Mathematics II

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You can find the last version of these course materials at

https://github.com/mbujosab/MatematicasII/tree/main/Eng

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Part II

Elementary transformations and elimination

LECTURE 4: Elimination

Lecture 4

(**Lecture** 4) S-1 Highlights of Lesson 4

Highlights of *Lesson 4*

- Elementary transformations
- Indentifying singular matrices by elimination
- Matrix multiplication of Elementary matrices

F1

Resumen de la lección En esta lección se cuenta la mecánica del método de eliminación[1]. Ahora se presenta sin ningún propósito concreto (en la próxima lección se empleará para invertir una matriz, dos lecciones más tarde para resolver sistemas de ecuaciones, y en la segunda parte del curso para calcular determinantes y diagonalizar matrices).

La eliminación solo emplea dos tipos de operaciones, denominadas *transformaciones elementales*:

Transformaciones Elementales Tipo I. Suman a un vector un múltiplo de *otro vector*

Transformaciones Elementales Tipo II. Multiplican un vector por un número *distinto de cero*

Una vez introducidos los dos tipos de transformaciones elementales, se propone pre-escalonar una matriz tres por tres. Con este ejercicio se indica en qué consiste una *matriz (pre)escalonada* y a qué se denomina *pivote*. El pivote de una columna no nula es su primer componente no nulo. Una matriz está *pre-escalonada* si las componentes a la derecha de cada pivote son nulas, y está *escalonada* si además los pivotes de sus columnas describen una escalera descendente, dejando todas las columnas nulas (si hay) a la derecha de las columnas no nulas.

A continuación se propone a los estudiantes que intenten pre-escalonar varias matrices y comprobar si la forma escalonada tiene tres pivotes (o si falta alguno). La idea es darles unos minutos para que lo intenten y comprendan la mecánica (a lo largo del curso casi todo se resolverá mediante eliminación... así que es fundamental comprender la mecánica cuanto antes).

Después se muestra cómo realizar una transformación elemental $\boldsymbol{\tau}$ de las columnas de $\mathbf{A}$ mediante un producto de matrices $\mathbf{A}(\mathbf{I}_{\boldsymbol{\tau}})$; donde $(\mathbf{I}_{\boldsymbol{\tau}})$ es la *matriz elemental* resultante de aplicar una sola transformación elemental $\boldsymbol{\tau}$ a las columnas de la matriz identidad. Es importante explicar la notación usada para denotar las transformaciones elementales de la columna j-ésima (que también describen las correspondientes matrices elementales): $\mathbf{I}_{\underset{[(a)\boldsymbol{k}+\boldsymbol{j}]}{\boldsymbol{\tau}}}$ y $\mathbf{I}_{\underset{[(a)\boldsymbol{j}]}{\boldsymbol{\tau}}}$.

A continuacion veremos una de las ideas que más usaremos durante el curso: que una sucesión de transformaciones elementales de las columnas de $\mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}$ se puede expresar como el producto de $\mathbf{A}$ por una matriz que es el producto de todas las matrices elementales empleadas en la sucesión: $\mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k} = \mathbf{I}_{\boldsymbol{\tau}_1}\cdots\mathbf{I}_{\boldsymbol{\tau}_k}$.

$$\mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k} = \mathbf{A}\left(\mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right)$$

Ésta idea es el pilar del método de eliminiación.

Después destacaremos que las transformaciones elementales se pueden *deshacer*, es decir, que son *invertibles* (esta es una propiedad fundamental, y que será central en la lección sobre matrices invertibles).

También veremos que el intercambio de posición de dos vectores resulta ser equivalente a una sucesión de transformaciones *Tipo I* y *II* —se debe mostrar en la pizarra con la matriz identidad de orden 2, animando a los estudiantes a que vayan indicando los pasos necesarios... es interesante que vean que hay varias formas de llegar al mismo resultado. Llamaremos *permutación*, $\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}$, a cualquier sucesión arbitraria de intercambios; así $\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}$ es una reordenación de las columnas de $\mathbf{I}$. Veremos que puesto que $\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}$ es una sucesión de intercambios, $\mathbf{A}(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}})$ reordena las columnas de $\mathbf{A}$.

[1] en esta lección solo aplicaremos la eliminación "de izquierda a derecha".

La demostración de que toda matriz se puede escalonar mediante transformaciones elementales no se cuenta en clase, pero dicha demostración es básicamente una descripción del método de eliminación (véase el libro y la documentación de la librería de Python que lo acompaña).

(**Lecture** 4) $\boxed{\text{S-2}}$ **Elementary transformations of a matrix**

Type *I*: $\mathbf{A}_{\underset{[(\lambda)\boldsymbol{i}+\boldsymbol{j}]}{\boldsymbol{\tau}}}$ ***(with*** $i \neq j$***)***
add λ times i-th column ($\lambda\mathbf{A}_{|i}$) to j-th column ($\mathbf{A}_{|j}$)

$$\begin{bmatrix} 1 & -3 & 0 \\ 1 & -6 & 3 \end{bmatrix}_{\underset{[(-2)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}} = \begin{bmatrix} 1 & -3 & -2 \\ 1 & -6 & 1 \end{bmatrix}$$

Type *II*: $\mathbf{A}_{\underset{[(\alpha)\boldsymbol{i}]}{\boldsymbol{\tau}}}$ ***(with*** $\alpha \neq 0$***)***
multiply by α the i-th column

$$\begin{bmatrix} 1 & -3 & 0 \\ 1 & -6 & 3 \end{bmatrix}_{\underset{[(10)\mathbf{2}]}{\boldsymbol{\tau}}} = \begin{bmatrix} 1 & -30 & 0 \\ 1 & -60 & 3 \end{bmatrix}$$

$\boxed{\text{F3}}$

Transformación **Tipo I** de la matriz identidad de orden 4:

$$\mathbf{I}_{\underset{[(-3)\mathbf{2}+\mathbf{4}]}{\boldsymbol{\tau}}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformación **Tipo II** de la matriz identidad de orden 4:

$$\mathbf{I}_{\underset{[(5)\mathbf{3}]}{\boldsymbol{\tau}}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Pivotes y forma escalonada de una matriz $\mathbf{A}$: llamamos *pivote* de una columna *no nula* al primer componente no nulo de dicha columna; y llamaremos *posición de pivote* al índice de la fila en la que se encuentra dicho coeficiente *no nulo*.

Diremos que una matriz $\mathbf{K}$ está *pre-escalonada*, si los componentes que quedan a la derecha de cada pivote son nulos. Y diremos que una matriz $\mathbf{L}$ es *escalonada*, si está pre-escalonada, y toda columna que precede a una columna $\mathbf{A}_{|k}$ no nula es no nula y tiene una posición de pivote anterior a la posición de pivote de $\mathbf{A}_{|k}$. Por tanto, en una matriz escalonada no puede haber columnas nulas a la izquierda de columnas no nulas (es decir, o no hay columnas nulas, o todas ellas están en el lado derecho de la matriz). Además, conforme nos movemos de izquierda a derecha, los pivotes cada vez aparecen más abajo (i.e., en filas con índices cada vez mayores). Así, recorriendo la matriz de izquierda a derecha, los pivotes describen una escalera descendente (de ahí el nombre de matriz "escalonada").

(**Lecture** 4) S-3 **Elimination and pre-echelon form of a matrix**

- *Pivot* is the first non-zero component of each column.
- *Elimination*: modifies a matrix until all *components at the right-hand side of each pivot are zeros*

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 8 & 4 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[{[(-3)\mathbf{1}+\mathbf{2}]}]{\tau} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix} \xrightarrow[{[(-2)\mathbf{2}+\mathbf{3}]}]{\tau} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & -2 & 5 \end{bmatrix} = \mathbf{L}$$

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(**Lecture** 4) S-4 **Elimination**

Elimination algorithm on A
modifies **A** using a sequence of *elementary transformations*

Goal
to get a *(pre)echelon* form

- *pre-echelon*: all components on the right side of each pivot are zero.
- *echelon*: if any column before a non-null column $\mathbf{A}_{|j}$ is non-null column and its pivot is above the pivot of $\mathbf{A}_{|j}$.

It is always possible to find a (pre)echelon form by elimination
Rank (rg): the number of pivots in any of its pre-echelon forms
$\underset{n\times n}{\mathbf{A}}$ is *singular* if its pre-echelon forms have null-columns (rg $< n$)

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(**Lecture** 4) S-5 **Elimination: When can't we find n pivots?**

$n \times n$ matrices are **singular** if less than n pivots after elimination

$$\begin{bmatrix} 0 & 1 & 3 \\ 4 & 2 & 8 \\ 1 & 1 & 1 \end{bmatrix}$$

Has this matrix n pivots? $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ 1 & 1 & 1 \end{bmatrix}$

and this one? $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

and this one? $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & -4 \end{bmatrix}$

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(**Lecture** 4) S-6 Matrix multiplication: elementary matrices

$$\underbrace{\begin{bmatrix} 1 & 3 & 0 \\ 2 & 8 & 4 \\ 1 & 1 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}_{\mathbf{I}_{\tau}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix}}_{\mathbf{A}_{\tau}}$$

We call $\mathbf{I}_{\tau}$ "Elementary matrix":

$$\mathbf{A}(\mathbf{I}_{\tau}) = \mathbf{A}_{\tau}$$

This specific elementary matrix $\mathbf{I}_{\tau}$ is written as $\mathbf{I}_{\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\tau}}$

$$\mathbf{A}\left(\mathbf{I}_{\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\tau}}\right) = \mathbf{A}_{\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\tau}}$$

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(**Lecture** 4) S-7 Matrix multiplication: elementary matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & -2 & 5 \end{bmatrix}$$

This specific elementary matrix $\mathbf{I}_{\tau}$ is written as $\mathbf{I}_{\underset{[(-2)\mathbf{2}+\mathbf{3}]}{\tau}}$

$$\mathbf{A}\left(\mathbf{I}_{\underset{[(-2)\mathbf{2}+\mathbf{3}]}{\tau}}\right) = \mathbf{A}_{\underset{[(-2)\mathbf{2}+\mathbf{3}]}{\tau}}$$

