Mathematics II

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You can find the last version of these course materials at

https://mbujosab.github.io/MatematicasII/

1 Highlights of Lesson 11

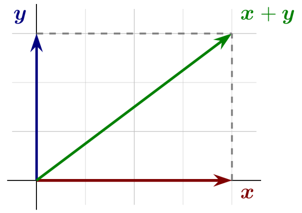
Highlights of Lesson 11

- Orthogonal vectors and subspaces
- Nullspace $\perp$ row space
$$\mathcal{N}\left(\mathbf{A}\right) \perp \mathcal{C}\left(\mathbf{A}^{\intercal}\right)$$
- left nullspace $\perp$ column space
$$\mathcal{N}\left(\mathbf{A}^{\intercal}\right) \perp \mathcal{C}\left(\mathbf{A}\right)$$

2 Some definitions

- Dot product
$$\boldsymbol{a}\cdot\boldsymbol{b} = \sum_{i=1}^{n} a_i b_i$$
- Length of a vector $\|\boldsymbol{a}\| = \sqrt{\boldsymbol{a}\cdot\boldsymbol{a}}$ $\quad\boldsymbol{a}\cdot\boldsymbol{a} = \|\boldsymbol{a}\|^2.$
- Unit vector: $\|\boldsymbol{a}\| = 1$ $\quad\frac{1}{\|\boldsymbol{x}\|}\cdot\boldsymbol{x}$
- Orthogonal (perpendicular) vectors: $\boldsymbol{x}\cdot\boldsymbol{y} = 0.$

3 Orthogonal vectors



$$\boldsymbol{x}\cdot\boldsymbol{y} = 0 \iff \boldsymbol{x}\perp\boldsymbol{y}$$

Pythagoras Thm.: $\boldsymbol{x}\cdot\boldsymbol{y} = 0 \iff \|\boldsymbol{x}\|^2+\|\boldsymbol{y}\|^2 = \|\boldsymbol{x}+\boldsymbol{y}\|^2$

$$\boldsymbol{x}\cdot\boldsymbol{x}+\boldsymbol{y}\cdot\boldsymbol{y} = (\boldsymbol{x}+\boldsymbol{y})\cdot(\boldsymbol{x}+\boldsymbol{y}).$$

4 Squared length of a vector

$$\|\boldsymbol{v}\|^2 = \boldsymbol{v}\cdot\boldsymbol{v}$$

$$\boldsymbol{x} = \begin{pmatrix}1\\2\\3\end{pmatrix} \to \|\boldsymbol{x}\|^2 = \quad ; \qquad \boldsymbol{y} = \begin{pmatrix}2\\-1\\0\end{pmatrix} \to \|\boldsymbol{y}\|^2 = \quad ;$$

Are these vectors orthogonal?

$$\boldsymbol{x}+\boldsymbol{y} = \begin{pmatrix} \\ \\ \end{pmatrix}; \qquad \|\boldsymbol{x}+\boldsymbol{y}\|^2 = \quad ;$$

(Pythagoras) (Orthogonality)

$$\boldsymbol{x}\cdot\boldsymbol{x}+\boldsymbol{y}\cdot\boldsymbol{y} = (\boldsymbol{x}+\boldsymbol{y})\cdot(\boldsymbol{x}+\boldsymbol{y}) \iff \boldsymbol{x}\cdot\boldsymbol{y} = 0.$$

5 Orthogonal subspaces

When subspace $\mathcal{S}$ is orthogonal to subspace $\mathcal{T}$:

$\boxed{\text{Every vector in } \mathcal{S} \text{ is orthogonal to every vector in } \mathcal{T}}$

Are the plane of the *blackboard* and the floor orthogonal?

6 Nullspace orthogonal to row space

- $\mathcal{N}(\mathbf{A}) \perp$ rows of $\mathbf{A}$

$$\mathbf{A}\boldsymbol{x} = \mathbf{0} \implies \begin{pmatrix} ({}_{1|}\mathbf{A})\cdot\boldsymbol{x} \\ \vdots \\ ({}_{m|}\mathbf{A})\cdot\boldsymbol{x} \end{pmatrix} = \begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix}$$

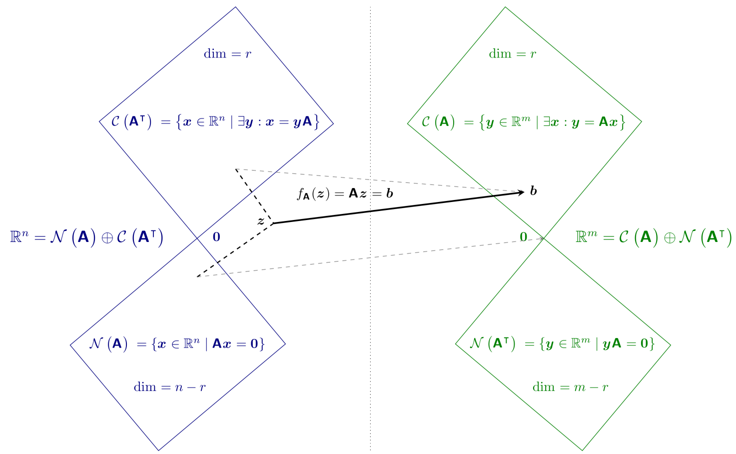
- $\mathcal{N}(\mathbf{A}) \perp \boldsymbol{d}\mathbf{A}, \quad \forall \boldsymbol{d}\in\mathbb{R}^m$ (any linear combination of the rows)

$$\boldsymbol{x}\in\mathcal{N}(\mathbf{A}) \Rightarrow \boldsymbol{d}\mathbf{A}\boldsymbol{x} = \boldsymbol{d}\cdot\mathbf{0} = 0.$$

