

Mathematics II

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L-11

L-12

1 Highlights of Lesson 11

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- Orthogonal vectors and subspaces
- Nullspace \perp row space
 $\mathcal{N}(\mathbf{A}) \perp \mathcal{C}(\mathbf{A}^T)$
- left nullspace \perp column space
 $\mathcal{N}(\mathbf{A}^T) \perp \mathcal{C}(\mathbf{A})$
- From parametric to Cartesian (or implicit) equations

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L-12

2 Some definitions

- Dot product

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^n a_i b_i$$

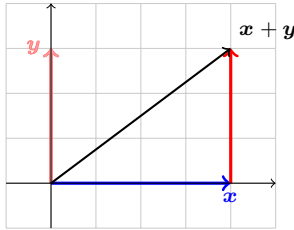
- Length of a vector $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$ $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$.

- Unit vector: $\|\mathbf{a}\| = 1$ $\frac{1}{\|\mathbf{x}\|} \cdot \mathbf{x}$

- Orthogonal (perpendicular) vectors: $\mathbf{x} \cdot \mathbf{y} = 0$.

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3 Orthogonal vectors



$$\mathbf{x} \cdot \mathbf{y} = 0 \iff \mathbf{x} \perp \mathbf{y}$$

Pythagoras Thm.: $\mathbf{x} \cdot \mathbf{y} = 0 \iff \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = \|\mathbf{x} + \mathbf{y}\|^2$

$$\mathbf{x} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} = (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}).$$

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4 Squared length of a vector

$$\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$$

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \rightarrow \|\mathbf{x}\|^2 = \quad ; \quad \mathbf{y} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \rightarrow \|\mathbf{y}\|^2 = \quad ;$$

Are these vectors orthogonal?

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}; \quad \|\mathbf{x} + \mathbf{y}\|^2 = \quad ;$$

(Pythagoras)

(Orthogonality)

$$\mathbf{x} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} = (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) \iff \mathbf{x} \cdot \mathbf{y} = 0.$$

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5 Orthogonal subspaces

When subspace \mathcal{S} is **orthogonal** to subspace \mathcal{T} :

Every vector in \mathcal{S} is orthogonal to every vector in \mathcal{T}

Are the plane of the *blackboard* and the floor orthogonal?

6 Nullspace orthogonal to row space

- $\mathcal{N}(\mathbf{A}) \perp$ rows of \mathbf{A}

$$\mathbf{A}\mathbf{x} = \mathbf{0} \implies \begin{pmatrix} (\mathbf{1}|\mathbf{A}) \cdot \mathbf{x} \\ \vdots \\ (\mathbf{m}|\mathbf{A}) \cdot \mathbf{x} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

- $\mathcal{N}(\mathbf{A}) \perp d\mathbf{A}, \quad \forall d \in \mathbb{R}^m$ (any **linear combination of the rows**)

$$\mathbf{x} \in \mathcal{N}(\mathbf{A}) \implies d\mathbf{A}\mathbf{x} = d \cdot \mathbf{0} = 0.$$

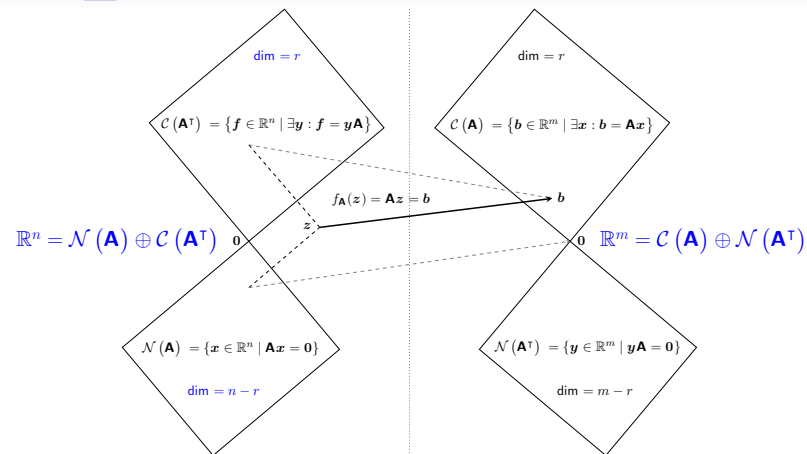
$$\text{nullspace} \perp \text{row space} \quad \mathcal{N}(\mathbf{A}) \perp \mathcal{C}(\mathbf{A}^\top)$$