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(**Lecture** 4) S-8 Elimination by elementary matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 8 & 4 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\tau}} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix} \xrightarrow{\underset{[(-2)\mathbf{2}+\mathbf{3}]}{\tau}} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & -2 & 5 \end{bmatrix} = \mathbf{L}$$

$$\mathbf{A}_{\underset{\substack{[(-3)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{2}+\mathbf{3}]}}{\tau}} = \mathbf{A}_{\underset{[(-3)\mathbf{1}+\mathbf{2}][(-2)\mathbf{2}+\mathbf{3}]}{\tau}} = \left(\mathbf{A}\left(\mathbf{I}_{\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\tau}}\right)\right)\left(\mathbf{I}_{\underset{[(-2)\mathbf{2}+\mathbf{3}]}{\tau}}\right) = \mathbf{L}$$

there is a matrix that does the whole job *at once*

$$\mathbf{A}_{\underset{\substack{[(-3)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{2}+\mathbf{3}]}}{\tau}} = \mathbf{A}\left(\left(\mathbf{I}_{\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\tau}}\right)\left(\mathbf{I}_{\underset{[(-2)\mathbf{2}+\mathbf{3}]}{\tau}}\right)\right) = \mathbf{A}\mathbf{I}_{\underset{\substack{[(-3)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{2}+\mathbf{3}]}}{\tau}} = \mathbf{L}$$

$$\boxed{\mathbf{A}_{\tau_1\cdots\tau_k} = \mathbf{A}\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)}$$

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(**Lecture** 4) S-9 how do I get from **L** back to **A**? Inverses

How do I reverse the first step? (it was subtract 3 times $\mathbf{A}_{|1}$ from $\mathbf{A}_{|2}$)

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\mathbf{I}_{\underset{[(-\lambda)i+j]}{\tau}} \quad \text{“undo”} \quad \mathbf{I}_{\underset{[(\lambda)i+j]}{\tau}}$$

How to undo $\mathbf{I}_{\underset{[(\alpha)i]}{\tau}}$?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

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(**Lecture** 4) S-10 Interchange or swap matrices

Which matrix exchanges the columns?

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix} = \begin{bmatrix} c & a \\ d & b \end{bmatrix}$$

Which matrix exchanges the rows? where do we put that matrix?

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \qquad = \begin{bmatrix} b & d \\ a & c \end{bmatrix}$$

Matrix multiplication is not commutative!

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(**Lecture** 4) S-11 Interchange of columns

Interchange of columns:

$\mathbf{A}_{\underset{[i\rightleftharpoons j]}{\tau}} \rightarrow$ swicht columns i and j of $\mathbf{A}$

$$\begin{bmatrix} 1 & -3 & 0 \\ 1 & -6 & 3 \end{bmatrix}_{\underset{[\mathbf{2}\rightleftharpoons \mathbf{3}]}{\tau}} = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 3 & -6 \end{bmatrix}$$

We can switch two columns by a sequence of elementary transformations

Matrix $\mathbf{I}_{\underset{[i\rightleftharpoons j]}{\tau}}$ is call a exchange matrix

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(**Lecture** 4) S-**12** Permutation matrices

Product between exchange matrices $\underset{[\cdot\rightleftharpoons\cdot]}{\mathbf{I}_{\boldsymbol{\tau}}}$ is a permutation matrix $\underset{[\mathfrak{S}]}{\mathbf{I}_{\boldsymbol{\tau}}}$.

$$\underset{[\mathfrak{S}]}{\mathbf{I}_{\boldsymbol{\tau}}} = \text{ Identity matrix } \mathbf{I} \text{ with rearranged columns}$$

Let's see the 3×3 case

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad \underset{[\mathbf{1}\rightleftharpoons\mathbf{2}]}{\mathbf{I}_{\boldsymbol{\tau}}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

How many 3×3 pemutations can we find?

what happens if I multiply two permutation matrices?

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The lecture ends here

Questions of the Lecture 4

(L-4) Question 1.

(a) Which three matrices $\underset{[(x)\mathbf{1}+\mathbf{2}]}{\mathbf{I}_{\boldsymbol{\tau}}}$, $\underset{[(y)\mathbf{1}+\mathbf{3}]}{\mathbf{I}_{\boldsymbol{\tau}}}$ and $\underset{[(z)\mathbf{2}+\mathbf{3}]}{\mathbf{I}_{\boldsymbol{\tau}}}$ put $\mathbf{A} = \begin{bmatrix} 1 & 4 & -2 \\ 1 & 6 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ into an echelon form?

(b) Multiply those $\mathbf{I}_{\boldsymbol{\tau}_i}$ to get one matrix $\mathbf{E}$ that does elimination: $\mathbf{AE} = \mathbf{K}$.

Based on (?, exercise 24 from section 1.4.)

(L-4) Question 2. Consider the matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ -1 & -3 & -2 \\ 0 & 1 & c \end{bmatrix}$$

For what value(s) of c the matrix is singular (we can't find three pivots)?

(L-4) Question 3. Consider the following 3 by 3 matrices.

(a) $\Big(\underset{[(-1)\mathbf{1}+\mathbf{2}]}{\mathbf{I}_{\boldsymbol{\tau}}}\Big)$ subtracts column 1 from column 2 and then $\Big(\underset{[\mathbf{2}\rightleftharpoons\mathbf{3}]}{\mathbf{I}_{\boldsymbol{\tau}}}\Big)$ exchanges columns 2 and 3. What matrix $\mathbf{E}$ does both steps at once?

(b) $\Big(\underset{[\mathbf{2}\rightleftharpoons\mathbf{3}]}{\mathbf{I}_{\boldsymbol{\tau}}}\Big)$ exchanges columns 2 and 3 and then $\underset{[(-1)\mathbf{1}+\mathbf{3}]}{\mathbf{I}_{\boldsymbol{\tau}}}$ subtracts column 1 from column 3. What matrix $\mathbf{N} = \Big(\underset{[\mathbf{2}\rightleftharpoons\mathbf{3}]}{\mathbf{I}_{\boldsymbol{\tau}}}\Big)\Big(\underset{[(-1)\mathbf{1}+\mathbf{3}]}{\mathbf{I}_{\boldsymbol{\tau}}}\Big)$ does both steps at once? Explain why $\mathbf{M}$ and $\mathbf{N}$ are the same but the $\mathbf{I}_{\boldsymbol{\tau}}$'s are different.

Based on (?, exercise 28 from section 1.4.)

(L-4) Question 4. Elimination matrices $\underset{[(?)\mathbf{1}+\mathbf{2}]}{\mathbf{I}_{\boldsymbol{\tau}}}$ and $\underset{[(?)\mathbf{2}+\mathbf{3}]}{\mathbf{I}_{\boldsymbol{\tau}}}$ will reduce $\mathbf{A}$ to triangular form. Find $\mathbf{E}$ so that $\mathbf{AE} = \mathbf{L}$ is lower triangular (echelon), if $\mathbf{A}$ is

$$\begin{bmatrix} 2 & 2 & 0 \\ 1 & 4 & 9 \\ 1 & 3 & 9 \end{bmatrix}$$

(L-4) Question 5. Although we will only consider as elementary the *Type I* and *II* transformations, in most of the Linear Algebra books appears a third type: the *exchange* of columns

$$\underset{[\boldsymbol{p}\rightleftharpoons\boldsymbol{s}]}{\mathbf{A}_{\boldsymbol{\tau}}} \to \text{Exchanges columns } p \text{ and } s \text{ of } \mathbf{A}.$$

Prove that a column exchange is, in fact, a sequence of *Type I* and *II* elementary transformations. Try transforming $\underset{2\times 2}{\mathbf{I}}$ in $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ by elementary transformations of the columns.

(L-4) QUESTION 6. Write down the 3 by 3 matrices that produce these elimination steps:
(a) $\mathbf{I}_{\underset{[(-5)\mathbf{1}+\mathbf{2}]}{\tau}}$ substracts 5 times column 1 from column 2,
(b) $\mathbf{I}_{\underset{[(-7)\mathbf{2}+\mathbf{3}]}{\tau}}$ substracts 7 times column 2 from column 3,
(c) $\mathbf{I}_{\underset{[\circlearrowleft]}{\tau}}$ exchanges columns 1 and 2, and then columns 2 and 3.
(?, exercise 1 from section 2.3.)

(L-4) QUESTION 7. Consider the matrices of QUESTION 6:
(a) when multiplying by $\mathbf{I}_{\underset{[(-5)\mathbf{1}+\mathbf{2}]}{\tau}}$ and then by $\mathbf{I}_{\underset{[(-7)\mathbf{2}+\mathbf{3}]}{\tau}}$ the matrix $\mathbf{A} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}$ we get $\mathbf{A}_{\underset{\substack{[(-5)\mathbf{1}+\mathbf{2}]\\ [(-7)\mathbf{2}+\mathbf{3}]}}{\tau}} = [\quad;\quad;\quad]$.
(b) But, when multiplying by $\mathbf{I}_{\underset{[(-5)\mathbf{1}+\mathbf{2}]}{\tau}}$ before and then by $\mathbf{I}_{\underset{[(-7)\mathbf{2}+\mathbf{3}]}{\tau}}$ we get $\mathbf{A}_{\underset{\substack{[(-7)\mathbf{2}+\mathbf{3}]\\ [(-5)\mathbf{1}+\mathbf{2}]}}{\tau}} = [\quad;\quad;\quad]$.
(c) When $\underset{[(-7)\mathbf{2}+\mathbf{3}]}{\tau}$ comes first, the column ____ feels no effect from column ____. This property will become very important in the LU factorization!
(?, exercise 2 from section 2.3.)

(L-4) QUESTION 8. What matrix $\mathbf{M}$ sends $\boldsymbol{v} = (1, \quad 0,)$ to $(0, \quad 1,)$, es decir $\boldsymbol{v}\mathbf{M} = (0, \quad 1,)$; and also sends $\boldsymbol{w} = (0, \quad 1,)$ to $(1, \quad 0,)$, es decir $\boldsymbol{w}\mathbf{M} = (1, \quad 0,)$?

(L-4) QUESTION 9. Consider a permutation (interchange) matrix $\mathbf{I}_{\underset{[i \rightleftharpoons j]}{\tau}}$, if we compute the product $\mathbf{A}(\mathbf{I}_{\underset{[i \rightleftharpoons j]}{\tau}})$, we get a new matrix like $\mathbf{A}$, but with exchanged columns. What happen if we compute the product $(\mathbf{I}_{\underset{[i \rightleftharpoons j]}{\tau}})\mathbf{A}$? Check your answer with a 2 by 2 example.