$\boxed{\text{nullspace} \perp \text{row space} \qquad \mathcal{N}(\mathbf{A}) \perp \mathcal{C}(\mathbf{A}^\intercal)}$

Also: $\boldsymbol{x}\mathbf{A} = \mathbf{0} \Rightarrow \mathcal{N}(\mathbf{A}^\intercal) \perp \mathcal{C}(\mathbf{A})$

7 The big picture: direct sum of orthogonal complements



$$\mathcal{C}(\mathbf{A}^\intercal) \perp \mathcal{N}(\mathbf{A}) \qquad \mathcal{C}(\mathbf{A}) \perp \mathcal{N}(\mathbf{A}^\intercal)$$

$$\boldsymbol{f} \cdot \boldsymbol{x} = \boldsymbol{y}\mathbf{A}\boldsymbol{x} = \boldsymbol{y} \cdot \mathbf{0} \qquad \boldsymbol{y} \cdot \boldsymbol{b} = \boldsymbol{y}\mathbf{A}\boldsymbol{x} = \mathbf{0} \cdot \boldsymbol{x}$$

8 Revisiting the Gaussian elimination

It's an algorithm to find a basis for the orthogonal complement

Give me some vectors (I write them as rows of $\mathbf{M}$) and ...

$$\left[\frac{\mathbf{M}}{\mathbf{I}}\right] = \left[\begin{array}{cccc} 1 & -3 & 0 & -1 \\ 0 & -1 & 1 & 1 \\ 1 & -4 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{2}+\mathbf{4}]}]{\substack{\tau \\ [(3)\mathbf{1}+\mathbf{2}] \\ [(1)\mathbf{1}+\mathbf{4}] \\ [(1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 1 & 3 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\frac{\mathbf{L}}{\mathbf{E}}\right] = \left[\begin{array}{cc} \mathbf{C} & \mathbf{0} \\ \hline \mathbf{D} & \mathbf{N} \end{array}\right]$$

Basis for the span of the given (row) vectors: $\mathcal{V}$

Basis for orthogonal complement: $\mathcal{V}^\perp$ $\qquad \mathbf{M}\mathbf{N} = \mathbf{0}$

If you had given me $\mathbf{N}_{|1}$ and $\mathbf{N}_{|2}$, after Gaussian elimination would have obtained a basis for...

Questions of the Lecture 11

(L-11) QUESTION 1. Describe the set of vectors in $\mathbb{R}^3$ orthogonal to this one $\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

(Hefferon, 2008, exercise 2.15 from section II.2.)

(L-11) QUESTION 2. Is there any vector perpendicular to itself?

(L-11) QUESTION 3. Find the length of each vector

(a) $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$. (b) $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$. (c) $\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$.

(d) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. (e) $\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$.

(Hefferon, 2008, exercise 2.11 from section II.2.)

(L-11) QUESTION 4. Find a unit vector with the same direction as $\boldsymbol{v} = (2, -1, 0, 4, -2)$.

(L-11) QUESTION 5. Find k so that these two vectors are perpendicular.

$$(k, \quad 1,), \qquad (4, \quad 3,).$$

(Hefferon, 2008, exercise 2.14 from section II.2.)

(L-11) QUESTION 6. Construc a matrix with the required property or say why that is impossible:

(a) Column space contains $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$, nullspace contains $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(b) Row space contains $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$, and nullspace contains $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(c) $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ has a solution and $\mathbf{A}^\intercal \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(d) Every row is orthogonal to every column ($\mathbf{A}$ is not the zero matrix)

(e) Columns add up to a column of zeros, rows add up to a row of 1's.

(Strang, 2003, exercise 3 from section 4.1.)

(L-11) QUESTION 7. If $\mathbf{AB} = \mathbf{0}$, the columns of $\mathbf{B}$ are in the ________ of $\mathbf{A}$. The rows of $\mathbf{A}$ are in the __________ of $\mathbf{B}$. Why can't $\mathbf{A}$ and $\mathbf{B}$ be 3 by 3 matrices of rank 2?

(Strang, 2003, exercise 4 from section 4.1.)

(L-11) QUESTION 8. Suppose that $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{w}$ and $\boldsymbol{u} \neq \mathbf{0}$. Must $\boldsymbol{v} = \boldsymbol{w}$?

(Hefferon, 2008, exercise 2.20 from section II.2.)

(L-11) QUESTION 9.

(a) If $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution and $\mathbf{A}^\intercal \boldsymbol{y} = \mathbf{0}$, then $\boldsymbol{y}$ is perpendicular to ____.
(b) If $\mathbf{A}^\intercal \boldsymbol{y} = \boldsymbol{c}$ has a solution and $\mathbf{A}\boldsymbol{x} = \mathbf{0}$, then $\boldsymbol{x}$ is perpendicular to ____.

(Strang, 2003, exercise 5 from section 4.1.)

(L-11) QUESTION 10. Demuestre, in $\mathbb{R}^n$, that if $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular then $||\boldsymbol{u}+\boldsymbol{v}||^2 = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2$.
(Hefferon, 2008, exercise 2.33 from section II.2.)

(L-11) QUESTION 11. Find a 1 by 3 matrix whose nullspace consists of all vectors in $\mathbb{R}^3$ such that $x_1 + 2x_2 + 4x_3 = 0$. Find a 3 by 3 matrix with that same nullspace.
(Strang, 2006, exercise 9 from section 2.4.)

