Quiz 1 Review

Last Revision Date: April 26, 2025

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1. General comments

Exam 1 covers lectures 1–10. The topics covered are (very briefly summarized):

1. Elimination and Matrix Operations (elimination, pivots, etcetera; different viewpoints of $\mathbf{AB}$ and $\mathbf{A}\boldsymbol{x}$ and $\boldsymbol{x}\mathbf{A}$, e.g. as linear combinations of rows or columns).
2. Elimination Matrices and Matrix Inverses (row operations = multiplying on left by elementary matrices, column operations = multiplying on right by elementary matrices, elimination, gaussian elimnation, Gauss-Jordan elimination and what happens when you repeat the elimination steps on $\mathbf{I}$).
3. Permutations, Dot Products, and Transposes.
4. Vector Spaces and Subspaces (for example, the column space and nullspace, what is and isn't a subspace in general, and other vector spaces/subspaces e.g. using matrices and functions).
5. Solving $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ (the nullspace), pre-echelon form $\mathbf{K}$, (rank, free columns, pivot columns, special solutions, etcetera).
6. Solving $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ for nonsquare $\mathbf{A}$ (particular solutions, relationship of rank/nullspace/columnspace to existence and uniqueness of solutions). Geometric interpretation. .
7. Linear independence [key point: the columns of a matrix $\mathbf{A}$ are independent if $\mathcal{N}\left(\mathbf{A}\right) = \{\mathbf{0}\}$], bases (an independent set of vectors that spans a space), and dimension of subspaces (the number of vectors in any basis).
8. The four fundamental subspaces [key points: their dimensions for a given rank r and $m \times n$ matrix $\mathbf{A}$, their relationship to the solutions (if any) of $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, and how/why we can find bases for them via the elimination process].

If there is one central technique in all of these lectures, it is elimination. You should know elimination forwards and backwards. Literally: we might give you the final steps and ask you to work backwards, or ask you what properties of $\mathbf{A}$ you can infer from certain results in elimination. Know how elimination relates to nullspaces and column spaces: column elimination doesn't change the column space...so you can check that $\boldsymbol{b}$ is in the column space of $\mathbf{A}$ by elimination (if elimination produces a last zero column at the augmented matrix $\left[\mathbf{A}| - \boldsymbol{b}\right]$). And column elimination neither change the left nullspace, which is why we can solve $\boldsymbol{x}\mathbf{R} = \mathbf{0}$ to get the left nullspace.[1] Understand why elimination works, not just how. Know how/why elimination corresponds to matrix operations (elimination matrices and $\mathbf{L}$ and $\mathbf{U}$).

As usual, the exam questions may turn these concepts around a bit, e.g. giving the answer and asking you to work backwards towards the question, or ask about the same concept in a slightly changed context. We want to know that you have really internalized these concepts, not just memorizing an algorithm but knowing why the method works and where it came from.

Summary of topics that must be known and <u>understood</u>

- A system is an ordered list of objects.
- A vector $\boldsymbol{a}$ in $\mathbb{R}^n$ is a system of n real numbers: $\quad \boldsymbol{a} = (a_1, \ldots, a_n,); \quad \boldsymbol{a}_{|k} = {}_{k|}\boldsymbol{a} = a_k \in \mathbb{R}$.
- $(\boldsymbol{a} + \boldsymbol{b})_{|k} = \boldsymbol{a}_{|k} + \boldsymbol{b}_{|k}; \quad \left(a\boldsymbol{b}\right)_{|k} = a(\boldsymbol{b}_{|k})$.
- A matrix $\mathbf{A}$ in $\mathbb{R}^{m\times n}$ is a system of n vectors in $\mathbb{R}^m$: $\quad \mathbf{A} = \left[\boldsymbol{a}_1; \ldots \boldsymbol{a}_n;\right]; \quad \mathbf{A}_{|k} = \boldsymbol{a}_k \in \mathbb{R}^m$ (those vectors $\boldsymbol{a}_k$ are call columns of $\mathbf{A}$).
- $(\mathbf{A} + \mathbf{B})_{|k} = \mathbf{A}_{|k} + \mathbf{B}_{|k}; \quad \left(a\mathbf{B}\right)_{|k} = a(\mathbf{B}_{|k}); \quad {}_{i|}\mathbf{A} = \Big({}_{i|}(\mathbf{A}_{|1}), \ldots {}_{i|}(\mathbf{A}_{|n}),\Big)$.
- $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^{n} (\boldsymbol{a}_{|i})(\boldsymbol{b}_{|i})$.
- $\mathbf{A}\boldsymbol{x} = x_1(\mathbf{A}_{|1}) + \cdots + x_n(\mathbf{A}_{|n}) \in \mathbb{R}^m \quad \Rightarrow \quad {}_{i|}(\mathbf{A}\boldsymbol{x}) = ({}_{i|}\mathbf{A}) \cdot \boldsymbol{x}$.

[1] you should be aware that if you study row elimination, you will see that it doesn't change the nullspace, which is why we can solve $\mathbf{R}\boldsymbol{x} = \mathbf{0}$ to get the nullspace, while it does change the column space...but you can check that $\boldsymbol{b}$ is in the column space of $\mathbf{A}$ by elimination (if elimination produces a zero row from $\mathbf{A}$, the same steps should produce a zero row from $\boldsymbol{b}$ if $\boldsymbol{b}$ is in the column space).

- $\boldsymbol{x}\mathbf{A} = x_1({}_{1|}\mathbf{A}) + \cdots + x_m({}_{m|}\mathbf{A}) \in \mathbb{R}^n \quad \Rightarrow \quad (\boldsymbol{x}\mathbf{A})_{|j} = \boldsymbol{x}\cdot(\mathbf{A}_{|j})$.
- $(\mathbf{AB})_{|j} = \mathbf{A}(\mathbf{B}_{|j})$ y ${}_{i|}(\mathbf{AB}) = ({}_{i|}\mathbf{A})\mathbf{B}$.
- ${}_{j|}(\mathbf{A}^{\intercal}) = \mathbf{A}_{|j}$. Therefore $\boldsymbol{x}(\mathbf{A}^{\intercal}) = \mathbf{A}\boldsymbol{x}$ y $(\mathbf{AB})^{\intercal} = (\mathbf{B}^{\intercal})(\mathbf{A}^{\intercal})$.
- The following are elementary transformations only if $\boxed{i \neq j}$ and $\boxed{\alpha \neq 0}$:
  - $\left(\mathbf{I}_{\underset{[(\lambda)\boldsymbol{i}+\boldsymbol{j}]}{\boldsymbol{\tau}}}\right)_{|k} = \begin{cases} \text{If } k=j: & \left(\mathbf{I}_{\underset{[(\lambda)\boldsymbol{i}+\boldsymbol{j}]}{\boldsymbol{\tau}}}\right)_{|j} = \lambda\mathbf{I}_{|i} + \mathbf{I}_{|j} \quad \text{(new } j\text{-th column)} \\ \text{If } k\neq j: & \left(\mathbf{I}_{\underset{[(\lambda)\boldsymbol{i}+\boldsymbol{j}]}{\boldsymbol{\tau}}}\right)_{|k} = \mathbf{I}_{|k} \quad \text{(the others don't change)} \end{cases}$.
  - $\left(\mathbf{A}_{\underset{[(\lambda)\boldsymbol{i}+\boldsymbol{j}]}{\boldsymbol{\tau}}}\right)_{|k} = \left\{\begin{array}{lll} \lambda\mathbf{A}_{|i} + \mathbf{A}_{|j} = \mathbf{A}(\lambda\mathbf{I}_{|i} + \mathbf{I}_{|j}) & = \mathbf{A}\left(\mathbf{I}_{\underset{[(\lambda)\boldsymbol{i}+\boldsymbol{j}]}{\boldsymbol{\tau}}}\right)_{|j} & \text{if } k=j \\ \mathbf{A}_{|k} = \mathbf{A}(\mathbf{I}_{|k}) & = \mathbf{A}\left(\mathbf{I}_{\underset{[(\lambda)\boldsymbol{i}+\boldsymbol{j}]}{\boldsymbol{\tau}}}\right)_{|k} & \text{if } k\neq j \end{array}\right\} = \mathbf{A}\left(\mathbf{I}_{\underset{[(\lambda)\boldsymbol{i}+\boldsymbol{j}]}{\boldsymbol{\tau}}}\right)_{|k}$.
  - $\left(\mathbf{I}_{\underset{[(\alpha)\boldsymbol{i}]}{\boldsymbol{\tau}}}\right)_{|k} = \begin{cases} \text{when } k=i: & \left(\mathbf{I}_{\underset{[(\alpha)\boldsymbol{i}]}{\boldsymbol{\tau}}}\right)_{|i} = \alpha\mathbf{I}_{|i} \quad \text{(new } j\text{-th column)} \\ \text{when } k\neq i: & \left(\mathbf{I}_{\underset{[(\alpha)\boldsymbol{i}]}{\boldsymbol{\tau}}}\right)_{|k} = \mathbf{I}_{|k} \quad \text{(the others don't change)} \end{cases}$.
  - $\left(\mathbf{A}_{\underset{[(\alpha)\boldsymbol{i}]}{\boldsymbol{\tau}}}\right)_{|k} = \left\{\begin{array}{lll} \alpha\mathbf{A}_{|k} & = \mathbf{A}(\alpha\mathbf{I}_{|k}) & \text{when } k=i \\ \mathbf{A}_{|k} & = \mathbf{A}(\mathbf{I}_{|k}) & \text{for the other columns} \end{array}\right\} = \mathbf{A}\left(\mathbf{I}_{\underset{[(\alpha)\boldsymbol{i}]}{\boldsymbol{\tau}}}\right)_{|k}$.
- Therefore, whether $\boldsymbol{\tau}$ is of type $\underset{[(\lambda)\boldsymbol{i}+\boldsymbol{j}]}{\boldsymbol{\tau}}$ or of type $\underset{[(\alpha)\boldsymbol{i}]}{\boldsymbol{\tau}}$ we have that:

$$\mathbf{A}_{\boldsymbol{\tau}} = \mathbf{A}(\mathbf{I}_{\boldsymbol{\tau}}) \quad \Rightarrow \quad (\mathbf{AB})_{\boldsymbol{\tau}} = \mathbf{AB}(\mathbf{I}_{\boldsymbol{\tau}}) = \mathbf{A}(\mathbf{B}_{\boldsymbol{\tau}}); \quad \text{es más,} \quad \mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k} = \mathbf{A}(\mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}).$$

- Besides, ${}_{\boldsymbol{\tau}}\mathbf{I} = (\mathbf{I}_{\boldsymbol{\tau}})^{\intercal}$ and ${}_{\boldsymbol{\tau}}\mathbf{A} = ({}_{\boldsymbol{\tau}}\mathbf{I})\mathbf{A}$; therefore we also have that:

$${}_{\boldsymbol{\tau}}\mathbf{A} = ({}_{\boldsymbol{\tau}}\mathbf{I})\mathbf{A} \quad \Rightarrow \quad {}_{\boldsymbol{\tau}}(\mathbf{AB}) = ({}_{\boldsymbol{\tau}}\mathbf{I})\mathbf{AB} = ({}_{\boldsymbol{\tau}}\mathbf{A})\mathbf{B}; \quad \text{es más,} \quad {}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\mathbf{A} = ({}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\mathbf{I})\mathbf{A}.$$

- Interchange (swap): $\underset{[\boldsymbol{i}\rightleftharpoons\boldsymbol{j}]}{\boldsymbol{\tau}} = \underset{[(-1)\boldsymbol{j}]}{\boldsymbol{\tau}}\underset{[(-1)\boldsymbol{j}+\boldsymbol{i}]}{\boldsymbol{\tau}}\underset{[(1)\boldsymbol{i}+\boldsymbol{j}]}{\boldsymbol{\tau}}\underset{[(-1)\boldsymbol{j}+\boldsymbol{i}]}{\boldsymbol{\tau}}$. Permutation: $\underset{[\mathfrak{S}]}{\boldsymbol{\tau}} = \underset{[\boldsymbol{i}\rightleftharpoons\boldsymbol{j}]}{\boldsymbol{\tau}}\underset{[\boldsymbol{k}\rightleftharpoons\boldsymbol{s}]}{\boldsymbol{\tau}}\cdots\underset{[\boldsymbol{m}\rightleftharpoons\boldsymbol{h}]}{\boldsymbol{\tau}}$.
- Since $\boxed{i \neq j}$ and $\boxed{\alpha \neq 0}$; every elementary transformation $\boldsymbol{\tau}$ is invertible. Its inverse, $\boldsymbol{\tau}^{-1}$, is also an elementary transformation: $\left(\underset{[(\lambda)\boldsymbol{i}+\boldsymbol{j}]}{\boldsymbol{\tau}}\right)^{-1} = \underset{[(-\lambda)\boldsymbol{i}+\boldsymbol{j}]}{\boldsymbol{\tau}}$; and $\left(\underset{[(\alpha)\boldsymbol{i}]}{\boldsymbol{\tau}}\right)^{-1} = \underset{[(\frac{1}{\alpha})\boldsymbol{i}]}{\boldsymbol{\tau}}$.
- Hence, $(\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k)^{-1} = \boldsymbol{\tau}_k^{-1}\cdots\boldsymbol{\tau}_1^{-1}$; so that $\mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k\boldsymbol{\tau}_k^{-1}\cdots\boldsymbol{\tau}_1^{-1}} = \mathbf{A}$; and also ${}_{\boldsymbol{\tau}_k^{-1}\cdots\boldsymbol{\tau}_1^{-1}\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\mathbf{A} = \mathbf{A}$.
- If $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$ then $\mathbf{A}^{-1} = \mathbf{B}$ and $\mathbf{B}^{-1} = \mathbf{A}$.
- The following statements about $\mathbf{A}$ of order n are equivalent:
  - Any of its pre-echelon forms $\mathbf{A}_{\boldsymbol{\tau}_1^*\cdots\boldsymbol{\tau}_p^*} = \mathbf{K}$ has no null columns.
  - $\mathbf{A}$ is invertible.
  - $\mathbf{A} = \mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}$.
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ and $(\mathbf{A}^{-1})^{\intercal} = (\mathbf{A}^{\intercal})^{-1}$.
- $\left(\mathbf{I}_{\underset{[\boldsymbol{j}\rightleftharpoons\boldsymbol{k}]}{\boldsymbol{\tau}}}\right)^{-1} = \mathbf{I}_{\underset{[\boldsymbol{j}\rightleftharpoons\boldsymbol{k}]}{\boldsymbol{\tau}}}$ and $\left(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}\right)^{\intercal}\left(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}\right) = \mathbf{I}$.
- $\mathcal{C}(\mathbf{A}) = \{\boldsymbol{b} \in \mathbb{R}^m \mid \exists \boldsymbol{x} \in \mathbb{R}^n;\ \boldsymbol{b} = \mathbf{A}\boldsymbol{x}\}$; $\quad \mathcal{C}(\mathbf{A}^{\intercal}) = \{\boldsymbol{f} \in \mathbb{R}^n \mid \exists \boldsymbol{x} \in \mathbb{R}^m;\ \boldsymbol{f} = \boldsymbol{x}\mathbf{A}\}$.
- $\mathcal{N}(\mathbf{A}) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \mathbf{A}\boldsymbol{x} = \mathbf{0}\}$; $\quad \mathcal{N}(\mathbf{A}^{\intercal}) = \{\boldsymbol{y} \in \mathbb{R}^m \mid \boldsymbol{x}\mathbf{A} = \mathbf{0}\}$.
- Since $\left(\mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right)_{|j} = \mathbf{A}\left(\mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right)_{|j}$ where $\left(\mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right)$ is invertible:
  - $\mathcal{C}(\mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}) = \mathcal{C}(\mathbf{A})$.

- $\left(\mathbf{A}_{\tau_1\cdots\tau_k}\right)_{|j} = \mathbf{0} \iff \left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)_{|j} \in \mathcal{N}\left(\mathbf{A}\right)$.
- $\mathcal{N}\left(\mathbf{A}^\intercal\right) = \mathcal{N}\left(\left(\mathbf{A}_{\tau_1\cdots\tau_k}\right)^\intercal\right)$; since $\boldsymbol{x}\mathbf{A} = \mathbf{0} \iff \boldsymbol{x}\left(\mathbf{A}_{\tau_1\cdots\tau_k}\right) = \mathbf{0}\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right) = \mathbf{0}$.

- Since ${}_{i|}\left({}_{\tau_1\cdots\tau_k}\mathbf{A}\right) = {}_{i|}\left({}_{\tau_1\cdots\tau_k}\mathbf{I}\right)\mathbf{A}$ where $\left({}_{\tau_1\cdots\tau_k}\mathbf{I}\right)$ is invertible:
  - $\mathcal{C}\left(\left({}_{\tau_1\cdots\tau_k}\mathbf{A}\right)^\intercal\right) = \mathcal{C}\left(\mathbf{A}^\intercal\right)$.
  - ${}_{i|}\left({}_{\tau_1\cdots\tau_k}\mathbf{A}\right) = \mathbf{0} \iff {}_{i|}\left({}_{\tau_1\cdots\tau_k}\mathbf{I}\right) \in \mathcal{N}\left(\mathbf{A}^\intercal\right)$.
  - $\mathcal{N}\left(\mathbf{A}\right) = \mathcal{N}\left({}_{\tau_1\cdots\tau_k}\mathbf{A}\right)$; since $\mathbf{A}\boldsymbol{x} = \mathbf{0} \iff \left({}_{\tau_1\cdots\tau_k}\mathbf{A}\right)\boldsymbol{x} = \left({}_{\tau_1\cdots\tau_k}\mathbf{I}\right)\mathbf{0} = \mathbf{0}$.

- Pivot is the first non-zero number in each column.

- When, in the row of each pivot in $\mathbf{K} = \mathbf{A}_{\tau_1\cdots\tau_k}$, all entries to the right of the pivot are zero, $\mathbf{K}$ is a pre-echelon form of $\mathbf{A}$.

- $\operatorname{rg}\left(\mathbf{A}\right)$ is the number of pivots in any pre-echelon form of $\mathbf{A}$.

- A vector space is a non-empty set $\mathcal{V}$ together with two operation: sum $(+)$ and scalar multiplication $(\cdot)$, that satisfy the following eight axioms: for every $\vec{u}$, $\vec{v}$ and $\vec{w}$ in $\mathcal{V}$, and a and b in $\mathbb{R}$
  1. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
  2. $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$
  3. There exists an element $\vec{0} \in \mathcal{V}$, called the zero vector, such that $\vec{x} + \vec{0} = \vec{x}$, for all $\vec{x} \in \mathcal{V}$.
  4. For every $\vec{x} \in \mathcal{V}$, there exists an element $-\vec{x}$, called the opposite vector of $\vec{x}$, such that $\vec{x} + (-\vec{x}) = \vec{0}$.
  5. $a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$
  6. $(a + b)\vec{x} = a\vec{x} + b\vec{x}$
  7. $(ab)\vec{x} = a(b\vec{x})$
  8. $1\vec{x} = \vec{x}$

- A subset $\mathcal{W}$ of a vector space $\mathcal{V}$ is a vector subspace if that set $\mathcal{W}$ is closed under linear combinations.

- If $\vec{v}_1, \dots \vec{v}_n \in \mathcal{V}$; $\mathcal{L}\left(\left[\vec{v}_1; \dots \vec{v}_n;\right]\right) =$ The smallest subspace that contains $\vec{v}_1, \dots \vec{v}_n$
  $= \left\{\vec{b} \in \mathcal{V} \mid \exists \boldsymbol{x} \in \mathbb{R}^n;\ \vec{b} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n\right\}$.

  Therefore, $\mathcal{C}\left(\mathbf{A}\right) = \mathcal{L}\left(\mathbf{A}\right)$ and $\mathcal{C}\left(\mathbf{A}^\intercal\right) = \mathcal{L}\left(\mathbf{A}^\intercal\right)$.
  $\mathcal{L}\left(\left[\vec{v}_1; \dots \vec{v}_n;\right]\right)$ is called the Span of $\left[\vec{v}_1; \dots \vec{v}_n;\right]$.

- $\mathcal{N}\left(\left[\vec{v}_1; \dots \vec{v}_n;\right]\right) = \left\{\boldsymbol{x} \in \mathbb{R}^n \mid x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}\right\}$.

- The system of vectors $\left[\vec{v}_1; \dots \vec{v}_n;\right]$ is *linearly independent* $\iff \mathcal{N}\left(\left[\vec{v}_1; \dots \vec{v}_n;\right]\right) = \{\mathbf{0}\}$.

  Therefore, columns of $\mathbf{A}$ are linearly independent $\iff \mathcal{N}\left(\mathbf{A}\right) = \{\mathbf{0}\}$.

- If $\vec{v}_1, \dots \vec{v}_n \in \mathcal{V}$ are linearly independent, $\left[\vec{v}_1; \dots \vec{v}_n;\right]$ is a basis for $\mathcal{L}\left(\left[\vec{v}_1; \dots \vec{v}_n;\right]\right)$.

- The number of vectors in a basis is the dimension of the space spanned by the basis.

- For $\underset{m\times n}{\mathbf{A}}$: $\dim\mathcal{C}\left(\mathbf{A}\right) = \dim\mathcal{C}\left(\mathbf{A}^\intercal\right) = \operatorname{rg}\left(\mathbf{A}\right)$; $\dim\mathcal{N}\left(\mathbf{A}^\intercal\right) = m - \operatorname{rg}\left(\mathbf{A}\right)$ $\dim\mathcal{N}\left(\mathbf{A}\right) = n - \operatorname{rg}\left(\mathbf{A}\right)$.

- Eliminatión by columns: $\left[\frac{\mathbf{A}}{\mathbf{I}}\right]_{\tau_1\cdots\tau_k} = \left[\frac{\mathbf{A}_{\tau_1\cdots\tau_k}}{\mathbf{I}_{\tau_1\cdots\tau_k}}\right]$ hence $\left[\frac{\mathbf{A}_{\tau_1\cdots\tau_k}}{\mathbf{I}_{\tau_1\cdots\tau_k}}\right]_{|j} = \left(\frac{\left(\mathbf{A}_{\tau_1\cdots\tau_k}\right)_{|j}}{\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)_{|j}}\right)$.

- When the elimination is from left to right, $\mathbf{I}_{\tau_1\cdots\tau_k}$ is upper triangular: ${}_{(j:n)|}\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)_{|j} = \mathbf{0}$.

  Hence, if $\left(\mathbf{A}_{\tau_1\cdots\tau_k}\right)_{|j} = \mathbf{0}$ then $\mathbf{A}_{|j} \in \mathcal{L}\left(\mathbf{A}_{|(1:j-1)}\right)$; in that case, vector $\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)_{|j}$ is call special solution.

- The list of special solutionsis a basis for, $\mathcal{N}\left(\mathbf{A}\right)$.

Note A basis and the space spanned by it are distinct objects. For example, $\boxed{\mathcal{C}(\mathbf{A}) \neq \text{a basis of } \mathcal{C}(\mathbf{A}).}$ Therefore:

- Don't answer by providing a basis when asked to express a subspace.
- Don't answer by expressing a subspace when asked to provide a basis.
- If asked for both, correctly indicate which is the basis and which is the subspace.

About row elimination:

- ${}_{\tau_1\cdots\tau_k}[\mathbf{A} \mid \mathbf{I}] = \left[{}_{\tau_1\cdots\tau_k}\mathbf{A} \,\middle|\, {}_{\tau_1\cdots\tau_k}\mathbf{I}\right]$ hence ${}_{i|}\left[{}_{\tau_1\cdots\tau_k}\mathbf{A} \,\middle|\, {}_{\tau_1\cdots\tau_k}\mathbf{I}\right] = \left({}_{i|}\left({}_{\tau_1\cdots\tau_k}\mathbf{A}\right) \,\middle|\, {}_{i|}\left({}_{\tau_1\cdots\tau_k}\mathbf{I}\right)\right)$.
- When the elimination is from top to bottom: ${}_{\tau_1\cdots\tau_k}\mathbf{I}$ is lower triangular: ${}_{i|}\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)_{|(i:n)} = \mathbf{0}$.

  Hence, if ${}_{i|}\left({}_{\tau_1\cdots\tau_k}\mathbf{A}\right) = \mathbf{0}$ then ${}_{i|}\mathbf{A} \in \mathcal{L}\left({}_{(1:i-1)|}\mathbf{A}\right)$; in that case, vector ${}_{j|}\left({}_{\tau_1\cdots\tau_k}\mathbf{I}\right)$ is call special solution of $\boldsymbol{x}\mathbf{A} = \mathbf{0}$.

2. Past exams

Below you can find the intermediate examinations from the past years. Until the academic year 12/13 I explained gaussian elimination by rows (as in the book). Therefore some questions were designed assuming row elimination. For this reason here I have introduced some variations of the exercises in order to solve everything by column reduction.

Read the instructions carefully

Quiz 1 Review

NAME: ______________________________

- Extra paper is available for calculations and/or drafting your responses; but...
- **Put your name** in the blanks above (*and also in all the paper provided*).
- Show your work and explain your answers clearly for full credit.
- **Answer exactly what is asked**.
- You will lose points for adding false statements to a correct argument.
- **No electronic devices, books or notes of any form are allowed**.

Grading

1.-

2.-

3.-

4.-

2.1. Grupo C curso 24/25

EXERCISE 1.
Suppose $\mathbf{A}$ is reduced by the usual column operations to

$$\mathbf{A}_{\tau_1\cdots\tau_k} = \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 0 \end{bmatrix}; \quad \text{where} \quad \mathbf{E} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & -2 & 3 \end{bmatrix} = \mathbf{I}_{\tau_1\cdots\tau_k}.$$

(a) (1^{pts}) Write a basis of $\mathcal{C}(\mathbf{A})$ and express $\mathcal{C}(\mathbf{A})$ using that basis.
(b) (1^{pts}) Write a basis of $\mathcal{N}(\mathbf{A})$ and express $\mathcal{N}(\mathbf{A})$ using that basis.
(c) (1^{pts}) (If possible) find the complete solution for the following system:

$$\mathbf{A}\boldsymbol{x} = \text{sum of the columns of } \mathbf{A}.$$

(d) (1^{pts}) Write a basis of $\mathcal{N}\left(\mathbf{A}^\intercal\right)$
(e) (1^{pts}) Find the inverse of $\mathbf{E}$.
(f) (1^{pts}) Write $\mathbf{A}$.
(g) (1^{pts}) Write a basis of $\mathcal{C}\left(\mathbf{A}^\intercal\right)$.

Basado en MIT Course 18.06 Quiz 1. Spring, 2005

EXERCISE 2.
(a) (0.5^{pts}) Suppose $\mathbf{A}$ and $\mathbf{B}$ have the same column space. Give an example where $\mathbf{A}$ and $\mathbf{B}$ have different nullspaces —or say why this is impossible.
(b) (0.5^{pts}) Again $\mathbf{A}$ and $\mathbf{B}$ have the same column space. Give an example where $\mathbf{A}$ and $\mathbf{B}$ have different ranks r – or say why this is impossible.

MIT Course 18.06. Exam I. Professor Strang. March 2, 2015

EXERCISE 3. *Determine whether the following statements are true or false, and explain your reasoning (to receive full credit you must explain your answers in a clear and concise way).*
(a) (0.5^{pts}) If $\mathbf{A}^2 = \mathbf{A}$, then $\mathbf{A} = \mathbf{0}$ or $\mathbf{A} = \mathbf{I}$.
(b) (0.5^{pts}) There is no 3×3 matrix whose column space equals its nullspace.
(c) (0.5^{pts}) If $\mathbf{A}$ is symetric, then so it is $\mathbf{A}^2$.

(a), (b): MIT Course 18.06 Quiz 1, October 3, 2007

EXERCISE 4. (0.5^{pts}) How many solutions (0, 1, ∞) does have $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$?

Please put your answers into the table below.

	$\mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$	$\mathcal{N}(\mathbf{A}) \neq \{\mathbf{0}\}$
$\boldsymbol{b} \in \mathcal{C}(\mathbf{A})$		
$\boldsymbol{b} \notin \mathcal{C}(\mathbf{A})$		

(0.5 points if all correct, 0 points otherwise.)

2.2. Grupo E curso 24/25

Exercise 1. Suppose $\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 4 & 5 & 2 \\ 7 & 8 & 2 \end{bmatrix}$.

(a) (1^{pts}) Find a linear combination of the columns of $\mathbf{A}$ that is $\mathbf{0}$.
(b) (1^{pts}) Explain why for any 3×3 matrix $\mathbf{B}$, the product $\mathbf{AB}$ cannot be invertible.

MIT Course 18.06 Exam 1, Spring, 2021

Exercise 2. (0.5^{pts}) Give an example of a 5×4 matrix with rank 3.

Exercise 3. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$

(a) (0.5^{pts}) What is the rank of $\mathbf{A}$?
(b) (1^{pts}) Find a matrix $\mathbf{B}$ such that the column space $\mathcal{C}(\mathbf{A})$ equals the nullspace $\mathcal{N}(\mathbf{B})$.
(c) (1^{pts}) Which of the following vectors belong(s) to the column space $\mathcal{C}(\mathbf{A})$:

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 2 \\ 4 \\ 8 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}?$$

Based on MIT Course 18.06 Quiz 1, March 5, 2007

Exercise 4. Let $\underset{3\times 3}{\mathbf{A}}$ be such that: $\mathbf{A}_{\underset{[(-2)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\underset{[(1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(-1)\mathbf{3}]}{\boldsymbol{\tau}}} = \mathbf{I}$.

(a) (1^{pts}) Find $\mathbf{A}^{-1}$.
(b) (1^{pts}) Find $\mathbf{A}$?

Based on MIT Course 18.06 Quiz 1, March 10, 1995

Exercise 5. Suppose $\mathbf{A}$ is the matrix:

$$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & -4 & 0 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}; \quad \Big(\text{so } \mathbf{A}(\mathbf{U}^{-1}) = \mathbf{L}\Big).$$

(a) (1^{pts}) Give a basis for the row space of $\mathbf{A}$ and a basis for the column space of $\mathbf{A}$.
(b) (1^{pts}) Describe explicitly all solutions to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.
(c) (1^{pts}) Find all solutions (if any, depending on c) to $\mathbf{A}\boldsymbol{x} = \begin{pmatrix}1, & 2, & 6, & c,\end{pmatrix}$.

Based on MIT Course 18.06 Quiz 1, March 10, 1995

2.3. Grupo C curso 23/24

EXERCISE 1. *Determine whether the following statements are true or false, and explain your reasoning (to receive full credit you must explain your answers in a clear and concise way).*

(a) (0.5^{pts}) If A is $n \times m$ and $m \neq n$, then $\left(\mathsf{A}^{\mathsf{T}}\right)^{\mathsf{T}} = \mathsf{A}$.

(b) (0.5^{pts}) If A^{-1} exists, then it is possible to find an *reduced echelon* form of A by columns without permuting columns.

MIT Course 18.06 Exam 1, October 5, 2015

EXERCISE 2. (1^{pts}) Solve the following linear system: $\begin{cases} a - 2b + 6c = 1 \\ -2a + 3b - 11c = -3 \end{cases}$

MIT Course 18.06 Exam 1, Spring, 2021

EXERCISE 3. (1^{pts}) Give an explanation of why a matrix A of order greater than one, whose first and last columns are formed by ones, does not have an inverse.
Basado en MIT Course 18.06 Quiz 1 February 28, 2020

EXERCISE 4. (3^{pts}) Suppose that we are solving $\mathsf{A}\boldsymbol{x} = \begin{pmatrix}1\\2\\3\end{pmatrix}$.

In each of the parts below, a complete solution set is proposed. For each possibility, say impossible if that could not be a complete solution to such an equation (and explain), or give the the size $m \times n$ and the rank of the matrix A if the solution set is possible.

(a) $\left\{ \begin{pmatrix}1\\2\\3\\4\end{pmatrix} \right\}$

(b) $\left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2,\ \boldsymbol{v} = \begin{pmatrix}1\\2\\3\\4\end{pmatrix} + \begin{bmatrix} -16 & 1 \\ -1 & 0 \\ 5 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}$

(c) $\left\{ \boldsymbol{v} \in \mathbb{R}^2 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{pmatrix}1\\2\end{pmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \boldsymbol{p} \right\}$

(d) $\left\{ \boldsymbol{v} \in \mathbb{R}^2 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{pmatrix}1\\2\end{pmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \boldsymbol{p} \right\}$

(e) $\left\{ \boldsymbol{v} \in \mathbb{R}^2 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2,\ \boldsymbol{v} = \begin{pmatrix}1\\2\end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}$

(f) $\left\{ \begin{pmatrix}1\\2\end{pmatrix} \right\}$

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EXERCISE 5. (1^{pts}) Consider the vectors $\boldsymbol{u} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix}1\\2\\1\end{pmatrix}$, and $\boldsymbol{w} = \begin{pmatrix}0\\0\\1\end{pmatrix}$. Let $\underset{3\times 3}{\mathsf{A}}$ be a matrix such that $\mathsf{A}\boldsymbol{u} = \boldsymbol{0}$, $\mathsf{A}\boldsymbol{v} = \boldsymbol{0}$, and $\mathsf{A}\boldsymbol{w} = \boldsymbol{w}$. What is the rank of A?

EXERCISE 6. (1^{pts}) Find a subset T of the set $S = \left\{ \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}1\\2\\-1\end{pmatrix} \right\}$ such that T is a basis for the subspace $\mathcal{L}(S)$ spanned by S.

EXERCISE 7. For each set below, decide if it is or is not a vector subspace. Explain why or why not.

(a) (0.5^{pts}) All $(n+1) \times n$ matrices of the form $\begin{bmatrix} 0 & 0 & \cdots & 0 \\ v_1 & 0 & \cdots & 0 \\ 0 & v_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v_n \end{bmatrix}$

(b) (0.5^{pts}) The vectors $(x, y, z,)$ in $\mathbb{R}^3$ where $x^2 + y^2 + z^2 \leq 1$.

(c) (0.5^{pts}) All 2×3 matrices whose 6 elements sum to 6.

(d) (0.5^{pts}) All 3×3 rank 1 matrices and the 3×3 zero matrix.

MIT Course 18.06 Quiz 1 March 1, 2019

2.4. Grupo E curso 23/24

Exercise 1. *Determine whether the following statements are true or false, and explain your reasoning (to receive full credit you must explain your answers in a clear and concise way).*

(a) (0.5^{pts}) If $\mathbf{A}^{-1}$ exists, then $\mathbf{A}^{\mathsf{T}}$ is invertible.

(b) (0.5^{pts}) $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ has infinitely many solutions if $\mathbf{A}$ is 3×4 and $\dim \mathcal{N}\left(\mathbf{A}\right) = 2$.

MIT Course 18.06 Exam 1, October 5, 2015

Exercise 2. (1^{pts}) Suppose $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 2 & 5 \\ 1 & 2 & 1 & 1 \end{bmatrix}$

(a) (1^{pts}) Write down the solution set (the complete solution is not a basis) for $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.

(b) (1^{pts}) For what value or values (if any) of a does $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 2a \\ a \end{pmatrix}$ have any solution $\boldsymbol{x}$?

MIT Course 18.06 Exam 1, Spring, 2022

Exercise 3. Give a *basis* for the **nullspace** and also for the **column space** of each of the following matrices:

(a) (1^{pts}) The one-column matrix $\mathbf{A} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

(b) (1^{pts}) The one-row matrix $\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$

(c) (1^{pts}) The 100-row matrix $\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & 4 \end{bmatrix}$ in which every row is $\begin{pmatrix} 1, & 2, & 3, & 4, \end{pmatrix}$.

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Exercise 4. (2^{pts}) Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 1 & 3 & 7 \\ 2 & 2 & 2 \end{bmatrix}$$

What condition(s) must $\boldsymbol{b} \in \mathbb{R}^4$ satisfy to be in the column space of $\mathbf{A}$? (Your answer should be one or more equations of the form $?b_1 + ?b_2 + ?b_3 + ?b_4 = ?$)

Exercise 5. For each set below, decide if it is or is not a vector subspace. Explain why or why not.

(a) (0.5^{pts}) All 4×2 matrices whose second column is both non-zero and also two times the first column.

(b) (0.5^{pts}) All functions of two variables $f(x, y)$ of the form $f(x, y) = ax^2 + bxy + c$ such that $f(7.03, 2024) = 0$.

(c) (0.5^{pts}) All of $\mathbb{R}^3$ except those vectors along the x-axis with $x > 0$. (This means $(x, 0, 0,)$ is excluded if $x > 0$.)

(d) (0.5^{pts}) All 5×5 symmetric matrices $\mathbf{A}$ (meaning $\mathbf{A} = \mathbf{A}^{\mathsf{T}}$).

MIT Course 18.06 Quiz 1 February 28, 2020

2.5. Grupo B curso 22/23

EXERCISE 1.

(a) (0.5^{pts}) Using elimination Gauss-Jordan, find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ (and check).

(b) (0.5^{pts}) Without doing elimination, write down the inverse of $\mathbf{B} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$.

MIT Course 18.06 Exam 1, October 5, 2015

EXERCISE 2. Consider $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 1 & 4 & 2 & 5 \\ 1 & 6 & 4 & 11 \end{bmatrix}$.

(a) (0.5^{pts}) Write a basis for $\mathcal{C}\left(\mathbf{A}\right)$.
(b) (0.5^{pts}) Describe $\mathcal{N}\left(\mathbf{A}\right)$.
(c) (1^{pts}) Solve $\mathbf{A}\boldsymbol{x} = \left(1, \quad 2, \quad 3,\right)$. (If you pay attention on $\mathbf{A}$, you probably don't need to do more calculations).

MIT Course 18.06 Exam 1, October 5, 2015

EXERCISE 3.

