Mathematics II

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You can find the last version of these course materials at

https://github.com/mbujosab/MatematicasII/tree/main/Eng

1 Highlights of Lesson 4

Highlights of Lesson 4

- Elementary transformations
- Indentifying singular matrices by elimination
- Matrix multiplication of Elementary matrices

2 Elementary transformations of a matrix

Type *I*: $\mathbf{A}_{\underset{[(\lambda)\boldsymbol{i}+\boldsymbol{j}]}{\boldsymbol{\tau}}}$ (with $i \neq j$)

add λ times i-th column ($\lambda\mathbf{A}_{|i}$) to j-th column ($\mathbf{A}_{|j}$)

$$\begin{bmatrix} 1 & -3 & 0 \\ 1 & -6 & 3 \end{bmatrix}_{\underset{[(-2)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}} = \begin{bmatrix} 1 & -3 & -2 \\ 1 & -6 & 1 \end{bmatrix}$$

Type *II*: $\mathbf{A}_{\underset{[(\alpha)\boldsymbol{i}]}{\boldsymbol{\tau}}}$ (with $\alpha \neq 0$)

multiply by α the i-th column

$$\begin{bmatrix} 1 & -3 & 0 \\ 1 & -6 & 3 \end{bmatrix}_{\underset{[(10)\mathbf{2}]}{\boldsymbol{\tau}}} = \begin{bmatrix} 1 & -30 & 0 \\ 1 & -60 & 3 \end{bmatrix}$$

3 Elimination and pre-echelon form of a matrix

- *Pivot* is the first non-zero component of each column.
- *Elimination*: modifies a matrix until all components at the right-hand side of each pivot are zeros

$$\mathbf{A} = \begin{bmatrix}1&3&0\\2&8&4\\1&1&1\end{bmatrix} \xrightarrow{\overset{\tau}{[(-3)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix}1&0&0\\2&2&4\\1&-2&1\end{bmatrix} \xrightarrow{\overset{\tau}{[(-2)\mathbf{2}+\mathbf{3}]}} \begin{bmatrix}1&0&0\\2&2&0\\1&-2&5\end{bmatrix} = \mathbf{L}$$

4 Elimination

Elimination algorithm on **A**

modifies **A** using a sequence of *elementary transformations*

Goal

to get a *(pre)echelon* form

- *pre-echelon*: all components on the right side of each pivot are zero.
- *echelon*: if any column before a non-null column $\mathbf{A}_{|j}$ is non-null column and its pivot is above the pivot of $\mathbf{A}_{|j}$.

It is always possible to find a (pre)echelon form by elimination

Rank (rg): the number of pivots in any of its pre-echelon forms

$\underset{n\times n}{\mathbf{A}}$ is *singular* if its pre-echelon forms have null-columns $(\mathrm{rg} < n)$

5 Elimination: When can't we find n pivots?

$n \times n$ matrices are **singular** if less than n pivots after elimination

$$\begin{bmatrix}0&1&3\\4&2&8\\1&1&1\end{bmatrix}$$

Has this matrix n pivots? $\begin{bmatrix}1&3&0\\2&6&4\\1&1&1\end{bmatrix}$

and this one? $\begin{bmatrix}1&3&0\\2&6&0\\1&1&1\end{bmatrix}$

and this one? $\begin{bmatrix}1&2&1\\3&8&1\\0&4&-4\end{bmatrix}$

6 Matrix multiplication: elementary matrices

$$\underbrace{\begin{bmatrix}1&3&0\\2&8&4\\1&1&1\end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{bmatrix}\quad&\quad&\quad\\ & & \\ & & \end{bmatrix}}_{\mathbf{I}_{\tau}} = \underbrace{\begin{bmatrix}1&0&0\\2&2&4\\1&-2&1\end{bmatrix}}_{\mathbf{A}_{\tau}}$$

