

Mathematics II

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1 Highlights of Lesson 1

Highlights of Lesson 1

- Vector and matrix operations
 - Addition and scalar multiplication
 - Some properties of these operations

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You can find the last version of these course materials at

<https://mbujosab.github.io/MatematicasII/>

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2 Vectors in \mathbb{R}^n

Vector in \mathbb{R}^n is an ordered list of n real numbers

Example

$v \in \mathbb{R}^3$: first component: 5, the second: 1 and the third: 10

$$v = \begin{cases} v_1 = 5 \\ v_2 = 1 \\ v_3 = 10 \end{cases} ; \quad v = (5, 1, 10) = \begin{pmatrix} 5 \\ 1 \\ 10 \end{pmatrix}.$$

Notation

- $a, x, 0$
- $\text{elem}_3(v) \equiv {}_3|v \equiv v|_3 \equiv v_3 = 10$

A parenthesis around a list of numbers denotes a vector.

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3 Basic operations with vectors

Vector addition: $(\mathbf{a} + \mathbf{b})_{|i} = \mathbf{a}_{|i} + \mathbf{b}_{|i}$

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \text{ add to } \mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}.$$

Scalar multiplication: $(\lambda \mathbf{a})_{|i} = \lambda(\mathbf{a}_{|i})$

$$2\mathbf{a} = \begin{pmatrix} 2a_1 \\ 2a_2 \end{pmatrix} \quad \text{and} \quad (-1)\mathbf{a} = \begin{pmatrix} -a_1 \\ -a_2 \end{pmatrix} \equiv -\mathbf{a}$$

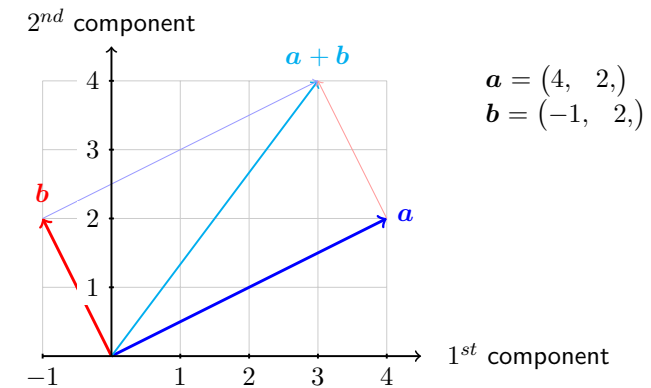
(Hence, the operator “ $|i$ ” is linear)

\mathbf{a} and \mathbf{b} (with n components) are equal when:

$$\mathbf{a}_{|i} = \mathbf{b}_{|i}, \quad i = 1 : n.$$

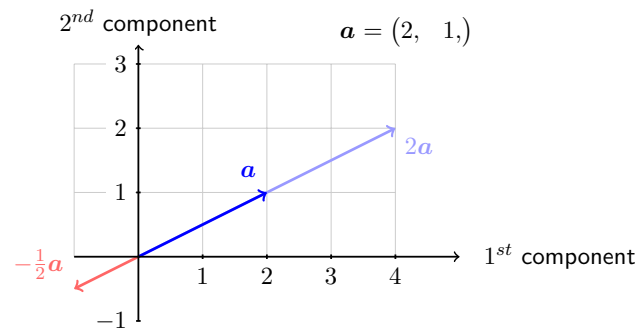
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4 Vector addition



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5 Scalar multiplication



What is the picture of all multiples of \mathbf{a} ?

Is $\mathbf{0}$ a multiple of \mathbf{a} ?

