

Mathematics II

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15/01/2025

1 / 44

Always squared matrices in this topic

Highlights of Lesson 16

- **Eigenvalues, eigenvectors** (prefix eigen is the German word for innate, distinct, self)
- $|\mathbf{A} - \lambda \mathbf{I}| = 0$ *Characteristic equation*
- $\text{tr}(\mathbf{A})$, $\det \mathbf{A}$ (demo in the next lesson)

2 / 44

You can find the last version of these course materials at

<https://github.com/mbujosab/MatematicasII/tree/main/Eng>

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1 / 44

Consider the equation

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \quad (\text{with } \mathbf{x} \neq \mathbf{0})$$

- *Eigenvalue* is any λ such that there are solutions.
- Such *non-null* solutions \mathbf{x} are called *eigenvectors*.
 $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{A}\mathbf{x}$ is *multiple* \mathbf{x}

When λ is 0, What are the eigenvectors?

3 / 44

3 Example: projection matrix

- Orthogonal projection
- Which vectors are eigenvectors?
What vectors are projected in the same starting direction?
- What are the eigenvalues of those eigenvectors?
- are there any other eigenvectors? with what eigenvalue?
- Two eigen-spaces

4 / 44

4 Another example: Interchange or swap matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- A vector that does not change after interchange?
- What is the eigenvalue?
- Is there an eigenvector corresponding to $\lambda_2 = -1$?

$$\mathbf{A}\mathbf{x}_2 = -\mathbf{x}_2$$

$$\text{Note: } \text{tr}(\mathbf{A}) = 0 = \lambda_1 + \lambda_2; \quad \det \mathbf{A} = -1 = \lambda_1 \cdot \lambda_2.$$

5 / 44

5 how to find eigenvalues and eigenvectors?

How to solve

$$\mathbf{A}\mathbf{x} = \underbrace{\lambda}_{?} \underbrace{\mathbf{x}}_{?}$$

Here's the trick (simple idea). Bring the \mathbf{x} s onto the same side ...

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} =$$

idea If $\mathbf{x} \neq \mathbf{0}$ what kind of matrix must be $(\mathbf{A} - \lambda \mathbf{I})$?

and then its determinant must be? $|\mathbf{A} - \lambda \mathbf{I}| =$

6 / 44

6 how to find eigenvalues and eigenvectors?

1. Eigenvalues are λ 's such that: $|\mathbf{A} - \lambda \mathbf{I}| =$
(Characteristic polynomial $P_{\mathbf{A}}(\lambda)$)
2. How to compute \mathbf{x} so that $(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$?

Eigenspace (Set of eigenvectors + $\mathbf{0}$):

$$\mathcal{E}_{\lambda}(\mathbf{A}) = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \lambda \mathbf{x} \right\}$$

Spectrum: set $\{\lambda_1, \dots, \lambda_k\}$ of eigenvalues (roots of $P_{\mathbf{A}}(\lambda)$)

7 / 44

7 Example (we must compute the eigenvalues first!)

We are looking for a null determinant (Characteristic polynomial)

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}; \quad \det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{vmatrix} = (3 - \lambda)^2 - 1 = 0$$

Note: $\text{tr}(\mathbf{A}) = 6 = \lambda_1 + \lambda_2$; $\det \mathbf{A} = 8 = \lambda_1 \cdot \lambda_2$.

8 / 44

8 Example (...and then the eigenspaces)

And now we compute the null space $\mathcal{N}(\mathbf{A} - \lambda \mathbf{I})$... for each λ .

For $\lambda_1 = 4$

$$(\mathbf{A} - 4\mathbf{I}) = \begin{bmatrix} 3-4 & 1 \\ 1 & 3-4 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow$$

For $\lambda_2 = 2$

$$(\mathbf{A} - 2\mathbf{I}) = \begin{bmatrix} 3-2 & 1 \\ 1 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow$$

Are they the only two eigenvectors?

$$\mathbf{A}\mathbf{x}_i = \lambda\mathbf{x}_i; \quad \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \mathbf{x}_i = \lambda\mathbf{x}_i.$$

9 / 44

9 Another example: 90° rotation matrix

$$\mathbf{Q} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- How much do the eigenvalues add up to?
- What is the determinant?

Difficulties

$$\lambda_1 + \lambda_2 = 0 \quad \text{and} \quad \lambda_1 \cdot \lambda_2 = 1 \quad (+) \cdot (-) = (+)?$$

What kind of vector can be parallel to itself after a 90° rotation?

$$\det(\mathbf{Q} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 =$$

10 / 44

10 There are even worse examples

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

- Eigenvalues

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 3 - \lambda & 1 \\ 0 & 3 - \lambda \end{vmatrix} = (3 - \lambda)(3 - \lambda) = 0 \quad \begin{cases} \lambda_1 = 3 \\ \lambda_2 = 3 \end{cases}$$

- Eigenvectors

- for λ_1 : $(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}_1$; $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- for λ_2 :

$\lambda = 3$ is repeated twice, but $\dim \mathcal{E}_3(\mathbf{A}) = 1$

$$\mu(3) = 2 \neq 1 = \gamma(3)$$

11 / 44

Summary:

1. The eigenvalues are those numbers λ that makes the matrix $(\mathbf{A} - \lambda\mathbf{I})$ singular. In other words, they are the roots of the Characteristic polynomial: $\det(\mathbf{A} - \lambda\mathbf{I})$.
2. Any n by n matrix has a characteristic polynomial of degree n
3. A polynomial of degree n has n roots (perhaps some repeated roots).
4. The sum of eigenvalues of a matrix equals its trace
5. The product of eigenvalues of a matrix equals its determinant
6. The eigenvectors associated with λ are the **non-zero** vectors in $\mathcal{N}(\mathbf{A} - \lambda\mathbf{I})$.

12 / 44

Questions of the Lecture 16

(L-16) QUESTION 1. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} -3 & 4 & -4 \\ -3 & 5 & -3 \\ -1 & 2 & 0 \end{bmatrix}$$

(a) The three eigenvalues of \mathbf{A} are -1 , 1 and 2 ; and two of its eigenvectors are

$$\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}; \quad \mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Check that both vectors are eigenvectors of \mathbf{A} . What are the corresponding eigenvalues?

(b) Find a third linearly independent eigenvector.

(L-16) QUESTION 2. Find the eigenvalues and eigenvectors of

(a)

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

12 / 44

(b)

$$\mathbf{B} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

(Strang, 2006, exercise 12 from section 5.1.)

(L-16) QUESTION 3. If \mathbf{B} has eigenvalues 1, 2, 3, \mathbf{C} has eigenvalues 4, 5, 6, and \mathbf{D} has eigenvalues 7, 8, 9, what are the eigenvalues of the 6 by 6 matrix $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}$? where \mathbf{B} , \mathbf{C} , \mathbf{D} are upper triangular matrices.
(Strang, 2006, exercise 13 from section 5.1.)

(L-16) QUESTION 4. Find the eigenvalues and eigenvectors of

(a)

$$\mathbf{A} = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

(b)

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

(Strang, 2006, exercise 5 from section 5.1.)

12 / 44

(L-16) QUESTION 5. The eigenvalues of \mathbf{A} equal the eigenvalues of \mathbf{A}^T . This is because $\det(\mathbf{A} - \lambda\mathbf{I})$ equals $\det(\mathbf{A}^T - \lambda\mathbf{I})$.

(a) That is true because _____

(b) Show by an example that, nevertheless, the eigenvectors of \mathbf{A} and \mathbf{A}^T are not the same.

(Strang, 2006, exercise 11 from section 5.1.)