We call $\mathbf{I}_{\tau}$ "Elementary matrix":

$$\mathbf{A}\left(\mathbf{I}_{\tau}\right) = \mathbf{A}_{\tau}$$

This specific elementary matrix $\mathbf{I}_{\tau}$ is written as $\mathbf{I}_{\overset{\tau}{[(-3)\mathbf{1}+\mathbf{2}]}}$

$$\mathbf{A}\left(\mathbf{I}_{\overset{\tau}{[(-3)\mathbf{1}+\mathbf{2}]}}\right) = \mathbf{A}_{\overset{\tau}{[(-3)\mathbf{1}+\mathbf{2}]}}$$

7 Matrix multiplication: elementary matrices

$$\begin{bmatrix}1 & 0 & 0\\ 2 & 2 & 4\\ 1 & -2 & 1\end{bmatrix}\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 2 & 2 & 0\\ 1 & -2 & 5\end{bmatrix}$$

This specific elementary matrix $\mathbf{I}_{\tau}$ is written as $\mathbf{I}_{\overset{\tau}{[(-2)\mathbf{2}+\mathbf{3}]}}$

$$\mathbf{A}\left(\mathbf{I}_{\overset{\tau}{[(-2)\mathbf{2}+\mathbf{3}]}}\right) = \mathbf{A}_{\overset{\tau}{[(-2)\mathbf{2}+\mathbf{3}]}}$$

8 Elimination by elementary matrices

$$\mathbf{A} = \begin{bmatrix}1 & 3 & 0\\ 2 & 8 & 4\\ 1 & 1 & 1\end{bmatrix} \xrightarrow{\overset{\tau}{[(-3)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix}1 & 0 & 0\\ 2 & 2 & 4\\ 1 & -2 & 1\end{bmatrix} \xrightarrow{\overset{\tau}{[(-2)\mathbf{2}+\mathbf{3}]}} \begin{bmatrix}1 & 0 & 0\\ 2 & 2 & 0\\ 1 & -2 & 5\end{bmatrix} = \mathbf{L}$$

$$\mathbf{A}_{\substack{\overset{\tau}{[(-3)\mathbf{1}+\mathbf{2}]}\\ [(-2)\mathbf{2}+\mathbf{3}]}} = \mathbf{A}_{\overset{\tau}{[(-3)\mathbf{1}+\mathbf{2}]}\overset{\tau}{[(-2)\mathbf{2}+\mathbf{3}]}} = \left(\mathbf{A}\left(\mathbf{I}_{\overset{\tau}{[(-3)\mathbf{1}+\mathbf{2}]}}\right)\right)\left(\mathbf{I}_{\overset{\tau}{[(-2)\mathbf{2}+\mathbf{3}]}}\right) = \mathbf{L}$$

there is a matrix that does the whole job at once

$$\mathbf{A}_{\substack{\overset{\tau}{[(-3)\mathbf{1}+\mathbf{2}]}\\ [(-2)\mathbf{2}+\mathbf{3}]}} = \mathbf{A}\left(\left(\mathbf{I}_{\overset{\tau}{[(-3)\mathbf{1}+\mathbf{2}]}}\right)\left(\mathbf{I}_{\overset{\tau}{[(-2)\mathbf{2}+\mathbf{3}]}}\right)\right) = \mathbf{A}\mathbf{I}_{\substack{\overset{\tau}{[(-3)\mathbf{1}+\mathbf{2}]}\\ [(-2)\mathbf{2}+\mathbf{3}]}} = \mathbf{L}$$

$$\boxed{\mathbf{A}_{\tau_1\cdots\tau_k} = \mathbf{A}\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)}$$

9 how do I get from L back to A? Inverses

How do I reverse the first step? (it was subtract 3 times $\mathbf{A}_{|1}$ from $\mathbf{A}_{|2}$)

$$\begin{bmatrix}1 & -3 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

$$\mathbf{I}_{\overset{\tau}{[(-\lambda)\boldsymbol{i}+\boldsymbol{j}]}} \text{ “undo” } \mathbf{I}_{\overset{\tau}{[(\lambda)\boldsymbol{i}+\boldsymbol{j}]}}$$

How to undo $\mathbf{I}_{\overset{\tau}{[(\alpha)\boldsymbol{i}]}}$?