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6 Addition and scalar multiplication

$$(\mathbf{a} + \mathbf{b})_{|i} = \mathbf{a}_{|i} + \mathbf{b}_{|i}$$

$$(\lambda \mathbf{a})_{|i} = \lambda(\mathbf{a}_{|i})$$

Let us recall some properties of scalars

Scalars

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $\mathbf{a}\mathbf{b} = \mathbf{b}\mathbf{a}$
6. $\mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{a}\mathbf{b} + \mathbf{a}\mathbf{c}$
7. $\mathbf{a}(\mathbf{b}\mathbf{c}) = (\mathbf{a}\mathbf{b})\mathbf{c}$
8. $1\mathbf{a} = \mathbf{a}$

Vectors

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$
6. $(\lambda + \eta)\mathbf{a} = \lambda\mathbf{a} + \eta\mathbf{a}$
7. $\lambda(\eta\mathbf{a}) = (\lambda\eta)\mathbf{a}$
8. $1\mathbf{a} = \mathbf{a}$

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7 Matrices

Matrix in $\mathbb{R}^{m \times n}$ is an ordered list of n vectors in \mathbb{R}^m

Example

Three vectors in \mathbb{R}^2 : $\mathbf{a} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 0 \\ 7 \end{pmatrix}$

$$\mathbf{A} = [\mathbf{a}; \mathbf{b}; \mathbf{c}] = \begin{bmatrix} 4 & -1 & 0 \\ 2 & 2 & 7 \end{bmatrix} \neq [\mathbf{c}; \mathbf{b}; \mathbf{a}]$$

Two vectors in \mathbb{R}^3 : $\mathbf{x} = (4, -1, 0)$ and $\mathbf{y} = (2, 2, 7)$

$$\mathbf{B} = [\mathbf{x}; \mathbf{y}]$$

Notation

- $\mathbf{A}, \mathbf{B}, \mathbf{0}$
- $\mathbf{A}, \mathbf{B};$
 $\begin{matrix} 2 \times 3 & 3 \times 2 \end{matrix}$ $\mathbf{A} \neq \mathbf{B}$
 $\begin{matrix} 2 \times 3 & 3 \times 2 \end{matrix}$

A bracket around a vector list denotes a matrix.

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8 More notation

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 7 & 0 & 3 \end{bmatrix}$$

Picking operators

- $\text{elem}_{21}(\mathbf{A}) = {}_2|\mathbf{A}|_1 = a_{21} : 7$
- $\text{row}_1(\mathbf{A}) = {}_1|\mathbf{A} : (1, 2, 1) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$
- $\text{col}_1(\mathbf{A}) = \mathbf{A}|_1 : \begin{pmatrix} 1 \\ 7 \end{pmatrix} = (1, 7)$

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9 Basic operations with matrices

Matrix addition: $(\mathbf{A} + \mathbf{B})|_j = \mathbf{A}|_j + \mathbf{B}|_j$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \text{ add to } \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

Scalar multiplication: $(\lambda \mathbf{A})|_j = \lambda(\mathbf{A}|_j)$

$$7\mathbf{A} = \begin{bmatrix} 7a_{11} & 7a_{12} \\ 7a_{21} & 7a_{22} \end{bmatrix} \text{ and } (-1)\mathbf{A} = \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix} = -\mathbf{A}.$$

(Hence, the operator “ $|_j$ ” is linear)

\mathbf{A} and \mathbf{B} (with same order) are equal when: $\mathbf{A}|_j = \mathbf{B}|_j$

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10 Addition and scalar multiplication

$$(\mathbf{A} + \mathbf{B})|_j = \mathbf{A}|_j + \mathbf{B}|_j;$$

$$(\lambda \mathbf{A})|_j = \lambda(\mathbf{A}|_j)$$

Vectors

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{0} + \mathbf{a} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$
6. $(\lambda + \eta)\mathbf{a} = \lambda\mathbf{a} + \eta\mathbf{a}$
7. $\lambda(\eta\mathbf{a}) = (\lambda\eta)\mathbf{a}$
8. $1\mathbf{a} = \mathbf{a}$

Matrices

1. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
2. $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$
3. $\mathbf{0} + \mathbf{A} = \mathbf{A}$
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7. $\lambda(\eta\mathbf{A}) = (\lambda\eta)\mathbf{A}$
8. $1\mathbf{A} = \mathbf{A}$

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11 Rewriting Rules

Distributive rules

$$\begin{aligned}(a+b)|_i &= a|_i + b|_i & {}_i|(a+b) &= {}_i|a + {}_i|b \\ (A+B)|_j &= A|_j + B|_j & {}_i|(A+B) &= {}_i|A + {}_i|B\end{aligned}$$

