicar el método de eliminación.

$\square$

(L-16) Question 12. Basta con igualar λ a cero.

$\square$

(L-16) Question 13. Para $\mathbf{A}$, la suma de λ_1 y λ_2 es -1 (la traza) y el producto es -6 (el determinante), por tanto $\lambda_1 = -3$ y $\lambda_2 = 2$

Para $\lambda_1 = -3$; una base del espacio nulo de $(\mathbf{A} + 3\mathbf{I}) = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$ es $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$. Sus múltiplos no nulos son los correspondientes autovectores.

Y para $\lambda_2 = 2$; una base del espacio nulo de $(\mathbf{A} - 2\mathbf{I}) = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}$ es $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Sus múltiplos no nulos son los correspondientes autovectores.

Por otra parte, para $\mathbf{A}^2$, la suma de λ_1 y λ_2 es 13 (la traza) y el producto es 36 (el determinante), por tanto $\lambda_1 = 9$ y $\lambda_2 = 4$

Para $\lambda_1 = 9$; una base del espacio nulo de $(\mathbf{A} - 9\mathbf{I}) = \begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix}$ es $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$. Sus múltiplos no nulos son los correspondientes autovectores.

Y para $\lambda_2 = 4$; una base del espacio nulo de $(\mathbf{A} - 4\mathbf{I}) = \begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix}$ es $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Sus múltiplos no nulos son los correspondientes autovectores.

$\mathbf{A}^2$ tiene los mismos **autovectores** que $\mathbf{A}$. Si los autovalores de $\mathbf{A}$ son λ_1 y λ_2, los autovalores de $\mathbf{A}^2$ son **el cuadrado de los anteriores** (λ_1^2 y λ_2^2).

$\square$

(L-16) Question 14. Los autovalores de $\mathbf{A}^2$ son el cuadrado de los autovalores de $\mathbf{A}$, por lo que
$$\operatorname{tr}\left(\mathbf{A}^2\right) = 1^2 + 2^2 + 4^2 = 21$$

Razonando de la misma manera
$$\left|(\mathbf{A}^{-1})^{\intercal}\right| = \det \mathbf{A}^{-1} = (1^{-1})(2^{-1})(4^{-1}) = 1 \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}.$$

$\square$

(L-16) Question 15. The condition says that $(\mathbf{A}^2 - 4\mathbf{I})$ is singular. But we know that, if $\lambda_1, \dots \lambda_n$ are eigenvalues of $\mathbf{A}$, then the eigenvalues of $(\mathbf{A}^2 - 4\mathbf{I})$ are $\lambda_1^2 - 4, \lambda_1^2 - 4, \dots, \lambda_n^2 - 4$. The condition $(\mathbf{A}^2 - 4\mathbf{I})$ being singular says that one of $\lambda_i^2 - 4$ is zero, and hence $\lambda_i = 2$ or $\lambda_i = -2$. That is to say $\mathbf{A}$ has an eigenvalue 2 or -2.

□

(L-16) Question 16. First, the last two columns of $\mathbf{A}$ are the same. Hence $\mathbf{A}$ is singular and it must have an eigenvalue $\lambda_1 = 0$. Also, we observe that $\mathbf{A}$ is a Markov matrix. This means that $\lambda_2 = 1$ is an eigenvalue of $\mathbf{A}$. Finally, we know the trace of $\mathbf{A}$ is the sum of its three eigenvalues. So, $\operatorname{tr}\big(\mathbf{A}\big) = 0.5 + 0.5 + 0.3 = 1.3$ and the last eigenvalue is $\lambda_3 = 1.3 - 1 - 0 = 0.3$.

□

(L-17) Question 1(a) From QUESTION 5 on page 7 (L-19) we know the eigenvalues and eigenvectors of $\mathbf{A}$; therefore we know

$$\mathbf{S} = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}; \qquad \mathbf{D} = \begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}.$$

We only need to compute $\mathbf{S}^{-1}$:

$$\left[\begin{array}{c}\mathbf{S}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{cc}1 & 1\\ 0 & 1\\ \hline 1 & 0\\ 0 & 1\end{array}\right] \xrightarrow[\left[(-1)\mathbf{1}+\mathbf{2}\right]]{\boldsymbol{\tau}} \left[\begin{array}{cc}1 & 0\\ 0 & 1\\ \hline 1 & -1\\ 0 & 1\end{array}\right] = \left[\begin{array}{c}\mathbf{I}\\ \hline \mathbf{S}^{-1}\end{array}\right]$$

Therefore

$$\mathbf{A} = \begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}.$$

O aplicando el algoritmo de diagonalización por bloques (que en este caso sabemos que arrojará una matriz diagonal, puesto que los autovalores no se repiten)

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] \xrightarrow[3\mathbf{I}]{(-)} \left[\begin{array}{cc}-2 & 2\\ 0 & 0\\ \hline 1 & 0\\ 0 & 1\end{array}\right] \xrightarrow[\left[(1)\mathbf{1}+\mathbf{2}\right]]{\boldsymbol{\tau}} \left[\begin{array}{cc}-2 & 0\\ 0 & 0\\ \hline 1 & 1\\ 0 & 1\end{array}\right] \xrightarrow[\substack{\boldsymbol{\tau}\\ \left[(-1)\mathbf{2}+\mathbf{1}\right]}]{} \left[\begin{array}{cc}-2 & 0\\ 0 & 0\\ \hline 1 & 1\\ 0 & 1\end{array}\right] \xrightarrow[3\mathbf{I}]{(+)} \left[\begin{array}{cc}1 & 0\\ 0 & 3\\ \hline 1 & 1\\ 0 & 1\end{array}\right] \xrightarrow[1\mathbf{I}]{(-)} \left[\begin{array}{cc}0 & 0\\ 0 & 2\\ \hline 1 & 1\\ 0 & 1\end{array}\right] \xrightarrow[1\mathbf{I}]{(+)} \left[\begin{array}{cc}1 & 0\\ 0 & 3\\ \hline 1 & 1\\ 0 & 1\end{array}\right]$$

```
A = Matrix([[1,2],[0,3]])
lambdas = [1,3]
D = DiagonalizaS(A,lambdas,1)
display(D)
display(D.S)
```

□

(L-17) Question 1(b) First we are going to solve the characteristic equation $\det(\mathbf{B} - \lambda\mathbf{I}) = 0$ in order to find the eigenvalues

$$\begin{vmatrix}1-\lambda & 1\\ 2 & 2-\lambda\end{vmatrix} = (1-\lambda)(2-\lambda) - 2 = \lambda^2 - 3\lambda = 0; \quad \Rightarrow \quad \begin{cases}\lambda_1 & = 0\\ \lambda_2 & = 3\end{cases}$$

(Why should we know one eigenvalue is zero before solving the characteristic equation?)

- for $\lambda_1 = 0$, the null space of $(\mathbf{B} - 0\mathbf{I}) = \mathbf{B} = \begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}$ consists of all multiples of the eigenvector

$$\boldsymbol{x}_1 = \begin{pmatrix}-1\\ 1\end{pmatrix}$$

- for $\lambda_2 = 3$, the null space of $(\mathbf{B} - \lambda\mathbf{I}) = \begin{bmatrix}-2 & 1\\ 2 & -1\end{bmatrix}$ consists of all the multiples of

$$\boldsymbol{x}_2 = \begin{pmatrix}1\\ 2\end{pmatrix}$$

Therefore:
$$\mathbf{S} = \begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix}; \qquad \mathbf{D} = \begin{bmatrix}3 & 0\\ 0 & 0\end{bmatrix}.$$

O aplicando el algoritmo de diagonalización por bloques (que en este caso sabemos que arrojará una matriz diagonal, puesto que los autovalores no se repiten)

$$\begin{bmatrix}\mathbf{A}\\ \hline \mathbf{I}\end{bmatrix} \xrightarrow[0\mathbf{I}]{(-)} \left[\begin{array}{cc}1 & 1\\ 2 & 2\\ \hline 1 & 0\\ 0 & 1\end{array}\right] \xrightarrow{\tau_{[(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc}1 & 0\\ 2 & 0\\ \hline 1 & -1\\ 0 & 1\end{array}\right] \xrightarrow[\tau_{[(1)\mathbf{2}+\mathbf{1}]}]{} \left[\begin{array}{cc}3 & 0\\ 2 & 0\\ \hline 1 & -1\\ 0 & 1\end{array}\right] \xrightarrow[0\mathbf{I}]{(+)} \left[\begin{array}{cc}3 & 0\\ 2 & 0\\ \hline 1 & -1\\ 0 & 1\end{array}\right]$$
$$\xrightarrow[3\mathbf{I}]{(-)} \left[\begin{array}{cc}0 & 0\\ \hline 2 & -3\\ \hline 1 & -1\\ 0 & 1\end{array}\right] \xrightarrow{\substack{\tau\\ [(3)\mathbf{1}]\\ [(2)\mathbf{2}+\mathbf{1}]}} \left[\begin{array}{cc}0 & 0\\ \hline 0 & -3\\ \hline 1 & -1\\ 2 & 1\end{array}\right] \xrightarrow[\substack{\tau\\ [(1/3)\mathbf{1}]\\ [(-2)\mathbf{1}+\mathbf{2}]}]{} \left[\begin{array}{cc}0 & 0\\ \hline 0 & -3\\ \hline 1 & -1\\ 2 & 1\end{array}\right] \xrightarrow[3\mathbf{I}]{(+)} \left[\begin{array}{cc}3 & 0\\ 0 & 0\\ \hline 1 & -1\\ 2 & 1\end{array}\right] = \begin{bmatrix}\mathbf{D}\\ \hline \mathbf{S}\end{bmatrix}$$

Finally we compute $\mathbf{S}^{-1}$:

$$\begin{bmatrix}\mathbf{S}\\ \hline \mathbf{I}\end{bmatrix} = \left[\begin{array}{cc}1 & -1\\ 2 & 1\\ \hline 1 & 0\\ 0 & 1\end{array}\right] \xrightarrow{\tau_{[(1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc}1 & 0\\ 2 & 3\\ \hline 1 & 1\\ 0 & 1\end{array}\right] \xrightarrow{\substack{\tau\\ [(3)\mathbf{1}]\\ [(-2)\mathbf{2}+\mathbf{1}]}} \left[\begin{array}{cc}3 & 0\\ 0 & 3\\ \hline 1 & 1\\ -2 & 1\end{array}\right] \xrightarrow{\substack{\tau\\ [(\frac{1}{3})\mathbf{1}]\\ [(\frac{1}{3})\mathbf{2}]}} \left[\begin{array}{cc}1 & 0\\ 0 & 1\\ \hline \frac{1}{3} & \frac{1}{3}\\ \frac{-2}{3} & \frac{1}{3}\end{array}\right] = \begin{bmatrix}\mathbf{I}\\ \hline \mathbf{S}^{-1}\end{bmatrix}$$

The factorization is
$$\mathbf{B} = \begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix} = \begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1/3 & 1/3\\ -2/3 & 1/3\end{bmatrix} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}.$$

```
B = Matrix([[1,1],[2,2]])
l = sympy.symbols('\lambda')
d = Determinante( (B-l*I(B.n)) ,1)
p = sympy.poly(d.valor)
e = sympy.real_roots(p)
display(Sistema([d, e]))
S = Sistema([])
D = tuple()
for i in set(e):
    caso = "\lambda = %d\n" % i
    display(Math(caso))
    L   = Elim( (B-i*I(B.n)).apila(I(B.n),1), 1)
    cL0 = tuple([c+1 for c,v in enumerate(slice(1,B.n)|L) if v.es_nulo()])
    S = S.concatena(slice(B.n+1,None)|L|cL0).sis()
    D = D+(i,)*e.count(i)

S = Matrix(S)
D = Vector(D).diag()
display(Sistema([D,S,S**-1]))
display(Sistema([(S**-1)*B*S, S*D*(S**-1)]))
```

□

(L-17) Question 2(a)

$$\begin{cases}\lambda_1 + \lambda_2 = 0\\ \lambda_1 \cdot \lambda_2 = 0\end{cases} \quad \Rightarrow \quad \lambda_1 = \lambda_2 = 0.$$

This matrix can be diagonalized if the null space of $(\mathbf{A}_1 - 0\mathbf{I}) = \mathbf{A}_1$ has dimension 2. Since $\mathbf{A}_1$ will have only one free column $\dim \mathcal{N}\left(\mathbf{A}\right) = 1$; therefore this matrix cannot be diagonalized.

