Mathematics II

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You can find the last version of these course materials at

https://mbujosab.github.io/MatematicasII/

Contents

Part V
Determinants

LECTURE 14: The properties of Determinants

Lecture 14

Highlights of *Lesson 14*

- Determinant: $\det(\mathbf{A}) \equiv |\mathbf{A}|$ $[\det: \quad \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}]$
  - Volume vs determinant
  - Properties: **1**, **2**, **3**
- We will deduce properties: **4** – **9**

F1

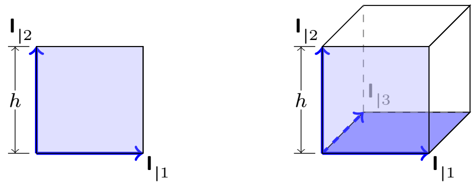
"Pinche aquí" y vea el notebook de Jupyter de la Lección 14. (¡Ojo! están mal numerados los Notebooks)

Definition of determinant function by three properties related to volume function

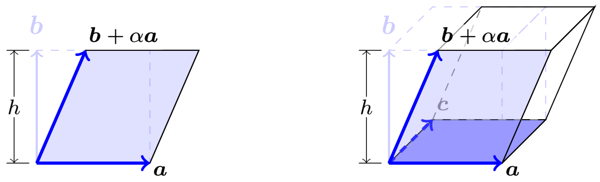
Three properties for the area (volume) function of a parallelepiped

(**Lecture** 14) S-**2** Area or volume

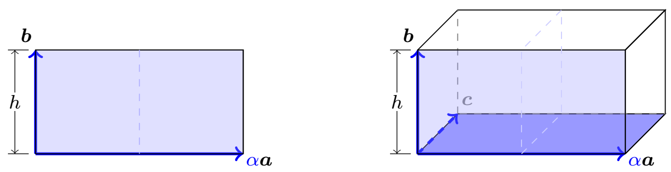
1. $\mathrm{Vol}\big(\underset{n\times n}{\mathbf{I}}\big) = 1.$



2. $\mathrm{Vol}\big(\mathbf{A}\big) \;=\; \mathrm{Vol}\Big(\mathbf{A}_{\substack{\boldsymbol{\tau}\\ [(\alpha)\boldsymbol{k}+\boldsymbol{j}]}}\Big)$ for $j \neq k$.



3. $|\alpha| \cdot \mathrm{Vol}\big(\mathbf{A}\big) \;=\; |\alpha| \cdot \mathrm{Vol}\big[\ldots; \mathbf{A}_{|k}; \ldots\big] \;=\; \mathrm{Vol}\big[\ldots; \alpha\mathbf{A}_{|k}; \ldots\big]$



F2

The first three properties (P-1 to P-3)

Here we define the *Determinant* as:

Definition 1. *The* Determinant *is any function that assigns to each system of n vectors in $\mathbb{R}^n$ (or to each squared matrix of order n) a real number*

$$\det : \quad \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$$

such that it satisfies the following three properties:

(**Lecture** 14) $\boxed{\text{S-3}}$ Determinant: 3 properties that define the function

$\boxed{\text{P-1}}$ **Determinant of identity matrices:** **P-1**

$$\det \underset{n\times n}{\mathbf{I}} = 1$$

$\boxed{\text{P-2}}$ ***Type I*** **elemen. transf. do not change the determinant**: **P-2**

$$\det \mathbf{A} = \det\left(\mathbf{A}_{\underset{[(\alpha)\boldsymbol{k}+\boldsymbol{j}]}{\boldsymbol{\tau}}}\right)$$

$\boxed{\text{P-3}}$ **Multiplying a column by an scalar multiplies the det.** **P-3**

$$\alpha\cdot\det\mathbf{A} \;=\; \det\left[\ldots;\alpha\mathbf{A}_{|k};\ldots\right] \text{ for any } k\in\{1:n\} \text{ and } \alpha\in\mathbb{R}$$

$$\boxed{\text{Absolute value of }\det\mathbf{A} = \operatorname{Vol}\mathbf{A}}$$

F3

Therefore, the absolute value of the determinant is the volume function. We will use two alternative notations to denote the *determinant* a matrix $\mathbf{A}$:

$$\text{determinant of } \mathbf{A} \equiv \det(\mathbf{A}) \equiv |\mathbf{A}|$$

Advertencia: Una barra vertical a cada lado de una matriz $|\mathbf{A}|$ significa determinante de la matriz. Una barra vertical a cada lado de un número $|a|$ significa valor absoluto del número. Es decir, el significado de las barras viene dado por el objeto encerrado: si es un número es el *valor absoluto*, y si es una matriz es el *determinante*. Jugando con esto, podemos decir que

$$\operatorname{Vol}\mathbf{A} = \text{Absolute value of }\det\mathbf{A} = \left|\det\mathbf{A}\right| = \left||\mathbf{A}|\right|.$$

Example 1. Then, we know that in $\mathbb{R}^3$:

$$\begin{vmatrix} a_1 & (b_1+\alpha c_1) & c_1 \\ a_2 & (b_2+\alpha c_2) & c_2 \\ a_3 & (b_3+\alpha c_3) & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix};$$

$$\det\left[\boldsymbol{a};\ (\boldsymbol{b}+\alpha\boldsymbol{c});\ \boldsymbol{c};\right] = \det\left[\boldsymbol{a};\ \boldsymbol{b};\ \boldsymbol{c};\right];$$

and also

$$\begin{vmatrix} a_1 & \alpha b_1 & c_1 \\ a_2 & \alpha b_2 & c_2 \\ a_3 & \alpha b_3 & c_3 \end{vmatrix}; = \alpha \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix};$$

$$\det\left[\boldsymbol{a};\ \alpha\boldsymbol{b};\ \boldsymbol{c};\right] = \alpha \det\left[\boldsymbol{a};\ \boldsymbol{b};\ \boldsymbol{c};\right];$$

The other properties (P-4 to P-11)

Determinant of a matrix with a zero column

(**Lecture** 14) S-4 Determinant of a matrix with a zero column

P-4 **Det. of a matrix $\mathbf{A}$ with a zero column** P-4

If $\mathbf{A}$ has a zero column $\mathbf{0}$, then

$$\boxed{\det(\mathbf{A}) = 0}$$

F5

To solve in classroom

EXERCISE 1. Consider a matrix $\mathbf{A}$ of order n with a zero column $\mathbf{0}$. Prove its determinant is zero:

Elementary transformations by columns.

(**Lecture** 14) S-5 Elementary matrices

We already know

$$\det\left(\mathbf{A}_{\underset{[(\alpha)\boldsymbol{k}+\boldsymbol{j}]}{\boldsymbol{\tau}}}\right) = |\mathbf{A}|; \qquad \det\left(\mathbf{A}_{\underset{[(\alpha)\boldsymbol{k}]}{\boldsymbol{\tau}}}\right) = \alpha|\mathbf{A}|.$$

Determinant of elementary matrices

$$\det\left(\mathbf{I}_{\underset{[(\alpha)\boldsymbol{k}+\boldsymbol{j}]}{\boldsymbol{\tau}}}\right) = 1 \qquad \text{and} \qquad \det\left(\mathbf{I}_{\underset{[(\alpha)\boldsymbol{j}]}{\boldsymbol{\tau}}}\right) = \alpha.$$

Hence, since $\mathbf{A}_{\boldsymbol{\tau}} = \mathbf{A}(\mathbf{I}_{\boldsymbol{\tau}})$, then

$$\left|\mathbf{A}(\mathbf{I}_{\boldsymbol{\tau}})\right| = |\mathbf{A}| \cdot |\mathbf{I}_{\boldsymbol{\tau}}| \tag{1}$$

where $\mathbf{I}_{\boldsymbol{\tau}}$ is an elementary matrix

F6

Sequence of elementary transformations by columns.

EXERCISE 2. Prove the following propositions

(a) $\det(\mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}) = |\mathbf{A}| \cdot |\mathbf{I}_{\boldsymbol{\tau}_1}| \cdots |\mathbf{I}_{\boldsymbol{\tau}_k}|$.

(b) If $\mathbf{B}$ is a full rank matrix, i.e., if $\mathbf{B} = \mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}$, then $|\mathbf{B}| = |\mathbf{I}_{\boldsymbol{\tau}_1}| \cdots |\mathbf{I}_{\boldsymbol{\tau}_k}|$, and therefore $|\mathbf{B}| \neq 0$.