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}1 & & \\ & 1 & \\ & & 1/2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

10 Interchange or swap matrices

Which matrix exchanges the columns?

$$\begin{bmatrix}a & c\\ b & d\end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix} = \begin{bmatrix}c & a\\ d & b\end{bmatrix}$$

Which matrix exchanges the rows? where do we put that matrix?

$$\begin{bmatrix}a & c\\ b & d\end{bmatrix} \qquad = \begin{bmatrix}b & d\\ a & c\end{bmatrix}$$

Matrix multiplication is not commutative!

11 Interchange of columns

Interchange of columns:

$\mathbf{A}_{\underset{[i\rightleftharpoons j]}{\tau}} \to$ swicht columns i and j of $\mathbf{A}$

$$\begin{bmatrix} 1 & -3 & 0 \\ 1 & -6 & 3 \end{bmatrix}_{\underset{[2\rightleftharpoons 3]}{\tau}} = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 3 & -6 \end{bmatrix}$$

We can switch two columns by a sequence of elementary transformations

Matrix $\mathbf{I}_{\underset{[i\rightleftharpoons j]}{\tau}}$ is call a exchange matrix

12 Permutation matrices

Product between exchange matrices $\mathbf{I}_{\underset{[\cdot\rightleftharpoons\cdot]}{\tau}}$ is a permutation matrix $\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}}$.

$$\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}} = \text{Identity matrix } \mathbf{I} \text{ with rearranged columns}$$

Let's see the 3×3 case

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad \mathbf{I}_{\underset{[1\rightleftharpoons 2]}{\tau}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

How many 3×3 pemutations can we find?

what happens if I multiply two permutation matrices?

Questions of the Lecture 4

(L-4) QUESTION 1.

(a) Which three matrices $\mathbf{I}_{\underset{[(x)1+2]}{\tau}}$, $\mathbf{I}_{\underset{[(y)1+3]}{\tau}}$ and $\mathbf{I}_{\underset{[(z)2+3]}{\tau}}$ put $\mathbf{A} = \begin{bmatrix} 1 & 4 & -2 \\ 1 & 6 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ into an echelon form?

(b) Multiply those $\mathbf{I}_{\tau_i}$ to get one matrix $\mathbf{E}$ that does elimination: $\mathbf{AE} = \mathbf{K}$.

Based on (Strang, 1988, exercise 24 from section 1.4.)

(L-4) QUESTION 2. Consider the matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ -1 & -3 & -2 \\ 0 & 1 & c \end{bmatrix}$$

For what value(s) of c the matrix is singular (we can't find three pivots)?

(L-4) QUESTION 3. Consider the following 3 by 3 matrices.

(a) $(\mathbf{I}_{\underset{[(-1)1+2]}{\tau}})$ subtracts column 1 from column 2 and then $(\mathbf{I}_{\underset{[2\rightleftharpoons 3]}{\tau}})$ exchanges columns 2 and 3. What matrix $\mathbf{E}$ does both steps at once?

(b) $(\mathbf{I}_{\underset{[2\rightleftharpoons 3]}{\tau}})$ exchanges columns 2 and 3 and then $\mathbf{I}_{\underset{[(-1)1+3]}{\tau}}$ subtracts column 1 from column 3. What matrix $\mathbf{N} = (\mathbf{I}_{\underset{[2\rightleftharpoons 3]}{\tau}})(\mathbf{I}_{\underset{[(-1)1+3]}{\tau}})$ does both steps at once? Explain why $\mathbf{M}$ and $\mathbf{N}$ are the same but the $\mathbf{I}_{\tau}$'s are different.

Based on (Strang, 1988, exercise 28 from section 1.4.)