□

(L-17) Question 2(b) There are no repeated eigenvalues; therefore this matrix is diagonalizable.

□

(L-17) Question 2(c) There is a repeated eigenvalue $\lambda = 2$; therefore, this matriz is diagonalizable if the null space of $(\mathbf{A}_3 - 2\mathbf{I})$ has dimension 2 (two linearly independent eigenvectors), but $\dim \mathcal{N}\left(\mathbf{A_3} - \mathbf{2I}\right) = 1$. This matrix can not be diagonalized.

□

(L-17) Question 3. Characteristic equation: $(4-\lambda)(2-\lambda) - 3 = \lambda^2 - 6\lambda + 5 = 0; \quad \Rightarrow \quad \lambda_1 = 5; \quad \lambda_2 = 1.$

$$\lambda_1 = 5 \qquad \Longrightarrow \qquad (\mathbf{A} - 5\mathbf{I}) = \begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{x} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\lambda_1 = 1 \qquad \Longrightarrow \qquad (\mathbf{A} - \mathbf{I}) = \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Or in an alternative way:

$$\left[\begin{array}{c} \mathbf{A} \\ \hline \mathbf{I} \end{array}\right] \xrightarrow[1\mathbf{I}]{(-)} \left[\begin{array}{cc} 3 & 3 \\ 1 & 1 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc} 3 & 0 \\ 1 & 0 \\ \hline 1 & -1 \\ 0 & 1 \end{array}\right] \xrightarrow[\substack{\boldsymbol{\tau} \\ [(1)\mathbf{2}+\mathbf{1}]}]{} \left[\begin{array}{cc} 4 & 0 \\ 1 & 0 \\ \hline 1 & -1 \\ 0 & 1 \end{array}\right] \xrightarrow[1\mathbf{I}]{(+)} \left[\begin{array}{cc} 5 & 0 \\ 1 & 1 \\ \hline 1 & -1 \\ 0 & 1 \end{array}\right]$$

$$\xrightarrow[5\mathbf{I}]{(-)} \left[\begin{array}{cc} 0 & 0 \\ \hline 1 & -4 \\ \hline 1 & -1 \\ 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(4)\mathbf{1}] \\ [(1)\mathbf{2}+\mathbf{1}]}} \left[\begin{array}{cc} 0 & 0 \\ \hline 0 & -4 \\ \hline 3 & -1 \\ 1 & 1 \end{array}\right] \xrightarrow[\substack{\boldsymbol{\tau} \\ [(1/4)\mathbf{1}] \\ [(-1)\mathbf{1}+\mathbf{2}]}]{} \left[\begin{array}{cc} 0 & 0 \\ \hline 0 & -4 \\ \hline 3 & -1 \\ 1 & 1 \end{array}\right] \xrightarrow[5\mathbf{I}]{(+)} \left[\begin{array}{cc} 5 & 0 \\ 0 & 1 \\ \hline 3 & -1 \\ 1 & 1 \end{array}\right]$$

Hence,

$$\mathbf{A}^{100} = \mathbf{S}\mathbf{D}^{100}\mathbf{S}^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 5^{100} & \\ & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 5 \cdot 5^{100} & 1 \\ 5^{100} & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \frac{1}{4} = \frac{1}{4} \begin{bmatrix} 3 \cdot 5^{100} - 1 & 3 \cdot 5^{100} + 3 \\ 5^{100} + 1 & 5^{100} - 3 \end{bmatrix}$$

□

(L-17) Question 4(a) True. $|\mathbf{A}| = 1 \times 1 \times 2 = 2 \neq 0$, therefore $\mathbf{A}$ is invertible.

□

(L-17) Question 4(b) We don't know. It has a repeated eigenvalue (might have 2 or 3 independent eigenvectors)

□

(L-17) Question 4(c) We don't know. It has a repeated eigenvalue (might have 2 or 3 independent eigenvectors)

□

(L-17) Question 5(a) Since the matrix is triangular, the elements on its main diagonal ($\lambda = 4$ and $\lambda = 2$) are the eigenvalues (both with algebraic multiplicity two):

$$\operatorname{rg}\left(\mathbf{A} - 4\mathbf{I}\right) = \operatorname{rg}\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix}\right) = 2; \quad \operatorname{rg}\left(\mathbf{A} - 2\mathbf{I}\right) = \operatorname{rg}\left(\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right) = 2.$$

Por tanto ya sabemos que $\mathbf{A}$ es diagonalizable.

Observando la matriz $(\mathbf{A} - 4\mathbf{I})$, es fácil ver que dos autovectores asociados a $\lambda = 4$ son $\begin{pmatrix} 2, & 0, & 0, & 1, \end{pmatrix}$ and $\begin{pmatrix} 0, & 1, & 0, & 0, \end{pmatrix}$; y observando la matriz $(\mathbf{A} - 2\mathbf{I})$, que dos autovectores asociados a $\lambda = 2$ son $\begin{pmatrix} 0, & 0, & 1, & 0, \end{pmatrix}$ and $\begin{pmatrix} 0, & 0, & 0, & 1, \end{pmatrix}$. Así pues, $\mathbf{D} = \begin{bmatrix} 4 & & & \\ & 4 & & \\ & & 2 & \\ & & & 2 \end{bmatrix}$; and $\mathbf{S} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$.

□

(L-17) Question 5(b) Puesto que hemos visto que $\boldsymbol{v}$ es un autovector de $\mathbf{A}$ asociado al autovalor $\lambda = 2$, sabemos que $\mathbf{A}\boldsymbol{v} = 2\boldsymbol{v}$, y por tanto:

$$\begin{aligned}
\left(\mathbf{A}^6\right)\boldsymbol{v} =& \mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\boldsymbol{v}\\
=& \mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\lambda\boldsymbol{v}\\
=& \mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\lambda^2\boldsymbol{v}\\
=& \mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\lambda^3\boldsymbol{v}\\
=& \mathbf{A}\cdot\mathbf{A}\cdot\lambda^4\boldsymbol{v}\\
=& \mathbf{A}\cdot\lambda^5\boldsymbol{v}\\
=& \lambda^6\boldsymbol{v} = 2^6\boldsymbol{v} = 64\boldsymbol{v} = \begin{pmatrix}0, & 0, & 0, & 64,\end{pmatrix}.
\end{aligned}$$

□

(L-17) Question 5(c) Puesto que ningún autovalor es cero, la matriz es de rango completo, es decir, invertible.

□

(L-17) Question 5(d) Puesto que $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$, entonces

$$\mathbf{A}^{-1} = \left(\mathbf{S}\mathbf{D}\mathbf{S}^{-1}\right)^{-1} = \left(\mathbf{D}\mathbf{S}^{-1}\right)^{-1}\mathbf{S}^{-1} = \left(\mathbf{S}^{-1}\right)^{-1}\mathbf{D}^{-1}\mathbf{S}^{-1} = \mathbf{S}\mathbf{D}^{-1}\mathbf{S}^{-1};$$

es decir, los autovectores $\mathbf{S}$ son los mismos, pero los autovalores $\mathbf{D}^{-1}$, son los inversos de los autovalores de la matriz $\mathbf{A}$.

□

(L-17) Question 6.

$$\mathbf{A}^3 = (\mathbf{S})(\mathbf{D}^3)(\mathbf{S}^{-1}); \quad \text{y} \quad \mathbf{A}^{-1} = (\mathbf{S})(\mathbf{D}^{-1})(\mathbf{S}^{-1}).$$

□

(L-17) Question 7(a) Debemos resolver la ecuación característica

$$\begin{bmatrix}1-\lambda & 0\\ -1 & 2-\lambda\end{bmatrix} = (1-\lambda)(2-\lambda) = 0.$$

Así $\lambda = 1,\ 2$ (por ser una matriz triangular).

□

(L-17) Question 7(b) Para encontrar los auto-vectores correspondientes a λ, debemos encontrar el espacio nulo de $(\mathbf{A} - \lambda\mathbf{I})$. Hay dos casos:

- $\lambda = 1$. Aquí debemos resolver $\begin{bmatrix}0 & 0\\ -1 & 1\end{bmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$. Así, un auto-vector es $\begin{pmatrix}1\\ 1\end{pmatrix}$.
- $\lambda = 2$. Aquí tenemos que resolver $\begin{bmatrix}-1 & 0\\ -1 & 0\end{bmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$. En este caso un auto-vector es $\begin{pmatrix}0\\ 1\end{pmatrix}$.

□

(L-17) Question 7(c) De los apartados anteriores concluimos que:

$$\mathbf{S} = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}; \qquad \mathbf{D} = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}.$$

Por ser $\mathbf{S}$ una matriz elemental, sabemos que su inversa es:

$$\mathbf{S}^{-1} = \begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}$$

Por tanto

$$\mathbf{A} = \begin{bmatrix}1 & 0\\ -1 & 2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}.$$

Pero también podiamos haberlo hecho así

$$\left[\begin{array}{c}\mathbf{A}\\\hline\mathbf{I}\end{array}\right]\xrightarrow[2\mathbf{I}]{(-)}\left[\begin{array}{cc}-1&0\\-1&0\\\hline 1&0\\0&1\end{array}\right]\xrightarrow[2\mathbf{I}]{(+)}\left[\begin{array}{cc}1&0\\-1&2\\\hline 1&0\\0&1\end{array}\right]\xrightarrow[1\mathbf{I}]{(-)}\left[\begin{array}{cc}0&0\\\hline -1&1\\\hline 1&0\\0&1\end{array}\right]\xrightarrow{\underset{[(1)\mathbf{2}+\mathbf{1}]}{\boldsymbol{\tau}}}\left[\begin{array}{cc}0&0\\\hline 0&1\\\hline 1&0\\1&1\end{array}\right]\xrightarrow[\underset{[(-1)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}]{}\left[\begin{array}{cc}0&0\\\hline 0&1\\\hline 1&0\\1&1\end{array}\right]\xrightarrow[1\mathbf{I}]{(+)}\left[\begin{array}{cc}1&0\\0&2\\\hline 1&0\\1&1\end{array}\right]$$

$\square$

(L-17) Question 8(a) **Verdadero**. Puesto que ninguno de los autovalores es nulo, el determinante (que es igual al producto de los autovalores) es necesariamente distinto de cero, y por tanto la matriz es invertible.