(L-4) QUESTION 10. If every column of $\mathbf{A}$ is a multiple of $(1, \quad 1, \quad 1,)$, then $\mathbf{A}\boldsymbol{x}$ is always a multiple of $(1, \quad 1, \quad 1,)$. Do a 3 by 3 example. How many pivots are produced by elimination?
(?, exercise 26 from section 1.4.)

End of Questions of the Lecture 4

LECTURE 5: Inverse matrices

Lecture 5

(**Lecture** 5) S-1 Highlights of Lesson 5

Highlights of *Lesson 5*

- Inverse of $\mathbf{A}$
- Gauss-Jordan elimination / finding $\mathbf{A}^{-1}$
- Inverse of $\mathbf{AB}$, $\mathbf{A}^{\intercal}$

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Resumen de la lección: La inversa de una matriz cuadrada $\mathbf{A}$ es una matriz $\mathbf{B}$ tal que

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}.$$

Es importante indicar que no toda matriz cuadrada tiene inversa. . . y expondremos dos argumentos: por una parte, es imposible de combinar las columnas de $\mathbf{A}$ para generar las columnas de $\mathbf{I}$ si las columnas de $\mathbf{A}$ están alineadas. Por otra, y de forma más general, es imposible que se puedan combinar las columnas de $\mathbf{A}$ de manera que $\mathbf{A}\boldsymbol{x} = \mathbf{0}$, con $\boldsymbol{x} \neq \mathbf{0}$, y que al mismo tiempo exista $\mathbf{A}^{-1}$.

Veremos que encontrar la inversa de $\mathbf{A}$ de orden n implica resolver n sistemas de ecuaciones con n incógnitas (pues hemos de encontrar n combinaciones de las columnas de $\mathbf{A}$ que generen las n columnas de $\mathbf{I}$, una combinación por cada columna de $\mathbf{I}$).

La aparente "magia" del método de Gauss consiste en que es posible resolver simultáneamente los n sistemas de ecuaciones. Para hacerlo basta alargar las columnas de $\mathbf{A}$ con las columnas de la matriz identidad y aplicar después la eliminación Gauss-Jordan; para entenderlo hay que recordar que aplicar una secuencia de transformaciones elementales es equivalente a multiplicar la matriz por la matriz resultante de aplicar dicha sucesión sobre la matriz identidad:

$$\mathbf{A}_{\tau_1\cdots\tau_k} = \mathbf{A}\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)$$

También deduciremos cómo es la inversa tanto del producto de matrices invertibles, $(\mathbf{AB})^{-1}$, como de la transpuesta de una matriz invertible, $(\mathbf{A}^{\intercal})^{-1}$.

Para finalizar, daremos una caracterización de las matrices invertibles relacionada con el método de eliminación y las matrices elementales (la demo formal aparece en el libro, pero tras haber aplicado Gauss-Jordan con un ejemplo en la pizarra, se puede explicar la idea que hay debajo de esta caracterización).

(**Lecture** 5) S-2 Inverse of a matrix (square matrices)

$\mathbf{A}$ squared of order n has inverse (is *invertible*) if exists $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}.$$

Then

$$\mathbf{B} = \mathbf{A}^{-1} \qquad \text{and} \qquad \mathbf{A} = \mathbf{B}^{-1}.$$

Not all matrices have inverse

Squared matrices with no inverse are called *singular* matrices

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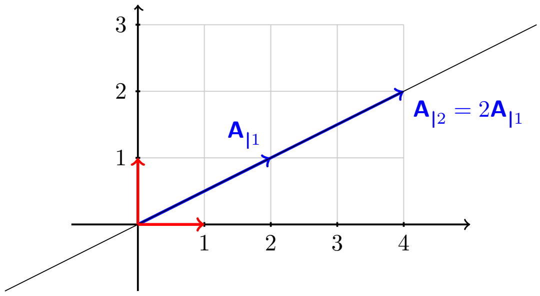
(**Lecture** 5) S-3 **Singular case (no inverse)**

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

Is it possible to find a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I}$?
...columns of $\mathbf{I}$ should be linear combinations of columns of $\mathbf{A}$... but both columns lie on the same line.



So

$$\boxed{\mathbf{A} \text{ is singular}}$$

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(**Lecture** 5) S-4 **Singular case (no inverse)**

Can we find $\boldsymbol{x} \neq \mathbf{0}$ such that $\mathbf{A}\boldsymbol{x} = \mathbf{0}$?

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} \\ \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ and $\boldsymbol{x} \neq \mathbf{0}$ $\Rightarrow$ there is no $\mathbf{A}^{-1}$

$$\boxed{\text{The existence of } \mathbf{A}^{-1} \text{ leads to a } \textit{contradiction}}$$

$$\text{If } \mathbf{A}\boldsymbol{x} = \mathbf{0} \text{ and } \boldsymbol{x} \neq \mathbf{0} \quad \Rightarrow \quad \mathbf{A}^{-1}\mathbf{A}\boldsymbol{x} = \mathbf{A}^{-1}\mathbf{0} \quad \Rightarrow \quad \boldsymbol{x} = \mathbf{0}.$$

When $\mathbf{A}^{-1}$ does exist

the only solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$.

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(**Lecture** 5) S-5 **Calculating the inverse matrix**

$$\mathbf{A}(\mathbf{A}^{-1}) = \mathbf{I}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So... we are solving m systems (of m equations each)

$$\begin{bmatrix} 1 & 3 \\ 2 & \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{bmatrix} 1 & 3 \\ 2 & \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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(**Lecture** 5) S-6 Gauss-Jordan: solving two linear systems at once

Gauss-Jordan elimination (obtaining a reduced echelon form R)
apply elementary transformations until a echelon matrix with only zeros to the left of each pivot (and all pivots equal to 1) is achieved

Let's solve the linear systems

$$\begin{bmatrix}1&3\\2&7\end{bmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix} \qquad \begin{bmatrix}1&3\\2&7\end{bmatrix}\begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$

applying Gauss-Jordan elimination on $\mathbf{A}$ stacked with $\mathbf{I}$

$$\begin{bmatrix}\mathbf{A}\\ \hline \mathbf{I}\end{bmatrix}=\begin{bmatrix}1&3\\2&7\\ \hline 1&0\\0&1\end{bmatrix}\to \qquad \to \qquad =$$