(L-16) QUESTION 6. Consider the matrix \mathbf{B} and its eigenvector \mathbf{x} associated to the eigenvalue λ , that is $\mathbf{B}\mathbf{x} = \lambda\mathbf{x}$; and also consider the matrix $\mathbf{A} = (\mathbf{B} + \alpha\mathbf{I})$. Prove that \mathbf{x} is also an eigenvector of \mathbf{A} with eigenvalue $(\lambda + \alpha)$.

(L-16) QUESTION 7.

(a) Encuentre los autovalores y los auto-vectores de la matriz $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$.

Compruebe que la traza es igual a la suma de los autovalores, y que el determinante es igual a su producto.

(b) Si consideramos una nueva matriz, generada a partir de la anterior como

$$\mathbf{B} = (\mathbf{A} - 7\mathbf{I}) = \begin{bmatrix} -6 & -1 \\ 2 & -3 \end{bmatrix}.$$

¿Cuáles son los autovalores y auto-vectores de la nueva matriz, y como están relacionados con los de \mathbf{A} ?

(Strang, 2006, exercise 1 and 3 from section 5.1.)

12 / 44

(L-16) QUESTION 8. Suponga que λ es un auto-valor de \mathbf{A} , y que \mathbf{x} es un auto-vector tal que $\mathbf{Ax} = \lambda\mathbf{x}$.

- (a) Demuestre que ese mismo \mathbf{x} es un auto-vector de $\mathbf{B} = \mathbf{A} - 7\mathbf{I}$, y encuentre el correspondiente auto-valor de \mathbf{B} .
- (b) Suponga que $\lambda \neq 0$ (y que \mathbf{A} es invertible), demuestre que \mathbf{x} también es un auto-vector de \mathbf{A}^{-1} , y encuentre el correspondiente auto-valor. ¿Qué relación tiene con λ ?

(Strang, 2006, exercise 7 from section 5.1.)

(L-16) QUESTION 9. Suponga que \mathbf{A} es una matriz de dimensiones $n \times n$, y que $\mathbf{A}^2 = \mathbf{A}$. ¿Qué posibles valores pueden tomar los autovalores de \mathbf{A} ?

(L-16) QUESTION 10. Suponga la matriz \mathbf{A} con autovalores 1, 2 y 3. Si \mathbf{v}_1 es un auto-vector asociado al auto-valor 1, \mathbf{v}_2 al auto-valor 2 y \mathbf{v}_3 al auto-valor 3; entonces ¿cuanto es $\mathbf{A}(\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3)$?

(L-16) QUESTION 11. Proporcione un ejemplo que muestre que los auto-valores pueden cambiar cuando un múltiplo de una columna se resta de otra. ¿Por qué los pasos de eliminación no modifican los autovalores nulos?

(Strang, 2006, exercise 6 from section 5.1.)

(L-16) QUESTION 12. El polinomio característico de una matriz \mathbf{A} se puede factorizar como

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda).$$

Demuestre, partiendo de esta factorización, que el determinante de \mathbf{A} es igual al producto de sus valores propios (autovalores). Para ello haga una elección inteligente del valor de λ .

(Strang, 2006, exercise 8 from section 5.1.)

(L-16) QUESTION 13. Calcule los valores característicos (autovalores o valores propios) y los vectores característicos de \mathbf{A} y \mathbf{A}^2 :

$$\mathbf{A} = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{y} \quad \mathbf{A}^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}$$

\mathbf{A}^2 tiene los mismos _____ que \mathbf{A} . Cuando los autovalores de \mathbf{A} son λ_1 y λ_2 , los autovalores de \mathbf{A}^2 son _____.

(Strang, 2006, exercise 22 from section 5.1.)

(L-16) QUESTION 14. Suponga que los valores característicos de \mathbf{A} son 1, 2 y 4, ¿cuál es la traza de \mathbf{A}^2 ? ¿Cuál es el determinante de $(\mathbf{A}^{-1})^T$?

(Strang, 2006, exercise 10 from section 5.2.)

(L-16) QUESTION 15. The equation $(\mathbf{A}^2 - 4\mathbf{I})\mathbf{x} = \mathbf{b}$ has no solution for some right-hand side \mathbf{b} . Give as much information as possible about the eigenvalues of the matrix \mathbf{A} (the matrix \mathbf{A} is diagonalizable).

(L-16) QUESTION 16. You are given the matrix

$$\mathbf{A} = \begin{bmatrix} 0.5 & 0.2 & 0.2 \\ 0.1 & 0.5 & 0.5 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$$

One of the eigenvalues is $\lambda = 1$. What are the eigenvalues of \mathbf{A} ? [Hint: Very little calculation required! You should be able to see another eigenvalue by inspection of the form of \mathbf{A} , and the third by an easy calculation. You shouldn't need to compute $\det(\mathbf{A} - \lambda\mathbf{I})$ unless you really want to do it the hard way.]

1 Highlights of Lesson 17

Highlights of Lesson 17

- Similar matrices: $\mathbf{C} = \mathbf{S}^{-1}\mathbf{AS}$
- Triangular block diagonalizing a matrix

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{I} \end{bmatrix} \xrightarrow[\text{esp}(\tau_p^{-1} \dots \tau_1^{-1})]{\tau_1 \dots \tau_p} \begin{bmatrix} \mathbf{C} \\ \mathbf{S} \end{bmatrix} \quad \text{where} \quad \mathbf{S} = \mathbf{I}_{\tau_1 \dots \tau_p}.$$

- Diagonalizable matrices: when \mathbf{C} is diagonal.

2 Similar matrices

Similarity

A and **C** are *similar* if there is an invertible **S** such that

$$\mathbf{C} = \mathbf{S}^{-1} \mathbf{A} \mathbf{S}$$

If **A** and **C** are similar (see demos in the book):

- The same determinant: $\det \mathbf{A} = \det \mathbf{C}$
- The same characteristic polynomial: $|\mathbf{A} - \lambda \mathbf{I}| = |\mathbf{C} - \lambda \mathbf{I}|$
- The same eigenvalues (same *algebraic* and *geometric* multiplicities).
- The same trace.

Mirror inverse transf.: $(\mathbf{I}_{(\tau_1 \dots \tau_k)})^{-1} = \text{esp}(\tau_k^{-1} \dots \tau_1^{-1}) \mathbf{I}$

$$\mathbf{I} = \begin{bmatrix} \mathbf{I} & \\ & \mathbf{I} \end{bmatrix} \xrightarrow{[(\alpha_j)j+i]} \begin{bmatrix} \mathbf{I} & \\ & \mathbf{I} \end{bmatrix} \xrightarrow{[(\frac{1}{\alpha_j})j]} \Rightarrow \mathbf{A} \text{ similar to } \text{esp}(\tau_1 \dots \tau_k)^{-1} \mathbf{A}_{\tau_1 \dots \tau_k}$$

14 / 44

3 Block diagonalizing a matrix (toothed matrix)

Consider $\mathbf{A} = \left[\begin{array}{c|c} \mathbf{C} & \\ \hline * & \mathbf{L} \end{array} \right] \in \mathbb{C}^{n \times n}$ where

C (of order m) is *singular* and **L** is *full rank lower triangular*; then there exists an invertible **R** such that

$$\mathbf{R}^{-1} \mathbf{A} \mathbf{R} = \left[\begin{array}{c|c} \begin{array}{c} * \\ \vdots \\ 0 \end{array} & \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \\ \hline * & \begin{array}{c} d_{m+1} \\ d_{m+2} \\ \vdots \\ d_n \end{array} \end{array} \right] \begin{array}{c} \beta_{m+1} \\ * \beta_{m+2} \\ * \quad * \quad \ddots \\ * \quad * \quad \dots \beta_n \end{array}$$

$$(\dots [(-\alpha_j)_{m+j}] \dots) \mathbf{A} (\dots [(\alpha_j)_{j+m}] \dots); \quad j = 1, \dots, m-1.$$