Suppose the nonzero vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ point in diferent directions in $\mathbb{R}^3$ but $3\boldsymbol{a} + 2\boldsymbol{b} + \boldsymbol{c} = \mathbf{0}$. The matrix $\mathbf{A}$ has those vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ in its columns.

(a) (1^{pts}) Describe the nullspace of $\mathbf{A}$.
(b) (1^{pts}) Which are the pivot columns of $\mathbf{A}$?
(c) (1^{pts}) Prove that all 3 by 3 matrices $\mathbf{A}$ such that $3\mathbf{A}_{|1} + 2\mathbf{A}_{|2} + \mathbf{A}_{|3} = \mathbf{0}$, that is

$$\mathcal{S} = \left\{ \mathbf{A} \in \mathbb{R}^{3\times 3} \;\middle|\; 3\mathbf{A}_{|1} + 2\mathbf{A}_{|2} + \mathbf{A}_{|3} = \mathbf{0} \right\},$$

is a subspace of the space $\mathbb{R}^{3\times 3}$ of 3 by 3 matrices.

EXERCISE 4. $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{A}^{-1}$ exists.

(a) (0.5^{pts}) What is $\mathcal{C}\left(\mathbf{A}\right)$ and $\mathcal{N}\left(\mathbf{A}\right)$? (You are not asked for definitions of these subspaces).
(b) (0.5^{pts}) Write down a basis for $\mathcal{C}\left(\mathbf{A}\right)$ and a basis for $\mathcal{N}\left(\mathbf{A}\right)$.

MIT Course 18.06 Exam 1, October 5, 2015

EXERCISE 5. (1^{pts}) If $\mathbf{A}^2 = \mathbf{0}$, the zero matrix, explain why $\mathbf{A}$ is not invertible. *MIT Course 18.06 Exam 1, March 1, 2000*

EXERCISE 6.

(a) (1^{pts}) Assume $\mathbf{A}$ and $\mathbf{B}$ are commuting matrices (that is, $\mathbf{AB} = \mathbf{BA}$). It they both are also nonsingular, show that $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ also commute.
(b) (1^{pts}) Which are true and wich are false? (Give a good reason!!!).

Let $\mathbf{A}$ be an m by n matrix. Then $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ has always a non-zero solution if:

1. $\operatorname{rg}\left(\mathbf{A}\right) < m$.
2. $\operatorname{rg}\left(\mathbf{A}\right) < n$.
3. $m = n$ and $\mathbf{A}^2 = \mathbf{0}$.

MIT Course 18.06 Exam 1, October 1, 2001

2.6. Grupo E curso 22/23

EXERCISE 1. (1^{pts}) Solve $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$.

MIT Course 18.06 Exam 1, October 5, 2015

EXERCISE 2. Let $\mathbf{A} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 1 & 0 & 2 \end{bmatrix}$.

(a) (0.5^{pts}) Find $\mathbf{E}$ such that $\mathbf{AE} = \mathbf{K}$ is preechelon and where $\mathbf{E}$ is upper triangular.
(b) (1^{pts}) Factor $\mathbf{A}$ as $\mathbf{A} = \mathbf{KU}$, where $\mathbf{K}$ is preechelon and $\mathbf{U}$ is upper triangular.
(c) (0.5^{pts}) Find a basis for the column space of $\mathbf{A}$.

MIT Course 18.06 Exam 1, March 1, 2000

EXERCISE 3. Supose $\mathbf{A} = \mathbf{EU} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 4 & 5 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$.

(a) (1^{pts}) What is the rank of $\mathbf{A}$?
(b) (1^{pts}) Without compute $\mathbf{A}$, find a basis for $\mathcal{N}(\mathbf{A})$.
(c) (1^{pts}) Without compute $\mathbf{A}$, find the complete solution to $\mathbf{A}\boldsymbol{x} = (3, \ -8, \ -7,)$ using an equivalent linear system $\mathbf{U}\boldsymbol{x} = \boldsymbol{c}$ whose coeficient matrix is $\mathbf{U}$.

MIT Course 18.06 Exam 1, March 1, 2000

EXERCISE 4.

If $\mathbf{A}$ is a 5×6 matrix and $[\mathbf{A}\ \mathbf{A}]$ is the 5×12 matrix that results from the concatenation of $\mathbf{A}$ with $\mathbf{A}$.

(a) (1^{pts}) How is $\mathcal{C}([\mathbf{A}\ \mathbf{A}])$ related to $\mathcal{C}(\mathbf{A})$? (explain).
(b) (1^{pts}) What is the dimension of $\mathcal{N}([\mathbf{A}\ \mathbf{A}])$ (explain).

MIT Course 18.06 Exam 1 review, Fall 2015

EXERCISE 5. (1^{pts}) Suppose $\mathbf{A}^n = \mathbf{0}$. Show that $(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{n-1}$.

MIT Course 18.06 Exam 1, October 1, 2001

EXERCISE 6. (1^{pts}) *Determine whether the following statements are true or false, and explain your reasoning (to receive full credit you must explain your answers in a clear and concise way).*

Let $\mathbf{A}$ and $\mathbf{B}$ be squared but not symmetric. If $\mathbf{AB}$ is a symmetric matrix, then $\mathbf{AB} = \mathbf{B}^{\intercal}\mathbf{A}^{\intercal}$.

MIT Course 18.06 Exam 1, October 5, 2015

2.7. Grupo D curso 21/22

EXERCISE 1. Suppose $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \\ 4 & 2 & 1 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 7 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$.

(a) (1^{pts}) What is the rank of $\mathbf{A}$?
(b) (1^{pts}) Write a basis (or a generating system) for $\mathcal{N}(\mathbf{A})$ (please specify if your answer is a basis or if it is a generating system).
(c) (1^{pts}) Write a basis for $\mathcal{N}(\mathbf{A}^{\intercal})$.
(d) (1^{pts}) Solve this *"unusual"* system of equations: $\boldsymbol{x}\mathbf{A} = (1, \quad 1, \quad 7,)$.

Based on MIT Course 18.06 Exam 1, March 1, 2000

EXERCISE 2.

(a) (1^{pts}) Find the number c that makes $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 6 \\ 2 & 6 & c \end{bmatrix}$ singular.
(b) (1^{pts}) If $c = 20$ what are $\mathcal{C}(\mathbf{A})$ and $\mathcal{N}(\mathbf{A})$? Describe then in this specific case (not just repeat their definitions). Also describe $\mathcal{C}(\mathbf{A}^{-1})$ and $\mathcal{N}(\mathbf{A}^{-1})$ (for the inverse matrix!)
(c) (0.5^{pts}) Find an upper triangular matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{L}$, where $\mathbf{L}$ is an echelon matrix.
(d) (0.5^{pts}) Factor $\mathbf{A}$ into $\mathbf{A} = \mathbf{LU}$ (lower triangular $\mathbf{L}$ and upper triangular $\mathbf{U}$).

Based on MIT Course 18.06 Quiz 1, October 2, 2000

EXERCISE 3. Suppose $\mathbf{A}$ is an m by n matrix of rank r.

(a) (0.5^{pts}) If $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution for every right hand side vector $\boldsymbol{b}$, what is $\mathcal{C}(\mathbf{A})$? Describe it in this specific case (not just repeat their definitions).
(b) (0.5^{pts}) In part (a), what are all equations or inequalities that must hold between the numbers m, n and r.
(c) (1^{pts}) Give a specific example of matrix $\mathbf{A}$ of rank 1 with first row $(2, 5,)$. Describe $\mathcal{C}(\mathbf{A})$ and $\mathcal{N}(\mathbf{A})$ completely (i.e, write parametric equations for both subspaces).
(d) (1^{pts}) In your example (part c); find the complete solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{b} = \mathbf{A}_{|1}$.

MIT Course 18.06 Quiz 1, October 2, 2000

2.8. Grupo E curso 21/22

EXERCISE 1. Suppose a 3×3 matrix A such that: $\;_{1|}\mathsf{A} + {}_{2|}\mathsf{A} = {}_{3|}\mathsf{A}$.

(a) (1^{pts}) Explain why $\mathsf{A}\boldsymbol{x} = \left(1, \quad 0, \quad 0,\right)$ can not have a solution.
(b) (0.5^{pts}) Which righthand side vector $\boldsymbol{b} = (b_1,\; b_2,\; b_3,)$ migth allow a solution for $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$? (Give the best answer you can, based on the information provided).
(c) (0.5^{pts}) Why A is not invertible?

MIT Course 18.06 Exam 1, March 1, 2000

EXERCISE 2. Let $\mathbb{R}^{2\times 2}$ be the vector space of 2×2 matrices and let $\mathsf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathsf{B} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.

(a) (0.5^{pts}) Give a basis for $\mathbb{R}^{2\times 2}$.
(b) (1^{pts}) Describe a subspace of $\mathbb{R}^{2\times 2}$ that contains A but does not contain B.
(c) (1^{pts}) Describe a subspace of $\mathbb{R}^{2\times 2}$ which contains no diagonal matrices except for the zero matrix.
(d) (1^{pts}) *Determine whether the following statements are true or false, and explain your reasoning (to receive full credit you must explain your answers in a clear and concise way).*

If a subspace of $\mathbb{R}^{2\times 2}$ contains A and B, it must contain the identity matrix.

MIT Course 18.06 Exam 1, March 1, 2000

EXERCISE 3. Suppose $\mathsf{A} = \begin{bmatrix} 1 & 2 & ? \\ 2 & a & ? \\ 1 & 1 & ? \\ b & 8 & ? \end{bmatrix}$; $\mathsf{B} = \begin{bmatrix} -1 & 2 & -3 \\ 1 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathsf{AB} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix} = \mathsf{R}$.

(a) (0.5^{pts}) What can you say immediately about $\mathsf{A}_{|3}$?
(b) (0.5^{pts}) What are the numbers a and b?
(c) (1^{pts}) Describe the solution set of $\mathsf{A}\boldsymbol{x} = \boldsymbol{0}$.
(d) (1^{pts}) Describe the solution set of $\boldsymbol{x}\mathsf{A} = \boldsymbol{0}$.
(e) (0.5^{pts}) Which of the four fundamental spaces are the same for A and R?

MIT Course 18.06 Quiz 1, October 2, 2000

EXERCISE 4. Let $\mathsf{A} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 1 & 0 & 2 \end{bmatrix}$.

(a) (0.5^{pts}) Find a basis for $\mathcal{C}\left(\mathsf{A}\right)$.
(b) (0.5^{pts}) What is the dimension of $\mathcal{N}\left(\mathsf{A}\right)$?

Based on MIT Course 18.06 Exam 1, March 1, 2000

2.9. Grupo D curso 20/21

EXERCISE 1. $\mathbf{A}$ is m by n. Suppose $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has at least one solution for every $\boldsymbol{b} \in \mathbb{R}^m$.

(a) (0.5^{pts}) The rank of $\mathbf{A}$ is ________.
(b) (0.5^{pts}) Describe all vectors in the nullspace of $\mathbf{A}^\intercal$.
(c) (1^{pts}) The equation $\mathbf{A}^\intercal\boldsymbol{x} = \boldsymbol{c}$ has (0 or 1) (1 or ∞) (0 or ∞) (1) solution for every $\boldsymbol{c} \in \mathbb{R}^n$.

MIT Course 18.06 Quiz 1, March 10, 1999

EXERCISE 2. Find the reduced echelon form of the following matrices:

(a) (0.5^{pts}) The 4×3 matrix $\mathbf{B}$ with ${}_{i|}\mathbf{B}_{|j} = i - j + 1$.
(b) (0.5^{pts}) $\mathbf{C} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 0 & 3 & 6 \end{bmatrix}$

MIT Course 18.06 Quiz 1, October 1, 1998

EXERCISE 3. Suppose $\left[\boldsymbol{u};\ \boldsymbol{v};\ \boldsymbol{w};\right]$ are a basis for a subspace of $\mathbb{R}^4$; and $\mathbf{A} = \left[\boldsymbol{u}\ \boldsymbol{v}\ \boldsymbol{w}\right]$.

(a) (1^{pts}) How do you know that $\mathbf{A}^\intercal\boldsymbol{y} = \mathbf{0}$ has a solution $\boldsymbol{y} \neq \mathbf{0}$?

MIT Course 18.06 Quiz 1, March 10, 1999

EXERCISE 4. Please briefly but clearly explain your answers.

(a) (1^{pts}) Are the set of invertible 2×2 matrices a subspace?
(b) (1^{pts}) Consider the set of matrices in $\mathbb{R}^{m\times 2}$ (for a given $m \geq 2$) whose nullspace contains $\boldsymbol{x}_0 = (1,\ 1,)$. Is this a subspace?

Based on MIT Course 18.06 Quiz 1, October 1, 1998

EXERCISE 5. Consider $\mathbf{A} = \begin{bmatrix} c & c & 1 \\ c & c & 2 \\ 3 & 6 & 9 \end{bmatrix}$. Which values of c lead to each of these possibilities?

(a) (1^{pts}) $\mathbf{A}$ is singular (less than three pivots).
(b) (1^{pts}) For each c, what is the rank of $\mathbf{A}$?
(c) (1^{pts}) For each c, describe exactly the nullspace of $\mathbf{A}$.
(d) (1^{pts}) For each c, give a basis for the column space of $\mathbf{A}$.

MIT Course 18.06 Quiz 1, March 10, 1999

2.10. Grupo E curso 20/21

Exercise 1.

(a) (0.5^{pts}) Find a reduced echelon form $\mathbf{R}$ of $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 3 & 6 & 3 \\ 3 & 6 & 3 \end{bmatrix}$.

(b) (1^{pts}) Find the matrix $\mathbf{E}^{-1}$ such that $\mathbf{A} = \mathbf{R}\mathbf{E}^{-1}$.

(c) (0.5^{pts}) Check that

$$\mathbf{A} = \left[\mathbf{R}_{|1}\right]\left[{}_{1|}(\mathbf{E}^{-1})\right]^{\intercal} + \left[\mathbf{R}_{|2}\right]\left[{}_{2|}(\mathbf{E}^{-1})\right]^{\intercal} + \left[\mathbf{R}_{|3}\right]\left[{}_{3|}(\mathbf{E}^{-1})\right]^{\intercal}.$$

Give the rank of the three matrices, $\left[\mathbf{R}_{|k}\right]\left[{}_{k|}(\mathbf{E}^{-1})\right]^{\intercal}$, in the sum.

Based on MIT Course 18.06 Quiz 1, March 6, 1998

Exercise 2. Suppose $\boldsymbol{u}$; $\boldsymbol{v}$; $\boldsymbol{w}$ are a basis for a subspace of $\mathbb{R}^4$; and $\mathbf{A} = \left[\boldsymbol{u}\ \boldsymbol{v}\ \boldsymbol{w}\right]$.

(a) (1^{pts}) How do you know that $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ has only the solution $\boldsymbol{x} = \mathbf{0}$?

MIT Course 18.06 Quiz 1, March 10, 1999

Exercise 3. Please briefly but clearly explain your answers.

(a) (1^{pts}) Are the set of singular 2×2 matrices a subspace?

(b) (1^{pts}) Consider $\mathcal{V} = \left\{\mathbf{A} \in \mathbb{R}^{3\times3} \mid \text{whose diagonal entries sum to zero}\right\}$. Is $\mathcal{V}$ a subspace?

Based on MIT Course 18.06 Quiz 1, October 1, 1998

Exercise 4.

(a) (0.5^{pts}) To find the third column of $\mathbf{A}^{-1}$ (3 by 3), what linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ would you solve?

(b) (0.5^{pts}) For $\mathbf{A} = \begin{bmatrix} 0 & 3 & 2 \\ 0 & b & 2 \\ 2 & a & 1 \end{bmatrix}$, find the third column of $\mathbf{A}^{-1}$ (if it exists).

(c) (1^{pts}) For each a and b, find the rank of this matrix $\mathbf{A}$ and say why.

(d) (1^{pts}) For each a and b, find a basis for the column space of $\mathbf{A}$.

MIT Course 18.06 Quiz 1, March 10, 1999

Exercise 5. (2^{pts}) Consider $\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 0 \end{bmatrix}$. Find bases for the four fundamental subspaces. *MIT Course 18.06 Quiz 2, October 23, 1998*

2.11. Grupo B curso 19/20

EXERCISE 1. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \\ 3 & 6 & 10 \end{bmatrix}$.

(a) (0.5^{pts}) Give a basis for the column space of $\mathbf{A}$.
(b) (0.5^{pts}) Give a basis for the nullspace of $\mathbf{A}$.
(c) (1^{pts}) Give the complete solution to $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 3 \\ 4 \\ 7 \\ 7 \end{pmatrix}$.

MIT Course 18.06 Quiz 1, Fall 1997

EXERCISE 2. (1^{pts}) Can you find a matrix $\mathbf{A}$ for which the system $\left[\ (1,\quad 2,\quad 1,);\ \right]$ is a basis for $\mathcal{C}\left(\mathbf{A}\right)$, and $\left[\ (1,\quad 1,\quad 1,);\ \right]$ is a basis for $\mathcal{N}\left(\mathbf{A}\right)$? If "yes", give a matrix $\mathbf{A}$. If "no", explain why the matrix $\mathbf{A}$ can not exist.
MIT Course 18.06 Hour exam I, Fall 1996

EXERCISE 3. Suppose the complete solution to

$$\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} \quad \text{is} \quad \left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2,\ \boldsymbol{v} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}$$

(a) (1^{pts}) What is the dimension of the row space of $\mathbf{A}$?
(b) (1^{pts}) What is the matrix $\mathbf{A}$?
(c) (1^{pts}) Describe the set of vectors $\boldsymbol{b}$ for which $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ can be solved (Don't just say that $\boldsymbol{b}$ must be in $\mathcal{C}\left(\mathbf{A}\right)$).

MIT Course 18.06 Quiz 1, Spring 1998

EXERCISE 4. Suppose $\mathbf{A}$ is some 3×3 matrix. We will transform this into a new 3×3 matrix $\mathbf{B}$ by doing operations on the rows or columns of $\mathbf{A}$ as follows. For each part, (i) explain how to express $\mathbf{B}$ as $\mathbf{B} = \mathbf{AE}$ or $\mathbf{B} = \mathbf{EA}$ (say which!) for some matrix $\mathbf{E}$ (write down $\mathbf{E}$!). Also, (ii) say whether $\mathbf{E}$ is invertible (that is, whether the transformation is reversible). (You don't need to compute $\mathbf{E}^{-1}$, just say whether the inverse exists!)

(a) (1^{pts}) Swap the first and second rows of $\mathbf{A}$.
(b) (1^{pts}) Keep the first row the same, then add the second row to the third row, then replace the second row with the sum of the first and third rows.
(c) (1^{pts}) Subtract the first column from the second and third columns.

MIT 18.06 - Quiz 1, Fall 2017

EXERCISE 5. (1^{pts}) Given that: $\underset{3\times 3}{\mathbf{A}} \begin{bmatrix} 4 & 3 & 3 \\ -1 & -1 & -1 \\ -3 & 0 & 1 \end{bmatrix} = \mathbf{I}$, find the inverse of $\mathbf{A}^{\intercal}$.
MIT Course 18.06 Hour exam I, Fall 1996

2.12. Grupo E curso 19/20

Exercise 1. Consider $\mathsf{A} = \begin{bmatrix} 1 & 3 & 2 & -1 \\ 2 & 7 & 4 & -2 \\ 3 & 9 & 6 & 7 \end{bmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 13 \end{pmatrix}$.

(a) (1^{pts}) Solve $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ (if solution exists). Describe the set of solutions geometrically. Is this set of solutions a subspace? (Explain).

(b) (1^{pts}) If the unknowns are $\boldsymbol{x} = (x, y, z, w)$, then your solution is expressed in terms of what variable? (what variable is "free"). Write the set of solutions in terms of a different "free" variable.

(c) (0.5^{pts}) Is $\mathsf{A}\boldsymbol{x} = \boldsymbol{c}$ solvable for any $\boldsymbol{c} \in \mathbb{R}^3$? (explain). Change the entry "7" in $\mathsf{A}_{|2}$ to a different number that gives a smaller column space for the new matrix (call it M). The new entry is ____.

(d) (0.5^{pts}) Give a new right hand side $\boldsymbol{c}$ so that $\mathsf{M}\boldsymbol{x} = \boldsymbol{c}$ has a solution and a right hand side $\boldsymbol{d}$ so that $\mathsf{M}\boldsymbol{x} = \boldsymbol{d}$ has no solution.

Based on MIT Course 18.06 Quiz 1, Spring 1997

Exercise 2. (1^{pts}) Given that $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 1 & 0 & 8 \end{bmatrix} \mathsf{A} = \underbrace{\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}}_{\mathsf{P}}$, find A^{-1}.

MIT Course 18.06 Quiz 1, Fall 1997

Exercise 3.

(a) (1^{pts}) If possible, give a matrix which has $\left[(1, 2, 1,);\right]$ as a basis for the column space, and $\left[(0, 3, 2, -1,);\right]$ as a basis for its row space. If not possible, give your reason.

(b) (1^{pts}) Are the vectors $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ a basis for the vector space $\mathbb{R}^3$?

MIT Course 18.06 Quiz 1, Fall 1997

Exercise 4. Suppose A is the product:

$$\mathsf{A} = \mathsf{B}\mathsf{C} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 6 & 6 \end{bmatrix}.$$

$\boxed{\text{Then, without computing } \mathsf{A}.}$

(a) (1^{pts}) Find a basis for the row space of A.

(b) (1^{pts}) Find a basis for the column space of A.

(c) (1^{pts}) Find a basis for the space of all solutions to $\mathsf{A}\boldsymbol{x} = \boldsymbol{0}$.

(d) (1^{pts}) The dimension of all these subspaces will be different if you correctly change one entry in B. Tell me the new matrix B.

MIT Course 18.06 Quiz 1, Spring 1998

2.13. Grupo B curso 18/19

EXERCISE 1.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 8 & 2 \\ 5 & 12 & 2 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

(a) (1^{pts}) Perform elimination to determine the condition that $\boldsymbol{b}$ is in the column space of $\mathbf{A}$.
(b) (0.5^{pts}) What is the rank of $\mathbf{A}$?
(c) (1^{pts}) Find a non-null vector in the nullspace of $\mathbf{A}^{\intercal}$ (this space is known as the left nullspace).

Based on MIT Course 18.06 Quiz 1, Problem 1. March 2, 2018

EXERCISE 2. (1^{pts}) Let $\mathbf{A}$ be a 3×3 matrix such that the equation $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ has both $\boldsymbol{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ as solutions. Find another solution to this equation. Explain.

EXERCISE 3. (1^{pts}) Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$. Find $\mathbf{A}^{-1}$.

EXERCISE 4. (1^{pts}) Give a matrix $\mathbf{A}$ whose null space is spanned by $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

Based on MIT Course 18.06 Exam 1, Problem 4. Fall 2018

EXERCISE 5. When solving $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ (with $\mathbf{A}$ m by n) we find only one special solution: $\boldsymbol{x} = \begin{pmatrix} c \\ 1 \\ 0 \\ d \end{pmatrix}$.

(a) (0.5^{pts}) Describe all the possibilities for the number of columns of $\mathbf{A}$.
(b) (0.5^{pts}) Describe all the possibilities for the rank of $\mathbf{A}$.
(c) (0.5^{pts}) Describe all the possibilities for the number of rows of $\mathbf{A}$.

Briefly explain your answers.

EXERCISE 6. *Determine whether the following statements are true or false, and explain your reasoning (to receive full credit you must explain your answers in a clear and concise way).*

$\mathcal{V}$ is a subspace

(a) (0.5^{pts}) $\mathcal{V} = \left\{ \text{all solutions } \boldsymbol{x} \text{ to } \mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$, where $\mathbf{A}$ is a 3×6.
(b) (0.5^{pts}) $\mathcal{V} = \left\{ \text{all } 6 \times 2 \text{ matrices } \mathbf{X} \text{ where } \mathbf{A}\mathbf{X} = \underset{3\times 2}{\mathbf{0}} \right\}$, where $\mathbf{A}$ is a 3×6.
(c) (0.5^{pts}) $\mathcal{V} = \left\{ \text{all } 3 \times 3 \text{ singular matrices } \mathbf{A} \right\}$.

Based on MIT Course 18.06 Exam 1, Problem 4. Fall 2018

EXERCISE 7.

(a) (0.5^{pts}) $\mathbf{A}$ and $\mathbf{B}$ are any matrices with the same number of rows. What can you say (*and explain why it is true*) about the comparison of

$$\text{rank of } \mathbf{A} \qquad \text{and} \qquad \text{rank of the block matrix } \left[\mathbf{A} \,|\, \mathbf{B}\right].$$

(b) (0.5^{pts}) Suppose $\mathbf{B} = \mathbf{A}^2$. How do those ranks compare? Explain your reasoning.
(c) (0.5^{pts}) If $\mathbf{A}$ is m by n of rank r, what are the dimensions of these nullspaces?

$$\text{Nullspace of } \mathbf{A} \qquad \text{and} \qquad \text{Nullspace of } \left[\mathbf{A} \,|\, \mathbf{A}\right]$$

MIT Course 18.06 Final, Fall 2006

2.14. Grupo E curso 18/19

EXERCISE 1. $\mathbf{A}$ is a 5×3 matrix. One of your Harvard friends performed column operations on $\mathbf{A}$ to convert it to an reduced echelon form, but did something weird—instead of getting usual $\mathbf{R} = \begin{bmatrix} \mathbf{I} \\ \mathbf{H} \end{bmatrix}$, they reduced it to a matrix in the form $\begin{bmatrix} \mathbf{H} \\ \mathbf{I} \end{bmatrix}$ instead. In particular, their column operations gave:

$$\mathbf{A} \longrightarrow \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) (1^{pts}) Find a basis for $\mathcal{N}\left(\mathbf{A}^\intercal\right)$.
(b) (1^{pts}) Give a matrix $\mathbf{M}$ so that if you multiply $\mathbf{A}$ by $\mathbf{M}$ (on the **left** or **right**?) then the **same** column operations as the ones used by your Harvard friend will give a matrix in the usual reduced echelon form:

$$\begin{pmatrix} \text{either } \mathbf{AM} \\ \text{or } \mathbf{MA} \end{pmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$$

Based on MIT Course 18.06 Exam 1, Problem 2. Fall 2018

EXERCISE 2. Consider the 5×5 matrices

$$\mathbf{E}_1 = \begin{bmatrix} 1 & a & b & c & d \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}; \quad \mathbf{E}_2 = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & x \\ & & & & 1 \end{bmatrix}; \quad \text{and} \quad \mathbf{E} = \mathbf{E}_1 \cdot \mathbf{E}_2.$$

You don't need too much computantions if you realized that $\mathbf{E}_1$ is a product of elementary matrices (a sequence of elementary tranformations on $\mathbf{I}$).

(a) (1^{pts}) Solve $\left(\mathbf{E}_1\right)\boldsymbol{x} = \begin{pmatrix}1, & 1, & 1, & 1, & 1,\end{pmatrix}$.
(b) (0.5^{pts}) Solve $\left(\mathbf{E}_1^\intercal\right)\boldsymbol{x} = \begin{pmatrix}1, & 1, & 1, & 1, & 1,\end{pmatrix}$.
(c) (0.5^{pts}) Compute $\mathbf{E}$.
(d) (1^{pts}) Compute $\mathbf{E}^{-1}$. Check your answer.
(e) (1^{pts}) Compute $(\mathbf{E}_1)^{10} = \mathbf{E}_1 \cdot \mathbf{E}_1 \cdot \mathbf{E}_1 \cdot \mathbf{E}_1 \cdot \mathbf{E}_1 \cdot \mathbf{E}_1 \cdot \mathbf{E}_1 \cdot \mathbf{E}_1 \cdot \mathbf{E}_1 \cdot \mathbf{E}_1$.

Based on MIT Course 18.06 Quiz 1, Problem 2. March 2, 2018

EXERCISE 3. (1^{pts}) Give a non-zero matrix $\mathbf{A}$ whose column space is in $\mathbb{R}^3$ but does not include $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

Based on MIT Course 18.06 Exam 1, Problem 4. Fall 2018

EXERCISE 4. (1^{pts}) Find the complete solution (also called the *general solution*) to

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

(Strang, 2003, exercise 4 from section 3.4.)

EXERCISE 5. If $\mathbf{A}$ is a 5 by 4 matrix with linearly independent columns, find each of these **explicitly**:

(a) (0.5^{pts}) The nullspace of $\mathbf{A}$.
(b) (0.5^{pts}) The dimension of the left nullspace $\mathcal{N}\left(\mathbf{A}^\intercal\right)$.
(c) (0.5^{pts}) One particular solution, $\boldsymbol{x}_p$, to $\mathbf{A}\boldsymbol{x} = 2\mathbf{A}_{|2}$.
(d) (0.5^{pts}) The general (complete) solution to $\mathbf{A}\boldsymbol{x} = 2\mathbf{A}_{|2}$.

2.15. Grupo E curso 17/18

Exercise 1. Suppose $\mathbf{A}$ is the 4×4 matrix $\begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{bmatrix}$

(a) (0.5^{pts}) What is the rank of $\mathbf{A}$? (Hint: doing elimination is okay. You should notice a simple pattern.)
(b) (0.5^{pts}) Give a *basis* for $\mathcal{N}(\mathbf{A})$.
(c) (1^{pts}) For what vectors $\boldsymbol{b} \in \mathbb{R}^4$ does $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ have a solution? Give an equation in terms of the entries $b_1, \dots b_4$.

Exercise 2. Which of the following (if any) are subspaces? For any that are not a subspace, give an example of how they violate a property of subspaces.
(a) (0.5^{pts})Given 3×5 matrix $\mathbf{A}$ with full row rank, the set of all solutions to

$$\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(b) (0.5^{pts}) All vectors $\boldsymbol{x}$ with $\boldsymbol{x}\left[\boldsymbol{y}\right] = \mathbf{0}$ and $\boldsymbol{x}\left[\boldsymbol{z}\right] = \mathbf{0}$ for some given vectors $\boldsymbol{y}$ and $\boldsymbol{z}$.
(c) (0.5^{pts}) All 3×5 matrices with $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in their column space.
(d) (0.5^{pts}) All 5×3 matrices with $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in their nullspace.

MIT Course 18.06 Quiz 1, Spring, 2009

Exercise 3. Suppose that we do row operations on the matrix $\mathbf{A}$ to transform it to another matrix $\mathbf{B}$:

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 0 \\ 4 & 1 & -1 \\ 6 & 10 & 3 \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 0 \\ 0 & -5 & -1 \\ 0 & 1 & 3 \end{bmatrix} = \mathbf{B}.$$

For example, we subtracted twice the first row of $\mathbf{A}$ from the second row of $\mathbf{A}$ to get the second row of $\mathbf{B}$.
(a) (0.5^{pts}) Write $\mathbf{B}$ as a matrix product involving $\mathbf{A}$ and some other matrix.
(b) (1^{pts}) Which of $\mathcal{C}\left(\mathbf{A}^\intercal\right)$ and $\mathcal{N}\left(\mathbf{A}^\intercal\right)$ are the same as $\mathcal{C}\left(\mathbf{B}^\intercal\right)$ and $\mathcal{N}\left(\mathbf{B}^\intercal\right)$, if any? (No computation should be required! You don't have to compute these subspaces explicitly!)

Exercise 4. Consider the following 5 by 4 matrix $\mathbf{A}$ such that, after gaussian elimination by columns (multiplying by $\mathbf{E}$), its echelon form $\mathbf{L}$ has just only one column of zeros, in particular the last one.

$$\mathbf{A} = \begin{bmatrix} a & b & 3 & 3 \\ 1 & 2 & 3 & 3 \\ c & d & 4 & 5 \\ 1 & 1 & 1 & 2 \\ 1 & 3 & 0 & 4 \end{bmatrix}; \qquad \mathbf{E} = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \qquad \mathbf{AE} = \mathbf{L}.$$

Note the structure of $\mathbf{E}$ to decide if it has been applied a column exchange and then answer the following questions.

If you have to use some of the rows or columns of $\mathbf{A}$ with letters, treat those vectors as if the letters were known numbers. ***You are not asked to find the values*** *a*, *b*, *c* ***and*** *d*).

Note!: In some questions you are asked to write a ***basis***, in others you are asked to describe a ***vector space***. They are not the same thing! If you write a ***basis*** instead of describing a ***subspace***, or vice versa, the answer will be considered incorrect.

(a) (0.5^{pts}) Can you find a basis for the column space of $\mathbf{A}$? If yes, write down a ***basis*** for $\mathcal{C}\left(\mathbf{A}\right)$.
(b) (0.5^{pts}) Can you find a basis for the row space of $\mathbf{A}$? If yes, write down a ***basis*** for $\mathcal{C}\left(\mathbf{A}^\intercal\right)$. Otherwise, indicate what additional information about $\mathbf{L}$ would be sufficient to find such a basis.
(c) (0.5^{pts}) Can you find a basis for the null space of $\mathbf{A}$? If yes, describe the ***subspace*** $\mathcal{N}\left(\mathbf{A}\right)$.

(d) (0.5$^{\text{pts}}$) Can you find a basis for the left null space of $\mathbf{A}$? If yes, describe the ***subspace*** $\mathcal{N}\left(\mathbf{A}^{\mathsf{T}}\right)$.

EXERCISE 5. The 3 by 3 matrix $\mathbf{A}$ reduces to $\mathbf{I}$ by the following column operations:

1. Subtract column 1 from column 2
2. Add $2 \times$ (column 1) to column 3
3. Subtract $2 \times$ (column 3) from column 2
4. Add column 2 to column 1.

(a) (1$^{\text{pts}}$) What is $\mathbf{A}^{-1}$?
(b) (1$^{\text{pts}}$) What is $\mathbf{A}$?

Based on MIT Course 18.06 Quiz 1, March 10, 1995

EXERCISE 6. (0.5$^{\text{pts}}$) If $\mathbf{A}$ is 5×3, $\mathbf{B}$ is 4×5, and $\mathcal{C}\left(\mathbf{A}\right) = \mathcal{N}\left(\mathbf{B}\right)$, then what is $\mathbf{BA}$?

2.16. Grupo F curso 17/18

EXERCISE 1.

(a) (1^{pts}) Elimination matrices $\left(\mathsf{I}_{[(a)\overset{\boldsymbol{\tau}}{\mathbf{1}}+\mathbf{2}]}\right)$ and $\left(\mathsf{I}_{[(b)\overset{\boldsymbol{\tau}}{\mathbf{2}}+\mathbf{3}]}\right)$ will reduce A to triangular form. Find E so that $\mathsf{AE} = \mathsf{L}$ is lower triangular (echelon), when A is $\begin{bmatrix} 2 & 2 & 0 \\ 1 & 4 & 9 \\ 1 & 3 & 9 \end{bmatrix}$.

(b) (1^{pts}) Find a matrix U so that $\mathsf{A} = \mathsf{LU}$.

Based on MIT Course 18.06 Quiz 1, October 13, 1993

EXERCISE 2. Consider a 5 by 4 matrix A with the following reduced column echelon form R:

$$\mathsf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -2 & 3 & 0 \\ -4 & 5 & 6 & 0 \end{bmatrix}$$

Note!: In some questions you are asked to write a ***basis***, in others you are asked to describe a ***vector space***. They are not the same thing! If you write a ***basis*** instead of describing a ***subspace***, or vice versa, the answer will be considered incorrect.

(a) (0.5^{pts}) Can you find a basis for the column space of A? If yes, write down a ***basis*** for $\mathcal{C}\left(\mathsf{A}\right)$.

(b) (0.5^{pts}) Can you find a basis for the row space of A? If yes, write down a ***basis*** for $\mathcal{C}\left(\mathsf{A}^{\mathsf{T}}\right)$.

(c) (0.5^{pts}) Can you find a basis for the null space of A? If yes, describe the ***subspace*** $\mathcal{N}\left(\mathsf{A}\right)$.

(d) (0.5^{pts}) Can you find a basis for the left null space of A? If yes, describe the ***subspace*** $\mathcal{N}\left(\mathsf{A}^{\mathsf{T}}\right)$.

(e) (0.5^{pts}) Can we guess A? If so, write down A.

EXERCISE 3. Which of the following statements might possibly be true? Give an example of a possible matrix A for each possibly true statement.

(a) (0.5^{pts}) $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for a 5×3 matrix A.

(b) (0.5^{pts}) $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for a 3×5 matrix A.

(c) (0.5^{pts}) $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ is not solvable for *any* $\boldsymbol{b}$ (i.e., it is imposible to find $\boldsymbol{b}$ such that the system is solvable.

(d) (0.5^{pts}) $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ is not solvable for any $\boldsymbol{b} \neq \mathbf{0}$.

EXERCISE 4. You are given three vectors $\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$. Your goal is to find *a linear combination of these three vectors* (that is, multiply them by some numbers x_1, x_2, x_3 and add them) to give the vector $\boldsymbol{b} = \begin{pmatrix} 2, & -2, & 12, \end{pmatrix}$.

(a) (0.5^{pts}) Write the equation in matrix form.

(b) (1^{pts}) Solve it to find the correct linear combination $(x_1, x_2, x_3,)$ of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$.

(c) (0.5^{pts}) Change one number in $\boldsymbol{v}_3$ to make the problem have no solution for most vectors $\boldsymbol{b}$, but give a new vector $\boldsymbol{b}'$ for which there is still a solution. This new $\boldsymbol{b}'$ is in the ________________ space of the ________________ matrix.

(There are multiple correct answers for your new $\boldsymbol{v}'_3$ and your new $\boldsymbol{b}'$.)

MIT 18.06 - Quiz 1, Fall 2017

EXERCISE 5. Suppose you are given the $\mathsf{A} = \mathsf{L}\dot{\mathsf{U}}$ factorization of an invertible $n \times n$ matrix A. Now, suppose we want to solve

$$\begin{bmatrix} \mathsf{A} & \mathsf{B} \\ \mathbf{0} & \mathsf{A} \end{bmatrix} \boldsymbol{x} = \boldsymbol{b}$$

for some $n \times n$ matrix B, where $\mathbf{0}$ denotes an $n \times n$ block of zeros in the lower-left corner.