If $\mathbf{R}=\mathbf{I}$, we have found $\mathbf{A}^{-1}$

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```
A = Matrix([[1,3],[2,7]])
B = ElimGJ( A.apila(I(2),1) , 1 )
(3,4)|B
```

(**Lecture** 5) S-7 Gauss-Jordan: Why does it work?

$$\begin{bmatrix}\mathbf{A}\\ \hline \mathbf{I}\end{bmatrix}=\begin{bmatrix}1&3\\2&7\\ \hline 1&0\\0&1\end{bmatrix}\xrightarrow[{[(-3)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \qquad \xrightarrow[{[(-2)\mathbf{2}+\mathbf{1}]}]{\boldsymbol{\tau}}$$

that is, since $\mathbf{A}_{\tau_1\cdots\tau_k}=\mathbf{A}\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)$:

$$\begin{bmatrix}\mathbf{A}\\ \hline \mathbf{I}\end{bmatrix}_{\tau_1\cdots\tau_k}=\begin{bmatrix}\mathbf{A}_{\tau_1\cdots\tau_k}\\ \hline \mathbf{I}_{\tau_1\cdots\tau_k}\end{bmatrix}=\begin{bmatrix}\mathbf{A}(\mathbf{I}_{\tau_1\cdots\tau_k})\\ \hline \mathbf{I}_{\tau_1\cdots\tau_k}\end{bmatrix}=\begin{bmatrix}\mathbf{I}\\ \hline \mathbf{I}_{\tau_1\cdots\tau_k}\end{bmatrix},$$

$$\boxed{\textit{who is } \mathbf{I}_{\tau_1\cdots\tau_k}\textit{?}}$$

therefore $\mathbf{A}^{-1}=$

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$$\text{Since}\quad \mathbf{I}_{\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}=\begin{bmatrix}1&-3\\0&1\end{bmatrix},\quad\text{and}\quad \mathbf{I}_{\underset{[(-2)\mathbf{2}+\mathbf{1}]}{\boldsymbol{\tau}}}=\begin{bmatrix}1&0\\-2&1\end{bmatrix};$$

then

$$\mathbf{I}_{\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(-2)\mathbf{2}+\mathbf{1}]}{\boldsymbol{\tau}}}=\left(\mathbf{I}_{\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}\right)\left(\mathbf{I}_{\underset{[(-2)\mathbf{2}+\mathbf{1}]}{\boldsymbol{\tau}}}\right)=\begin{bmatrix}1&-3\\0&1\end{bmatrix}\begin{bmatrix}1&0\\-2&1\end{bmatrix}=\begin{bmatrix}7&-3\\-2&1\end{bmatrix}.$$

Since $\mathbf{A}\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)=\mathbf{I}$, it follows that $\mathbf{I}_{\tau_1\cdots\tau_k}$ is the inverse matrix of $\mathbf{A}$.

$$\boxed{\text{In other words, if } \mathbf{A}_{\tau_1\cdots\tau_k}=\mathbf{I} \text{ then } \mathbf{I}_{\tau_1\cdots\tau_k}=\mathbf{A}^{-1}.}$$

We can also operate with the rows (the usual way in most of the textbooks). Here we multiply on the left to operate with the rows .

$$[\mathbf{A}|\mathbf{I}]=\left[\begin{array}{cc|cc}1&3&1&0\\2&7&0&1\end{array}\right]\xrightarrow[{\underset{[(-2)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}]{}\left[\begin{array}{cc|cc}1&3&1&0\\0&1&-2&1\end{array}\right]\xrightarrow[{\underset{[(-3)\mathbf{2}+\mathbf{1}]}{\boldsymbol{\tau}}}]{}\left[\begin{array}{cc|cc}1&0&7&-3\\0&1&-2&1\end{array}\right]=\left[{}_{\tau_1\cdots\tau_k}\mathbf{A}\,|\,{}_{\tau_1\cdots\tau_k}\mathbf{I}\right]=\left[\mathbf{I}\,|\,{}_{\tau_1\cdots\tau_k}\mathbf{I}\right].$$

What is the right hand side matrix ${}_{\tau_1\cdots\tau_k}\mathbf{I}$? The argument is exactly the same:

$$ {}_{\tau_1\cdots\tau_k}\left[\mathbf{A}\,|\,\mathbf{I}\right] = \left[({}_{\tau_1\cdots\tau_k}\mathbf{I})\mathbf{A}\,|\,{}_{\tau_1\cdots\tau_k}\mathbf{I}\right] = \left[\mathbf{I}\,|\,{}_{\tau_1\cdots\tau_k}\mathbf{I}\right] $$

Since $({}_{\tau_1\cdots\tau_k}\mathbf{I})\mathbf{A} = \mathbf{I}$, then ${}_{\tau_1\cdots\tau_k}\mathbf{I} = \mathbf{A}^{-1}$.

$$\boxed{\text{In other words, if } {}_{\tau_1\cdots\tau_k}\mathbf{A} = \mathbf{I} \text{ then } {}_{\tau_1\cdots\tau_k}\mathbf{I} = \mathbf{A}^{-1}.}$$

(**Lecture** 5) S-8 **Inverse of a product**

When $\mathbf{A}$ and $\mathbf{B}$, of order n, are invertible, $(\mathbf{AB})$ is invertible.
what matrix gives me the inverse of $\mathbf{AB}$? lets try with $(\mathbf{B}^{-1}\mathbf{A}^{-1})$:

$$\mathbf{AB}\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right) =$$

$$\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right)\mathbf{AB} =$$

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(**Lecture** 5) S-9 **Inverse of a transpose matrix**

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

let me transpose both sides

$$\left((\mathbf{A}^{-1})^{\mathsf{T}}\right)\mathbf{A}^{\mathsf{T}} = \mathbf{I}$$

then

$$\boxed{\text{the inverse of } \mathbf{A}^{\mathsf{T}} \text{ is} \qquad\qquad}$$

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(**Lecture** 5) S-10 **Interchanges and permutations**

Are interchange matrices $\mathbf{I}_{\substack{\tau\\ [i \rightleftharpoons j]}}$, invertible?
It is easy to check that

$$\left(\mathbf{I}_{\substack{\tau\\ [\mathfrak{S}]}}\right)^{\mathsf{T}}\left(\mathbf{I}_{\substack{\tau\\ [\mathfrak{S}]}}\right) = \mathbf{I} \qquad \Longrightarrow$$

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Exercise 1. Prove $\left(\mathbf{I}_{\substack{\tau\\ [\mathfrak{S}]}}\right)^{\mathsf{T}}\left(\mathbf{I}_{\substack{\tau\\ [\mathfrak{S}]}}\right) = \mathbf{I}$.

(**Lecture** 5) S-**11** Caracterización of invertible matrices

Given $\mathbf{A}$ of order n, the following statements are equivalent

1. No zero columns in $\mathbf{A}_{\tau_1\cdots\tau_p} = \mathbf{K}$ (pre-echelon matrix).
2. $\mathbf{A}$ has inverse.
3. $\mathbf{A}$ is product of elementary matrices.

$$\mathbf{A}_{\tau_1\cdots\tau_k} = \mathbf{A}\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right) = \mathbf{I} \qquad \Rightarrow \qquad \mathbf{A} = \left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)^{-1}$$

where

$$\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)^{-1} = \left((\mathbf{I}_{\tau_1})\cdots(\mathbf{I}_{\tau_k})\right)^{-1} = (\mathbf{I}_{\tau_k^{-1}})\cdots(\mathbf{I}_{\tau_1^{-1}}) = \mathbf{I}_{\tau_k^{-1}\cdots\tau_1^{-1}}$$

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Resumen de la lección

- Si algunas columnas de $\underset{n\times n}{\mathbf{A}}$ están alineadas (si unas son combinación lineal de las otras) la matriz es singular.
- Si el sistema de ecuaciones $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ tiene más de una solución, entonces la matriz cuadrada $\mathbf{A}$ es singular.
- Una matriz cuadrada $\mathbf{A}$ es singular si cualquiera de sus formas pre-escalonadas tienen columnas nulas.
- Encontrar la matriz inversa de $\underset{n\times n}{\mathbf{A}}$ es resolver n sistemas de ecuaciones.
- La inversa de $\mathbf{A}$ es el producto de las matrices elementales (operaciones elementales) que transforman $\mathbf{A}$ en $\mathbf{I}$:

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right]_{\tau_1\cdots\tau_k} = \left[\begin{array}{c}\mathbf{A}(\mathbf{I}_{\tau_1\cdots\tau_k})\\ \hline \mathbf{I}_{\tau_1\cdots\tau_k}\end{array}\right]; \qquad \text{si} \quad \mathbf{A}\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right) = \mathbf{I} \quad \text{entonces} \quad \mathbf{I}_{\tau_1\cdots\tau_k} = \mathbf{A}^{-1}.$$

- $\left(\mathbf{A}\mathbf{B}\right)^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
- $(\mathbf{A}^{-1})^{\mathsf{T}} = \left(\mathbf{A}^{\mathsf{T}}\right)^{-1}$
- $\left(\mathbf{I}_{\underset{[j\rightleftharpoons k]}{\tau}}\right)^{-1} = \mathbf{I}_{\underset{[j\rightleftharpoons k]}{\tau}}$
- $\left(\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}}\right)^{\mathsf{T}}\left(\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}}\right) = \mathbf{I} \qquad \Rightarrow \qquad \left(\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}}\right)^{\mathsf{T}} = \left(\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}}\right)^{-1}$

The lecture ends here

Questions of the Lecture 5

(L-5) Question 1. Use the Gauss-Jordan method to invert

(a) $\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) $\mathbf{A}_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

(c) $\mathbf{A}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

(?, exercise 6 from section 1.6.)

(L-5) QUESTION 2.
(a) If $\mathbf{A}$ is invertible and $\mathbf{AB} = \mathbf{AC}$, prove quickly that $\mathbf{B} = \mathbf{C}$.
(b) If $\mathbf{A} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, find an example with $\mathbf{AB} = \mathbf{AC}$, but $\mathbf{B} \neq \mathbf{C}$.

(?, exercise 4 from section 1.6.)

(L-5) QUESTION 3. Use the Gauss-Jordan method to invert the generic matrix 2×2

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The matrix is invertible (not singular) only when ...

(L-5) QUESTION 4. Use the Gauss-Jordan method to invert the following matrices.

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 6 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 4 & -2 \\ 1 & 3 & 1 \end{bmatrix}$$

(L-5) QUESTION 5. If the 3 by 3 matrix $\mathbf{A}$ has $\mathbf{A}_{|1} + \mathbf{A}_{|2} = \mathbf{A}_{|3}$, show that $\mathbf{A}$ is not invertible, by two different methods:
(a) Find a nonzero solution $\boldsymbol{x}$ to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.
(b) Elimination keeps *column* 1 + *column* 2 = *column* 3. Explain why there is no third pivot.

(?, exercise 26 from section 1.6.)

(L-5) QUESTION 6. Find the inverses of

(a) $\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}.$

(b) $\mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}.$

(c) $\mathbf{A}_3 = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}.$

(?, exercise 10 from section 1.6.)

(L-5) QUESTION 7. Find the inverse of

$$\mathbf{A} = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}$$

What values of a and b make the matrix singular?
(?, exercise 42 from section 1.6.)

(L-5) QUESTION 8. Find $\mathbf{E}^2$, $\mathbf{E}^8$ and $\mathbf{E}^{-1}$ if $\quad \mathbf{E} = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$

(?, exercise 6 from section 1.5.)

(L-5) Question 9. Consider the following permutation matrix:

$$\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Find $\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}^{-1}$. Can you say something else about the relationship between $\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}$ and $\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}^{-1}$?

(L-5) Question 10. The 3 by 3 matrix $\mathbf{A}$ reduces to the identity matrix $\mathbf{I}$ by the following three column operations (in order):

$\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}$: Subtract 4 times column 1 from column 2.

$\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}$: Subtract 3 times column 1 from column 3.

$\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}$: Subtract column 3 from column 2.

(a) Write $\mathbf{A}^{-1}$ in terms of elementary matrices $\mathbf{I}_{\boldsymbol{\tau}}$. **Then compute $\mathbf{A}^{-1}$.**
(b) What is the original matrix $\mathbf{A}$?

(Based on *MIT Course 18.06 Quiz 1, October 4, 2006*)

(L-5) Question 11. The 3 by 3 matrix $\mathbf{A}$ reduces to the identity matrix $\mathbf{I}$ by the following three **row** operations (in order):

$\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}$: Subtract 4 times row 1 from row 2.

$\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}$: Subtract 3 times row 1 from row 3.

$\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}$: Subtract row 3 from row 2.

(a) Write $\mathbf{A}^{-1}$ in terms of the $\mathbf{E}$'s. **Then compute $\mathbf{A}^{-1}$.**
(b) What is the original matrix $\mathbf{A}$?

(*MIT Course 18.06 Quiz 1, October 4, 2006*)

(L-5) Question 12.

(a) Find the inverse of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
(b) Find the inverse of the following matrix **using the Gauss-Jordan method**

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & d \end{bmatrix}$$

(**?**, exercise 36, 38 and 59 from section 3.3.)

(L-5) Question 13. Consider the squared matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. True or false?
(a) If $\mathbf{AB} = \mathbf{I}$ and $\mathbf{CA} = \mathbf{I}$ then $\mathbf{B} = \mathbf{C}$.
(b) $(\mathbf{AB})^2 = \mathbf{A}^2\mathbf{B}^2$.

(L-5) QUESTION 14. Consider the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & a & 0 & 2a \\ a & 0 & 1 & 0 \\ 1 & 0 & a & 1 \end{bmatrix}$

(a) Prove that $\mathbf{A}$ is invertible for any value of a.
(b) Compute $\mathbf{A}^{-1}$ when $a = 0$.

(L-5) QUESTION 15. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$. Find $\mathbf{A}^{-1}$.

(L-5) QUESTION 16. Find (if it is possible) the inverse of the following inverses

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}.$$

(L-5) QUESTION 17. There is a finite number $(n!)$ of $n \times n$ permutation matrices. In addition, any power of a permutation matrix is a another permutation matrix. Use these facts to prove that $\left(\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}}\right)^r = \mathbf{I}$ for some integer numbers r.

End of Questions of the Lecture 5

Solutions

(L-4) Question 1(a)

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & -2 \\ 1 & 6 & 2 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[{[(2)\mathbf{1}+\mathbf{3}]}]{\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{\underset{[(-2)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & -2 \end{bmatrix} = \mathbf{L}$$

so

$$\mathbf{I}_{\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}} = \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \mathbf{I}_{\underset{[(2)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \mathbf{I}_{\underset{[(-2)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix};$$