15 / 44

4 Block diagonalizing a matrix (toothed matrix)

Consider $\mathbf{A} = \left[\begin{array}{c|c} \mathbf{C} & \\ \hline * & \mathbf{L} \end{array} \right] \in \mathbb{C}^{n \times n}$ where

C (of order m) is *singular* and **L** is *full rank lower triangular*, then there exists **S** = **RP** (invertible) such that

$$\mathbf{P}^{-1} \mathbf{R}^{-1} \mathbf{A} \mathbf{R} \mathbf{P} = \left[\begin{array}{c|c} \begin{array}{c} * \\ \vdots \\ 0 \end{array} & \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \\ \hline * & \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \end{array} \end{array} \right] \begin{array}{c} \beta_{m+1} \\ * \beta_{m+2} \\ * \quad * \quad \ddots \\ * \quad * \quad \dots \beta_n \end{array}$$

$$(\dots [(-\alpha_j)_{m+j}] \dots) \mathbf{R}^{-1} \mathbf{A} \mathbf{R} (\dots [(\alpha_j)_{j+m}] \dots); \quad j = m+1, \dots, n.$$

16 / 44

5 A very simple example

Example

Consider $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ with eigenvalues 0, 1 and 1.

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{I} \end{bmatrix} \xrightarrow{\begin{smallmatrix} (-) \\ \text{OI} \end{smallmatrix}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\begin{smallmatrix} \tau \\ [(1)1+2] \\ [(2)3+2] \\ [2 \Rightarrow 3] \end{smallmatrix}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\begin{smallmatrix} \tau \\ [2 \Rightarrow 3] \\ [(-2)2+3] \\ [(-1)2+1] \end{smallmatrix}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\begin{smallmatrix} (+) \\ \text{OI} \end{smallmatrix}} \begin{bmatrix} \mathbf{C} \\ \mathbf{S} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{diagonal}}$$

17 / 44

6 A not so simple example

Example

Consider $\mathbf{A} = \begin{bmatrix} -2 & 0 & 3 \\ 3 & -2 & -9 \\ -1 & 2 & 6 \end{bmatrix}$ with eigenvalues 1, 1 and 0.

$$\begin{array}{c}
 \xrightarrow[11]{(-)} \begin{bmatrix} -3 & 0 & 3 \\ 3 & -3 & -9 \\ -1 & 2 & 5 \end{bmatrix} \xrightarrow[\tau]{[(1)1+3], [(-2)2+3]} \begin{bmatrix} -3 & 0 & 0 \\ 3 & -3 & 0 \\ -1 & 2 & 0 \end{bmatrix} \xrightarrow[\tau]{[(2)3+2], [(-1)3+1]} \begin{bmatrix} -2 & -2 & 0 \\ 1 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix} \xrightarrow[11]{(+)} \begin{bmatrix} -2 & -2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \\
 \xrightarrow[11]{(-)} \begin{bmatrix} -2 & -2 & 0 \\ 1 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix} \xrightarrow[\tau]{[(-1)1+2]} \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 3 & 0 \end{bmatrix} \xrightarrow[\tau]{[(1)2+1]} \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow[11]{(+)} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \\
 \xrightarrow[01]{(-)} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \xrightarrow[\tau]{[(-1)2+1], [(4)3+1]} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow[\tau]{[(-4)1+3], [(1)1+2]} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow[01]{(+)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & -1 & 1 \\ -9 & 1 & -2 \\ 4 & 0 & 1 \end{bmatrix}
 \end{array}$$

18 / 44

7 Every matrix is similar to a toothed matrix

For every \mathbf{A} there exists \mathbf{S} such that

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{C} \Rightarrow \mathbf{A}\mathbf{S} = \mathbf{S}\mathbf{C}$$

where \mathbf{C} , toothed, has the eigenvalues on the diagonal

Example

$$\begin{bmatrix} 6 & -1 & 1 \\ -9 & 1 & -2 \\ 4 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -2 & 0 & 3 \\ 3 & -2 & -9 \\ -1 & 2 & 6 \end{bmatrix} \begin{bmatrix} 6 & -1 & 1 \\ -9 & 1 & -2 \\ 4 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}}_{\text{toothed}}$$

Consequences

- $\sum \lambda_i = \text{tr}(\mathbf{A})$ and $\prod \lambda_i = \det \mathbf{A}$
- $\mathbf{A}\mathbf{S}_{|j} = \mathbf{S}\mathbf{C}_{|j} \Rightarrow$ for j such that $\mathbf{C}_{|j} = \lambda_i \mathbf{I}_{|j}$:
 $\mathbf{A}(\mathbf{S}_{|j}) = \lambda_i(\mathbf{S}_{|j}) \Rightarrow \mathbf{S}_{|j}$ is an eigenvector.

19 / 44

8 Back to the simple, "toothless" example

Consider $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ with eigenvalues 0, 1 and 1.

$$\begin{array}{c}
 \xrightarrow[01]{(-)} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow[\tau]{[(1)1+2], [(2)3+2], [2 \Rightarrow 3]} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[\tau]{[2 \Rightarrow 3], [(-2)2+3], [(-1)2+1]} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[01]{(+)} \begin{bmatrix} \mathbf{C} \\ \mathbf{S} \end{bmatrix}
 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}(\mathbf{S}_{|j}) = \lambda_i(\mathbf{S}_{|j}) \Rightarrow \mathbf{S}_{|j} \text{ is an eigenvector.}$$

20 / 44

9 Diagonalizable matrices

- A matrix is diagonalizable **if and only if** algebraic and geometric multiplicities are equal for each eigenvalue
- If there are no repeated eigenvalues, there are no "teeth" either
- When there are no repeated eigenvalues \mathbf{A} is diagonalizable
 $n \times n$
(is sure to have n independent eigenvectors)

21 / 44

10 Diagonalizing a matrix

- Find the spectrum: $\{\lambda_1, \lambda_2, \dots\}$
- Find the *algebraic multiplicity* of each eigenvalue: $\mu(\lambda_i)$

then choose one of these alternatives:

- teething the matrix (implemented in NAcAL)
- ...or for every λ_i
 - find the eigenspace

$$\mathcal{E}_{\lambda_i}(\mathbf{A}) = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \lambda_i \mathbf{x} \right\} = \mathcal{N}(\mathbf{A} - \lambda_i \mathbf{I}).$$

- check $\mu(\lambda_i) = \dim \mathcal{E}_{\lambda_i}(\mathbf{A})$ (algebraic and geometric multiplicities are equal)

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix}; \quad \mathbf{S} = [\text{Basis for } \mathcal{E}_{\lambda_1}(\mathbf{A}) \parallel \dots \parallel \text{Basis for } \mathcal{E}_{\lambda_k}(\mathbf{A})]$$

$$\mathbf{S}^{-1} \mathbf{A} \mathbf{S} = \mathbf{D} \quad \Leftrightarrow \quad \mathbf{A} = \mathbf{S} \mathbf{D} \mathbf{S}^{-1}$$

22 / 44

11 Matrix powers

If $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ then $\mathbf{A}^2\mathbf{x} = \mathbf{A}\mathbf{A}\mathbf{x} = \mathbf{A}(\lambda\mathbf{x}) = \lambda\mathbf{A}\mathbf{x} =$

- What can I say about the eigenvectors?
- What is the relationship between the eigenvalues of \mathbf{A} and those of \mathbf{A}^2

In a matrix form (if \mathbf{A} is diagonalizable, $\mathbf{A} = \mathbf{S} \mathbf{D} \mathbf{S}^{-1}$):

$$\mathbf{A}^2 = \mathbf{S} \mathbf{D} \mathbf{S}^{-1} \mathbf{S} \mathbf{D} \mathbf{S}^{-1} = \mathbf{S} \mathbf{D}^2 \mathbf{S}^{-1}$$

In general, for, $n \in \mathbb{Z}$, $n \geq 0$... $\mathbf{A}^n =$
what about \mathbf{A} both diagonalizable and invertible?