(a) (1^{pts}) Suppose we express $\boldsymbol{x} = \begin{pmatrix} \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix}$ where $\boldsymbol{y}$ and $\boldsymbol{z}$ are n-component vectors. Similarly, we express $\boldsymbol{b} = \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix}$ in terms of n-component vectors $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$. That is

$$\begin{bmatrix} \mathsf{A} & \mathsf{B} \\ \mathbf{0} & \mathsf{A} \end{bmatrix} \boldsymbol{x} = \boldsymbol{b} \qquad \text{is} \qquad \begin{bmatrix} \mathsf{A} & \mathsf{B} \\ \mathbf{0} & \mathsf{A} \end{bmatrix} \begin{pmatrix} \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix}.$$

Write the solution $\boldsymbol{y}$ and $\boldsymbol{z}$ in terms of A, B, $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ (or the inverses of those matrices).

(b) (0.5^{pts}) Write the solution $\boldsymbol{y}$ and $\boldsymbol{z}$ in terms of L, $\dot{\mathsf{U}}$, B, $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ (or the inverses of those matrices).

2.17. Grupo B curso 16/17

EXERCISE 1.

(a) (1^{pts} if all answers correct) Complete these sentences appropriately for a matrix $\underset{3\times 3}{\mathbf{A}}$.

- If the column space is a plane, the nullspace is a ______________________________
- If the column space is a line, the nullspace is a ______________________________
- If the column space is all $\mathbb{R}^3$, the nullspace is ______________________________
- If the column space is the zero vector, the nullspace is ______________________________

(b) (1^{pts}) Find a 7×7 matrix $\mathbf{A}$ whose column space equals its nullspace, or argue briefly it can not exist. (Hint: part (a) might provide a clue.)

MIT Course 18.06 Quiz 1, October 4, 2013

EXERCISE 2. For a 3 by 3 matrix $\mathbf{A}$, suppose we take the following sequence of elementary transformations: $\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}$, $\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}$, and $\underset{[(-3)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}$.

(a) (1^{pts}) What is $\mathbf{A}$, if that column elimination sequence reaches

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & g \end{bmatrix}?$$

(b) (1^{pts}) Find the $\mathbf{L}\dot{\mathbf{U}}$ factorization of $\mathbf{A}$.

(c) (1^{pts}) In case $g = 0$, the three columns of $\mathbf{A}$ must be dependent. Find the nullspace (a vector space) of $\mathbf{A}$.

(d) (0.5^{pts}) In case $g \neq 0$, what is the colum space of $\mathbf{L}$? What is the column space of the original matrix $\mathbf{A}$? How do you know?

based on MIT Course 18.06 Quiz 1, March 4, 2013

EXERCISE 3. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 2 & 2 \\ -1 & -2 & 0 & 0 & -1 \\ 1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

(a) (1^{pts}) Find the complete solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.

(b) (1^{pts}) Find the complete solution of the equation $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

(c) (1^{pts}) Find all vectors $\boldsymbol{b}$ such that the equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution.

(d) (0.5^{pts}) Find a matrix $\mathbf{B}$ such that $\mathcal{N}\left(\mathbf{A}\right) = \mathcal{C}\left(\mathbf{B}\right)$.

(e) (1^{pts}) Find bases of the four fundamental subspaces for the matrix $\mathbf{A}$.

MIT Course 18.06 Quiz 1, October 5, 2009

2.18. Grupo E curso 16/17

EXERCISE 1.

(a) (1^{pts}) By elimination find the **rank** of $\mathbf{A}$ and a basis for de column space of $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 6 & 3 & 9 \\ 2 & 4 & 2 & 9 \end{bmatrix}.$$

(b) (1^{pts}) Find the special solutions to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ and then find **all solutions** to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.

(c) (1^{pts}) For which number b_3 does $\boldsymbol{b} = \begin{pmatrix} 3 \\ 9 \\ b_3 \end{pmatrix}$ have a solution? Write the **complete solution** $\boldsymbol{x}$ (the general solution) with that value of b_3.

MIT Course 18.06 Quiz 1, Spring 2011

EXERCISE 2. Which of the following are subspaces? Explain why.

(a) (0.5^{pts}) All vectors $\boldsymbol{x}$ in $\mathbb{R}^3$ such that $\boldsymbol{x}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0$.

(b) (0.5^{pts}) All vectors $(x, y,)$ in $\mathbb{R}^2$ such that $x^2 - y^2 = 0$.

(c) (0.5^{pts}) All vectors $(x, y,)$ in $\mathbb{R}^2$ such that $x^2 - y^2 = 2$.

(d) (0.5^{pts}) All vectors $\boldsymbol{x}$ in $\mathbb{R}^3$ which are in the column space AND in the nullspace of the matrix $\begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix}$.

(e) (0.5^{pts}) All vectors $\boldsymbol{x}$ in $\mathbb{R}^3$ which are in the column space OR in the nullspace of the matrix $\begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix}$.

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EXERCISE 3. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix}$

(a) (1^{pts}) Find the factorization $\mathbf{A} = \mathbf{L}\dot{\mathbf{U}}$.

(b) (1^{pts}) Find the inverse of $\mathbf{A}$.

(c) (0.5^{pts}) For which values of c is the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 1 & c \end{bmatrix}$ invertible?

basado en MIT Course 18.06 Quiz 1, October 5, 2009

EXERCISE 4.

Let $\mathbf{A}$ be an m by n matrix. Let $\mathbf{B}$ be an n by m matrix. Suppose that $\mathbf{AB} = \underset{m\times m}{\mathbf{I}}$ is the m by m identity matrix.

(a) (1^{pts}) Let $r = rank(\mathbf{A})$ denote the rank of the matrix $\mathbf{A}$. Choose one answer and be sure to justify it.

a) $r \geq m$

b) $r \leq m$

c) $r = m$

d) $r > m$

(b) (1^{pts}) Is $m \leq n$ or is $n \leq m$? Why.

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2.19. Grupo E curso 15/16

EXERCISE 1. (2^{pts}) Find all solutions of the following system:

$$\begin{cases} x_1 + x_2 \qquad + x_4 = 7 \\ x_1 + x_2 + x_3 + x_4 = 10 \\ x_1 \qquad + x_3 + x_4 = 9 \end{cases}$$

EXERCISE 2. (1^{pts}) Are the following three vectors in $\mathbb{R}^4$ linearly independent or linearly dependent? Show your work and explain your answer.

$$\begin{bmatrix}\boldsymbol{u};\ \boldsymbol{v};\ \boldsymbol{w};\end{bmatrix} \quad \text{where} \quad \boldsymbol{u} = \begin{pmatrix}1\\2\\3\\-1\end{pmatrix}, \ \boldsymbol{v} = \begin{pmatrix}-1\\2\\-2\\4\end{pmatrix}, \ \boldsymbol{w} = \begin{pmatrix}3\\2\\8\\2\end{pmatrix}.$$

EXERCISE 3. Parts (a), (b) and (c) of this question all refer to the matrix $\mathbf{A}$ and the following gaussian elimination by columns

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{rrrrr} 1 & 2 & -3 & 1 & -3 \\ 2 & 4 & 1 & -3 & 5 \\ 1 & 2 & 2 & 1 & 12 \\ -3 & -6 & -2 & -1 & -20 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\tau_1,\ldots,\tau_n} \left[\begin{array}{rrrrr} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & -5 & 0 & 0 \\ 1 & 0 & 0 & 25 & 0 \\ -3 & 0 & 2 & -41 & 0 \\ \hline 1 & -2 & -1 & 8 & -20 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & -15 \\ 0 & 0 & 1 & 7 & -10 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right] = \left[\frac{\mathbf{L}}{\mathbf{E}}\right]$$

(you do not need to check this).

(a) (0.5^{pts}) Find a basis for the column space $\mathcal{C}\left(\mathbf{A}\right)$ and $\dim \mathcal{C}\left(\mathbf{A}\right)$.

(b) (0.5^{pts}) Find a basis for the null space $\mathcal{N}\left(\mathbf{A}\right)$ and $\dim \mathcal{N}\left(\mathbf{A}\right)$.

(c) (0.5^{pts}) If $\boldsymbol{v} = \begin{pmatrix}2\\-1\\-4\\0\\1\end{pmatrix}$ then $\mathbf{A}\boldsymbol{v} = \begin{pmatrix}9\\1\\4\\-12\end{pmatrix}$ (you do not need to check this).

Find all solutions of $\mathbf{A}\boldsymbol{x} = \begin{pmatrix}9\\1\\4\\-12\end{pmatrix}$.

EXERCISE 4. (1^{pts}) Find a lower triangular matrix $\mathbf{L}$ and an upper triangular matrix $\dot{\mathbf{U}}$ such that $\mathbf{A} = \mathbf{L}\dot{\mathbf{U}}$, where

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 0 \\ 1 & 1 & 3 \\ -1 & -3 & 2 \end{bmatrix}$$

EXERCISE 5.

(a) (0.5^{pts}) Suppose the matrices $\mathbf{A}$ and $\mathbf{B}$ have the same column space. Give an example where $\mathbf{A}$ and $\mathbf{B}$ have different nullspaces —or say why this is impossible.

(b) (0.5^{pts}) Again $\mathbf{A}$ and $\mathbf{B}$ have the same column space. Give an example where $\mathbf{A}$ and $\mathbf{B}$ have different ranks r – or say why this is impossible.

MIT Course 18.06. Exam I. Professor Strang. March 2, 2015

EXERCISE 6. Let $\mathbf{A}$ be a 3×4 matrix (3 rows, 4 columns). Let $\boldsymbol{v} = \begin{pmatrix}1\\2\\3\\4\end{pmatrix} \in \mathbb{R}^4$.

(a) (0.5^{pts}) Express $\mathbf{A}\boldsymbol{v}$ in terms of the columns $\mathbf{A}_{|j}$ of $\mathbf{A}$.
(b) (0.5^{pts}) Express $\mathbf{A}\boldsymbol{v}$ in terms of the rows ${}_{j|}\mathbf{A}$ of $\mathbf{A}$.
(c) (0.5^{pts}) Can the columns of $\mathbf{A}$ be linearly independent? Explain.
(d) (0.5^{pts}) How many solutions $\boldsymbol{x}$ are there to the system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$? None, one, infinitely many, or does it depend on $\mathbf{A}$?
(e) (0.5^{pts}) Find all possible pairs of numbers (p, q) so that p is the dimension of the null space of $\mathbf{A}$ and q is the dimension of the column space of $\mathbf{A}$.

Exercise 7. (1^{pts}) Show that for every linearly independent system $\left[\boldsymbol{u};\, \boldsymbol{v};\, \boldsymbol{w};\right]$ of vectors in $\mathbb{R}^n$, the system $\left[\boldsymbol{u};\, (\boldsymbol{u}+\boldsymbol{v});\, (\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w});\right]$ is also linearly independent.

2.20. Grupo H curso 15/16

EXERCISE 1. (2^{pts}) Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 1 & 3 & 7 \\ 2 & 2 & 2 \end{bmatrix}$$

What condition(s) must $\boldsymbol{b} \in \mathbb{R}^4$ satisfy to be in the column space of $\mathbf{A}$? (Your answer should be one or more equations of the form $?b_1 + ?b_2 + ?b_3 + ?b_4 = ?$)

EXERCISE 2.

(a) (1^{pts}) Compute the inverse of the matrix

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & -1 \end{bmatrix}$$

(b) (1^{pts}) For which value(s) of x is the matrix below not invertible? Explain your answer.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 5 & x & 6 \end{bmatrix}$$

EXERCISE 3. Suppose $\mathbf{A}$ is a 5×5 matrix with reduced column echelon form

$$\mathbf{AE} = \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 4 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}; \quad \text{where} \quad \mathbf{E} = \begin{bmatrix} 1 & -1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For each question below, give the answer when possible. Otherwise answer “not enough information”. (Please note that two issues are required in some parts below.)

(a) (0.5^{pts}) Find a basis for $\mathcal{C}\left(\mathbf{A}\right)$ and dimension: $\dim \mathcal{C}\left(\mathbf{A}\right)$).
(b) (0.5^{pts}) Find a basis for $\mathcal{N}\left(\mathbf{A}\right)$ and $\dim \mathcal{N}\left(\mathbf{A}\right)$.
(c) (0.5^{pts}) Find a basis for $\mathcal{C}\left(\mathbf{A}^{\intercal}\right)$ and $\dim \mathcal{C}\left(\mathbf{A}^{\intercal}\right)$.
(d) (0.5^{pts}) Find a basis for $\mathcal{N}\left(\mathbf{A}^{\intercal}\right)$ and $\dim \mathcal{N}\left(\mathbf{A}^{\intercal}\right)$.
(e) (0.5^{pts}) Find a vector $\boldsymbol{b} \in \mathbb{R}^5$ such that $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has no solutions.
(f) (0.5^{pts}) Find a vector $\boldsymbol{b} \in \mathbb{R}^5$ such that $\boldsymbol{x}\mathbf{A} = \boldsymbol{b}$ has no solutions.
(g) (0.5^{pts}) Are there vectors $\boldsymbol{b} \in \mathbb{R}^5$ such that $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution?

EXERCISE 4.

(a) (0.5^{pts}) Does there exist a matrix $\mathbf{B}$ whose column space is spanned by $(1, 2, 3,)$ and $(1, 0, 1,)$ and whose nullspace is spanned by $(1, 2, 3, 6,)$? If so, construct $\mathbf{B}$. If not, explain why not.
(b) (0.5^{pts}) Consider the set of all 3 by 3 matrices with the column vector $(1, 1, 1,)$ in their column space. Is this set of matrices a vector space or not? Yes or No with a reason.
(c) (0.5^{pts}) If the columns of a 5 by 3 matrix $\mathbf{M}$ are linearly independent and $\boldsymbol{x} \in \mathbb{R}^3$ is not the zero vector, then you know that $\mathbf{M}\boldsymbol{x}$ is not ____________________.
(I am looking for an answer that uses independence of columns and $\boldsymbol{x} \neq \mathbf{0}$).

MIT Course 18.06 Quiz 1. Spring, 2015

EXERCISE 5. (0.5^{pts}) How many solutions $(0, 1, \infty)$ does have $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$?

Please put your answers into the table below.

	$\mathcal{N}\left(\mathbf{A}\right) = \{\mathbf{0}\}$	$\mathcal{N}\left(\mathbf{A}\right) \neq \{\mathbf{0}\}$
$\boldsymbol{b} \in \mathcal{C}\left(\mathbf{A}\right)$		
$\boldsymbol{b} \notin \mathcal{C}\left(\mathbf{A}\right)$		

(0.5 points if all correct, 0 points otherwise.)

Exercise 6. *Determine whether the following statements are true or false, and explain your reasoning (to receive full credit you must explain your answers in a clear and concise way).*

(a) (0.5^{pts}) If the columns of a matrix $\underset{m\times n}{\mathbf{A}}$ are linearly independent, then there exist $\mathbf{A}^{-1}$.

2.21. Grupo A curso 14/15

EXERCISE 1. True or false (to receive full credit you must explain your answers in a clear and concise way)

(a) (0.5^{pts}) The columns of a 4×5 matrix must be linearly dependent.

(b) (0.5^{pts}) If a 3×3 matrix $\mathbf{A}$ is non-invertible, then the last column of $\mathbf{A}$ must be a linear combination of the first two columns of $\mathbf{A}$.

(c) (0.5^{pts}) Let $\mathbf{A}$ and $\mathbf{B}$ be $m \times n$ matrices. If $\mathbf{A}$ is row-equivalent to $\mathbf{B}$ (so $\mathbf{EA} = \mathbf{B}$, where $\mathbf{E}$ is invertible —i.e., $\mathbf{E}$ is product of elementary matrices) then $\mathcal{C}\left(\mathbf{A}\right) = \mathcal{C}\left(\mathbf{B}\right)$.

(d) (0.5^{pts}) If $5\mathbf{A}^4 - \mathbf{A}^2 + 2\mathbf{A} + 6\mathbf{I} = \mathbf{0}$, then $\mathbf{A}$ is invertible.

EXERCISE 2.

(a) (1^{pts}) Find the inverse of the following matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(b) (1^{pts}) Find the inverse of the following matrix **using the Gauss-Jordan method**

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & d \end{bmatrix}$$

Ejercicios 36, 38 y 59 de la sección 3.3 del manual de Poole

EXERCISE 3. Start with the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & -2 & 4 & 0 \\ 3 & -3 & 7 & 0 \end{bmatrix}$$

(a) (1^{pts}) Choose a basis of $\mathcal{C}\left(\mathbf{A}\right)$ and describe $\mathcal{C}\left(\mathbf{A}\right)$ using that basis. Write another basis for $\mathbf{A}$.

(b) (1^{pts}) Choose a basis of $\mathcal{N}\left(\mathbf{A}\right)$ and describe $\mathcal{N}\left(\mathbf{A}\right)$ using that basis

(c) (1^{pts}) Find a basis for the row space $\mathcal{C}\left(\mathbf{A}^{\intercal}\right)$

(d) (1^{pts}) Write the complete solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & -2 & 4 & 0 \\ 3 & -3 & 7 & 0 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

MIT Course 18.06 Quiz 1. Spring, 2015

EXERCISE 4. Which of the following subsets are subspaces in $\mathbb{R}^3$? (*Explain your answer*).

(a) (1^{pts}) $S_1 = \left\{\boldsymbol{x} \in \mathbb{R}^3 \text{ such that } x_1 = x_3\right\}$.

(b) (1^{pts}) $S_2 = \left\{\boldsymbol{x} \in \mathbb{R}^3 \text{ such that } x_1 = 2\right\}$.

2.22. Grupo E curso 14/15

EXERCISE 1. True or false (to receive full credit you must explain your answers in a clear and concise way)

(a) (0.5^{pts}) If $\mathbf{A}$ is a 4×3 matrix, and $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ are linearly *dependent* vectors in $\mathbb{R}^3$, then the vectors $\mathbf{A}\boldsymbol{v}_1$, $\mathbf{A}\boldsymbol{v}_2$, $\mathbf{A}\boldsymbol{v}_3$ must be dependent as well.

(b) (0.5^{pts}) If $\mathbf{A}$ is a 4×3 matrix, and $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ are linearly *independent* vectors in $\mathbb{R}^3$, then the vectors $\mathbf{A}\boldsymbol{v}_1$, $\mathbf{A}\boldsymbol{v}_2$, $\mathbf{A}\boldsymbol{v}_3$ must be independent as well.

(c) (0.5^{pts}) If $\mathbf{A}$ is a 3×5 matrix such that $\mathcal{N}\left(\mathbf{A}\right)$ is 2 dimensional, then the equation $\mathbf{A}\boldsymbol{x} = \begin{pmatrix}3\\1\\4\end{pmatrix}$ has infinitely many solutions.

EXERCISE 2. (0.5^{pts}) Give an example of a 5×4 matrix with rank 3.

EXERCISE 3.

Suppose $\mathbf{A}$ is reduced by the usual column operations to

$$\mathbf{R} = \begin{bmatrix}1&0&0\\4&0&0\\0&1&0\\2&2&0\end{bmatrix}; \quad \text{where} \quad \mathbf{E} = \begin{bmatrix}1&-1&3\\1&1&-2\\1&0&1\end{bmatrix}$$

is the product of the elementary matrices used in the column reduction process.

(a) (1^{pts}) Choose a basis of $\mathcal{C}\left(\mathbf{A}\right)$ and describe $\mathcal{C}\left(\mathbf{A}\right)$ using that basis.

(b) (1^{pts}) Choose a basis of $\mathcal{N}\left(\mathbf{A}\right)$ and describe $\mathcal{N}\left(\mathbf{A}\right)$ using that basis.

(c) (1^{pts}) Find the complete solution (if a solution exists) to this system involving the original $\mathbf{A}$:

$$\mathbf{A}\boldsymbol{x} = \text{ suma de las columnas de}\mathbf{A}.$$

(d) (1^{pts}) Write a basis of $\mathcal{N}\left(\mathbf{A}^{\intercal}\right)$

(e) (1^{pts}) Find the inverse of $\mathbf{E}$.

(f) (1^{pts}) Write $\mathbf{A}$.

(g) (1^{pts}) Write a basis of $\mathcal{C}\left(\mathbf{A}^{\intercal}\right)$.

(h) (1^{pts}) Find the $\mathbf{L}\dot{\mathbf{U}}$ factorization of $\mathbf{A}$.

Basado en MIT Course 18.06 Quiz 1. Spring, 2005

2.23. Grupo H curso 14/15

EXERCISE 1.

(a) (0.5^{pts}) If $\mathbf{A}$ is invertible, must the column space of $\mathbf{A}^{-1}$ be the same as $\mathcal{C}\left(\mathbf{A}\right)$?

(b) (0.5^{pts}) If $\mathbf{A}$ is square, must the column space of $\mathbf{A}^2$ be the same as $\mathcal{C}\left(\mathbf{A}\right)$?

MIT Course 18.06 Quiz 2. Fall, 2008

EXERCISE 2. $\mathbf{A}$ is a matrix with a nullspace $\mathcal{N}\left(\mathbf{A}\right) = \left\{\boldsymbol{x} \in \mathbb{R}^4 : \mathbf{A}\boldsymbol{x} = \mathbf{0}\right\}$ spanned by the following three vectors

$$\begin{pmatrix}1\\2\\-1\\3\end{pmatrix}, \quad \begin{pmatrix}0\\1\\1\\4\end{pmatrix}, \quad \begin{pmatrix}-1\\-1\\3\\1\end{pmatrix}.$$

(a) (1^{pts}) Give a matrix $\mathbf{B}$ such that its column space $\mathcal{C}\left(\mathbf{B}\right)$ is the same as $\mathcal{N}\left(\mathbf{A}\right)$. (There is more than one correct answer.) [Thus, any vector $\boldsymbol{y}$ in the nullspace of $\mathbf{A}$ satisfies $\mathbf{B}\boldsymbol{u} = \boldsymbol{y}$ for some $\boldsymbol{u}$.]

(b) (1^{pts}) Give a different possible answer to (a): another $\mathbf{B}$ with $\mathcal{C}\left(\mathbf{B}\right) = \mathcal{N}\left(\mathbf{A}\right)$.

(c) (1^{pts})For some vector $\boldsymbol{b}$, you are told that a particular solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is

$$\boldsymbol{x}_p = \begin{pmatrix}1\\2\\3\\4\end{pmatrix}$$

Now, your classmate Zarkon tells you that a second solution is:

$$\boldsymbol{x}_Z = \begin{pmatrix}1\\1\\3\\0\end{pmatrix}$$

while your other classmate Hastur tells you "No, Zarkon's solution can't be right, but here's a second solution that is correct:"

$$\boldsymbol{x}_H = \begin{pmatrix}1\\1\\3\\1\end{pmatrix}$$

Is Zarkon's solution correct, or Hastur's solution, or are both correct? (Hint: what should be true of $\boldsymbol{x} - \boldsymbol{x}_p$ if $\boldsymbol{x}$ is a valid solution?)

MIT Course 18.06 Quiz 1, Spring, 2009

EXERCISE 3. (1^{pts}) I was looking for an m by n matrix $\mathbf{A}$ and vectors $\boldsymbol{b}$, $\boldsymbol{c}$ such that $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has no solution and $\mathbf{A}^{\mathsf{T}}\boldsymbol{y} = \boldsymbol{c}$ has exactly one solution. Why can I not find $\mathbf{A}$, $\boldsymbol{b}$, and $\boldsymbol{c}$?

EXERCISE 4.

(a) (1^{pts}) Reduce $\mathbf{A} = \begin{bmatrix}-1 & -1 & 1\\-6 & -7 & 3\\1 & 1 & -1/2\end{bmatrix}$ to an lower triangular matrix $\mathbf{L}$;

(b) (1^{pts}) Factor the 3 by 3 matrix $\mathbf{A}$ into $\mathbf{L}\dot{\mathbf{U}}$ = (lower triangular)(upper triangular).

(c) (1^{pts}) If you change the last entry in $\mathbf{A}$ from -1/2 to ________ (what number gives $\mathbf{A}_{new}$?) then $\mathbf{A}_{new}$ becomes singular. Describe its column space exactly.

(d) (1^{pts}) In that singular case from part (c), what condition(s) on b_1, b_2, b_3 allow the system $(\mathbf{A}_{new})\boldsymbol{x} = \boldsymbol{b}$ to be solved?

(e) (1^{pts}) Write down the complete solution to $(\mathbf{A}_{new})\boldsymbol{x} = \begin{pmatrix}-1\\-6\\1\end{pmatrix}$ (the first column of $\mathbf{A}$).

Basado en MIT Course 18.06 Quiz 1. Spring, 2005

2.24. Grupo E curso 13/14

EXERCISE 1. Suppose that $\mathbf{A}$ is the matrix $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 6 & 5 \\ 2 & 4 \end{bmatrix}$.

(a) (0.5^{pts}) Explain in words how knowing all solutions to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ decides if a given vector $\boldsymbol{b}$ is in the column space of $\mathbf{A}$.

(b) (1^{pts}) Is the vector $\boldsymbol{b} = \begin{pmatrix} 8 \\ 28 \\ 14 \end{pmatrix}$ in the column space of $\mathbf{A}$?

MIT Course 18.06 Quiz 1, Spring 2010

EXERCISE 2. The 3 by 3 matrix $\mathbf{A}$ reduces to $\mathbf{I}$ by the following column operations:

1. Subtract $2 \times$ (column 1) from column 2
2. Subtract $3 \times$ (column 1) from column 3
3. Subtract column 3 from column 2
4. Subtract $3 \times$ (column 2) from column 1.

(a) (1^{pts}) What is $\mathbf{A}^{-1}$?

(b) (1^{pts}) What is $\mathbf{A}$?

Based on MIT Course 18.06 Quiz 1, March 10, 1995

EXERCISE 3. Suppose $\mathbf{A}$ is the matrix:

$$\mathbf{A} = \mathbf{L}\dot{\mathbf{U}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 1 & 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) (1^{pts}) Give a basis for the row space of $\mathbf{A}$ and a basis for the column space of $\mathbf{A}$.

(b) (1^{pts}) Describe explicitly all solutions to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.

(c) (1^{pts}) Find all solutions (if any, depending on c) to $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 4 \\ c \end{pmatrix}$.

Based on MIT Course 18.06 Quiz 1, March 10, 1995

EXERCISE 4. This question is about an m by n matrix for which

$$\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{has no solution; and} \quad \mathbf{A}\boldsymbol{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{has exactly one solution.}$$

(a) (1^{pts}) Give all possible information about m and n and the rank r of $\mathbf{A}$.

(b) (1^{pts}) If $\mathbf{A}\boldsymbol{x} = \mathbf{0}$, state one special fact about $\boldsymbol{x}$ (not just that $\boldsymbol{x}$ is in the nullspace).

(c) (1^{pts}) Write down an example of a matrix $\mathbf{A}$ that fits the description in this question.

(d) (0.5^{pts}) *(Not related to parts (a)–(c))* How do you know that the rank of a matrix stays the same if its first and last columns are exchanged?

MIT Course 18.06 Quiz 1, March 10, 1995

2.25. Grupo E curso 12/13

Exercise 1. (2^{pts}) Find all solutions to the linear system $\begin{cases} x+2y+\ \ z-2w=5 \\ 2x+4y+\ \ z+\ \ w=9 \\ 3x+6y+2z-\ \ w=14 \end{cases}$.

MIT Course 18.06 Quiz 1, October 3, 2007

Exercise 2. In class, we learned how to do "rightwards" elimination to put a matrix $\mathbf{A}$ in lower-triangular (or echelon) form $\mathbf{L}$: not counting column swaps, we subtract multiples of pivot columns from subsequent columns to put zeros to the right of the pivots, corresponding to right multiplying $\mathbf{A}$ by elimination matrices.

Instead, we could do elimination "upwards" by subtracting multiples of pivot rows from upwards rows, again to get a lower-triangular matrix $\mathbf{L}$ (left multiplying by elimination matrices). For example, let:

$$\mathbf{A} = \begin{bmatrix} 7 & 6 & 2 \\ 6 & 3 & 0 \\ 4 & 12 & 1 \end{bmatrix};$$

we could subtract twice the third row from the first row to eliminate the 2, so that we get zeros above the "pivot" 1 at the lower right.

(a) (1^{pts}) Continue this "upwards" elimination to obtain a lower-triangular matrix $\mathbf{L}$ from the $\mathbf{A}$ above, and write $\mathbf{L}$ in terms of $\mathbf{A}$ multiplied by a sequence of matrices corresponding to each upwards-elimination step.
(b) (1^{pts}) Suppose we followed this process for an arbitrary $\mathbf{A}$ (not necessarily square or invertible) to get an echelon matrix $\mathbf{L}$. Which of the column space and null space, if any, are the same between $\mathbf{A}$ and $\mathbf{L}$, and why?
(c) (1^{pts}) Is the $\mathbf{L}$ that we get by upwards elimination always the same as the $\mathbf{L}$ we get from ordinary "rightwards" elimination? Why or why not?

Based on MIT Course 18.06 Quiz 1, October 3, 2007

Exercise 3. Are the following sets of vectors in $\mathbb{R}^3$ vector subspaces? Explain your answer.

(a) (0.5^{pts}) vectors $(x, y, z,)$ such that $2x - 2y + z = 0$
(b) (0.5^{pts}) vectors $(x, y, z,)$ such that $x^2 - y^2 + z = 0$
(c) (0.5^{pts}) vectors $(x, y, z,)$ such that $2x - 2y + z = 1$
(d) (0.5^{pts}) vectors $(x, y, z,)$ such that $x = y$ AND $x = 2z$
(e) (0.5^{pts}) vectors $(x, y, z,)$ such that $x = y$ OR $x = 2z$

MIT Course 18.06 Quiz 1, March 5, 2007

Exercise 4. Let $\mathbf{A}$ be a 4×3 matrix with linearly independent columns.

(a) (0.5^{pts}) What are the dimensions of the four fundamental subspaces $\mathcal{C}(\mathbf{A})$, $\mathcal{N}(\mathbf{A})$, $\mathcal{C}(\mathbf{A}^\intercal)$, and $\mathcal{N}(\mathbf{A}^\intercal)$?
(b) (0.5^{pts}) Describe explicitly the nullspace $\mathcal{N}(\mathbf{A})$ and the row space $\mathcal{C}(\mathbf{A}^\intercal)$ of $\mathbf{A}$. (I do not ask the definition of these subspaces. I ask you to describe what vectors are contained in this particular case).
(c) (0.5^{pts}) Suppose that $\mathbf{B}$ is a 4×3 matrix such that the matrices $\mathbf{A}$ and $\mathbf{B}$ have exactly the same column spaces $\mathcal{C}(\mathbf{A}) = \mathcal{C}(\mathbf{B})$; and the same nullspaces $\mathcal{N}(\mathbf{A}) = \mathcal{N}(\mathbf{B})$; and the same row spaces $\mathcal{C}(\mathbf{A}^\intercal) = \mathcal{C}(\mathbf{B}^\intercal)$; and same left nullspaces $\mathcal{N}(\mathbf{A}^\intercal) = \mathcal{N}(\mathbf{B}^\intercal)$.

Are you sure that in this case $\mathbf{A} = \mathbf{B}$? YES NO . Prove that $\mathbf{A} = \mathbf{B}$ or give a counterexample where $\mathbf{A} \neq \mathbf{B}$.

MIT Course 18.06 Quiz 1, March 5, 2007

Exercise 5. Let $\mathbf{A} = \begin{bmatrix} 1 & a & 0 & d & 0 & g \\ 0 & b & 1 & e & 0 & h \\ 0 & c & 0 & f & 1 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}$.

(a) (0.5^{pts}) Find the complete solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{v}$, if $s = 1$.
(b) (0.5^{pts}) Find the complete solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{v}$, if $s = 0$.

Hint: Best if you don't work too hard.

MIT Course 18.06 Quiz 1, March 6, 2006

2.26. Grupo H curso 12/13

EXERCISE 1. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 5 \\ 1 & 3 & 5 & 9 \end{bmatrix}$.

(a) (0.5^{pts}) What is the rank of $\mathbf{A}$?
(b) (1^{pts}) Find a matrix $\mathbf{B}$ such that the column space $\mathcal{C}(\mathbf{A})$ equals the nullspace $\mathcal{N}(\mathbf{B})$.
(c) (0.5^{pts}) Which of the following vectors belong(s) to the column space $\mathcal{C}(\mathbf{A})$:

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 2 \\ 4 \\ 8 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \ ?$$

MIT Course 18.06 Quiz 1, March 5, 2007

EXERCISE 2. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 4 & k \end{bmatrix}$.

(a) (0.5^{pts}) For which values of k will the system $\mathbf{A}\boldsymbol{x} = (2, 3, 7,)$ have a unique solution?
(b) (0.5^{pts}) For which values of k will the system $\mathbf{A}\boldsymbol{x} = (2, 3, 7,)$ have an infinite number of solutions?
(c) (1^{pts}) For $k = 4$, find a lower triangular matriz $\mathbf{L}$ and an upper triangular matrix $\mathbf{U}$ such as $\mathbf{LU} = \mathbf{A}$. (*Hint:* the inverse of an upper triangular matrix is upper triangular)
(d) (0.5^{pts}) For all values of k, find the complete solution to the system $\mathbf{A}\boldsymbol{x} = (2, 3, 7,)$. (You might need to consider several cases).

MIT Course 18.06 Quiz 1, March 5, 2007

EXERCISE 3. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 & 0 & 0 \\ 1 & 2 & 0 & 2 & 2 \\ 1 & 2 & -1 & 0 & 0 \\ 2 & 4 & 0 & 4 & 4 \end{bmatrix}$.

(a) (0.5^{pts}) Find a basis of the column space $\mathcal{C}(\mathbf{A})$.
(b) (0.5^{pts}) Find a basis of the nullspace $\mathcal{N}(\mathbf{A})$.
(c) (0.5^{pts}) Find linear conditions on b_1, b_2, b_3, b_4 that guarantee that $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution.
(d) (0.5^{pts}) Find the complete solution for the system $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}$.

MIT Course 18.06 Quiz 1, March 5, 2007

EXERCISE 4. *Determine whether the following statements are true or false, and explain your reasoning (to receive full credit you must explain your answers in a clear and concise way).*

(a) (0.5^{pts}) If $\mathbf{A}^2 = \mathbf{A}$, then $\mathbf{A} = \mathbf{0}$ or $\mathbf{A} = \mathbf{I}$.
(b) (0.5^{pts}) All the 2×2 matrices that commute with $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$ (i.e. all 2×2 matrices $\mathbf{B}$ with $\mathbf{AB} = \mathbf{BA}$) form a vector space (with the usual formulas for addition andscalar multiplication of matrices).
(c) (0.5^{pts}) There is no 3×3 matrix whose column space equals its nullspace.
(d) (0.5^{pts}) If $\mathbf{A}$ is symetric, then so it is $\mathbf{A}^2$.
(e) (0.5^{pts}) If the columns of a matrix $\underset{m\times n}{\mathbf{A}}$ with $m \neq n$ are linearly independent, then there exist $\mathbf{A}^{-1}$.

(a), (b), (c): MIT Course 18.06 Quiz 1, October 3, 2007

EXERCISE 5. Suppose the columns of a 7 by 4 matrix $\mathbf{A}$ are linearly independent.

(a) (0.5^{pts}) After elementary operations reduce $\mathbf{A}$ to $\mathbf{L}$ or $\mathbf{R}$, how many columns will be $\mathbf{0}$ (or is it impossible to tell)?
(b) (0.5^{pts}) What is the row space of $\mathbf{A}$? Explain why this equation will surely be solvable:

$$\mathbf{A}^{\mathsf{T}}\boldsymbol{y} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Based on MIT Course 18.06 Quiz 1 February 28, 2005

2.27. Grupo E curso 11/12

EXERCISE 1.

(a) (1^{pts}) Elimination matrices $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ will reduce $\mathbf{A}$ to triangular form. Find $\mathbf{E}$ so that $\mathbf{AE} = \mathbf{L}$ is lower triangular (echelon), if $\mathbf{A}$ is

$$\begin{bmatrix} 2 & 2 & 0 \\ 1 & 4 & 9 \\ 1 & 3 & 9 \end{bmatrix}$$

(b) (1^{pts}) Find a matrix $\mathbf{U}$ so that $\mathbf{A} = \mathbf{LU}$.

Based on MIT Course 18.06 Quiz 1, October 13, 1993

EXERCISE 2.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 5 & 4 \\ 3 & 3 & 9 & 6 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 3 \\ 6 \\ c \end{pmatrix}$$

(a) (1^{pts}) Find a basis for the nullspace of $\mathbf{A}$.
(b) (1^{pts}) For which number c is the vector $\boldsymbol{b}$ in the column space of $\mathbf{A}$?
(c) (1^{pts}) Find the complete (general) solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ when c is chosen so that this equation is solvable.

MIT Course 18.06 Quiz 1, October 13, 1993

EXERCISE 3.

(a) (1^{pts}) Find a basis for the space of all vectors in $\mathbb{R}^4$ that are orthogonal (perpendicular) to both of these vectors:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}; \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}.$$

(b) (1^{pts}) If $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are three nonzero vectors in $\mathbb{R}^7$, what are the possible dimensions of the subspace they span?

MIT Course 18.06 Quiz 1, October 13, 1993

EXERCISE 4. Given a 5 by 3 matrix $\mathbf{A}$.