In addition, if we allow $\lambda a = a\lambda$ and $\lambda A = A\lambda$, then we get

Associative rules (moving parentheses)

$$\begin{aligned}(\lambda b)|_i &= \lambda(b|_i) & {}_i|(b\lambda) &= ({}_i|b)\lambda \\ (\lambda A)|_j &= \lambda(A|_j) & {}_i|(A\lambda) &= ({}_i|A)\lambda\end{aligned}$$

Scalar and operator interchange

$$\begin{aligned}(b\lambda)|_i &= (b|_i)\lambda & {}_i|(\lambda b) &= \lambda({}_i|b) \\ (A\lambda)|_j &= (A|_j)\lambda & {}_i|(\lambda A) &= \lambda({}_i|A)\end{aligned}$$

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Questions of the Lecture 1 You should always complete the exercises in the theoretical sections previous to each lecture

- Prove the properties of vector operations.
- Prove the properties of matrix operations.

(L-1) **QUESTION 1.** Give 3 by 3 examples (not just the zero matrix) of:

- (a) A diagonal matrix: ${}_i|A|_j = 0$ if $i \neq j$. (b) A symmetric matrix: $A|_j = {}_j|A$.
 (c) An upper triangular matrix: ${}_i|A|_j = 0$ if $i > j$. (d) A skew-symmetric matrix: ${}_i|A|_j = -{}_j|A|_i$.

(Strang, 1988, exercise 7 from section 1.4.)

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1 Highlights of Lesson 2

Highlights of Lesson 2

- Dot product
- linear combinations
- The column picture of the Geometry of linear equations

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2 Dot product

$$x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3 + \cdots + x_n y_n = \sum_{i=1}^n x_i y_i.$$

Symmetric

$$x \cdot y = y \cdot x$$

Linear in the first argument

$$\begin{aligned}(ax) \cdot y &= a(x \cdot y) \\ (x+y) \cdot z &= x \cdot z + y \cdot z\end{aligned}$$

Positive

$$x \cdot x \geq 0$$

Definite

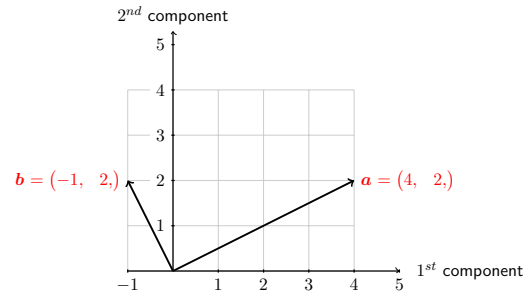
$$x \cdot x = 0 \Leftrightarrow x = 0$$

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3 Linear combinations

The sum of xa and yb is a **linear combination** of a and b

$$xa + yb = x \begin{pmatrix} 4 \\ 2 \end{pmatrix} + y \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{bmatrix} a & b \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{MATRIX} \times \mathbf{vector}$$

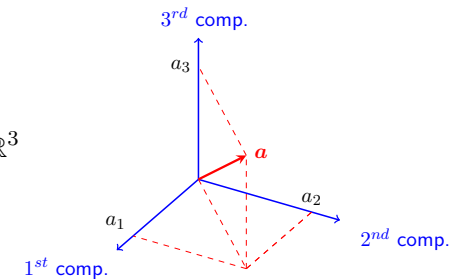


Is $\mathbf{0}$ a linear combination of a and b ? What is the picture of **all** linear combinations of a and b ?

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4 Linear combinations in \mathbb{R}^3

$$a = (a_1, a_2, a_3) \in \mathbb{R}^3$$



What is the picture of all multiples of a ?