(L-11) QUESTION 12. Consider $\underset{4\times 2}{\mathbf{A}}$ with exactly two special solutions for $\boldsymbol{x}\mathbf{A} = \mathbf{0}$:

$$\boldsymbol{s}_1 = (3, \quad 1, \quad 0, \quad 0,), \quad \text{and} \quad \boldsymbol{s}_2 = (6, \quad 0, \quad 2, \quad 1,).$$

(a) Find the reduced row echelon form $\mathbf{R}$ of $\mathbf{A}$.
(b) What is the row space of $\mathbf{A}$?
(c) What is the complete solution to $\boldsymbol{x}\mathbf{R} = (3, \quad 6,)$?
(d) Find a combination of rows 2, 3, 4 that equals $\mathbf{0}$. (Not OK to use $0({}_{2|}\mathbf{A}) + 0({}_{3|}\mathbf{A}) + 0({}_{4|}\mathbf{A})$. The problem is to show that these rows are dependent.)

(L-11) QUESTION 13. Suppose $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution (maybe many solutions). It can be shown that any solution $\boldsymbol{x}$ of this system can be decomposed as the sum of two vectors ($\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n$) where $\boldsymbol{x}_r$ is a combination of the rows of $\mathbf{A}$ and $\boldsymbol{x}_n$ belongs to the solution set of $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.

(a) (0.5^{pts}) Prove that $\mathbf{A}(\boldsymbol{x}_r) = \boldsymbol{b}$.
(b) (1^{pts}) Suppose that $\boldsymbol{v}_r$ is a linear combination of the rows of $\mathbf{A}$ and furthermore $\mathbf{A}(\boldsymbol{v}_r) = \boldsymbol{b}$. What vector subspaces does the difference $(\boldsymbol{v}_r - \boldsymbol{x}_r)$ belong to? Show that $\boldsymbol{x}_r$ and $\boldsymbol{v}_r$ are equal.
(c) (1^{pts}) Compute the solution $\boldsymbol{x}_r$ in the row space of this matrix $\mathbf{A}$, by solving for c and d

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \end{bmatrix} \boldsymbol{x}_r = \begin{pmatrix} 14 \\ 9 \end{pmatrix} \quad \text{with} \quad \boldsymbol{x}_r = c\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + d\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

1 Highlights of Lesson 12

Highlights of Lesson 12

- From parametric to Cartesian (or implicit) equations
- Choosing a,omg parametric equations

2 Cartesian (implicit) and parametric equations of lines and planes

Cartesian (implicit) equations $\{\boldsymbol{x} \in \mathbb{R}^n \mid \mathbf{A}\boldsymbol{x} = \boldsymbol{b}\}$:

For example

$$\left\{\boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \text{sol. set of } \begin{cases} x_1 - x_2 + x_3 = 1 \\ x_3 = 1 \end{cases}$$

Parametric equations:

for the above set

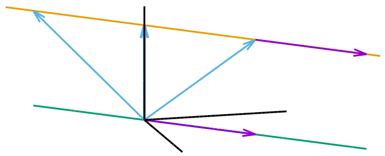
$$\left\{\boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1 : \boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \boldsymbol{p} \right\}$$

In this case *dimension* 1 line A line (there is only one parameter a) line or

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1 : \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \boldsymbol{p} \right\}$$

or

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1 : \boldsymbol{x} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \boldsymbol{p} \right\}$$



3 Cartesian (implicit) and parametric equations of lines and planes

Cartesian (implicit) equations $\{\boldsymbol{x} \in \mathbb{R}^n \mid \mathbf{A}\boldsymbol{x} = \boldsymbol{b}\}$:

For example

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \boldsymbol{x} = (1,) \right\} = \text{sol. set of } \{x_1 - x_2 + x_3 = 1$$

Parametric equations:

for the above set

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 : \boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}$$

In this case *dimension* 2 plane

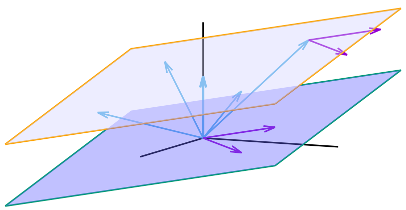
A plane (two parameters a and b) plane

or

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 : \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}$$

but also

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 : \boldsymbol{x} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}$$



4 From parametric to Cartesian equations

$$\boxed{\mathcal{C}\left(\mathbf{A}^\mathsf{T}\right) \perp \mathcal{N}\left(\mathbf{A}\right)}$$

Consider

$$H = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^k : \boldsymbol{x} = \boldsymbol{s} + [\boldsymbol{n}_1; \; \ldots \; \boldsymbol{n}_k;] \boldsymbol{p} \right\}.$$

If we find $\mathbf{A}$ such that $\mathbf{A}\boldsymbol{n}_i = \mathbf{0}$ then if $\boldsymbol{x} \in H$

$$\mathbf{A}\boldsymbol{x} \;=\; \mathbf{A}\boldsymbol{s} + \underbrace{\mathbf{A}[\boldsymbol{n}_1; \; \ldots \; \boldsymbol{n}_k;]}_{\mathbf{0}} \boldsymbol{p} \quad \Rightarrow \quad \mathbf{A}\boldsymbol{x} \;=\; \boldsymbol{b}, \quad \text{where } \boldsymbol{b} = \mathbf{A}\boldsymbol{s}.$$

Therefore

$$H = \{\boldsymbol{x} \in \mathbb{R}^n \mid \mathbf{A}\boldsymbol{x} = \boldsymbol{b}\}.$$

5 From the set of solution to a linear system

Find the implicit equations of the plane P parallel to the spam of $(1,2,0,-2)$ and $(0,0,1,3)$, that goes through $\boldsymbol{s}=(1,3,1,1)$.