Also: $\mathbf{x}\mathbf{A} = \mathbf{0} \implies \mathcal{N}(\mathbf{A}^\top) \perp \mathcal{C}(\mathbf{A})$

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7 The big picture: direct sum of orthogonal complements



$$C(A^T) \perp N(A) \\ f \cdot x = yAx = y \cdot 0$$

$$C(A) \perp N(A^T) \\ y \cdot b = yAx = 0 \cdot x$$

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8 Revisiting the Gaussian elimination

It's an algorithm to find a basis for the orthogonal complement

Give me some vectors (I write them as rows of **M**) and ...

$$\begin{bmatrix} \mathbf{M} \\ \mathbf{I} \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & -1 \\ 0 & -1 & 1 & 1 \\ 1 & -4 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\begin{matrix} \tau \\ [(3)1+2] \\ [(1)1+4] \\ [(1)2+3] \\ [(1)2+4] \end{matrix}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 3 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{L} \\ \mathbf{E} \end{bmatrix} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{D} & \mathbf{N} \end{bmatrix}$$

Basis for the span of the given (row) vectors: \mathcal{V}

Basis for orthogonal complement: \mathcal{V}^\perp

$$\mathbf{M}\mathbf{N} = \mathbf{0}$$

If you had given me \mathbf{N}_{11} and \mathbf{N}_{12} , after Gaussian elimination would have obtained a basis for...

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9 Cartesian (implicit) and parametric equations of lines and planes

Cartesian (implicit) equation $\{x \in \mathbb{R}^n \mid Ax = b\}$:

For example

$$\left\{ x \in \mathbb{R}^3 \mid \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \text{sol. set of } \begin{cases} x_1 - x_2 + x_3 = 1 \\ x_3 = 1 \end{cases}$$

Parametric equation:

for the above set

$$\left\{ x \in \mathbb{R}^3 \mid \exists p \in \mathbb{R}^1 : x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} p \right\}$$

In this case *dimension 1* A **line** (there is only one parameter *a*)
line

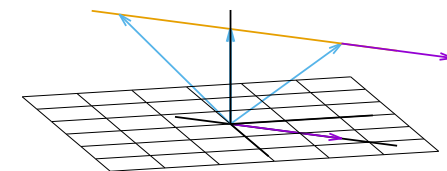
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or

$$\left\{ x \in \mathbb{R}^3 \mid \exists p \in \mathbb{R}^1 : x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} p \right\}$$

or

$$\left\{ x \in \mathbb{R}^3 \mid \exists p \in \mathbb{R}^1 : x = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} p \right\}$$



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10 Cartesian (implicit) and parametric equations of lines and planes

Cartesian (implicit) equation $\{x \in \mathbb{R}^n \mid Ax = b\}$:

For example

$$\{x \in \mathbb{R}^3 \mid [1 \ -1 \ 1] x = (1,)\} = \text{sol. set of } \{x_1 - x_2 + x_3 = 1\}$$

Parametric equation:

for the above set

$$\left\{ x \in \mathbb{R}^3 \mid \exists p \in \mathbb{R}^2 : x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} p \right\}$$

In this case *dimension 2*
plane

A plane (two parameters a and b)
plane

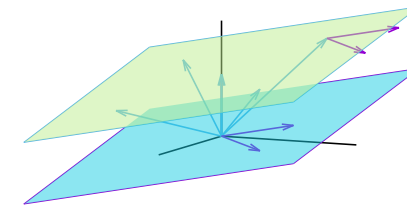
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or

$$\left\{ x \in \mathbb{R}^3 \mid \exists p \in \mathbb{R}^2 : x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} p \right\}$$

but also

$$\left\{ x \in \mathbb{R}^3 \mid \exists p \in \mathbb{R}^2 : x = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} p \right\}$$