(c) *If $\mathbf{A}$ and $\mathbf{B}$ have order n and $\mathbf{B}$ is full rank, then*

$$\det(\mathbf{AB}) = |\mathbf{A}| \cdot |\mathbf{B}| \tag{2}$$

(**Lecture** 14) S-6 Determinant after a sequence of elementary transformations

Example 2. a sequence $\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k$ of *Type I* elementary transformations does not change the determinant.

$$|\mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}| = |\mathbf{A}(\mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k})| = |\mathbf{A}| \cdot |\mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}| = |\mathbf{A}| \cdot 1 = |\mathbf{A}|$$

Example 3. but a sequence of *Type II* can.

$$\begin{vmatrix} 2a & 3c \\ 2b & 3d \end{vmatrix} = \underline{\ ?\ } \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

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Antisymetric property. Singular matrices. Inverse of a matrix. Determinant of a product of matrices

Permutation of columns

(**Lecture** 14) S-7 Antisymmetric property

P-5 **[Antisymmetric property]** P-5
Column exchange changes the sign of the determinant.

Proof. Column exchange is a sequence of *Type I* transformation and just only one *Type II* transformation that multiplies a column by -1 □

Therefore:

$$\begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix} = (-1) \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

F8

Singular matrices. Inverse of a matrix

Note 1. *By elementary transformations, you can reduced* $\underset{n\times n}{\mathbf{A}}$ *to its reduced echelon form* $\mathbf{R}$.

There are two cases: if the matrix is singular (rank$<n$*) then the determinant is zero; if the matrix is full rank, then* $\mathbf{R} = \mathbf{I}$*; and we only need to take account of the* Type II *elementary operations that we have used in order to get* $\mathbf{I}$ *(the* Type I *ones do not matter!...)*

(**Lecture** 14) S-8 Singular matrices. Inverse of a matrix

P-6 **If $\mathbf{A}$ is singular then $|\mathbf{A}| = 0$.** P-6
P-7 $\det(\mathbf{A}^{-1}) = \big(\det \mathbf{A}\big)^{-1}$. P-7

Proof. Let $\mathbf{A}_{\tau_1\cdots\tau_k} = \underset{n\times n}{\mathbf{R}}$ be a reduced equelon form (and $\mathbf{E} = \mathbf{I}_{\tau_1\cdots\tau_k}$).

$$\text{Since } \mathbf{AE} = \mathbf{R}, \text{ then:} \quad |\mathbf{A}|\cdot|\mathbf{E}| = |\mathbf{R}|; \text{ with only two cases:}$$

$$\begin{cases} \mathbf{A} \text{ singular } (\mathbf{R}_{|n} = \mathbf{0}\ ): & |\mathbf{A}|\cdot|\mathbf{E}| = 0 \ \Rightarrow\ |\mathbf{A}| = 0 \\[2ex] \mathbf{A} \text{ not singular } (\mathbf{R} = \mathbf{I}): & |\mathbf{A}|\cdot|\mathbf{E}| = 1 \ \Rightarrow\ |\mathbf{E}| = |\mathbf{A}^{-1}| = \big(|\mathbf{A}|\big)^{-1} \end{cases}.$$

□

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Determinant of the inverse of a matrix

We can calculate the determinat of $\mathbf{A}^{-1}$ by gaussian elimination. If $\mathbf{R} = \mathbf{I}$ then $|\mathbf{A}|\cdot|\mathbf{E}| = |\mathbf{A}|\cdot|\mathbf{A}^{-1}| = |\mathbf{I}| = 1$ and therefore

$$\boxed{|\mathbf{A}^{-1}| = \big(|\mathbf{A}|\big)^{-1}.}$$

Example 4. For $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$:

$$\left[\begin{array}{cc}1&2\\2&2\\\hline 1&0\\0&1\end{array}\right] \xrightarrow[Type I]{\underset{[(-2)\mathbf{1}+\mathbf{2}]}{\tau}} \left[\begin{array}{cc}1&0\\2&-2\\\hline 1&-2\\0&1\end{array}\right] \xrightarrow[Type II]{\underset{[(-1/2)\mathbf{2}]}{\tau}} \left[\begin{array}{cc}1&0\\2&1\\\hline 1&1\\0&-1/2\end{array}\right] \xrightarrow[Type I]{\underset{[(-2)\mathbf{2}+\mathbf{1}]}{\tau}} \left[\begin{array}{cc}1&0\\0&1\\\hline -1&1\\1&-1/2\end{array}\right]$$

So

$$\left|\mathbf{A}^{-1}\right| = \left|\mathbf{I}_{\underset{[(-2)\mathbf{1}+\mathbf{2}]}{\tau}}\right| \cdot \left|\mathbf{I}_{\underset{[(-1/2)\mathbf{2}]}{\tau}}\right| \cdot \left|\mathbf{I}_{\underset{[(-2)\mathbf{2}+\mathbf{1}]}{\tau}}\right| = 1 \cdot \frac{-1}{2} \cdot 1 = \frac{-1}{2};$$

that is $\boxed{|\mathbf{A}| = -2.}$

Determinant of matrix multiplication.

Remember that for any $\mathbf{B}$ of order n, there is $\mathbf{E} = \mathbf{I}_{\tau_1 \cdots \tau_k}$ (full rank) such that $\mathbf{BE} = \mathbf{L}$ is an echelon form of $\mathbf{B}$. If $\mathbf{B}$ is singular then $\mathbf{L}_{|n} = \mathbf{0}$. Hence, when we apply the same elementary transformations on $\mathbf{AB}$ we get

$$\mathbf{ABE}_{|n} = \mathbf{AL}_{|n} = \mathbf{0};$$

since $\mathbf{E}_{|n} \neq \mathbf{0}$ then $(\mathbf{AB})$ is singular .

(**Lecture** 14) S-9 Determinant of a product

P-8 **[Determinant of a product of matrices]** **P-8**

$$\boxed{\det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B}).} \tag{3}$$

$$\begin{cases} \mathbf{B} \text{ singular, then so it is } \mathbf{AB} \Rightarrow & \det(\mathbf{AB}) = 0 = \det(\mathbf{A}) \cdot \det(\mathbf{B}) \\ \mathbf{B} = \mathbf{I}_{\tau_1 \cdots \tau_k} \Rightarrow & \det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B}) \end{cases}$$

F10

Determinant of transposed matrix.

To solve in classroom

EXERCISE 3. **[Transposed matrices]**

(a) What is the relation between the determinant of an elementary matrix $\mathbf{I}_{\tau}$ and the determinant of its transpose ${}_{\tau}\mathbf{I}$?

(b) Consider $\mathbf{B}$, a full rank matrix, proof that $|\mathbf{B}| = |\mathbf{B}^{\mathsf{T}}|$.

(**Lecture** 14) S-10 Determinant of a transpose

P-9 **Determinant of a transpose** P-9

$$|\mathbf{A}| = |\mathbf{A}^\intercal|.$$

Proof.

$$\begin{cases} \text{if } \mathbf{A} \text{ singular:} & \mathbf{A}^\intercal \text{ singular } \Rightarrow \det \mathbf{A}^\intercal = \det \mathbf{A} = 0 \\ \text{if } \mathbf{A} \text{ NO singular:} & \mathbf{A} = \mathbf{I}_{\tau_1 \cdots \tau_k} \Rightarrow \det \mathbf{A}^\intercal = \det \mathbf{A} \end{cases}.$$

□

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The lecture ends here

Questions of the Lecture 14

(L-14) Question 1. Complete the proofs of this lecture.

(L-14) Question 2. Knowing that $|\mathbf{BC}| = |\mathbf{B}||\mathbf{C}|$; prove that for any invertible matrix $\mathbf{A}$ (so $\det \mathbf{A} \neq 0$)

$$\det(\mathbf{A}^{-1}) = \Big(\det(\mathbf{A})\Big)^{-1}.$$

(L-14) Question 3. Consider $\underset{3\times3}{\mathbf{A}}$ and $\underset{3\times3}{\mathbf{B}}$ such that $\det(\mathbf{A}) = 2$ and $\det(\mathbf{B}) = -2$

(a) (0.5^{pts}) Compute the determinants of $\mathbf{A}(\mathbf{B})^2$ and $(\mathbf{AB})^{-1}$
(b) (0.5^{pts}) Is it possible to compute the rank of $\mathbf{A}+\mathbf{B}$? and the rank of $\mathbf{AB}$?