23 / 44

Questions of the Lecture 17

(L-17) QUESTION 1. Factor these two matrices into $\mathbf{S} \mathbf{D} \mathbf{S}^{-1}$;

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

(Strang, 2006, exercise 15 from section 5.2.)

(L-17) QUESTION 2. Which of these matrices cannot be diagonalized?

(a)

$$\mathbf{A}_1 = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$$

(b)

$$\mathbf{A}_2 = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}$$

(c)

$$\mathbf{A}_3 = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

(Strang, 2006, exercise 5 from section 5.2.)

23 / 44

(L-17) QUESTION 3. If $\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ find \mathbf{A}^{100} by diagonalizing \mathbf{A} .

(Strang, 2006, exercise 7 from section 5.2.)

(L-17) QUESTION 4. If the eigenvalues of \mathbf{A} are 1, 1 and 2, which of the following are certain to be true? Give a reason if true or a counterexample if false:

- (a) \mathbf{A} is invertible.
 (b) \mathbf{A} is diagonalizable.
 (c) \mathbf{A} is not diagonalizable

(Strang, 2006, exercise 11 from section 5.2.)

(L-17) QUESTION 5. Let \mathbf{A} be the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

- (a) (1^{pts}) Determine if \mathbf{A} is diagonalizable, and if so, diagonalize it.
 (b) (0.5^{pts}) Compute $(\mathbf{A}^6)\mathbf{v}$, where $\mathbf{v} = (0, 0, 0, 1)^T$.
 (c) (0.5^{pts}) Using the the eigenvalues found in part (a) justify that \mathbf{A} is invertible.
 (d) (0.5^{pts}) What is the relation between the eigenvalues of \mathbf{A} and the eigenvalues of \mathbf{A}^{-1} ?

23 / 44

(L-17) QUESTION 6. Si $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$; entonces $\mathbf{A}^3 = (\quad)(\quad)(\quad)$ y $\mathbf{A}^{-1} = (\quad)(\quad)(\quad)$.
(Strang, 2006, exercise 16 from section 5.2.)

(L-17) QUESTION 7. Considere la matriz

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

- (a) Encuentre los autovalores de \mathbf{A}
- (b) Encuentre los auto-vectores de \mathbf{A}
- (c) Diagonalice \mathbf{A} : escríbalo como $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$.

(L-17) QUESTION 8. ¿Falso o verdadero? Si los autovalores de \mathbf{A} son 2, 2 y 3 entonces sabemos que la matriz es

- (a) Invertible
- (b) Diagonalizable
- (c) No diagonalizable.

(L-17) QUESTION 9. Sean las matrices

$$\mathbf{A}_1 = \begin{bmatrix} 8 & \\ & 2 \end{bmatrix}; \quad \mathbf{A}_2 = \begin{bmatrix} 9 & 4 \\ & 1 \end{bmatrix}; \quad \mathbf{A}_3 = \begin{bmatrix} 10 & 5 \\ -5 & \end{bmatrix}$$

- (a) Encuentre los autovalores y auto-vectores de la matriz $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.
- (b) Explique por qué (o por qué no) la matriz \mathbf{A} es diagonalizable.

(L-17) QUESTION 14. Sea \mathbf{A} una matriz 3×3 . Asuma que sus autovalores son 1 y 0, que una base de los autovectores asociados a $\lambda = 1$ son $[1, 0, 1]$ y $[0, 0, 1]$; mientras que los asociados a $\lambda = 0$ son paralelos a $[1, 1, 2]$.

- (a) ¿Es \mathbf{A} diagonalizable? En caso afirmativo escriba la matriz diagonal \mathbf{D} y la matriz \mathbf{S} tales que $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$.
- (b) Encuentre \mathbf{A} .

(L-17) QUESTION 15. Let \mathbf{A} be a 2×2 matrix such that $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ is an eigenvector for \mathbf{A} with eigenvalue 2, and $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is another eigenvector for \mathbf{A} with eigenvalue -2. If $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, compute $(\mathbf{A}^3)\mathbf{v}$.

- (a) Complete dichas matrices de modo que en los tres casos $\det \mathbf{A}_i = 25$. Así, la traza es en todos los casos igual a 10, y por tanto para las tres matrices el único auto-valor $\lambda = 5$ está repetido dos veces ($\lambda^2 = 25$ y $\lambda + \lambda = 10$ implica $\lambda = 5$).
- (b) Encuentre un vector característico con $\mathbf{A}\mathbf{x} = 5\mathbf{x}$. Estas tres matrices no son diagonalizable porque no hay un segundo auto-vector linealmente independiente del primero.

(Strang, 2006, exercise 27 from section 5.2.)

(L-17) QUESTION 10. Factorice las siguientes matrices en $\mathbf{S}\mathbf{D}\mathbf{S}^{-1}$

$$(a) \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad (b) \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

(Strang, 2006, exercise 1 from section 5.2.)

(L-17) QUESTION 11. Encuentre la matriz \mathbf{A} cuyos autovalores son 1 y 4, cuyos autovectores son $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ y $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ respectivamente.

(Strang, 2006, exercise 2 from section 5.2.)

(L-17) QUESTION 12. Si los elementos diagonales de una matriz triangular superior de orden 3×3 son 1, 2 y 7, ¿puede saber si la matriz es diagonalizable? ¿Quién es \mathbf{D} ?
(Strang, 2006, exercise 4 from section 5.2.)

(L-17) QUESTION 13.

1 Highlights of Lesson 18

Highlights of Lesson 18

- Symetric matrices $\mathbf{A} = \mathbf{A}^T$
 - Eigenvalues and eigenvectors
- Introd. positive Definiteness matrices

2 Symmetric matrices $\mathbf{A} = \mathbf{A}^T$

what's special about $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ when \mathbf{A} is symmetric?
 $n \times n$

1. A symmetric matrix has only **REAL EIGENVALUES**
2. n **EIGENVECTORS** can be chosen **ORTHOGONAL**
 (always diagonalizable)

The usual diagonalizable case:

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D} \iff \mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$$

Symmetric case:

I can choose perpendicular unit eigenvectors (**orthonormal** columns of $\mathbf{S} = \mathbf{Q}$)

$$(\text{if } \mathbf{A} = \mathbf{A}^T) \quad \mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}^{-1} = \mathbf{Q}\mathbf{D}\mathbf{Q}^T \quad \text{Spectral thm.}$$

Orthogonally diagonalizable.

25 / 44

3 Eigenspaces are orthogonal for symmetric matrices

Eigenvectors (corresponding to different eigenvalues) of a symmetric matrix are orthogonal.

Proof.