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11 From parametric to Cartesian equations

$$\mathcal{C}(A^T) \perp \mathcal{N}(A)$$

Consider

$$C = \left\{ x \in \mathbb{R}^n \mid \exists p \in \mathbb{R}^k : x = s + [n_1; \dots; n_k] p \right\}.$$

If we find A such that $An_i = 0$ then if $x \in C$

$$Ax = As + \underbrace{A[n_1; \dots; n_k]}_0 p \Rightarrow Ax = b, \text{ where } b = As.$$

Therefore

$$C = \{x \in \mathbb{R}^n \mid Ax = b\}.$$

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12 From the set of solution to a linear system

Find the implicit equations of the plane P parallel to the span of $(1, 2, 0, -2)$ and $(0, 0, 1, 3)$, that goes through $s = (1, 3, 1, 1)$.

$$P = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \mid \exists a, b \in \mathbb{R} : \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix} + a \begin{pmatrix} 1 \\ 2 \\ 0 \\ -2 \end{pmatrix} + b \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \end{pmatrix} \right\}$$

$$= \left\{ x \in \mathbb{R}^4 \mid \exists p \in \mathbb{R}^2 : x = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \\ -2 & 3 \end{bmatrix} p \right\}$$

We need vectors perpendicular to $(1, 2, 0, -2)$ and $(0, 0, 1, 3)$

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13 From the set of solution to a linear system

$$\mathbf{x} = (x, y, z, w,); \quad \mathbf{s} = (1, 3, 1, 1,).$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -2 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array} \right] \xrightarrow{\substack{[(-2)\mathbf{I}+2] \\ [(2)\mathbf{I}+4]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array} \right] \xrightarrow{[(-3)\mathbf{I}+4]} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

So $\mathbf{A} = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 2 & 0 & -3 & 1 \end{bmatrix}$; and then $\mathbf{A}\mathbf{x} = \begin{pmatrix} -2x + y \\ 2x + w - 3z \end{pmatrix}$ and

$\mathbf{b} = \mathbf{A}\mathbf{s} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Hence $\begin{cases} -2x + y = 1 \\ 2x - 3z + w = 0 \end{cases}$

$$P = \left\{ \mathbf{x} \in \mathbb{R}^4 \mid \begin{bmatrix} -2 & 1 & 0 & 0 \\ 2 & 0 & -3 & 1 \end{bmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

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Questions of the Lecture 11

(L-11) QUESTION 1. Describe the set of vectors in \mathbb{R}^3 orthogonal to this one $\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

(Hefferon, 2008, exercise 2.15 from section II.2.)

(L-11) QUESTION 2.

(a) Find a parametric representation for the line passing through the points

$$\mathbf{x}_P = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ y } \mathbf{x}_Q = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

(b) Find a implicit representation for the same line.

(L-11) QUESTION 3.

(a) Find a parametric representation for the line passing through the points

$$\mathbf{x}_P = (1, -3, 1) \text{ and } \mathbf{x}_Q = (-2, 4, 5).$$

(b) Find a implicit representation (Cartesian equations) for the same line.

(L-11) QUESTION 4. Is there any vector perpendicular to itself?

(L-11) QUESTION 5.

(a) Parametric equation of a line parallel to $2x - 3y = 5$ that goes through $(1, 1)$.

(b) Find a implicit representation for the line.

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(L-11) QUESTION 6. Find the length of each vector

(a) $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$. (b) $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$. (c) $\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$.

(d) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. (e) $\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$.

(Hefferon, 2008, exercise 2.11 from section II.2.)

(L-11) QUESTION 7. Find a unit vector with the same direction as $\mathbf{v} = (2, -1, 0, 4, -2)$.

(L-11) QUESTION 8. Find k so that these two vectors are perpendicular.

$$(k, 1), \quad (4, 3).$$

(Hefferon, 2008, exercise 2.14 from section II.2.)