What is the picture of all linear combinations of two vectors in \mathbb{R}^3 ?
(linear combination)

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5 Matrix times vector

$$\begin{aligned} \mathbf{A}b &= b_1 \mathbf{A}_{|1} + b_2 \mathbf{A}_{|2} + \cdots + b_n \mathbf{A}_{|n} \\ &= b_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{i1} \\ \vdots \\ a_{m1} \end{pmatrix} + b_2 \begin{pmatrix} a_{12} \\ \vdots \\ a_{i2} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + b_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{in} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} (1|\mathbf{A}) \cdot b \\ \vdots \\ (i|\mathbf{A}) \cdot b \\ \vdots \\ (n|\mathbf{A}) \cdot b \end{pmatrix} \end{aligned}$$

Hence,

$$\boxed{{}_i(\mathbf{A}b) = ({}_i\mathbf{A}) \cdot b}$$

if we omit the period, we can simply write: ${}_i\mathbf{A}b$

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6 Matrix times vector

If $b \in \mathbb{R}^n$ then $(\mathbf{A}b) \in \mathbb{R}^m$; where ${}_i(\mathbf{A}b) = ({}_i\mathbf{A}) \cdot b$

Matrix times vector

1. $\mathbf{I}a = a$
2. $\mathbf{A}(\mathbf{I}_{|j}) = \mathbf{A}_{|j}$
3. $\mathbf{A}(b + c) = \mathbf{A}b + \mathbf{A}c$
4. $\mathbf{A}(\lambda b) = \lambda(\mathbf{A}b)$
5. $\mathbf{A}(\lambda b) = (\lambda\mathbf{A})b$
6. $\mathbf{A}(\mathbf{B}c) = [\mathbf{A}(\mathbf{B}_{|1}); \dots \mathbf{A}(\mathbf{B}_{|n});] c$
7. $(\mathbf{A} + \mathbf{B})c = \mathbf{A}c + \mathbf{B}c$

Prove the above propositions (Book exercises: 14 to 18)
(follow the rewriting rules and properties of the dot product)

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7 Example of linear system: 2 equations and 2 unknowns

$$\begin{cases} 2x - y = 0 \\ -x + 2y = 3 \end{cases}$$

$$\begin{bmatrix} & \end{bmatrix} \begin{pmatrix} & \end{pmatrix} = \begin{pmatrix} & \end{pmatrix}$$

$$\mathbf{Ax} = \mathbf{b}$$

$$\underbrace{\mathbf{Ax}}_{\text{which linear combination}} = \underbrace{\mathbf{b}}_{\text{equals this vector?}}$$

$$x(\mathbf{A}_{|1}) + y(\mathbf{A}_{|2}) = \mathbf{b}$$

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8 Geometry of linear systems: Linear combination of columns

$$\begin{cases} 2x - y = 0 \\ -x + 2y = 3 \end{cases}$$

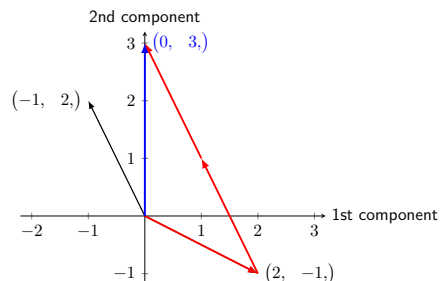
$$x \begin{pmatrix} & \end{pmatrix} + y \begin{pmatrix} & \end{pmatrix} = \begin{pmatrix} & \end{pmatrix}$$

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9 2 equations and 2 unknowns: Column picture

$$x \begin{pmatrix} 2 \\ -1 \end{pmatrix} + y \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

Which linear combination of $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ gives $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$?



What is the set of all possible combinations?

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10 Example: 3 equations and 3 unknowns

$$\begin{cases} 2x - y = 0 \\ -x + 2y - z = -1 \\ -3y + 4z = 4 \end{cases}$$

$$\begin{bmatrix} & & \end{bmatrix} \begin{pmatrix} & & \end{pmatrix} = \begin{pmatrix} & & \end{pmatrix}$$

$$\mathbf{Ax} = \mathbf{b}$$

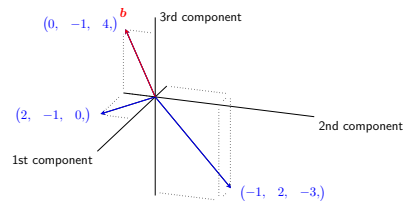
$$\underbrace{\mathbf{Ax}}_{\text{which linear combination}} = \underbrace{\mathbf{b}}_{\text{equals this vector?}}$$