Consider $\mathbf{A}\mathbf{x} = \lambda_1\mathbf{x}$ and $\mathbf{A}\mathbf{y} = \lambda_2\mathbf{y}$ (with $\lambda_1 \neq \lambda_2$). then

$$\lambda_1\mathbf{x} \cdot \mathbf{y} = \mathbf{A}\mathbf{x} \cdot \mathbf{y} = \mathbf{x}(\mathbf{A}^T)\mathbf{y} = \mathbf{x}\mathbf{A}\mathbf{y} = (\mathbf{x} \cdot \mathbf{y})\lambda_2.$$

Since $\lambda_1 \neq \lambda_2$ then:

$$\lambda_1(\mathbf{x} \cdot \mathbf{y}) - \lambda_2(\mathbf{x} \cdot \mathbf{y}) = 0 \implies (\lambda_1 - \lambda_2)\mathbf{x} \cdot \mathbf{y} = 0 \implies \mathbf{x} \cdot \mathbf{y} = 0.$$

□

26 / 44

4 Quadratic forms

Quadratic form:

$$\mathbf{x}\mathbf{A}\mathbf{x}; \quad \text{with } \mathbf{A}^T = \mathbf{A}$$

Since $\mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}^T$ (with $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{I}$), then

$$\mathbf{x}\mathbf{A}\mathbf{x} = \mathbf{x}\mathbf{Q}\mathbf{D}\mathbf{Q}^T\mathbf{x} = (\mathbf{Q}^T\mathbf{x})\mathbf{D}(\mathbf{Q}^T\mathbf{x}) \quad (\text{weighted sum of squares})$$

Positive definite quadratic form:

$$\mathbf{x}\mathbf{A}\mathbf{x} > 0 \quad \forall \mathbf{x} \neq \mathbf{0} \iff \lambda_i > 0, \quad i = 1 : n.$$

then we also say \mathbf{A} is positive definite.

27 / 44

5 Positive definite matrices

Meaning:

$$\mathbf{x}\mathbf{A}\mathbf{x} > 0 \quad (\text{except for } \mathbf{x} = \mathbf{0})$$

Some properties

Consider a positive definite symmetric \mathbf{A} : What about \mathbf{A}^{-1} ?

$$\mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}^{-1} = \mathbf{Q}\mathbf{D}\mathbf{Q}^T$$

Consider two positive definite symmetric matrices \mathbf{A} , \mathbf{B} : What about $\mathbf{A} + \mathbf{B}$?

the answer must be...

28 / 44

6 The matrix product $\mathbf{A}^T \mathbf{A}$

Consider the rectangular matrix \mathbf{A} . Is $\mathbf{A}^T \mathbf{A}$ positive definite?
 $m \times n$

$$x(\mathbf{A}^T \mathbf{A})x =$$

It can only be 0 when $\mathbf{A}x$ is 0

How can we guarantee that $\mathbf{A}x \neq 0$ when $x \neq 0$?

29 / 44

8 Positive definite symmetric matrices

- All eigenvalues are:
- All pivots are:

$$\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$$

Pivots:

What is the sign of each eigenvalue?

$$\lambda^2 - 8\lambda + 11 = 0 \rightarrow \lambda = 4 \pm \sqrt{5} > 0$$

31 / 44

7 Symmetric matrices: signs of eigenvalues

are all λ_i positive? are they negative?

Computing eigenvalues of \mathbf{A} is impossible in general! (5th degree polynomial)
 5×5

Good news: The signs of the pivots of echelon form are the same as the signs of the eigenvalues λ_i
 (if we do not change the sign of the determinant with *Type II* elementary transformations)

num. of positive pivots = num. of positive eigenvalues

30 / 44

Summary (for symmetric matrices):

1. Symmetric matrices have *real eigenvalues* and *perpendicular eigenvectors* can be chosen
2. $\mathbf{A} = \mathbf{Q} \mathbf{D} \mathbf{Q}^T$ where \mathbf{Q} is orthogonal
3. \mathbf{A} is symmetric if and only if it is *orthogonally* diagonalizable
4. The signs of the pivots in the echelon form are same as the signs of the eigenvalues λ_i (only if we do not change the sign of the determinant with *Type II* elementary transformations)

32 / 44

Questions of the Lecture 18

(L-18) QUESTION 1. Write \mathbf{A} , \mathbf{B} and \mathbf{C} in the form \mathbf{QDQ}^T of the spectral theorem:

(a) $\mathbf{A} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\mathbf{C} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

(Strang, 2006, exercise 11 from section 5.5.)

(L-18) QUESTION 2. Find the eigenvalues and the unit eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(Strang, 2003, exercise 3 from section 6.4.)

(L-18) QUESTION 3. Find an orthonormal \mathbf{Q} that diagonalizes this symmetric matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$$

32 / 44

(L-18) QUESTION 6. Sean

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{B} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

- (a) Encuentre los valores característicos de \mathbf{A} (recuerde que $i^2 = -1$).
 (b) Encuentre los valores característicos de \mathbf{B} (en este caso quizá le resulte más sencillo encontrar primero los autovectores, y deducir entonces los autovalores).

(Strang, 2006, exercise 14 from section 5.5.)

(L-18) QUESTION 7. Si $\mathbf{A}^3 = \mathbf{0}$ entonces los autovalores de \mathbf{A} deben ser _____. De un ejemplo tal que $\mathbf{A} \neq \mathbf{0}$. Ahora bien, si \mathbf{A} es además simétrica, demuestre que entonces \mathbf{A}^3 es necesariamente $\mathbf{0}$.

32 / 44

(Strang, 2003, exercise 5 from section 6.4.)

(L-18) QUESTION 4. Suppose \mathbf{A} is a symmetric 3 by 3 matrix with eigenvalues 0, 1, 2.

- (a) What properties can be guaranteed for the corresponding unit eigenvectors \mathbf{u} , \mathbf{v} and \mathbf{w} ?
 (b) In terms of \mathbf{u} , \mathbf{v} , \mathbf{w} , describe the nullspace, left nullspace, row space, and column space of \mathbf{A} .
 (c) Find a vector \mathbf{x} that satisfies $\mathbf{Ax} = \mathbf{v} + \mathbf{w}$. Is \mathbf{x} unique?
 (d) Under what conditions on \mathbf{b} does $\mathbf{Ax} = \mathbf{b}$ have a solution?
 (e) If \mathbf{u} , \mathbf{v} , \mathbf{w} are the columns of \mathbf{S} , what are \mathbf{S}^{-1} and $\mathbf{S}^{-1}\mathbf{AS}$.

(Strang, 2006, exercise 13 from section 5.5.)

(L-18) QUESTION 5. Escriba un hecho destacado sobre los valores característicos de cada uno de estos tipos de matrices:

- (a) Una matriz simétrica real.
 (b) Una matriz diagonalizable tal que $\mathbf{A}^n \rightarrow \mathbf{0}$ cuando $n \rightarrow \infty$.
 (c) Una matriz no diagonalizable
 (d) Una matriz singular

(Strang, 2006, exercise 16 from section 5.5.)

32 / 44

(L-18) QUESTION 8. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} a & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

- (a) Prove that \mathbf{A} is not diagonalizable when $a = 3$.
 (b) Is \mathbf{A} diagonalizable when $a = 2$? (explain). If it is diagonalizable, find an eigenvalue diagonal matrix \mathbf{D} and an eigenvector matrix \mathbf{S} such as $\mathbf{A} = \mathbf{SDS}^{-1}$.
 (c) Is $\mathbf{A}^T\mathbf{A}$ diagonalizable for any value a ? Is it possible to find a full set of orthonormal eigenvectors of $\mathbf{A}^T\mathbf{A}$?
 (d) Find all possible values a such as \mathbf{A} is invertible and diagonalizable.

(L-18) QUESTION 9. Sea la matriz

$$\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};$$

- (a) Expresé \mathbf{B} en la forma $\mathbf{B} = \mathbf{A} = \mathbf{QDQ}^T$ del teorema espectral.
 (b) ¿Es \mathbf{B} diagonalizable? Si no lo es, diga las razones; y en caso contrario genere una matriz \mathbf{S} que diagonalice a \mathbf{B} .