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(L-11) QUESTION 9. Construc a matrix with the required property or say why that is impossible:

(a) Column space contains $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$, nullspace contains $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(b) Row space contains $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$, and nullspace contains $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(c) $\mathbf{A}\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ has a solution and $\mathbf{A}^T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(d) Every row is orthogonal to every column (\mathbf{A} is not the zero matrix)

(e) Columns add up to a column of zeros, rows add up to a row of 1's.

(Strang, 2003, exercise 3 from section 4.1.)

(L-11) QUESTION 10. If $\mathbf{AB} = \mathbf{0}$, the columns of \mathbf{B} are in the _____ of \mathbf{A} . The rows of \mathbf{A} are in the _____ of \mathbf{B} . Why can't \mathbf{A} and \mathbf{B} be 3 by 3 matrices of rank 2?

(Strang, 2003, exercise 4 from section 4.1.)

(L-11) QUESTION 11. Suppose that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ and $\mathbf{u} \neq \mathbf{0}$. Must $\mathbf{v} = \mathbf{w}$?

(Hefferon, 2008, exercise 2.20 from section II.2.)

(L-11) QUESTION 12.

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- (a) If $\mathbf{Ax} = \mathbf{b}$ has a solution and $\mathbf{A}^T \mathbf{y} = \mathbf{0}$, then \mathbf{y} is perpendicular to ____.
- (b) If $\mathbf{A}^T \mathbf{y} = \mathbf{c}$ has a solution and $\mathbf{Ax} = \mathbf{0}$, then \mathbf{x} is perpendicular to ____.

(Strang, 2003, exercise 5 from section 4.1.)

(L-11) QUESTION 13. Demuestre, in \mathbb{R}^n , that if \mathbf{u} and \mathbf{v} are perpendicular then $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

(Hefferon, 2008, exercise 2.33 from section II.2.)

(L-11) QUESTION 14.

- (a) Find parametric equations of the plane that goes through the point (0,1,1) and parallel to the vectors (0,1,2) and (1,1,0)
- (b) Write the implicit equation of the same plane.

(L-11) QUESTION 15.

- (a) Find a parametric equation of the plane through the point (2, 1, 3) with normal vector (3, 1, 1).
- (b) Write the implicit equation of the same plane.

(L-11) QUESTION 16. Find a 1 by 3 matrix whose nullspace consists of all vectors in \mathbb{R}^3 such that $x_1 + 2x_2 + 4x_3 = 0$. Find a 3 by 3 matrix with that same nullspace. (Strang, 2006, exercise 9 from section 2.4.)

(L-11) QUESTION 17. Consider the system $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}.$$

- (a) (1^{pts}) Find the solution to the system.
- (b) (0.5^{pts}) Explain why the solution set is a line in \mathbb{R}^5 . Find a direction vector (a vector parallel to the line) and any point on that line.
- (c) (1^{pts}) Find the set of vectors perpendicular to the solution set. Prove that set is a four dimensional subspace. Find a basis for that subspace.

(L-11) QUESTION 18. Consider \mathbf{A} with exactly two special solutions for $\mathbf{x}\mathbf{A} = \mathbf{0}$:

$$\mathbf{s}_1 = (3, 1, 0, 0), \text{ and } \mathbf{s}_2 = (6, 0, 2, 1).$$

- (a) Find the reduced row echelon form \mathbf{R} of \mathbf{A} .
- (b) What is the row space of \mathbf{A} ?
- (c) What is the complete solution to $\mathbf{x}\mathbf{R} = (3, 6)$?
- (d) Find a combination of rows 2, 3, 4 that equals $\mathbf{0}$. (Not OK to use $0_{(2)}\mathbf{A} + 0_{(3)}\mathbf{A} + 0_{(4)}\mathbf{A}$). The problem is to show that these rows are dependent.)