(a) (0.5^{pts}) How would you decide if the column $\boldsymbol{c} = \begin{pmatrix} 1, & 1, & 1, & 1, & 1, \end{pmatrix}$ is a linear combination of the columns of $\mathbf{A}$? One sentence please.
(b) (0.5^{pts}) How would you decide by row operations (not fair to transpose $\mathbf{A}$) if the vector $\boldsymbol{r} = \begin{pmatrix} 1, & 1, & 1, \end{pmatrix}$ is a combination of the rows of $\mathbf{A}$?
(c) (1^{pts}) If the decisions in (a) and (b) are both yes, what information do you have about the rank of $\mathbf{A}$? Full information please.
(d) (1^{pts}) If the decisions in (a) and (b) are both no, what information do you have about the rank of $\mathbf{A}$? Give reason also.

MIT Course 18.06 Quiz 1, March 10, 1995

2.28. Grupo H curso 11/12

EXERCISE 1. The 3 by 3 matrix $\mathbf{A}$ reduces to $\mathbf{I}$ by the following column operations:

1. Subtract $2 \times$ (column 1) from column 2
2. Subtract $3 \times$ (column 1) from column 3
3. Subtract column 3 from column 2
4. Subtract $3 \times$ (column 2) from column 1.

(a) (1^{pts}) What is $\mathbf{A}^{-1}$?
(b) (1^{pts}) What is $\mathbf{A}$?

Based on MIT Course 18.06 Quiz 1, March 10, 1995

EXERCISE 2. Suppose $\mathbf{A}$ is the matrix:

$$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) (1^{pts}) Give a basis for the row space of $\mathbf{A}$ and a basis for the column space of $\mathbf{A}$.
(b) (1^{pts}) Describe explicitly all solutions to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.
(c) (1^{pts}) Find all solutions (if any, depending on c) to $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 4 \\ c \end{pmatrix}$.

Based on MIT Course 18.06 Quiz 1, March 10, 1995

EXERCISE 3. This question is about an m by n matrix for which

$$\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{has no solution; and} \quad \mathbf{A}\boldsymbol{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{has exactly one solution.}$$

(a) (1^{pts}) Give all possible information about m and n and the rank r of $\mathbf{A}$.
(b) (1^{pts}) If $\mathbf{A}\boldsymbol{x} = \mathbf{0}$, state one special fact about $\boldsymbol{x}$ (not just that $\boldsymbol{x}$ is in the nullspace).
(c) (1^{pts}) Write down an example of a matrix $\mathbf{A}$ that fits the description in this question.
(d) (0.5^{pts}) *(Not related to parts (a)–(c))* How do you know that the rank of a matrix stays the same if its first and last columns are exchanged?

MIT Course 18.06 Quiz 1, March 10, 1995

EXERCISE 4. Suppose the rows of a 4 by 7 matrix A are linearly independent.

(a) (0.5^{pts}) After some column operations reduce $\mathbf{A}$ to $\mathbf{L}$ or $\mathbf{R}$, how many columns will be all zero (or is it impossible to tell)?
(b) (1^{pts}) What is the column space of $\mathbf{A}$? Explain why this equation will surely be solvable:

$$\mathbf{A}\boldsymbol{y} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

2.29. Grupo A curso 10/11

EXERCISE 1. Suppose $\mathbf{A}$ is the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 7 \\ 0 & 0 & 4 & 2 \end{bmatrix}.$$

(a) (1^{pts}) Find all special solutions to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ and describe in words the shape of the whole nullspace of $\mathbf{A}$.
(b) (1^{pts}) Describe the column space of this particular matrix $\mathbf{A}$. "All combinations of the four columns" is not a sufficient answer.
(c) (1^{pts}) What is the reduced row echelon form $\mathbf{R}^* = \text{rref}(\mathbf{B})$ when $\mathbf{B}$ is the 6 by 8 block matrix

$$\mathbf{B} = \begin{bmatrix} \mathbf{A} & \mathbf{A} \\ \mathbf{A} & \mathbf{A} \end{bmatrix}, \text{ using the same } \mathbf{A}?$$

MIT Course 18.06 Quiz 1, Spring 2010

EXERCISE 2. This question is about an m by n matrix A for which

$$\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ has } \textbf{no solutions} \quad \text{and} \quad \mathbf{A}\boldsymbol{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ has } \textbf{exactly one solution}.$$

(a) (1^{pts}) Give all possible information about m and n and the rank r of $\mathbf{A}$.
(b) (1^{pts}) Find all solutions to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ and **explain your answer**.
(c) (1^{pts}) Write down an example of a matrix $\mathbf{A}$ that fits the description in part (a).

MIT Course 18.06 Quiz 1, Fall 2008

EXERCISE 3. Suppose a 3 by 5 matrix $\mathbf{A}$ has rank $r = 3$.

(a) (0.75^{pts}) **Circle the words** that correctly complete the following sentence:

Then the equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ (always / sometimes but not always)
has (a unique solution / many solutions / no solution) .

(b) (0.75^{pts}) What is the column space of $\mathbf{A}$? Describe the nullspace of $\mathbf{A}$.

MIT Course 18.06 Quiz 1, Spring 2010

EXERCISE 4.

(a) (1^{pts}) Find a parametric representation for the line passing through the points $\boldsymbol{p} = (2, 2,)$ and $\boldsymbol{q} = (-1, 4,)$.
(b) (1^{pts}) Find a implicit representation for the same line.

2.30. Grupo E curso 10/11

El ejercicio siguiente está pensado en función de la eliminación por filas y no tiene mucho sentido transformalo a un ejercicio de eliminación por columnas, salvo que se quiera plantear un sistema del tipo $\boldsymbol{x}\mathbf{A} = \boldsymbol{b}$. Como un sistema así es inusual, lo dejo tal cual...

EXERCISE 1. Forward ROW elimination ("the Gaussian part") changes $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ to a row reduced equivalent system $\mathbf{R}\boldsymbol{x} = \boldsymbol{d}$: the complete solution is

$$\boldsymbol{x} = \begin{pmatrix}4\\0\\0\end{pmatrix} + c_1\begin{pmatrix}2\\1\\0\end{pmatrix} + c_2\begin{pmatrix}5\\0\\1\end{pmatrix}$$

(a) (1^{pts}) What is the 3 by 3 reduced row echelon matrix $\mathbf{R}$ and what is $\boldsymbol{d}$?
(b) (1.5^{pts}) If the process of elimination subtracted 3 times row 1 from row 2 and then 5 times row 1 from row 3, what matrix connects $\mathbf{R}$ and $\boldsymbol{d}$ to the original $\mathbf{A}$ and $\boldsymbol{b}$? Use this matrix to find $\mathbf{A}$ and $\boldsymbol{b}$.

MIT Course 18.06 Quiz 1, Spring 2010

EXERCISE 2. Consider $\mathbf{A} = \begin{bmatrix}1&2&3&-1\\-1&1&0&-1\\5&4&9&-1\\1&2&3&-1\\0&1&1&2\end{bmatrix}$ with its reduced echelon form: $\mathbf{R} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\3&-2&0&0\\1&0&0&0\\0&0&0&1\end{bmatrix}$.

(a) (0.75^{pts}) What is the rank of $\mathbf{A}$? What are the dimensions of the column space $\mathcal{C}\left(\mathbf{A}\right)$, the row space $\mathcal{C}\left(\mathbf{A}^\intercal\right)$ and the nullspace $\mathcal{N}\left(\mathbf{A}\right)$?
(b) (0.75^{pts}) Find a basis for the column space $\mathcal{C}\left(\mathbf{A}\right)$.
(c) (0.75^{pts}) Find a basis of the row space $\mathcal{C}\left(\mathbf{A}^\intercal\right)$.
(d) (1^{pts}) Find a basis for the left nullspace $\mathcal{N}\left(\mathbf{A}^\intercal\right)$.
(e) (0.75^{pts}) Write down the third row of $\mathbf{A}$ as a linear combination of ${}_{1|}\mathbf{A}$, ${}_{2|}\mathbf{A}$, ${}_{4|}\mathbf{A}$ and ${}_{5|}\mathbf{A}$.

EXERCISE 3. Suppose a 3 by 5 matrix $\mathbf{A}$ has rank $r = 3$.

(a) (0.75^{pts}) **Circle the words** that correctly complete the following sentence:

Then the equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ (always / sometimes but not always)
has (a unique solution / many solutions / no solution) .

(b) (0.75^{pts}) What is the column space of $\mathbf{A}$? Describe the nullspace of $\mathbf{A}$.

MIT Course 18.06 Quiz 1, Spring 2010

EXERCISE 4.

(a) (1^{pts}) Find a parametric representation for the line passing through the points $\boldsymbol{p} = (2, 4,)$ y $\boldsymbol{q} = (1, 3,)$.
(b) (1^{pts}) Find a implicit representation for the same line.

2.31. Grupo G curso 10/11

EXERCISE 1.

(a) (1^{pts}) By elimination find the **rank** of $\mathbf{A}$ and the pivot columns of $\mathbf{A}$ (in its column space):

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 6 & 3 & 9 \\ 2 & 4 & 2 & 9 \end{bmatrix}.$$

(b) (1^{pts}) Find the special solutions to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ and then find **all solutions** to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.

(c) (1^{pts}) For which number b_3 does $\boldsymbol{b} = \begin{pmatrix} 3 \\ 9 \\ b_3 \end{pmatrix}$ have a solution? Write the **complete solution** $\boldsymbol{x}$ (the general solution) with that value of b_3.

MIT Course 18.06 Quiz 1, Spring 2011

EXERCISE 2. Suppose $\mathbf{A}$ is a 3 by 5 matrix and the equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution for every $\boldsymbol{b}$. What are (a)(b)(c)(d)? (If you don't have enough information to answer, tell as much about the answer as you can).

(a) (1^{pts}) Column space of $\mathbf{A}$
(b) (1^{pts}) Nullspace of $\mathbf{A}$
(c) (0.75^{pts}) Rank of $\mathbf{A}$
(d) (0.75^{pts}) Rank of the 6 by 5 matrix: $\mathbf{B} = \begin{bmatrix} \mathbf{A} \\ \mathbf{A} \end{bmatrix}$.

MIT Course 18.06 Quiz 1, Spring 2011

EXERCISE 3. Suppose a 3 by 5 matrix $\mathbf{A}$ has rank $r = 3$.

(a) (0.75^{pts}) **Circle the words** that correctly complete the following sentence:

Then the equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ (always / sometimes but not always)
has (a unique solution / many solutions / no solution) .

(b) (0.75^{pts}) What is the column space of $\mathbf{A}$? Describe the nullspace of $\mathbf{A}$.

MIT Course 18.06 Quiz 1, Spring 2010

EXERCISE 4.

(a) (1^{pts}) Find a parametric representation for the line passing through the points $\boldsymbol{p} = (1, -1,)$ y $\boldsymbol{q} = (0, 1,)$.
(b) (1^{pts}) Find a implicit representation for the same line.

2.32. Grupo F curso 09/10

EXERCISE 1.

(a) (1^{pts}) Find the inverse of the following matrices $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(b) (1^{pts}) Find the inverse of the following matrix **using the Gauss-Jordan method**

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & d \end{bmatrix}$$

Ejercicios 36, 38 y 59 de la sección 3.3 del manual de Poole

EXERCISE 2. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 & 0 & 0 \\ 1 & 2 & 0 & 2 & 2 \\ 1 & 2 & -1 & 0 & 0 \\ 2 & 4 & 0 & 4 & 4 \end{bmatrix}$

(a) (0.5^{pts}) Find a basis of the column space $\mathcal{C}(\mathbf{A})$.
(b) (0.5^{pts}) Find a basis of the nullspace $\mathcal{N}(\mathbf{A})$.
(c) (0.5^{pts}) Find linear conditions on b_1, b_2, b_3, b_4 that guarantee that the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution.
(d) (0.5^{pts}) Find the complete solution for the system $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}$.

MIT Course 18.06 Quiz 1, March 5, 2007

EXERCISE 3. For each of these statements, say whether the claim is true or false and give a brief justification.

(a) **True/False:** The set of 3×3 non-invertible matrices forms a subspace of the set of all 3×3 matrices.
(b) **True/False:** If the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has no solution then $\mathbf{A}$ does not have full row rank.
(c) **True/False:** There exist $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{B}$ is not invertible but $\mathbf{AB}$ is invertible.
(d) **True/False:** For any permutation matrix $\mathbf{P}$, we have that $\mathbf{P}^2 = \mathbf{I}$.

MIT Course 18.06 Quiz 1, October 4, 2004

EXERCISE 4. $\mathbf{A}$ is a matrix with a nullspace $\mathcal{N}(\mathbf{A}) = \{\boldsymbol{x} \in \mathbb{R}^4 : \mathbf{A}\boldsymbol{x} = \mathbf{0}\}$ spanned by the following system

$$\left[\begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \\ 1 \\ 4 \end{pmatrix}; \begin{pmatrix} -1 \\ -1 \\ 3 \\ 1 \end{pmatrix}; \right].$$

(a) (1^{pts}) Give a matrix $\mathbf{B}$ such that its column space $\mathcal{C}(\mathbf{B})$ is the same as $\mathcal{N}(\mathbf{A})$. (There is more than one correct answer.) [Thus, any vector $\boldsymbol{y}$ in the nullspace of $\mathbf{A}$ satisfies $\mathbf{B}\boldsymbol{u} = \boldsymbol{y}$ for some $\boldsymbol{u}$.]
(b) (1^{pts}) Give a different possible answer to (a): another $\mathbf{B}$ with $\mathcal{C}(\mathbf{B}) = \mathcal{N}(\mathbf{A})$.
(c) (1^{pts}) For some vector $\boldsymbol{b}$, you are told that a particular solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is

$$\boldsymbol{x}_p = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

Now, your classmate Zarkon tells you that a second solution is:

$$\boldsymbol{x}_Z = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \end{pmatrix}$$

while your other classmate Hastur tells you "No, Zarkon's solution can't be right, but here's a second solution that is correct:"

$$\boldsymbol{x}_H = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \end{pmatrix}$$

Is Zarkon's solution correct, or Hastur's solution, or are both correct? (Hint: what should be true of $\boldsymbol{x} - \boldsymbol{x}_p$ if $\boldsymbol{x}$ is a valid solution?)

MIT Course 18.06 Quiz 1, Spring, 2009

2.33. Grupo H curso 09/10

EXERCISE 1. The 3 by 3 matrix $\mathbf{A}$ reduces to the identity matrix $\mathbf{I}$ by the following three *row operations* (in order):

$$\begin{array}{cl} \underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}: & \text{Subtract } 4 \times (\text{row } 1) \text{ from row } 2. \\ \underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}: & \text{Subtract } 3 \times (\text{row } 1) \text{ from row } 3. \\ \underset{[(-1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}: & \text{Subtract row } 3 \text{ from row } 2. \end{array}$$

(a) Write the inverse matrix $\mathbf{A}^{-1}$ in terms of the corresponding elementary matrices. **Then compute $\mathbf{A}^{-1}$.**
(b) What is the original matrix $\mathbf{A}$?

MIT Course 18.06 Quiz 1, October 4, 2006

EXERCISE 2. Which of the following (if any) are subspaces? For any that are not a subspace, give an example of how they violate a property of subspaces.
(a) Given 3×5 matrix $\mathbf{A}$ with full row rank, the set of all solutions to

$$\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(b) All vectors $\boldsymbol{x}$ with $[\boldsymbol{x}]^{\intercal}[\boldsymbol{y}] = 0$ and $[\boldsymbol{x}]^{\intercal}[\boldsymbol{z}] = 0$ for some given vectors $\boldsymbol{y}$ and $\boldsymbol{z}$.
(c) All 3×5 matrices with $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in their column space.
(d) All 5×3 matrices with $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in their nullspace.

MIT Course 18.06 Quiz 1, Spring, 2009

EXERCISE 3. Find the complete solution (also called the *general solution*) to

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

Exercise 4, section 3.4 del manual de Strang 2005

EXERCISE 4. By performing column eliminations (and possibly permutations) on the following 4×8 matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & -2 & 2 & 0 \\ -2 & 1 & -2 & 2 \\ 1 & -2 & 1 & 2 \\ -5 & 3 & -5 & 4 \\ 0 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 \\ -3 & 0 & 1 & 2 \end{bmatrix} \quad \text{we got the following matrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) What is the rank of $\mathbf{A}$?
(b) What are the dimensions of the 4 fundamental subspaces?
(c) How many solutions does $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ have? Does it depend on $\boldsymbol{b}$? Justify.
(d) Are the columns of $\mathbf{A}$ linearly independent? Why?
(e) Do rows 4, 5, 6 and 7 of $\mathbf{A}$ form a basis of $\mathbb{R}^4$? Why?
(f) Give a basis of $\mathcal{N}(\mathbf{A})$.
(g) Give a basis of $\mathcal{N}(\mathbf{A}^{\intercal})$.
(h) (Again calculations are not necessary for this part.) Let $\mathbf{B} = \mathbf{AE}$. Is $\mathbf{E}$ invertible? If so, what is the inverse of $\mathbf{E}$?

MIT Course 18.06 Quiz 1, October 4, 2004

References

Strang, G. (). 18.06 linear algebra. Massachusetts Institute of Technology: MIT OpenCourseWare. License: Creative Commons BY-NC-SA.
URL http://ocw.mit.edu

Strang, G. (2003). *Introduction to Linear Algebra.* Wellesley-Cambridge Press, Wellesley, Massachusetts. USA, third ed. ISBN 0-9614088-9-8. 21

Solutions to Exercises

(Grupo C curso 24/25) Exercise 1(a)

$$\text{A basis of } \mathcal{C}\left(\mathbf{A}\right) = \left[\begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \end{pmatrix}; \right]. \quad \mathcal{C}\left(\mathbf{A}\right) = \left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2,\ \boldsymbol{v} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 2 & -2 \end{bmatrix} \boldsymbol{p} \right\}$$

□

(Grupo C curso 24/25) Exercise 1(b)

$$\text{A basis of } \mathcal{N}\left(\mathbf{A}\right) = \left[\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}; \right]. \qquad \mathcal{N}\left(\mathbf{A}\right) = \left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \boldsymbol{p} \right\}.$$

The basis cames from the last column of $\mathbf{E}$.

□

(Grupo C curso 24/25) Exercise 1(c) Since $\mathbf{A}\left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right) = \mathbf{A}_{|1} + \mathbf{A}_{|2} + \mathbf{A}_{|3}$, the particular solution $(1, 1, 1,)$ gives $\mathbf{A}\boldsymbol{x} =$ sum of the columns of $\mathbf{A}$. Hence, the set of solutions is

$$\left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \boldsymbol{p} \right\}.$$

□

(Grupo C curso 24/25) Exercise 1(d) Since $\mathcal{N}\left(\mathbf{A}^{\intercal}\right) = \mathcal{N}\left(\mathbf{R}^{\intercal}\right)$; and since two rows of $\mathbf{R}$ have no pivots, we can search two linear combinations of rows of $\mathbf{R}$:

$$\text{Since} \quad \begin{pmatrix} 1, & 1, & 0, & 0, \end{pmatrix} \mathbf{R} = \mathbf{0} \quad \text{and since} \quad \begin{pmatrix} -2, & 0, & 2, & 1, \end{pmatrix} \mathbf{R} = \mathbf{0}: \qquad \text{Basis: } \left[\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} -2 \\ 0 \\ 2 \\ 1 \end{pmatrix}; \right].$$

□

(Grupo C curso 24/25) Exercise 1(e)

$$\left[\begin{array}{ccc} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & -2 & 3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ [(2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 3 \\ \hline 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \\ \hline 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(2)\mathbf{3}+\mathbf{2}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 4 & 1 \\ 0 & 3 & 1 \\ 0 & 2 & 1 \end{array}\right] = \left[\begin{array}{c} \mathbf{I} \\ \hline \mathbf{E}^{-1} \end{array}\right]$$

$$\text{So } \mathbf{E}^{-1} = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 3 & 1 \\ 0 & 2 & 1 \end{bmatrix}; \text{ since } \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 1 \\ 0 & 3 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

(Grupo C curso 24/25) Exercise 1(f) Since $\mathbf{AE} = \mathbf{R}$, then $\mathbf{A} = \mathbf{R}(\mathbf{E}^{-1})$:

$$\mathbf{A} = \mathbf{R}(\mathbf{E}^{-1}) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 & 1 \\ 0 & 3 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 1 \\ -1 & -4 & -1 \\ 0 & 3 & 1 \\ 2 & 2 & 0 \end{bmatrix}$$

□

(Grupo C curso 24/25) Exercise 1(g) The pivot rows (rows 1 and 3) of $\mathbf{A}$ are a basis of $\mathcal{C}\left(\mathbf{A}^{\intercal}\right)$.

$$\text{Basis} = \left[\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}; \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}; \right]$$

□

(Grupo C curso 24/25) Exercise 2(a) The simplest way to provide an example is to add dependent columns to matrix $\mathbf{A}$. For example, matrices $\begin{bmatrix}1\end{bmatrix}$, $\begin{bmatrix}1 & 0\end{bmatrix}$, and $\begin{bmatrix}0 & 1\end{bmatrix}$ have the same column space and different null spaces.

$\square$

(Grupo C curso 24/25) Exercise 2(b) The rank is the dimension of the column space. That means the rank is the same for both matrices.

$\square$

(Grupo C curso 24/25) Exercise 3(a) **False.** Counterexample: if $\mathbf{A} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, then $\mathbf{A}^2 = \mathbf{A}$ but $\mathbf{A} \neq \mathbf{I}$ and $\mathbf{A} \neq \mathbf{0}$.

Note that if we assume $\mathbf{A}$ is invertible, then the only solution is $\mathbf{A} = \mathbf{I}$ (multiply both sides of $\mathbf{A}^2 = \mathbf{A}$ by $\mathbf{A}^{-1}$), but this assumption is not warranted here.

$\square$

(Grupo C curso 24/25) Exercise 3(b) **True.** Suppose the rank of $\mathbf{A}$ is r, then the dimension of column space is r, and the dimension of null space is $3 - r$. Obviously no matter $r = 0, 1, 2, 3$, we always have $r \neq 3 - r$. (Equivalently, $r = 3 - r$ would imply a fractional rank $r = 3/2$!) This shows that the two spaces are not the same, since they must have different dimensions. Note that if the matrix is of even order n, it can be true. For example: $\left[\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right]$.

$\square$

(Grupo C curso 24/25) Exercise 3(c) **True**: $\left(\mathbf{A}^2\right)^{\intercal} = \left(\mathbf{A}\mathbf{A}\right)^{\intercal} = (\mathbf{A}^{\intercal})(\mathbf{A}^{\intercal}) = \mathbf{A}\mathbf{A} = \mathbf{A}^2$

$\square$

(Grupo C curso 24/25) Exercise 4.

	$\mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$	$\mathcal{N}(\mathbf{A}) \neq \{\mathbf{0}\}$
$\boldsymbol{b} \in \mathcal{C}(\mathbf{A})$	1	∞
$\boldsymbol{b} \notin \mathcal{C}(\mathbf{A})$	0	0

$\square$

(Grupo E curso 24/25) Exercise 1(a) A possible answer is $\mathbf{A}\begin{pmatrix}2\\-2\\1\end{pmatrix} = \mathbf{0}$; since

$$\left[\begin{array}{ccc}1&2&2\\4&5&2\\7&8&2\\\hline 1&0&0\\0&1&0\\0&0&1\end{array}\right] \xrightarrow[{[(-2)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau}\\{[(-2)\mathbf{1}+\mathbf{2}]}}} \left[\begin{array}{ccc}1&0&0\\4&-3&-6\\7&-6&-12\\\hline 1&-2&-2\\0&1&0\\0&0&1\end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\{[(-2)\mathbf{2}+\mathbf{3}]}}} \left[\begin{array}{ccc}1&0&0\\4&-3&0\\7&-6&0\\\hline 1&-2&2\\0&1&-2\\0&0&1\end{array}\right]$$

However, since they only ask for a linear combination, it would suffice to answer $\mathbf{A}\left(\begin{smallmatrix}0\\0\\0\end{smallmatrix}\right) = \mathbf{0}$.

$\square$

(Grupo E curso 24/25) Exercise 1(b) Since the columns of the 3×3 matrix $\mathbf{A}$ are linearly dependent, they span a subspace of dimension less than 3. The columns of $\mathbf{AB}$, being linear combinations of $\mathbf{A}$'s columns, also span a space of dimension less than 3. Thus, the 3×3 matrix $\mathbf{AB}$ cannot be invertible.

Alternatively, since $\mathbf{A}$ is singular, there exists a $\boldsymbol{x} \neq \mathbf{0}$ such that $\boldsymbol{x}\mathbf{A} = \mathbf{0}$. If $\mathbf{AB}$ were invertible, this would imply $\boldsymbol{x} = \boldsymbol{x}\mathbf{AB}(\mathbf{AB})^{-1} = \mathbf{0B}(\mathbf{AB})^{-1} = \mathbf{0}$, a contradiction, since $\boldsymbol{x} \neq \mathbf{0}$.

$\square$

(Grupo E curso 24/25) Exercise 2. The simplest example is $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$

$\square$

(Grupo E curso 24/25) Exercise 3(a) By elimination we can check that $\mathrm{rg}\,(\mathbf{A}) = 2$ (Elimination on the augmented matrix $[\mathbf{A}| - \boldsymbol{b}]$ will help answer the second question.)

$$\left[\begin{array}{ccc|c}1&1&1&-b_1\\1&2&3&-b_2\\1&3&5&-b_3\end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\{[(-1)\mathbf{1}+\mathbf{2}]}\\{[(-1)\mathbf{1}+\mathbf{3}]}\\{[(b_1)\mathbf{1}+\mathbf{4}]}}} \left[\begin{array}{ccc|c}1&0&0&0\\1&1&2&b_1-b_2\\1&2&4&b_1-b_3\end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\{[(-1)\mathbf{2}+\mathbf{1}]}\\{[(-2)\mathbf{2}+\mathbf{3}]}\\{[(-b_1+b_2)\mathbf{2}+\mathbf{4}]}}} \left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\-1&2&0&-b_1+2b_2-b_3\end{array}\right]$$

$\square$

(Grupo E curso 24/25) Exercise 3(b) We see from the last row of the pre-echelon for of the augmented matrix that the condition for a vector to be in the column space is $b_1 - 2b_2 + b_3 = 0$. We easily reach the same conclusion if we use the non-null columns of the reduced echelon form of $\mathbf{A}$ as a basis:

$$\boldsymbol{b} \in \mathcal{C}\left(\mathbf{A}\right) \iff \text{there exist scalars } b_1 \text{ and } b_2 \text{ such that } \boldsymbol{b} = b_1\begin{pmatrix}1\\0\\-1\end{pmatrix} + b_2\begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}b_1\\b_2\\-b_1+2b_2\end{pmatrix}.$$

Thus, for $\boxed{\mathbf{B} = \begin{bmatrix}1 & -2 & 1\end{bmatrix}}$ we have

$$\mathcal{C}\left(\mathbf{B}\right) = \left\{\boldsymbol{b} \in \mathbb{R}^3 \mid b_1 - 2b_2 + b_3 = 0\right\} = \left\{\boldsymbol{x} \in \mathbb{R}^3 \,\middle|\, \begin{bmatrix}1 & -2 & 1\end{bmatrix}\boldsymbol{x} = \mathbf{0}\right\} = \mathcal{N}\left(\mathbf{B}\right).$$

Another way to find such a matrix $\mathbf{B} = \begin{bmatrix}a & b & c\end{bmatrix}$ is by solving $\mathbf{BA} = \mathbf{0}$, that is, $\begin{bmatrix}a & b & c\end{bmatrix}\mathbf{A} = \begin{bmatrix}0 & 0 & 0\end{bmatrix}$.

□

(Grupo E curso 24/25) Exercise 3(c) The last two vectors can't belong to the column space because they are in $\mathbb{R}^4$. From the condition in part (b), we see that $\left(2, \; 0, \; -2,\right)$ is in the column space $\mathcal{C}\left(\mathbf{A}\right)$, since $\mathbf{B}\left(2, \; 0, \; -2,\right) = \left(0,\right)$ but $\left(1, \; -2, \; 1,\right)$ is not, since. $\mathbf{B}\left(1, \; -2, \; 1,\right) = \left(6,\right)$. Another way to see it is by elimination:

$$\left[\begin{array}{ccc|c|c}1&1&1&2&1\\1&2&3&0&-2\\1&3&5&-2&1\end{array}\right] \xrightarrow[\substack{[(-1)\mathbf{1}+\mathbf{3}]\\ [(-2)\mathbf{1}+\mathbf{4}]\\ [(-1)\mathbf{1}+\mathbf{5}]}]{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc|c|c}1&0&0&0&0\\1&1&2&-2&-3\\1&2&4&-4&0\end{array}\right] \xrightarrow[\substack{[(2)\mathbf{2}+\mathbf{4}]\\ [(3)\mathbf{2}+\mathbf{5}]}]{\substack{\boldsymbol{\tau}\\ [(-2)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc|c|c}1&0&0&0&0\\1&1&0&0&0\\1&2&0&0&6\end{array}\right]$$

□

(Grupo E curso 24/25) Exercise 4(a) The four elementary column operations in the Gauss-Jordan elimination process are $\mathbf{A}\Big(\mathbf{I}_{\underset{[(-2)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\underset{[(1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(-1)\mathbf{3}]}{\boldsymbol{\tau}}}\Big) = \mathbf{I}$; hence, $\mathbf{A}^{-1} = \Big(\mathbf{I}_{\underset{[(-2)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\underset{[(1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(-1)\mathbf{3}]}{\boldsymbol{\tau}}}\Big)$. So

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{\underset{[(-2)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}} \begin{bmatrix}1&0&-2\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{\underset{[(1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}} \begin{bmatrix}1&-2&-2\\0&1&0\\0&1&1\end{bmatrix} \xrightarrow{\underset{[(-1)\mathbf{3}]}{\boldsymbol{\tau}}} \begin{bmatrix}1&-2&2\\0&1&0\\0&1&-1\end{bmatrix} = \mathbf{A}^{-1}$$

Or, using products of elementary matrices:

$$\mathbf{A}^{-1} = \begin{bmatrix}1&0&-2\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix} = \begin{bmatrix}1&-2&-2\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix} = \begin{bmatrix}1&-2&2\\0&1&0\\0&1&-1\end{bmatrix}.$$

□

(Grupo E curso 24/25) Exercise 4(b) Computing the inverse elementary operations on $\mathbf{I}$ (but in the reverse order) we can compute de inverse of $\mathbf{A}^{-1}$ in order to get $\mathbf{A}$:

$$\left[\begin{array}{c}\mathbf{A}^{-1}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{ccc}1&-2&2\\0&1&0\\0&1&-1\\ \hline 1&0&0\\0&1&0\\0&0&1\end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{3}]\\ [(-1)\mathbf{3}+\mathbf{2}]\\ [(2)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc}1&0&0\\0&1&0\\0&0&1\\ \hline 1&0&2\\0&1&0\\0&1&-1\end{array}\right] = \left[\begin{array}{c}\mathbf{I}\\ \hline \mathbf{A}\end{array}\right],$$

or, also, computing the following product: $\Big(\mathbf{I}_{\underset{[(-1)\mathbf{3}]}{\boldsymbol{\tau}}}\Big)\Big(\mathbf{I}_{\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}}\Big)\Big(\mathbf{I}_{\underset{[(2)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}}\Big) = \mathbf{A}$;

□

(Grupo E curso 24/25) Exercise 5(a) Since $\mathbf{A} = \mathbf{LU} = \begin{bmatrix}1&0&0\\2&0&0\\2&-4&0\\1&-2&0\end{bmatrix}\begin{bmatrix}1&0&3\\0&1&1\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&3\\2&0&6\\2&-4&2\\1&-2&1\end{bmatrix}$,

and since $\mathbf{L}$ has its first pivot in first row, and the second pivot in third row; then

$$\text{A basis for } \mathcal{C}\left(\mathbf{A}\right): \left[\begin{pmatrix}1\\2\\2\\1\end{pmatrix}; \begin{pmatrix}0\\0\\-4\\-2\end{pmatrix};\right]. \qquad \text{A basis for } \mathcal{C}\left(\mathbf{A}^{\intercal}\right): \left[\begin{pmatrix}1\\0\\3\end{pmatrix}; \begin{pmatrix}2\\-4\\2\end{pmatrix};\right].$$

□

(Grupo E curso 24/25) Exercise 5(b) Since $\mathsf{L}_{|3} = \mathsf{A}(\mathsf{U}^{\text{-}1})_{|3}$ is the only null column, $[(\mathsf{U}^{\text{-}1})_{|3};]$ is a basis for $\mathcal{N}(\mathsf{A})$. Since

$$\mathsf{U}(\mathsf{U}^{\text{-}1}) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \mathsf{I}; \quad \text{then} \quad (\mathsf{U}^{\text{-}1})_{|3} = \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}.$$

Hence, the set of solutions to $\mathsf{A}\boldsymbol{x} = \boldsymbol{0}$ is $\mathcal{N}(\mathsf{A}) = \left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \boldsymbol{p} \right\}$.

□

(Grupo E curso 24/25) Exercise 5(c)
$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 2 & 0 & 6 & -2 \\ 2 & -4 & 2 & -6 \\ 1 & -2 & 1 & -c \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{1}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(-3)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & -4 & -4 & -4 \\ 1 & -2 & -2 & 1-c \\ \hline 1 & 0 & -3 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{2}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}+\mathbf{3}]}}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & -4 & 0 & 0 \\ 1 & -2 & 0 & 3-c \\ \hline 1 & 0 & -3 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

If $c \neq 3$ the system has no solution. When $c = 3$ the solution to the system is:

$$\left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \boldsymbol{p} \right\}$$

□

(Grupo C curso 23/24) Exercise 1(a) **True**. Since $(\mathsf{A}^{\intercal})_{|j} = {}_{j|}\mathsf{A}$ for any matrix, then $\big((\mathsf{A}^{\intercal})^{\intercal}\big)_{|j} = {}_{j|}(\mathsf{A}^{\intercal}) = \mathsf{A}_{|j}$.

□

(Grupo C curso 23/24) Exercise 1(b) **False**. For example $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

□

(Grupo C curso 23/24) Exercise 2.

$$\left[\begin{array}{ccc|c} 1 & -2 & 6 & -1 \\ -2 & 3 & -11 & 3 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{1}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(2)\mathbf{1}+\mathbf{2}] \\ [(-6)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -2 & -1 & 1 & 1 \\ \hline 1 & 2 & -6 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{2}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ \hline 1 & 2 & -4 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Solution set (*we have several ways to express the same set*):

$$\left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} \boldsymbol{p} \right\} = \left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists a \in \mathbb{R},\ \boldsymbol{v} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + a \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

□

(Grupo C curso 23/24) Exercise 3. A square matrix is invertible if any of its pre-echelon forms *do not have null columns*. However, when reducing all numbers to the right of the first 1 in the first row, we subtract the first column ($\mathbf{A}_{|1}$) from the last one ($\mathbf{A}_{|n}$), and since they are identical, we will transform the last column in a zero column, obtaining a pre-echelon matrix *whose last column is null.*

□

(Grupo C curso 23/24) Exercise 4(a) **Impossible**. $\mathbf{A}$ would need to be a 3×4 matrix, but such a matrix would have rank ≤ 3 (could not be full column rank) and hence the linear system could not have unique solution.

□

(Grupo C curso 23/24) Exercise 4(b) **Possible**. $\mathbf{A}$ must be a 3×4 matrix of rank 2, in order to have $\dim \mathcal{N}\left(\mathbf{A}\right) = 2$.

□

(Grupo C curso 23/24) Exercise 4(c) **Possible**. $\mathbf{A}$ must be a 3×2 matrix of rank 1, in order to have $\dim \mathcal{N}\left(\mathbf{A}\right) = 1$.

□

(Grupo C curso 23/24) Exercise 4(d) **Impossible**. For $\boldsymbol{p} = (-1,)$ we get the vector $\mathbf{0}$, which can't be a solution with a nonzero right-hand side vector.

□

(Grupo C curso 23/24) Exercise 4(e) **Impossible**: $\mathbf{A}$ would need to be a 3×2 matrix, but to have $\dim \mathcal{N}\left(\mathbf{A}\right) = 2$ it would need to have rank 0 (so $\mathbf{A} = \mathbf{0}$), which means that no non-zero right-hand-side could have a solution. Equivalently, since $\mathcal{N}\left(\mathbf{A}\right) = \mathbb{R}^2$, vector $(-1, -2,) \in \mathcal{N}\left(\mathbf{A}\right)$ cancels the particular solution $(1, 2,)$ and gives $\mathbf{0}$ as another solution, which cannot be with a nonzero right-hand side vector.

□

(Grupo C curso 23/24) Exercise 4(f) **Possible**. $\mathbf{A}$ must be a 3×2 matrix with rank 2 in order to have a unique solution.

□

(Grupo C curso 23/24) Exercise 5. Since $\boldsymbol{w} \neq \mathbf{0}$ and $\boldsymbol{w} \in \mathcal{C}\left(\mathbf{A}\right)$ (because $\mathbf{A}\boldsymbol{w} = \boldsymbol{w}$), we know that $\operatorname{rg}\left(\mathbf{A}\right) \geq 1$. But $\boldsymbol{u}$ and $\boldsymbol{v}$ belong to $\mathcal{N}\left(\mathbf{A}\right)$ (because $\mathbf{A}\boldsymbol{u} = \mathbf{0}$ and $\mathbf{A}\boldsymbol{v} = \mathbf{0}$) and they are linearly independent (because neither of the two vectors is a multiple of the other), so $\dim \mathcal{N}\left(\mathbf{A}\right) \geq 2$ and, since $\mathbf{A}$ has three columns, $\operatorname{rg}\left(\mathbf{A}\right) \leq 1 = 3 - 2$. Therefore. $\boxed{\operatorname{rg}\left(\mathbf{A}\right) = 1}$.