$$P=\left\{\begin{pmatrix}x\\y\\z\\w\end{pmatrix}\;\middle|\;\exists a,b\in\mathbb{R}:\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}1\\3\\1\\1\end{pmatrix}+a\begin{pmatrix}1\\2\\0\\-2\end{pmatrix}+b\begin{pmatrix}0\\0\\1\\3\end{pmatrix}\right\}$$

$$=\left\{\boldsymbol{x}\in\mathbb{R}^4\;\middle|\;\exists\boldsymbol{p}\in\mathbb{R}^2:\boldsymbol{x}=\begin{pmatrix}1\\3\\1\\1\end{pmatrix}+\begin{bmatrix}1&0\\2&0\\0&1\\-2&3\end{bmatrix}\boldsymbol{p}\right\}$$

We need vectors perpendicular to $(1,2,0,-2)$ and $(0,0,1,3)$

6 From the set of solution to a linear system

$$\boldsymbol{x}=(x,y,z,w,);\qquad \boldsymbol{s}=(1,3,1,1,).$$

$$\left[\begin{array}{cccc}1&2&0&-2\\0&0&1&3\\\hline x&y&z&w\\\hline 1&3&1&1\end{array}\right]\xrightarrow[{[(2)\mathbf{1}+\mathbf{4}]}]{\boldsymbol{\tau}\,[(-2)\mathbf{1}+\mathbf{2}]}\left[\begin{array}{cccc}1&0&0&0\\0&0&1&3\\\hline x&y-2x&z&w+2x\\\hline 1&1&1&3\end{array}\right]\xrightarrow{\boldsymbol{\tau}\,[(-3)\mathbf{3}+\mathbf{4}]}\left[\begin{array}{cccc}1&0&0&0\\0&0&1&0\\\hline x&y-2x&z&w+2x-3z\\\hline 1&1&1&0\end{array}\right]$$

So $\mathbf{A}=\begin{bmatrix}-2&1&0&0\\2&0&-3&1\end{bmatrix}$; and then $\mathbf{A}\boldsymbol{x}=\begin{pmatrix}-2x+y\\2x+w-3z\end{pmatrix}$ and $\boldsymbol{b}=\mathbf{A}\boldsymbol{s}=\begin{pmatrix}1\\0\end{pmatrix}$. Hence $\begin{cases}-2x+y&=1\\2x\quad-3z+w&=0\end{cases}$

$$P=\left\{\boldsymbol{x}\in\mathbb{R}^4\;\middle|\;\begin{bmatrix}-2&1&0&0\\2&0&-3&1\end{bmatrix}\boldsymbol{x}=\begin{pmatrix}1\\0\end{pmatrix}\right\}.$$

7 A problem from Microeconomics

Solve Y in terms of X to get PPF

$$\begin{cases}X&&=4L_x\\&Y&=3L_y\\&&L_x+L_y=80\end{cases}\quad\to\quad\begin{cases}X&-4L_x&&=0\\&Y&-3L_y&=0\\&L_x+&L_y&=80\end{cases}$$

("in terms of" X means X free)

$$\left[\begin{array}{cccc|c}1&0&-4&0&0\\0&1&0&-3&0\\0&0&1&1&-80\\\hline 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\\hline 0&0&0&0&1\end{array}\right]\xrightarrow[{[(3)\mathbf{2}+\mathbf{4}]}]{\boldsymbol{\tau}\,[(4)\mathbf{1}+\mathbf{3}]}\left[\begin{array}{cccc|c}1&0&0&0&0\\0&1&0&0&0\\0&0&1&1&-80\\\hline 1&0&4&0&0\\0&1&0&3&0\\0&0&1&0&0\\0&0&0&1&0\\\hline 0&0&0&0&1\end{array}\right]\xrightarrow[{[(80)\mathbf{3}+\mathbf{5}]}]{\boldsymbol{\tau}\,[(-1)\mathbf{3}+\mathbf{4}]}\left[\begin{array}{cccc|c}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\\hline 1&0&4&-4&320\\0&1&0&3&0\\0&0&1&-1&80\\0&0&0&1&0\\\hline 0&0&0&0&1\end{array}\right]$$

$$\begin{pmatrix}X\\Y\\L_x\\L_y\end{pmatrix}=\begin{pmatrix}320\\0\\80\\0\end{pmatrix}+a\begin{pmatrix}-4\\3\\-1\\1\end{pmatrix}\Rightarrow a=L_y\Rightarrow\begin{pmatrix}X\\Y\\L_x\\L_y\end{pmatrix}=\begin{pmatrix}320-4L_y\\3L_y\\80-L_y\\L_y\end{pmatrix}\quad\text{"in terms of" }L_y$$

8 Free variable

$$\left[\begin{array}{cccc|c}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\\hline 1&0&4&-4&320\\0&1&0&3&0\\0&0&1&-1&80\\0&0&0&1&0\\\hline 0&0&0&0&1\end{array}\right]\xrightarrow[{[(-320)\mathbf{4}+\mathbf{5}]}]{\boldsymbol{\tau}\,[(\frac{-1}{4})\mathbf{4}]}\left[\begin{array}{cccc|c}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\\hline 1&0&4&1&0\\0&1&0&-3/4&240\\0&0&1&1/4&0\\0&0&0&-1/4&80\\\hline 0&0&0&0&1\end{array}\right]$$

$$\begin{pmatrix}X\\Y\\L_x\\L_y\end{pmatrix}=\begin{pmatrix}0\\240\\0\\80\end{pmatrix}+a\begin{pmatrix}1\\-\frac{3}{4}\\\frac{1}{4}\\-\frac{1}{4}\end{pmatrix}\Rightarrow a=X\Rightarrow\begin{pmatrix}X\\Y\\L_x\\L_y\end{pmatrix}=\begin{pmatrix}X\\240-\frac{3}{4}X\\\frac{1}{4}X\\80-\frac{1}{4}X\end{pmatrix}$$