1 Highlights of Lesson 12

Highlights of Lesson 12

- Projections
- Projection matrices

2 Direct sum of subspaces

\mathbb{R}^n is a *direct sum* of \mathcal{A} and \mathcal{B} ($\mathbb{R}^n = \mathcal{A} \oplus \mathcal{B}$)

if every $\mathbf{x} \in \mathbb{R}^n$ has a **unique** representation $\mathbf{x} = \mathbf{a} + \mathbf{b}$,

with $\mathbf{a} \in \mathcal{A}$ and $\mathbf{b} \in \mathcal{B}$.

Example

$$\mathbb{R}^n = \mathcal{C}(\mathbf{A}^T) \oplus \mathcal{N}(\mathbf{A})$$

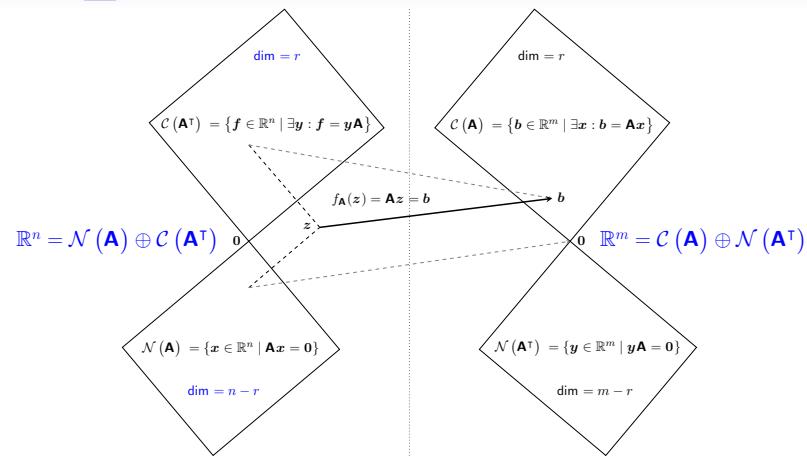
$$\begin{bmatrix} \mathbf{A} \\ \mathbf{I} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & -2 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{Basis of } \mathbb{R}^3; \left[\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}; \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} \right]$$

$$\forall \mathbf{x} \in \mathbb{R}^3, \exists c_1, c_2, c_3 \left| \mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} = \mathbf{a} + \mathbf{b} \right.$$

where $\mathbf{a} \in \mathcal{C}(\mathbf{A}^T)$ and $\mathbf{b} \in \mathcal{N}(\mathbf{A})$.

$$\text{Tambi3n } \mathbb{R}^m = \mathcal{C}(\mathbf{A}) \oplus \mathcal{N}(\mathbf{A}^T)$$

3 The big picture: direct sum of orthogonal complements



$$\mathcal{C}(\mathbf{A}^T) \perp \mathcal{N}(\mathbf{A})$$

$$\mathbf{f} \cdot \mathbf{x} = \mathbf{y} \mathbf{A} \mathbf{x} = \mathbf{y} \cdot \mathbf{0}$$

$$\mathcal{C}(\mathbf{A}) \perp \mathcal{N}(\mathbf{A}^T)$$

$$\mathbf{y} \cdot \mathbf{b} = \mathbf{y} \mathbf{A} \mathbf{x} = \mathbf{0} \cdot \mathbf{x}$$

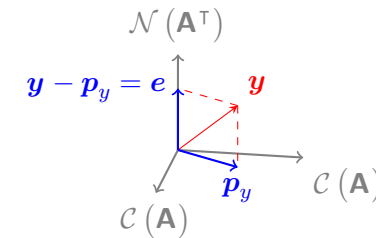
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4 Orthogonal Projection onto $\mathcal{C}(\mathbf{A})$

Consider \mathbf{A} ; since $\mathbb{R}^m = \mathcal{C}(\mathbf{A}) \oplus \mathcal{N}(\mathbf{A}^T)$, for any $\mathbf{y} \in \mathbb{R}^m$

$$\mathbf{y} = \mathbf{p}_y + \mathbf{e}; \quad (\mathbf{e} = \mathbf{y} - \mathbf{p}_y)$$

where $\mathbf{p}_y \in \mathcal{C}(\mathbf{A})$ and $\mathbf{e} \perp \mathbf{p}_y$, so $\mathbf{e} \in \mathcal{N}(\mathbf{A}^T)$.