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11 3 equations and 3 unknowns: Column picture

$$x \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + y \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} + z \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$$

Which linear combination of the columns gives \mathbf{b} ?



$$\left\{ x = \quad ; \quad y = \quad ; \quad z = \quad \right\}$$

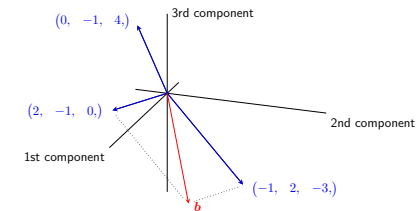
What happens with a different \mathbf{b} ?... let's see

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12 3 equations and 3 unknowns: Column picture

$$x \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + y \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} + z \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$$

Which linear combination of the columns gives this new \mathbf{b} ?



$$\left\{ x = \quad ; \quad y = \quad ; \quad z = \quad \right\}$$

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13 What does $\mathbf{Ax}=\mathbf{b}$ mean?

\mathbf{Ax} is a linear combination of columns of \mathbf{A} :

Example

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \end{pmatrix}$$

" $\mathbf{Ax} = \mathbf{b}$ " is asking for a particular linear combination:

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} \quad \\ \quad \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \end{pmatrix}$$

To solve linear systems we will first learn to transform coefficient matrices by elimination (next lectures)

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Questions of the Lecture 2

You must complete the exercises from the corresponding sections of the book

(L-2) QUESTION 1. Working a column at a time, compute the following products

(a)

$$\begin{bmatrix} 4 & 1 \\ 4 & 1 \\ 6 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

(b)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

(c)

$$\begin{bmatrix} 4 & 3 \\ 6 & 6 \\ 8 & 9 \end{bmatrix} \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix}$$

(Strang, 1988, exercise 2 from section 1.4.)

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(L-2) QUESTION 2. Can the three equations be solved simultaneously?

$$x + 2y = 2$$

$$x - y = 2$$

$$y = 1.$$

What happens if all right hand sides are zero? Is there any non-zero choice of right hand sides which allows the three equations to have a solution? How many non-zero choices have we?

(Strang, 1988, exercise 4 from section 1.2.)

(L-2) QUESTION 3. Compute the product $\mathbf{A}x$ with

$$\mathbf{A} = \begin{bmatrix} 3 & -6 & 0 \\ 0 & 2 & -2 \\ 1 & -1 & -1 \end{bmatrix} \quad \text{and} \quad x = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

For this matrix \mathbf{A} , find a solution vector x to the system $\mathbf{A}x = \mathbf{0}$, with zeros on the right side of all three equations. Can you find more than one solution?

(Strang, 1988, exercise 5 from section 1.4.)

(L-2) QUESTION 4. Suppose $\mathbf{A}x = b$ has two solutions v and w (with $b \neq 0$). Then show that $\frac{1}{2}(v + w)$ is also a solution, although $v + w$ is not.

Hint

Use the following properties: $\mathbf{A}(b + c) = \mathbf{A}b + \mathbf{A}c$ and $\mathbf{A}(cb) = c(\mathbf{A}b)$.

(L-2) QUESTION 5. "It is impossible for a system of linear equations to have exactly two solutions". Explain why (answering the next question):

(a) If v y w are two solutions, what is another one?

(Strang, 2003, exercise 19 from section 2.2.)

(L-2) QUESTION 6. Draw $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, along with $v + w$, $2v + w$, and $v - w$ in a plane (first component on the horizontal axis and second component on the vertical axis).

(L-2) QUESTION 7. draw the column picture of the following system with solution $x = 3$ and $y = -1$.

$$\begin{cases} 2x + y = 5 \\ x - 3y = 6 \end{cases}$$

(L-2) QUESTION 8. draw the column picture of the following system.

$$\begin{cases} 2x - y = 3 \\ x + y = 1 \end{cases} \quad ; \quad \left(\text{the solution is :} \quad x = 1 + \frac{1}{3}, \quad y = -\frac{1}{3} \right).$$

No deje de hacer los ejercicios del libro.