"in terms of" X

9 Free variables

$$
\begin{cases}
x + 2y - z + w = -1 \\
-x - 2y + 3z + 5w = -5 \\
-x - 2y - z - 7w = 7
\end{cases}
$$

1. Solve in terms of y and w
2. Solve in terms of x and w
3. Solve in terms of x and z
4. Solve in terms of x and y

$$
\left[\begin{array}{cccc|c}
1 & 2 & -1 & 1 & -1 \\
-1 & -2 & 3 & 5 & -5 \\
-1 & -2 & -1 & -7 & 7 \\
\hline
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 1
\end{array}\right]
\xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(1)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{4}] \\ [(1)\mathbf{1}+\mathbf{5}]}}
\left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 2 & 6 & -6 \\
-1 & 0 & -2 & -6 & 6 \\
\hline
1 & -2 & 1 & -1 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

$$
\xrightarrow{\substack{\boldsymbol{\tau} \\ [(-3)\mathbf{3}+\mathbf{4}] \\ [(3)\mathbf{3}+\mathbf{5}]}}
\left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 2 & 0 & 0 \\
-1 & 0 & -2 & 0 & 0 \\
\hline
1 & -2 & 1 & -4 & 4 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -3 & 3 \\
0 & 0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

$$
\left[\begin{array}{cc|c}
-2 & -4 & 4 \\
1 & 0 & 0 \\
0 & -3 & 3 \\
0 & 1 & 0
\end{array}\right]
\begin{cases}
\xrightarrow{\substack{\boldsymbol{\tau} \\ [(\frac{-1}{2})\mathbf{1}] \\ [(4)\mathbf{1}+\mathbf{2}] \\ [(-4)\mathbf{1}+\mathbf{3}]}}
\left[\begin{array}{cc|c}
1 & 0 & 0 \\
\frac{-1}{2} & -2 & 2 \\
0 & -3 & 3 \\
0 & 1 & 0
\end{array}\right]
\xrightarrow{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{2}+\mathbf{3}] \\ [(\frac{-1}{3})\mathbf{2}]}}
\left[\begin{array}{cc|c}
1 & 0 & 0 \\
\frac{-1}{2} & \frac{2}{3} & 0 \\
0 & 1 & 0 \\
0 & \frac{-1}{3} & 1
\end{array}\right]
\\
\xrightarrow{\substack{\boldsymbol{\tau} \\ [(\frac{-1}{2})\mathbf{1}] \\ [(4)\mathbf{1}+\mathbf{2}] \\ [(-4)\mathbf{1}+\mathbf{3}]}}
\left[\begin{array}{cc|c}
1 & 0 & 0 \\
\frac{-1}{2} & -2 & 2 \\
0 & -3 & 3 \\
0 & 1 & 0
\end{array}\right]
\xrightarrow{\substack{\boldsymbol{\tau} \\ [(\frac{-1}{2})\mathbf{2}] \\ [(\frac{1}{2})\mathbf{2}+\mathbf{1}] \\ [(-2)\mathbf{2}+\mathbf{3}]}}
\left[\begin{array}{cc|c}
1 & 0 & 0 \\
0 & 1 & 0 \\
\frac{3}{4} & \frac{3}{2} & 0 \\
\frac{-1}{4} & \frac{-1}{2} & 1
\end{array}\right]
\end{cases}
$$

Questions of the Lecture 12

(L-12) Question 1.

(a) Find a parametric representation for the line passing through the points $\boldsymbol{x}_P = \begin{pmatrix}1\\2\end{pmatrix}$ y $\boldsymbol{x}_Q = \begin{pmatrix}3\\1\end{pmatrix}$.

(b) Find a implicit representation for the same line.

(L-12) Question 2.

(a) Find a parametric representation for the line passing through the points $\boldsymbol{x}_P = (1, \quad -3, \quad 1,)$ and $\boldsymbol{x}_Q = (-2, \quad 4, \quad 5,)$.

(b) Find a implicit representation (Cartesian equations) for the same line.

(L-12) Question 3.

(a) Parametric equation of a line parallel to $2x - 3y = 5$ that goes through $(1, 1)$.

(b) Find a implicit representation for the line.

(L-12) Question 4.

(a) Find parametric equations of the plane that goes through the point (0,1,1) and parallel to the vectors (0,1,2) and (1,1,0)

(b) Write the implicit equation of the same plane.

(L-12) Question 5.

(a) Find a parametric equation of the plane through the point $(2, \quad 1, \quad 3,)$ with normal vector $(3, \quad 1, \quad 1,)$.

(b) Write the implicit equation of the same plane.

(L-12) Question 6. Consider the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}.$$

(a) (1^{pts}) Find the solution to the system.

(b) (0.5^{pts}) Explain why the solution set is a line in $\mathbb{R}^5$. Find a direction vector (a vector parallel to the line) and any point on that line.

(c) (1^{pts}) Find the set of vectors perpendicular to the solution set. Prove that set is a four dimensional subspace. Find a basis for that subspace.