(L-4) QUESTION 4. Elimination matrices $\mathbf{I}_{\underset{[(?)1+2]}{\tau}}$ and $\mathbf{I}_{\underset{[(?)2+3]}{\tau}}$ will reduce $\mathbf{A}$ to triangular form. Find $\mathbf{E}$ so that $\mathbf{AE} = \mathbf{L}$ is lower triangular (echelon), if $\mathbf{A}$ is

$$\begin{bmatrix} 2 & 2 & 0 \\ 1 & 4 & 9 \\ 1 & 3 & 9 \end{bmatrix}$$

(L-4) QUESTION 5. Although we will only consider as elementary the *Type I* and *II* transformations, in most of the Linear Algebra books appears a third type: the *exchange* of columns

$$\mathbf{A}_{\substack{\tau\\ [\mathbf{p}\rightleftharpoons\mathbf{s}]}} \to \text{Exchanges columns } p \text{ and } s \text{ of } \mathbf{A}.$$

Prove that a column exchange is, in fact, a sequence of *Type I* and *II* elementary transformations. Try transforming $\underset{2\times 2}{\mathbf{I}}$ in $\begin{bmatrix}0&1\\1&0\end{bmatrix}$ by elementary transformations of the columns.

(L-4) QUESTION 6. Write down the 3 by 3 matrices that produce these elimination steps:

(a) $\mathbf{I}_{\substack{\tau\\ [(-5)\mathbf{1}+\mathbf{2}]}}$ substracts 5 times column 1 from column 2,

(b) $\mathbf{I}_{\substack{\tau\\ [(-7)\mathbf{2}+\mathbf{3}]}}$ substracts 7 times column 2 from column 3,

(c) $\mathbf{I}_{\substack{\tau\\ [\mathfrak{S}]}}$ exchanges columns 1 and 2, and then columns 2 and 3.

(Strang, 2003, exercise 1 from section 2.3.)

(L-4) QUESTION 7. Consider the matrices of QUESTION 6:

(a) when multiplying by $\mathbf{I}_{\substack{\tau\\ [(-5)\mathbf{1}+\mathbf{2}]}}$ and then by $\mathbf{I}_{\substack{\tau\\ [(-7)\mathbf{2}+\mathbf{3}]}}$ the matrix $\mathbf{A} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}$ we get $\mathbf{A}_{\substack{\tau\\ [(-5)\mathbf{1}+\mathbf{2}]\\ [(-7)\mathbf{2}+\mathbf{3}]}} = [\quad;\quad;\quad].$

(b) But, when multiplying by $\mathbf{I}_{\substack{\tau\\ [(-5)\mathbf{1}+\mathbf{2}]}}$ before and then by $\mathbf{I}_{\substack{\tau\\ [(-7)\mathbf{2}+\mathbf{3}]}}$ we get $\mathbf{A}_{\substack{\tau\\ [(-7)\mathbf{2}+\mathbf{3}]\\ [(-5)\mathbf{1}+\mathbf{2}]}} = [\quad;\quad;\quad].$

(c) When $\substack{\tau\\ [(-7)\mathbf{2}+\mathbf{3}]}$ comes first, the column ____ feels no effect from column ____.

This property will become very important in the LU factorization!

(Strang, 2003, exercise 2 from section 2.3.)

(L-4) QUESTION 8. What matrix $\mathbf{M}$ sends $\boldsymbol{v} = (1,\ 0,)$ to $(0,\ 1,)$, es decir $\boldsymbol{v}\mathbf{M} = (0,\ 1,)$; and also sends $\boldsymbol{w} = (0,\ 1,)$ to $(1,\ 0,)$, es decir $\boldsymbol{w}\mathbf{M} = (1,\ 0,)$?

(L-4) QUESTION 9. Consider a permutation (interchange) matrix $\mathbf{I}_{\substack{\tau\\ [\boldsymbol{i}\rightleftharpoons\boldsymbol{j}]}}$, if we compute the product $\mathbf{A}(\mathbf{I}_{\substack{\tau\\ [\boldsymbol{i}\rightleftharpoons\boldsymbol{j}]}})$, we get a new matrix like $\mathbf{A}$, but with exchanged columns. What happen if we compute the product $(\mathbf{I}_{\substack{\tau\\ [\boldsymbol{i}\rightleftharpoons\boldsymbol{j}]}})\mathbf{A}$? Check your answer with a 2 by 2 example.

(L-4) QUESTION 10. If every column of $\mathbf{A}$ is a multiple of $(1,\ 1,\ 1,)$, then $\mathbf{A}\boldsymbol{x}$ is always a multiple of $(1,\ 1,\ 1,)$. Do a 3 by 3 example. How many pivots are produced by elimination?
(Strang, 1988, exercise 26 from section 1.4.)