(L-14) Question 4. Use the Gauss-Jordan method to compute the determinant

(a) $\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\mathbf{A}_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

(c) $\mathbf{A}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(L-14) Question 5. The 3 by 3 matrix $\mathbf{A}$ reduces to the identity matrix $\mathbf{I}$ by the following three column operations (in order):

$\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\tau}$: Subtract 4 times column 1 from column 2.

$\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\tau}$: Subtract 3 times column 1 from column 3.

$\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\tau}$: Subtract column 3 from column 2.

Find the determinant of $\mathbf{A}$.

(L-14) Question 6.

(a) Find the determinant of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(b) Find the determinant of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & d \end{bmatrix}$ **using Gauss-Jordan**.

End of Questions of the Lecture 14

LECTURE 15: Determinant Formulas and Cofactors

$\boxed{\textit{Lecture 15}}$

(**Lecture** 15) S-1 Highlights of Lesson 15

Highlights of *Lesson 15*

- Computing $|\mathbf{A}|$ by gaussian elimination
- **P-10** — Multilinear property
- Expansion of det $\mathbf{A}$ in Cofactors (Laplace expansion).
- Application of determinants
  - Cramer's rule for solving linear equations
  - Computing the inverse of $\mathbf{A}$

F12

"Pinche aquí" y vea el notebook de Jupyter de la Lección 15

(**Lecture** 15) S-2 Extended matrix

$$\text{Extended matrix of } \mathbf{B} : \begin{bmatrix} \mathbf{B} & \\ & 1 \end{bmatrix}$$

1. Given $\boldsymbol{\tau}$: $\begin{bmatrix} \mathbf{B}_{\boldsymbol{\tau}} & \\ & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \\ & 1 \end{bmatrix}_{\boldsymbol{\tau}}$.

2. Since $\begin{bmatrix} \mathbf{I} & \\ & 1 \end{bmatrix}_{\boldsymbol{\tau}}$ and $\mathbf{I}_{\boldsymbol{\tau}}$ same *type* Elem. Mat. $\Rightarrow$ same det.

Applying 1. k times, and then 2.

$$\left|\begin{bmatrix} \mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k} & \\ & 1 \end{bmatrix}\right| = \left|\begin{bmatrix} \mathbf{I} & \\ & 1 \end{bmatrix}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right| = \left|\begin{bmatrix} \mathbf{I} & \\ & 1 \end{bmatrix}_{\boldsymbol{\tau}_1}\cdots\begin{bmatrix} \mathbf{I} & \\ & 1 \end{bmatrix}_{\boldsymbol{\tau}_k}\right| = \left|\mathbf{I}_{\boldsymbol{\tau}_1}\right|\cdots\left|\mathbf{I}_{\boldsymbol{\tau}_k}\right| = \left|\mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right|.$$

$$\text{If } \mathbf{A} \text{ is the extended matrix of } \mathbf{B} \begin{cases} \text{If } \mathbf{B} \text{ singular} & |\mathbf{B}| = 0 = |\mathbf{A}| \\ \text{If } \mathbf{B} \text{ invertible} & |\mathbf{B}| = |\mathbf{A}| \end{cases}$$

F13

Triangular matrices

EXERCISE 4. [**Triangular matrices**]
(a) Find the determinant of a full rank lower triangular matrix $\mathbf{L}$
(b) Find the determinant of a triangular matrix with a zero entry in the main diagonal
(c) Find the determinant of an upper triangular matrix $\mathbf{U}$

$$\text{In addition} \quad \begin{vmatrix} \mathbf{A} & \underset{m\times n}{\mathbf{0}} \\ \underset{n\times m}{\mathbf{C}} & \mathbf{B} \end{vmatrix} = \left|\mathbf{A}\right| \cdot \left|\mathbf{B}\right|.$$

Cálculo del determinante por eliminación Gaussiana

(**Lecture** 15) S-3 Computing by Gaussian elimination

Example 5. $\mathbf{A} = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$: $\left[\begin{array}{cc|c} 1 & 5 & \\ 2 & 3 & \\ \hline & & 1 \end{array}\right] \xrightarrow{\overset{\boldsymbol{\tau}}{[(-5)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc|c} 1 & 0 & \\ 2 & -7 & \\ \hline & & 1 \end{array}\right]$ $\boxed{|\mathbf{A}| = -7}$

Example 6.

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 9 & 6 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\overset{\boldsymbol{\tau}}{[(\frac{1}{2})\mathbf{4}]}]{\overset{\boldsymbol{\tau}}{\substack{[(2)\mathbf{3}] \\ [(-1)\mathbf{2}+\mathbf{3}]}}} \left[\begin{array}{ccc|c} 0 & 2 & 0 & 0 \\ 9 & 6 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{2} \end{array}\right] \xrightarrow[\overset{\boldsymbol{\tau}}{[(-1)\mathbf{4}]}]{\overset{\boldsymbol{\tau}}{[\mathbf{1}\rightleftharpoons\mathbf{2}]}} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 6 & 9 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & -\frac{1}{2} \end{array}\right]$$

$$\begin{vmatrix} 0 & 2 & 1 \\ 9 & 6 & 3 \\ 0 & 1 & 1 \end{vmatrix} = -9,$$

F14

```
A =  Matrix([ [0,2,1], [9,6,3], [0,1,1] ])
Determinante(A,1)
```

Easy formulas for matrices of order less than 4

Matrices of order 1, $\mathbf{A} = \begin{bmatrix} a \end{bmatrix}$:

$$\left[\begin{array}{c|c} a & 0 \\ \hline 0 & 1 \end{array}\right] \Rightarrow |\mathbf{A}| = a.$$

Matrices of order 2:

$$\left[\begin{array}{cc|c} a & b & 0 \\ c & d & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow{\overset{\boldsymbol{\tau}}{[(-\frac{b}{a})\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc|c} a & 0 & 0 \\ c & d-\frac{bc}{a} & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

$$|\mathbf{A}| = ad - bc = a\det\begin{bmatrix} d \end{bmatrix} - b\det\begin{bmatrix} c \end{bmatrix}.$$

Matrices of order 3:

$$\left[\begin{array}{ccc|c} a & b & c & 0 \\ d & e & f & 0 \\ g & h & i & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\overset{\boldsymbol{\tau}}{\substack{[(-\frac{b}{a})\mathbf{1}+\mathbf{2}] \\ [(-\frac{c}{a})\mathbf{1}+\mathbf{3}]}}} \left[\begin{array}{ccc|c} a & 0 & 0 & 0 \\ d & e-\frac{bd}{a} & f-\frac{cd}{a} & 0 \\ g & h-\frac{bg}{a} & i-\frac{cg}{a} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\overset{\boldsymbol{\tau}}{[(\frac{-af+cd}{ae-bd})\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc|c} a & 0 & 0 & 0 \\ d & e-\frac{bd}{a} & 0 & 0 \\ g & h-\frac{bg}{a} & \frac{aei-afh-bdi+bfg+cdh-ceg}{ae-bd} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$|\mathbf{A}| = \underbrace{aei - afh - bdi + bfg + cdh - ceg}_{\text{(Rule of Sarrus)}} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

Matrices of order 4:

$$\left[\begin{array}{cccc|c} a & b & c & d & 0 \\ e & f & g & h & 0 \\ i & j & k & l & 0 \\ m & n & o & p & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\overset{\boldsymbol{\tau}}{\substack{[(-\frac{b}{a})\mathbf{1}+\mathbf{2}] \\ [(-\frac{c}{a})\mathbf{1}+\mathbf{3}] \\ [(-\frac{d}{a})\mathbf{1}+\mathbf{4}] \\ [(\frac{-ag+ce}{af-be})\mathbf{2}+\mathbf{3}] \\ [(\frac{-ah+de}{af-be})\mathbf{2}+\mathbf{4}] \\ [(\frac{-afl+ahj+bel-bhi-dej+dfi}{afk-agj-bek+bgi+cej-cfi})\mathbf{3}+\mathbf{4}]}}}$$

$$\left[\begin{array}{cccc|c} a & 0 & 0 & 0 & 0 \\ e & f-\frac{be}{a} & 0 & 0 & 0 \\ i & j-\frac{bi}{a} & \frac{afk-agj-bek+bgi+cej-cfi}{af-be} & 0 & 0 \\ m & n-\frac{bm}{a} & \frac{afo-agn-beo+bgm+cen-cfm}{af-be} & p+\frac{\left(-l+\frac{\left(h-\frac{de}{a}\right)\left(j-\frac{bi}{a}\right)}{f-\frac{be}{a}}+\frac{di}{a}\right)\left(-o+\frac{\left(g-\frac{ce}{a}\right)\left(n-\frac{bm}{a}\right)}{f-\frac{be}{a}}+\frac{cm}{a}\right)}{-k+\frac{\left(g-\frac{ce}{a}\right)\left(j-\frac{bi}{a}\right)}{f-\frac{be}{a}}+\frac{ci}{a}} - \frac{\left(h-\frac{de}{a}\right)\left(n-\frac{bm}{a}\right)}{f-\frac{be}{a}} - \frac{dm}{a} & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$|\mathbf{A}| = afkp - aflo - agjp + agln + ahjo - ahkn - bekp + belo + bgip - bglm - bhio + bhkm + cejp - celn - cfip + cflm + chin - chjm - dejo + dekn + dfio - dfkm - dgin + dgjm = a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c\begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$

Expansion by cofactors (Laplace expansion).