1 Highlights of Lesson 13

Highlights of Lesson 13

- Projections
- Projection matrices

2 Direct sum of subspaces

$\mathbb{R}^n$ is a *direct sum* of $\mathcal{A}$ and $\mathcal{B}$ $\qquad (\mathbb{R}^n = \mathcal{A} \oplus \mathcal{B})$

if every $\boldsymbol{x} \in \mathbb{R}^n$ has a **unique** representation $\boldsymbol{x} = \boldsymbol{a} + \boldsymbol{b}$,

with $\boldsymbol{a} \in \mathcal{A}$ and $\boldsymbol{b} \in \mathcal{B}$.

Example $\boxed{\mathbb{R}^n = \mathcal{C}\left(\mathbf{A}^\intercal\right) \oplus \mathcal{N}\left(\mathbf{A}\right)}$

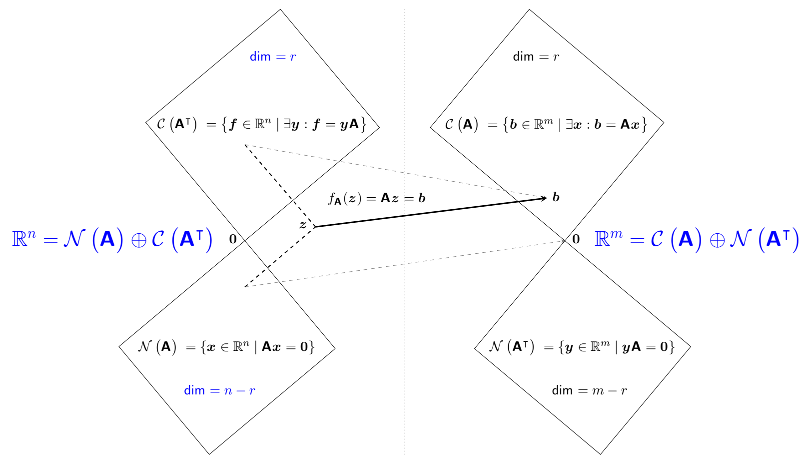
$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{ccc} 1 & 2 & 5 \\ 2 & 4 & 10 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc} 1 & 0 & 0 \\ 2 & 0 & 0 \\ \hline 1 & -2 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \Rightarrow \text{Basis of } \mathbb{R}^3; \left[\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}; \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}\right]$$

$$\forall \boldsymbol{x} \in \mathbb{R}^3, \exists c_1, c_2, c_3 \;\Big|\; \boldsymbol{x} = c_1 \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} = \boldsymbol{a} + \boldsymbol{b}$$

where $\boldsymbol{a} \in \mathcal{C}\left(\mathbf{A}^\intercal\right)$ and $\boldsymbol{b} \in \mathcal{N}\left(\mathbf{A}\right)$.

Also $\boxed{\mathbb{R}^m = \mathcal{C}\left(\mathbf{A}\right) \oplus \mathcal{N}\left(\mathbf{A}^\intercal\right)}$

3 The big picture: direct sum of orthogonal complements



$$\mathcal{C}\left(\mathbf{A}^\intercal\right) \perp \mathcal{N}\left(\mathbf{A}\right) \qquad \mathcal{C}\left(\mathbf{A}\right) \perp \mathcal{N}\left(\mathbf{A}^\intercal\right)$$

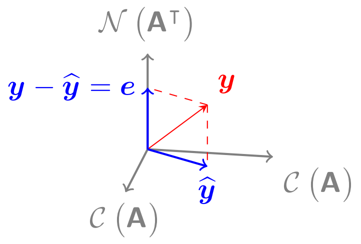
$$\boldsymbol{f} \cdot \boldsymbol{x} = \boldsymbol{y}\mathbf{A}\boldsymbol{x} = \boldsymbol{y} \cdot \mathbf{0} \qquad \boldsymbol{y} \cdot \boldsymbol{b} = \boldsymbol{y}\mathbf{A}\boldsymbol{x} = \mathbf{0} \cdot \boldsymbol{x}$$

4 Orthogonal Projection onto $\mathcal{C}(\mathbf{A})$

Consider $\underset{m\times n}{\mathbf{A}}$; since $\mathbb{R}^m = \mathcal{C}(\mathbf{A}) \oplus \mathcal{N}(\mathbf{A}^\intercal)$, for any $\boldsymbol{y} \in \mathbb{R}^m$

$$\boldsymbol{y} = \widehat{\boldsymbol{y}} + \boldsymbol{e}; \qquad (\boldsymbol{e} = \boldsymbol{y} - \widehat{\boldsymbol{y}})$$

where $\boxed{\widehat{\boldsymbol{y}} \in \mathcal{C}(\mathbf{A}) \text{ and } \boldsymbol{e} \perp \widehat{\boldsymbol{y}}}$, so $\boldsymbol{e} \in \mathcal{N}(\mathbf{A}^\intercal)$.



How to compute $\widehat{\boldsymbol{y}} \in \mathcal{C}(\mathbf{A})$?

5 Normal equations

Consider $\underset{m\times n}{\mathbf{A}}$. We want to find the descoposition $\boxed{\boldsymbol{y} = \widehat{\boldsymbol{y}} + \boldsymbol{e}}$ where

$$\widehat{\boldsymbol{y}} \in \mathcal{C}(\mathbf{A}) \quad \text{and} \quad (\widehat{\boldsymbol{y}} - \boldsymbol{y}) \in \mathcal{N}(\mathbf{A}^\intercal)$$

Then

$$\mathbf{A}\widehat{\boldsymbol{x}} = \widehat{\boldsymbol{y}} \quad \Leftrightarrow \quad (\mathbf{A}\widehat{\boldsymbol{x}} - \boldsymbol{y}) \in \mathcal{N}(\mathbf{A}^\intercal)$$