```
A = Matrix([ [1, 4,-2], [1, 6, 2], [0, 1, 0] ])
L = Elim(A,1)
Tr = L.TrC
display(Tr)
Sistema([ Sistema([t, I(3)&t]) for t in Tr])
```

□

(L-4) Question 1(b)

$$\mathbf{E} = \mathbf{I}_{\substack{\boldsymbol{\tau} \\ [(-4)\mathbf{1}+\mathbf{2}] \\ [(2)\mathbf{1}+\mathbf{3}] \\ [(-2)\mathbf{2}+\mathbf{3}]}} = \left(\mathbf{I}_{\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}\right)\left(\mathbf{I}_{\underset{[(2)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}}\right)\left(\mathbf{I}_{\underset{[(-2)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}}\right) = \begin{bmatrix} 1 & -4 & 10 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix};$$

$$\mathbf{AE} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & -2 \end{bmatrix} = \mathbf{K}.$$

```
ME = Sistema( [ I(3) & t for t in Tr])
E  = (ME|1)*(ME|2)*(ME|3)
display(E)
display(A*E)
A & Tr
```

□

(L-4) Question 2. Substracting 2 times first column from second and 4 times from third, and then adding 2 times the second to the third

$$\mathbf{A} \quad = \quad \begin{bmatrix} 1 & 2 & 4 \\ -1 & -3 & -2 \\ 0 & 1 & c \end{bmatrix} \xrightarrow[{[(-4)\mathbf{1}+\mathbf{3}]}]{\underset{[(-2)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}} \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 2 \\ 0 & 1 & c \end{bmatrix} \xrightarrow{\underset{[(2)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}} \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 1 & c+2 \end{bmatrix} \quad = \quad \mathbf{L}$$

We will find only two pivots if the last column of $\mathbf{L}$ becomes a zero column, and this happens when $c = -2$.

```
c = sympy.symbols('c')
Elim( Matrix([[1,2,4],[-1,-3,-2],[0,1,c]]) )
```

□

(L-4) Question 3(a) $\mathbf{E} = \left(\mathbf{I}_{\underset{[(-1)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}\right)\left(\mathbf{I}_{\underset{[\mathbf{2}\rightleftharpoons\mathbf{3}]}{\boldsymbol{\tau}}}\right) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \mathbf{I}_{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{2}] \\ [\mathbf{2}\rightleftharpoons\mathbf{3}]}}$

□

(L-4) Question 3(b) $\mathbf{N} = \left(\mathbf{I}_{\underset{[\mathbf{2}\rightleftharpoons\mathbf{3}]}{\boldsymbol{\tau}}}\right)\left(\mathbf{I}_{\underset{[(-1)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \mathbf{I}_{\substack{\boldsymbol{\tau} \\ [\mathbf{2}\rightleftharpoons\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{3}]}}$

the E and N are the same because the transformations are equivalent: in the first case we subtract column 1 from column 2, and then we put the result in the third column; in the second case we first exchange column 2 and 3, and then we operate on the third column.

□

(L-4) Question 4.

$$\begin{bmatrix} 2&2&0\\1&4&9\\1&3&9 \end{bmatrix} \xrightarrow[{[(-1)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \begin{bmatrix} 2&0&0\\1&3&9\\1&2&9 \end{bmatrix} \xrightarrow[{[(-3)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \begin{bmatrix} 2&0&0\\1&3&0\\1&2&3 \end{bmatrix}$$

Therefore

$$\mathsf{I}_{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{1}+\mathbf{2}]\\ [(-3)\mathbf{2}+\mathbf{3}]}} = \begin{bmatrix} 1&-1&0\\0&1&0\\0&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0\\0&1&-3\\0&0&1 \end{bmatrix} = \begin{bmatrix} 1&-1&3\\0&1&-3\\0&0&1 \end{bmatrix} = \mathsf{E}.$$

```
A      = Matrix([[2,2,0],[1,4,9],[1,3,9]])
TrfCol = Elim(A).TrC
display(TrfCol)
I(3) & TrfCol
```

□

(L-4) Question 5. $\begin{bmatrix} 1&0\\0&1 \end{bmatrix} \xrightarrow[{[(1)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \begin{bmatrix} 1&1\\0&1 \end{bmatrix} \xrightarrow[{[(-1)\mathbf{2}+\mathbf{1}]}]{\boldsymbol{\tau}} \begin{bmatrix} 0&1\\-1&1 \end{bmatrix} \xrightarrow[{[(1)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \begin{bmatrix} 0&1\\-1&0 \end{bmatrix} \xrightarrow[{[(-1)\mathbf{1}]}]{\boldsymbol{\tau}} \begin{bmatrix} 0&1\\1&0 \end{bmatrix}$

```
pasos = [ T((1,1,2)), T((-1,2,1)), T((1,1,2)),  T((-1,1))]
Math(rprElim(I(2),[[],pasos]))
# I(2) & T((1,1,2)) & T((-1,2,1)) & T((1,1,2)) & T((-1,1))
```

□

(L-4) Question 6(a)

$$\mathsf{I}_{\substack{\boldsymbol{\tau}\\ [(-5)\mathbf{1}+\mathbf{2}]}} = \begin{bmatrix} 1&-5&0\\0&1&0\\0&0&1 \end{bmatrix}$$

```
display( T((-5,1,2)) )
I(3) & T((-5,1,2))
```

□

(L-4) Question 6(b)

$$\mathsf{I}_{\substack{\boldsymbol{\tau}\\ [(-7)\mathbf{2}+\mathbf{3}]}} = \begin{bmatrix} 1&0&0\\0&1&-7\\0&0&1 \end{bmatrix}$$

```
display( T((-7,2,3)) )
I(3) & T((-7,2,3))
```

□

(L-4) Question 6(c)

$$\mathsf{I}_{\substack{\boldsymbol{\tau}\\ [\mathbf{1}\rightleftharpoons\mathbf{2}]}} = \begin{bmatrix} 0&1&0\\1&0&0\\0&0&1 \end{bmatrix}; \qquad \mathsf{I}_{\substack{\boldsymbol{\tau}\\ [\mathbf{2}\rightleftharpoons\mathbf{3}]}} = \begin{bmatrix} 1&0&0\\0&0&1\\0&1&0 \end{bmatrix}$$

The matrix that does all the exchanges at once is

$$\mathsf{I}_{\substack{\boldsymbol{\tau}\quad\boldsymbol{\tau}\\ [\mathbf{1}\rightleftharpoons\mathbf{2}][\mathbf{2}\rightleftharpoons\mathbf{3}]}} = \begin{bmatrix} 0&1&0\\1&0&0\\0&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0\\0&0&1\\0&1&0 \end{bmatrix} = \begin{bmatrix} 0&0&1\\1&0&0\\0&1&0 \end{bmatrix} = \mathsf{I}_{\substack{\boldsymbol{\tau}\\ [\mathfrak{S}]}}$$

El producto de matrices intercambio es siempre una matriz cuyas columnas son como las de la matriz identidad, pero en general reordenadas en una disposición distinta; a dichas matrices la llamamos *matrices permutación* y las denotamos con: $\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}}$.

```
display( T([ {1,2}, {2,3} ]) )
I(3) & T([ {1,2}, {2,3} ])
```

□

(L-4) Question 7(a)

$$\mathbf{A}_{\underset{[(-5)\mathbf{1}+\mathbf{2}]}{\tau}\underset{[(-7)\mathbf{2}+\mathbf{3}]}{\tau}} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & -5 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & -7\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & -5 & 35\\0 & 1 & -7\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -5 & 35\end{bmatrix}.$$

```
TrfC = T([ (-5,1,2), (-7,2,3) ])
display( I(3) & TrfC )
Matrix([ [1,0,0]  ]) & TrfC
```

□

(L-4) Question 7(b)

$$\mathbf{A}_{\underset{[(-7)\mathbf{1}+\mathbf{2}]}{\tau}\underset{[(-5)\mathbf{2}+\mathbf{3}]}{\tau}} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & -7\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & -5 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & -5 & 0\\0 & 1 & -7\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -5 & 0\end{bmatrix}.$$

```
TrfC = T([ (-7,2,3), (-5,1,2) ])
display( I(3) & TrfC )
Matrix([ [1,0,0]  ]) & TrfC
```

□

(L-4) Question 7(c) Then column 3 feels no effect from column 1.

□

(L-4) Question 8. On the one hand $(1, \quad 0,)\,\mathbf{M} = (0, \quad 1,) \quad \Rightarrow {}_{1|}\mathbf{M} = (0, \quad 1,)$; on the other $(0, \quad 1,)\,\mathbf{M} = (1, \quad 0,) \quad \Rightarrow {}_{2|}\mathbf{M} = (1, \quad 0,)$. Hence, the matrix is $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$.

□

(L-4) Question 9. The product $\mathbf{A}\Big(\mathbf{I}_{\underset{[i\rightleftharpoons j]}{\tau}}\Big)$ exchange the columns of $\mathbf{A}$, but $\Big({}_{\underset{[i\rightleftharpoons j]}{\tau}}\mathbf{I}\Big)\mathbf{A}$ exchange the rows. For example:

$$\Big({}_{\underset{[\mathbf{1}\rightleftharpoons\mathbf{2}]}{\tau}}\mathbf{I}\Big)\mathbf{A} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}c & d\\a & b\end{bmatrix}.$$

$$\mathbf{A}\Big(\mathbf{I}_{\underset{[\mathbf{1}\rightleftharpoons\mathbf{2}]}{\tau}}\Big) = \begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}b & a\\d & c\end{bmatrix}.$$

Note that $\Big(\mathbf{I}_{\underset{[i\rightleftharpoons j]}{\tau}}\Big)$ is symmetric, so that $\Big(\mathbf{I}_{\underset{[i\rightleftharpoons j]}{\tau}}\Big) = \Big({}_{\underset{[i\rightleftharpoons j]}{\tau}}\mathbf{I}\Big)$.