$\square$

(L-17) Question 8(b) **Falso**. Si no hay autovalores repetidos, sabemos que necesariamente la matriz es diagonalizable, pues existen suficientes auto-vectores linealmente independientes, como para generar una matriz $\mathbf{S}$ invertible. Puesto que el auto-valor 2 está repetido, no podemos saber si existen suficientes auto-vectores linealmente independientes.

$\square$

(L-17) Question 8(c) **Falso**. No lo podemos saber. Necesitamos conocer si hay tres auto-vectores linealmente independientes (lo sabríamos si no hubiera autovalores repetidos).

$\square$

(L-17) Question 9(a)

$$\mathbf{A}_1=\begin{bmatrix}8&a\\b&2\end{bmatrix},\quad a=\frac{-9}{b};\qquad \mathbf{A}_2=\begin{bmatrix}9&4\\-4&1\end{bmatrix};\qquad \mathbf{A}_3=\begin{bmatrix}10&5\\-5&0\end{bmatrix}$$

$\square$

(L-17) Question 9(b) El espacio nulo de la matriz $(\mathbf{A}_1-5\mathbf{I})$; es decir, de $\begin{bmatrix}3&3\\-3&-3\end{bmatrix}$; es el conjunto de vectores múltiplos de $\boldsymbol{x}=\begin{pmatrix}-1\\1\end{pmatrix}$, y el es mismo que el espacio nulo de las matrices $(\mathbf{A}_2-5\mathbf{I})=\begin{bmatrix}4&4\\-4&-4\end{bmatrix}$ y $(\mathbf{A}_2-5\mathbf{I})=\begin{bmatrix}5&5\\-5&-5\end{bmatrix}$. Por tanto, en los tres casos no podemos encontrar dos auto-vectores linealmente independientes, y en consecuencia estas matrices no son diagonalizables.

$\square$

(L-17) Question 10(a) $\begin{bmatrix}1&1\\1&1\end{bmatrix}=\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}2&0\\0&0\end{bmatrix}\begin{bmatrix}1&1\\1&-1\end{bmatrix}^{-1}.$

$\square$

(L-17) Question 10(b) $\begin{bmatrix}2&1\\0&0\end{bmatrix}=\begin{bmatrix}1&1\\0&-2\end{bmatrix}\begin{bmatrix}2&0\\0&0\end{bmatrix}\begin{bmatrix}1&1\\0&-2\end{bmatrix}^{-1}.$

$\square$

(L-17) Question 11.

$$\begin{bmatrix}3&2\\1&1\end{bmatrix}^{-1}=\begin{bmatrix}1&-2\\-1&3\end{bmatrix};\qquad \begin{bmatrix}3&2\\1&1\end{bmatrix}\begin{bmatrix}1&0\\0&4\end{bmatrix}\begin{bmatrix}1&-2\\-1&3\end{bmatrix}=\begin{bmatrix}-5&18\\-3&10\end{bmatrix}$$

Comprobación: Traza $=5$; determinante $=4$.

$\square$

(L-17) Question 12. Sabemos que es diagonalizable, puesto que no tiene autovalores repetidos. La matriz de autovalores $\mathbf{D}$ es una matriz diagonal; con los valores 1, 2 y 7 es su diagonal principal (en cualquier orden).

$\square$

(L-17) Question 13(a) Los tres autovalores son iguales a 1 (son los elementos de la diagonal, por ser $\mathbf{A}$ triangular). Los auto-vectores para el único auto-valor (triple) $\lambda=1$ se calculan partiendo de la matriz $[\mathbf{A}-1\mathbf{I}]$:

$$\mathbf{A}-1\mathbf{I}=\begin{bmatrix}0&0&0\\-2&0&0\\1&0&0\end{bmatrix}$$

que tiene sólo dos columnas libres; por tanto debemos encontrar dos soluciones especiales (cuyas combinaciones lineales son el espacio nulo de la matriz anterior y que constituyen los auto-vectores asociados al auto-valor $\lambda = 1$):

$$\boldsymbol{x}_a = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \qquad \boldsymbol{x}_b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$\square$

(L-17) Question 13(b) La matriz no es diagonalizable ya que como máximo podemos encontrar 2 auto-vectores linealmente independientes (para que lo fuera necesitaríamos encontrar 3).

$\square$

(L-17) Question 14(a) Es diagonalizable, puesto que tiene tres auto-vectores linealmente independientes. Así pues:

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}; \qquad \mathbf{D} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}.$$

$\square$

(L-17) Question 14(b)

$$\mathbf{A} = \mathbf{SDS}^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$\square$

(L-17) Question 15.

$$\begin{aligned}\left(\mathbf{A}^3\right)\boldsymbol{v} &= \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & \\ & -2 \end{bmatrix}^3 \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2^3 & \\ & -2^3 \end{bmatrix} \begin{bmatrix} 1/2 & 1 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{bmatrix} 8 & 32 \\ 0 & -8 \end{bmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -24 \\ 8 \end{pmatrix}.\end{aligned}$$

$\square$

(L-18) Question 1(a) Characteristic equation:

$$0 = |\mathbf{A} - \lambda\mathbf{I}| = \lambda^2 - \lambda,$$

then $\lambda_1 = 0$ and $\lambda_2 = 1$

- $\lambda_1 = 0$

  $$\left(\mathbf{A} - \mathbf{0}\right) = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

  The corresponding unit eigenvector is $\boldsymbol{x}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

- $\lambda_1 = 1$

  $$\left(\mathbf{A} - \mathbf{I}\right) = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

  The corresponding unit eigenvector is $\boldsymbol{x}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Therefore

$$\mathbf{A} = \mathbf{QDQ}^{\intercal} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & \\ & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}}$$

$\square$

(L-18) Question 1(b) Characteristic equation:

$$0 = |\mathbf{B} - \lambda \mathbf{I}| = \lambda^2 - 1,$$

then $\lambda = \pm 1$

- $\lambda_1 = 1$

$$(\mathbf{B} - \mathbf{I}) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

The corresponding unit eigenvector is $\boldsymbol{x}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- $\lambda_2 = -1$

$$(\mathbf{B} + \mathbf{I}) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The corresponding unit eigenvector is $\boldsymbol{x}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Therefore

$$\mathbf{B} = \mathbf{Q}\mathbf{D}\mathbf{Q}^\mathsf{T} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}}$$

□

(L-18) Question 1(c) Characteristic equation:

$$0 = |\mathbf{C} - \lambda \mathbf{I}| = (3 - \lambda)(-3 - \lambda) - 16 = \lambda^2 - 25,$$

therefore $\lambda = \pm 5$

- $\lambda_1 = 5$

$$(\mathbf{C} - 5\mathbf{I}) = \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix}$$

The corresponding unit eigenvector is $\boldsymbol{x}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

- $\lambda_1 = -5$

$$(\mathbf{C} + 5\mathbf{I}) = \begin{bmatrix} 8 & 4 \\ 4 & -2 \end{bmatrix}$$

The corresponding unit eigenvector is $\boldsymbol{x}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Therefore

$$\mathbf{C} = \mathbf{Q}\mathbf{D}\mathbf{Q}^\mathsf{T} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & \\ & -5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \frac{1}{\sqrt{5}}$$

□

(L-18) Question 2. Characteristic equation:

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda^3 + \lambda^2 + 2\lambda = 0 \quad \Rightarrow \quad \begin{cases} \lambda = 0 \\ -\lambda^2 + \lambda + 2 \Rightarrow \begin{cases} \lambda = 2 \\ \lambda = -1 \end{cases} \end{cases}$$

$\lambda = 0$, 2, -1; with unit eigenvectors

$$\boldsymbol{x}_1 = \pm \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}; \qquad \boldsymbol{x}_2 = \pm \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}; \qquad \boldsymbol{x}_3 = \pm \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

since

$$\left[\begin{array}{c}\mathbf{A}-{\color{blue}0}\mathbf{I}\\ \hline \mathbf{I}\end{array}\right]=\left[\begin{array}{ccc}1&1&1\\1&0&0\\1&0&0\\ \hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow{\substack{\tau\\ [(-1)\mathbf{1}+\mathbf{2}]\\ [(-1)\mathbf{1}+\mathbf{3}]}}\left[\begin{array}{ccc}1&0&0\\1&-1&-1\\1&-1&-1\\ \hline 1&-1&-1\\0&1&0\\0&0&1\end{array}\right]\xrightarrow{\substack{\tau\\ [(-1)\mathbf{2}+\mathbf{3}]}}\left[\begin{array}{ccc}1&0&0\\1&-1&0\\1&-1&0\\ \hline 1&-1&\mathbf{0}\\0&1&\mathbf{-1}\\0&0&\mathbf{1}\end{array}\right]$$

$$\left[\begin{array}{c}\mathbf{A}-{\color{blue}2}\mathbf{I}\\ \hline \mathbf{I}\end{array}\right]=\left[\begin{array}{ccc}-1&1&1\\1&-2&0\\1&0&-2\\ \hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow{\substack{\tau\\ [(1)\mathbf{1}+\mathbf{2}]\\ [(1)\mathbf{1}+\mathbf{3}]}}\left[\begin{array}{ccc}-1&0&0\\1&-1&1\\1&1&-1\\ \hline 1&1&1\\0&1&0\\0&0&1\end{array}\right]\xrightarrow{\substack{\tau\\ [(1)\mathbf{2}+\mathbf{3}]}}\left[\begin{array}{ccc}-1&0&0\\1&-1&0\\1&1&0\\ \hline 1&1&\mathbf{2}\\0&1&\mathbf{1}\\0&0&\mathbf{1}\end{array}\right]$$

$$\left[\begin{array}{c}\mathbf{A}+{\color{blue}1}\mathbf{I}\\ \hline \mathbf{I}\end{array}\right]=\left[\begin{array}{ccc}2&1&1\\1&1&0\\1&0&1\\ \hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow{\substack{\tau\\ [(2)\mathbf{2}]\\ [(-1)\mathbf{1}+\mathbf{2}]\\ [(2)\mathbf{3}]\\ [(-1)\mathbf{1}+\mathbf{3}]}}\left[\begin{array}{ccc}2&0&0\\1&1&-1\\1&-1&1\\ \hline 1&-1&-1\\0&2&0\\0&0&2\end{array}\right]\xrightarrow{\substack{\tau\\ [(1)\mathbf{2}+\mathbf{3}]}}\left[\begin{array}{ccc}2&0&0\\1&1&0\\1&-1&0\\ \hline 1&-1&\mathbf{-2}\\0&2&\mathbf{2}\\0&0&\mathbf{2}\end{array}\right]$$