1 Highlights of Lesson 3

Highlights of Lesson 3

- Matrix multiplication: $(\mathbf{AB})_{ij} = \mathbf{A}(\mathbf{B}_{ij})$
 - Properties
- Transpose of a matrix
- $\mathbf{A}x$ and $x\mathbf{A}$ (linear combinations)
- Other ways to compute the product
- Transpose of \mathbf{AB}

2 Matrix multiplication (by columns)

Column j of $\begin{pmatrix} \mathbf{A} & \text{times} & \mathbf{B} \end{pmatrix}$ is:

 $m \times p$ $p \times n$

$$(\mathbf{AB})_{|j} = \mathbf{A}(\mathbf{B}_{|j}) \longrightarrow \mathbf{AB}_{|j}$$

Each column of \mathbf{AB} is a linear combination of the p columns of \mathbf{A}

Example

$$\begin{bmatrix} 2 & 1 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{A}(\mathbf{B}_{|1}); & \mathbf{A}(\mathbf{B}_{|2}); \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}; & \begin{bmatrix} 2 & 1 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{pmatrix} 6 \\ 0 \end{pmatrix}; \end{bmatrix} = \begin{bmatrix} 3 & 12 \\ 11 & 18 \\ 13 & 24 \end{bmatrix}$$

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3 Matrix multiplication properties

MATRIX \times MATRIX = MATRIX

1. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ remember $\mathbf{A}(\mathbf{Bc}) = [\mathbf{A}(\mathbf{B}_{|1}); \dots \mathbf{A}(\mathbf{B}_{|n});] c$
2. $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$.
3. $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$.
4. $\mathbf{A}(\lambda\mathbf{B}) = (\lambda\mathbf{A})\mathbf{B} = \lambda(\mathbf{AB})$.
5. $\mathbf{IA} = \mathbf{A}$.
6. $\mathbf{AI} = \mathbf{A}$.

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4 Transposing a matrix

Transpose

$$(\text{column } i \text{ of } \mathbf{A}^T) = (\text{row } i \text{ of } \mathbf{A}) \leftrightarrow (\mathbf{A}^T)_{|i} = {}_i\mathbf{A}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 4 & 1 \end{bmatrix}; \quad \mathbf{A}^T =$$

$${}_i\mathbf{A}_{|j} = {}_{j|}(\mathbf{A}^T)_{|i}; \quad (\mathbf{A}^T)^T = \mathbf{A}; \quad {}_{j|}(\mathbf{A}^T) = \mathbf{A}_{|j}$$

Symmetric matrices $\mathbf{A}^T = \mathbf{A}$

$$\begin{bmatrix} 3 & 1 & 7 \\ & 2 & 9 \\ & & 1 \end{bmatrix}$$

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5 Vectors, row matrices, column matrices

$$(1, 3, -10) = \begin{pmatrix} 1 \\ 3 \\ -10 \end{pmatrix}; \quad \text{but} \quad [1 \ 3 \ -10] \neq \begin{bmatrix} 1 \\ 3 \\ -10 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}; \quad {}_2\mathbf{A} = (2, 3) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}; \quad \mathbf{A}_{|1} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = (1, 2, 4)$$

When writing vectors between "square brackets" we get a matrix whose columns are those vectors

$$[{}_3\mathbf{A}; \ {}_1\mathbf{A}] = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}; \quad \mathbf{A}^T = [{}_1\mathbf{A}; \ {}_2\mathbf{A}; \ {}_3\mathbf{A}]$$

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6 Linear combination of rows and columns

Linear combination of columns

$$\begin{bmatrix} \diamond & \clubsuit \\ \heartsuit & \spadesuit \\ \diamond & \clubsuit \end{bmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3 \begin{pmatrix} \diamond \\ \heartsuit \\ \diamond \end{pmatrix} + 4 \begin{pmatrix} \clubsuit \\ \spadesuit \\ \clubsuit \end{pmatrix}$$