(L-18) QUESTION 10. Suppose the vectors \mathbf{q}_1 , \mathbf{q}_2 , \mathbf{q}_3 form an orthonormal basis for \mathbb{R}^3 and the matrix \mathbf{A} satisfies $\mathbf{A}\mathbf{q}_1 = (1, 0, 0)$, $\mathbf{A}\mathbf{q}_2 = (0, 1, 0)$, and $\mathbf{A}\mathbf{q}_3 = (0, 0, 1)$.

32 / 44

- (a) (0.5^{pts}) Write the matrix **A** explicitly in terms of the vectors q_1, q_2, q_3 .
- (b) (1^{pts}) Write down all possibilities for $\det \mathbf{A}$.
- (c) Which of the following statements are correct: The eigenvalues of **A** must...
- be real numbers.
 - be positive real numbers.
 - be imaginary numbers.
 - have absolute value $|\lambda| = 1$.

32 / 44

1 Highlights of Lesson 19

Highlights of Lesson 19

- Positive and Negative (semi)definite matrices
- Completing the squares
- Diagonalization by congruence

33 / 44

2 Quadratic forms

- Positive definite: $\forall x \neq 0 \Rightarrow x\mathbf{A}x > 0$.
- Positive semi-definite: $\forall x \neq 0 \Rightarrow x\mathbf{A}x \geq 0$.
- Negative definite: $\forall x \neq 0 \Rightarrow x\mathbf{A}x < 0$.
- Negative semi-definite: $\forall x \neq 0 \Rightarrow x\mathbf{A}x \leq 0$.
- Indefinite: neither positive semi-definite, nor negative semi-definite.

34 / 44

Example

What number do I have to put there for the matrix **A** to be singular?

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & \end{bmatrix}$$

- Eigenvalues:
- Leading principal minors:
- For the following quadratic form

$$q_{\mathbf{A}}(x) = x\mathbf{A}x = (x, y) \begin{bmatrix} 2 & 6 \\ 6 & \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 12xy + y^2$$

Is there a $x \neq 0$ such that $x\mathbf{A}x = 0$?

35 / 44

Example

If $\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$ then $(x, y) \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 12xy + ?y^2$

- Are there numbers x and y that make $x\mathbf{A}x$ negative?
- Does the function go through the origin?
- When $y = 0$ and $x = 1$, is it positive? (and when $x = -1$?)
- When $x = 0$ and $y = 1$, is it positive? (and when $y = -1$?)
- Is it always positive?

$(0, 0,)$ **saddle point**: minimum in some directions, maximum in others.

$$\lambda_1 = -2, \quad \begin{pmatrix} -6 \\ 4 \end{pmatrix}; \quad \lambda_2 = 11, \quad \begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

36 / 44

Example

If $\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$ then $(x, y) \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 12xy + 20y^2$

Positive definite.

Does it pass the tests?

- Are the leading principal minors positive?
- Are the eigenvalues positive?

$$q_{\mathbf{A}}(x) = x\mathbf{A}x > 0 \quad \text{for all } x \neq 0$$

37 / 44

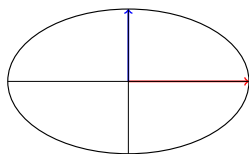
3 Completing the squares

If we could express $q(x)$ as a sum of squares, we would know whether $q(x)$ is positive definite.

Let's complete the square!

- $q(x, y) = 2x^2 + 12xy + 20y^2 = 2(x + ?y)^2 + ?$
- $q(x, y) = 2x^2 + 12xy + 7y^2$
- $q(x, y) = 2x^2 + 12xy + 18y^2$
- $q(x, y) = 2x^2 + 12xy + 20y^2$ (graph)

If positive definite: $q(x, y) = a; \quad a > 0$: ellipse



is

$$q(x, y, z, w, t) = 2t^2 - 2tx - 2tz + w^2 - 2wy + 2x^2 - 2xy + 2y^2 + z^2$$

positive definite? 😞😞😞😞😞!!!!???

38 / 44

4 Congruent matrices

\mathbf{A} and \mathbf{C} are congruent if there exists an invertible \mathbf{B} such that

$$\mathbf{C} = \mathbf{B}^T \mathbf{A} \mathbf{B}$$

Diagonalization by congruence

For each \mathbf{A} (symmetric) exists $\mathbf{B} = \mathbf{I}_{\tau_1 \dots \tau_k}$ (invertible) such that

$$\mathbf{D} = \mathbf{B}^T \mathbf{A} \mathbf{B} \quad \text{is diagonal} \quad (\mathbf{B}^T = \tau_k \dots \tau_1 \mathbf{I})$$

Spectral Theorem: |Diagonalization by similarity and congruence!

$$\mathbf{D} = \mathbf{Q}^{-1} \mathbf{A} \mathbf{Q} = \mathbf{Q}^T \mathbf{A} \mathbf{Q}.$$

Hence, every quadratic form can be written as a sum of squares

$$x\mathbf{A}x = x(\mathbf{B}^{-1})^T \mathbf{D} \mathbf{B}^{-1} x = y\mathbf{D}y; \quad \text{where} \quad y = \mathbf{B}^{-1}x.$$

39 / 44

5 Completing the squares

$$2x^2 + 12xy + 20y^2$$

$$\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \xrightarrow{[(-3)1+2]} \begin{bmatrix} 2 & 0 \\ 6 & 2 \end{bmatrix} \xrightarrow{[(-3)1+2]} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix};$$

therefore, we get:

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \mathbf{D} = \mathbf{E}^T \mathbf{A} \mathbf{E} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

hence $\mathbf{A} = (\mathbf{E}^T)^{-1} \mathbf{D} \mathbf{E}^{-1}$ so

$$\mathbf{x} \mathbf{A} \mathbf{x} = (x, y) \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (\mathbf{x} (\mathbf{E}^{-1})^T) \mathbf{D} (\mathbf{E}^{-1} \mathbf{x})$$

$$= ((x + 3y), y) \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{pmatrix} x + 3y \\ y \end{pmatrix} = 2(x + 3y)^2 + 2y^2$$

40 / 44

6 example 3 by 3

$$\text{Is } \mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \text{ positive definite?}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{[(\frac{1}{2})1+2]} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{[(\frac{2}{3})2+3]} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

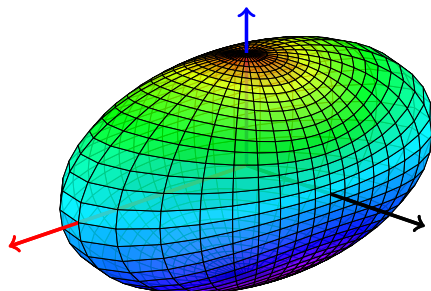
$$\mathbf{x} \mathbf{A} \mathbf{x} = 2x^2 + 2y^2 + 2z^2 - 2xy - 2yz > 0$$

$$\mathbf{x} \mathbf{A} \mathbf{x} = 1 : (\text{ellipsoid}) \text{ axes are eigenvectors } \mathbf{A} = \mathbf{Q}^T \boldsymbol{\lambda} \mathbf{Q}$$

41 / 44

7 Positive definite matrices and ellipsoids: example 3 by 3

- The region $(\mathbf{x} \mathbf{A} \mathbf{x} = a)$ is an (ellipsoid).
- The eigenvectors of \mathbf{Q} are in the direction of the three principal axes.
- Lengths of axes determined by the eigenvalues