How to compute $\mathbf{p}_y \in \mathcal{C}(\mathbf{A})$?

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5 Normal equations

Consider \mathbf{A} . We want to find the decomposition $\mathbf{y} = \mathbf{p}_y + \mathbf{e}$ where

$$\mathbf{p}_y \in \mathcal{C}(\mathbf{A}) \quad \text{and} \quad (\mathbf{p}_y - \mathbf{y}) \in \mathcal{N}(\mathbf{A}^T)$$

Then

$$\mathbf{A} \hat{\mathbf{x}} = \mathbf{p}_y \quad \Leftrightarrow \quad (\mathbf{A} \hat{\mathbf{x}} - \mathbf{y}) \in \mathcal{N}(\mathbf{A}^T)$$

Therefore

$$\mathbf{A} \hat{\mathbf{x}} = \mathbf{p}_y \quad \Leftrightarrow \quad \mathbf{A}^T (\mathbf{A} \hat{\mathbf{x}} - \mathbf{y}) = \mathbf{0} \quad \Leftrightarrow \quad (\mathbf{A}^T \mathbf{A}) \hat{\mathbf{x}} = \mathbf{A}^T \mathbf{y}$$

Equivalent systems! $\Rightarrow \mathcal{N}(\mathbf{A}) = \mathcal{N}(\mathbf{A}^T \mathbf{A}) \Rightarrow \text{rg}(\mathbf{A}) = \text{rg}(\mathbf{A}^T \mathbf{A})$

unique solution $\hat{\mathbf{x}}$ if and only if \mathbf{A} is full column rank

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6 The solution to the normal equations (full column rank)

$$\mathbf{A}^T \mathbf{A} \hat{\mathbf{x}} = \mathbf{A}^T \mathbf{y} \quad (\mathbf{A} \text{ is full column rank})$$

The solution

$$\hat{\mathbf{x}} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{y}$$

The projection

$$\mathbf{p} = \mathbf{A} \hat{\mathbf{x}} = \mathbf{A} (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{y}$$

The projection matrix

$$\mathbf{P} = \mathbf{A} (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$$

$$\mathbf{p} = \mathbf{P} \mathbf{y}$$

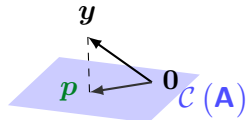
\mathbf{P} : Symmetric and idempotent.

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7 Projection matrix

$$\mathbf{P} = \mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$$

Projection $\mathbf{P}\mathbf{y}$ is the point \mathbf{p} of $\mathcal{C}(\mathbf{A})$ closest to \mathbf{y}

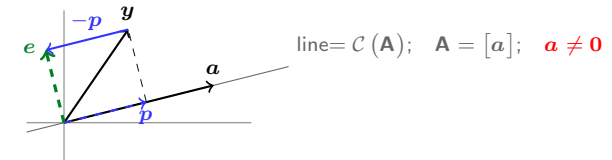


Extreme cases:

- If $\mathbf{y} \in \mathcal{C}(\mathbf{A})$ then $\mathbf{P}\mathbf{y} = \mathbf{y}$
- If $\mathbf{y} \perp \mathcal{C}(\mathbf{A})$ then $\mathbf{P}\mathbf{y} = \mathbf{0}$

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8 Projection onto a line



I'd like to find the point \mathbf{p} on that line closest to \mathbf{y}

$$\mathbf{p} \in \mathcal{C}([a]) \quad \perp \quad \mathbf{e} = (\mathbf{y} - \mathbf{p}) \in \mathcal{N}([a]^T).$$

\mathbf{p} is some multiple of \mathbf{a} : $\mathbf{p} = [\mathbf{a}](\hat{x},)$