□

(Grupo C curso 23/24) Exercise 6. Since

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & -1 \end{bmatrix} \xrightarrow[\substack{[(-3)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{4}]}]{\boldsymbol{\tau}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -4 & -2 & 0 \\ 1 & -2 & -1 & -2 \end{bmatrix} \xrightarrow[\substack{[(2)\mathbf{3}] \\ [(-1)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -4 & 0 & 0 \\ 1 & -2 & 0 & -2 \end{bmatrix},$$

a basis for $\mathcal{L}\left(S\right)$ consists of the vectors of the subset $T = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}$

□

(Grupo C curso 23/24) Exercise 7(a) On one hand, any combination of matrices whose first row is null is a matrix whose first row is null. On the other hand, the i-th row of any combination of matrices whose i-th row is a multiple of the i-th row of the identity matrix of order n is $\alpha\Big(a({}_{i|}\mathbf{I})\Big) + \beta\Big(b({}_{i|}\mathbf{I})\Big) = ab\alpha\beta({}_{i|}\mathbf{I})$, meaning, a multiple of the i-th row of the identity matrix of order n.

Therefore, **the set is a subspace**.

□

(Grupo C curso 23/24) Exercise 7(b) **Not a subspace** because, even though $(1, 0, 0,)$ belongs to the set, $2(1, 0, 0,)$ does not belong.

□

(Grupo C curso 23/24) Exercise 7(c) **Not a subspace**: the zero matrix is not in the set.

□

(Grupo C curso 23/24) Exercise 7(d) **NOT a subspace**; The set is not closed under addition:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$\square$

(Grupo E curso 23/24) Exercise 1(a) **True**. Since $\mathbf{I}$ is symmetric and since $\left(\mathbf{BC}\right)^{\intercal} = (\mathbf{C}^{\intercal})(\mathbf{B}^{\intercal})$ for any squared matrices $\mathbf{B}$ and $\mathbf{C}$ with the same order, then $\mathbf{I} = (\mathbf{I})^{\intercal} = \left(\mathbf{A}\mathbf{A}^{-1}\right)^{\intercal} = \left((\mathbf{A}^{-1})^{\intercal}\right)(\mathbf{A}^{\intercal})$.

$\square$

(Grupo E curso 23/24) Exercise 1(b) **False**. Since $\dim \mathcal{N}\left(\mathbf{A}\right) = 2$, the rank of $\mathbf{A}$ is 2 (only two of the four columns of its row-echelon forms have pivots). Therefore, $\mathbf{A}$ does not have full row rank (i.e., $\mathcal{C}\left(\mathbf{A}\right) \neq \mathbb{R}^3$). Thus, when $\boldsymbol{b} \in \mathcal{C}\left(\mathbf{A}\right)$, the system has infinitely many solutions, but when $\boldsymbol{b} \notin \mathcal{C}\left(\mathbf{A}\right)$, *the system has none*, and since we cannot know if $(3, 1, 4)$ is in $\mathcal{C}\left(\mathbf{A}\right)$, we cannot assert that the system is solvable.

$\square$

(Grupo E curso 23/24) Exercise 2(a) Since

$$\left[\begin{array}{cccc} 1 & 2 & 1 & 2 \\ 2 & 4 & 2 & 5 \\ 1 & 2 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(-2)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ \hline 1 & -2 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right],$$

the solution set (the complete solution) is (*we have several ways to express the same set*)

$$\left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2,\ \boldsymbol{v} = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{p} \right\} = \left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \exists a, b \in \mathbb{R},\ \boldsymbol{v} = a \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$= \mathcal{L}\left(\left[\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \right] \right) = \text{The set of all linear combinations of } \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

$\square$

(Grupo E curso 23/24) Exercise 2(b) Since $\left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & -1 \\ 2 & 4 & 2 & 5 & -2a \\ 1 & 2 & 1 & 1 & -a \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(-2)\mathbf{1}+\mathbf{4}] \\ [(1)\mathbf{1}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 2-2a \\ 1 & 0 & 0 & -1 & 1-a \end{array}\right]$; for $a = 1$.

$\square$

(Grupo E curso 23/24) Exercise 3(a) Since $\mathbf{A}$ is a full column rank matrix,

$$\text{Basis for } \mathcal{N}\left(\mathbf{A}\right): [\,]; \qquad \text{Basis for } \mathcal{C}\left(\mathbf{A}\right): \left[\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}; \right]$$

$\square$

(Grupo E curso 23/24) Exercise 3(b) Since $\left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(-3)\mathbf{1}+\mathbf{3}] \\ [(-4)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ \hline 1 & -2 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

$$\text{Basis for } \mathcal{N}\left(\mathbf{B}\right): \left[\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}; \right]; \qquad \text{Basis for } \mathcal{C}\left(\mathbf{B}\right): \left[\ (1,);\ \right]$$

$\square$

(Grupo E curso 23/24) Exercise 3(c) From last question

$$\text{Basis for } \mathcal{N}\left(\mathbf{C}\right): \left[\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}; \begin{pmatrix}-3\\0\\1\\0\end{pmatrix}; \begin{pmatrix}-4\\0\\0\\1\end{pmatrix}; \right]; \qquad \text{Basis for } \mathcal{C}\left(\mathbf{C}\right): \left[\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}; \right].$$

□

(Grupo E curso 23/24) Exercise 4.

$$\left[\begin{array}{ccc|c}1&2&4&-b_1\\0&1&3&-b_2\\1&3&7&-b_3\\2&2&2&-b_4\end{array}\right] \xrightarrow[\substack{[(-2)\mathbf{1}+\mathbf{2}]\\ [(-4)\mathbf{1}+\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc|c}1&0&0&-b_1\\0&1&3&-b_2\\1&1&3&-b_3\\2&-2&-6&-b_4\end{array}\right] \xrightarrow[{[(-3)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc|c}1&0&0&-b_1\\0&1&0&-b_2\\1&1&0&-b_3\\2&-2&0&-b_4\end{array}\right]$$

$$\xrightarrow[{[(b_1)\mathbf{1}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&-b_2\\1&1&0&b_1-b_3\\2&-2&0&2b_1-b_4\end{array}\right] \xrightarrow[{[(b_2)\mathbf{2}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\1&1&0&b_1+b_2-b_3\\2&-2&0&2b_1-2b_2-b_4\end{array}\right]$$

So,

$$\boxed{\begin{cases} b_1 + \;\; b_2 - b_3 \qquad = 0\\ 2b_1 - 2b_2 \qquad - b_4 = 0\end{cases}}.$$

Altenative answer: We need $\boldsymbol{b} \in \mathcal{C}\left(\mathbf{A}\right)$, so:

$$\begin{pmatrix}b_1\\b_2\\b_3\\b_4\end{pmatrix} = a\begin{pmatrix}1\\0\\1\\2\end{pmatrix} + c\begin{pmatrix}0\\1\\1\\-2\end{pmatrix}; \qquad \text{assigning } a = b_1 \text{ and } c = b_2, \qquad \begin{pmatrix}b_1\\b_2\\b_3\\b_4\end{pmatrix} = b1\begin{pmatrix}1\\0\\1\\2\end{pmatrix} + b2\begin{pmatrix}0\\1\\1\\-2\end{pmatrix}$$

we get

$$\begin{cases} b_3 = b_1 + b_2\\ b_4 = 2b_1 - 2b_2\end{cases} \quad \rightarrow \boxed{\begin{cases} b_1 + \;\; b_2 - b_3 \qquad = 0\\ 2b_1 - 2b_2 \qquad - b_4 = 0\end{cases}}.$$

□

(Grupo E curso 23/24) Exercise 5(a) **NOT a subspace**. The zero matrix does not belong to the set.

□

(Grupo E curso 23/24) Exercise 5(b) **It is a subspace**. For any $f(x,y)$ and $g(x,y)$ in the set

$$a \cdot f(7.03, 2024) + b \cdot f(7.03, 2024) = a0 + b0 = 0.$$

□

(Grupo E curso 23/24) Exercise 5(c) **NOT a subspace**. $(-1,0,0,)$ is in the set but $-1\cdot(-1,0,0,)$ is not.

□

(Grupo E curso 23/24) Exercise 5(d) **It is a subspace**. For any $\mathbf{A}$ and $\mathbf{B}$ in the set

$$\left(a\mathbf{A} + b\mathbf{B}\right)^{\intercal} = \left(a\mathbf{A}^{\intercal}\right) + \left(b\mathbf{B}^{\intercal}\right) = a(\mathbf{A}^{\intercal}) + b(\mathbf{B}^{\intercal}) = a\mathbf{A} + b\mathbf{B}.$$

alternatively

$$\left(a\mathbf{A} + b\mathbf{B}\right)_{|j} = \left(a\mathbf{A}\right)_{|j} + \left(b\mathbf{B}\right)_{|j} = {}_{j|}\left(a(\mathbf{A}^{\intercal})\right) + {}_{j|}\left(b(\mathbf{B}^{\intercal})\right) = {}_{j|}\left(a\mathbf{A} + b\mathbf{B}\right).$$

□

(Grupo B curso 22/23) Exercise 1(a) Since

$$\left[\begin{array}{cc}1&2\\2&1\\\hline 1&0\\0&1\end{array}\right]\xrightarrow[{[(-2)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}}\left[\begin{array}{cc}1&0\\2&-3\\\hline 1&-2\\0&1\end{array}\right]\xrightarrow[{\substack{[(3)\mathbf{1}]\\[(2)\mathbf{2}+\mathbf{1}]}}]{\boldsymbol{\tau}}\left[\begin{array}{cc}3&0\\0&-3\\\hline -1&-2\\2&1\end{array}\right]\xrightarrow[{\substack{[(\frac{1}{3})\mathbf{1}]\\[(-\frac{1}{3})\mathbf{2}]}}]{\boldsymbol{\tau}}\left[\begin{array}{cc}1&0\\0&1\\\hline -\frac{1}{3}&\frac{2}{3}\\\frac{2}{3}&-\frac{1}{3}\end{array}\right],$$

then $\mathbf{A}^{-1}=\frac{1}{3}\begin{bmatrix}-1&2\\2&-1\end{bmatrix}$. Check: $\frac{1}{3}\begin{bmatrix}-1&2\\2&-1\end{bmatrix}\begin{bmatrix}1&2\\2&1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}3&0\\0&3\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$.

□

(Grupo B curso 22/23) Exercise 1(b) From part a) $\begin{bmatrix}1&2&0&0\\2&1&0&0\\0&0&1&2\\0&0&2&1\end{bmatrix}\begin{bmatrix}-\frac{1}{3}&\frac{2}{3}&0&0\\\frac{2}{3}&-\frac{1}{3}&0&0\\0&0&-\frac{1}{3}&\frac{2}{3}\\0&0&\frac{2}{3}&-\frac{1}{3}\end{bmatrix}=\mathbf{I}$.

So $\mathbf{B}^{-1}=\begin{bmatrix}\mathbf{A}^{-1}&\\&\mathbf{A}^{-1}\end{bmatrix}$.

□

(Grupo B curso 22/23) Exercise 2(a) Since

$$\left[\begin{array}{cccc}1&2&0&-1\\1&4&2&5\\1&6&4&11\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[{\substack{[(-2)\mathbf{1}+\mathbf{2}]\\[(1)\mathbf{1}+\mathbf{4}]}}]{\boldsymbol{\tau}}\left[\begin{array}{cccc}1&0&0&0\\1&2&2&6\\1&4&4&12\\\hline 1&-2&0&1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[{\substack{[(-1)\mathbf{2}+\mathbf{3}]\\[(-3)\mathbf{2}+\mathbf{4}]}}]{\boldsymbol{\tau}}\left[\begin{array}{cccc}1&0&0&0\\1&2&0&0\\1&4&0&0\\\hline 1&-2&2&7\\0&1&-1&-3\\0&0&1&0\\0&0&0&1\end{array}\right],$$

$\left[\begin{pmatrix}1\\1\\1\end{pmatrix};\ \begin{pmatrix}0\\2\\4\end{pmatrix};\right]$ is a basis for $\mathcal{C}\left(\mathbf{A}\right)$ (but there are many other possible answers).

□

(Grupo B curso 22/23) Exercise 2(b) It is the plane in $\mathbb{R}^4$ that consists of the following set of vectors:

$$\left\{\boldsymbol{v}\in\mathbb{R}^4\;\middle|\;\exists\boldsymbol{p}\in\mathbb{R}^2,\ \boldsymbol{v}=\begin{bmatrix}2&7\\-1&-3\\1&0\\0&1\end{bmatrix}\boldsymbol{p}\right\}$$

□

(Grupo B curso 22/23) Exercise 2(c) Since the vector on the right hand side is $\frac{1}{2}\left(\mathbf{A}_{|2}\right)$, the solution set is

$$\left\{\boldsymbol{v}\in\mathbb{R}^4\;\middle|\;\exists\boldsymbol{p}\in\mathbb{R}^2,\ \boldsymbol{v}=\begin{pmatrix}0\\\frac{1}{2}\\0\\0\end{pmatrix}+\begin{bmatrix}2&7\\-1&-3\\1&0\\0&1\end{bmatrix}\boldsymbol{p}\right\}.$$

□

(Grupo B curso 22/23) Exercise 3(a) If $\mathbf{A}=\left[\boldsymbol{a};\boldsymbol{b};\boldsymbol{c}\right]$, since $3\boldsymbol{a}+2\boldsymbol{b}+\boldsymbol{c}=\mathbf{0}$, we have that $\mathbf{A}\begin{pmatrix}3\\2\\1\end{pmatrix}=\mathbf{0}$. So $\begin{pmatrix}3\\2\\1\end{pmatrix}\in\mathcal{N}\left(\mathbf{A}\right)$. Since those nonzero vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ point in different directions in $\mathbb{R}^3$, then $\boldsymbol{a}$, $\boldsymbol{b}$ are linearly independent ($\boldsymbol{b}$ is not a multiple of $\boldsymbol{a}$). Therefore $\operatorname{rg}\left(\mathbf{A}\right)=2$ and $\dim\mathcal{N}\left(\mathbf{A}\right)=1$. Hence $\mathcal{N}\left(\mathbf{A}\right)$ is the set of multiples of $\left(3,2,1,\right)$.

□

(Grupo B curso 22/23) Exercise 3(b) Since those nonzero vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ point in different directions in $\mathbb{R}^3$, then $\boldsymbol{a}$, $\boldsymbol{b}$ are linearly independent. And since $3\boldsymbol{a}+2\boldsymbol{b}+\boldsymbol{c}=\mathbf{0}$, the last column become $\mathbf{0}$ after elimination. Hence, the two first columns are pivot columns. The last column is a free column.

□

(Grupo B curso 22/23) Exercise 3(c) Since the zero matrix $\underset{3\times 3}{\mathbf{0}}$ belongs to the set, the set is not empty. We only need to prove that the set is closed under linear combinations. Let $\mathbf{B}\left(3,2,1,\right)=\mathbf{0}$ and $\mathbf{C}\left(3,2,1,\right)=\mathbf{0}$, and $x,y\in\mathbb{R}$; then

$$\left(x\mathbf{B}+y\mathbf{C}\right)\begin{pmatrix}3\\2\\1\end{pmatrix}=x\mathbf{B}\begin{pmatrix}3\\2\\1\end{pmatrix}+y\mathbf{C}\begin{pmatrix}3\\2\\1\end{pmatrix}=x\mathbf{0}+y\mathbf{0}=\mathbf{0}\quad\Longrightarrow\quad\left(x\mathbf{B}+y\mathbf{C}\right)\in\mathcal{S}.$$

Therefore $\mathcal{S}$ is a subspace of $\mathbb{R}^{3\times 3}$.

□

(Grupo B curso 22/23) Exercise 4(a) $\mathcal{C}\left(\mathbf{A}\right)=\mathbb{R}^n$ and $\mathcal{N}\left(\mathbf{A}\right)=\{\mathbf{0}\}$.

□

(Grupo B curso 22/23) Exercise 4(b) The system $[\mathbf{I}_{|1};\ldots\mathbf{I}_{|n}]$ of columns of the identity matrix $\underset{n\times n}{\mathbf{I}}$ is a basis for $\mathcal{C}\left(\mathbf{A}\right)$. The empty system, $[\;]$, is a basis for $\mathcal{N}\left(\mathbf{A}\right)$.

□

(Grupo B curso 22/23) Exercise 5. If we supose that $\mathbf{A}^{-1}$ exists we will find a contradiction. Let's see three ways to find a contradiction:

1. When we apply a sequence of elementary transformations to an invertible matrix *we cannot obtain a matrix with null columns*. But, if we (wrongly) assume that $\mathbf{A}$ is invertible, that is, if we (wrongly) assume $\mathbf{A}=\mathbf{I}_{\tau_1\cdots\tau_k}$, then

$$\begin{aligned}\mathbf{A}^2=\mathbf{A}\mathbf{A}=\mathbf{A}(\mathbf{I}_{\tau_1\cdots\tau_k})&=\mathbf{0}\\ \mathbf{A}_{\tau_1\cdots\tau_k}&=\mathbf{0}.\end{aligned}$$

2. An invertible matrix *cannot have null columns* but since $\mathbf{A}^2=\mathbf{A}\mathbf{A}=\mathbf{0}$, multiplying both sides by the (non-existent) $\mathbf{A}^{-1}$ we get

$$\begin{aligned}(\mathbf{A}^{-1})\mathbf{A}^2=(\mathbf{A}^{-1}\mathbf{A})\mathbf{A}&=(\mathbf{A}^{-1})\mathbf{0}\\ \mathbf{A}&=\mathbf{0}.\end{aligned}$$

3. If $\underset{n\times n}{\mathbf{A}}$ is full rank, then $\operatorname{rg}\left(\mathbf{A}\right)=n\neq 0$, and therefore the row space of $\mathbf{A}$ has dimension $n\neq 0$. But, since $\mathbf{A}^2=\mathbf{A}\mathbf{A}=\mathbf{0}$, multiplying both sides by the *impossible* $\mathbf{A}^{-1}$ we get

$$\mathbf{A}=(\mathbf{A}^{-1})\mathbf{0}\quad\Rightarrow\quad\mathcal{C}\left(\mathbf{A}^\intercal\right)\subset\mathcal{C}\left(\mathbf{0}^\intercal\right)\quad\Rightarrow\quad\operatorname{rg}\left(\mathbf{A}\right)=0=\dim\mathcal{C}\left(\mathbf{0}^\intercal\right).$$

Two more proofs

1. Since $\left(\mathbf{A}\mathbf{A}\right)_{|j}=\mathbf{A}\left(\mathbf{A}_{|j}\right)=\mathbf{0}_{|j}$, then: $\begin{cases}\mathbf{A}_{|j}=\mathbf{0}\ \Rightarrow\ \mathbf{A}\text{ singular since it has a null column}\\ \mathbf{A}_{|j}\neq\mathbf{0}\ \Rightarrow\ \mathbf{A}\text{ has linearly dependent columns}\end{cases}$

2. Since the product of invertible matrices is invertible, then $\mathbf{A}\mathbf{A}=\mathbf{A}^2$ would be invertible, but it is not since $\mathbf{A}^2$ has columns of zeros.

□

(Grupo B curso 22/23) Exercise 6(a)

$$\begin{aligned}\mathbf{A}\mathbf{B}&=\mathbf{B}\mathbf{A}\\ \left(\mathbf{A}\mathbf{B}\right)^{-1}&=\left(\mathbf{B}\mathbf{A}\right)^{-1}\\ (\mathbf{B}^{-1})(\mathbf{A}^{-1})&=(\mathbf{A}^{-1})(\mathbf{B}^{-1})\end{aligned}$$

□

(Grupo B curso 22/23) Exercise 6(b) $\mathbf{A}\boldsymbol{x}=\mathbf{0}$ has non-zero solutions when any of the echelon forms of $\mathbf{A}$ has at least one null column (so we can find at least one special solution). Hence, it is true when $\boxed{\operatorname{rg}\left(\mathbf{A}\right)<n}$ and also when $\boxed{\mathbf{A}^2=\mathbf{0}}$, since this implies that $\mathbf{A}$ is singular. Case 1 is false in general.

□

(Grupo E curso 22/23) Exercise 1.

$$\left[\begin{array}{ccc|c} 1&1&1&-2\\ 2&2&1&-3\\ 0&3&-1&-4\\ \hline 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ \hline 0&0&0&1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{1}+\mathbf{2}]\\ [(-1)\mathbf{1}+\mathbf{3}]\\ [(2)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1&0&0&0\\ 2&0&-1&1\\ 0&3&-1&-4\\ \hline 1&-1&-1&2\\ 0&1&0&0\\ 0&0&1&0\\ \hline 0&0&0&1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ [(1)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1&0&0&0\\ 2&0&-1&0\\ 0&3&-1&-5\\ \hline 1&-1&-1&1\\ 0&1&0&0\\ 0&0&1&1\\ \hline 0&0&0&1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ [(3)\mathbf{4}]\\ [(5)\mathbf{2}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1&0&0&0\\ 2&0&-1&0\\ 0&3&-1&0\\ \hline 1&-1&-1&-2\\ 0&1&0&5\\ 0&0&1&3\\ \hline 0&0&0&3 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ [(\frac{1}{3})\mathbf{4}]}}$$

$$\left[\begin{array}{ccc|c} 1&0&0&0\\ 2&0&-1&0\\ 0&3&-1&0\\ \hline 1&-1&-1&-\frac{2}{3}\\ 0&1&0&\frac{5}{3}\\ 0&0&1&1\\ \hline 0&0&0&1 \end{array}\right].$$ Hence, the solution set is $\left\{ \begin{pmatrix} -\frac{2}{3}\\ \frac{5}{3}\\ 1 \end{pmatrix} \right\}$.

□

(Grupo E curso 22/23) Exercise 2(a) Since
$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{ccc} 2&2&0\\ 2&5&3\\ 1&0&2\\ \hline 1&0&0\\ 0&1&0\\ 0&0&1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc} 2&0&0\\ 2&3&3\\ 1&-1&2\\ \hline 1&-1&0\\ 0&1&0\\ 0&0&1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} 2&0&0\\ 2&3&0\\ 1&-1&3\\ \hline 1&-1&1\\ 0&1&-1\\ 0&0&1 \end{array}\right] = \left[\begin{array}{c}\mathbf{K}\\ \hline \mathbf{E}\end{array}\right],$$

then $\boxed{\mathbf{E} = \begin{bmatrix} 1&-1&1\\ 0&1&-1\\ 0&0&1 \end{bmatrix}}$; check: $\begin{bmatrix} 2&2&0\\ 2&5&3\\ 1&0&2 \end{bmatrix}\begin{bmatrix} 1&-1&1\\ 0&1&-1\\ 0&0&1 \end{bmatrix} = \begin{bmatrix} 2&0&0\\ 2&3&0\\ 1&-1&3 \end{bmatrix}$.

□

(Grupo E curso 22/23) Exercise 2(b) Since $\mathbf{AE} = \mathbf{K}$ then $\mathbf{A}{=}\mathbf{KE}^{-1}$; and since

$$\left[\begin{array}{c}\mathbf{E}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{ccc} 1&-1&1\\ 0&1&-1\\ 0&0&1\\ \hline 1&0&0\\ 0&1&0\\ 0&0&1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ [(1)\mathbf{1}+\mathbf{2}]\\ [(-1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc} 1&0&0\\ 0&1&-1\\ 0&0&1\\ \hline 1&1&-1\\ 0&1&0\\ 0&0&1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ [(1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} 1&0&0\\ 0&1&0\\ 0&0&1\\ \hline 1&1&0\\ 0&1&1\\ 0&0&1 \end{array}\right] = \left[\begin{array}{c}\mathbf{I}\\ \hline \mathbf{E}^{-1}\end{array}\right],$$

then

$$\boxed{\mathbf{U} = \begin{bmatrix} 1&1&0\\ 0&1&1\\ 0&0&1 \end{bmatrix}}; \text{ check: } \mathbf{KU} = \begin{bmatrix} 2&0&0\\ 2&3&0\\ 1&-1&3 \end{bmatrix}\begin{bmatrix} 1&1&0\\ 0&1&1\\ 0&0&1 \end{bmatrix} = \begin{bmatrix} 2&2&0\\ 2&5&3\\ 1&0&2 \end{bmatrix} = \mathbf{A}.$$

□

(Grupo E curso 22/23) Exercise 2(c) Since the rank is three, the column space is all of $\mathbb{R}^3$. Therefore any three independent vectors in $\mathbb{R}^3$ will do. For example a list with the three columns of $\mathbf{I}$ of order 3, or a list with the three columns of $\mathbf{A}$, or a list with the three columns of $\mathbf{K}$.

□

(Grupo E curso 22/23) Exercise 3(a) *One answer is*: Since $\mathbf{E}$ is invertible then $\boxed{\operatorname{rg}(\mathbf{A}) = \operatorname{rg}(\mathbf{U}) = 3}$; since

$$\begin{cases} \mathbf{A} = \mathbf{EU} \ \Rightarrow \mathcal{C}\left(\mathbf{A}^\intercal\right) \subset \mathcal{C}\left(\mathbf{U}^\intercal\right)\\ \mathbf{U} = (\mathbf{E}^{-1})\mathbf{A} \ \Rightarrow \mathcal{C}\left(\mathbf{U}^\intercal\right) \subset \mathcal{C}\left(\mathbf{A}^\intercal\right) \end{cases}; \text{ and therefore } \mathcal{C}\left(\mathbf{A}^\intercal\right) = \mathcal{C}\left(\mathbf{U}^\intercal\right).$$

But the following reasoning will help us answer part b): Since $\mathbf{A} = \mathbf{EU}$ then $\mathbf{A}_{\tau_1\cdots\tau_k} = \mathbf{E}\big(\mathbf{U}_{\tau_1\cdots\tau_k}\big)$. Thus, by obtaining a pre-echelon form of $\mathbf{U}$ the range of $\mathbf{A}$ will become evident:

$$\left[\begin{array}{c}\mathbf{U}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{ccccc} 1&0&1&4&5\\ 0&1&2&2&1\\ 0&0&0&1&1\\ \hline 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{1}+\mathbf{3}]\\ [(-2)\mathbf{2}+\mathbf{3}]\\ [(-4)\mathbf{1}+\mathbf{4}]\\ [(-2)\mathbf{2}+\mathbf{4}]}} \left[\begin{array}{ccccc} 1&0&0&0&5\\ 0&1&0&0&1\\ 0&0&0&1&1\\ \hline 1&0&-1&-4&0\\ 0&1&-2&-2&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ [(-5)\mathbf{1}+\mathbf{5}]\\ [(-1)\mathbf{2}+\mathbf{5}]\\ [(-1)\mathbf{4}+\mathbf{5}]}} \left[\begin{array}{ccccc} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0\\ \hline 1&0&-1&-4&-1\\ 0&1&-2&-2&1\\ 0&0&1&0&0\\ 0&0&0&1&-1\\ 0&0&0&0&1 \end{array}\right] = \left[\begin{array}{c}\mathbf{U}_{\tau_1\cdots\tau_k}\\ \hline \mathbf{I}_{\tau_1\cdots\tau_k}\end{array}\right].$$

$$\mathbf{A}_{\tau_1\cdots\tau_k} = \mathbf{E}\mathbf{U}_{\tau_1\cdots\tau_k} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \boxed{\operatorname{rg}(\mathbf{A}) = 3}.$$

□

(Grupo E curso 22/23) Exercise 3(b) $\mathbf{A}_{\tau_1\cdots\tau_k} = \mathbf{A}(\mathbf{I}_{\tau_1\cdots\tau_k})$ is a pre-echelon form and columns 3 and 5 of $(\mathbf{A}_{\tau_1\cdots\tau_k})$ are zero, so columns 3 and 5 of $(\mathbf{I}_{\tau_1\cdots\tau_k})$ are special solutions. Hence $\left[\begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix};\right]$ is a basis for $\mathcal{N}(\mathbf{A})$.

□

(Grupo E curso 22/23) Exercise 3(c) $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is equivalent to $(\mathbf{E}^{-1})\mathbf{A}\boldsymbol{x} = (\mathbf{E}^{-1})\boldsymbol{b}$ where $(\mathbf{E}^{-1})\mathbf{A}=\mathbf{U}$, and we already known a reduced echelon form of $\mathbf{U}$. So, the (not so) hard part is to compute the right hand side vector $\boldsymbol{c} = (\mathbf{E}^{-1})\boldsymbol{b}$ to get an equivalent system of equations $\mathbf{U}\boldsymbol{x} = \boldsymbol{c}$.

$$\left[\frac{\mathbf{E}}{\mathbf{I}}\right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -3 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(3)\mathbf{2}+\mathbf{1}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{3}+\mathbf{2}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{array}\right] = \left[\frac{\mathbf{I}}{\mathbf{E}^{-1}}\right];$$

so $\mathbf{E}^{-1}\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{pmatrix} 3 \\ -8 \\ -7 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \boldsymbol{c}$. Checking the reduced echelon form of $\mathbf{U}$ in part (a) we see that three times $3\mathbf{U}_{|1}$ plus $\mathbf{U}_{|2}$ plus $\mathbf{U}_{|4}$ equals $\boldsymbol{c}$. Hence, a particular solution is

$$3(\mathbf{I}_{\tau_1\cdots\tau_k})_{|1} + (\mathbf{I}_{\tau_1\cdots\tau_k})_{|2} + (\mathbf{I}_{\tau_1\cdots\tau_k})_{|4} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -4 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Therefore, the solution is $\left\{ \boldsymbol{v} \in \mathbb{R}^5 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2,\ \boldsymbol{v} = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{bmatrix} -1 & -1 \\ -2 & 1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}.$

□

(Grupo E curso 22/23) Exercise 4(a) The columns added in the concatenation $[\mathbf{A}\ \mathbf{A}]$ are the same as those already in $\mathbf{A}$. Thus, after left-right elimination, the pre-echeloned forms of both $\mathcal{C}([\mathbf{A}\ \mathbf{A}])$ and $\mathcal{C}(\mathbf{A})$ will have the same pivot columns, which form a basis of their corresponding column spaces. That is, $\boxed{\mathcal{C}([\mathbf{A}\ \mathbf{A}]) = \mathcal{C}(\mathbf{A})}$.

□

(Grupo E curso 22/23) Exercise 4(b) Since both matrices have the same column space, they both have the same rank r. Therefore $\boxed{\dim \mathcal{N}([\mathbf{A}\ \mathbf{A}]) = 12 - r}$.

□

(Grupo E curso 22/23) Exercise 5.

$$\begin{aligned}(\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A}+\mathbf{A}^2+\cdots+\mathbf{A}^{n-1}) =& \mathbf{I}(\mathbf{I}+\mathbf{A}+\mathbf{A}^2+\cdots+\mathbf{A}^{n-1}) - \mathbf{A}(\mathbf{I}+\mathbf{A}+\mathbf{A}^2+\cdots+\mathbf{A}^{n-1}) \\ =& (\mathbf{I}+\mathbf{A}+\mathbf{A}^2+\cdots+\mathbf{A}^{n-1}) - (\mathbf{A}+\mathbf{A}^2+\cdots+\mathbf{A}^{n-1}+\mathbf{0}) = \mathbf{I}.\end{aligned}$$

□

(Grupo E curso 22/23) Exercise 6. **True.** $\mathbf{AB} = (\mathbf{AB})^\intercal = (\mathbf{B}^\intercal)(\mathbf{A}^\intercal)$.

□

(Grupo D curso 21/22) Exercise 1(a) The matrix on the right-hand side of the product is full rank (i.e., a product of elementary matrices $\mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}$); if we denote it as $\mathbf{E} = \mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}$ we have that the matrix on the left of the product is equal to $\mathbf{A}(\mathbf{E}^{-1}) = \mathbf{A}_{\boldsymbol{\tau}_k^{-1}\cdots\boldsymbol{\tau}_1^{-1}}$; there fore the left-hand matrix is an echelon form of MatA with three pivots $\Rightarrow \boxed{\text{rg}\left(\mathbf{A}\right) = 3}$. In other words, if $\mathbf{E}$ is full rank, $\text{rg}\left(\mathbf{BE}\right) = \text{rg}\left(\mathbf{B}\right)$.

□

(Grupo D curso 21/22) Exercise 1(b) Since $\mathcal{N}\left(\mathbf{A}\right) = \left\{\begin{pmatrix}0, & 0, & 0,\end{pmatrix}\right\}$, a generating system contains only the null vector or it is empty

$$\text{Generating system of } \mathcal{N}\left(\mathbf{A}\right): \quad \left[\begin{pmatrix}0\\0\\0\end{pmatrix};\right] \quad \text{or the empty list} \quad [\,] = \emptyset,$$

the basis is an empty list (zero dimension): Basis of $\mathcal{N}\left(\mathbf{A}\right) = [\,] = \emptyset$.

□

(Grupo D curso 21/22) Exercise 1(c) Since $\mathcal{N}\left(\mathbf{B}^{\intercal}\right) = \mathcal{N}\left((\mathbf{AE})^{\intercal}\right)$ when $\mathbf{E}$ is of full rank, it is enough to focus on the left-hand matrix: $\begin{bmatrix}1&0&0\\0&1&0\\1&2&0\\4&2&1\\5&1&1\end{bmatrix} \xrightarrow[{[(-2)\mathbf{3}+\mathbf{2}]}]{\substack{\boldsymbol{\tau}\\{[(-4)\mathbf{3}+\mathbf{1}]}}} \begin{bmatrix}1&0&0\\0&1&0\\1&2&0\\0&0&1\\1&-1&1\end{bmatrix} \Rightarrow$ a basis for $\mathcal{N}\left(\mathbf{A}^{\intercal}\right)$:

$$\left[\begin{pmatrix}-1\\-2\\1\\0\\0\end{pmatrix};\ \begin{pmatrix}-1\\1\\0\\-1\\1\end{pmatrix};\right].$$

□

(Grupo D curso 21/22) Exercise 1(d) Since the first row of $\mathbf{A}$ is $\begin{pmatrix}1, & 1, & 7,\end{pmatrix}$, the solution set is

$$\left\{ \boldsymbol{v} \in \mathbb{R}^5 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2,\ \boldsymbol{v} = \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} + \begin{bmatrix}-1&-1\\-2&1\\1&0\\0&-1\\0&1\end{bmatrix}\boldsymbol{p} \right\}.$$

□

(Grupo D curso 21/22) Exercise 2(a)

$$\begin{bmatrix}1&2&3\\1&5&6\\2&6&c\end{bmatrix} \xrightarrow[{[(-3)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau}\\{[(-2)\mathbf{1}+\mathbf{2}]}}} \begin{bmatrix}1&0&0\\1&3&3\\2&2&c-6\end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau}\\{[(-1)\mathbf{2}+\mathbf{3}]}}} \begin{bmatrix}1&0&0\\1&3&0\\2&2&c-8\end{bmatrix} = \mathbf{L};$$

Hence; $c = 8 \Rightarrow \mathbf{A}$ singular.

□

(Grupo D curso 21/22) Exercise 2(b) When $c \neq 8$ the matriz is invertible (rank 3). So its null space is just the zero vector, $\mathcal{N}\left(\mathbf{A}\right) = \{\mathbf{0}\}$, and its column space is all $\mathbb{R}^3$. The same logic and answers apply to $\mathbf{A}^{-1}$.

□

(Grupo D curso 21/22) Exercise 2(c) Applying the transformations of part (a) to the columns of $\mathbf{I}$:

$$\mathbf{I}_{\substack{\boldsymbol{\tau}\\{[(-2)\mathbf{1}+\mathbf{2}]}}\substack{\boldsymbol{\tau}\\{[(-3)\mathbf{1}+\mathbf{3}]}}\substack{\boldsymbol{\tau}\\{[(-1)\mathbf{2}+\mathbf{3}]}}} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow[{[(-3)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau}\\{[(-2)\mathbf{1}+\mathbf{2}]}}} \begin{bmatrix}1&-2&-3\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau}\\{[(-1)\mathbf{2}+\mathbf{3}]}}} \begin{bmatrix}1&-2&-1\\0&1&-1\\0&0&1\end{bmatrix} = \mathbf{B}.$$

So, $\mathbf{A}_{\substack{\boldsymbol{\tau}\\{[(-2)\mathbf{1}+\mathbf{2}]}}\substack{\boldsymbol{\tau}\\{[(-3)\mathbf{1}+\mathbf{3}]}}\substack{\boldsymbol{\tau}\\{[(-1)\mathbf{2}+\mathbf{3}]}}} = \mathbf{AB} = \mathbf{L}$.

□

(Grupo D curso 21/22) Exercise 2(d) Simply invert B; for example, by applying the inverse of the sequence of transformations in (a) to the columns of I..

$$\mathsf{B}^{\text{-}1} = \mathsf{I}_{\substack{[(1)\overset{\tau}{\mathbf{2}}+\mathbf{3}]\\ [(3)\mathbf{1}+\mathbf{3}]\\ [(2)\mathbf{1}+\mathbf{2}]}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{[(1)\overset{\tau}{\mathbf{2}}+\mathbf{3}]} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{[(3)\overset{\tau}{\mathbf{1}}+\mathbf{3}]} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{[(2)\overset{\tau}{\mathbf{1}}+\mathbf{2}]} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \mathsf{U};$$

so $\mathsf{A} = \mathsf{LU}$; since $\mathsf{LU} = \Big(\mathsf{A}_{[(-2)\overset{\tau}{\mathbf{1}}+\mathbf{2}][(-3)\overset{\tau}{\mathbf{1}}+\mathbf{3}][(-1)\overset{\tau}{\mathbf{2}}+\mathbf{3}]}\Big)\Big(\mathsf{I}_{[(1)\overset{\tau}{\mathbf{2}}+\mathbf{3}][(3)\overset{\tau}{\mathbf{1}}+\mathbf{3}][(2)\overset{\tau}{\mathbf{1}}+\mathbf{2}]}\Big) = \mathsf{A}$.