42 / 44

8 Another example 3 by 3

$$\text{Is } \mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ positive definite?}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{[(1)3+1]} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{[(\frac{1}{2})1+3]} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \xrightarrow{[2-3]} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Indefinite matrix

43 / 44

9 "Classification" of quadratic

$$x\mathbf{A}x \leq 0; \quad \text{for all } x \neq 0$$

Methods

Check the signs of

1. Elem. diag.: $\mathbf{D} = \mathbf{B}^T \mathbf{A} \mathbf{B}$ (Diagonalization by congruence) 😊
2. Computing eigenvalues: (Roots of a polynomial) 😞
3. Leading principal minors: (Sylvester's criterion) 😞

Law of inertia

the number of positive, negative and zero entries of the diagonal of \mathbf{D} is an invariant of \mathbf{A} , i.e. it does not depend on \mathbf{B}

(Orthogonal diagonalization $\mathbf{D} = \mathbf{Q}^T \mathbf{A} \mathbf{Q}$ is a special case)

44 / 44

Questions of the Lecture 19

(L-19) QUESTION 1. Decide for or against the positive definiteness of these matrices, and write out the corresponding quadratic form $f = x\mathbf{A}x$:

- (a) $\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$
 (b) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$
 (c) $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$
 (d) $\begin{bmatrix} -1 & 2 \\ 2 & -8 \end{bmatrix}$

(e) The determinant in (b) is zero; along what line is $f(x, y) = 0$?

(Strang, 2006, exercise 2 from section 6.1.)

(L-19) QUESTION 2. What is the quadratic $f = ax^2 + 2bxy + cy^2$ for each of these matrices? Complete the square to write f as a sum of one or two squares $d_1(\quad)^2 + d_2(\quad)^2$.

- (a) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix}$
 (b) $\mathbf{B} = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$

(Strang, 2006, exercise 15 from section 6.1.)

44 / 44

(L-19) QUESTION 3. Which one of the following matrices has two positive eigenvalues? Test $a > 0$ and $ac > b^2$, don't compute the eigenvalues. $x\mathbf{A}x < 0$.

- (a) $\mathbf{A} = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix}$
 (b) $\mathbf{B} = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix}$
 (c) $\mathbf{C} = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix}$
 (d) $\mathbf{D} = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}$

(Strang, 2006, exercise 14 from section 6.1.)

(L-19) QUESTION 4. Show that $f(x, y) = x^2 + 4xy + 3y^2$ does not have a minimum at $(0, 0)$ even though it has positive coefficients. Write $f(x, y)$ as a difference of squares and find a point (x, y) where $f(x, y)$ is negative.
 (Strang, 2006, exercise 16 from section 6.1.)

(L-19) QUESTION 5. Show from the eigenvalues that if \mathbf{A} is positive definite, so is \mathbf{A}^2 and so is \mathbf{A}^{-1} .

(Strang, 2006, exercise 4 from section 6.2.)

44 / 44

(L-19) QUESTION 6. Consider the following quadratic forms

$$\begin{aligned} q_1(x, y, z) &= x^2 + 4y^2 + 5z^2 - 4xy. \\ q_2(x, y, z) &= -x^2 + 4y^2 + z^2 + 2xy - 2axz. \end{aligned}$$

- (a) Show that $q_1(x, y, z)$ is positive semi-definite.
 (b) Find, if it is possible, any value of a such as $q_2(x, y, z)$ is negative definite.

(L-19) QUESTION 7. Decide for or against the positive definiteness of

- (a) $\mathbf{A} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$
 (b) $\mathbf{B} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$
 (c) $\mathbf{C} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}^2$

(Strang, 2006, exercise 2 from section 6.2.)

44 / 44

(L-19) QUESTION 8. Consider the following quadratic form

$$q(x, y, z) = x^2 + 6xy + y^2 + az^2;$$

Decide for which values a the quadratic form is positive definite, negative definite, semidefinite, or indefinite.

(L-19) QUESTION 9. Si $\mathbf{A} = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ es definida positiva, pruebe que \mathbf{A}^{-1} es definida positiva.

(Strang, 2006, exercise 8 from section 6.1.)

(L-19) QUESTION 10. Si una matriz simétrica de 2 por 2 satisface $a > 0$, y $ac > b^2$, demuestre que sus autovalores son reales y positivos (definida positiva). Emplee la ecuación característica y el hecho de que el producto de los autovalores es igual al determinante.

(Strang, 2006, exercise 3 from section 6.1.)

(L-19) QUESTION 11. Decida si las siguientes matrices son definidas positivas, definidas negativas, semi-definidas, o indefinidas.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 6 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 3 \end{bmatrix}$

(c) $\mathbf{C} = -\mathbf{B}$

(d) $\mathbf{D} = \mathbf{A}^{-1}$

(L-19) QUESTION 12. Una matriz definida positiva no puede tener un cero (o incluso peor; un número negativo) en su diagonal principal. Demuestre que esta matriz no cumple $\mathbf{x}\mathbf{A}\mathbf{x} > 0$, para todo $\mathbf{x} \neq \mathbf{0}$:

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{no es positiva cuando} \quad \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$$

(Strang, 2006, exercise 21 from section 6.2.)

(L-19) QUESTION 13. Demuestre que si \mathbf{A} y \mathbf{B} son definidas positivas entonces $\mathbf{A} + \mathbf{B}$ también es definida positiva. Para esta demostración los pivotes y los valores característicos no son convenientes. Es mejor emplear $\mathbf{x}(\mathbf{A} + \mathbf{B})\mathbf{x} > 0$

(L-19) QUESTION 14. Find the \mathbf{LDL}^T factorization for the following symmetric matrices.

(a)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

(L-19) QUESTION 15. La forma cuadrática $f(x, y) = 3(x + 2y)^2 + 4y^2$ es definida positiva. Encuentre la matriz \mathbf{A} , factorícela en \mathbf{LDL}^T , y relacione los elementos en \mathbf{D} y \mathbf{L} con 3, 2 y 4 en f .

(Strang, 2006, exercise 9 from section 6.1.)

(L-19) QUESTION 16. Consider the following matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & a & a \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- (a) (0.5pts) Compute the eigenvalues of \mathbf{A} .
- (b) (0.5pts) Prove that when $a = 2$ the matrix \mathbf{A} is not diagonalisable.
- (c) (1pts) For matrix \mathbf{B} , find a diagonal matrix \mathbf{D} and an orthonormal matrix \mathbf{P} such as $\mathbf{B} = \mathbf{PDP}^T$.
- (d) (0.5pts) Find the quadratic form $f(x, y, z)$ associated to \mathbf{B} , and prove it is positive defined.

(L-19) QUESTION 17. Given the matrix $\mathbf{A} = \begin{pmatrix} a & 3/5 \\ b & 4/5 \end{pmatrix}$, compute the values (if they exist) of a and b such as

- (a) (0.5pts) \mathbf{A} is ortho-normal.
- (b) (0.5pts) Columns of \mathbf{A} are linearly independent.
- (c) (0.5pts) $\lambda = 0$ is an eigenvalue of \mathbf{A} .
- (d) (0.5pts) \mathbf{A} is a symmetric definite negative matrix.

(L-19) QUESTION 18.

- (a) Consider the quadratic form $q(x, y, z) = x^2 + 2xy + ay^2 + 8z^2$ and find its corresponding symmetric matrix \mathbf{Q} ; determine if \mathbf{Q} is positive-definite, positive-semidefinite, negative-definite, negative-semidefinite or indefinite when the parameter a is equal to one ($a = 1$).
- (b) If $a \neq 1$, determine whether the matrix is positive-definite, positive-semidefinite, negative-definite, negative-semidefinite or indefinite.