Multilinear property

(**Lecture** 15) S-4 Multilinear property

P-10 **Multilinear property** P-10

$$\det\left[\dots;(\beta\boldsymbol{b}+\psi\boldsymbol{c});\dots\right] = \beta\det\left[\dots;\boldsymbol{b};\dots\right] + \psi\det\left[\dots;\boldsymbol{c};\dots\right]$$

Example 7. Then, in the 2 dimensional case $\mathbb{R}^2$

$$\begin{vmatrix} a+\alpha & c \\ b+\beta & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} + \begin{vmatrix} \alpha & c \\ \beta & d \end{vmatrix};$$

therefore

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = \begin{vmatrix} a & c \\ 0 & d \end{vmatrix} + \begin{vmatrix} & c \\ & d \end{vmatrix}.$$

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Proof. In the book (`https://mbujosab.github.io/CursoDeAlgebraLineal/libro.pdf`) □

Minors and cofactors

New notation and definition for minors and cofactors

Consider $\boldsymbol{q} = (q_1, \dots q_n,)$ in $\mathbb{R}^n$, if we remove the jth element, q_j, we will denote the new vector in $\mathbb{R}^{(n-1)}$ as

$$\boldsymbol{q}^{\urcorner j} = \left(q_1, \dots,\ q_{j-1},\quad q_{j+1},\ \dots, q_n\right) \in \mathbb{R}^{n-1}.$$

In the same way, if $\mathbf{A}$ is of order m by n, we denote the submatrix that results from removing the i-th row with ${}^{i\urcorner}\mathbf{A}$, and the submatrix that results from removing the j-th column with $\mathbf{A}^{\urcorner j}$. So, ${}^{i\urcorner}\mathbf{A}^{\urcorner j}$, is the $m-1$ by $n-1$ submatrix that results from removing the i-th row and the j-th column.

(**Lecture** 15) S-5 minors and cofactors

Definition 2 (minors and cofactors)**.** *We denote a submatrix of* $\mathbf{A}$ *obtained by deleting row* i *and column* j *of* $\mathbf{A}$ *by*

$${}^{i\urcorner}\mathbf{A}^{\urcorner j};$$

Its determinant is called the minor *of* a_{ij}*. And*

$$\operatorname{cof}_{ij}(\mathbf{A}) = (-1)^{i+j}\det\left({}^{i\urcorner}\mathbf{A}^{\urcorner j}\right)$$

is called the cofactor *of* a_{ij}*.*

F18

Example 8. For $\quad \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad$ we have

$$ {}^{1\urcorner}\mathbf{A}^{\ulcorner 2} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}, \qquad {}^{3\urcorner}\mathbf{A}^{\ulcorner 3} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} $$

hence

$$ \operatorname{cof}_{12}(\mathbf{A}) = (-1)^{1+2} \det\left({}^{1\urcorner}\mathbf{A}^{\ulcorner 2}\right) = (-1) \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}. $$

and

$$ \operatorname{cof}_{33}(\mathbf{A}) = (-1)^{3+3} \det\left({}^{3\urcorner}\mathbf{A}^{\ulcorner 3}\right) = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}. $$

Expansion by cofactors (Laplace expansion).

(**Lecture** 15) S-6 Expansion by cofactors

Theorem 0.1 ([**Laplace expansion**]). *For* $\mathbf{A}$ *n by n,* $\det(\mathbf{A})$ *may be computed as the sum of the products of the elements of any column (row) of* $\mathbf{A}$ *by their cofactors:*

$$ \det(\mathbf{A}) = \sum_{i=1}^{n} a_{ij} \operatorname{cof}_{ij}(\mathbf{A}), \qquad \textit{the expansion by the } j\textit{th column} $$

or

$$ \det(\mathbf{A}) = \sum_{j=1}^{n} a_{ij} \operatorname{cof}_{ij}(\mathbf{A}), \qquad \textit{the expansion by the } i\textit{th row} $$

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Proof. In the book (`https://mbujosab.github.io/CursoDeAlgebraLineal/libro.pdf`) □

To solve in classroom

EXERCISE 5. Compute $\det \mathbf{A} = \begin{vmatrix} 2 & 0 & 3 & 2 \\ 5 & 1 & 2 & 4 \\ 3 & 0 & 1 & 2 \\ 5 & 3 & 2 & 1 \end{vmatrix}$

First aplications

Solving linear equations. Cramer's Rule

(**Lecture** 15) S-7 Cramer's Rule

$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b}; \qquad |\mathbf{A}| \neq 0 \quad \text{then}$$

$$\boldsymbol{b} = (\mathbf{A}_{|1})x_1 + \cdots + (\mathbf{A}_{|j})x_j + \cdots + (\mathbf{A}_{|n})x_n.$$

$$\det\Big[\mathbf{A}_{|1};\ \ldots\ \overbrace{\boldsymbol{b}}^{\text{pos. } j};\ \ldots\ \mathbf{A}_{|n}\Big] = x_j \cdot \det(\mathbf{A}).$$

$$x_j = \frac{\det\Big[\mathbf{A}_{|1};\ \ldots\ \overbrace{\boldsymbol{b}}^{\text{pos. } j};\ \ldots\ \mathbf{A}_{|n}\Big]}{\det(\mathbf{A})}.$$

Computational issues when $\det \mathbf{A} \simeq 0$ (tiny angle between vectors) F21

Second aplication

Computing the inverse of $\mathbf{A}$

Definition 3. *For* $\underset{n\times n}{\mathbf{A}}$ *the matrix* $\mathbf{Adj}(\mathbf{A})$, *the adjoint of* $\mathbf{A}$, *is defined to be the transpose of the matrix obtained from* $\mathbf{A}$ *by replacing each element by its cofactor. That is,*

$$\mathbf{Adj}(\mathbf{A}) = \begin{bmatrix} \operatorname{cof}_{11}(\mathbf{A}) & \operatorname{cof}_{12}(\mathbf{A}) & \cdots & \operatorname{cof}_{1n}(\mathbf{A}) \\ \operatorname{cof}_{21}(\mathbf{A}) & \operatorname{cof}_{22}(\mathbf{A}) & \cdots & \operatorname{cof}_{2n}(\mathbf{A}) \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{cof}_{n1}(\mathbf{A}) & \operatorname{cof}_{n2}(\mathbf{A}) & \cdots & \operatorname{cof}_{nn}(\mathbf{A}) \end{bmatrix}^{\intercal}$$

What will we get if we multiply the adjoint of $\mathbf{A}$ by $\mathbf{A}$?

(**Lecture** 15) S-8 The inverse of a matrix

$[\mathbf{Adj(A)}] \cdot \mathbf{A} =$

$$\begin{bmatrix} \operatorname{cof}_{11}(\mathbf{A}) & \operatorname{cof}_{21}(\mathbf{A}) & \cdots & \operatorname{cof}_{n1}(\mathbf{A}) \\ \operatorname{cof}_{12}(\mathbf{A}) & \operatorname{cof}_{22}(\mathbf{A}) & \cdots & \operatorname{cof}_{n2}(\mathbf{A}) \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{cof}_{1n}(\mathbf{A}) & \operatorname{cof}_{2n}(\mathbf{A}) & \cdots & \operatorname{cof}_{nn}(\mathbf{A}) \end{bmatrix} \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{\mathbf{A}}$$

F22

The first element on the diagonal is the expansion of $|\mathbf{A}|$ along the first column of $\mathbf{A}$. The second element on the diagonal is the expansion of $|\mathbf{A}|$ by the second column of $\mathbf{A}$, etc.