$\square$

(Grupo D curso 21/22) Exercise 3(a) Since for any $\boldsymbol{b} \in \mathbb{R}^m$ there is a solution, we known that $\boldsymbol{b} \in \mathcal{C}\left(\mathsf{A}\right)$, then $\mathbb{R}^m \subset \mathcal{C}\left(\mathsf{A}\right)$. And since $\mathcal{C}\left(\mathsf{A}\right) \subset \mathbb{R}^m$ (because every column has m entries): $\boxed{\mathcal{C}\left(\mathsf{A}\right) = \mathbb{R}^m}$.

$\square$

(Grupo D curso 21/22) Exercise 3(b) Since every row has a pivot: $\boxed{n \geq m = r}$.

$\square$

(Grupo D curso 21/22) Exercise 3(c) $\mathsf{A} = \begin{bmatrix} 2 & 5 \\ 4 & 10 \\ 0 & 0 \end{bmatrix}$.

$\mathcal{C}\left(\mathsf{A}\right)$ is the line $\left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \boldsymbol{p} \right\}$.

$\mathcal{N}\left(\mathsf{A}\right)$ is the line $\left\{ \boldsymbol{v} \in \mathbb{R}^2 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{bmatrix} -5 \\ 2 \end{bmatrix} \boldsymbol{p} \right\}$.

$\square$

(Grupo D curso 21/22) Exercise 3(d) $\left\{ \boldsymbol{v} \in \mathbb{R}^2 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{bmatrix} -5 \\ 2 \end{bmatrix} \boldsymbol{p} \right\}$.

$\square$

(Grupo E curso 21/22) Exercise 1(a) All columns verify that their third component is the sum of the first two. $(a,\ b,\ (a+b),)$. As, $1 + 0 \neq 0$,we known that $(1,\ 0,\ 0,) \notin \mathcal{C}\left(\mathsf{A}\right)$.

$\square$

(Grupo E curso 21/22) Exercise 1(b) Any in the set

$$\left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists b_1, b_2 \in \mathbb{R},\ \boldsymbol{v} = \begin{pmatrix} b_1 \\ b_2 \\ b_1 + b_2 \end{pmatrix} \right\} = \left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} -1 & -1 & 1 \end{bmatrix} \boldsymbol{v} = (0,) \right\}.$$

$\square$

(Grupo E curso 21/22) Exercise 1(c) Because A has three rows but $\dim \mathcal{C}\left(\mathsf{A}^{\intercal}\right) < 3$, i.e., $\operatorname{rg}\big(\underset{3\times 3}{\mathsf{A}}\big) < 3$.

$\square$

(Grupo E curso 21/22) Exercise 2(a) For example: $\left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix};\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix};\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix};\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \right]$.

$\square$

(Grupo E curso 21/22) Exercise 2(b) For example, the set of multiples of A:

$$\left\{ \mathsf{M} \in \mathbb{R}^{2\times 2} \;\middle|\; \exists c \in \mathbb{R};\ \mathsf{M} = c \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\} = \mathcal{L}\left(\left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \right] \right).$$

Another example: the set of linear combinations of the first three matrices of the basis of (a) above:

$$\left\{ \mathsf{M} \in \mathbb{R}^{2\times 2} \;\middle|\; \exists \boldsymbol{b} \in \mathbb{R}^3;\ \mathsf{M} = \left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix};\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix};\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \right] \boldsymbol{b} \right\}$$
$$= \mathcal{L}\left(\left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix};\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix};\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \right] \right).$$

$\square$

(Grupo E curso 21/22) Exercise 2(c) For example:

$$\left\{ \mathbf{M} \in \mathbb{R}^{2\times 2} \,\middle|\, \exists a, b \in;\ \mathbf{M} = a \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\} = \mathcal{L}\left(\left[\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix};\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \right] \right).$$

□

(Grupo E curso 21/22) Exercise 2(d) True. If a subspace $\mathcal{V}$ contains $\mathbf{A}$ and $\mathbf{B}$, it also contains $\mathbf{A} - \mathbf{B} = \mathbf{I}$.

□

(Grupo E curso 21/22) Exercise 3(a) The columns of $\mathbf{A}$ are linearly dependent since $\mathbf{R}_{|3} = \mathbf{A}(\mathbf{B}_{|3}) = \mathbf{0}$. Since $\mathbf{B}_{|3}$ has only non-zero components, we know that $\mathbf{A}_{|3}$ is a linear combination of $\mathbf{A}_{|1}$ and $\mathbf{A}_{|2}$. In particular $\mathbf{A}_{|3} = 3\mathbf{A}_{|1} - 2\mathbf{A}_{|2}$.

□

(Grupo E curso 21/22) Exercise 3(b) After two steps of elimination we have

$$\begin{bmatrix} 1 & 2 & ? \\ 2 & a & ? \\ 1 & 1 & ? \\ b & 8 & ? \end{bmatrix} \xrightarrow[{[(-2)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \begin{bmatrix} 1 & 0 & ? \\ 2 & a-4 & ? \\ 1 & -1 & ? \\ b & -2\,(b-4) & ? \end{bmatrix} \xrightarrow[{[(-1)\mathbf{2}]}]{\boldsymbol{\tau}} \begin{bmatrix} 1 & 0 & ? \\ 2 & 4-a & ? \\ 1 & 1 & ? \\ b & 2\,(b-4) & ? \end{bmatrix}.$$

Since $4 - a = 0$ and since $2\,(b-4) = 2$ then $a = 4$ and $b = 5$.

□

(Grupo E curso 21/22) Exercise 3(c) $\left\{ \boldsymbol{v} \in \mathbb{R}^3 \,\middle|\, \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \boldsymbol{p} \right\}$.

□

(Grupo E curso 21/22) Exercise 3(d) It is the same set as for $\boldsymbol{x}\mathbf{R} = \mathbf{0}$. Hence,

$$\left\{ \boldsymbol{v} \in \mathbb{R}^4 \,\middle|\, \exists \boldsymbol{p} \in \mathbb{R}^2,\ \boldsymbol{v} = \begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}.$$

□

(Grupo E curso 21/22) Exercise 3(e) $\mathcal{C}\left(\mathbf{A}\right) = \mathcal{C}\left(\mathbf{R}\right)$ y $\mathcal{N}\left(\mathbf{A}^{\intercal}\right) = \mathcal{N}\left(\mathbf{R}^{\intercal}\right)$.

□

(Grupo E curso 21/22) Exercise 4(a) $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 1 & 0 & 2 \end{bmatrix} \xrightarrow[{[(-1)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \begin{bmatrix} 2 & 0 & 0 \\ 2 & 3 & 3 \\ 1 & -1 & 2 \end{bmatrix} \xrightarrow[{[(-1)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \begin{bmatrix} 2 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -1 & 3 \end{bmatrix}$.

Since $\operatorname{rg}\left(\mathbf{A}\right) = 3$, then $\mathcal{C}\left(\mathbf{A}\right) = \mathbb{R}^3$. Therefore any basis for $\mathbb{R}^3$ is valid; e.g. $\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix};\ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix};\ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \right]$.

□

(Grupo E curso 21/22) Exercise 4(b) $\dim \mathcal{N}\left(\mathbf{A}\right) = 0$.

□

(Grupo D curso 20/21) Exercise 1(a) Since system is solvable for $\boldsymbol{b} \in \mathbb{R}^m$, any the rank of $\mathbf{A}$ is equal to the number of rows, so $\boxed{rg(\mathbf{A}) = m.}$

□

(Grupo D curso 20/21) Exercise 1(b) Since rank is m then, $\mathcal{N}\left(\underset{n\times m}{\mathbf{A}^{\intercal}}\right) = \{\mathbf{0}\}$ where $\mathbf{0} \in \mathbb{R}^m$.

□

(Grupo D curso 20/21) Exercise 1(c) Since $rg(\mathbf{A}) = m \leq n$, then $\underset{n\times m}{\mathbf{A}^{\intercal}}$ is full column rank. So 0 or 1 solutions (if $m = n$ then always 1 solution).

□

(Grupo D curso 20/21) Exercise 2(a)

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \\ 4 & 3 & 2 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau}\\ [(1)\mathbf{1}+\mathbf{3}]}} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 2 & 4 \\ 4 & 3 & 6 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau}\\ [(-2)\mathbf{2}+\mathbf{1}] \\ [(-2)\mathbf{2}+\mathbf{3}]}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 0 \\ -2 & 3 & 0 \end{bmatrix} = \mathbf{R}$$

□

(Grupo D curso 20/21) Exercise 2(b)

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 0 & 3 & 6 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau}\\ [(-2)\mathbf{2}+\mathbf{3}]}} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 0 & 3 & 0 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau}\\ [(\frac{1}{3})\mathbf{2}] \\ [(\frac{1}{2})\mathbf{3}]}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau}\\ [\mathbf{1} \rightleftharpoons \mathbf{3}]}} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \mathbf{R}$$

□

(Grupo D curso 20/21) Exercise 3(a) Since it is an homogeneous system, it is always solvable. Since there are more columns than rows, any pre-echelon form of $\mathbf{A}^\intercal$ has one free column, so we can find one special solution $\neq \mathbf{0}$.

(Since $\boldsymbol{u}$; $\boldsymbol{v}$; $\boldsymbol{w}$ are a basis, they are linearly independent and $\mathbf{A}^\intercal$ (3×4) is full *row* rank. So the system $\mathbf{A}^\intercal\boldsymbol{y} = \boldsymbol{b}$ is solvable for any rigth hand side vector. Hence, the statement is true even when the rigth hand side is not zero).

□

(Grupo D curso 20/21) Exercise 4(a) No. The set is no closed under addition. For example: $\mathbf{I} + (-\mathbf{I}) = \mathbf{0}$ is not invertible. The set is no closed under scaling. For example: $0\mathbf{I} = \mathbf{0}$ is not invertible.

□

(Grupo D curso 20/21) Exercise 4(b) Yes, it is a subspace, since it is closed under linar combinations. Consider $a, b \in \mathbb{R}$ and $\mathbf{A}$ and $\mathbf{B}$ $(m \times 2)$ such that $\mathbf{A}(\boldsymbol{x}_0) = \mathbf{0}$ and $\mathbf{B}(\boldsymbol{x}_0) = \mathbf{0}$ then

$$(a\mathbf{A} + b\mathbf{B})\begin{pmatrix}1\\1\end{pmatrix} = a\mathbf{A}\begin{pmatrix}1\\1\end{pmatrix} + b\mathbf{B}\begin{pmatrix}1\\1\end{pmatrix} = a\mathbf{0} + b\mathbf{0} = \mathbf{0}.$$

□

(Grupo D curso 20/21) Exercise 5(a)

- If $c \neq 0$, $\mathbf{A}$ is full rank: $\begin{bmatrix} c & c & 1 \\ c & c & 2 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{1}+\mathbf{2}] \\ [(-\frac{1}{c})\mathbf{1}+\mathbf{3}]}} \begin{bmatrix} c & 0 & 0 \\ c & 0 & 1 \\ 3 & 3 & \frac{3\cdot(3c-1)}{c} \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau}\\ [(-\frac{3c-1}{c})\mathbf{2}+\mathbf{3}]}} \begin{bmatrix} c & 0 & 0 \\ c & 0 & 1 \\ 3 & 3 & 0 \end{bmatrix}.$
- $\boxed{\mathbf{A} \text{ is singular if and only if } c = 0}$ (2 pivots): $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau}\\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(-3)\mathbf{1}+\mathbf{3}]}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 3 & 0 & 0 \end{bmatrix}.$

□

(Grupo D curso 20/21) Exercise 5(b)

- When $c = 0$: rank 2
- When $c \neq 0$: rank 3

□

(Grupo D curso 20/21) Exercise 5(c)

- When $c = 0$: $\mathcal{N}(\mathbf{A}) = \left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \boldsymbol{p} \right\} = \mathcal{L}\left(\left[\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}; \right] \right)$
- When $c \neq 0$: $\mathcal{N}(\mathbf{A}) = \{\mathbf{0}\} \subset \mathbb{R}^3$.

□

(Grupo D curso 20/21) Exercise 5(d)

- When $c=0$: A basis for $\mathcal{C}\left(\mathbf{A}\right) = \left[\begin{pmatrix}0\\0\\3\end{pmatrix};\ \begin{pmatrix}1\\2\\9\end{pmatrix};\ \right]$

- When $c\neq 0$: A basis for $\mathcal{C}\left(\mathbf{A}\right)$ is any basis for $\mathbb{R}^3$; for example $\left[\begin{pmatrix}1\\0\\0\end{pmatrix};\ \begin{pmatrix}0\\1\\0\end{pmatrix};\ \begin{pmatrix}0\\0\\1\end{pmatrix};\ \right]$.

$\square$

(Grupo E curso 20/21) Exercise 1(a)

$$\mathbf{A} = \begin{bmatrix} 1&2&1\\0&0&1\\3&6&3\\3&6&3 \end{bmatrix} \xrightarrow[{[(-1)\mathbf{1}+\mathbf{3}]}]{\overset{\boldsymbol{\tau}}{[(-2)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix} 1&0&0\\0&0&1\\3&0&0\\3&0&0 \end{bmatrix} \xrightarrow{\overset{\boldsymbol{\tau}}{[\mathbf{2}\rightleftharpoons\mathbf{3}]}} \begin{bmatrix} 1&0&0\\0&1&0\\3&0&0\\3&0&0 \end{bmatrix} = \mathbf{R}$$

$\square$

(Grupo E curso 20/21) Exercise 1(b) Since $\mathbf{A}_{\substack{\boldsymbol{\tau}\\ [(-2)\mathbf{1}+\mathbf{2}]\\ [(-1)\mathbf{1}+\mathbf{3}]\\ [\mathbf{2}\rightleftharpoons\mathbf{3}]}} = \mathbf{R}$, then $\mathbf{E} = \mathbf{I}_{\substack{\boldsymbol{\tau}\\ [(-2)\mathbf{1}+\mathbf{2}]\\ [(-1)\mathbf{1}+\mathbf{3}]\\ [\mathbf{2}\rightleftharpoons\mathbf{3}]}}$. Therefore

$$\mathbf{E}^{\text{-}1} = \mathbf{I}_{\substack{\boldsymbol{\tau}\\ [\mathbf{2}\rightleftharpoons\mathbf{3}]\\ [(1)\mathbf{1}+\mathbf{3}]\\ [(2)\mathbf{1}+\mathbf{2}]}},\ \text{so, } \mathbf{E}^{\text{-}1} = \mathbf{I}_{\substack{\boldsymbol{\tau}\\ [\mathbf{2}\rightleftharpoons\mathbf{3}]\\ [(1)\mathbf{1}+\mathbf{3}]\\ [(2)\mathbf{1}+\mathbf{2}]}} = \begin{bmatrix} 1&0&0\\0&0&1\\0&1&0 \end{bmatrix} \begin{bmatrix} 1&0&1\\0&1&0\\0&0&1 \end{bmatrix} \begin{bmatrix} 1&2&0\\0&1&0\\0&0&1 \end{bmatrix} = \begin{bmatrix} 1&2&1\\0&0&1\\0&1&0 \end{bmatrix}.$$

$\square$

(Grupo E curso 20/21) Exercise 1(c) $\begin{bmatrix}1\\0\\3\\3\end{bmatrix}\begin{bmatrix}1&2&1\end{bmatrix} + \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\begin{bmatrix}0&0&1\end{bmatrix} + \begin{bmatrix}0\\0\\0\\0\end{bmatrix}\begin{bmatrix}0&1&0\end{bmatrix} =$

$\begin{bmatrix} 1&2&1\\0&0&0\\3&6&3\\3&6&3 \end{bmatrix} + \begin{bmatrix} 0&0&0\\0&0&1\\0&0&0\\0&0&0 \end{bmatrix} + \begin{bmatrix} 0&0&0\\0&0&0\\0&0&0\\0&0&0 \end{bmatrix} = \begin{bmatrix} 1&2&1\\0&0&1\\3&6&3\\3&6&3 \end{bmatrix}$. Rank 1, 1 and 0 respectively.

$\square$

(Grupo E curso 20/21) Exercise 2(a) Since $\boldsymbol{u}$; $\boldsymbol{v}$; $\boldsymbol{w}$ are a basis, they are linearly independent. Hence, (de orden 4×3) is full *column* rank. So, all columns of any pre-echelon form of $\mathbf{A}$ are pivot columns (no special solutions).

$\square$

(Grupo E curso 20/21) Exercise 3(a) No. The set is not closed under addition. For example: $\begin{bmatrix}1&0\\0&0\end{bmatrix} + \begin{bmatrix}0&0\\0&1\end{bmatrix} = \mathbf{I}$ is full rank.

$\square$

(Grupo E curso 20/21) Exercise 3(b) Yes. It is a vector space. Consider matrices $\mathbf{A}$ and $\mathbf{B}$ with diagonal entries $(a_{11},\ a_{22},\ a_{33})$ and $(b_{11},\ b_{22},\ b_{33})$, respectively, where $a_{11}+a_{22}+a_{33} = b_{11}+b_{22}+b_{33} = 0$. Any linear combination $\lambda\mathbf{A} + \mu\mathbf{B}$ will have diagonal entries $(\lambda a_{11}+\mu b_{11},\ \lambda a_{22}+\mu b_{22},\ \lambda a_{33}+\mu b_{33})$. Hence $(\lambda a_{11}+\mu b_{11}) + (\lambda a_{22}+\mu b_{22}) + (\lambda a_{33}+\mu b_{33}) = \lambda(a_{11}+a_{22}+a_{33}) + \mu(b_{11}+b_{22}+b_{33}) = 0+0 = 0$.
$\square$

(Grupo E curso 20/21) Exercise 4(a) The system $\mathbf{A}\boldsymbol{x} = \begin{pmatrix}0\\0\\1\end{pmatrix}$.

$\square$

(Grupo E curso 20/21) Exercise 4(b) $\mathbf{A}$ invertible if $b\neq 3$. And $\begin{bmatrix} 0&3&2\\0&b&2\\2&a&1 \end{bmatrix}\boldsymbol{x} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ gives $\boldsymbol{x} = \begin{pmatrix}\frac{1}{2}\\0\\0\end{pmatrix}$ by inspection.

$\square$

(Grupo E curso 20/21) Exercise 4(c)

- If $b = 3$ then $\operatorname{rg}(\mathbf{A}) = 2$ (Two equal rows, regardless of a).
- If $b \neq 3$ then $\operatorname{rg}(\mathbf{A}) = 3$ (Three linearly independent columns, regardless of a).

□

(Grupo E curso 20/21) Exercise 4(d)

- If $b = 3$ then a basis for $\mathcal{C}(\mathbf{A})$ is $\mathcal{C}(\mathbf{A}) = \left[\begin{pmatrix}0\\0\\2\end{pmatrix}; \begin{pmatrix}3\\3\\a\end{pmatrix}; \right]$.
- If $b \neq 3$ then a basis for $\mathcal{C}(\mathbf{A})$ is any basis for $\mathbb{R}^3$, for example $\left[\begin{pmatrix}1\\0\\0\end{pmatrix}; \begin{pmatrix}0\\1\\0\end{pmatrix}; \begin{pmatrix}0\\0\\1\end{pmatrix}; \right]$.

□

(Grupo E curso 20/21) Exercise 5.

$$\text{Basis for } \mathcal{C}(\mathbf{M}^\intercal): \left[\begin{pmatrix}1\\0\\0\end{pmatrix}; \begin{pmatrix}0\\1\\0\end{pmatrix}; \right], \qquad \text{Basis for } \mathcal{N}(\mathbf{M}): \left[\begin{pmatrix}0\\0\\1\end{pmatrix}; \right],$$

$$\text{Basis for } \mathcal{C}(\mathbf{M}): \left[\begin{pmatrix}0\\1\\2\\0\\4\end{pmatrix}; \begin{pmatrix}0\\0\\0\\1\\2\end{pmatrix}; \right], \qquad \text{Basis for } \mathcal{N}(\mathbf{M}^\intercal): \left[\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}; \begin{pmatrix}0\\-2\\1\\0\\0\end{pmatrix}; \begin{pmatrix}0\\-4\\0\\-2\\1\end{pmatrix}; \right].$$

□

(Grupo B curso 19/20) Exercise 1(a)

$$\left[\begin{array}{ccc|c} 1&2&3&-3\\ 2&4&7&-4\\ 3&6&10&-7\\ 3&6&10&-7\\ \hline 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ \hline 0&0&0&1 \end{array}\right] \xrightarrow[\substack{[(-2)\mathbf{1}+\mathbf{2}]\\ [(-3)\mathbf{1}+\mathbf{3}]\\ [(3)\mathbf{1}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc|c} 1&0&0&0\\ 2&0&1&2\\ 3&0&1&2\\ 3&0&1&2\\ \hline 1&-2&-3&3\\ 0&1&0&0\\ 0&0&1&0\\ \hline 0&0&0&1 \end{array}\right] \xrightarrow[{[(-2)\mathbf{3}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc|c} 1&0&0&0\\ 2&0&1&0\\ 3&0&1&0\\ 3&0&1&0\\ \hline 1&-2&-3&9\\ 0&1&0&0\\ 0&0&1&-2\\ \hline 0&0&0&1 \end{array}\right]$$

A basis for $\mathcal{C}(\mathbf{A})$: $\left[\begin{pmatrix}1\\2\\3\\3\end{pmatrix}; \begin{pmatrix}3\\7\\10\\10\end{pmatrix}; \right]$; another one: $\left[\begin{pmatrix}1\\2\\3\\3\end{pmatrix}; \begin{pmatrix}0\\1\\1\\1\end{pmatrix}; \right]$.

□

(Grupo B curso 19/20) Exercise 1(b) A basis for $\mathcal{N}(\mathbf{A})$: $\left[\begin{pmatrix}-2\\1\\0\end{pmatrix}; \right]$.

□

(Grupo B curso 19/20) Exercise 1(c) $\left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{pmatrix}9\\0\\-2\end{pmatrix} + \begin{bmatrix}-2\\1\\0\end{bmatrix} \boldsymbol{p} \right\}$.

□

(Grupo B curso 19/20) Exercise 2. $\boxed{\text{There is no such matrix}}$. Since $\mathcal{N}(\mathbf{A}) \subset \mathbb{R}^3$ and $\mathcal{C}(\mathbf{A}) \subset \mathbb{R}^3$, matrix $\mathbf{A}$ is of order 3 times 3; and since $\dim \mathcal{C}(\mathbf{A}) = 1$, the rank is one. Hence, $\dim \mathcal{N}(\mathbf{A})$ must be $3 - 1 = 2$, therefore the system $[\ (1,\ \ 2,\ \ 1,);\ ]$ can't be a basis for $\mathcal{N}(\mathbf{A})$, since it has only one vector.
□

(Grupo B curso 19/20) Exercise 3(a) The nullspace has dimension 2. Therefore $\operatorname{rg}(\mathbf{A}) = 3 - 2 = 1$, and then $\boxed{\dim \mathcal{C}(\mathbf{A}^\intercal) = 1}$.

□

(Grupo B curso 19/20) Exercise 3(b) Column $\mathbf{A}_{|1}$ comes from knowing a particular solution. The other columns come from knowing the two special solutions: $\boxed{\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}}$.

□

(Grupo B curso 19/20) Exercise 3(c) $\boldsymbol{b}$ must be multiples of $\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$. Hence

$$\mathcal{C}\left(\mathbf{A}\right) = \left\{ \boldsymbol{b} \in \mathbb{R}^3 \;\middle|\; \exists a \in \mathbb{R} \text{ such that } \boldsymbol{b} = a \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} \right\}.$$

□

(Grupo B curso 19/20) Exercise 4(a) Remember that left-multiplications do row operations and that right- multiplications do column operations. (i) Swapping the first and second rows is an elementary row operation, given by left-multiplication by the matrix:

$$\mathbf{B} = \mathbf{E}\mathbf{A} \quad \text{where} \quad \mathbf{E} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In this case we therefore have $\mathbf{B} = \mathbf{E}\mathbf{A}$ with $\mathbf{E}$ as above. (ii) $\mathbf{E}$ is invertible (in fact $\mathbf{E}^2 = \mathbf{I}$), since you can undo a row swap by swapping again.

□

(Grupo B curso 19/20) Exercise 4(b) (i) We are again performing row operations, so we'll have $\boxed{\mathbf{B} = \mathbf{E}\mathbf{A}}$. To find $\mathbf{E}$ E, we can simply apply the operations to the identity matrix $\mathbf{I}$. Keeping the first row the same doesn't change $\mathbf{I}$. Adding the second row to the third row yields the matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Replacing the second row with the sum the final answer:

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(ii) $\mathbf{E}$ is not invertible; its columns are linearly dependent. In fact, the last two columns are equal. This means that the vector $(0, 1, 1)$ is in the nullspace of $\mathbf{E}$. But the nullspace of an invertible matrix must include only the zero vector, hence $\mathbf{E}$ is singular.

□

(Grupo B curso 19/20) Exercise 4(c) (i) We are now operating on columns, so we'll have $\boxed{\mathbf{B} = \mathbf{A}\mathbf{E}}$. To compute $\mathbf{E}$, as usual we can just apply the operation in question to the identity matrix. Subtracting the first column from the second and third columns gives the matrix:

$$\mathbf{E} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(ii) $\mathbf{E}$ is invertible , because the corresponding column operation is invertible: just add the first column back to the other two! In fact, from this we can see that the inverse of $\mathbf{E}$ is

$$\mathbf{E}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

(Grupo B curso 19/20) Exercise 5. $\left(\mathbf{A}^{\intercal}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{\intercal} = \begin{bmatrix} 4 & 3 & 3 \\ -1 & -1 & -1 \\ -3 & 0 & 1 \end{bmatrix}^{\intercal} = \boxed{\begin{bmatrix} 4 & -1 & -3 \\ 3 & -1 & 0 \\ 3 & -1 & 1 \end{bmatrix}}.$

□

(Grupo E curso 19/20) Exercise 1(a)

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & -1 & -1 \\ 2 & 7 & 4 & -2 & -2 \\ 3 & 9 & 6 & 7 & -13 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-3)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{3}] \\ [(1)\mathbf{1}+\mathbf{4}] \\ [(1)\mathbf{1}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 10 & -10 \\ \hline 1 & -3 & -2 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(1)\mathbf{4}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 10 & 0 \\ \hline 1 & -3 & -2 & 1 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$S = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \exists a \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} + a \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

The set of solutions is a line but, since $(0, 0, 0, 0)$ is not in the set, the set of solutions is not a subspace.

□

(Grupo E curso 19/20) Exercise 1(b) In the previous solution, the "free" variable is z (corresponding to the third column of $\mathbf{A}$). But, by elementary trasformations we get

$$\left[\begin{array}{c|c} -2 & 2 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{c|c} -2 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(\frac{-1}{2})\mathbf{1}]}} \left[\begin{array}{c|c} 1 & 0 \\ 0 & 0 \\ \frac{-1}{2} & 1 \\ 0 & 1 \end{array}\right] \Rightarrow S = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \;\middle|\; \exists a \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + a \begin{pmatrix} 1 \\ 0 \\ \frac{-1}{2} \\ 0 \end{pmatrix} \right\}.$$

Where now the "free" variable is x.

□

(Grupo E curso 19/20) Exercise 1(c) $\boxed{\text{Yes}}$. Since $\mathcal{C}\left(\mathbf{A}\right) \subset \mathbb{R}^3$ and rank is 3, then $\boxed{\mathcal{C}\left(\mathbf{A}\right) = \mathbb{R}^3.}$

$\boxed{\text{The new entry is } \underline{{}_{2|}\mathbf{A}_{|2} = 6\,;}}$ so $\mathbf{M} = \begin{bmatrix} 1 & 3 & 2 & -1 \\ 2 & 6 & 4 & -2 \\ 3 & 9 & 6 & 7 \end{bmatrix}$; and $\operatorname{rg}\left(\mathbf{M}\right) = 2 = \dim \mathcal{C}\left(\mathbf{M}\right)$.

□

(Grupo E curso 19/20) Exercise 1(d) For example: $\boxed{\boldsymbol{c} = \mathbf{A}_{|1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}; \quad \boldsymbol{d} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.}$

□

(Grupo E curso 19/20) Exercise 2. Premultiply both sides of the original equation by $\mathbf{P}^{-1} = \mathbf{P}^{\intercal}$ we get

$$\begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 1 & 0 & 8 \end{bmatrix} \mathbf{A} = \mathbf{P}^{-1}\mathbf{P} = \mathbf{I} \quad \Longrightarrow \quad \mathbf{A}^{-1} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 1 & 0 & 8 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 8 \\ 1 & 2 & 3 \\ 2 & 5 & 6 \end{bmatrix}}.$$

□

(Grupo E curso 19/20) Exercise 3(a)

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \\ 0 & 1 & 2 & -1 \end{bmatrix}, \text{ or any nonzero multiple of it.}$$

□

(Grupo E curso 19/20) Exercise 3(b) $\boxed{\text{Yes.}}$

$$\mathbf{A} = \begin{bmatrix}0&1&1\\1&1&3\\1&1&2\end{bmatrix} \xrightarrow[{[(-1)\mathbf{2}+\mathbf{3}]}]{\tau} \begin{bmatrix}0&1&0\\1&1&2\\1&1&1\end{bmatrix} \xrightarrow[{\substack{[(-1)\mathbf{1}+\mathbf{2}]\\ [(-2)\mathbf{1}+\mathbf{3}]}}]{\tau} \begin{bmatrix}0&1&0\\1&0&0\\1&0&-1\end{bmatrix} \xrightarrow[{[\mathbf{1}\rightleftharpoons\mathbf{2}]}]{\tau} \begin{bmatrix}1&0&0\\0&1&0\\0&1&-1\end{bmatrix}$$

Therefore $\mathcal{C}\left(\mathbf{A}\right) = \mathbb{R}^3$ (columns span the space) and $\mathcal{N}\left(\mathbf{A}\right) = \{\mathbf{0}\}$; (are linearly independent) so, the columns of $\mathbf{A}$ form a basis for $\mathbb{R}^3$.

□

(Grupo E curso 19/20) Exercise 4(a) Since ${}_{i|}\Big(\mathbf{BC}\Big) = \Big({}_{i|}\mathbf{B}\Big)\mathbf{C}$, since ${}_{1|}\mathbf{C}$ and ${}_{3|}\mathbf{C}$ are linearly independent and $\mathbf{B}$ is full rank, then a basis for the row space is: $\left[{}_{1|}\mathbf{C};\quad {}_{3|}\mathbf{C};\right] = \left[(2,2,4,4,);\quad (2,2,6,6,);\right]$.

□

(Grupo E curso 19/20) Exercise 4(b) Since $\mathbf{A}_{|j} = \mathbf{B}\Big(\mathbf{C}_{|j}\Big)$; and since $\mathbf{B}$ is full rank and $\mathbf{C}_{|1}$ and $\mathbf{C}_{|3}$ are linearly independent, then $\mathbf{BC}_{|1} = \mathbf{B}\begin{pmatrix}2\\0\\4\end{pmatrix} = \begin{pmatrix}4\\6\\6\end{pmatrix}$ and $\mathbf{BC}_{|3} = \mathbf{B}\begin{pmatrix}2\\0\\6\end{pmatrix} = \begin{pmatrix}8\\12\\14\end{pmatrix}$ are also linearly independent. Hence, one basis for the column space is: $\left[\begin{pmatrix}2\\0\\6\end{pmatrix};\quad \begin{pmatrix}8\\12\\14\end{pmatrix};\right]$.

□

(Grupo E curso 19/20) Exercise 4(c) Since the rank of $\mathbf{A}$ is 2 and $\mathbf{C}_{|1} = \mathbf{C}_{|2}$ and $\mathbf{C}_{|3} = \mathbf{C}_{|4}$, a basis for $\mathcal{N}\left(\mathbf{A}\right)$ is $\left[\begin{pmatrix}-1\\1\\0\\0\end{pmatrix};\quad \begin{pmatrix}0\\0\\-1\\1\end{pmatrix};\right]$.

□

(Grupo E curso 19/20) Exercise 4(d) If $b_{33} = 0$ then $\mathbf{B} = \begin{bmatrix}2&0&0\\3&4&0\\2&0&0\end{bmatrix}$ and therefore, for the new $\mathbf{A} = \mathbf{BC}$ follows that $\dim\mathcal{C}\left(\mathbf{A}\right) = 1 = \dim\mathcal{C}\left(\mathbf{A}^{\intercal}\right)$ and $\dim\mathcal{N}\left(\mathbf{A}\right) = 3$.

□

(Grupo B curso 18/19) Exercise 1(a)

$$\left[\begin{array}{ccc|c}1&3&1&-b_1\\3&8&2&-b_2\\5&12&2&-b_3\end{array}\right] \xrightarrow[{\substack{[(-3)\mathbf{1}+\mathbf{2}]\\ [(-1)\mathbf{1}+\mathbf{3}]\\ [(b_1)\mathbf{1}+\mathbf{4}]}}]{\tau} \left[\begin{array}{ccc|c}1&0&0&0\\3&-1&-1&3b_1-b_2\\5&-3&-3&5b_1-b_3\end{array}\right] \xrightarrow[{\substack{[(-1)\mathbf{2}+\mathbf{3}]\\ [(3b_1-b_2)\mathbf{2}+\mathbf{4}]}}]{\tau} \left[\begin{array}{ccc|c}1&0&0&0\\3&-1&0&0\\5&-3&0&-4b_1+3b_2-b_3\end{array}\right]$$

hence, $\boxed{4b_1 - 3b_2 + b3 = 0}$.

□

(Grupo B curso 18/19) Exercise 1(b) There are two pivots in the echelon matrix, so the rank is $\boxed{2}$.

□

(Grupo B curso 18/19) Exercise 1(c) First we get a reduced echelon form

$$\begin{bmatrix}1&0&0\\3&-1&0\\5&-3&0\end{bmatrix} \xrightarrow[{[(3)\mathbf{2}+\mathbf{1}]}]{\tau} \begin{bmatrix}1&0&0\\0&-1&0\\-4&-3&0\end{bmatrix} \xrightarrow[{[(-1)\mathbf{2}]}]{\tau} \begin{bmatrix}1&0&0\\0&1&0\\-4&3&0\end{bmatrix} = \mathbf{R}.$$

One might have noticed from $\mathbf{R}$ that

$$\begin{pmatrix}4, & -3, & 1,\end{pmatrix}\mathbf{R} = \mathbf{0} \quad\Longrightarrow\quad \begin{pmatrix}4, & -3, & 1,\end{pmatrix}\mathbf{A} = \mathbf{0},$$

so clearly $\boxed{\begin{pmatrix}4, & -3, & 1,\end{pmatrix} \text{ is in the left nullspace}}$.

Using R we can also answer the first question, since every vector in $\mathcal{C}\left(\mathsf{A}\right)$ is a linear combination of the two first columns of R

$$b_1\begin{pmatrix}1\\0\\-4\end{pmatrix}+b_2\begin{pmatrix}0\\1\\3\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\-4b_1+3b_2\end{pmatrix}\quad\Rightarrow\quad b_3=-4b_1+3b_2$$

□

(Grupo B curso 18/19) Exercise 2. Since $\mathsf{A}\left(\boldsymbol{v}-\boldsymbol{w}\right)=\mathsf{A}\boldsymbol{v}-\mathsf{A}\boldsymbol{w}=\boldsymbol{0}$; then $\left(\boldsymbol{v}-\boldsymbol{w}\right)=\begin{pmatrix}1\\2\\3\end{pmatrix}-\begin{pmatrix}3\\1\\4\end{pmatrix}=\begin{pmatrix}-2\\1\\-1\end{pmatrix}\in\mathcal{N}\left(\mathsf{A}\right)$. Therefore $\begin{pmatrix}1\\2\\3\end{pmatrix}+\left(\boldsymbol{v}-\boldsymbol{w}\right)=\begin{pmatrix}1\\2\\3\end{pmatrix}+\begin{pmatrix}-2\\1\\-1\end{pmatrix}=\boxed{\begin{pmatrix}-1\\3\\2\end{pmatrix}\text{ is another solution}}$.

□

(Grupo B curso 18/19) Exercise 3.

$$\left[\begin{array}{cccc}1&1&0&1\\0&1&0&-1\\0&0&1&0\\0&0&0&-2\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[\substack{[(-1)\mathbf{1}+\mathbf{2}]\\ [(-1)\mathbf{1}+\mathbf{4}]\\ [(1)\mathbf{2}+\mathbf{4}]\\ [(\frac{-1}{2})\mathbf{4}]}]{\boldsymbol{\tau}}\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\\hline 1&-1&0&1\\0&1&0&\frac{-1}{2}\\0&0&1&0\\0&0&0&\frac{-1}{2}\end{array}\right];\quad\text{then}\quad\boxed{\mathsf{A}^{-1}=\begin{bmatrix}1&-1&0&1\\0&1&0&-1/2\\0&0&1&0\\0&0&0&-1/2\end{bmatrix}}.$$

□

(Grupo B curso 18/19) Exercise 4. We want to find a matrix whose null space is spanned by $\boldsymbol{v}=\begin{pmatrix}1&2&-1\end{pmatrix}$. Such a matrix must have three columns, so that this vector can be an element of the null space. It must have rank two, so that there are no other vectors in a basis for its null space: the null space must be one-dimensional to be spanned by $\boldsymbol{v}$. Examples of possible matrices are:

$$\begin{bmatrix}-1&0&-1\\0&2&4\end{bmatrix},\quad\begin{bmatrix}1&0&1\\0&1&2\\0&0&0\end{bmatrix};\quad\text{since}\quad\begin{bmatrix}-1&0&-1\\0&2&4\end{bmatrix}\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix};\ \begin{bmatrix}1&0&1\\0&1&2\\0&0&0\end{bmatrix}\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Applying *reverse engineering* (for example on a three row matrix) we can find more matrices; for example, is a and b are non-zero numbers, then:

$$\left[\begin{array}{ccc}a&0&0\\0&b&0\\c&d&0\\\hline 1&0&1\\0&1&2\\0&0&-1\end{array}\right]\xrightarrow[{[(-1)\mathbf{3}]}]{\boldsymbol{\tau}}\left[\begin{array}{ccc}a&0&0\\0&b&0\\c&d&0\\\hline 1&0&-1\\0&1&-2\\0&0&1\end{array}\right]\xrightarrow[\substack{[(1)\mathbf{1}+\mathbf{3}]\\ [(2)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}}\left[\begin{array}{ccc}a&0&a\\0&b&2b\\c&d&(c+2d)\\\hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]=\left[\begin{array}{c}\mathsf{A}\\\hline\mathsf{I}\end{array}\right]\quad\Rightarrow\quad\mathsf{A}=\begin{bmatrix}a&0&a\\0&b&2b\\c&d&(c+2d)\end{bmatrix}$$

You might be tempted to write a matrix with one row, like $\begin{bmatrix}1&0&1\end{bmatrix}$, which indeed has $\boldsymbol{v}$ in its nullspace, but such a matrix has other vectors in its null space also—this matrix is rank 1, so it has a two-dimensional nullspace that is not spanned by $\boldsymbol{v}$. A matrix of all zeros would be even worse—it would have rank 0, with a $3d$ nullspace that contains $\boldsymbol{v}$ but also contains every other 3-component vector.