How:

$$[\mathbf{a}]^T [\mathbf{a}] \hat{x} = [\mathbf{a}]^T \mathbf{y}$$

The solution

$$\hat{x} = ([\mathbf{a}]^T [\mathbf{a}])^{-1} [\mathbf{a}]^T \mathbf{y}$$

The projection

$$\mathbf{p} = [\mathbf{a}] \hat{x} = [\mathbf{a}] ([\mathbf{a}]^T [\mathbf{a}])^{-1} [\mathbf{a}]^T \mathbf{y}$$

The projection matrix

$$\mathbf{P} = [\mathbf{a}] ([\mathbf{a}]^T [\mathbf{a}])^{-1} [\mathbf{a}]^T$$

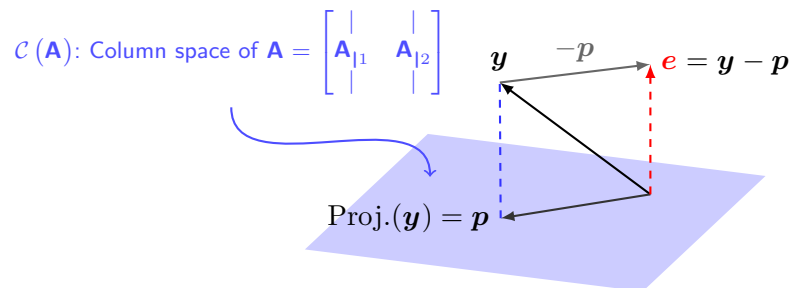
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9 Projection onto a plane

Why project?

So we will solve

$$\mathbf{A}\mathbf{x} = (\text{Proj. of } \mathbf{y} \text{ onto } \mathcal{C}(\mathbf{A})).$$

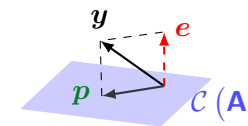


$$(\mathbf{y} - \mathbf{p}) = \mathbf{e} \perp \mathcal{C}(\mathbf{A}) \quad \dots \text{that's the crucial fact.}$$

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10 Normal equations

What's the projection of \mathbf{y} onto the column space of $\mathbf{A} = \begin{bmatrix} | & | \\ \mathbf{A}_{|1} & \mathbf{A}_{|2} \\ | & | \end{bmatrix}$?



$$\mathbf{p} = (\mathbf{A}_{|1})\hat{x}_1 + (\mathbf{A}_{|2})\hat{x}_2 = \mathbf{A}\hat{x}$$

"Find the right combination of the columns so $\mathbf{e} \perp \mathcal{C}(\mathbf{A})$ "

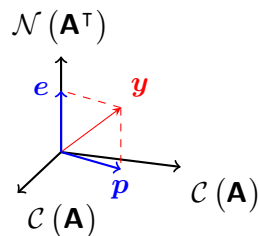
$$\mathbf{e} \perp \mathcal{C}(\mathbf{A}) \quad \Rightarrow \quad \mathbf{e} \in$$

$$\mathbf{A}^T \mathbf{e} = \mathbf{A}^T (\mathbf{y} - \mathbf{p}) = \mathbf{A}^T (\mathbf{y} - \mathbf{A}\hat{x}) = \mathbf{0} \quad \Leftrightarrow \quad (\mathbf{A}^T \mathbf{A})\hat{x} = \mathbf{A}^T \mathbf{y}$$

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11 Two projections

y has a component p in $\mathcal{C}(\mathbf{A})$, and another component e in $\mathcal{C}(\mathbf{A})^\perp$.



$$p + e = y$$

$$p = \mathbf{P}y \quad \text{projection onto } \mathcal{C}(\mathbf{A})$$

$$e = (\mathbf{I} - \mathbf{P})y \quad \text{projection onto } \mathcal{C}(\mathbf{A})^\perp$$

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(L-12) QUESTION 3. Although pictures guided our development, we are not restricted to spaces that we can draw. In \mathbb{R}^4 project this vector into this line.

$$\begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}; \quad \left\{ v \in \mathbb{R}^4 \mid \exists p \in \mathbb{R}^1, v = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} p \right\}.$$

(L-12) QUESTION 4.