1 Highlights of Lesson 5

Highlights of Lesson 5

- Inverse of $\mathbf{A}$
- Gauss-Jordan elimination / finding $\mathbf{A}^{-1}$
- Inverse of $\mathbf{AB}$, $\mathbf{A}^{\intercal}$

2 Inverse of a matrix (square matrices)

A squared of order n has inverse (is *invertible*) if exists **B** such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}.$$

Then

$$\mathbf{B} = \mathbf{A}^{-1} \qquad \text{and} \qquad \mathbf{A} = \mathbf{B}^{-1}.$$

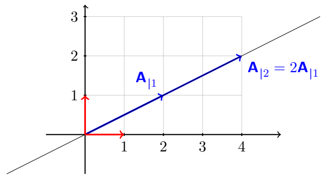
Not all matrices have inverse

Squared matrices with no inverse are called *singular* matrices

3 Singular case (no inverse)

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

Is it possible to find a matrix **B** such that **AB** =**I**?
...columns of **I** should be linear combinations of columns of **A**... but both columns lie on the same line.



So

$$\boxed{\mathbf{A} \text{ is singular}}$$

4 Singular case (no inverse)

Can we find $\boldsymbol{x} \neq \mathbf{0}$ such that $\mathbf{A}\boldsymbol{x} = \mathbf{0}$?

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} \quad \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If $\mathbf{A}\boldsymbol{x} = 0$ and $\boldsymbol{x} \neq 0 \quad \Rightarrow \quad$ there is no $\mathbf{A}^{-1}$

$$\boxed{\text{The existence of } \mathbf{A}^{-1} \text{ leads to a contradiction}}$$

If $\mathbf{A}\boldsymbol{x} = 0$ and $\boldsymbol{x} \neq \mathbf{0} \quad \Rightarrow \quad \mathbf{A}^{-1}\mathbf{A}\boldsymbol{x} = \mathbf{A}^{-1}\mathbf{0} \quad \Rightarrow \quad \boldsymbol{x} = \mathbf{0}.$

When $\mathbf{A}^{-1}$ does exist

the only solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$.

5 Calculating the inverse matrix

$$\mathbf{A}(\mathbf{A}^{-1}) = \mathbf{I}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So...we are solving m systems (of m equations each)

$$\begin{bmatrix} 1 & 3 \\ 2 & \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 2 & \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

6 Gauss-Jordan: solving two linear systems at once

Gauss-Jordan elimination (obtaining a reduced echelon form $\mathbf{R}$)
apply elementary transformations until a echelon matrix with only zeros to the left of each pivot (and all pivots equal to 1) is achieved

Let's solve the linear systems

$$\begin{bmatrix}1 & 3\\2 & 7\end{bmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix} \qquad \begin{bmatrix}1 & 3\\2 & 7\end{bmatrix}\begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$

applying Gauss-Jordan elimination on $\mathbf{A}$ stacked with $\mathbf{I}$

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right]=\left[\begin{array}{cc}1 & 3\\2 & 7\\\hline 1 & 0\\0 & 1\end{array}\right]\rightarrow \qquad \rightarrow \qquad =$$

If $\mathbf{R}=\mathbf{I}$, we have found $\mathbf{A}^{-1}$

7 Gauss-Jordan: Why does it work?

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right]=\left[\begin{array}{cc}1 & 3\\2 & 7\\\hline 1 & 0\\0 & 1\end{array}\right]\xrightarrow{\boldsymbol{\tau}_{[(-3)\mathbf{1}+\mathbf{2}]}} \qquad \xrightarrow{\boldsymbol{\tau}_{[(-2)\mathbf{2}+\mathbf{1}]}}$$

that is, since $\mathbf{A}_{\tau_1\cdots\tau_k}=\mathbf{A}\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)$:

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right]_{\tau_1\cdots\tau_k}=\left[\frac{\mathbf{A}_{\tau_1\cdots\tau_k}}{\mathbf{I}_{\tau_1\cdots\tau_k}}\right]=\left[\frac{\mathbf{A}\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)}{\mathbf{I}_{\tau_1\cdots\tau_k}}\right]=\left[\frac{\mathbf{I}}{\mathbf{I}_{\tau_1\cdots\tau_k}}\right],$$

$\boxed{\text{who is } \mathbf{I}_{\tau_1\cdots\tau_k}?}$

therefore $\mathbf{A}^{-1}=$

8 Inverse of a product

When $\mathbf{A}$ and $\mathbf{B}$, of order n, are invertible, $(\mathbf{AB})$ is invertible.

what matrix gives me the inverse of $\mathbf{AB}$? lets try with $(\mathbf{B}^{-1}\mathbf{A}^{-1})$:

$$\mathbf{AB}\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right)=$$

$$\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right)\mathbf{AB}=$$

9 Inverse of a transpose matrix

$$\mathbf{AA}^{-1}=\mathbf{I}$$

let me transpose both sides

$$\left((\mathbf{A}^{-1})^{\mathsf{T}}\right)\mathbf{A}^{\mathsf{T}}=\mathbf{I}$$

then

$\boxed{\text{the inverse of } \mathbf{A}^{\mathsf{T}} \text{ is}}$

10 Interchanges and permutations

Are interchange matrices $\mathbf{I}_{\underset{[i\rightleftharpoons j]}{\boldsymbol{\tau}}}$, invertible?

It is easy to check that

$$\left(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}\right)^{\mathsf{T}}\left(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}\right)=\mathbf{I} \qquad \Longrightarrow$$

11 Caracterización of invertible matrices

Given $\mathbf{A}$ of order n, the following statements are equivalent

1. No zero columns in $\mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_p}=\mathbf{K}$ (pre-echelon matrix).
2. $\mathbf{A}$ has inverse.
3. $\mathbf{A}$ is product of elementary matrices.

$$\mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}=\mathbf{A}\left(\mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right)=\mathbf{I} \qquad \Rightarrow \qquad \mathbf{A}=\left(\mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right)^{-1}$$

where

$$\left(\mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right)^{-1} = \left((\mathbf{I}_{\boldsymbol{\tau}_1})\cdots(\mathbf{I}_{\boldsymbol{\tau}_k})\right)^{-1} = (\mathbf{I}_{\boldsymbol{\tau}_k^{-1}})\cdots(\mathbf{I}_{\boldsymbol{\tau}_1^{-1}}) = \mathbf{I}_{\boldsymbol{\tau}_k^{-1}\cdots\boldsymbol{\tau}_1^{-1}}$$

Questions of the Lecture 5

(L-5) Question 1. Use the Gauss-Jordan method to invert

(a) $\mathbf{A}_1=\begin{bmatrix}1&0&0\\1&1&1\\0&0&1\end{bmatrix}$.

(b) $\mathbf{A}_2=\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}$.

(c) $\mathbf{A}_3=\begin{bmatrix}0&0&1\\0&1&1\\1&1&1\end{bmatrix}$.

(Strang, 1988, exercise 6 from section 1.6.)

(L-5) Question 2.

(a) If $\mathbf{A}$ is invertible and $\mathbf{AB}=\mathbf{AC}$, prove quickly that $\mathbf{B}=\mathbf{C}$.
(b) If $\mathbf{A}=\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]$, find an example with $\mathbf{AB}=\mathbf{AC}$, but $\mathbf{B}\neq\mathbf{C}$.

(Strang, 1988, exercise 4 from section 1.6.)

(L-5) Question 3. Use the Gauss-Jordan method to invert the generic matrix 2×2

$$\mathbf{M}=\begin{bmatrix}a&b\\c&d\end{bmatrix}.$$

The matrix is invertible (not singular) only when ...