□

(Grupo B curso 18/19) Exercise 5(a) Since $\boldsymbol{x}\in\mathbb{R}^4$, then $\boxed{n=4}$.

□

(Grupo B curso 18/19) Exercise 5(b) Since $n=4$ and there is only one special solution, then $\boxed{\operatorname{rg}\left(\mathsf{A}\right)=4-1=3}$.

□

(Grupo B curso 18/19) Exercise 5(c) Since $\operatorname{rg}\left(\mathsf{A}\right)=3$ then $\boxed{m\geq 3}$.

□

(Grupo B curso 18/19) Exercise 6(a) $\boxed{\text{This is not a subspace}}$, since it does not contain the zero vector $\mathbf{0} \in \mathbb{R}^6$.

□

(Grupo B curso 18/19) Exercise 6(b) $\boxed{\text{This is a subspace}}$ (it is closely related to, but is not the same as, the null space of $\mathbf{A}$). If we take any two matrices $\mathbf{C}$, $\mathbf{D}$ in our set $\mathcal{V}$, then the linear combination $a\mathbf{C} + b\mathbf{D}$ will still be in our set $\mathcal{V}$, since $\mathbf{A}(a\mathbf{C} + b\mathbf{D}) = a\mathbf{AC} + b\mathbf{AD} = a\mathbf{0} + b\mathbf{0} = \mathbf{0}$.

□

(Grupo B curso 18/19) Exercise 6(c) $\boxed{\text{This is not a vector space}}$. The set does contain the zero matrix (the zero matrix is singular), and it is closed under multiplication by scalars. However, consider two matrices

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which are both singular (neither has three pivots). Their sum is $\mathbf{I}$, and the identity matrix is not singular. Therefore the set is not closed under matrix addition.

□

(Grupo B curso 18/19) Exercise 7(a)

All you can say is that $\boxed{\text{rank }\mathbf{A} \le \text{rank }[\mathbf{A}\,|\,\mathbf{B}]}$. ($\mathbf{A}$ can have any number r of pivot columns, and these will all be pivot columns for $[\mathbf{A}\,|\,\mathbf{B}]$; but there could be more pivot columns among the columns of $\mathbf{B}$).

□

(Grupo B curso 18/19) Exercise 7(b)

Now $\boxed{\text{rank }\mathbf{A} = \text{rank }[\mathbf{A}\,|\,\mathbf{A}^2]}$. (Every column of $\mathbf{A}^2$ is a linear combination of columns of A. For instance, if we call $\mathbf{A}$'s first column $\mathbf{A}_{|1}$, then $\mathbf{A} \cdot \mathbf{A}_{|1}$ is the first column of $\mathbf{A}^2$. So there are no new pivot columns in the $\mathbf{A}^2$ part of $[\mathbf{A}\,|\,\mathbf{A}^2]$).

□

(Grupo B curso 18/19) Exercise 7(c)

The nullspace A has dimension $n-r$, as always. Since $[\mathbf{A}\ \mathbf{A}]$ only has r pivot columns — the n columns we added are all duplicates — $[\mathbf{A}\,|\,\mathbf{A}]$ is an m-by-$2n$ matrix of rank r, and its nullspace $N([\mathbf{A}\,|\,\mathbf{A}])$ has dimension $\boxed{2n - r}$.

□

(Grupo E curso 18/19) Exercise 1(a) Column operations always preserve the left null space $\mathcal{N}\left(\mathbf{A}^\mathsf{T}\right)$, i.e. any solution to $\boldsymbol{x}\mathbf{A} = \mathbf{0}$ will be preserved by column operations. Let $\mathbf{W}$ be the weird colummn-reduced matrix obtained by our Harvard friend. We can still seek special solutions to $\boldsymbol{x}\mathbf{W} = \mathbf{0}$ using the usual method. Rows 3, 4 and 5 are the pivot rows, while rows 1 and 2 are the free rows. We therefore look for two special solutions:

$$\boldsymbol{s}_1 = \begin{pmatrix}1, & 0, & x_3, & x_4, & x_5,\end{pmatrix}; \qquad \boldsymbol{s}_2 = \begin{pmatrix}0, & 1, & y_3, & y_4, & y_5,\end{pmatrix}.$$

We can then see that $\begin{pmatrix}x_3, & x_4, & x_5,\end{pmatrix} = \begin{pmatrix}-2, & -4, & -6,\end{pmatrix}$ and $\begin{pmatrix}y_3, & y_4, & y_5,\end{pmatrix} = \begin{pmatrix}-3, & -5, & -7,\end{pmatrix}$, i.e., the negative entries of each row of $\mathbf{H}$. This gives us a basis for the left null space of $\mathbf{A}$:

$$\boxed{\left\{ \begin{pmatrix}1, & 0, & -2, & -4, & -6,\end{pmatrix}; \quad \begin{pmatrix}0, & 1, & -3, & -5, & -7,\end{pmatrix} \right\}}.$$

□

(Grupo E curso 18/19) Exercise 1(b) We want to first reorder the rows of $\mathbf{W}$ so that it is in the usual reduced echelon form. Recall that row operations are equivalent to multiplying on the left by an appropriate matrix. A matrix that will put the rows of $\mathbf{W}$ in the correct order is the following permutation matrix

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The matrix $\mathbf{R} = \mathbf{MW}$ will then be in the usual reduced echelon form. Remember that our Harvard friend performed column operations to put our matrix $\mathbf{A}$ into the weird form $\mathbf{W}$, and column operations won't change the row order. In particular, recall that column operations are equivalent to multiplying by an appropriate matrix on the right, so there exists a matrix $\mathbf{E}$ so that $\mathbf{AE} = \mathbf{W}$. The product $\mathbf{R} = \mathbf{MW} = \mathbf{MAE} = \big(\mathbf{MA}\big)\mathbf{E}$ is then in the usual reduced echelon form. So peforming the same column operations as our Harvard friend on the matrix $\mathbf{MA}$ will give us a matrix in the usual reduced echelon form.

$\square$

(Grupo E curso 18/19) Exercise 2(a) The fastest way might be to recognize that $\mathbf{E}_1$ performs four elementary column transformations of adding a, b, c, d times the first column to the other columns, and $\mathbf{E}_1^{-1}$ undoes the same operations, i.e. subtracts a, b, c, d times the first column from the other columns, giving immediately that $\boldsymbol{x}$ is the sum of the columns of $\mathbf{E}_1^{-1}$:

$$\big(\mathbf{E}_1\big)\boldsymbol{x} = \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{x} = \big(\mathbf{E}_1^{-1}\big)\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} = \begin{bmatrix}1 & -a & -b & -c & -d\\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1\end{bmatrix}\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}(1-a-b-c-d)\\1\\1\\1\\1\end{pmatrix}.$$

$\square$

(Grupo E curso 18/19) Exercise 2(b) The fastest way is to recognize that $\boldsymbol{x}$ is the sum of the columns of $\big(\mathbf{E}_1^{\intercal}\big)^{-1}$. Hence:

$$\big(\mathbf{E}_1^{\intercal}\big)\boldsymbol{x} = \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{x} = \big(\mathbf{E}_1^{\intercal}\big)^{-1}\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} = \begin{bmatrix}1 & & & & \\ -a & 1 & & & \\ -b & & 1 & & \\ -c & & & 1 & \\ -b & & & & 1\end{bmatrix}\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\(1-a)\\(1-b)\\(1-c)\\(1-d)\end{pmatrix}.$$

$\square$

(Grupo E curso 18/19) Exercise 2(c)

$$\mathbf{E} = \mathbf{E}_1 \cdot \mathbf{E}_2 = \begin{bmatrix}1 & a & b & c & (d+cx)\\ & 1 & & & \\ & & 1 & & \\ & & & 1 & x\\ & & & & 1\end{bmatrix}.$$

$\square$

(Grupo E curso 18/19) Exercise 2(d)

$$\mathbf{E}^{-1} = \mathbf{E}_2^{-1} \cdot \mathbf{E}_1^{-1} = \begin{bmatrix}1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & -x\\ & & & & 1\end{bmatrix}\begin{bmatrix}1 & -a & -b & -c & -d\\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1\end{bmatrix} = \begin{bmatrix}1 & -a & -b & -c & -d\\ & 1 & & & \\ & & 1 & & \\ & & & 1 & -x\\ & & & & 1\end{bmatrix}.$$

To check the answer we multiply this by the matrix $\mathbf{E}$ of part c)

$$\begin{bmatrix}1 & -a & -b & -c & -d\\ & 1 & & & \\ & & 1 & & \\ & & & 1 & -x\\ & & & & 1\end{bmatrix}\begin{bmatrix}1 & a & b & c & (d+cx)\\ & 1 & & & \\ & & 1 & & \\ & & & 1 & x\\ & & & & 1\end{bmatrix} = \begin{bmatrix}1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1\end{bmatrix}.$$

$\square$

(Grupo E curso 18/19) Exercise 2(e) The best solution is the operator interpretation of adding a, b, c, d times column 1, 10 times and interpreting the answer. One can also notice the pattern by iteration: We do 2 multiplication steps to notice and pattern and then write the solution

$$\big(\mathbf{E}_1\big)^2 = \begin{bmatrix}1 & a & b & c & d\\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1\end{bmatrix}\begin{bmatrix}1 & a & b & c & d\\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1\end{bmatrix} = \begin{bmatrix}1 & 2a & 2b & 2c & 2d\\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1\end{bmatrix}$$

$$(\mathbf{E}_1)^3 = \begin{bmatrix} 1 & 2a & 2b & 2c & 2d \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & a & b & c & d \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3a & 3b & 3c & 3d \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

and thus iteratively we can see

$$(\mathbf{E}_1)^{10} = \begin{bmatrix} 1 & 10a & 10b & 10c & 10d \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}.$$

□

(Grupo E curso 18/19) Exercise 3. We want to find a matrix whose column space is a subspace of $\mathbb{R}$, but does not include $\begin{pmatrix}1, & 2, & -1,\end{pmatrix}$. Such a matrix must have three rows, but can have any number of columns, provided that $\begin{pmatrix}1, & 2, & -1,\end{pmatrix}$ is not in the span of the columns, which means that the matrix necessarily has rank less than three. Examples of possible matrices are

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that, to know that a vector is not in the column space, you must be sure that any linear combination of the columns of the matrix cannot give you that vector. So, for example, the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 2 \end{bmatrix}$$

does not work. Even though $\begin{pmatrix}1, & 2, & -1,\end{pmatrix}$ does not appear explicitly as one of its columns, this vector is in the column space because you can get it by the linear combination $3\mathbf{A}_{|1} - 2\mathbf{A}_{|2} = \begin{pmatrix}1, & 2, & -1,\end{pmatrix}$.

□

(Grupo E curso 18/19) Exercise 4.

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & -1 \\ 2 & 6 & 4 & 8 & -3 \\ 0 & 0 & 2 & 4 & -1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{1}+\mathbf{5}]}]{\substack{\tau \\ [(-3)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(-2)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 4 & -1 \\ 0 & 0 & 2 & 4 & -1 \\ \hline 1 & -3 & -1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-2)\mathbf{3}+\mathbf{4}] \\ [(\frac{1}{2})\mathbf{3}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ \hline 1 & -3 & -1 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\text{Conjunto de vectores:} \quad \left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 \text{ tal que } \boldsymbol{x} = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix} + \begin{bmatrix} -3 & 0 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}$$

□

(Grupo E curso 18/19) Exercise 5(a) The set with the zero vector of $\mathbb{R}^4$ only: $\{\mathbf{0}\}$.

□

(Grupo E curso 18/19) Exercise 5(b) $5 - 4 = 1$

□

(Grupo E curso 18/19) Exercise 5(c) $\boldsymbol{x}_p = \begin{pmatrix}0, & 1, & 0, & 0,\end{pmatrix}$

□

(Grupo E curso 18/19) Exercise 5(d) $\boldsymbol{x} = \boldsymbol{x}_p$ because $\mathcal{N}\left(\mathbf{A}\right) = \{\mathbf{0}\}$.

□

(Grupo E curso 17/18) Exercise 1(a)

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -b_1\\ -1 & 2 & -1 & 0 & -b_2\\ 0 & -1 & 2 & -1 & -b_3\\ 0 & 0 & -1 & 1 & -b_4\\ \hline 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{3}+\mathbf{4}]}]{\substack{\tau\\ {[(1)\mathbf{1}+\mathbf{2}]}\\ {[(1)\mathbf{2}+\mathbf{3}]}}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -b_1\\ -1 & 1 & 0 & 0 & -b_2\\ 0 & -1 & 1 & 0 & -b_3\\ 0 & 0 & -1 & 0 & -b_4\\ \hline 1 & 1 & 1 & 1 & 0\\ 0 & 1 & 1 & 1 & 0\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1 & 0\\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(b_1+b_2+b_3)\mathbf{3}+\mathbf{5}]}]{\substack{\tau\\ {[(b_1)\mathbf{1}+\mathbf{5}]}\\ {[(b_1+b_2)\mathbf{2}+\mathbf{5}]}}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0\\ -1 & 1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0\\ 0 & 0 & -1 & 0 & -b_1-b_2-b_3-b_4\\ \hline 1 & 1 & 1 & 1 & 3b_1+2b_2+b3\\ 0 & 1 & 1 & 1 & 2b_1+2b_2+b3\\ 0 & 0 & 1 & 1 & b_1+b_2+b_3\\ 0 & 0 & 0 & 1 & 0\\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

After these calculations we have a lot of information. In fact, there is more information than we need to answer the questions *(for example, we don't need the "blue" particular solution.)*

Then answer to this question: the rank is 3.

□

(Grupo E curso 17/18) Exercise 1(b) Since there is only one "special solution" after gaussian elimination, the null space is a *1-dimensional* space. Hence

$$\text{Basis for } \mathcal{N}\left(\mathbf{A}\right) = \left[\begin{pmatrix}1\\1\\1\\1\end{pmatrix};\right]$$

□

(Grupo E curso 17/18) Exercise 1(c) We need to find a condition on $\boldsymbol{b}$ that guarantees $\boldsymbol{b}$ is in the column space of $\mathbf{A}$, such a condition is

$$-b_1 - b_2 - b_3 - b_4 = 0.$$

In other words, the entries must add zero. Note that all columns of $\mathbf{A}$ meet this condition.

□

(Grupo E curso 17/18) Exercise 2(a) No. This is not a vector space because $\mathbf{0}$ is not in this subspace.

□

(Grupo E curso 17/18) Exercise 2(b) Yes. (This is actually just the left nullspace of the matrix whose columns are $\boldsymbol{y}$ and $\boldsymbol{z}$ ... also the null space of the matrix whose rows are $\boldsymbol{y}$ and $\boldsymbol{z}$).

$$\boldsymbol{x}\left[\boldsymbol{y}; \quad \boldsymbol{z};\right] = \mathbf{0}.$$

□

(Grupo E curso 17/18) Exercise 2(c) No. For example, the zero matrix $\mathbf{0}$ is not in this subset.

□

(Grupo E curso 17/18) Exercise 2(d) Yes. If the nullspaces of $\mathbf{A}$ and $\mathbf{B}$ contain $\begin{pmatrix}1\\2\\3\end{pmatrix}$ then any linear combination, $(a\mathbf{A}+b\mathbf{B})$, of these matrices does too:

$$\left(a\mathbf{A}+b\mathbf{B}\right)\begin{pmatrix}1\\2\\3\end{pmatrix} = a\mathbf{A}\begin{pmatrix}1\\2\\3\end{pmatrix} + b\mathbf{B}\begin{pmatrix}1\\2\\3\end{pmatrix} = a\mathbf{0} + b\mathbf{0} = \mathbf{0}.$$

□

(Grupo E curso 17/18) Exercise 3(a) Column operations are multiplications on the *right* by an invertible elimination matrix; similarly, row operations correspond to multiplications on the *left* by an invertible elimination matrix. So, we can write $\mathbf{B} = \mathbf{EA}$ via an elimination matrix $\mathbf{E}$. An easy way to get $\mathbf{E}$ is just to start with the 3×3 identity matrix $\mathbf{I}$ and do the desired row operations, since $\mathbf{EI} = \mathbf{E}$.

$$\underset{[(-2)\mathbf{1}+\mathbf{2}][(-3)\mathbf{1}+\mathbf{3}]}{\overset{\tau}{}\quad\overset{\tau}{}}\mathbf{I} = \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -3 & 0 & 1\end{bmatrix} = \mathbf{E}$$

hence

$$\mathbf{B} = \underset{[(-2)\mathbf{1}+\mathbf{2}][(-3)\mathbf{1}+\mathbf{3}]}{\overset{\tau}{}\quad\overset{\tau}{}}\mathbf{A} = \underset{[(-2)\mathbf{1}+\mathbf{2}][(-3)\mathbf{1}+\mathbf{3}]}{\overset{\tau}{}\quad\overset{\tau}{}}\mathbf{I}\cdot\mathbf{A} = \mathbf{E}\cdot\mathbf{A} = \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -3 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 3 & 0\\ 4 & 1 & -1\\ 6 & 10 & 3\end{bmatrix}.$$

□

(Grupo E curso 17/18) Exercise 3(b) Since $\mathbf{B} = \mathbf{EA}$ involves multiplication on the left by an invertible matrix, we haven't changed the row space $\mathcal{C}\left(\mathbf{B}^{\intercal}\right) = \mathcal{C}\left(\mathbf{A}^{\intercal}\right)$:

- For any $\boldsymbol{y} \in \mathcal{C}\left(\mathbf{B}^{\intercal}\right)$, $\boldsymbol{y} = \boldsymbol{x}\mathbf{B} = (\boldsymbol{x}\mathbf{E})\mathbf{A}$ for some $\boldsymbol{x}$, and hence $\boldsymbol{y} \in \mathcal{C}\left(\mathbf{A}^{\intercal}\right)$.
- Similarly, for any $\boldsymbol{y} \in \mathcal{C}\left(\mathbf{A}^{\intercal}\right)$, $\boldsymbol{y} = \boldsymbol{x}\mathbf{A} = (\boldsymbol{x}\mathbf{E}^{-1})\mathbf{EB} = \boldsymbol{z}\mathbf{B}$ is in the row space of $\mathbf{B}$ (we did something very similar in class.)

In general, multiplying $\mathbf{A}$ on the left will change the left null space, since if $\boldsymbol{x} \in \mathcal{N}\left(\mathbf{A}^{\intercal}\right)$ then $\boldsymbol{x}\mathbf{E}^{-1}$ (not $\boldsymbol{x}$) is in $\mathcal{N}\left(\mathbf{B}^{\intercal}\right)$:

$$\boldsymbol{x}\mathbf{E}^{-1}\mathbf{B} = \boldsymbol{x}\mathbf{E}^{-1}\mathbf{EA} = \boldsymbol{x}\mathbf{A} = \mathbf{0}.$$

However, in this particular case, the matrix is full row rank, so $\mathcal{N}\left(\mathbf{B}^{\intercal}\right) = \mathcal{N}\left(\mathbf{A}^{\intercal}\right) = \{\mathbf{0}\}$.

□

(Grupo E curso 17/18) Exercise 4(a) Yes. Using the three first columns we get (only the last column of $\mathbf{L}$ is cero!):

$$\text{Basis for } \mathcal{C}\left(\mathbf{A}\right) = \left[\begin{pmatrix} a\\1\\c\\1\\1 \end{pmatrix}; \begin{pmatrix} b\\2\\d\\1\\3 \end{pmatrix}; \begin{pmatrix} 3\\3\\4\\1\\0 \end{pmatrix};\right].$$

□

(Grupo E curso 17/18) Exercise 4(b) Since the rank is 3, and the three rows with no letters are linearly independent, those rows constitute a basis for the row space.

$$\text{Basis for } \mathcal{C}\left(\mathbf{A}^{\intercal}\right) = \Big[\begin{pmatrix}1 & 2 & 3 & 3\end{pmatrix}; \begin{pmatrix}1 & 1 & 1 & 2\end{pmatrix}; \begin{pmatrix}1 & 3 & 0 & 4\end{pmatrix};\Big].$$

□

(Grupo E curso 17/18) Exercise 4(c) The system that consists of the last column of $\mathbf{E}$ is a basis. Hence, the null space is the line formed by all the multiples of $\mathbf{E}_{|4}$.

$$\mathcal{N}\left(\mathbf{A}\right) = \left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \exists a \in \mathbb{R} \text{ such that } \boldsymbol{v} = a\begin{pmatrix}-1\\-1\\0\\1\end{pmatrix} \right\}$$

□

(Grupo E curso 17/18) Exercise 4(d) No. There is not enough information.

□

(Grupo E curso 17/18) Exercise 5(a) The four elementary operations in the Gauss-Jordan elimination process are

$$\mathbf{A}\Big(\mathbf{I}_{\underset{[(-1)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}\Big)\Big(\mathbf{I}_{\underset{[(2)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}}\Big)\Big(\mathbf{I}_{\underset{[(-2)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}}\Big)\Big(\mathbf{I}_{\underset{[(1)\mathbf{2}+\mathbf{1}]}{\boldsymbol{\tau}}}\Big) = \mathbf{AE} = \mathbf{I}$$

or

$$\mathbf{A}\cdot \underbrace{\begin{bmatrix}1 & -1 & \\ & 1 & \\ & & 1\end{bmatrix}\begin{bmatrix}1 & & 2\\ & 1 & \\ & & 1\end{bmatrix}}_{\begin{bmatrix}1 & -1 & 2\\ & 1 & \\ & & 1\end{bmatrix}} \underbrace{\begin{bmatrix}1 & & \\ & 1 & \\ & -2 & 1\end{bmatrix}\begin{bmatrix}1 & & \\ 1 & 1 & \\ & & 1\end{bmatrix}}_{\begin{bmatrix}1 & & \\ 1 & 1 & \\ -2 & -2 & 1\end{bmatrix}} = \mathbf{A}\begin{bmatrix}-4 & -5 & 2\\ 1 & 1 & 0\\ -2 & -2 & 1\end{bmatrix} = \mathbf{I}.$$

We can also write the same as:

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] \xrightarrow[{[(1)\mathbf{2}+\mathbf{1}]}]{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{1}+\mathbf{2}]\\ [(2)\mathbf{1}+\mathbf{3}]\\ [(-2)\mathbf{3}+\mathbf{2}]}} \left[\frac{\mathbf{A}_{\underset{[(-1)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(2)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\underset{[(-2)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(1)\mathbf{2}+\mathbf{1}]}{\boldsymbol{\tau}}}}{\mathbf{I}_{\underset{[(-1)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(2)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\underset{[(-2)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(1)\mathbf{2}+\mathbf{1}]}{\boldsymbol{\tau}}}}\right] = \left[\frac{\mathbf{I}}{\mathbf{A}^{-1}}\right] = \left[\begin{array}{ccc}1 & & \\ & 1 & \\ & & 1\\ \hline -4 & -5 & 2\\ 1 & 1 & 0\\ -2 & -2 & 1\end{array}\right] \Rightarrow \mathbf{A}^{-1} = \begin{bmatrix}-4 & -5 & 2\\ 1 & 1 & 0\\ -2 & -2 & 1\end{bmatrix}.$$

□

(Grupo E curso 17/18) Exercise 5(b) We can compute de inverse of $\mathbf{A}^{-1}$ in order to get $\mathbf{A}$, computing the inverse elementary operations on $\mathbf{I}$ (but in the reverse order).

$$\left[\frac{\mathbf{A}^{-1}}{\mathbf{I}}\right] = \left[\begin{array}{ccc} -2 & 1 & -3 \\ -3 & 1 & 0 \\ 3 & -1 & 1 \\ \hline 1 & & \\ & 1 & \\ & & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{1}+\mathbf{2}]}]{\substack{\tau \\ [(-1)\mathbf{2}+\mathbf{1}] \\ [(2)\mathbf{3}+\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc} 1 & & \\ & 1 & \\ & & 1 \\ \hline 1 & 1 & -2 \\ -1 & 0 & 2 \\ 0 & 2 & 1 \end{array}\right] = \left[\frac{\mathbf{I}}{\mathbf{A}}\right],$$

hence, it is the following product: $\left(\mathbf{I}_{\underset{[(-1)\mathbf{2}+\mathbf{1}]}{\tau}}\right)\left(\mathbf{I}_{\underset{[(2)\mathbf{3}+\mathbf{2}]}{\tau}}\right)\left(\mathbf{I}_{\underset{[(-2)\mathbf{1}+\mathbf{3}]}{\tau}}\right)\left(\mathbf{I}_{\underset{[(1)\mathbf{1}+\mathbf{2}]}{\tau}}\right) = \mathbf{A}.$

□

(Grupo E curso 17/18) Exercise 6. $\mathbf{BA}$ is a 4×3. matrix. Since $\mathcal{C}\left(\mathbf{A}\right) = \mathcal{N}\left(\mathbf{B}\right)$, then $\mathbf{BA}\boldsymbol{x}$ for any $\boldsymbol{x}$ gives $\mathbf{B}$ multiplied by something in $\mathcal{N}\left(\mathbf{B}\right)$, which gives zero. Since $\mathbf{BA}\boldsymbol{x} = \mathbf{0}$ for any $\boldsymbol{x}$, we must have $\mathbf{BA} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

□

(Grupo F curso 17/18) Exercise 1(a)

$$\left[\begin{array}{ccc} 2 & 2 & 0 \\ 1 & 4 & 9 \\ 1 & 3 & 9 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\underset{[(-1)\mathbf{1}+\mathbf{2}]}{\tau}} \left[\begin{array}{ccc} 2 & 0 & 0 \\ 1 & 3 & 9 \\ 1 & 2 & 9 \\ \hline 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\underset{[(-3)\mathbf{2}+\mathbf{3}]}{\tau}} \left[\begin{array}{ccc} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 2 & 3 \\ \hline 1 & -1 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{array}\right]$$

Therefore

$$\mathbf{E} = \mathbf{I}_{\underset{[(-1)\mathbf{1}+\mathbf{2}]}{\tau}\underset{[(-3)\mathbf{2}+\mathbf{3}]}{\tau}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

(Grupo F curso 17/18) Exercise 1(b) Since $\mathbf{AE} = \mathbf{L}$, then $\mathbf{A} = \mathbf{LE}^{-1}$; hence $\mathbf{U} = \mathbf{E}^{-1}$.

$$\mathbf{U} = \mathbf{E}^{-1} = \left(\mathbf{I}_{\underset{[(-1)\mathbf{1}+\mathbf{2}]}{\tau}\underset{[(-3)\mathbf{2}+\mathbf{3}]}{\tau}}\right)^{-1} = \left(\mathbf{I}_{\underset{[(-3)\mathbf{2}+\mathbf{3}]}{\tau}}\right)^{-1}\left(\mathbf{I}_{\underset{[(-1)\mathbf{1}+\mathbf{2}]}{\tau}}\right)^{-1} = \left(\mathbf{I}_{\underset{[(3)\mathbf{2}+\mathbf{3}]}{\tau}}\right)\left(\mathbf{I}_{\underset{[(1)\mathbf{1}+\mathbf{2}]}{\tau}}\right)$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

□

(Grupo F curso 17/18) Exercise 2(a) Yes. Using the "pivot" columns of $\mathbf{R}$

$$\text{Basis for } \mathcal{C}\left(\mathbf{A}\right) = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ -4 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 5 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 6 \end{pmatrix};\right].$$

□

(Grupo F curso 17/18) Exercise 2(b) No. There is not enough information. We would need to know the rows of $\mathbf{A}$.

□

(Grupo F curso 17/18) Exercise 2(c) No. There is not enough information. We would need to know how we have managed to make a column of zeros in $\mathbf{R}$.

□

(Grupo F curso 17/18) Exercise 2(d) We have to look at the last two rows of $\mathbf{R}$ (the rows with no pivot). Since $\mathcal{N}\left(\mathbf{A}^{\intercal}\right) = \mathcal{N}\left(\mathbf{R}^{\intercal}\right)$, vectors $\begin{pmatrix}1, & 2, & -3, & 1, & 0,\end{pmatrix}$ and $\begin{pmatrix}4, & -5, & -6, & 0, & 1,\end{pmatrix}$ form a basis for $\mathcal{N}\left(\mathbf{A}^{\intercal}\right)$

Hence, $\mathcal{N}\left(\mathbf{A}^{\intercal}\right)$ is the plane formed by all the linear combinations of those two vectors

$$\mathcal{N}\left(\mathbf{A}^{\intercal}\right) = \left\{ \boldsymbol{v} \in \mathbb{R}^5 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 \text{ tal que } \boldsymbol{v} = \begin{bmatrix} 1 & 4 \\ 2 & -5 \\ -3 & -6 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}$$

□

(Grupo F curso 17/18) Exercise 2(e) No. There is not enough information.

□

(Grupo F curso 17/18) Exercise 3(a) **Possible.**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

□

(Grupo F curso 17/18) Exercise 3(b) **Impossible.** 3 is the maximun possible rank. Hence, columns are linearly dependent.

□

(Grupo F curso 17/18) Exercise 3(c) **Impossible.** If $\boldsymbol{b}=\boldsymbol{0}$, then $\mathbf{A}\boldsymbol{x} = \boldsymbol{0}$ is always solvable: $\boldsymbol{x}=\boldsymbol{0}$.

□

(Grupo F curso 17/18) Exercise 3(d) **Possible.**

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

□

(Grupo F curso 17/18) Exercise 4(a)

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} \boldsymbol{v}_1; & \boldsymbol{v}_2; & \boldsymbol{v}_3; \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 2 \\ 3 & 5 & 4 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 12 \end{pmatrix}.$$

□

(Grupo F curso 17/18) Exercise 4(b)

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -2 \\ -2 & 0 & 2 & 2 \\ 3 & 5 & 4 & -12 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(2)\mathbf{1}+\mathbf{4}]}]{\substack{\tau \\ [(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -2 & 2 & 2 & -2 \\ 3 & 2 & 4 & -6 \\ \hline 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(2)\mathbf{3}+\mathbf{4}]}]{\substack{\tau \\ [(-1)\mathbf{2}+\mathbf{3}] \\ [(1)\mathbf{2}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 3 & 2 & 2 & 0 \\ \hline 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Hence, the solution is $(x_1, x_2, x_3,) = (3, -1, 2,)$ since

$$3\boldsymbol{v}_1 - \boldsymbol{v}_2 + 2\boldsymbol{v}_3 = \boldsymbol{b}.$$

□

(Grupo F curso 17/18) Exercise 4(c) We need a new $\boldsymbol{v'}_3$ such that it is a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Then, the new matrix is singular, and therefore, the sistem is not solvable for some $\boldsymbol{b}$. For example when $\boldsymbol{v'}_3 = \boldsymbol{v}_2 - \boldsymbol{v}_1$:

$$\boldsymbol{v'}_3 = \boldsymbol{v}_2 - \boldsymbol{v}_1 = \begin{bmatrix} 1 & 1 \\ -2 & 0 \\ 3 & 5 \end{bmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}.$$

and therefore, the column space of the new matrix $\mathbf{B} = \left[\boldsymbol{v}_1; \boldsymbol{v}_2; \boldsymbol{v'}_3;\right] = \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 2 \\ 3 & 5 & 2 \end{bmatrix}$ has dimensión 2 .

The new system $\mathbf{B}\boldsymbol{x} = \boldsymbol{b'}$ is solvable when $\boldsymbol{b'} \in \mathcal{C}\left(\mathbf{B}\right)$. So, we can choose any column of $\mathbf{B}$ or any linear combination od the columns of $\mathbf{B}$ (for example $\mathbf{0}$).

□

(Grupo F curso 17/18) Exercise 5(a) We are going to operate by blocks:

$$\left[\begin{array}{cc|c} \mathbf{A} & \mathbf{B} & -\boldsymbol{b}_1 \\ \mathbf{0} & \mathbf{A} & -\boldsymbol{b}_2 \\ \hline \mathbf{I} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{I} & 0 \end{array}\right] \xrightarrow[{[(\mathbf{A}^{-1})\mathbf{2}]}]{\overset{\tau}{[(\mathbf{A}^{-1})\mathbf{1}]}} \left[\begin{array}{cc|c} \mathbf{I} & \mathbf{B}\mathbf{A}^{-1} & -\boldsymbol{b}_1 \\ \mathbf{0} & \mathbf{I} & -\boldsymbol{b}_2 \\ \hline \mathbf{A}^{-1} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{A}^{-1} & 0 \end{array}\right] \xrightarrow{\overset{\tau}{[(-\mathbf{B}\mathbf{A}^{-1})\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc|c} \mathbf{I} & \mathbf{0} & -\boldsymbol{b}_1 \\ \mathbf{0} & \mathbf{I} & -\boldsymbol{b}_2 \\ \hline \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1} & 0 \\ \mathbf{0} & \mathbf{A}^{-1} & 0 \end{array}\right]$$

$$\xrightarrow{\overset{\tau}{[(\boldsymbol{b}_1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cc|c} \mathbf{I} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{I} & -\boldsymbol{b}_2 \\ \hline \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1} & \mathbf{A}^{-1}\boldsymbol{b}_1 \\ \mathbf{0} & \mathbf{A}^{-1} & 0 \end{array}\right] \xrightarrow{\overset{\tau}{[(\boldsymbol{b}_2)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{cc|c} \mathbf{I} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{I} & 0 \\ \hline \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1} & \mathbf{A}^{-1}\boldsymbol{b}_1 - \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}\boldsymbol{b}_2 \\ \mathbf{0} & \mathbf{A}^{-1} & \mathbf{A}^{-1}\boldsymbol{b}_2 \end{array}\right]$$

Therefore,

$$\boldsymbol{y} = \mathbf{A}^{-1}\boldsymbol{b}_1 - \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}\boldsymbol{b}_2 \quad = \quad \mathbf{A}^{-1}\left(\boldsymbol{b}_1 - \mathbf{B}\mathbf{A}^{-1}\boldsymbol{b}_2\right)$$

and

$$\boldsymbol{z} = \mathbf{A}^{-1}\boldsymbol{b}_2.$$

□

(Grupo F curso 17/18) Exercise 5(b) Since $\mathbf{A} = \mathbf{L}\dot{\mathbf{U}}$, we known that $\mathbf{A}^{-1} = \dot{\mathbf{U}}^{-1}\mathbf{L}^{-1}$; hence

$$\boldsymbol{y} = \dot{\mathbf{U}}^{-1}\mathbf{L}^{-1}\left(\boldsymbol{b}_1 - \mathbf{B}\dot{\mathbf{U}}^{-1}\mathbf{L}^{-1}\boldsymbol{b}_2\right)$$

and

$$\boldsymbol{z} = \dot{\mathbf{U}}^{-1}\mathbf{L}^{-1}\boldsymbol{b}_2.$$

□

(Grupo B curso 16/17) Exercise 1(a)

- If the column space is a plane, the nullspace is a line.
- If the column space is a line, the nullspace is a plane.
- If the column space is all $\mathbb{R}^3$, the nullspace is a point.
- If the column space is the zero vector, the nullspace is $\mathbb{R}^3$.

□

(Grupo B curso 16/17) Exercise 1(b)

Since $\dim \mathcal{C}\left(\mathbf{A}\right) + \dim \mathcal{N}\left(\mathbf{A}\right) = 7$, if $\mathcal{C}\left(\mathbf{A}\right) = \mathcal{N}\left(\mathbf{A}\right)$, then they both have dimension 7/2, but dimension must be an integer number. So the matrix $\mathbf{A}$ can not exist.

□

(Grupo B curso 16/17) Exercise 2(a)

$$\left[\frac{\mathbf{L}}{\mathbf{I}}\right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & g \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\overset{\tau}{[(3)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 2 & 6 \\ 1 & 1 & g+3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\overset{\tau}{[(3)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 3 \\ 1 & 2 & 9 \\ 1 & 1 & g+6 \\ \hline 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\overset{\tau}{[(3)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc} 1 & 3 & 3 \\ 1 & 5 & 9 \\ 1 & 4 & g+6 \\ \hline 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right] = \left[\frac{\mathbf{A}}{\mathbf{E}^{-1}}\right],$$

where $\mathbf{A}_{\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\tau}\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\tau}\underset{[(-3)\mathbf{2}+\mathbf{3}]}{\tau}} = \mathbf{AE} = \mathbf{L}$.