□

(L-18) Question 3. $|\mathbf{A}-\lambda\mathbf{I}|=\lambda\left(9-\lambda^2\right)=0\ \to\ \begin{cases}\lambda_1=0\\ \lambda_2=3\\ \lambda_3=-3\end{cases}.$

$$\begin{cases}
\lambda_1=-3:\ \left[\begin{array}{c}\mathbf{A}+3\mathbf{I}\\ \hline \mathbf{I}\end{array}\right]=\left[\begin{array}{ccc}4&0&2\\0&2&-2\\2&-2&3\\ \hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow{\substack{\tau\\ [(2)\mathbf{3}]\\ [(-1)\mathbf{1}+\mathbf{3}]}}\left[\begin{array}{ccc}4&0&0\\0&2&-4\\2&-2&4\\ \hline 1&0&-1\\0&1&0\\0&0&2\end{array}\right]\xrightarrow{\substack{\tau\\ [(2)\mathbf{2}+\mathbf{3}]}}\left[\begin{array}{ccc}4&0&0\\0&2&0\\2&-2&0\\ \hline 1&0&-1\\0&1&2\\0&0&2\end{array}\right]\\[2ex]
\lambda_1=0:\ \left[\begin{array}{c}\mathbf{A}-0\mathbf{I}\\ \hline \mathbf{I}\end{array}\right]=\left[\begin{array}{ccc}1&0&2\\0&-1&-2\\2&-2&0\\ \hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow{\substack{\tau\\ [(-2)\mathbf{1}+\mathbf{3}]}}\left[\begin{array}{ccc}1&0&0\\0&-1&-2\\2&-2&-4\\ \hline 1&0&-2\\0&1&0\\0&0&1\end{array}\right]\xrightarrow{\substack{\tau\\ [(-2)\mathbf{2}+\mathbf{3}]}}\left[\begin{array}{ccc}1&0&0\\0&-1&0\\2&-2&0\\ \hline 1&0&-2\\0&1&-2\\0&0&1\end{array}\right]\\[2ex]
\lambda_1=3:\ \left[\begin{array}{c}\mathbf{A}-3\mathbf{I}\\ \hline \mathbf{I}\end{array}\right]=\left[\begin{array}{ccc}-2&0&2\\0&-4&-2\\2&-2&-3\\ \hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow{\substack{\tau\\ [(1)\mathbf{1}+\mathbf{3}]}}\left[\begin{array}{ccc}-2&0&0\\0&-4&-2\\2&-2&-1\\ \hline 1&0&1\\0&1&0\\0&0&1\end{array}\right]\xrightarrow{\substack{\tau\\ [(2)\mathbf{3}]\\ [(-1)\mathbf{2}+\mathbf{3}]}}\left[\begin{array}{ccc}-2&0&0\\0&-4&0\\2&-2&0\\ \hline 1&0&2\\0&1&-1\\0&0&2\end{array}\right]
\end{cases}$$

Hence, for example

$$\text{If}\quad \mathbf{D}=\begin{bmatrix}0&0&0\\0&3&0\\0&0&-3\end{bmatrix}\quad\text{then}\quad \mathbf{Q}=\frac{1}{3}\begin{bmatrix}2&-2&-1\\2&1&2\\-1&-2&2\end{bmatrix}.$$

```
B = Matrix([[1,0,2],[0,-1,-2],[2,-2,0]])
l = sympy.symbols('\lambda')
```

```
d = Determinante( (B-1*I(B.n)) ,1)
p = sympy.poly(d.valor)
e = sympy.real_roots(p)
display(Sistema([d, e]))
S=Sistema([])
D = tuple()
for i in set(e):
    caso = "\lambda = %d\n" % i
    display(Math(caso))
    L = Elim( (B-i*I(B.n)).apila(I(B.n),1), 1)
    cL0 = tuple([c+1 for c,v in enumerate(slice(1,B.n)|L) if v.es_nulo()])
    S = S.concatena(slice(B.n+1,None)|L|cL0).sis()
    D = D+(i,)*e.count(i)

Q = Matrix([v.normalizado() for v in S]).GS()
D = Vector(D).diag()
display(Sistema([D,Q,~Q,Q**-1]))
display(Sistema([(~Q)*B*Q, Q*D*(~Q)]))
~Q*Q
```

O sencillamente

```
B = Matrix([[1,0,2],[0,-1,-2],[2,-2,0]])
espectro = [0,3,-3]
D = DiagonalizaO( B, espectro )
display(Sistema([D, D.Q]))
```

□

(L-18) Question 4(a) $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are orthogonal to each other.

□

(L-18) Question 4(b) The nullspace is spanned by $\boldsymbol{u}$; the left nullspace is the same as the nullspace; the row space is spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$; the column space is the same as the row space.

$$\mathcal{C}\left(\mathbf{A}\right) = \mathcal{C}\left(\mathbf{A}^{\intercal}\right) \perp \mathcal{N}\left(\mathbf{A}\right) = \mathcal{N}\left(\mathbf{A}^{\intercal}\right).$$

□

(L-18) Question 4(c) $\boldsymbol{x} = \boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$, since

$$\mathbf{A}(\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}) = \mathbf{A}\boldsymbol{v} + \frac{1}{2}\mathbf{A}\boldsymbol{w} = \boldsymbol{v} + \frac{1}{2}(2\boldsymbol{w}) = \boldsymbol{v} + \boldsymbol{w};$$

not unique, we can add any multiple of $\boldsymbol{u}$ to $\boldsymbol{x}$.

□

(L-18) Question 4(d) Since $\boldsymbol{b} \in \mathcal{C}\left(\mathbf{A}\right) = \mathcal{C}\left(\mathbf{A}^{\intercal}\right)$ and $\mathcal{N}\left(\mathbf{A}\right)$ es perpendicular a $\mathcal{C}\left(\mathbf{A}^{\intercal}\right)$ $=\mathcal{C}\left(\mathbf{A}\right)$, we need $\boldsymbol{b}\cdot\boldsymbol{u} = 0$.
□

(L-18) Question 4(e) $\mathbf{S}^{-1} = \mathbf{S}^{\intercal}$;

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 2 \end{bmatrix} = \mathbf{D}.$$

□

(L-18) Question 5(a) Autovalores reales.

□

(L-18) Question 5(b) Todos los autovalores son menores que uno en valor absoluto.

□

(L-18) Question 5(c) Tiene autovalores repetidos

□

(L-18) Question 5(d) Tiene al menos un autovalor igual a cero.

□

(L-18) Question 6(a) La ecuación característica es $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

$$\begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 & 0 \\ -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \end{vmatrix} = \lambda^4 - 1 = 0$$

por tanto $\lambda = \pm 1$ y $\lambda = \pm i$; es decir cuatro autovalores distintos.

□

(L-18) Question 6(b) La matriz tiene un espacio nulo de dimensión 3. Busquemos, por tanto, tres autovectores que sean base del espacio nulo de $\mathbf{B}$ (autovalor igual a cero):

$$\boldsymbol{x}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad \boldsymbol{x}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad \boldsymbol{x}_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Además,

$$\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Por tanto, hay un triple autovalor igual a 0, y otro autovalor igual a 1. Y puesto que hemos encontrado cuatro autovectores linealmente independientes, esta matriz es diagonalizable.

□

(L-18) Question 7. Si $\mathbf{A}^3 = \mathbf{0}$ entonces para todos los autovalores $\lambda^3 = 0$, es decir $\lambda = 0$ como en

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Si $\mathbf{A}$ es simétrica, entonces es diagonalizable, y por el Teorema espectral, se puede factorizar como

$$\mathbf{A}^3 = \mathbf{Q}\mathbf{D}^3\mathbf{Q}^\intercal = \mathbf{Q}\mathbf{0}\mathbf{Q}^\intercal = \mathbf{0}.$$

□

(L-18) Question 8(a) Los autovalores de

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

son los elementos de la diagonal; $\lambda = 3$ (con multiplicidad 2) y $\lambda = 2$. Pero el rango de

$$\mathbf{A} - 3\lambda = \begin{bmatrix} 3-3 & 1 & 1 \\ 0 & 3-3 & 1 \\ 0 & 0 & 2-3 \end{bmatrix}$$

es 2; por tanto la matriz no es diagonalizable.

□

(L-18) Question 8(b) Los autovalores de

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

son los elementos de la diagonal; $\lambda = 3$ y $\lambda = 2$ (con multiplicidad 2). El rango de

$$\mathbf{A} - 2\lambda = \begin{bmatrix} 2-2 & 1 & 1 \\ 0 & 3-2 & 1 \\ 0 & 0 & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

es 1; por tanto la matriz es diagonalizable.

Dos autovectores independientes correspondientes al autovalor $\lambda = 2$ son $\begin{pmatrix}1\\0\\0\end{pmatrix}$ y $\begin{pmatrix}0\\1\\-1\end{pmatrix}$.

Por otra parte

$$\mathbf{A} - 3\lambda = \begin{bmatrix}2-3 & 1 & 1\\ 0 & 3-3 & 1\\ 0 & 0 & 2-3\end{bmatrix} = \begin{bmatrix}-1 & 1 & 1\\ 0 & 0 & 1\\ 0 & 0 & -1\end{bmatrix}$$

Un autovector correspondiente al autovalor $\lambda = 3$ es $\begin{pmatrix}1\\1\\0\end{pmatrix}$.

Así pues

$$\mathbf{D} = \begin{bmatrix}2 & & \\ & 2 & \\ & & 3\end{bmatrix} \quad \text{and} \quad \mathbf{S} = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & -1 & 0\end{bmatrix}$$

son matrices tales que $\mathbf{A} = \mathbf{SDS^{-1}}$.

□

(L-18) Question 8(c) Sea como sea $\mathbf{A}$, la matriz $\mathbf{A^{\mathsf{T}}A}$ siempre es simétrica; y por tanto es diagonalizable, y es posible encontrar una base ortonormal de autovectores de $\mathbf{A^{\mathsf{T}}A}$.

□

(L-18) Question 8(d) Basta con encontrar los valores de a que hacen la matriz de rango completo; es decir, cualquier valor de a distinto de cero ($a \neq 0$) (para que la matriz sea invertible) y simultáneamente distinto de tres ($a \neq 3$) (para que la matriz sea diagonalizable).

□

(L-18) Question 9(a) Ecuación característica:

$$0 = |\mathbf{B} - \lambda\mathbf{I}| = \lambda^2 - 1,$$

por tanto $\lambda = \pm 1$

- $\lambda_1 = 1$

$$(\mathbf{B} - \mathbf{I}) = \begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$$

  Con autovector unitario $\boldsymbol{x}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

- $\lambda_1 = -1$

$$(\mathbf{B} + \mathbf{I}) = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

  Con autovector unitario $\boldsymbol{x}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}$

Por tanto

$$\mathbf{B} = \mathbf{QDQ^{\mathsf{T}}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & \\ & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\frac{1}{\sqrt{2}}$$

□

(L-18) Question 9(b) Puesto que los dos autovalores son distintos los dos autovectores encontrados son linealmente independientes, y la matriz es diagonalizable. Una matriz $\mathbf{S}$ que diagonaliza $\mathbf{B}$ es

$$\mathbf{S} = \begin{bmatrix}\boldsymbol{x}_1 & \boldsymbol{x}_2\end{bmatrix} = \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix};$$

ya que $\mathbf{BS} = \mathbf{SD}$, donde

$$\mathbf{D} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

Puesto que la matriz $\mathbf{S}$ es invertible, ya que ambos auto-vectores son linealmente independientes, podemos diagonalizar $\mathbf{B}$ del siguiente modo:

$$\mathbf{S}^{-1}\mathbf{B}\mathbf{S} = \mathbf{D};$$

donde la matriz diagonal contiene los autovalores de $\mathbf{B}$.