All elements outside the diagonal are determinant of matrices with two equal columns. For example, the second element in the first row of $[\mathbf{Adj(A)}] \cdot \mathbf{A}$ is the expansion along the first column of

$$\begin{vmatrix} a_{12} & a_{12} & \dots & a_{1n} \\ a_{22} & a_{22} & \dots & a_{2n} \\ a_{32} & a_{32} & \dots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{n2} & a_{n2} & \dots & a_{nn} \end{vmatrix} = a_{12}\operatorname{cof}_{11}(\mathbf{A}) + a_{22}\operatorname{cof}_{21}(\mathbf{A}) + a_{32}\operatorname{cof}_{31}(\mathbf{A}) + \cdots + a_{n2}\operatorname{cof}_{n1}(\mathbf{A}) = \sum_{i=1}^{n} a_{i2}\operatorname{cof}_{i1}(\mathbf{A}) = 0,$$

where we can find the second column twice (as a first and second columns), thus the determinant is zero.

And k-th component ($k \neq 1$) from the first row is

$$\begin{vmatrix} a_{1k} & a_{12} & \dots & a_{1n} \\ a_{2k} & a_{22} & \dots & a_{2n} \\ a_{3k} & a_{32} & \dots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{nk} & a_{n2} & \dots & a_{nn} \end{vmatrix} = a_{1k}\operatorname{cof}_{11}(\mathbf{A}) + a_{2k}\operatorname{cof}_{21}(\mathbf{A}) + a_{3k}\operatorname{cof}_{31}(\mathbf{A}) + \cdots + a_{nk}\operatorname{cof}_{n1}(\mathbf{A}) = \sum_{i=1}^{n} a_{ik}\operatorname{cof}_{i1}(\mathbf{A}) = 0,$$

where we can find the k-th column twice (as a first and k-th columns), thus the determinant is zero.

Therefore $[\mathbf{Adj(A)}] \cdot \mathbf{A} = |\mathbf{A}| \cdot \mathbf{I}$; and then:

$$\left[\frac{\mathbf{Adj(A)}}{|\mathbf{A}|}\right] \cdot \mathbf{A} = \mathbf{I};$$

where the first matrix (in brackets) is the inverse of $\mathbf{A}$

The lecture ends here

Questions of the Lecture 15

(L-15) QUESTION 1. Complete the proofs of the exercises of this lecture.

(L-15) QUESTION 2. Consider $\mathbf{A} = \left[\mathbf{A}_{|1}; \quad \mathbf{A}_{|2}; \quad \mathbf{A}_{|3};\right]$ with $\det \mathbf{A} = 2$.

(a) What are $\det(2\mathbf{A})$ and $\det \mathbf{A}^{-1}$?
(b) What is $\det\left[(3\mathbf{A}_{|1} + 2\mathbf{A}_{|2}); \quad \mathbf{A}_{|3}; \quad \mathbf{A}_{|2};\right]$

(L-15) QUESTION 3. The determinant of the 1000 by 1000 matrix $\mathbf{A}$ is 12. What is the determinant of $-\mathbf{A}^\intercal$? (Careful: No credit for the wrong sign.)
(*MIT Course 18.06 Quiz 2, Fall, 2008*)

(L-15) QUESTION 4. Consider the squared matrix $\mathbf{A}$. True or false? (to receive full credit you must explain your answer in a clear and concise way)
$|\mathbf{A}\mathbf{A}^\intercal| = |\mathbf{A}|^2$.

(L-15) QUESTION 5. We have a 3×3 matrix $\mathbf{A} = \begin{bmatrix} a & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$ with $\det \mathbf{A} = 3$. Compute the determinant of the following matrices:

(a) (0.5 pts) $\begin{bmatrix} a-2 & 1 & 2 \\ b-4 & 3 & 4 \\ c-6 & 5 & 6 \end{bmatrix}$
(b) (0.5 pts) $\begin{bmatrix} 7a & 7 & 14 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$
(c) (1 pts) $(2\mathbf{A})^{-1}\mathbf{A}^\intercal$
(d) (0.5 pts) $\begin{bmatrix} a-2 & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$

(L-15) QUESTION 6.

(a) Escalone la matriz $\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 5 & 2 \\ 4 & 6 & 0 \end{bmatrix}$.
(b) ¿Es $\mathbf{A}$ invertible?
(c) En caso afirmativo calcule $|\mathbf{A}^{-1}|$; en caso contrario calcule $|\mathbf{A}|$
(d) La matriz $\mathbf{C}$ es igual al producto de $\mathbf{A}$ con la *traspuesta* de la matriz $\mathbf{B}$, es decir

$$\mathbf{C} = \mathbf{A}\mathbf{B}^\intercal \qquad \text{donde} \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

¿Cuánto vale el determinante de $\mathbf{C}$? ¿Es $\mathbf{C}$ invertible?

(L-15) QUESTION 7. What is the determinant of the following matrices using Laplace expansions.

(a) $\begin{bmatrix} 1 & 2 \\ -4 & 3 \end{bmatrix}$
(b) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & -2 \end{bmatrix}$
(c) $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 1 & -2 \end{bmatrix}$

(L-15) Question 8. Compute the following determinant using Laplace expansions:

$$\begin{vmatrix} 0 & 0 & 0 & 3 & 0 \\ -2 & 0 & 0 & 2 & 0 \\ 8 & -1 & 0 & -7 & 2 \\ -1 & 2 & 2 & 3 & 2 \\ 2 & 2 & 3 & 6 & 4 \end{vmatrix}$$

(L-15) Question 9. Compute $\det \mathbf{A} = \begin{vmatrix} 2 & 2 & 0 & 0 \\ 5 & 5 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{vmatrix}$

(L-15) Question 10. Compute the value of $\det \mathbf{A}$ using Laplace expansion

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 3 & \cdots & 3 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & n \end{bmatrix}$$

(L-15) Question 11. Consider a n by n matrix $\mathbf{A}_n$ full of 3s in its diagonal, and twos just below the diagonal, and another 2 at the position $(1, n)$; for example, for $n = 4$:

$$\mathbf{A}_4 = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 2 & 3 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix}.$$

(a) Find, using the cofactors of the first row, the determinant of $\mathbf{A}_4$.
(b) Find the determinant of $\mathbf{A}_n$ for $n > 4$.

(L-15) Question 12. Consider the following block matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}$$

Prove $|\mathbf{A}| = |\mathbf{B}||\mathbf{C}|$.

Hint. $\begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}$

(L-15) Question 13. Solve the following linear systems using Cramer's Rule

(a) $\begin{cases} 2x + 5y = 1 \\ x + 4y = 2 \end{cases}.$

(b) $\begin{cases} 2x + y = 1 \\ x + 2y + z = 0 \\ y + 2z = 0 \end{cases}.$

(exercise 13 from section 4.4 of Strang (2006))

(L-15) Question 14. Find the inverse of the following matrices using the *adjoint matrix*

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

(exercise 18 from section 4.4 of Strang (2006))

(L-15) QUESTION 15. Consider the matrices $\mathbf{A} = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 2 & 3 & 3 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & a \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$; and the vector $\boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

(a) (0.5^{pts}) For wich values of a the matrix $\mathbf{A}$ is invertible?
(b) (1^{pts}) Consider $a = 5$. Using the Cramer's rule, compute the fourth coordinate x_4 of $\boldsymbol{x}$ for linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.
(c) (1^{pts}) Compute $\mathbf{B}^{-1}$. Use the matrix $\mathbf{B}^{-1}$ to solve $\mathbf{B}\boldsymbol{x} = \boldsymbol{b}$.

End of Questions of the Lecture 15

References

Strang, G. (2006). *Linear algebra and its applications*. Thomson Learning, Inc., fourth ed. ISBN 0-03-010567-6.