(a) Project the vector $b = (1, 1, 0, 0)$ onto the lines through $a_1 = (1, 0, 0, 0)$ and $a_2 = (1, 0, 0, 0)$. Add the projections: $p_1 + p_2$. The projections do not add to b because a_1 and a_2 are not orthogonal.

(b) The projection of b onto the plane of a_1 and a_2 will equal b . Find $\mathbf{P} = \mathbf{A}(\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top$ for $\mathbf{A} = [a_1; a_2]$.

(Strang, 2003, exercise 8–9 from section 4.2.)

(L-12) QUESTION 5.

(a) If $\mathbf{P}^2 = \mathbf{P}$ show that $(\mathbf{I} - \mathbf{P})^2 = \mathbf{I} - \mathbf{P}$. When \mathbf{P} projects onto the column space of \mathbf{A} , $(\mathbf{I} - \mathbf{P})$ projects onto the _____.

(b) If $\mathbf{P}^\top = \mathbf{P}$ show that $(\mathbf{I} - \mathbf{P})^\top = \mathbf{I} - \mathbf{P}$.

(Strang, 2003, exercise 17 from section 4.2.)

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Questions of the Lecture 12

(L-12) QUESTION 1. Project the first vector orthogonally into the line spanned by the second vector. Check that e is perpendicular to a . Find the projection matrix $\mathbf{P} = [\mathbf{a}][\mathbf{a}]^\top [\mathbf{a}]^\top [\mathbf{a}]^{-1} [\mathbf{a}]^\top$ onto the line through each vector a . Verify in each case that $\mathbf{P}^2 = \mathbf{P}$. Multiply $\mathbf{P}b$ in each case to compute the projection p .

(a) $b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}; a = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$

(b) $b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}; a = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$

(c) $b = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}; a = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$

(d) $b = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}; a = \begin{pmatrix} 3 \\ 3 \\ 12 \end{pmatrix}.$

(Hefferon, 2008, exercise 1.6 from section VI.1.)

(L-12) QUESTION 2. Project the vector orthogonally into the line.

(a) $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$, The line: $\left\{ v \in \mathbb{R}^3 \mid \exists p \in \mathbb{R}^1, v = \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix} p \right\}.$

(b) $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$, the line $y = 3x$.

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(L-12) QUESTION 6.

(a) Compute the projection matrices $\mathbf{P} = [\mathbf{a}][\mathbf{a}]^\top [\mathbf{a}]^\top [\mathbf{a}]^{-1} [\mathbf{a}]^\top$ onto the lines through $a_1 = (-1, 2, 2)$ and $a_2 = (2, 2, -1)$. Show that $a_1 \perp a_2$. Multiply those projection matrices and explain why their product $\mathbf{P}_1 \mathbf{P}_2$ is what it is.

(b) Project $b = (1, 0, 0)$ onto the lines through a_1 and a_2 and also onto $a_3 = (2, -1, 2)$. Add up the three projections $p_1 + p_2 + p_3$.

(c) Find the projection matrix \mathbf{P}_3 onto $\mathcal{L}([a_3; 1]) = \mathcal{L}([(2, -1, 2); 1])$. Verify that $\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 = \mathbf{I}$. The basis a_1, a_2, a_3 is orthogonal!

(Strang, 2003, exercise 5–7 from section 4.2.)

(L-12) QUESTION 7. Project b onto the column space of \mathbf{A} by solving $\mathbf{A}^\top \mathbf{A} \hat{x} = \mathbf{A}^\top b$ and $p = \mathbf{A} \hat{x}$. Find $e = b - p$.

(a) $\mathbf{A}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $b_1 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

(b) $\mathbf{A}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $b_2 = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix}$

(c) Compute the projection matrices \mathbf{P}_1 and \mathbf{P}_2 onto the column spaces. Verify that $\mathbf{P}_1 b_1$ gives the first projection p_1 . Also verify $(\mathbf{P}_2)^2 = \mathbf{P}_2$.

(Strang, 2003, exercise 11–12 from section 4.2.)

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URL

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