□

(L-18) Question 10(a) The three equations tell us that $\mathbf{A}\mathbf{Q} = \mathbf{I}$, where the columns of $\mathbf{Q}$ are $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$. So $\mathbf{A} = \mathbf{Q}^{-1}$, and since $\mathbf{Q}$ is an orthogonal matrix, we have $\mathbf{A} = \mathbf{Q}^{-1} = \mathbf{Q}^{\intercal} = \left[\boldsymbol{q}_1; \quad \boldsymbol{q}_2; \quad \boldsymbol{q}_3;\right]^{\intercal}$.

□

(L-18) Question 10(b) ComoSince $\mathbf{A} = \mathbf{Q}^{\intercal}$ then $\det \mathbf{A} = \det \mathbf{Q}$; and since $\mathbf{Q}^{\intercal}\mathbf{Q} = \mathbf{I}$

$$1 = \det \mathbf{I} = \det\left(\mathbf{Q}^{\intercal}\mathbf{Q}\right) = \det\left(\mathbf{Q}^{\intercal}\right)\det\left(\mathbf{Q}\right) = |\mathbf{Q}|^2 \quad \Rightarrow \quad \det \mathbf{A} \pm 1.$$

□

(L-18) Question 10(c) The eigenvalues of $\mathbf{A}$ must have absolute value $|\lambda| = 1$.

Proof: Let λ be an eigenvalue of $\mathbf{A}$ and let $\boldsymbol{v}$ be a corresponding eigenvector. Then we have

$$\mathbf{A}\boldsymbol{v} = \lambda\boldsymbol{v}.$$

It follows from this we have

$$\|\mathbf{A}\boldsymbol{v}\|^2 = \|\lambda\boldsymbol{v}\|^2 = |\lambda|^2\|\boldsymbol{v}\|^2.$$

And since $\mathbf{A}$ is orthogonal ($\mathbf{A}^{\intercal}\mathbf{A} = \mathbf{I}$), the left hand side becomes

$$\|\mathbf{A}\boldsymbol{v}\|^2 \;=\; \mathbf{A}\boldsymbol{v}\cdot\mathbf{A}\boldsymbol{v} \;=\; \boldsymbol{v}\mathbf{A}^{\intercal}\mathbf{A}\boldsymbol{v} \;=\; \boldsymbol{v}\cdot\boldsymbol{v} \;=\; \|\boldsymbol{v}\|^2$$

It follows that

$$|\lambda|^2\|\boldsymbol{v}\|^2 = \|\boldsymbol{v}\|^2.$$

Since $\boldsymbol{v}$ is an eigenvector, it is non-zero, and hence $\|\boldsymbol{v}\| \neq \mathbf{0}$. Canceling $\|\boldsymbol{v}\|$, we have $|\lambda|^2 = 1$. Since the absolute value is non-negative, we obtain $\boxed{|\lambda| = 1}$.

□

(L-19) Question 1(a) No. $\begin{bmatrix}1&3\\3&5\end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau}\\ [(-3)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix}1&0\\3&-4\end{bmatrix} \xrightarrow[\substack{\boldsymbol{\tau}\\ [(-3)\mathbf{1}+\mathbf{2}]}]{} \begin{bmatrix}1&0\\0&-4\end{bmatrix}$. $\qquad f(x,y) = x^2 + 6xy + 5y^2$.

```
A = Matrix([[1,3],[3,5]])
DiagonalizaC(A,1)
```

□

(L-19) Question 1(b) No. $\begin{bmatrix}1&-1\\-1&1\end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau}\\ [(1)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix}1&0\\-1&0\end{bmatrix} \xrightarrow[\substack{\boldsymbol{\tau}\\ [(1)\mathbf{1}+\mathbf{2}]}]{} \begin{bmatrix}1&0\\0&0\end{bmatrix}$. $\qquad f(x,y) = x^2 - 2xy + y^2 = (x-y)^2$.

□

(L-19) Question 1(c) Yes. $\begin{bmatrix}2&3\\3&5\end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau}\\ [(2)\mathbf{2}]\\ [(-3)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix}2&0\\3&1\end{bmatrix} \xrightarrow[\substack{\boldsymbol{\tau}\\ [(-3)\mathbf{1}+\mathbf{2}]\\ [(2)\mathbf{2}]}]{} \begin{bmatrix}2&0\\0&2\end{bmatrix}$. $\qquad f(x,y) = 2x^2 + 6xy + 5y^2$.

□

(L-19) Question 1(d) No, There are negative number in the main diagonal. $\begin{bmatrix}-1&2\\2&-8\end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau}\\ [(2)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix}-1&0\\2&-4\end{bmatrix} \xrightarrow[\substack{\boldsymbol{\tau}\\ [(2)\mathbf{1}+\mathbf{2}]}]{} \begin{bmatrix}-1&0\\0&-4\end{bmatrix}$. $\qquad f(x,y) = -x^2 + 4xy - 8y^2$.

□

(L-19) Question 1(e) The function $f(x,y)=(y-x)^2$ is zero along the line $y=x$. Note that (1,1) is an eigenvector corresponding to $\lambda=0$.

$\square$

(L-19) Question 2(a) $\begin{bmatrix}1&2\\2&9\end{bmatrix} \xrightarrow{\underset{[(-2)\mathbf{1}+\mathbf{2}]}{\tau}} \begin{bmatrix}1&0\\2&5\end{bmatrix} \xrightarrow[\underset{[(-2)\mathbf{1}+\mathbf{2}]}{\tau}]{} \begin{bmatrix}1&0\\0&5\end{bmatrix}$. Hence

$$\mathbf{D}=\begin{bmatrix}1&0\\0&5\end{bmatrix}=\begin{bmatrix}1&0\\-2&1\end{bmatrix}\begin{bmatrix}1&2\\2&9\end{bmatrix}\begin{bmatrix}1&-2\\0&1\end{bmatrix}=\mathbf{B}^\intercal\mathbf{A}\mathbf{B}.$$

Therefore, $\boldsymbol{x}\mathbf{A}\boldsymbol{x}=\boldsymbol{x}\left(\mathbf{B}^{-1}\right)^\intercal\mathbf{D}\mathbf{B}^{-1}\boldsymbol{x}=\begin{pmatrix}x&y\end{pmatrix}\begin{bmatrix}1&0\\2&1\end{bmatrix}\begin{bmatrix}1&0\\0&5\end{bmatrix}\begin{bmatrix}1&2\\0&1\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+2y&y\end{pmatrix}\begin{bmatrix}1&0\\0&5\end{bmatrix}\begin{pmatrix}x+2y\\y\end{pmatrix}.$

So, $f(x,y)=x^2+4\,x\,y+9\,y^2=(x+2y)^2+5(y)^2$.

$\square$

(L-19) Question 2(b) $\begin{bmatrix}1&3\\3&9\end{bmatrix} \xrightarrow{\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\tau}} \begin{bmatrix}1&0\\3&0\end{bmatrix} \xrightarrow[\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\tau}]{} \begin{bmatrix}1&0\\0&0\end{bmatrix}$. $\quad f(x,y)=x^2+6\,x\,y+9\,y^2=(x+3y)^2+0(y)^2.$

$\square$

(L-19) Question 3(a) Not this one. The determinant is negative.

$\square$

(L-19) Question 3(b) Not this one; since $a=-1$.

$\square$

(L-19) Question 3(c) Not this one; This one is singular ($\det\mathbf{C}=0$).

$\square$

(L-19) Question 3(d) $\mathbf{D}$ has two positive eigenvalues since $a=1$ and $\det\mathbf{A}=1$.

$\square$

(L-19) Question 4. $\begin{bmatrix}1&2\\2&3\end{bmatrix} \xrightarrow{\underset{[(-2)\mathbf{1}+\mathbf{2}]}{\tau}} \begin{bmatrix}1&0\\2&-1\end{bmatrix} \xrightarrow[\underset{[(-2)\mathbf{1}+\mathbf{2}]}{\tau}]{} \begin{bmatrix}1&0\\0&-1\end{bmatrix}$.

$$f(x,y)=x^2+4xy+3y^2=(x+2y)^2-y^2.$$

It is negative (because $-y^2$) at $(2,-1)$:

$$f(2,-1)=-1.$$

$\square$

(L-19) Question 5. If $\mathbf{A}\boldsymbol{x}=\lambda\boldsymbol{x} \quad\Rightarrow\quad \mathbf{A}^2\boldsymbol{x}=\mathbf{A}\mathbf{A}\boldsymbol{x}=\mathbf{A}(\lambda\boldsymbol{x})=\lambda^2\boldsymbol{x}$.

Si $\mathbf{A}$ tiene autovalores λ_i positivos entonces los autovalores de $\mathbf{A}^2$ son λ_i^2 y los de $\mathbf{A}^{-1}$ son $1/\lambda_i$, y por tanto también positivos.

$\square$

(L-19) Question 6(a) The corresponding matrix is

$$\begin{bmatrix}1&-2&0\\-2&4&0\\0&0&5\end{bmatrix} \xrightarrow[\underset{[(2)\mathbf{1}+\mathbf{2}]}{\tau}]{\underset{[(2)\mathbf{1}+\mathbf{2}]}{\tau}} \begin{bmatrix}1&0&0\\0&0&0\\0&0&5\end{bmatrix} \xrightarrow[\underset{[\mathbf{2}\rightleftharpoons\mathbf{3}]}{\tau}]{\underset{[\mathbf{2}\rightleftharpoons\mathbf{3}]}{\tau}} \begin{bmatrix}1&0&0\\0&5&0\\0&0&0\end{bmatrix}$$

Cuya diagonal está formada por dos números positivos y un cero.

```
A = Matrix([[1,-2,0],[-2,4,0],[0,0,5]])
DiagonalizaC(A,1)
```

$\square$

(L-19) Question 6(b) The corresponding matrix is

$$\mathbf{A}=\begin{bmatrix}-1 & 1 & -a\\ 1 & 4 & 0\\ -a & 0 & 1\end{bmatrix}\xrightarrow{\substack{\tau\\ [(1)\mathbf{1+2}]\\ [(-a)\mathbf{1+3}]}}\begin{bmatrix}-1 & 0 & 0\\ 1 & 5 & -a\\ -a & -a & a^2+1\end{bmatrix}\xrightarrow[\substack{\tau\\ [(-a)\mathbf{1+3}]\\ [(1)\mathbf{1+2}]}]{}\begin{bmatrix}-1 & 0 & 0\\ 0 & 5 & -a\\ 0 & -a & a^2+1\end{bmatrix}\dots$$

Since there is a 5 (positive) in the main diagonal, it can't be negative definite.