Solutions

Exercise 1. The zero vector $\mathbf{0}$ in $\mathbb{R}^n$ is a multiple of any other vector $\boldsymbol{x}$ in $\mathbb{R}^n$ since $\mathbf{0} = 0 \cdot \boldsymbol{x}$; it follows, by Property P-3, that $\det \mathbf{A} = 0$:

$$\det \left[\mathbf{A}_{|1} \ldots; \mathbf{0}; \ldots; \mathbf{A}_{|n}\right] = \det \left[\mathbf{A}_{|1} \ldots; 0\boldsymbol{x}; \ldots; \mathbf{A}_{|n}\right] = 0 \cdot \det \left[\mathbf{A}_{|1} \ldots; \boldsymbol{x}; \ldots; \mathbf{A}_{|n}\right] = 0$$

$\square$

Exercise 2(a) Since $\mathbf{A}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k} = \mathbf{A}(\mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k}) = \mathbf{A}(\mathbf{I}_{\boldsymbol{\tau}_1}) \cdots (\mathbf{I}_{\boldsymbol{\tau}_k})$, applying repeatedly (1) we get

$$\begin{aligned}
\det(\mathbf{A}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k}) &= \left|\mathbf{A}(\mathbf{I}_{\boldsymbol{\tau}_1}) \cdots (\mathbf{I}_{\boldsymbol{\tau}_k})\right| \\
&= \left|\mathbf{A}(\mathbf{I}_{\boldsymbol{\tau}_1}) \cdots (\mathbf{I}_{\boldsymbol{\tau}_{(k-1)}})\right| \cdot |\mathbf{I}_{\boldsymbol{\tau}_k}| \\
&= \left|\mathbf{A}(\mathbf{I}_{\boldsymbol{\tau}_1}) \cdots (\mathbf{I}_{\boldsymbol{\tau}_{(k-2)}})\right| \cdot |\mathbf{I}_{\boldsymbol{\tau}_{(k-1)}}| \cdot |\mathbf{I}_{\boldsymbol{\tau}_k}| \\
&\ \vdots \\
&= |\mathbf{A}| \cdot |\mathbf{I}_{\boldsymbol{\tau}_1}| \cdots |\mathbf{I}_{\boldsymbol{\tau}_k}|.
\end{aligned}$$

$\square$

Exercise 2(b)

$$|\mathbf{B}| = \det(\mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k}) = |\mathbf{I}_{\boldsymbol{\tau}_1}| \cdots |\mathbf{I}_{\boldsymbol{\tau}_k}|.$$

and, since the determinants of elementary matrices are not zero $|\mathbf{B}| \neq 0$.

$\square$

Exercise 2(c) If $\mathbf{B}$ is full rank then it is the product of k elementary matrices $\mathbf{B} = \mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k}$. Hence

$$\det(\mathbf{AB}) = \det(\mathbf{A}\mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k}) = \det(\mathbf{A}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k}) = |\mathbf{A}| \cdot \left(|\mathbf{I}_{\boldsymbol{\tau}_1}| \cdots |\mathbf{I}_{\boldsymbol{\tau}_k}|\right) = |\mathbf{A}| \cdot |\mathbf{B}|.$$

$\square$

Exercise 3(a) Since they are elementary matrices of the same type, $\det(\mathbf{I}_{\boldsymbol{\tau}}) = \det({}_{\boldsymbol{\tau}}\mathbf{I})$.

$\square$

Exercise 3(b) Since $\mathbf{B} = \mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k} = (\mathbf{I}_{\boldsymbol{\tau}_1}) \cdots (\mathbf{I}_{\boldsymbol{\tau}_k})$, the determinant is

$$|\mathbf{B}| = \det\left(\mathbf{I}_{\boldsymbol{\tau}_1}\right) \cdots \det\left(\mathbf{I}_{\boldsymbol{\tau}_k}\right) = \prod_{i=1}^{k} \det\left(\mathbf{I}_{\boldsymbol{\tau}_i}\right).$$

But we also know that $\mathbf{B}^{\mathsf{T}} = {}_{\boldsymbol{\tau}_k \cdots \boldsymbol{\tau}_1}\mathbf{I} = \left({}_{\boldsymbol{\tau}_k}\mathbf{I}\right) \cdots \left({}_{\boldsymbol{\tau}_1}\mathbf{I}\right)$, and then its determinant is

$$|\mathbf{B}^{\mathsf{T}}| = \prod_{i=1}^{k} \det\left({}_{\boldsymbol{\tau}_i}\mathbf{I}\right) = \prod_{i=1}^{k} \det\left(\mathbf{I}_{\boldsymbol{\tau}_i}\right) = |\mathbf{B}|.$$

$\square$

(L-14) Question 2. Since $\mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$, we know that

$$1 = |\mathbf{I}| = |\mathbf{A}||\mathbf{A}^{-1}|;$$

and therefore $\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$.

$\square$

(L-14) Question 3(a) $\left|\mathbf{A}(\mathbf{B})^2\right| = 2 \cdot (-2)^2 = 8$.
$\left|(\mathbf{AB})^{-1}\right| = \left(|\mathbf{AB}|\right)^{-1} = \frac{1}{-4}$.

$\square$

(L-14) Question 3(b) There is no enough information to compute the determinant of $\mathbf{A}+\mathbf{B}$. "Not enough information" means we can find two pairs of examples $(\mathbf{A}_1,\mathbf{B}_1)$ and $(\mathbf{A}_1,\mathbf{B}_1)$ that satisfies the hypothesis: $\det \mathbf{A}_1 = 2 = \det \mathbf{B}_1$ and $\det \mathbf{A}_2 = 2 = \det \mathbf{B}_2$; but $\operatorname{rg}\left(\mathbf{A}_1+\mathbf{B}_1\right) \neq \operatorname{rg}\left(\mathbf{A}_2+\mathbf{B}_2\right)$.

On the other hand, since $|\mathbf{AB}| = -4 \neq 0$, we know $\underset{3\times 3}{\mathbf{AB}}$ is a full rank matrix; therefore its rank is 3.

□

(L-14) Question 4(a)

$$[\mathbf{A}_1] = \begin{bmatrix}1&0&0\\1&1&1\\0&0&1\end{bmatrix} \xrightarrow[{[(-1)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix} \xrightarrow[{[(-1)\mathbf{2}+\mathbf{1}]}]{\boldsymbol{\tau}} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = [\mathbf{I}] \quad \Rightarrow \quad \det \mathbf{A}_1 = -1$$

```
A =  Matrix([ [1,0,0], [1,1,1], [0,0,1] ])
Determinante(A)     # esta es una opción
A.determinante()    # esta es otra opción
```

□

(L-14) Question 4(b)

$$\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix} \xrightarrow[{\substack{[(2)\mathbf{2}]\\ [(1)\mathbf{1}+\mathbf{2}]}}]{\boldsymbol{\tau}} \begin{bmatrix}2&0&0\\-1&3&-1\\0&-2&2\end{bmatrix} \xrightarrow[{\substack{[(3)\mathbf{3}]\\ [(1)\mathbf{2}+\mathbf{3}]}}]{\boldsymbol{\tau}} \begin{bmatrix}2&0&0\\-1&3&0\\0&-2&4\end{bmatrix} \xrightarrow[{\substack{[(3)\mathbf{1}]\\ [(1)\mathbf{2}+\mathbf{1}]}}]{\boldsymbol{\tau}} \begin{bmatrix}6&0&0\\0&3&0\\-2&-2&4\end{bmatrix}$$

$$\xrightarrow[{\substack{[(2)\mathbf{1}]\\ [(1)\mathbf{3}+\mathbf{1}]\\ [(2)\mathbf{2}]\\ [(1)\mathbf{3}+\mathbf{2}]}}]{\boldsymbol{\tau}} \begin{bmatrix}12&0&0\\0&6&0\\0&0&4\end{bmatrix} \xrightarrow[{\substack{[(\frac{1}{12})\mathbf{1}]\\ [(\frac{1}{6})\mathbf{2}]\\ [(\frac{1}{4})\mathbf{3}]}}]{\boldsymbol{\tau}} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Therefore $\det \mathbf{A}_2 = \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot 12\cdot 6\cdot 4 = 4$

□

(L-14) Question 4(c)

$$\mathbf{A}_3 = \begin{bmatrix}0&0&1\\0&1&1\\1&1&1\end{bmatrix} \xrightarrow[{[(-1)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \begin{bmatrix}0&0&1\\0&1&0\\1&1&0\end{bmatrix} \xrightarrow[{[(-1)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} \xrightarrow[{[\mathbf{1}\rightleftharpoons\mathbf{3}]}]{\boldsymbol{\tau}} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = [\mathbf{I}] \quad \Rightarrow \quad \det \mathbf{A}_3 = -1$$

□

(L-14) Question 5. Since all applied trasformations are *Type I*, then $\mathbf{AE} = \mathbf{I} \Rightarrow \det(\mathbf{A})\cdot 1 = 1$.