(L-5) Question 4. Use the Gauss-Jordan method to invert the following matrices.

$$\mathbf{A}=\begin{bmatrix}3&1&2\\-1&0&1\\0&1&6\end{bmatrix}; \qquad \mathbf{B}=\begin{bmatrix}1&2&1\\-1&4&-2\\1&3&1\end{bmatrix}$$

(L-5) Question 5. If the 3 by 3 matrix $\mathbf{A}$ has $\mathbf{A}_{|1}+\mathbf{A}_{|2}=\mathbf{A}_{|3}$, show that $\mathbf{A}$ is not invertible, by two different methods:

(a) Find a nonzero solution $\boldsymbol{x}$ to $\mathbf{A}\boldsymbol{x}=\mathbf{0}$.
(b) Elimination keeps $column\,1+column\,2=column\,3$. Explain why there is no third pivot.

(Strang, 1988, exercise 26 from section 1.6.)

(L-5) QUESTION 6. Find the inverses of

(a) $\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}$.

(b) $\mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}$.

(c) $\mathbf{A}_3 = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$.

(Strang, 1988, exercise 10 from section 1.6.)

(L-5) QUESTION 7. Find the inverse of

$$\mathbf{A} = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}$$

What values of a and b make the matrix singular?
(Strang, 1988, exercise 42 from section 1.6.)

(L-5) QUESTION 8. Find $\mathbf{E}^2$, $\mathbf{E}^8$ and $\mathbf{E}^{-1}$ if $\mathbf{E} = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$
(Strang, 1988, exercise 6 from section 1.5.)

(L-5) QUESTION 9. Consider the following permutation matrix:

$$\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Find $\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}}^{-1}$. Can you say something else about the relationship between $\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}}$ and $\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}}^{-1}$?

(L-5) QUESTION 10. The 3 by 3 matrix **A** reduces to the identity matrix **I** by the following three column operations (in order):

$\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\tau}$:	Subtract 4 times column 1 from column 2.
$\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\tau}$:	Subtract 3 times column 1 from column 3.
$\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\tau}$:	Subtract column 3 from column 2.

(a) Write $\mathbf{A}^{-1}$ in terms of elementary matrices $\mathbf{I}_{\tau}$. **Then compute $\mathbf{A}^{-1}$.**
(b) What is the original matrix **A**?

(Based on *MIT Course 18.06 Quiz 1, October 4, 2006*)

(L-5) QUESTION 11. The 3 by 3 matrix **A** reduces to the identity matrix **I** by the following three **row** operations (in order):

$\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\tau}$:	Subtract 4 times row 1 from row 2.
$\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\tau}$:	Subtract 3 times row 1 from row 3.
$\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\tau}$:	Subtract row 3 from row 2.

(a) Write $\mathbf{A}^{-1}$ in terms of the **E**'s. **Then compute $\mathbf{A}^{-1}$.**
(b) What is the original matrix **A** ?

(*MIT Course 18.06 Quiz 1, October 4, 2006*)

(L-5) QUESTION 12.

(a) Find the inverse of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(b) Find the inverse of the following matrix **using the Gauss-Jordan method**

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & d \end{bmatrix}$$

(Poole, 2004, exercise 36, 38 and 59 from section 3.3.)

(L-5) QUESTION 13. Consider the squared matrices **A**, **B**, and **C**. True or false?

(a) If $\mathbf{AB} = \mathbf{I}$ and $\mathbf{CA} = \mathbf{I}$ then $\mathbf{B} = \mathbf{C}$.
(b) $(\mathbf{AB})^2 = \mathbf{A}^2\mathbf{B}^2$.

(L-5) QUESTION 14. Consider the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & a & 0 & 2a \\ a & 0 & 1 & 0 \\ 1 & 0 & a & 1 \end{bmatrix}$

(a) Prove that **A** is invertible for any value of a.
(b) Compute $\mathbf{A}^{-1}$ when $a = 0$.

(L-5) QUESTION 15. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$. Find $\mathbf{A}^{-1}$.

(L-5) QUESTION 16. Find (if it is possible) the inverse of the following inverses

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}.$$

(L-5) QUESTION 17. There is a finite number ($n!$) of $n \times n$ permutation matrices. In addition, any power of a permutation matrix is a another permutation matrix. Use these facts to prove that $\left(\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}}\right)^r = \mathbf{I}$ for some integer numbers r.

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