```
a, b, c, d  = sympy.symbols('a b c d')
A = Matrix([ [a,b], [c,d] ])
display( (T({1,2}) & I(2)) * A )
display( A * (I(2) & T({1,2})) )
(I(2) & T({1,2})) == (T({1,2}) & I(2))
```

□

(L-4) Question 10. For example, the first column is $\begin{pmatrix}a, & a, & a,\end{pmatrix}$, the second is $\begin{pmatrix}b, & b, & b,\end{pmatrix}$ and the third is $\begin{pmatrix}c, & c, & c,\end{pmatrix}$; and where $a \neq 0$, then

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix}a & b & c\\ a & b & c\\ a & b & c\end{bmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = x_1\begin{pmatrix}a\\ a\\ a\end{pmatrix} + x_2\begin{pmatrix}b\\ b\\ b\end{pmatrix} + x_3\begin{pmatrix}c\\ c\\ c\end{pmatrix}$$
$$= ax_1\begin{pmatrix}1\\ 1\\ 1\end{pmatrix} + bx_2\begin{pmatrix}1\\ 1\\ 1\end{pmatrix} + cx_3\begin{pmatrix}1\\ 1\\ 1\end{pmatrix} = (ax_1 + bx_2 + cx_3)\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$$

By Gaussian elimination we get

$$\begin{bmatrix}a & b & c\\ a & b & c\\ a & b & c\end{bmatrix} \xrightarrow{\underset{[(-\frac{b}{a})\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}} \begin{bmatrix}a & 0 & c\\ a & 0 & c\\ a & 0 & c\end{bmatrix} \xrightarrow{\underset{[(-\frac{c}{a})\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}} \begin{bmatrix}a & 0 & 0\\ a & 0 & 0\\ a & 0 & 0\end{bmatrix};$$

only one pivot.

□

Exercise 1. Una permutación $\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}$ es el producto de k matrices intercambio, $\mathbf{I}_{\underset{[\cdot\rightleftharpoons\cdot]}{\boldsymbol{\tau}}}$, es decir, una sucesión de k intercambios entre las columnas de la matriz identidad, $\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}} = \Big(\mathbf{I}_{\underset{[h\rightleftharpoons i]}{\boldsymbol{\tau}}}\Big)\Big(\mathbf{I}_{\underset{[j\rightleftharpoons k]}{\boldsymbol{\tau}}}\Big)\cdots\Big(\mathbf{I}_{\underset{[p\rightleftharpoons q]}{\boldsymbol{\tau}}}\Big)$, por lo que su transpuesta es $\Big(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}\Big)^{\intercal} = \Big({}_{\underset{[p\rightleftharpoons q]}{\boldsymbol{\tau}}}\mathbf{I}\Big)\cdots\Big({}_{\underset{[j\rightleftharpoons k]}{\boldsymbol{\tau}}}\mathbf{I}\Big)\Big({}_{\underset{[h\rightleftharpoons i]}{\boldsymbol{\tau}}}\mathbf{I}\Big)$; donde cada matriz intercambio es simétrica, $\Big({}_{\underset{[i\rightleftharpoons j]}{\boldsymbol{\tau}}}\mathbf{I}\Big) = \Big(\mathbf{I}_{\underset{[i\rightleftharpoons j]}{\boldsymbol{\tau}}}\Big)$ y su cuadrado es la matriz identidad (si se intercambian las mismas columnas dos veces, se vuelve al punto de partida), $\Big(\mathbf{I}_{\underset{[i\rightleftharpoons j]}{\boldsymbol{\tau}}}\Big)\Big(\mathbf{I}_{\underset{[i\rightleftharpoons j]}{\boldsymbol{\tau}}}\Big) = \mathbf{I}$. Así

$$\Big(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}\Big)^{\intercal}\Big(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}\Big) = \Big({}_{\underset{[p\rightleftharpoons q]}{\boldsymbol{\tau}}}\mathbf{I}\Big)\cdots\overbrace{\Big({}_{\underset{[j\rightleftharpoons k]}{\boldsymbol{\tau}}}\mathbf{I}\Big)\underbrace{\Big({}_{\underset{[h\rightleftharpoons i]}{\boldsymbol{\tau}}}\mathbf{I}\Big)\Big(\mathbf{I}_{\underset{[h\rightleftharpoons i]}{\boldsymbol{\tau}}}\Big)}_{\mathbf{I}}\Big(\mathbf{I}_{\underset{[j\rightleftharpoons k]}{\boldsymbol{\tau}}}\Big)}^{\mathbf{I}}\cdots\Big(\mathbf{I}_{\underset{[p\rightleftharpoons q]}{\boldsymbol{\tau}}}\Big) = \mathbf{I}.$$

□

(L-5) Question 1(a) $\left[\begin{array}{c}\mathbf{A}_1\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{ccc}1 & 0 & 0\\ 1 & 1 & 1\\ 0 & 0 & 1\\ \hline 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right] \xrightarrow{\underset{[(-1)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}} \left[\begin{array}{ccc}1 & 0 & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\\ \hline 1 & 0 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{array}\right] \xrightarrow{\underset{[(-1)\mathbf{2}+\mathbf{1}]}{\boldsymbol{\tau}}} \left[\begin{array}{ccc}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ \hline 1 & 0 & 0\\ -1 & 1 & -1\\ 0 & 0 & 1\end{array}\right].$

Therefore $\mathbf{A}_1^{-1} = \begin{bmatrix}1 & 0 & 0\\ -1 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}$

```
A1 = Matrix([[1,0,0],[1,1,1],[0,0,1]])
InvMat(A1, 1)
```

□

(L-5) Question 1(b)

$$
\begin{bmatrix}\mathbf{A}_2\\ \hline \mathbf{I}\end{bmatrix} = \left[\begin{array}{rrr} 2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\\ \hline 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right] \xrightarrow{\substack{\tau\\ [(2)\mathbf{2}]\\ [(1)\mathbf{2+1}]}} \left[\begin{array}{rrr} 2 & 0 & 0\\ -1 & 3 & -1\\ 0 & -2 & 2\\ \hline 1 & 1 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{array}\right] \xrightarrow{\substack{\tau\\ [(3)\mathbf{3}]\\ [(1)\mathbf{2+3}]}} \left[\begin{array}{rrr} 2 & 0 & 0\\ -1 & 3 & 0\\ 0 & -2 & 4\\ \hline 1 & 1 & 1\\ 0 & 2 & 2\\ 0 & 0 & 3\end{array}\right] \xrightarrow{\substack{\tau\\ [(2)\mathbf{2}]\\ [(1)\mathbf{3+2}]}}
$$

$$
\left[\begin{array}{rrr} 2 & 0 & 0\\ -1 & 6 & 0\\ 0 & 0 & 4\\ \hline 1 & 3 & 1\\ 0 & 6 & 2\\ 0 & 3 & 3\end{array}\right] \xrightarrow{\substack{\tau\\ [(6)\mathbf{1}]\\ [(1)\mathbf{2+1}]}} \left[\begin{array}{rrr} 12 & 0 & 0\\ 0 & 6 & 0\\ 0 & 0 & 4\\ \hline 9 & 3 & 1\\ 6 & 6 & 2\\ 3 & 3 & 3\end{array}\right] \xrightarrow{\substack{\tau\\ [(\frac{1}{3})\mathbf{1}]\\ [(\frac{2}{3})\mathbf{2}]}} \left[\begin{array}{rrr} 4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\\ \hline 3 & 2 & 1\\ 2 & 4 & 2\\ 1 & 2 & 3\end{array}\right] \xrightarrow{\substack{\tau\\ [(\frac{1}{4})\mathbf{1}]\\ [(\frac{1}{4})\mathbf{2}]\\ [(\frac{1}{4})\mathbf{3}]}} \frac{1}{4}\left[\begin{array}{ccc} & 4\mathbf{I} & \\ \hline 3 & 2 & 1\\ 2 & 4 & 2\\ 1 & 2 & 3\end{array}\right]
$$

Therefore $\mathbf{A}_2^{-1} = \frac{1}{4}\begin{bmatrix}3 & 2 & 1\\ 2 & 4 & 2\\ 1 & 2 & 3\end{bmatrix}$

□

(L-5) Question 1(c) $\begin{bmatrix}\mathbf{A}_3\\ \hline \mathbf{I}\end{bmatrix} = \left[\begin{array}{rrr} 0 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 1\\ \hline 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right] \xrightarrow{\substack{\tau\\ [\mathbf{1}\rightleftharpoons\mathbf{3}]}} \left[\begin{array}{rrr} 1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\\ \hline 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{array}\right] \xrightarrow{\substack{\tau\\ [(-1)\mathbf{2+1}]\\ [(-1)\mathbf{3+2}]}} \left[\begin{array}{rrr} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ \hline 0 & -1 & 1\\ -1 & 1 & 0\\ 1 & 0 & 0\end{array}\right].$

Therefore $\mathbf{A}_3^{-1} = \begin{bmatrix}0 & -1 & 1\\ -1 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}$

□

(L-5) Question 2(a)

$$
\begin{aligned}
\mathbf{AB} &= \mathbf{AC}\\
\mathbf{A}^{-1}\mathbf{AB} &= \mathbf{A}^{-1}\mathbf{AC}\\
\mathbf{IB} &= \mathbf{IC}\\
\mathbf{B} &= \mathbf{C}.
\end{aligned}
$$

□

(L-5) Question 2(b)

$$
\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\ 5 & 5\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\ 2 & 2\end{bmatrix}
$$

□

(L-5) Question 3.

$$
\left[\begin{array}{cc} a & b\\ c & d\\ \hline 1 & 0\\ 0 & 1\end{array}\right] \xrightarrow{\substack{\tau\\ [(\frac{-b}{a})\mathbf{1+2}]}} \left[\begin{array}{cc} a & 0\\ c & d-\frac{bc}{a}\\ \hline 1 & -b/a\\ 0 & 1\end{array}\right] = \left[\begin{array}{cc} a & 0\\ c & \frac{ad-bc}{a}\\ \hline 1 & -b/a\\ 0 & 1\end{array}\right] \xrightarrow{\substack{\tau\\ [(\frac{a}{ad-bc})\mathbf{2}]}} \left[\begin{array}{cc} a & 0\\ c & 1\\ \hline 1 & \frac{-b}{ad-bc}\\ 0 & \frac{a}{ad-bc}\end{array}\right]
$$

$$
\xrightarrow{\substack{\tau\\ [(-c)\mathbf{2+1}]}} \left[\begin{array}{cc} a & 0\\ 0 & 1\\ \hline 1+\frac{bc}{ad-bc} & \frac{-b}{ad-bc}\\ \frac{-ac}{ad-bc} & \frac{a}{ad-bc}\end{array}\right] = \left[\begin{array}{cc} a & 0\\ 0 & 1\\ \hline \frac{ad}{ad-bc} & \frac{-b}{ad-bc}\\ \frac{-ac}{ad-bc} & \frac{a}{ad-bc}\end{array}\right] \xrightarrow{\substack{\tau\\ [(\frac{1}{a})\mathbf{2}]}} \left[\begin{array}{cc} 1 & 0\\ 0 & 1\\ \hline \frac{d}{ad-bc} & \frac{-b}{ad-bc}\\ \frac{-c}{ad-bc} & \frac{a}{ad-bc}\end{array}\right]
$$

so

$$\mathbf{M}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The matrix is invertible only when $ad \neq bc$.