□

(L-14) Question 6(a) The first one is an elementary matrix, its determinant is 1.

The second one is a permutation matrix that exchanges two vectors, its determinant is -1.

□

(L-14) Question 6(b) $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\a&b&c&d\end{bmatrix} \xrightarrow[{[(\frac{1}{d})\mathbf{4}]}]{\boldsymbol{\tau}} \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\a&b&c&1\end{bmatrix} \xrightarrow[{\substack{[(-c)\mathbf{4}+\mathbf{3}]\\ [(-b)\mathbf{4}+\mathbf{2}]\\ [(-a)\mathbf{4}+\mathbf{1}]}}]{\boldsymbol{\tau}} \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$ Hence, the determinant is d.

□

Exercise 4(a) Como la matriz de orden n es de rango completo, los n elementos de la diagonal principal son pivotes (i.e., distintos de cero).

$$\mathbf{L} = \begin{bmatrix}*_1 & & & \\ \vdots & *_2 & & \\ \vdots & \vdots & \ddots & \\ \vdots & \vdots & \vdots & *_n\end{bmatrix} \qquad \text{donde } *_j \text{ son números distintos de cero.}$$

Dividiendo cada columna j-ésima por su pivote $*_j$ para normalizar los pivotes (y compensando dichas transformaciones multiplicado la última fila por cada pivote); y aplicando, en una segunda fase, la eliminación de izquierda a derecha con transformaciones de Tipo I para anular todo lo que queda a la izquierda de los pivotes (ahora basta multiplicar la última fila por 1), llegamos a:

$$\left[\begin{array}{cccc|c} *_1 & & & & \\ \vdots & *_2 & & & \\ \vdots & \vdots & \ddots & & \\ \vdots & \vdots & \vdots & *_n & \\ \hline & & & & 1 \end{array}\right] \xrightarrow[\substack{\boldsymbol{\tau} \\ [(*_1)(\mathbf{n+1})] \\ \vdots \\ [(*_n)(\mathbf{n+1})]}]{\substack{\boldsymbol{\tau} \\ [(\frac{1}{*_1})\mathbf{1}] \\ \vdots \\ [(\frac{1}{*_n})\mathbf{n}]}} \left[\begin{array}{cccc|c} 1 & & & & \\ \vdots & 1 & & & \\ \vdots & \vdots & \ddots & & \\ \vdots & \vdots & \vdots & 1 & \\ \hline & & & & *_1 \cdot *_2 \cdots *_n \end{array}\right] \xrightarrow[\substack{\boldsymbol{\tau} \\ [(1)(\mathbf{n+1})]}]{\substack{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_q \\ \text{(de Tipo I)}}} \left[\begin{array}{cccc|c} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ \hline & & & & *_1 \cdot *_2 \cdots *_n \end{array}\right]$$

por tanto, si la matriz es triangular inferior es de rango completo, su determinante es igual al producto de sus pivotes; es decir, al producto de los elementos de la diagonal.

$$\det(\mathbf{L}) = \text{ producto de los elementos de la diagonal}$$

$\square$

Exercise 4(b) Una matriz de orden n y triangular solo puede ser de rango completo si los n elementos de la diagonal son distintos de cero. Por tanto, si la matriz tringular es singular, necesariamente tiene algún cero en su diagonal principal. Como su determinante es cero, por ser singular, su determinatne es igual al producto de los elementos de la diagonal (donde uno de ellos es cero).

$\square$

Exercise 4(c)

$$\det(\mathbf{U}) = \det\left(\mathbf{U}^{\intercal}\right) = \text{ producto de los elementos de la diagonal}$$

por ser $\mathbf{U}^{\intercal}$ triangular inferior.

$\square$

Exercise 5. Expanding by the second column we get

$$\det \mathbf{A} \quad = \quad -0\begin{vmatrix}5 & 2 & 4\\3 & 1 & 2\\5 & 2 & 1\end{vmatrix} + \begin{vmatrix}2 & 3 & 2\\3 & 1 & 2\\5 & 2 & 1\end{vmatrix} - 0\begin{vmatrix}2 & 3 & 2\\5 & 2 & 4\\5 & 2 & 1\end{vmatrix} + 3\begin{vmatrix}2 & 3 & 2\\5 & 2 & 4\\3 & 1 & 2\end{vmatrix} \quad = \quad 0 \;+\; 17 \;-\; 0 \;+\; 3 \;\times\; 4 \quad = \quad 29$$

$\square$

(L-15) Question 2(a) $\det\big(2 \underset{3\times 3}{\mathbf{A}}\big) = 2^3 \cdot \det \mathbf{A} = 16.$ $\qquad \det \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} = 1/2.$

$\square$

(L-15) Question 2(b)

$$\det\left[(3\mathbf{A}_{|1} + 2\mathbf{A}_{|2});\quad \mathbf{A}_{|3};\quad \mathbf{A}_{|2};\right] \quad = \quad \det\left[3\mathbf{A}_{|1};\quad \mathbf{A}_{|3};\quad \mathbf{A}_{|2};\right] \quad = \quad 3\det\left[\mathbf{A}_{|1};\quad \mathbf{A}_{|3};\quad \mathbf{A}_{|2};\right] \quad = \quad -3\det \mathbf{A} \quad = \quad -6.$$

$\square$

(L-15) Question 3.

$$\det(-\mathbf{A}^{\intercal}) = \det(-\mathbf{A}) = (-1)^n\mathbf{A} = \det(\mathbf{A})$$

since n is an even number.

$\square$

(L-15) Question 4. True, since

$$|\mathbf{A}\mathbf{A}^{\intercal}| = |\mathbf{A}||\mathbf{A}^{\intercal}| = |\mathbf{A}||\mathbf{A}| = |\mathbf{A}|^2.$$

$\square$

(L-15) Question 5(a)

$$\begin{vmatrix}a-2 & 1 & 2\\b-4 & 3 & 4\\c-6 & 5 & 6\end{vmatrix} = \begin{vmatrix}a & 1 & 2\\b & 3 & 4\\c & 5 & 6\end{vmatrix} = 3$$

□

(L-15) Question 5(b)

$$\begin{vmatrix} 7a & 7 & 14 \\ b & 3 & 4 \\ c & 5 & 6 \end{vmatrix} = 7 \begin{vmatrix} a & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{vmatrix} = 7 \times 3 = 21$$

□

(L-15) Question 5(c)

$$\left|(2\mathbf{A})^{-1}\mathbf{A}^{\intercal}\right| = \frac{1}{\det 2\mathbf{A}} \det \mathbf{A} = \frac{1}{2^3 \det \mathbf{A}} \det \mathbf{A} = \frac{1}{8}.$$

□

(L-15) Question 5(d)

$$\begin{vmatrix} a-2 & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{vmatrix} = \begin{vmatrix} a & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{vmatrix} - \begin{vmatrix} 2 & 1 & 2 \\ 0 & 3 & 4 \\ 0 & 5 & 6 \end{vmatrix} = 3 + 4 = 7$$

□

(L-15) Question 6(a)

$$\left[\begin{array}{c} \mathbf{A} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{ccc} 2 & 4 & 4 \\ 3 & 5 & 6 \\ 1 & 2 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc} 2 & 0 & 0 \\ 3 & -1 & 0 \\ 1 & 0 & -2 \\ \hline 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c} \mathbf{L} \\ \hline \mathbf{E} \end{array}\right]$$

□

(L-15) Question 6(b) Puesto que $\mathbf{L}$ tiene tres pivotes, $\mathbf{A}$ es invertible.

□

(L-15) Question 6(c)

$$|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|} = \frac{1}{\text{producto de los pivotes de } \mathbf{L}} = \frac{1}{2(-1)(-2)} = \frac{1}{4}.$$

□

(L-15) Question 6(d)

$$|\mathbf{C}| = |\mathbf{A}\mathbf{B}^{\intercal}| = |\mathbf{A}||\mathbf{B}^{\intercal}| = 4 \cdot 0 = 0;$$

ya que $\mathbf{B}$ tiene dos filas iguales. Por tanto $\mathbf{C}$ no es invertible.