Therefore

$$\mathbf{A}\widehat{\boldsymbol{x}} = \widehat{\boldsymbol{y}} \quad \Leftrightarrow \quad \mathbf{A}^\intercal(\mathbf{A}\widehat{\boldsymbol{x}} - \boldsymbol{y}) = \mathbf{0} \quad \Leftrightarrow \quad \boxed{(\mathbf{A}^\intercal\mathbf{A})\widehat{\boldsymbol{x}} = \mathbf{A}^\intercal\boldsymbol{y}}$$

Equivalent systems! $\Rightarrow \mathcal{N}(\mathbf{A}) = \mathcal{N}(\mathbf{A}^\intercal\mathbf{A}) \Rightarrow \operatorname{rg}(\mathbf{A}) = \operatorname{rg}(\mathbf{A}^\intercal\mathbf{A})$

$\boxed{\text{unique solution } \widehat{\boldsymbol{x}} \text{ if and only if } \mathbf{A} \text{ is full column rank}}$

6 The solution to the normal equations (full column rank)

$\boxed{\mathbf{A}^\intercal\mathbf{A}\widehat{\boldsymbol{x}} = \mathbf{A}^\intercal\boldsymbol{y}}$ ($\mathbf{A}$ is full column rank)

The solution $\widehat{\boldsymbol{x}} = (\mathbf{A}^\intercal\mathbf{A})^{-1}\mathbf{A}^\intercal\boldsymbol{y}$

The projection $\widehat{\boldsymbol{y}} = \mathbf{A}\widehat{\boldsymbol{x}} = \mathbf{A}(\mathbf{A}^\intercal\mathbf{A})^{-1}\mathbf{A}^\intercal\boldsymbol{y}$

The projection matrix $\mathbf{P} = \mathbf{A}(\mathbf{A}^\intercal\mathbf{A})^{-1}\mathbf{A}^\intercal$

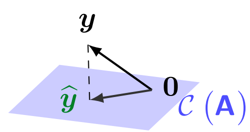
$$\widehat{\boldsymbol{y}} = \mathbf{P}\boldsymbol{y}$$

$\mathbf{P}$: Symetric and idempotent.

7 Projection matrix

$$\boxed{\mathbf{P} = \mathbf{A}\left(\mathbf{A}^\intercal\mathbf{A}\right)^{-1}\mathbf{A}^\intercal}$$

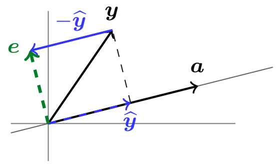
Projection $\mathbf{P}\boldsymbol{y}$ *is the point* $\widehat{\boldsymbol{y}}$ *of* $\mathcal{C}(\mathbf{A})$ *closest to* $\boldsymbol{y}$



Extreme cases:

- If $\boldsymbol{y} \in \mathcal{C}(\mathbf{A})$ then $\mathbf{P}\boldsymbol{y} =$
- If $\boldsymbol{y} \perp \mathcal{C}(\mathbf{A})$ then $\mathbf{P}\boldsymbol{y} =$

8 Projection onto a line



line= $\mathcal{C}(\mathbf{A})$; $\mathbf{A} = [a]$; $a \neq \mathbf{0}$

I'd like to find the point $\widehat{y}$ on that line closest to y

$\widehat{y} \in \mathcal{C}([a]) \quad \perp \quad e = (y - \widehat{y}) \in \mathcal{N}([a]^\top)$.

$\widehat{y}$ is some multiple of a: $\quad \widehat{y} = [a](\widehat{x},)$

How: $\quad [a]^\top [a] \widehat{x} = [a]^\top y$

The solution $\quad \widehat{x} = ([a]^\top [a])^{-1} [a]^\top y$

The projection $\quad \widehat{y} = [a]\widehat{x} = [a]([a]^\top [a])^{-1} [a]^\top y$

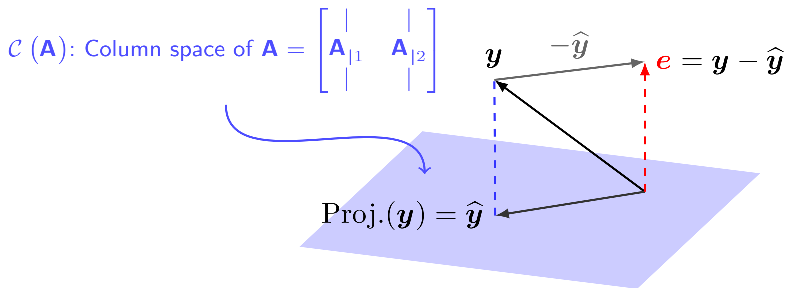
The projection matrix $\quad \mathbf{P} = [a]([a]^\top [a])^{-1} [a]^\top$

9 Projection onto a plane

Why project?

So we will solve

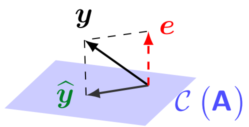
$$\mathbf{A}x = \left(\text{Proj. of } y \text{ onto } \mathcal{C}(\mathbf{A})\right).$$



$(y - \widehat{y}) = e \perp \mathcal{C}(\mathbf{A})$ ...that's the crucial fact.