```
a, b, c, d = sympy.symbols('a b c d')
A = Matrix([ [a,b], [c,d] ])
InvMat(A, 1)
```

$\square$

(L-5) Question 4.

$$\left[\begin{array}{rrr} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 6 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [\mathbf{1}\rightleftharpoons\mathbf{2}]}} \left[\begin{array}{rrr} 1 & 3 & 2 \\ 0 & -1 & 1 \\ 1 & 0 & 6 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-3)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 4 \\ \hline 0 & 1 & 0 \\ 1 & -3 & -2 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-1)\mathbf{2}] \\ [(-1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \\ \hline 0 & -1 & 1 \\ 1 & 3 & -5 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-3)\mathbf{3}+\mathbf{2}] \\ [(-1)\mathbf{3}+\mathbf{1}]}} \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline -1 & -4 & 1 \\ 6 & 18 & -5 \\ -1 & -3 & 1 \end{array}\right] = \left[\begin{array}{c} \mathbf{I} \\ \hline \mathbf{A}^{-1} \end{array}\right]$$

$$\left[\begin{array}{c} \mathbf{B} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{rrr} 1 & 2 & 1 \\ -1 & 4 & -2 \\ 1 & 3 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{rrr} 1 & 0 & 0 \\ -1 & 6 & -1 \\ 1 & 1 & 0 \\ \hline 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [\mathbf{2}\rightleftharpoons\mathbf{3}] \\ [(-1)\mathbf{2}]}} \left[\begin{array}{rrr} 1 & 0 & 0 \\ -1 & 1 & 6 \\ 1 & 0 & 1 \\ \hline 1 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{\substack{\tau \\ [(1)\mathbf{2}+\mathbf{1}] \\ [(-6)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ \hline 2 & 1 & -8 \\ 0 & 0 & 1 \\ -1 & -1 & 6 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-1)\mathbf{3}+\mathbf{1}]}} \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 10 & 1 & -8 \\ -1 & 0 & 1 \\ -7 & -1 & 6 \end{array}\right] = \left[\begin{array}{c} \mathbf{I} \\ \hline \mathbf{B}^{-1} \end{array}\right]$$

$\square$

(L-5) Question 5(a)

$$\begin{bmatrix} a_1 & b_1 & a_1+b_1 \\ a_2 & b_2 & a_2+b_2 \\ a_3 & b_3 & a_3+b_3 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\square$

(L-5) Question 5(b) When we sustract the first and second columns from the third we get a zero column (no pivot in that column)

$$\begin{bmatrix} a_1 & b_1 & a_1+b_1 \\ a_2 & b_2 & a_2+b_2 \\ a_3 & b_3 & a_3+b_3 \end{bmatrix} \xrightarrow{\substack{\tau \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{2}+\mathbf{3}]}} \begin{bmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{bmatrix}$$

$\square$

(L-5) Question 6(a)

$$\left[\begin{array}{c} \mathbf{A}_1 \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [\mathbf{1}\rightleftharpoons\mathbf{4}] \\ [\mathbf{2}\rightleftharpoons\mathbf{3}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\substack{\tau \\ [(1/2)\mathbf{2}] \\ [(1/3)\mathbf{3}] \\ [(1/4)\mathbf{4}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1/4 \\ 0 & 0 & 1/3 & 0 \\ 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$$

□

(L-5) Question 6(b)

$$\begin{bmatrix}\mathbf{A}_2\\ \hline \mathbf{I}\end{bmatrix}=\left[\begin{array}{cccc}1&0&0&0\\-1/2&1&0&0\\0&-2/3&1&0\\0&0&-3/4&1\\ \hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[{[(1/2)\mathbf{2}+\mathbf{1}]}]{\substack{\boldsymbol{\tau}\\{[(3/4)\mathbf{4}+\mathbf{3}]}\\{[(2/3)\mathbf{3}+\mathbf{2}]}}}\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\ \hline 1&0&0&0\\1/2&1&0&0\\1/3&2/3&1&0\\1/4&1/2&3/4&1\end{array}\right]=\begin{bmatrix}\mathbf{I}\\ \hline \mathbf{A}_2^{-1}\end{bmatrix}$$

□

(L-5) Question 6(c) We just need to repeat, for each block, the steps of Exercise 3:

$$\left[\begin{array}{cccc}a&b&0&0\\c&d&0&0\\0&0&a&b\\0&0&c&d\\ \hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow{\substack{\boldsymbol{\tau}\\{[(-b/a)\mathbf{1}+\mathbf{2}]}\\{[(\frac{a}{ad-bc})\mathbf{2}]}\\{[(-c)\mathbf{2}+\mathbf{1}]}\\{[(\frac{1}{a})\mathbf{2}]}}}\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\0&0&a&b\\0&0&c&d\\ \hline \frac{d}{ad-bc}&\frac{-b}{ad-bc}&0&0\\\frac{-c}{ad-bc}&\frac{a}{ad-bc}&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow{\substack{\boldsymbol{\tau}\\{[(-b/a)\mathbf{3}+\mathbf{4}]}\\{[(\frac{a}{ad-bc})\mathbf{4}]}\\{[(-c)\mathbf{4}+\mathbf{3}]}\\{[(\frac{1}{a})\mathbf{4}]}}}\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\ \hline \frac{d}{ad-bc}&\frac{-b}{ad-bc}&0&0\\\frac{-c}{ad-bc}&\frac{a}{ad-bc}&0&0\\0&0&\frac{d}{ad-bc}&\frac{-b}{ad-bc}\\0&0&\frac{-c}{ad-bc}&\frac{a}{ad-bc}\end{array}\right]=\begin{bmatrix}\mathbf{I}\\ \hline \mathbf{A}_3^{-1}\end{bmatrix}$$

□

(L-5) Question 7.

$$\left[\begin{array}{ccc}a&b&b\\a&a&b\\a&a&a\\ \hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow{\substack{\boldsymbol{\tau}\\{[(\frac{-b}{a})\mathbf{1}+\mathbf{2}]}\\{[(\frac{-b}{a})\mathbf{1}+\mathbf{3}]}}}\left[\begin{array}{ccc}a&0&0\\a&a-b&0\\a&a-b&a-b\\ \hline 1&\frac{-b}{a}&\frac{-b}{a}\\0&1&0\\0&0&1\end{array}\right]\xrightarrow{\substack{\boldsymbol{\tau}\\{[(\frac{1}{a})\mathbf{1}]}\\{[(\frac{1}{a-b})\mathbf{2}]}\\{[(\frac{1}{a-b})\mathbf{3}]}}}\left[\begin{array}{ccc}1&0&0\\1&1&0\\1&1&1\\ \hline \frac{1}{a}&\frac{-b/a}{a-b}&\frac{-b/a}{a-b}\\0&\frac{1}{a-b}&0\\0&0&\frac{1}{a-b}\end{array}\right]\xrightarrow{\substack{\boldsymbol{\tau}\\{[(-1)\mathbf{2}+\mathbf{1}]}\\{[(-1)\mathbf{3}+\mathbf{2}]}}}\left[\begin{array}{ccc}1&0&0\\0&1&0\\0&0&1\\ \hline \frac{1}{a-b}&0&\frac{-b/a}{a-b}\\\frac{-1}{a-b}&\frac{1}{a-b}&0\\0&\frac{-1}{a-b}&\frac{1}{a-b}\end{array}\right]$$

Hence $\quad \mathbf{A}^{-1}=\frac{1}{a-b}\begin{bmatrix}1&0&-b/a\\-1&1&0\\0&-1&1\end{bmatrix}$

There is no inverse when $a=b$, since the denominator $(a-b)$ is zero (in that case all rows in $\mathbf{A}$ are the same; and therefore there is only one pivot).

There is not inverse, either, when $a=0$; since the third element on the first row is not defined (in that case $\mathbf{A}$ has a zero column and a zero row).

□

(L-5) Question 8. Since $\mathbf{E}$ is an elementary matrix

$$\mathbf{E}^2=\begin{bmatrix}1&0\\12&1\end{bmatrix};\qquad \mathbf{E}^8=\begin{bmatrix}1&0\\48&1\end{bmatrix};\qquad \mathbf{E}^{-1}=\begin{bmatrix}1&0\\-6&1\end{bmatrix};\qquad \mathbf{E}^n=\begin{bmatrix}1&0\\6n&1\end{bmatrix}$$

□

(L-5) Question 9. $\begin{bmatrix}0&0&0&1\\0&1&0&0\\1&0&0&0\\0&0&1&0\end{bmatrix}$. One is the transpose matrix of the other.

□

(L-5) Question 10(a) Since $\mathsf{A}\left(\mathbf{I}_{\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}}\right) = \mathbf{I}$, then $\left(\mathbf{I}_{\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}}\right) = \mathsf{A}^{-1}$:

$$\mathbf{I}_{\substack{\boldsymbol{\tau}\\ [(-4)\mathbf{1}+\mathbf{2}]\\ [(-3)\mathbf{1}+\mathbf{3}]\\ [(-1)\mathbf{3}+\mathbf{2}]}} = \begin{bmatrix}1&-4&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-3\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix} = \begin{bmatrix}1&-1&-3\\0&1&0\\0&-1&1\end{bmatrix} = \mathsf{A}^{-1}$$