□

(L-15) Question 7(a)

$$\begin{vmatrix} 1 & 2 \\ -4 & 3 \end{vmatrix} = 1 \cdot 3 - (-4) \cdot 2 = 11$$

□

(L-15) Question 7(b) Expanding by the first row

$$\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & -2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} - (-1) \begin{vmatrix} 0 & 2 \\ 2 & -2 \end{vmatrix} + 0 = -8$$

□

(L-15) Question 7(c) Expanding by the second column we get a minor equal to the determinant in the previous exercise:

$$\begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 1 & -2 \end{vmatrix} = -0 + 1 \cdot \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & -2 \end{vmatrix} - 0 + 0 = -8$$

□

(L-15) Question 8.

$$\begin{vmatrix} 0 & 0 & 0 & 3 & 0 \\ -2 & 0 & 0 & 2 & 0 \\ 8 & -1 & 0 & -7 & 2 \\ -1 & 2 & 2 & 3 & 2 \\ 2 & 2 & 3 & 6 & 4 \end{vmatrix} = -3 \begin{vmatrix} -2 & 0 & 0 & 0 \\ 8 & -1 & 0 & 2 \\ -1 & 2 & 2 & 2 \\ 2 & 2 & 3 & 4 \end{vmatrix} = (-3)\cdot(-2)\begin{vmatrix} -1 & 0 & 2 \\ 2 & 2 & 2 \\ 2 & 3 & 4 \end{vmatrix} = (-3)\cdot(-2)\cdot(2) = 12$$

□

(L-15) Question 9.

$$\det \mathbf{A} = \begin{vmatrix} 2 & 2 & 0 & 0 \\ 5 & 5 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 2 & 0 \\ 5 & 5 & 0 \\ 3 & 0 & 1 \end{vmatrix} = 1 \cdot 1 \cdot \begin{vmatrix} 2 & 2 \\ 5 & 5 \end{vmatrix} = 1 \cdot 1 \cdot 0 = 0.$$

Note that the first column is a linear combination of the others.

□

(L-15) Question 10. Expanding by the first column

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 3 & \cdots & 3 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & n \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 2 & \cdots & 2 \\ 0 & 3 & \cdots & 3 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & n \end{vmatrix} - 0 + 0 - \cdots 0$$

expanding by the first column again

$$= 1 \cdot 2 \cdot \begin{vmatrix} 3 & \cdots & 3 \\ \vdots & \ddots & \\ 0 & \cdots & n \end{vmatrix} - 0 + 0 - \cdots 0$$

and again... and again...

$$1 \cdot 2 \cdot 3 \cdots (n-2) \cdot \begin{vmatrix} (n-1) & (n-1) \\ 0 & n \end{vmatrix} = n!.$$

□

(L-15) Question 11(a)

$$|\mathbf{A}_4| = \begin{vmatrix} 3 & 0 & 0 & 2 \\ 2 & 3 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 3 \end{vmatrix} = 3 \begin{vmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 2 & 3 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{vmatrix} = 3 \cdot 27 - 2 \cdot 8 = 65$$

□

(L-15) Question 11(b) In general $|\mathbf{A}_n| = 3^n + (-1)^{n-1}2^n$.

□

(L-15) Question 12. Expanding by the last column the first matrix, and expanding by the first column the second matrix, and repeating the expansions with the minors (as in the previous exercise), we get

$$\begin{vmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{vmatrix} = |\mathbf{B}|$$

$$\begin{vmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{vmatrix} = |\mathbf{C}|.$$

Therefore

$$\det \mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} = \begin{vmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{vmatrix} \begin{vmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{vmatrix} = \det \mathbf{B} \det \mathbf{C}.$$

□

(L-15) Question 13(a) On the one hand,

$$\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}; \qquad \boldsymbol{b} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \end{bmatrix},$$

on the other hand,

$$\det(\mathbf{A}) = 3; \qquad \begin{vmatrix} \mathbf{1} & 5 \\ \mathbf{2} & 4 \end{vmatrix} = -6; \qquad \begin{vmatrix} 2 & \mathbf{1} \\ 1 & \mathbf{2} \end{vmatrix} = 3.$$

Therefore

$$x = \frac{\begin{vmatrix} \mathbf{1} & 5 \\ \mathbf{2} & 4 \end{vmatrix}}{\det(\mathbf{A})} = \frac{-6}{3} = -2; \qquad y = \frac{\begin{vmatrix} 2 & \mathbf{1} \\ 1 & \mathbf{2} \end{vmatrix}}{\det(\mathbf{A})} = \frac{3}{3} = 1.$$

□

(L-15) Question 13(b) On the one hand,

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}; \qquad \boldsymbol{b} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

on the other hand,

$$\det(\mathbf{A}) = 4; \qquad \begin{vmatrix} \mathbf{1} & 1 & 0 \\ \mathbf{0} & 2 & 1 \\ \mathbf{0} & 1 & 2 \end{vmatrix} = 3; \qquad \begin{vmatrix} 2 & \mathbf{1} & 0 \\ 1 & \mathbf{0} & 1 \\ 0 & \mathbf{0} & 2 \end{vmatrix} = -2; \qquad \begin{vmatrix} 2 & 1 & \mathbf{1} \\ 1 & 2 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \end{vmatrix} = 1.$$

Therefore, $x = \frac{3}{4}$; $\qquad y = \frac{-2}{4} = \frac{-1}{2}$; $\qquad z = \frac{1}{4}$.

□

(L-15) Question 14(a) $\quad \mathbf{Adj(A)} = \begin{bmatrix} 3 & -2 & 0 \\ -0 & 1 & -0 \\ 0 & -4 & 3 \end{bmatrix}; \qquad \det(\mathbf{A}) = 3;$

$$\mathbf{A}^{-1} = \frac{\mathbf{Adj(A)}}{|\mathbf{A}|} = \frac{1}{3} \begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 3 \end{bmatrix}.$$

□

(L-15) Question 14(b) $\quad \mathbf{Adj(B)} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}; \qquad \det(\mathbf{B}) = 4$

$$\mathbf{B}^{-1} = \frac{\mathbf{Adj(B)}}{|\mathbf{B}|} = \frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Note the inverse of a symmetric matrix is also symmetric.

□

(L-15) Question 15(a)

$$\begin{bmatrix} 1 & 4 & 2 & 3 \\ 2 & 3 & 3 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & a \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & -5 & -1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & a \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & -5 & -1 & 1 \\ 0 & 2 & 0 & a \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & 16 \\ 0 & 0 & 0 & a-6 \end{bmatrix}$$

In order to have a full rank matrix, the parameter a must be different from 6.

$\square$

(L-15) Question 15(b) On the one hand

$$\det \mathbf{A} = \begin{vmatrix} 1 & 4 & 2 & 3 \\ 2 & 3 & 3 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 5 \end{vmatrix} = 1 \begin{vmatrix} 3 & 3 & 7 \\ 1 & 0 & 3 \\ 2 & 0 & 5 \end{vmatrix} - 2 \begin{vmatrix} 4 & 2 & 3 \\ 1 & 0 & 3 \\ 2 & 0 & 5 \end{vmatrix} = -3 \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} - 2 \cdot (-2) \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = -1.$$

On the other hand

$$\begin{vmatrix} 1 & 4 & 2 & 1 \\ 2 & 3 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{vmatrix} = -1 \cdot \begin{vmatrix} 2 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{vmatrix} = -2 \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} = 0$$

Consequently, $x_4 = \frac{0}{-1} = 0$.

$\square$

(L-15) Question 15(c)

$$\left[\begin{array}{cccc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{4}]}]{\substack{\tau \\ [(-1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ \hline 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[\mathbf{3} \rightleftharpoons \mathbf{4}]}]{\substack{\tau \\ [\mathbf{1} \rightleftharpoons \mathbf{2}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ \hline 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[{[(-1)\mathbf{4}]}]{\substack{\tau \\ [(1)\mathbf{4}+\mathbf{1}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Hence $\mathbf{B}^{-1} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ and thus, multiplying by $\mathbf{B}^{-1}$ we get $\mathbf{B}\boldsymbol{x} = \boldsymbol{b} \Rightarrow \mathbf{B}^{-1}\mathbf{B}\boldsymbol{x} = \mathbf{B}^{-1}\boldsymbol{b} \Rightarrow \boldsymbol{x} = \mathbf{B}^{-1}\boldsymbol{b}$:

$$\boldsymbol{x} = \mathbf{B}^{-1}\boldsymbol{b} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

$\square$