44 / 44

- (f) If 1 is the only eigenvalue of a 2×2 matrix \mathbf{A} , then \mathbf{A} must be the identity matrix \mathbf{I} .

(L-OPT-2) QUESTION 3. complete los blancos, o responda Verdadero/Falso.

- (a) Cualquier sistema generador de un espacio vectorial contiene una base del espacio (V/F)

_____.

- (b) Que los vectores $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ sean linealmente independientes significa que

_____.

- (c) El conjunto que sólo contiene el vector $\mathbf{0}$ es un conjunto linealmente independiente. (V/F)

_____.

- (d) Una matriz cuadrada de orden n por n es diagonalizable cuando:

_____.

- (e) Si $\mathbf{u} = (1, 2, -1, 1)$, entonces $\|\mathbf{u}\| =$ _____.

- (f) Si $\mathbf{u} = (1, 2, -1, 1)$ y $\mathbf{v} = (-2, 1, 0, 0)$, entonces $\mathbf{u} \cdot \mathbf{v} =$ _____.

44 / 44

Questions of the Optional Lecture 2

(L-OPT-2) QUESTION 1. Consider the following matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- (a) (0.5pts) Prove \mathbf{A} is invertible if and only if $a \neq 0$.
- (b) (0.5pts) Is \mathbf{A} positive definite when $a = 1$? Explain your answer.
- (c) (1pts) Compute \mathbf{A}^{-1} when $a = 2$.
- (d) (0.5pts) How many variables can be chosen as pivot (or exogenous) variables in the system $\mathbf{A}\mathbf{x} = \mathbf{o}$ when $a = 0$? Which ones?

(L-OPT-2) QUESTION 2. True or false (to receive full credit you must explain your answers in a clear and concise way)

- (a) If \mathbf{A} is symmetric, then so is \mathbf{A}^2 .
- (b) If $\mathbf{A}^2 = \mathbf{A}$ then $(\mathbf{I} - \mathbf{A})^2 = (\mathbf{I} - \mathbf{A})$ where \mathbf{I} is the identity matrix.
- (c) If $\lambda = 0$ is an eigenvalue of the squared matrix \mathbf{A} , then the linear system $\mathbf{A}\mathbf{x} = \mathbf{0}$ is always solvable and has only one solution.
- (d) If $\lambda = 0$ is an eigenvalue of the squared matrix \mathbf{A} , then the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ could be unsolvable.
- (e) If a matrix is orthogonal (perpendicular columns of norm one), then so is the inverse of that matrix.

44 / 44

(L-OPT-2) QUESTION 4. En las preguntas siguientes \mathbf{A} y \mathbf{B} son matrices $n \times n$. Indique si las siguientes afirmaciones son verdaderas o falsas (incluya una breve explicación, o un contra ejemplo que justifique su respuesta):

- (a) Si \mathbf{A} no es cero entonces $\det(\mathbf{A}) \neq 0$
- (b) Si $\det(\mathbf{AB}) \neq 0$ entonces \mathbf{A} es invertible.
- (c) Si intercambio las dos primeras filas de \mathbf{A} sus autovalores cambian.
- (d) Si \mathbf{A} es real y simétrica, entonces sus autovalores son reales (**aquí no es necesaria una justificación**).
- (e) Si la forma reducida de echelon de $(\mathbf{A} - 5\mathbf{I})$ es la matriz identidad, entonces 5 no es un autovalor de \mathbf{A} .
- (f) Sea \mathbf{b} un vector columna de \mathbb{R}^n . Si el sistema $\mathbf{A}\mathbf{x} = \mathbf{b}$ no tiene solución, entonces $\det(\mathbf{A}) \neq 0$
- (g) Sea \mathbf{C} de orden 3×5 . El rango de \mathbf{C} puede ser 4.
- (h) Sea \mathbf{C} de orden $n \times m$, y \mathbf{b} un vector columna de \mathbb{R}^n . Si $\mathbf{C}\mathbf{x} = \mathbf{b}$ no tiene solución, entonces $\text{rg}(\mathbf{C}) < n$.
- (i) Toda matriz diagonalizable es invertible.
- (j) Si \mathbf{A} es invertible, entonces su forma reducida de echelon es la matriz identidad.

44 / 44

(L-OPT-2) QUESTION 5. Sean

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & 5 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 0 & 4 \\ 0 & 0 & 6 \end{bmatrix}; \quad \mathbf{D} = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Los autovalores de \mathbf{B} son 0 y 2. Use esta información para responder a las siguientes cuestiones. Para cada matriz debe dar una explicación. Puede haber más de una matriz que cumpla la condición:

- (a) ¿Qué matrices son invertibles?
- (b) ¿Qué matrices tienen un autovalor repetido?
- (c) ¿Qué matrices tienen rango menor a tres?
- (d) ¿Qué matrices son diagonalizables?
- (e) ¿Para qué matrices diagonalizables podemos encontrar tres autovectores ortogonales entre sí?

(L-OPT-2) QUESTION 6. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

- (a) Find the eigenvalues and eigenvectors of \mathbf{A} .
- (b) Is \mathbf{A} diagonalizable?

44 / 44

- (c) Is it possible to find a matrix \mathbf{P} such as $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T$, where \mathbf{D} is diagonal?
- (d) Find $|\mathbf{A}^{-1}|$.

(L-OPT-2) QUESTION 7. Consider a 3 by 3 matrix \mathbf{A} with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = -1$; and let $\mathbf{v}_1 = (1, 0, 1)^T$ and $\mathbf{v}_2 = (1, 1, 1)^T$ be the corresponding eigenvectors to λ_1 and λ_2 .

- (a) Is \mathbf{A} diagonalizable?
- (b) Is $\mathbf{v}_3 = (-1, 0, -1)^T$ an eigenvector associated to the eigenvalue $\lambda_3 = -1$?
- (c) Compute $\mathbf{A}(\mathbf{v}_1 - \mathbf{v}_2)$.

(L-OPT-2) QUESTION 8.

- (a) (0.5pts) Find an homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$ such as its solutions set is

$$\left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \mid \exists \alpha, \beta, \gamma \in \mathbb{R} \text{ such that } \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

- (b) (0.5pts) If the characteristic polynomial of a matrix \mathbf{A} is $p(\lambda) = \lambda^5 + 3\lambda^4 - 24\lambda^3 + 28\lambda^2 - 3\lambda + 10$, find the rank of \mathbf{A} .

(L-OPT-2) QUESTION 9. Suponga una matriz cuadrada e invertible \mathbf{A} . $n \times n$

44 / 44

- (a) ¿Cuáles son sus espacios columna $\mathcal{C}(\mathbf{A})$ y espacio nulo $\mathcal{N}(\mathbf{A})$? (no responda con la definición, diga qué conjunto de vectores compone cada espacio).
- (b) Suponga que \mathbf{A} puede ser factorizada en $\mathbf{A} = \mathbf{L}\mathbf{U}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 7 & 3 & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{12} & u_{13} \\ 0 & 0 & u_{13} \end{bmatrix}$$

Describe el primer paso de eliminación en la reducción de \mathbf{A} a \mathbf{U} . ¿porqué sabe que \mathbf{U} es también una matriz invertible? ¿Cuanto vale el determinante de \mathbf{A} ?

- (c) Encuentre una matriz particular de dimensiones 3×3 e invertible \mathbf{A} que no pueda ser factorizada en la forma $\mathbf{L}\mathbf{U}$ (sin permutar previamente las filas). ¿Qué factorización es todavía posible en su ejemplo? (no es necesario que realice la factorización). ¿Cómo sabe que su matriz \mathbf{A} es invertible?

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