$\square$

(L-19) Question 7(a) $\mathbf{A}$ is not positive definite since $\begin{bmatrix}1 & 1 & 1\end{bmatrix}^{\intercal}$ is in its nullspace. In fact, diagonalizing by congruence we see it is positive semidefinite

$$\begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}\xrightarrow{\substack{\tau\\ [(2)\mathbf{2}]\\ [(1)\mathbf{1+2}]\\ [(2)\mathbf{3}]\\ [(1)\mathbf{1+3}]}}\begin{bmatrix}2 & 0 & 0\\ -1 & 3 & -3\\ -1 & -3 & 3\end{bmatrix}\xrightarrow[\substack{\tau\\ [(1)\mathbf{1+3}]\\ [(2)\mathbf{3}]\\ [(1)\mathbf{1+2}]\\ [(2)\mathbf{2}]}]{}\begin{bmatrix}2 & 0 & 0\\ 0 & 6 & -6\\ 0 & -6 & 6\end{bmatrix}\xrightarrow{\substack{\tau\\ [(1)\mathbf{2+3}]}}\begin{bmatrix}2 & 0 & 0\\ 0 & 6 & 0\\ 0 & -6 & 0\end{bmatrix}\xrightarrow[\substack{\tau\\ [(1)\mathbf{2+3}]}]{}\begin{bmatrix}2 & 0 & 0\\ 0 & 6 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$\square$

(L-19) Question 7(b) $\mathbf{B}$ is positive definite

$$\begin{bmatrix}2 & -1 & -1\\ -1 & 2 & 1\\ -1 & 1 & 2\end{bmatrix}\xrightarrow{\substack{\tau\\ [(2)\mathbf{2}]\\ [(1)\mathbf{1+2}]\\ [(2)\mathbf{3}]\\ [(1)\mathbf{1+3}]}}\begin{bmatrix}2 & 0 & 0\\ -1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}\xrightarrow[\substack{\tau\\ [(1)\mathbf{1+3}]\\ [(2)\mathbf{3}]\\ [(1)\mathbf{1+2}]\\ [(2)\mathbf{2}]}]{}\begin{bmatrix}2 & 0 & 0\\ 0 & 6 & 2\\ 0 & 2 & 6\end{bmatrix}\xrightarrow{\substack{\tau\\ [(3)\mathbf{3}]\\ [(-1)\mathbf{2+3}]}}\begin{bmatrix}2 & 0 & 0\\ 0 & 6 & 0\\ 0 & 2 & 16\end{bmatrix}\xrightarrow[\substack{\tau\\ [(-1)\mathbf{2+3}]\\ [(3)\mathbf{3}]}]{}\begin{bmatrix}2 & 0 & 0\\ 0 & 6 & 0\\ 0 & 0 & 48\end{bmatrix}$$

Hence, Sylvester's method works in this case!... $\det(2)=2;\qquad \begin{vmatrix}2 & -1\\ -1 & 2\end{vmatrix}=3;\qquad \det(\mathbf{B})=4.$

$\square$

(L-19) Question 7(c) The eigenvalues of $\mathbf{C}$ are the square eigenvalues of

$$\begin{bmatrix}0 & 1 & 2\\ 1 & 0 & 1\\ 2 & 1 & 0\end{bmatrix};$$

hence, if this matrix is full rank, then $\mathbf{C}$ is positive definite. Let's see it by diagonalization by congruence:

$$\begin{bmatrix}0 & 1 & 2\\ 1 & 0 & 1\\ 2 & 1 & 0\end{bmatrix}\xrightarrow{\substack{\tau\\ [(1)\mathbf{2+1}]}}\begin{bmatrix}1 & 1 & 2\\ 1 & 0 & 1\\ 3 & 1 & 0\end{bmatrix}\xrightarrow[\substack{\tau\\ [(1)\mathbf{2+1}]}]{}\begin{bmatrix}2 & 1 & 3\\ 1 & 0 & 1\\ 3 & 1 & 0\end{bmatrix}\xrightarrow{\substack{\tau\\ [(2)\mathbf{2}]\\ [(-1)\mathbf{1+2}]\\ [(2)\mathbf{3}]\\ [(-3)\mathbf{1+3}]}}\begin{bmatrix}2 & 0 & 0\\ 1 & -1 & -1\\ 3 & -1 & -9\end{bmatrix}$$

$$\xrightarrow[\substack{\tau\\ [(-3)\mathbf{1+3}]\\ [(2)\mathbf{3}]\\ [(-1)\mathbf{1+2}]\\ [(2)\mathbf{2}]}]{}\begin{bmatrix}2 & 0 & 0\\ 0 & -2 & -2\\ 0 & -2 & -18\end{bmatrix}\xrightarrow{\substack{\tau\\ [(-1)\mathbf{2+3}]}}\begin{bmatrix}2 & 0 & 0\\ 0 & -2 & 0\\ 0 & -2 & -16\end{bmatrix}\xrightarrow[\substack{\tau\\ [(-1)\mathbf{2+3}]}]{}\begin{bmatrix}2 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -16\end{bmatrix}.$$

Since there are three pivots, zero is not an eigenvalue of that matrix nor of $\mathbf{C}$. Therefore $\mathbf{C}$ is positive definite.

$\square$

(L-19) Question 8.

$$\begin{bmatrix}1&3&0\\3&1&0\\0&0&a\end{bmatrix}\xrightarrow[{}]{\substack{\tau\\ [(-3)\mathbf{1}+\mathbf{2}]}}\begin{bmatrix}1&0&0\\3&-8&0\\0&0&a\end{bmatrix}\xrightarrow[\substack{\tau\\ [(-3)\mathbf{1}+\mathbf{2}]}]{}\begin{bmatrix}1&0&0\\0&-8&0\\0&0&a\end{bmatrix};$$

la forma cuadrática no es definida (sea cual sea el valor de a).

□

(L-19) Question 9. $\mathbf{A}$ es simétrica y por lo tanto diagonalizable:

$$\mathbf{A}=\mathbf{SDS}^{-1};$$

y sabemos que en este caso

$$\mathbf{A}^n=\mathbf{SD}^n\mathbf{S}^{-1};$$

Así pues, como $\mathbf{A}$ es definida positiva, sus autovalores son positivos $\lambda_1>0$ y $\lambda_2>0$ y entonces $\lambda_1^{-1}>0$ y $\lambda_2^{-1}>0$ que son los autovalores de $\mathbf{A}^{-1}$.

□

(L-19) Question 10. La ecuación característica es:

$$|\mathbf{A}-\lambda\mathbf{I}|=\begin{vmatrix}a-\lambda&b\\b&c-\lambda\end{vmatrix}=\lambda^2-(a+c)\lambda+ac-b^2=0;$$

cuyas soluciones son

$$\lambda=\frac{(a+c)\pm\sqrt{(a+c)^2-4(ac-b^2)}}{2}$$

pero

$$(a+c)^2-4(ac-b^2)=a^2+c^2+2ac-4ac+4b^2=a^2+c^2-2ac+4b^2=(a-c)^2+(2b)^2$$

que es una suma de cuadrados, y por lo tanto, lo que hay dentro de la raíz cuadrada es mayor o igual a cero. Así pues, los **autovalores son reales** (no hay raíces cuadradas de números negativos).

Por otra parte, si $a>0$ y $ac>b^2$ necesariamente $c\geq 0$. Así que sabemos que $(a+c)>0$ y por tanto

$$\lambda_1=\frac{(a+c)+\sqrt{(a-c)^2+(2b)^2}}{2}>0$$

Además, puesto que $ac>b^2$, sabemos que $\det\mathbf{A}=ac-b^2>0$; pero, puesto que $\lambda_1\lambda_2=\det\mathbf{A}>0$ y $\lambda_1>0$, necesariamente $\lambda_2>0$.

□

(L-19) Question 11(a) Miremos los signos de los pivotes

$$\begin{bmatrix}1&2&3\\2&5&4\\3&4&9\end{bmatrix}\xrightarrow{\substack{\tau\\ [(-2)\mathbf{1}+\mathbf{2}]\\ [(-3)\mathbf{1}+\mathbf{3}]}}\begin{bmatrix}1&0&0\\2&1&-2\\3&-2&0\end{bmatrix}\xrightarrow[\substack{\tau\\ [(-3)\mathbf{1}+\mathbf{3}]\\ [(-2)\mathbf{1}+\mathbf{2}]}]{}\begin{bmatrix}1&0&0\\0&1&-2\\0&-2&0\end{bmatrix}\xrightarrow{\substack{\tau\\ [(2)\mathbf{2}+\mathbf{3}]}}\begin{bmatrix}1&0&0\\0&1&0\\0&-2&-4\end{bmatrix}\xrightarrow[\substack{\tau\\ [(2)\mathbf{2}+\mathbf{3}]}]{}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-4\end{bmatrix}$$

Puesto que hay tanto pivotes positivos como negativos, la matriz **es indefinida.**

□

(L-19) Question 11(b)

$$\begin{bmatrix}1&2&0&0\\2&6&-2&0\\0&-2&5&-2\\0&0&-2&3\end{bmatrix}\xrightarrow{\substack{\tau\\ [(-2)\mathbf{1}+\mathbf{2}]}}\begin{bmatrix}1&0&0&0\\2&2&-2&0\\0&-2&5&-2\\0&0&-2&3\end{bmatrix}\xrightarrow{\substack{\tau\\ [(1)\mathbf{2}+\mathbf{3}]}}\begin{bmatrix}1&0&0&0\\2&2&0&0\\0&-2&3&-2\\0&0&-2&3\end{bmatrix}\xrightarrow{\substack{\tau\\ [(\frac{2}{3})\mathbf{3}+\mathbf{4}]}}\begin{bmatrix}1&0&0&0\\2&2&0&0\\0&-2&3&0\\0&0&-2&\frac{5}{3}\end{bmatrix}$$

Puesto que aplicando eliminación usando *solo* transformaciones *Tipo I* hemos llegado a una matriz triangular cuyos pivotes son positivos → **Definida positiva.**

$\square$

(L-19) Question 11(c) Since $\mathbf{B}$ is positive definite, $-\mathbf{B}$ is **Negative Definite**.

$\square$

(L-19) Question 11(d) Puesto que $\mathbf{A}$ tiene dos pivotes positivos $\lambda_1 > 0$ y $\lambda_2 > 0$ y uno negativo $\lambda_3 < 0$; los pivotes de $\mathbf{D}$ —que son los inversos de los de $\mathbf{A}$— conservan los signos. Por tanto es: **indefinida.**

(ejercicio 14 del conjunto de problemas 6.2 del libro de texto)

$\square$

(L-19) Question 12. $\begin{pmatrix} 0 & a & 0 \end{pmatrix}$, with $a \neq 0$

$\square$

(L-19) Question 13. $\boldsymbol{x}(\mathbf{A}+\mathbf{B})\boldsymbol{x} = (\boldsymbol{x}\mathbf{A}+\boldsymbol{x}\mathbf{B})\boldsymbol{x} = \boldsymbol{x}\mathbf{A}\boldsymbol{x}+\boldsymbol{x}\mathbf{B}\boldsymbol{x} > 0,$ ya que $\mathbf{A}$ y $\mathbf{B}$ son definidas positivas.

$\square$

(L-19) Question 14(a)

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 5 & 9 \\ 1 & 3 & 9 & 19 \\ \hline 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-2)\mathbf{2}+\mathbf{3}] \\ [(-3)\mathbf{2}+\mathbf{4}]}]{\boldsymbol{\tau}}$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 3 \\ 1 & 3 & 3 & 10 \\ \hline 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-3)\mathbf{3}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \\ \hline 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\frac{\mathbf{L}}{\dot{\mathbf{E}}}\right]$$

Hence, the Gauss transformations are

$$\mathbf{G}_{1\triangleright} = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad \mathbf{G}_{2\triangleright} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad \mathbf{G}_{3\triangleright} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$\dot{\mathbf{E}} = \dot{\mathbf{G}}_{1\triangleright}\ \dot{\mathbf{G}}_{2\triangleright}\ \dot{\mathbf{G}}_{3\triangleright}$. Matrix $\dot{\mathbf{U}}$ is the inverse of $\dot{\mathbf{E}}^{-1}$, so: $\dot{\mathbf{U}} = \dot{\mathbf{E}}^{-1} = (\dot{\mathbf{G}}_{1\triangleright}\dot{\mathbf{G}}_{2\triangleright}\dot{\mathbf{G}}_{3\triangleright})^{-1} = \dot{\mathbf{G}}_{3\triangleright}^{-1}\dot{\mathbf{G}}_{2\triangleright}^{-1}\dot{\mathbf{G}}_{1\triangleright}^{-1}$, and the $\dot{\mathbf{L}}\mathbf{D}\dot{\mathbf{U}}$ factorization is:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & 2 & 3 \\ & & 1 & 3 \\ & & & 1 \end{bmatrix} = \dot{\mathbf{L}}\mathbf{D}\dot{\mathbf{U}}$$

that is equal to $\mathbf{A} = \dot{\mathbf{L}}\mathbf{D}\dot{\mathbf{L}}^{\mathsf{T}}$, since $\mathbf{A}$ is symmetric.