```
Tr = T((-4,1,2)) & T((-3,1,3)) & T((-1,3,2))
display(Tr)
Ainv = I(3) & Tr
Ainv
```

□

(L-5) Question 10(b) Since $\left(\mathbf{I}_{\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}}\right)^{-1} = \mathsf{A}$ then $\mathsf{A} = \left(\mathbf{I}_{\underset{[(1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\underset{[(4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}\right)$.

$$\mathbf{I}_{\substack{\boldsymbol{\tau}\\ [(1)\mathbf{3}+\mathbf{2}]\\ [(3)\mathbf{1}+\mathbf{3}]\\ [(4)\mathbf{1}+\mathbf{2}]}} = \begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&3\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&4&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&4&3\\0&1&0\\0&1&1\end{bmatrix} = \mathsf{A}$$

$$\text{Check:}\quad \mathsf{A}^{-1}\mathsf{A} = \begin{bmatrix}1&-1&-3\\0&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}1&4&3\\0&1&0\\0&1&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

```
display(Tr**-1)
A    = I(3) & Tr**-1
Sistema([A, A*Ainv] )
```

□

(L-5) Question 11(a) Remember that ${}_{\boldsymbol{\tau}}\mathbf{I}$ is the transpose of $\mathbf{I}_{\boldsymbol{\tau}}$.

Since $\left({}_{\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}\mathbf{I}\right)\mathsf{A} = \mathbf{I}$, then $\left({}_{\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}\mathbf{I}\right) = \mathsf{A}^{-1}$:

$${}_{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{3}+\mathbf{2}]\\ [(-3)\mathbf{1}+\mathbf{3}]\\ [(-4)\mathbf{1}+\mathbf{2}]}}\mathbf{I} = \begin{bmatrix}1&0&0\\0&1&-1\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-3&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-4&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\-1&1&-1\\-3&0&1\end{bmatrix} = \mathsf{A}^{-1}$$

```
Tr   = T((-4,1,2)) & T((-3,1,3)) & T((-1,3,2))
display(Tr)
display( Sistema([t & I(3) for t in ~Tr]) )
Ainv = ~Tr & I(3)
Ainv
```

□

(L-5) Question 11(b) Since $\left({}_{\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}\mathbf{I}\right)^{-1} = \mathsf{A}$ then $\mathsf{A} = \left({}_{\underset{[(4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\underset{[(1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}}\mathbf{I}\right)$.

$${}_{\substack{\boldsymbol{\tau}\\ [(4)\mathbf{1}+\mathbf{2}]\\ [(3)\mathbf{1}+\mathbf{3}]\\ [(1)\mathbf{3}+\mathbf{2}]}}\mathbf{I} = \begin{bmatrix}1&0&0\\4&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\3&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\4&1&1\\3&0&1\end{bmatrix} = \mathsf{A}$$

$$\text{Check:}\quad \mathsf{A}^{-1}\mathsf{A} = \begin{bmatrix}1&0&0\\-1&1&-1\\-3&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\4&1&1\\3&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

```
display(~Tr**-1)
display( Sistema([t & I(3) for t in ~Tr**-1]) )
A    = ~Tr**-1 & I(3)
Sistema([A, Ainv*A] )
```

□

(L-5) Question 12(a) The first one is an elementary matrix, its inverse is $\begin{bmatrix}1&0&0\\0&1&-c\\0&0&1\end{bmatrix}$; the second one is a permutation matrix, its inverse is its transpose.

□

(L-5) Question 12(b)

$$\begin{bmatrix}\mathbf{A}\\\hline\mathbf{I}\end{bmatrix}=\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\0&0&1&0\\a&b&c&d\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[{[(\frac{1}{d})\mathbf{4}]}]{\boldsymbol{\tau}}\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\0&0&1&0\\a&b&c&1\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&\frac{1}{d}\end{array}\right]\xrightarrow[\substack{[(-c)\mathbf{4}+\mathbf{3}]\\[(-b)\mathbf{4}+\mathbf{2}]\\[(-a)\mathbf{4}+\mathbf{1}]}]{\boldsymbol{\tau}}\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\-\frac{a}{d}&-\frac{b}{d}&-\frac{c}{d}&\frac{1}{d}\end{array}\right]=\begin{bmatrix}\mathbf{I}\\\hline\mathbf{A}^{-1}\end{bmatrix}$$

```
a, b, c, d = sympy.symbols('a b c d')
C = Matrix([ [1,0,0,0], [0,1,0,0], [0,0,1,0], [a,b,c,d] ])
C.apila(I(4),1) & T((fracc(1,d),4)) & T([(-c,4,3),(-b,4,2),(-a,4,1)])
```

□

(L-5) Question 13(a) True, since

$$\mathbf{AB}=\mathbf{I}\quad\Rightarrow\quad\mathbf{B}=\mathbf{A}^{-1}.$$

and

$$\mathbf{CA}=\mathbf{I}\quad\Rightarrow\quad\mathbf{C}=\mathbf{A}^{-1}.$$

Therefore $\mathbf{B}$ and $\mathbf{C}$ are the same matrix $\mathbf{A}^{-1}$.

□

(L-5) Question 13(b) False:

$$(\mathbf{AB})^2=(\mathbf{AB})\,(\mathbf{AB})=\mathbf{ABAB}$$

is in general different from

$$\mathbf{A}^2\mathbf{B}^2=\mathbf{AABB}.$$

Example:

$$\mathbf{A}=\begin{bmatrix}1&1\\0&1\end{bmatrix};\quad\mathbf{B}=\begin{bmatrix}1&1\\-1&-1\end{bmatrix}.\qquad\mathbf{ABAB}=\begin{bmatrix}0&0\\1&1\end{bmatrix};\quad\mathbf{AABB}=\begin{bmatrix}0&0\\0&0\end{bmatrix}.$$

□

(L-5) Question 14(a) Lets check if the rank is 4 (using column reduction):

$$\begin{bmatrix}\mathbf{A}\\\hline\mathbf{I}\end{bmatrix}=\left[\begin{array}{cccc}0&1&0&2\\1&a&0&2a\\a&0&1&0\\1&0&a&1\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[{[(-2)\mathbf{2}+\mathbf{4}]}]{\boldsymbol{\tau}}\left[\begin{array}{cccc}0&1&0&0\\1&a&0&0\\a&0&1&0\\1&0&a&1\\\hline 1&0&0&0\\0&1&0&-2\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[{[\mathbf{1}\rightleftharpoons\mathbf{2}]}]{\boldsymbol{\tau}}\left[\begin{array}{cccc}1&0&0&0\\a&1&0&0\\0&a&1&0\\0&1&a&1\\\hline 0&1&0&0\\1&0&0&-2\\0&0&1&0\\0&0&0&1\end{array}\right]=\begin{bmatrix}\mathbf{L}\\\hline\mathbf{E}\end{bmatrix}$$

This matrix has rank 4 for any value a; therefore it is invertible.

□

(L-5) Question 14(b)

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] \to \left[\frac{\mathbf{L}}{\mathbf{E}}\right] = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{4}+\mathbf{2}]}]{\tau} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right] = \left[\frac{\mathbf{I}}{\mathbf{A}^{-1}}\right]$$

```
a    = sympy.symbols('a')
A    = Matrix([ [0,1,0,2],[1,a,0,2*a],[a,0,1,0],[1,0,a,1] ])
Ainv = InvMat(A, 1)
Ainv.subs([(a,0)])
```

□

(L-5) Question 15. $\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}$

```
A = Matrix([ [1,1,0,1], [0,1,0,-1], [0,0,1,0], [0,0,0,-2] ])
InvMat(A, 1)
```

□

(L-5) Question 16.

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{ccc} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{3}]}]{\tau} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \\ \hline 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{\substack{[(-1)\mathbf{2}+\mathbf{1}] \\ [(1)\mathbf{2}+\mathbf{3}]}}]{\tau} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 2 \\ \hline 1 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{\substack{[(\frac{1}{2})\mathbf{3}] \\ [(-1)\mathbf{3}+\mathbf{2}] \\ [(1)\mathbf{3}+\mathbf{1}]}}]{\tau} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{array}\right] = \left[\frac{\mathbf{I}}{\mathbf{A}^{-1}}\right]$$

$$\left[\frac{\mathbf{B}}{\mathbf{I}}\right] = \left[\begin{array}{ccc} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{\substack{[(-1)\mathbf{1}+\mathbf{2}] \\ [(2)\mathbf{1}+\mathbf{3}]}}]{\tau} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & -3 & 3 \\ -2 & 3 & -3 \\ \hline 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{2}+\mathbf{3}]}]{\tau} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & -3 & 0 \\ -2 & 3 & 0 \\ \hline 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right] = \left[\frac{\mathbf{L}}{\mathbf{E}}\right]$$

This matrix is singular (no inverse).

□

(L-5) Question 17. Since any power of a permutation matrix is a permutation matrix, and there is a finite number of permutation matrices; the sequence $(\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}})$, $(\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}})^2$, $(\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}})^3$, $\ldots$, $(\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}})^r, \ldots$ must repeat itself. Therefore, for some m and n, $(\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}})^m = (\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}})^n$. And, since permutation matrices are invertible, there exist $(\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}})^{-n}$, such as $(\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}})^{-n}(\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}})^n = \mathbf{I}$. Then:

$$\begin{aligned} (\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}})^m &= (\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}})^n \\ (\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}})^{-n}(\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}})^m &= (\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}})^{-n}(\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}})^n \\ (\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}})^{m-n} &= \mathbf{I}. \end{aligned}$$

In words, the $m - n$ power of $\mathsf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}$ is the identity matrix.

$\square$