10 Normal equations

What's the projection of y onto the column space of $\mathbf{A} = \begin{bmatrix} | & | \\ \mathbf{A}_{|1} & \mathbf{A}_{|2} \\ | & | \end{bmatrix}$?



$$\widehat{y} = (\mathbf{A}_{|1})\widehat{x_1} + (\mathbf{A}_{|2})\widehat{x_2} = \mathbf{A}\widehat{x}$$

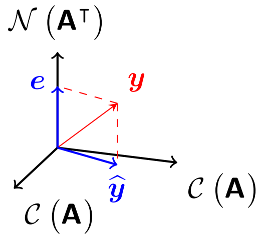
"Find the right combination of the columns so $e \perp \mathcal{C}(\mathbf{A})$"

$$e \perp \mathcal{C}(\mathbf{A}) \quad \Rightarrow \quad e \in$$

$$\mathbf{A}^\top e = \mathbf{A}^\top (y - \widehat{y}) \quad = \quad \mathbf{A}^\top (y - \mathbf{A}\widehat{x}) = \mathbf{0} \quad \Leftrightarrow \quad \boxed{(\mathbf{A}^\top \mathbf{A})\widehat{x} = \mathbf{A}^\top y}$$

11 Two projections

y has a component $\widehat{y}$ in $\mathcal{C}(\mathbf{A})$, and another component e in $\mathcal{C}(\mathbf{A})^\perp$.



$\widehat{y} + e = y$

$\widehat{y} = \mathbf{P}y$ — projection onto $\mathcal{C}(\mathbf{A})$

$e = (\mathbf{I} - \mathbf{P})y$ — projection onto $\mathcal{C}(\mathbf{A})^\perp$

Questions of the Lecture 13

(L-13) QUESTION 1. Project the first vector orthogonally into the line spanned by the second vector. Check that e is perpendicular to a. Find the projection matrix $\mathbf{P} = [a]\left([a]^\intercal [a]\right)^{-1} [a]^\intercal$ onto the line through each vector a. Verify in each case that $\mathbf{P}^2 = \mathbf{P}$. Multiply $\mathbf{P}b$ in each case to compute the projection $\widehat{b}$.

(a) $b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$; $a = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

(b) $b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$; $a = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

(c) $b = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$; $a = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

(d) $b = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$; $a = \begin{pmatrix} 3 \\ 3 \\ 12 \end{pmatrix}$.

(Hefferon, 2008, exercise 1.6 from section VI.1.)

(L-13) QUESTION 2. Project the vector orthogonally into the line.

(a) $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$, The line : $\left\{ v \in \mathbb{R}^3 \;\middle|\; \exists p \in \mathbb{R}^1,\ v = \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix} p \right\}$.

(b) $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$, the line $y = 3x$.

(L-13) QUESTION 3. Although pictures guided our development, we are not restricted to spaces that we can draw. In $\mathbb{R}^4$ project this vector into this line.

$$\begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}; \quad \left\{ v \in \mathbb{R}^4 \;\middle|\; \exists p \in \mathbb{R}^1,\ v = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} p \right\}.$$

(L-13) QUESTION 4.

(a) Project the vector $b = (1, \quad 1,)$ onto the lines through $a_1 = (1, \quad 0,)$ and $a_2 = (1, \quad 2,)$. Add the projections: $\widehat{b_1} + \widehat{b_2}$. The projections do not add to b because a_1 and a_2 are not orthogonal.

(b) The projection of b onto the plane of a_1 and a_2 will equal b. Find $\mathbf{P} = \mathbf{A}(\mathbf{A}^\intercal\mathbf{A})^{-1}\mathbf{A}^\intercal$ for $\mathbf{A} = [a_1; a_2;]$.

(Strang, 2003, exercise 8–9 from section 4.2.)

(L-13) QUESTION 5.

(a) If $\mathbf{P}^2 = \mathbf{P}$ show that $(\mathbf{I} - \mathbf{P})^2 = \mathbf{I} - \mathbf{P}$. When $\mathbf{P}$ projects onto the column space of $\mathbf{A}$, $(\mathbf{I} - \mathbf{P})$ projects onto the ________.

(b) If $\mathbf{P}^\intercal = \mathbf{P}$ show that $(\mathbf{I} - \mathbf{P})^\intercal = \mathbf{I} - \mathbf{P}$.

(Strang, 2003, exercise 17 from section 4.2.)

(L-13) QUESTION 6.

(a) Compute the projection matrices $\mathbf{P} = [a]\left([a]^\intercal [a]\right)^{-1} [a]^\intercal$ onto the lines through $a_1 = (-1, \quad 2, \quad 2,)$ and $a_2 = (2, \quad 2, \quad -1,)$. Show that $a_1 \perp a_2$. Multiply those projection matrices and explain why their product $\mathbf{P}_1\mathbf{P}_2$ is what it is.

(b) Project $b = (1, \quad 0, \quad 0,)$ onto the lines through a_1, and a_2 and also onto $a_3 = (2, \quad -1, \quad 2,)$. Add up the three projections $\widehat{b_1} + \widehat{b_2} + \widehat{b_3}$.

(c) Find the projection matrix $\mathbf{P}_3$ onto $\mathcal{L}([a_3;]) = \mathcal{L}([(2, \quad -1, \quad 2,);])$. Verify that $\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 = \mathbf{I}$. The basis a_1, a_2, a_3 is orthogonal!

(Strang, 2003, exercise 5–7 from section 4.2.)

(L-13) QUESTION 7. Project b onto the column space of $\mathbf{A}$ by solving $\mathbf{A}^\intercal\mathbf{A}\widehat{x} = \mathbf{A}^\intercal b$ and then computing $\widehat{b} = \mathbf{A}\widehat{x}$. Find $e = b - \widehat{b}$.

(a) $\mathbf{A}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $b_1 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

(b) $\mathbf{A}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $b_2 = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix}$

(c) Compute the projection matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ onto the column spaces. Verify that $\mathbf{P}_1 b_1$ gives the first projection $\widehat{b_1}$. Also verify $(\mathbf{P}_2)^2 = \mathbf{P}_2$.

(Strang, 2003, exercise 11–12 from section 4.2.)

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