$\square$

(L-19) Question 14(b)

$$\begin{bmatrix}\mathbf{A}\\ \hline \mathbf{I}\end{bmatrix} = \left[\begin{array}{cccc} 1 & 1 & 1 & 1\\ 1 & 2 & 2 & 2\\ 1 & 2 & 3 & 3\\ 1 & 2 & 3 & 4\\ \hline 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-1)\mathbf{1}+\mathbf{2}]\\ [(-1)\mathbf{1}+\mathbf{3}]\\ [(-1)\mathbf{1}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0\\ 1 & 1 & 1 & 1\\ 1 & 1 & 2 & 2\\ 1 & 1 & 2 & 3\\ \hline 1 & -1 & -1 & -1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-1)\mathbf{2}+\mathbf{3}]\\ [(-1)\mathbf{2}+\mathbf{4}]}]{\boldsymbol{\tau}}$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 2\\ \hline 1 & -1 & 0 & 0\\ 0 & 1 & -1 & -1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{3}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 1 & 1 & 1 & 0\\ 1 & 1 & 1 & 1\\ \hline 1 & -1 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix}\mathbf{L}\\ \hline \dot{\mathbf{E}}\end{bmatrix}$$

Hence, the Gauss transformations are

$$\dot{\mathbf{G}}_{1\triangleright} = \begin{bmatrix} 1 & -1 & -1 & -1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad \dot{\mathbf{G}}_{2\triangleright} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & -1 & -1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad \dot{\mathbf{G}}_{3\triangleright} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$\dot{\mathbf{E}} = \dot{\mathbf{G}}_{1\triangleright}\,\dot{\mathbf{G}}_{2\triangleright}\,\dot{\mathbf{G}}_{3\triangleright}$. Matrix $\dot{\mathbf{U}}$ is the inverse of $\dot{\mathbf{E}}^{-1}$, so: $\dot{\mathbf{U}} = \dot{\mathbf{E}}^{-1} = \dot{\mathbf{G}}_{3\triangleright}^{-1}\dot{\mathbf{G}}_{2\triangleright}^{-1}\dot{\mathbf{G}}_{1\triangleright}^{-1}$, and the $\mathbf{L}\mathbf{D}\dot{\mathbf{U}}$ factorization is:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1\\ 1 & 2 & 2 & 2\\ 1 & 2 & 3 & 3\\ 1 & 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1\\ & 1 & 1 & 1\\ & & 1 & 1\\ & & & 1 \end{bmatrix} = \dot{\mathbf{L}}\mathbf{D}\dot{\mathbf{U}}$$

that is equal to $\mathbf{A} = \dot{\mathbf{L}}\mathbf{D}\dot{\mathbf{L}}^{\intercal}$, since $\mathbf{A}$ is symmetric.

□

(L-19) Question 15.

$$\begin{bmatrix} 3 & 6\\ 6 & 16 \end{bmatrix} \xrightarrow[{[(-2)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \begin{bmatrix} 3 & 0\\ 6 & 4 \end{bmatrix} \xrightarrow[{\boldsymbol{\tau}\atop [(-2)\mathbf{1}+\mathbf{2}]}]{} \begin{bmatrix} 3 & 0\\ 0 & 4 \end{bmatrix}$$

por tanto

$$\mathbf{A} = \begin{bmatrix} 3 & 6\\ 6 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0\\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0\\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2\\ 0 & 1 \end{bmatrix}$$

Los coeficientes de los cuadrados son los pivotes en $\mathbf{D}$, y los coeficientes dentro de los cuadrados son las columnas de $\mathbf{L}$.

□

(L-19) Question 16(a)

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 2-\lambda & 1 & 0 & 0\\ 1 & 2-\lambda & 0 & 0\\ 0 & 0 & a-\lambda & 0\\ 0 & 0 & a & a-\lambda \end{vmatrix} = (a-\lambda)^2 \begin{vmatrix} 2-\lambda & 1\\ 1 & 2-\lambda \end{vmatrix} = 0.$$

Therefore, the eigenvalue $\lambda = a$ is repeated twice. We can get the other two eigenvalues solving

$$\begin{vmatrix} 2-\lambda & 1\\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = 0; \qquad \Rightarrow \lambda^2 - 4\lambda + 3 = 0$$

Thus, the other two eigenvalues are 1 and 3.

□

(L-19) Question 16(b) When $\lambda = a = 2$, the rank of the matrix

$$\mathbf{A} - 2\mathbf{I} = \begin{bmatrix} 2-\lambda & 1 & 0 & 0 \\ 1 & 2-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 2 & 2-\lambda \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

is 3. Therefore $\dim \mathcal{N}\left(\mathbf{A}\right) = 1$ (only one free column); hence it is not possible to find two linearly independent eigenvectors for the repeated eigenvalue $\lambda = 2$. It follows that THE MATRIX IS NOT DIAGONALISABLE.

□

(L-19) Question 16(c)

$$|\mathbf{B} - \lambda\mathbf{I}| = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\cdot\left((2-\lambda)^2 - 1\right) = 0$$

Clearly one eigenvector is $\lambda = 1$. The other two are the roots of

$$\left((2-\lambda)^2 - 1\right) = 4 + \lambda^2 - 4\lambda - 1 = \lambda^2 - 4\lambda + 3 = 0.$$

that is, $\lambda = 3$ and $\lambda = 1$. Thus, $\boxed{\mathbf{D} = \left[\begin{smallmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right].}$

- For $\lambda = 3$

$$\mathbf{A} - 3\lambda\mathbf{I} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix};$$

therefore $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector. Since its norm is $\sqrt{2}$, then $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is a normalised eigenvector for $\lambda = 3$.

- For $\lambda = 1$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

because the last column of $(\mathbf{A} - \lambda\mathbf{I})$, the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector (with norm 1); besides, from the other two columns, it is ease tho check that $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ is another eigenvector for $\lambda = 3$. Normalising the vector we get $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

It is not difficult to see that those three vector are orthogonal. Therefore:

$$\boxed{\mathbf{P} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.}$$

□

(L-19) Question 16(d) The quadratic form is

$$f(x,y,z) = \boldsymbol{b}\mathbf{X}\boldsymbol{b} = \begin{pmatrix} x & y & z \end{pmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2x^2 + 2y^2 + z^2 + 2xy,$$

and we already know it is positive defined, since the three eigenvalues of the symmetric matrix $\mathbf{B}$ are positive.

□

(L-19) Question 17(a) There are two cases:

- $a = -4/5$ and $b = 3/5$
- $a = 4/5$ and $b = -3/5$.

□

(L-19) Question 17(b) Any values of a and b such as the first column is not a multiple of the second; for example, $a = 1$ and $b = 0$.

□

(L-19) Question 17(c) This is just the opposite case. . . , here we need a singular matrix; therefore we can use any multiple of the second column; for example: $a = 3$ and $b = 4$.

□

(L-19) Question 17(d) By symmetry, $b = 3/5$. In addition, we need $a < 0$ and $\det \mathbf{A} > 0$; that is $a \cdot 4/5 - (3/5)^2 > 0$, or

$$a \cdot 4/5 > (3/5)^2$$

something impossible if $a < 0$. Therefore, THERE ISN'T SUCH VALUES OF a AND b.

□

(L-19) Question 18(a)

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & a & 0 \\ 0 & 0 & 8 \end{bmatrix} \xrightarrow[\;]{\underset{[(-1)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}} \begin{bmatrix} 1 & 0 & 0 \\ 1 & a-1 & 0 \\ 0 & 0 & 8 \end{bmatrix} \xrightarrow[\underset{[(-1)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & a-1 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Si $a = 1$ la matriz $\mathbf{Q}$ es semidefinida positiva.

□

(L-19) Question 18(b)

- Si $a > 1$ la matriz $\mathbf{Q}$ es definida positiva.
- Si $a < 1$ la matriz $\mathbf{Q}$ es indefinida.

□

(L-Opt-2) Question 1(a)

$$\det \mathbf{A} = \begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix} = -a;$$

therefore $|\mathbf{A}|$ is no zero if and only if $a \neq 0$.

□

(L-Opt-2) Question 1(b) No, since:

$$|1| = 1; \qquad \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0; \qquad \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0; \quad \begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -1,$$

when all subdeterminants should be possitive. It follows that the matrix is not definite.

□

(L-Opt-2) Question 1(c)

$$\left[\begin{array}{cccc} 1 & 1 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{4}] \\ [(1)\mathbf{2}+\mathbf{4}]}]{\tau} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & -2 \\ \hline 1 & -1 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-1)\mathbf{2}+\mathbf{1}] \\ [(1)\mathbf{4}+\mathbf{1}] \\ [(2)\mathbf{2}] \\ [(-1)\mathbf{4}+\mathbf{2}]}]{\tau} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \\ \hline 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(\frac{1}{2})\mathbf{2}] \\ [(\frac{-1}{2})\mathbf{4}]}]{\tau} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & \frac{-1}{2} \\ 0 & 0 & 1 & 0 \\ 1 & \frac{-1}{2} & 0 & \frac{-1}{2} \end{array}\right]$$

Hence,

$$\mathbf{A}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & 0 \\ 1 & -1/2 & 0 & -1/2 \end{bmatrix}$$

□

(L-Opt-2) Question 1(d) From the steps given in the gaussian elimination when solving the first part of the exercise, it's easy to check that rg $\left(\mathbf{A}\right) = 3$ when $a = 0$; and therefore, there are three pivot variables. Hence, only one variable can be chosen as a free variable.

When $a = 0$ the second and third columns are equal (and hence dependent); it follows that we can take as free variable either the second or the third one.

□

(L-Opt-2) Question 2(a) *Es verdadero.* Si $\mathbf{A}$ es simétrica, estonces $\mathbf{A}^\intercal = \mathbf{A}$, por tanto

$$\left(\mathbf{A}^2\right)^\intercal = \left(\mathbf{A}\mathbf{A}\right)^\intercal = \mathbf{A}^\intercal\mathbf{A}^\intercal = \left(\mathbf{A}^\intercal\right)^2 = \mathbf{A}^2,$$

es decir, que $\mathbf{A}^2$ también es simétrica.

□

(L-Opt-2) Question 2(b) *Es verdadero.* Veamoslo:

$$(\mathbf{I} - \mathbf{A})^2 = (\mathbf{I} - \mathbf{A})(\mathbf{I} - \mathbf{A}) = \mathbf{I} - \mathbf{A} - \mathbf{A} + \mathbf{A}^2 = \mathbf{I} - \mathbf{A} - \mathbf{A} + \mathbf{A} = \mathbf{I} - \mathbf{A}$$

A las matrices con esta propiedad se las denomina *"matrices idempotentes"*.