MATRIX \times **vector** = **vector**

Linear combination of rows

$$(1, 2, 7) \begin{bmatrix} \diamond & \clubsuit \\ \heartsuit & \spadesuit \\ \diamond & \clubsuit \end{bmatrix} = 1 (\diamond, \clubsuit) + 2 (\heartsuit, \spadesuit) + 7 (\diamond, \clubsuit)$$

vector \times **MATRIX** = **vector**

Linear combinations

\longrightarrow

$$\mathbf{aB} = (\mathbf{B}^T \mathbf{a})$$

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7 Vector times matrix

Remember that ${}_i | (\mathbf{A} \mathbf{b}) = ({}_i | \mathbf{A}) \cdot \mathbf{b}$; hence

$$(\mathbf{aB})_{|j} = {}_j | (\mathbf{aB}) = {}_j | ((\mathbf{B}^T \mathbf{a})) = ({}_j | (\mathbf{B}^T)) \cdot \mathbf{a} = (\mathbf{B}_{|j}) \cdot \mathbf{a} = \mathbf{a} \cdot (\mathbf{B}_{|j})$$

Rewriting rules

$${}_i | (\mathbf{A} \mathbf{b}) = ({}_i | \mathbf{A}) \cdot \mathbf{b}$$

and

$$(\mathbf{aB})_{|j} = \mathbf{a} \cdot (\mathbf{B}_{|j})$$

Thus, if we omit the period, we can simply write:

$${}_i | \mathbf{A} \mathbf{b} \quad \text{and} \quad \mathbf{aB}_{|j}$$

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8 Matrix multiplication: rows times columns

Consider \mathbf{A} and \mathbf{B} , then:

$m \times p$ $p \times n$

$${}_i | (\mathbf{AB})_{|j} = ({}_i | \mathbf{A}) \cdot (\mathbf{B}_{|j})$$

Proof.

Remember that ${}_i | (\mathbf{A} \mathbf{b}) = ({}_i | \mathbf{A}) \cdot \mathbf{b}$, hence

$${}_i | (\mathbf{AB})_{|j} = {}_i | ((\mathbf{AB})_{|j}) = {}_i | (\mathbf{A} (\mathbf{B}_{|j})) = ({}_i | \mathbf{A}) \cdot (\mathbf{B}_{|j})$$

□

Thus, if we omit the period, we can simply write:

$${}_i | \mathbf{AB}_{|j}$$

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9 Matrix multiplication (by rows)

Consider \mathbf{A} and \mathbf{B} , then:

$m \times p$ $p \times n$

$${}_i | (\mathbf{AB}) = ({}_i | \mathbf{A}) \mathbf{B}$$

Proof.

Let's see the j th components are equal:

$$({}_i | (\mathbf{AB}))_{|j} = {}_i | ((\mathbf{AB})_{|j}) = ({}_i | \mathbf{A}) \cdot (\mathbf{B}_{|j}) = (({}_i | \mathbf{A}) \mathbf{B})_{|j}$$

so ${}_i | (\mathbf{AB}) = ({}_i | \mathbf{A}) \mathbf{B}$.

□

Thus, we can simply write:

$${}_i | \mathbf{AB}$$

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10 Matrix multiplication (by rows)

Each row of \mathbf{AB} is a linear combination of the p rows of \mathbf{B}

$$\begin{bmatrix} 2 & 1 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 12 \\ 11 & 18 \\ 13 & 24 \end{bmatrix} \quad \text{where} \quad \begin{cases} (2, 1) \begin{bmatrix} 1 & 6 \\ 1 & 0 \end{bmatrix} = (3, 12) \\ (3, 8) \begin{bmatrix} 1 & 6 \\ 1 & 0 \end{bmatrix} = (11, 18) \\ (4, 9) \begin{bmatrix} 1 & 6 \\ 1 & 0 \end{bmatrix} = (13, 24) \end{cases}$$

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Librería `nacl` para Python

Revise la implementación de las operaciones del álgebra matricial en la librería `nacl` para Python que acompaña al curso:

Sección 1.3 de la documentación (o estudie directamente el código).