□

(L-Opt-2) Question 2(c) *Es falso.* El determinate de una matriz es igual al producto de sus autovaloes; si uno de ellos es cero, necesariamente la matriz es singular. En tal caso sus columnas son linealmente dependientes y es posible encontrar una solución distinta a la trivial ($\boldsymbol{x} = \mathbf{0}$) para dicho sistema homogeneo; así que hay más de una solución y el sistema es necesariamente *indeterminado.*

□

(L-Opt-2) Question 2(d) *Verdadero.* Por el mismo motivo de antes, $\underset{m\times m}{\mathbf{A}}$ es singular, lo que quiere decir que el subespacio generado por las columnas de $\mathbf{A}$ (*que llamaremos espacio columna de* $\mathbf{A}$, $\mathcal{C}\left(\mathbf{A}\right)$) es de dimensión menor que m, pero eso quiere decir que existen vectores de $\mathbb{R}^m$ que no pertenencen a $\mathcal{C}\left(\mathbf{A}\right)$. Si $\boldsymbol{b}$ fuera uno de ellos, entonces no existiría una combinación lineal de las columnas de $\mathbf{A}$ igual a $\boldsymbol{b}$, es decir, que $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ será incompatible para dicho $\boldsymbol{b} \notin \mathcal{C}\left(\mathbf{A}\right)$.

□

(L-Opt-2) Question 2(e) *True.* If $\mathbf{Q}$ is orthogonal, then $\mathbf{Q}^\intercal\mathbf{Q} = \mathbf{I}$; therefore the inverse of $\mathbf{Q}$ is its transpose ($\mathbf{Q}^\intercal = \mathbf{Q}^{-1}$), and then $\mathbf{Q}\mathbf{Q}^{-1} = \mathbf{I} = \mathbf{Q}\mathbf{Q}^\intercal$; but this means that the columns of $\mathbf{Q}^{-1}$ are orthogonal (since all the elements of $\mathbf{Q}\mathbf{Q}^\intercal = \mathbf{I}$ outside the main diagonal are zero) with norm equal to one (since $\mathbf{Q}\ \mathbf{Q}^\intercal$ has only ones in the main diagonal).

□

(L-Opt-2) Question 2(f) *False.* For the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

1 is the only eigenvalue, but $\mathbf{A}$ is not the identity matrix.

□

(L-Opt-2) Question 3(a) Verdadero.

□

(L-Opt-2) Question 3(b) $a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_n\boldsymbol{v}_n = \boldsymbol{0}$ si y sólo si todos los coeficientes a_j son iguales a cero. Es decir, si la única solución a $\left[\boldsymbol{v}_1, \ldots \boldsymbol{v}_n\right]\boldsymbol{x} = \boldsymbol{0}$ es un vector de ceros (es decir, si $\mathcal{N}\left(\left[\boldsymbol{v}_1, \ldots \boldsymbol{v}_n\right]\right)$ es $\{\boldsymbol{0}\}$.

□

(L-Opt-2) Question 3(c) Falso, ya que $a_1\boldsymbol{0} = \boldsymbol{0}$ incluso para $a_1 \neq 0$.

□

(L-Opt-2) Question 3(d) podemos encontrar n autovectores linealmente independientes.

□

(L-Opt-2) Question 3(e) $\sqrt{7}$

□

(L-Opt-2) Question 3(f) $\boldsymbol{u} \cdot \boldsymbol{v} = 0$

□

(L-Opt-2) Question 4(a) Falso. $\left|\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right| = 0$

□

(L-Opt-2) Question 4(b) Verdadero $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}) \neq 0 \Rightarrow \det(\mathbf{A}) \neq 0 \neq \det(\mathbf{B})$

□

(L-Opt-2) Question 4(c) Verdadero. El determinante cambia de signo, y el producto de los autovalores es igual al determinante.

□

(L-Opt-2) Question 4(d) Verdadero

□

(L-Opt-2) Question 4(e) Verdadero. En tal caso $(\mathbf{A} - 5\mathbf{I})$ es de rango completo, y $\dim \mathcal{N}(\mathbf{A} - 5\mathbf{I}) = 0$, por tanto no hay autovectores asociados al autovalor 5.

□

(L-Opt-2) Question 4(f) Falso. $\left(\begin{smallmatrix} -1 & 1 \\ -1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$ no tiene solución y $|\mathbf{A}| = 0$.

□

(L-Opt-2) Question 4(g) Falso. Como máximo puede tener tres pivotes.

□

(L-Opt-2) Question 4(h) Verdadero. $\boldsymbol{b} \in \mathbb{R}^n$ pero $\boldsymbol{b} \notin \mathcal{C}\left(\mathbf{C}\right)$; por tanto no todos los vectores de $\mathbb{R}^n$ están en $\mathcal{C}\left(\mathbf{C}\right)$. Es decir, $\operatorname{rg}\left(\mathbf{C}\right) = \dim \mathcal{C}\left(\mathbf{C}\right) < n = \dim \mathbb{R}^n$.

□

(L-Opt-2) Question 4(i) Falso. Si algún autovalor es igual a cero, la matriz no es invertible. Por ejemplo una matriz nula y cuadrada.

□

(L-Opt-2) Question 4(j) Verdadero. Si es invertible tiene rango completo —n pivotes iguales a uno con ceros por encima y por debajo... es decir la identidad).

□

(L-Opt-2) Question 5(a) $\mathbf{A}$ y $\mathbf{D}$ por tener tres pivotes (para $\mathbf{A}$ se ve directamente, y para $\mathbf{D}$ tras el primer paso de eliminación).

□

(L-Opt-2) Question 5(b) Sólo $\mathbf{B}$ (puesto que sólo tiene dos autovalores 0 y 2, necesariamente alguno esta repetido).

□

(L-Opt-2) Question 5(c) $\mathbf{B}$ y $\mathbf{C}$ por ser no invertibles (ambas tienen un autovalor igual a cero)

□

(L-Opt-2) Question 5(d) $\mathbf{A}$, $\mathbf{C}$ y $\mathbf{D}$ por tener autovalores distintos (además $\mathbf{D}$ es simétrica).
Para $\mathbf{B}$, el autovalor $\lambda = 2$ está repetido:

$$(\mathbf{B} - 2\mathbf{I}) = \begin{bmatrix} -1 & 1 & 3 \\ 1 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

sólo una columna libre, y por tanto sólo podemos encontrar una dirección asociada al autovalor repetido $\lambda = 2$. Es decir, *la matriz* $\mathbf{B}$ *no es diagonalizable.*

□

(L-Opt-2) Question 5(e) Para la matriz simétrica $\mathbf{D}$.

□

(L-Opt-2) Question 6(a) Since the matrix is triangular, the eigenvalues are the numbers on the main diagonal: $\lambda_1 = 1$ and $\lambda_2 = 2$.

For $\lambda = 1$

$$(\mathbf{A} - \lambda\mathbf{I}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

The following are three linearly independent eigenvectors

$$\boldsymbol{x}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad \boldsymbol{x}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad \boldsymbol{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda = 2$

$$(\mathbf{A} - 2\lambda\mathbf{I}) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$$

The following is an eigenvector

$$\boldsymbol{x}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

□

(L-Opt-2) Question 6(b) Yes, since there are 4 linearly independent eigenvectors

□

(L-Opt-2) Question 6(c) This factorization $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{\intercal}$ implies that $\mathbf{A}$ must be symetric; but $\mathbf{A}$ is not. Therefore, it is not possible.

□

(L-Opt-2) Question 6(d)

$$|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|} = \frac{1}{\text{product of eigenvalues of } \mathbf{A}} = \frac{1}{2}.$$

□

(L-Opt-2) Question 7(a) Yes, it is. A 3 by 3 matrix with 3 different eigenvalues.

□

(L-Opt-2) Question 7(b) No, it is not. Since $\boldsymbol{v}_3 = -\boldsymbol{v}_1$, then $\boldsymbol{v}_3$ must be an eigenvector associated to λ_1.

□

(L-Opt-2) Question 7(c)

$$\mathbf{A}(\boldsymbol{v}_1 - \boldsymbol{v}_2) = \mathbf{A}\boldsymbol{v}_1 - \mathbf{A}\boldsymbol{v}_2 = \lambda_1\boldsymbol{v}_1 - \lambda_2\boldsymbol{v}_2 = 1\begin{pmatrix}1\\0\\1\end{pmatrix} - 2\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\-2\\-1\end{pmatrix}$$

□

(L-Opt-2) Question 8(a) Since the first two vectors are the same, the dimension of $\mathcal{N}\left(\mathbf{A}\right)$ is 2. The number of the columns is 4, therefore the rank of $\mathbf{A}$ is 2.

The last vector is telling us that the last column of $\mathbf{A}$ is zero vector; and the first vector means that the first column of $\mathbf{A}$ is the opposite of the third. Then, one possibility is:

$$\begin{bmatrix}1 & 0 & -1 & 0\\0 & 1 & 0 & 0\end{bmatrix}\begin{pmatrix}x\\y\\z\\t\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \qquad \Longrightarrow \qquad \begin{cases}x - z & = 0\\ y & = 0\end{cases}.$$

But that is not the only possible answer; for example we can add zero rows below.

The coefficient matrix $\underset{m\times 4}{\mathbf{A}}$ must be rank 2, with a fourth column full of zeros, and a first column opposite to the third one.

□

(L-Opt-2) Question 8(b) Since $\mathbf{A}$ has a characteristic polynomial of degree 5, we know that $\mathbf{A}$ is a 5×5 matrix. Since 0 is not a root of $p(\cdot)$ and so is not an eigenvalue, we know $\mathbf{A}$ is invertible so $rank(A) = 5$.

□

(L-Opt-2) Question 9(a) Puesto que la matriz es invertible, el rango es 3; y el espacio columna $\mathcal{C}\left(\mathbf{A}\right)$ es todo $\mathbb{R}^3$. Por ello no hay columnas libres, es decir, el espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$ sólo contiene el vector cero $\mathbf{0}$.

□

(L-Opt-2) Question 9(b) Puesto que $\mathbf{L}$ es $\mathbf{E}^{-1}$ (la inversa del producto de matrices elementales necesarias para triangularizar $\mathbf{A}$), el 5 en la primera posición de la segunda fila de $\mathbf{L}$ nos dice que el primer paso fue *"restar a la segunda fila cinco veces la primera"*.

Puesto que $\mathbf{U}$ tiene tres pivotes, es de rango completo; y por tanto es invertible.

El determinante es $\det(\mathbf{A}) = u_{11}\cdot u_{22}\cdot u_{33}$.

□

(L-Opt-2) Question 9(c) La matriz

$$\mathbf{A} = \begin{bmatrix}0 & 0 & 1\\0 & 1 & 1\\1 & 1 & 1\end{bmatrix}$$

no puede ser triangularizada sin permutar las filas previamente.

Sin embargo, permutando la primera fila con la tercera, la matriz ya es triangular. De hecho toda matriz invertible admite la siguiente factorización:

$$\mathbf{PA} = \mathbf{LU}.$$

La matriz $\mathbf{A}$ es invertible ya que una vez se han permutado las filas, aparecen tres pivotes; es decir, la matriz es de rango completo.

□