<https://github.com/mbujosab/nacllib>

Verá que el código es una traducción literal de las *definiciones* vistas aquí; pero que **no hay ni una línea de código que describa las propiedades** que hemos demostrado en estas tres lecciones. ¡No es necesario! Las definiciones implican las propiedades (como hemos comprobado teóricamente con las demostraciones de estas lecciones). **Verifique con ejemplos que todas las propiedades se cumplen.** Estudie los **notebooks de Jupyter** correspondientes a las tres primeras lecciones.

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11 Transposing a product of matrices

Since

- $(\mathbf{A}^T)_{|j} = {}_j|\mathbf{A}$
- $\mathbf{aB} = (\mathbf{B}^T)\mathbf{a}$

it follows that:

$$(\mathbf{AB})^T = (\mathbf{B}^T)(\mathbf{A}^T)$$

Proof.

$$(\mathbf{AB})^T_{|j} = {}_j|\mathbf{AB} = (\mathbf{B}^T)({}_j|\mathbf{A}) = (\mathbf{B}^T)(\mathbf{A}^T)_{|j}.$$

□

Matrix times its transpose is always symmetric

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Questions of the Lecture 3

No deje de hacer los ejercicios del libro.

(L-3) QUESTION 1. Multiply these matrices in the orders \mathbf{EF} , \mathbf{FE} and \mathbf{E}^2

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}; \quad \mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}.$$

(Strang, 1988, exercise 34 from section 1.4.)

(L-3) QUESTION 2. True or false; give a specific counterexample when false.

- If the first and third columns of \mathbf{B} are the same, so are the first and third columns of \mathbf{AB} .
- If the first and third rows of \mathbf{B} are the same, so are the first and third rows of \mathbf{AB} .
- If the first and third rows of \mathbf{A} are the same, so are the first and third rows of \mathbf{AB} .
- $(\mathbf{AB})^2 = \mathbf{A}^2\mathbf{B}^2$.

(Strang, 1988, exercise 10 from section 1.4.)

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(L-3) QUESTION 3. Consider the vectors

$\mathbf{a} = (1, -2, 7)$ and $\mathbf{b} = (3, 5, 1)$. Compute the following products

(a) $\mathbf{a} \cdot \mathbf{a}$ (b) $\mathbf{a} \cdot \mathbf{b}$ (c) $[\mathbf{a}][\mathbf{b}]^T$

(Strang, 1988, exercise 3 from section 1.4.)

(L-3) QUESTION 4. Write down the 2 by 2 matrices \mathbf{A} and \mathbf{B} that have entries $a_{ij} = i + j$ and $b_{ij} = (-1)^{i+j}$. Multiply them to find \mathbf{AB} and \mathbf{BA} .

(Strang, 1988, exercise 6 from section 1.4.)

(L-3) QUESTION 5. The product of two lower triangular matrices is again lower triangular (all its entries above the main diagonal are zero). Confirm this with a 3 by 3 example, and then explain how it follows from the laws of matrix multiplication. (Strang, 1988, exercise 12 from section 1.4.)

(L-3) QUESTION 6. Consider the matrices \mathbf{A} , \mathbf{B} , \mathbf{C} , \mathbf{D} , \mathbf{E} and \mathbf{F} .

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & 4 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

Compute (in particular, note that $\mathbf{EF} \neq \mathbf{FE}$!)

(a) $\mathbf{B} + \mathbf{D}$ (b) $2\mathbf{E} - \mathbf{F}$ (c) \mathbf{AC}
 (d) \mathbf{BC} (e) \mathbf{CB} (f) \mathbf{ACD}
 (g) \mathbf{EF} (h) \mathbf{FE} (i) \mathbf{CEF}

Strang, G. (1988). *Linear algebra and its applications*. Thomson Learning, Inc., third ed. ISBN 0-15-551005-3.

Strang, G. (2003). *Introduction to Linear Algebra*. Wellesley-Cambridge Press, Wellesley, Massachusetts. USA, third ed. ISBN 0-9614088-9-8.