$\square$

(Grupo B curso 16/17) Exercise 2(b)

Since $\mathbf{AE} = \mathbf{L}$, then $\mathbf{A} = \mathbf{LE}^{-1} = \mathbf{L}\dot{\mathbf{U}}$; so

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 5 & 9 \\ 1 & 4 & g+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & g \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{L}\dot{\mathbf{U}}.$$

$\square$

(Grupo B curso 16/17) Exercise 2(c)

Since $\mathbf{E} = \left(\mathbf{I}_{\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\tau}}\right)\left(\mathbf{I}_{\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\tau}}\right)\left(\mathbf{I}_{\underset{[(-3)\mathbf{2}+\mathbf{3}]}{\tau}}\right) = \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$; the last column of $\mathbf{E}$ is a basis of $\mathcal{N}\left(\mathbf{A}\right)$, therefore

$$\mathcal{N}\left(\mathbf{A}\right) = \left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists a \in \mathbb{R} \text{ such that } a \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} \right\}$$

$\square$

(Grupo B curso 16/17) Exercise 2(d)

If $g \neq 0$, then rank is three and therefore $\mathcal{C}\left(\mathbf{L}\right) = \mathbb{R}^3$. Since we have used column elimination $\mathcal{C}\left(\mathbf{L}\right) = \mathcal{C}\left(\mathbf{A}\right) = \mathbb{R}^3$.

$\square$

(Grupo B curso 16/17) Exercise 3(a)

In preparation for the next problems, lets first column reduce this matrix with an arbitrary vector augmented.

$$\left[\begin{array}{c|c} \mathbf{A} & -\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 2 & -a \\ -1 & -2 & 0 & 0 & -1 & -b \\ 1 & 2 & 0 & 0 & 1 & -c \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(-2)\mathbf{1}+\mathbf{4}] \\ [(-2)\mathbf{1}+\mathbf{5}] \\ [(a)\mathbf{1}+\mathbf{6}]}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 2 & 1 & -b-a \\ 1 & 0 & -1 & -2 & -1 & -c+a \\ \hline 1 & -2 & -1 & -2 & -2 & a \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{\tau \\ [(-2)\mathbf{3}+\mathbf{4}] \\ [(-1)\mathbf{3}+\mathbf{5}] \\ [(b+a)\mathbf{3}+\mathbf{6}] \\ [(1)\mathbf{3}+\mathbf{1}]}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -b-c \\ \hline 0 & -2 & -1 & 0 & -1 & -b \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 & -1 & b+a \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{R} & \boldsymbol{c} \\ \hline \mathbf{E} & \boldsymbol{x}_p \\ \hline \mathbf{0} & 1 \end{array}\right]$$

So a general solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ is just any linear combination of the nullspace basis vectors

$$\left\{ \boldsymbol{v} \in \mathbb{R}^5 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^3 \text{ tal que } \boldsymbol{v} = \begin{bmatrix} -2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}$$

$\square$

(Grupo B curso 16/17) Exercise 3(b)

To find a general solution, we set $a = 2$, $b = 1$ and $c = -1$ in our augmented matrix column reduction performed above. So we solve directly and find our particular solution $\boldsymbol{x}_p = \begin{pmatrix} -1, & 0, & 3, & 0, & 0, \end{pmatrix}$.

Thus a general solution is just this particular solution plus the general solution for a nullspace vector given in part (a) :

$$\left\{ \boldsymbol{x} \in \mathbb{R}^5 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^3 \text{ tal que } \boldsymbol{x} = \begin{pmatrix} -1 \\ 0 \\ 3 \\ 0 \\ 0 \end{pmatrix} + \begin{bmatrix} -2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}$$

□

(Grupo B curso 16/17) Exercise 3(c)

Finding all vectors $\boldsymbol{b}$ such that $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution is asking for condition on $\boldsymbol{b}$ so that it is in the column space. From our original computation, we see that we must have $b + c = 0$, so an arbitrary vector in the column space looks like $\begin{pmatrix} a \\ b \\ -b \end{pmatrix}$ where b and c are any real numbers. In otherwords, the column space is spanned by vectors of the form:

$$a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\-1\end{pmatrix}.$$

□

(Grupo B curso 16/17) Exercise 3(d)

To find a matrix $\mathbf{B}$ such that $\mathcal{N}\left(\mathbf{A}\right) = \mathcal{C}\left(\mathbf{B}\right)$, we can simply take a matrix whose column vectors are a basis for the nullspace of $\mathbf{A}$. In otherwords, the matrix:

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

(Grupo B curso 16/17) Exercise 3(e)

Observe that the three vectors in part (a) form a basis for the nullspace of $\mathbf{A}$, the two vectors in part (c) form a basis for the column space of $\mathbf{A}$. Thus all that is left is to find basis vectors for the row space, which we can take to be the two independent row vectors corresponding to the pivot rows of $\mathbf{A}$:

$$\Big[(1, \quad 2, \quad 1, \quad 2, \quad 2,)\,;\;(-1, \quad -2, \quad 0, \quad 0, \quad -1,)\,;\Big].$$

We can deduce a basis for the left null space from $\mathbf{R}$:

$$\Big[(0, \quad 1, \quad 1,)\,;\Big].$$

□

(Grupo E curso 16/17) Exercise 1(a)

In preparation for the next problems, lets first column reduce the augmented matrix with the left hand side vector described in part (c).

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & -3 \\ 3 & 6 & 3 & 9 & -9 \\ 2 & 4 & 2 & 9 & -b_3 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-2)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(-4)\mathbf{1}+\mathbf{4}] \\ [(3)\mathbf{1}+\mathbf{5}]}]{\tau} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & -3 & 0 \\ 2 & 0 & 0 & 1 & -b_3+4 \\ \hline 1 & -2 & -1 & -4 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$r = \operatorname{rg}\left(\mathbf{A}\right) = 2$, columns $\begin{pmatrix}1\\3\\2\end{pmatrix}$ and $\begin{pmatrix}4\\9\\9\end{pmatrix}$ are a basis of $\mathcal{C}\left(\mathbf{A}\right)$.

The column space $\mathcal{C}\left(\mathbf{A}\right)$ is a plane in $\mathbb{R}^3$ spaned by the two pivot columns.

□

(Grupo E curso 16/17) Exercise 1(b)

Special sol.: $\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}$. $\mathcal{N}\left(\mathbf{A}\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 \text{ tal que } \boldsymbol{x} = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{p} \right\}.$

□

(Grupo E curso 16/17) Exercise 1(c)
To have a solution we need $\boxed{b_3 = 6}$

For $b_3 = 3$, a particular solution is given by $\boldsymbol{x}_p = \begin{pmatrix}3\\0\\0\\0\end{pmatrix}$.

Complete solution: $\left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 \text{ tal que } \boldsymbol{x} = \begin{pmatrix}3\\0\\0\\0\end{pmatrix} + \begin{bmatrix}-2 & -1\\1 & 0\\0 & 1\\0 & 0\end{bmatrix} \boldsymbol{p} \right\}$.

□

(Grupo E curso 16/17) Exercise 2(a)

Yes. This equation describes the left nullspace of the matrix $\begin{bmatrix}1\\2\\3\end{bmatrix}$.

□

(Grupo E curso 16/17) Exercise 2(b)
No. Consider the vectors $(1, 1,)$ and $(1, -1,)$, both of which satisfy this equation. The sum $(1, 1,) + (1, -1,) = (2, 0,)$ does not satisfy the equation since $2^2 - 0 = 4$.

□

(Grupo E curso 16/17) Exercise 2(c) No. This set does not contain $(0, 0,)$.

□

(Grupo E curso 16/17) Exercise 2(d) Yes. It is the intersection of two subspaces. Note that this column space of this matrix is the span of the vector $\begin{pmatrix}1\\1\\1\end{pmatrix}$; and the nullspace is spanned by the vectors: $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}2\\1\\0\end{pmatrix}$. Since the sum of the two nullspace basis vectors is the vector $\begin{pmatrix}1\\1\\1\end{pmatrix}$, then $\mathcal{C}\left(\mathbf{A}\right) \subset \mathcal{N}\left(\mathbf{A}\right)$ so $\mathcal{C}\left(\mathbf{A}\right) \cap \mathcal{N}\left(\mathbf{A}\right) = \mathcal{C}\left(\mathbf{A}\right)$, and we know that $\mathcal{C}\left(\mathbf{A}\right)$ is a subspace.

□

(Grupo E curso 16/17) Exercise 2(e)
Yes. We have already seen in part (d) that the column space is a subspace of the nullspace. Thus the vectors that are in the column space or the nullspace are just the columns in the nullspace, which is a vector space.

□

(Grupo E curso 16/17) Exercise 3(a)
Since

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{ccc}1 & 1 & 0\\2 & 1 & 1\\3 & 1 & 3\\\hline 1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{array}\right] \xrightarrow{\overset{\tau}{[(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc}1 & 0 & 0\\2 & -1 & 1\\3 & -2 & 3\\\hline 1 & -1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{array}\right] \xrightarrow{\overset{\tau}{[(1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc}1 & 0 & 0\\2 & -1 & 0\\3 & -2 & 1\\\hline 1 & -1 & -1\\0 & 1 & 1\\0 & 0 & 1\end{array}\right] = \left[\frac{\mathbf{L}}{\mathbf{E}}\right]$$

then

$$\mathbf{A} = \begin{bmatrix}1 & 0 & 0\\2 & -1 & 0\\3 & -2 & 1\end{bmatrix} \begin{bmatrix}1 & 1 & 0\\0 & 1 & -1\\0 & 0 & 1\end{bmatrix} = \mathbf{L\dot{U}}$$

□

(Grupo E curso 16/17) Exercise 3(b)

Note that $\mathbf{A}^{-1} = \Big(\mathbf{LU}\Big)^{-1} = \mathbf{U}^{-1}\mathbf{L}^{-1}$. Since $\mathbf{U}^{-1} = \mathbf{E}$, we only need to compute $\mathbf{L}^{-1}$:

$$\left[\frac{\mathbf{L}}{\mathbf{I}}\right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 3 & -2 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(2)\mathbf{2}+\mathbf{1}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & -2 & 1 \\ \hline 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(1)\mathbf{3}+\mathbf{1}] \\ [(2)\mathbf{3}+\mathbf{2}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{2}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 1 \end{array}\right] = \left[\frac{\mathbf{I}}{\mathbf{L}^{-1}}\right].$$

Therefore

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3 & -1 \\ 3 & -3 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

We get the same result if we reduce $\mathbf{A}$ to $\mathbf{R}$.

$\square$

(Grupo E curso 16/17) Exercise 3(c)

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{ccc} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 1 & c \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 2 & -1 & 1 \\ 3 & -2 & c \\ \hline 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 3 & -2 & c-2 \\ \hline 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

Thus $\mathbf{A}$ is invertible when $c \neq 2$.

$\square$

(Grupo E curso 16/17) Exercise 4(a)

$r = m$. The rank of $\mathbf{A}$ is equal to the dimension of the column space $\mathcal{C}\left(\mathbf{A}\right)$. Now the column space of $\mathbf{A}$ is the subspace of $\mathbb{R}^m$ that can be written as $\mathbf{A}\boldsymbol{x}$, where $\boldsymbol{x}$ is a vector in $\mathbb{R}^n$. I claim that $\mathcal{C}\left(\mathbf{A}\right) = \mathbb{R}^m$. This follows because $\mathcal{C}\left(\mathbf{AB}\right)$ is the column space of the identity matrix, so it is all of $\mathbb{R}^m$. In particular, any vector $\boldsymbol{v}$ in $\mathbb{R}^m$ can be written as $\mathbf{AB}\boldsymbol{v} = \mathbf{I}\boldsymbol{v} = \boldsymbol{v}$. Thus any vector $\boldsymbol{v}$ in $\mathbb{R}^m$ is in the column space of $\mathbf{A}$ because it can be written as $\boldsymbol{v} = \mathbf{A}\boldsymbol{x}$ by simply setting $\boldsymbol{x} = \mathbf{B}\boldsymbol{v}$. Therefore we have seen that the dimension of the column space is m , and thus the answer is (c).

$\square$

(Grupo E curso 16/17) Exercise 4(b)

We know that the rank of $\mathbf{A}$ must be less than or equal to the smallest dimension of $\mathbf{A}$. Since $r = m$, it must be the case that $m \leq n$.

$\square$

(Grupo E curso 15/16) Exercise 1.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & -7 \\ 1 & 1 & 1 & 1 & -10 \\ 1 & 0 & 1 & 1 & -9 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{4}] \\ [(7)\mathbf{1}+\mathbf{5}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -3 \\ 1 & -1 & 1 & 0 & -2 \\ \hline 1 & -1 & 0 & -1 & 7 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(3)\mathbf{3}+\mathbf{5}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 \\ \hline 1 & -1 & 0 & -1 & 7 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(1)\mathbf{2}+\mathbf{3}] \\ [(1)\mathbf{2}+\mathbf{5}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ \hline 1 & -1 & -1 & -1 & 6 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

So, the solutions set is

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists a \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} 6 \\ 1 \\ 3 \\ 0 \end{pmatrix} + a \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\square$

(Grupo E curso 15/16) Exercise 2.

Let the three vectors be the columns of matrix $\mathbf{A}$, then

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 2 & 2 \\ 3 & -2 & 8 \\ -1 & 4 & 2 \end{bmatrix} \xrightarrow[\substack{[(1)\mathbf{1}+\mathbf{2}] \\ [(-3)\mathbf{1}+\mathbf{3}]}]{\boldsymbol{\tau}} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & -4 \\ 3 & 1 & -1 \\ -1 & 3 & 5 \end{bmatrix} \xrightarrow[{[(1)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 1 & 0 \\ -1 & 3 & 8 \end{bmatrix} = \mathbf{L}.$$

Since there are three pivots, the only linear combination of the columns of $\mathbf{A}$ that is zero is $0\boldsymbol{u}+0\boldsymbol{v}+0\boldsymbol{w}$, hence, these vectors are linearly independent.

□

(Grupo E curso 15/16) Exercise 3(a)

$$\left[\begin{pmatrix}1\\2\\1\\-3\end{pmatrix};\begin{pmatrix}0\\-5\\0\\2\end{pmatrix};\begin{pmatrix}0\\0\\25\\-41\end{pmatrix};\right]$$

form a basis for $\mathcal{C}\left(\mathbf{A}\right)$ (also columns 1, 3, and 4 of $\mathbf{A}$, since the only permutation is $\underset{[\mathbf{3}\rightleftharpoons\mathbf{4}]}{\boldsymbol{\tau}}$). So $\dim\mathcal{C}\left(\mathbf{A}\right)=3$

□

(Grupo E curso 15/16) Exercise 3(b)

$$\left[\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix};\begin{pmatrix}-4\\0\\-3\\-2\\1\end{pmatrix};\right]$$

form a basis for $\mathcal{N}\left(\mathbf{A}\right)$. So $\dim\mathcal{N}\left(\mathbf{A}\right)=2$.

□

(Grupo E curso 15/16) Exercise 3(c)

Since the vector $\boldsymbol{v}$ is one solution, every solution takes the form $\boldsymbol{v}+\boldsymbol{x}_n$, where $\boldsymbol{x}_n$ is in $\mathcal{N}\left(\mathbf{A}\right)$. Thus the solutions are

$$\left\{\boldsymbol{x}\in\mathbb{R}^5 \;\middle|\; \exists\boldsymbol{p}\in\mathbb{R}^2 \text{ tal que } \boldsymbol{x}=\begin{pmatrix}2\\-1\\-4\\0\\1\end{pmatrix}+\begin{bmatrix}-2&-4\\1&0\\0&-3\\0&-2\\0&1\end{bmatrix}\boldsymbol{p}\right\}.$$

□

(Grupo E curso 15/16) Exercise 4.

$$\left[\begin{array}{ccc}2&4&0\\1&1&3\\-1&-3&2\\\hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow[{[(-2)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}}\left[\begin{array}{ccc}2&0&0\\1&-1&3\\-1&-1&2\\\hline 1&-2&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow[{[(3)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}}\left[\begin{array}{ccc}2&0&0\\1&-1&0\\-1&-1&-1\\\hline 1&-2&-6\\0&1&3\\0&0&1\end{array}\right]$$

So,

$$\mathbf{L}=\begin{bmatrix}2&0&0\\1&-1&0\\-1&-1&-1\end{bmatrix};\qquad \dot{\mathbf{U}}=\begin{bmatrix}1&2&0\\0&1&-3\\0&0&1\end{bmatrix}.$$

□

(Grupo E curso 15/16) Exercise 5(a) The simplest way to provide an example is to add dependent columns to matrix $\mathbf{A}$. For example, matrices $\begin{bmatrix}1\end{bmatrix}$ and $\begin{bmatrix}1&0\end{bmatrix}$ have the same column space and different null spaces.

□

(Grupo E curso 15/16) Exercise 5(b) The rank is the dimension of the column space. That means the rank is the same for both matrices.

□

(Grupo E curso 15/16) Exercise 6(a)

$\mathbf{A}\boldsymbol{v}=\left(\mathbf{A}_{|1}\right)+2\left(\mathbf{A}_{|2}\right)+3\left(\mathbf{A}_{|3}\right)+4\left(\mathbf{A}_{|4}\right)$.

□

(Grupo E curso 15/16) Exercise 6(b) $\mathbf{A}\boldsymbol{v} = \begin{bmatrix} ({}_{1|}\mathbf{A}) \cdot \boldsymbol{v} \\ ({}_{2|}\mathbf{A}) \cdot \boldsymbol{v} \\ ({}_{3|}\mathbf{A}) \cdot \boldsymbol{v} \\ ({}_{4|}\mathbf{A}) \cdot \boldsymbol{v} \end{bmatrix}$

□

(Grupo E curso 15/16) Exercise 6(c)

The answer is "no". Any four vectors in $\mathbb{R}^3$ are linearly dependent because if you put those vectors in the columns of a 3 by 4 matrix, like $\mathbf{A}$ here, then reduced echelon form $\mathbf{R}$ has at most three pivots, therefore there are free variables and hence non-zero vectors in $\mathcal{N}\left(\mathbf{A}\right)$. But any non-zero vector in $\mathcal{N}\left(\mathbf{A}\right)$ gives rise to a linear dependence relation between the columns of $\mathbf{A}$, using the formula for $\mathbf{A}\boldsymbol{v}$ as a linear combination of columns of $\mathbf{A}$.

□

(Grupo E curso 15/16) Exercise 6(d)

Infinitely many. The explanation just above in (c) explains why the null space of $\mathbf{A}$ contains at least a line in $\mathbb{R}^4$. There is at least one free variable, maybe more.

□

(Grupo E curso 15/16) Exercise 6(e)

For a k by n matrix, the number of pivot columns plus the number of non-pivot columns is the total number of columns, namely n. The number of non-pivot columns (free variables) is the dimension of the null space of $\mathbf{A}$, called p here. The number of pivot columns is the dimension of the column space of $\mathbf{A}$, called q here. So $p + q = n$. In our case, $n = 4$. However, we also know $p = 1$, 2, 3 or 4, since there must be free variables. That is, p cannot be 0.

The answer is therefore $(p, q) = (1, 3)$ or $(2, 2)$ or $(3, 1)$ or $(4, 0)$. (The last case happens only when every entry of the 3 by 4 matrix $\mathbf{A}$ is 0.)

□

(Grupo E curso 15/16) Exercise 7.

Suppose

$$a\boldsymbol{u} + b(\boldsymbol{u} + \boldsymbol{v}) + c(\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}) = \mathbf{0}$$

Then

$$(a + b + c)\boldsymbol{u} + (b + c)\boldsymbol{v} + c\boldsymbol{w} = \mathbf{0}$$

Since $\left[\boldsymbol{u};\boldsymbol{v};\boldsymbol{w};\right]$ is linearly independent, this implies $\begin{cases} a+b+c=0 \\ \quad\; b+c=0 \\ \qquad\;\; c=0 \end{cases}$. This system has only one solutions $c = 0$, $b = 0$ and $a = 0$. Hence $\left[\boldsymbol{u};(\boldsymbol{u}+\boldsymbol{v});(\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w});\right]$ is linearly independent.

Another proof:

Since $\left[\boldsymbol{u};\ \boldsymbol{v};\ \boldsymbol{w};\right]$ is a linearly independent system of vectors in $\mathbb{R}^n$, matrix

$$\mathbf{A} = \left[\boldsymbol{u}; \quad \boldsymbol{v}; \quad \boldsymbol{w};\right]$$

is full column rank (3 pivots). Hence, matrix $\mathbf{M} = \left[\boldsymbol{u};\ (\boldsymbol{u}+\boldsymbol{v});\ (\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w});\right]$

$$\left[\frac{\mathbf{M}}{\mathbf{I}}\right] = \left[\begin{array}{ccc} | & | & | \\ \boldsymbol{u} & (\boldsymbol{u}+\boldsymbol{v}) & (\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}) \\ | & | & | \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{2}]}]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} | & | & | \\ \boldsymbol{u} & \boldsymbol{v} & \boldsymbol{w} \\ | & | & | \\ \hline 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right] = \left[\frac{\mathbf{A}}{\mathbf{E}}\right]$$

is also full column rank.

□

(Grupo H curso 15/16) Exercise 1.

$$\left[\begin{array}{ccc|c}1&2&4&-b_1\\0&1&3&-b_2\\1&3&7&-b_3\\2&2&2&-b_4\end{array}\right]\xrightarrow[{[(-4)\mathbf{1}+\mathbf{3}]}]{\overset{\boldsymbol{\tau}}{[(-2)\mathbf{1}+\mathbf{2}]}}\left[\begin{array}{ccc|c}1&0&0&-b_1\\0&1&3&-b_2\\1&1&3&-b_3\\2&-2&-6&-b_4\end{array}\right]\xrightarrow{\overset{\boldsymbol{\tau}}{[(-3)\mathbf{2}+\mathbf{3}]}}\left[\begin{array}{ccc|c}1&0&0&-b_1\\0&1&0&-b_2\\1&1&0&-b_3\\2&-2&0&-b_4\end{array}\right]$$

$$\xrightarrow{\overset{\boldsymbol{\tau}}{[(b_1)\mathbf{1}+\mathbf{4}]}}\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&-b_2\\1&1&0&b_1-b_3\\2&-2&0&2b_1-b_4\end{array}\right]\xrightarrow{\overset{\boldsymbol{\tau}}{[(b_2)\mathbf{2}+\mathbf{4}]}}\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\1&1&0&b_1+b_2-b_3\\2&-2&0&2b_1-2b_2-b_4\end{array}\right]$$

So,

$$\boxed{\begin{cases}b_1+\ \ b_2-b_3 &=0\\2b_1-2b_2\qquad -b_4&=0\end{cases}}.$$

Altenative answer: We need $\boldsymbol{b}\in\mathcal{C}\left(\mathbf{A}\right)$, so:

$$\begin{pmatrix}b_1\\b_2\\b_3\\b_4\end{pmatrix}=a\begin{pmatrix}1\\0\\1\\2\end{pmatrix}+c\begin{pmatrix}0\\1\\1\\-2\end{pmatrix};\qquad\text{assigning } a=b_1 \text{ and } c=b_2,\qquad\begin{pmatrix}b_1\\b_2\\b_3\\b_4\end{pmatrix}=b1\begin{pmatrix}1\\0\\1\\2\end{pmatrix}+b2\begin{pmatrix}0\\1\\1\\-2\end{pmatrix}$$

we get

$$\begin{cases}b_3=b_1+b_2\\b_4=2b_1-2b_2\end{cases}\quad\rightarrow\boxed{\begin{cases}b_1+\ \ b_2-b_3 &=0\\2b_1-2b_2\qquad -b_4&=0\end{cases}}.$$

□

(Grupo H curso 15/16) Exercise 2(a)

$$\left[\begin{array}{ccc}1&3&-2\\0&2&4\\0&0&-1\\\hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow[{[(2)\mathbf{1}+\mathbf{3}]}]{\overset{\tau}{[(-3)\mathbf{1}+\mathbf{2}]}}\left[\begin{array}{ccc}1&0&0\\0&2&4\\0&0&-1\\\hline 1&-3&2\\0&1&0\\0&0&1\end{array}\right]\xrightarrow{\overset{\tau}{[(-2)\mathbf{2}+\mathbf{3}]}}\left[\begin{array}{ccc}1&0&0\\0&2&0\\0&0&-1\\\hline 1&-3&8\\0&1&-2\\0&0&1\end{array}\right]\xrightarrow[{[(-1)\mathbf{3}]}]{\overset{\tau}{[(\frac{1}{2})\mathbf{2}]}}\left[\begin{array}{ccc}1&0&0\\0&1&0\\0&0&1\\\hline 1&-3/2&-8\\0&1/2&2\\0&0&-1\end{array}\right]$$

□

(Grupo H curso 15/16) Exercise 2(b)

$$\left[\begin{array}{ccc}1&1&1\\0&1&2\\5&x&6\\\hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow[{[(-1)\mathbf{1}+\mathbf{3}]}]{\overset{\boldsymbol{\tau}}{[(-1)\mathbf{1}+\mathbf{2}]}}\left[\begin{array}{ccc}1&0&0\\0&1&2\\5&x-5&1\\\hline 1&-1&-1\\0&1&0\\0&0&1\end{array}\right]\xrightarrow{\overset{\boldsymbol{\tau}}{[(-2)\mathbf{2}+\mathbf{3}]}}\left[\begin{array}{ccc}1&0&0\\0&1&0\\5&x-5&11-2x\\\hline 1&-1&1\\0&1&-2\\0&0&1\end{array}\right]$$

this matrix is not invertible if and only if $11-2x=0$, that is, $x=\frac{11}{2}$. Hence, this matrix is invertible for any $x\neq\frac{11}{2}$.

□

(Grupo H curso 15/16) Exercise 3(a)

$$\left[\begin{pmatrix}1\\0\\-1\\4\\0\end{pmatrix};\begin{pmatrix}0\\1\\2\\3\\0\end{pmatrix};\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix};\right]$$

$\dim\mathcal{C}\left(\mathbf{A}\right))=3$

□

(Grupo H curso 15/16) Exercise 3(b)

$$\left[\begin{pmatrix}-1\\1\\2\\1\\0\end{pmatrix};\;\begin{pmatrix}4\\-1\\-2\\0\\1\end{pmatrix};\right]$$

$\dim \mathcal{N}\left(\mathbf{A}\right) = 2.$

□

(Grupo H curso 15/16) Exercise 3(c)

$\dim \mathcal{C}\left(\mathbf{A}^{\intercal}\right) = 3.$

Cuando escribí el examen no caí en la cuenta de que era posible obtener $\mathbf{A}$. A cada alumno que ha calculado correctamente $\mathbf{A}$ le he otorgado un punto y medio adicional a su calificación:

$$\left[\frac{\mathbf{E}}{\mathbf{I}}\right] = \left[\begin{array}{ccccc} 1 & -1 & 2 & -1 & 4\\ 0 & 1 & 0 & 1 & -1\\ 0 & 0 & 1 & 2 & -2\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ \hline 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-4)\mathbf{1}+\mathbf{5}]}]{\substack{\tau\\ {[(1)\mathbf{1}+\mathbf{2}]}\\ {[(-2)\mathbf{1}+\mathbf{3}]}\\ {[(1)\mathbf{1}+\mathbf{4}]}}} \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 1 & -1\\ 0 & 0 & 1 & 2 & -2\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ \hline 1 & 1 & -2 & 1 & -4\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[{[(1)\mathbf{2}+\mathbf{5}]}]{\substack{\tau\\ {[(-1)\mathbf{2}+\mathbf{4}]}}} \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 2 & -2\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ \hline 1 & 1 & -2 & 0 & -3\\ 0 & 1 & 0 & -1 & 1\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(2)\mathbf{3}+\mathbf{5}]}]{\substack{\tau\\ {[(-2)\mathbf{3}+\mathbf{4}]}}} \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ \hline 1 & 1 & -2 & 4 & -7\\ 0 & 1 & 0 & -1 & 1\\ 0 & 0 & 1 & -2 & 2\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\frac{\mathbf{I}}{\mathbf{E}^{-1}}\right].$$

Así pues, $\mathbf{A} = \mathbf{R}\cdot\mathbf{E}^{-1}$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ -1 & 2 & 0 & 0 & 0\\ 4 & 3 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 & 4 & -7\\ 0 & 1 & 0 & -1 & 1\\ 0 & 0 & 1 & -2 & 2\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 & 4 & -7\\ 0 & 1 & 0 & -1 & 1\\ -1 & 1 & 2 & -6 & 9\\ 4 & 7 & -8 & 13 & -25\\ 0 & 0 & 1 & -2 & 2 \end{bmatrix}$$

$$\text{Basis of } \mathcal{C}\left(\mathbf{A}^{\intercal}\right) = \Big[\big(1, \;\; 1, \;\; -2, \;\; 4, \;\; -7,\big);\; \big(0, \;\; 1, \;\; 0, \;\; -1, \;\; 1,\big);\; \big(0, \;\; 0, \;\; 1, \;\; -2, \;\; 2,\big);\Big].$$

□

(Grupo H curso 15/16) Exercise 3(d)

$$\Big[\big(1, \;\; -2, \;\; 1, \;\; 0, \;\; 0,\big);\; \big(-4, \;\; -3, \;\; 0, \;\; 1, \;\; 0,\big);\Big].$$

$\dim \mathcal{N}\left(\mathbf{A}^{\intercal}\right) = 2.$

□

(Grupo H curso 15/16) Exercise 3(e)

$$\begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix}.$$

□

(Grupo H curso 15/16) Exercise 3(f)

Not enough information … salvo que haya calculado $\mathbf{A}$. En tal caso, sabemos que la segunda y última fila son independientes (son filas pivote). Su suma es $\big(0, \;\; 1, \;\; 1, \;\; -3, \;\; 3,\big)$. Basta cambiar el último número para obtener un vector fuera del espacio fila. Por ejemplo $\big(0, \;\; 1, \;\; 1, \;\; -3, \;\; 0,\big)$.

□

(Grupo H curso 15/16) Exercise 3(g)
No. There are free columns.

□

(Grupo H curso 15/16) Exercise 4(a)
Such matrix does not exist. The dimensions of such a matrix must be 3 by 4 ($m = 3$ and $n = 4$). The dimension of the column space is 2, because the given vectors are independent. That means the dimension of the nullspace must be $4 - 2 = 2$. The null space cannot be spanned by 1 vector.

□

(Grupo H curso 15/16) Exercise 4(b)
Consider matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

They contain $(1, 1, 1,)$ in their column space, but there sum does not:

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}.$$

□

(Grupo H curso 15/16) Exercise 4(c)
$\mathbf{M}\boldsymbol{x}$ is a non zero vector in $\mathbb{R}^5$.

□

(Grupo H curso 15/16) Exercise 5.

	$\mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$	$\mathcal{N}(\mathbf{A}) \neq \{\mathbf{0}\}$
$\boldsymbol{b} \in \mathcal{C}(\mathbf{A})$	1	∞
$\boldsymbol{b} \notin \mathcal{C}(\mathbf{A})$	0	0

□

(Grupo H curso 15/16) Exercise 6(a)
False. That would be true if you knew the matrix $\mathbf{A}$ was square.

□

(Grupo A curso 14/15) Exercise 1(a)
True. The columns belong to the four dimensonal space $\mathbb{R}^4$ (only 4 rows), so the maximun number of linearly independent vectors in $\mathbb{R}^4$ is 4.
Another reasoning is that the maximun number of pivots after gaussian elimination is 4 (one pivot in each row), so there must be at least one free column (so the columns are linearly dependent).

□

(Grupo A curso 14/15) Exercise 1(b)
False. Example:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

□

(Grupo A curso 14/15) Exercise 1(c)
False.

$$\mathbf{EA} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{B}.$$

□

(Grupo A curso 14/15) Exercise 1(d)
True. If $5\mathbf{A}^4 - \mathbf{A}^2 + 2\mathbf{A} + 6\mathbf{I} = \mathbf{0}$, then $\mathbf{I} = (-1/6)[5\mathbf{A}^3 - \mathbf{A}^2 + 2\mathbf{A}] = (-1/6)[5\mathbf{A}^3 - \mathbf{A} + 2\mathbf{I}]\mathbf{A}$, so we can set $\mathbf{B} = (-1/6)[5\mathbf{A}^3 - \mathbf{A} + 2\mathbf{I}]$ and have $\mathbf{BA} = \mathbf{I}$ so $\mathbf{B} = \mathbf{A}^{-1}$.

□

(Grupo A curso 14/15) Exercise 2(a)

The first one is an elementary matrix, its inverse is

$$\begin{bmatrix}1&0&0\\0&1&-c\\0&0&1\end{bmatrix}.$$

The second one is a permutation matrix, its inverse is its transpose.

□

(Grupo A curso 14/15) Exercise 2(b)

$$\left[\begin{array}{c}\mathsf{A}\\\hline\mathsf{I}\end{array}\right]=\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\0&0&1&0\\a&b&c&d\\\hline1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[{[(1/d)\mathbf{4}]}]{\boldsymbol{\tau}}\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\0&0&1&0\\a&b&c&1\\\hline1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&\frac{1}{d}\end{array}\right]\xrightarrow[{\substack{[(-c)\mathbf{4}+\mathbf{3}]\\[(-b)\mathbf{4}+\mathbf{2}]\\[(-a)\mathbf{4}+\mathbf{1}]}}]{\boldsymbol{\tau}}\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\\hline1&0&0&0\\0&1&0&0\\0&0&1&0\\-\frac{a}{d}&-\frac{b}{d}&-\frac{c}{d}&\frac{1}{d}\end{array}\right]=\left[\begin{array}{c}\mathsf{I}\\\hline\mathsf{A}^{-1}\end{array}\right]$$

Then, the inverse of the matrix is

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-\frac{a}{d}&-\frac{b}{d}&-\frac{c}{d}&\frac{1}{d}\end{bmatrix}$$

□

(Grupo A curso 14/15) Exercise 3(b)

$$\left[\begin{array}{cccc|c}1&-1&2&0&-1\\2&-2&4&0&-2\\3&-3&7&0&-4\\\hline1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\\hline0&0&0&0&1\end{array}\right]\xrightarrow[{\substack{[(1)\mathbf{1}+\mathbf{2}]\\[(-2)\mathbf{1}+\mathbf{3}]\\[(1)\mathbf{1}+\mathbf{5}]}}]{\tau}\left[\begin{array}{cccc|c}1&0&0&0&0\\2&0&0&0&0\\3&0&1&0&-1\\\hline1&1&-2&0&1\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\\hline0&0&0&0&1\end{array}\right]\xrightarrow[{[(1)\mathbf{3}+\mathbf{5}]}]{\tau}\left[\begin{array}{cccc|c}1&0&0&0&0\\2&0&0&0&0\\3&0&1&0&0\\\hline1&1&-2&0&-1\\0&1&0&0&0\\0&0&1&0&1\\0&0&0&1&0\\\hline0&0&0&0&1\end{array}\right]=\left[\begin{array}{c|c}\mathsf{K}&\mathbf{0}\\\hline\mathsf{E}&\boldsymbol{x}_p\end{array}\right]$$

□

(Grupo A curso 14/15) Exercise 3(a)

$$\text{A basis}=\left[\begin{pmatrix}1\\2\\3\end{pmatrix};\begin{pmatrix}0\\0\\1\end{pmatrix};\right].\quad\mathcal{C}\left(\mathsf{A}\right)=\left\{\boldsymbol{v}\in\mathbb{R}^3\;\middle|\;\exists\boldsymbol{p}\in\mathbb{R}^2\text{ such that }\boldsymbol{v}=\begin{bmatrix}1&0\\2&0\\3&1\end{bmatrix}\boldsymbol{p}\right\}.$$

$$\text{Another basis}=\left[\begin{pmatrix}1\\2\\3\end{pmatrix};\begin{pmatrix}2\\4\\7\end{pmatrix};\right].$$

□

(Grupo A curso 14/15) Exercise 3(b)

$$\text{Basis}=\left[\begin{pmatrix}1\\1\\0\\0\end{pmatrix};\begin{pmatrix}0\\0\\0\\1\end{pmatrix};\right].\qquad\mathcal{N}\left(\mathsf{A}\right)=\left\{\vec{x}\in\mathbb{R}^4\;\middle|\;\exists a,b\in\mathbb{R}\text{ such that }\boldsymbol{x}=a\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+b\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$$

The special solutions came from columns 2 and 4 of E.

□

(Grupo A curso 14/15) Exercise 3(c)

The pivot rows of $\mathbf{A}$:

$$\text{Basis} = \left[\begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}; \begin{pmatrix} 3 \\ -3 \\ 7 \\ 0 \end{pmatrix}; \right].$$

$\square$

(Grupo A curso 14/15) Exercise 3(d)
Set of solutions to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 \text{ tal que } \boldsymbol{x} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}$$

where we have used the particular solution $\boldsymbol{x}_p$.

$\square$

(Grupo A curso 14/15) Exercise 4(a)
It is a subspace, since it is close under addition

$$\begin{pmatrix} a, & b, & a, \end{pmatrix} + \begin{pmatrix} c, & d, & c, \end{pmatrix} = \begin{pmatrix} a+c, & b+d, & a+c, \end{pmatrix}$$

and it is close under scalar multiplication

$$\lambda \begin{pmatrix} b, & c, & d, \end{pmatrix} = \begin{pmatrix} \lambda b, & \lambda c, & \lambda b, \end{pmatrix}.$$

The set S_1 (a line through the origin in $\mathbb{R}^3$) is the set of solutions to

$$\mathbf{A}\boldsymbol{x} = \mathbf{0}; \qquad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0};$$

so, $S_1 = \mathcal{N}\left(\mathbf{A}\right)$, with $\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.

$\square$

(Grupo A curso 14/15) Exercise 4(b)
It is not a subspace. For example $\boldsymbol{x} = \begin{pmatrix} 2, & 0, & 0, \end{pmatrix}$ belongs to S_2, but $2\boldsymbol{x} = \boldsymbol{x} + \boldsymbol{x}$ does not. Hence, S_2 is neither close under scalar multiplication nor close under addition.

$\square$

(Grupo E curso 14/15) Exercise 1(a)
True. Since $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ are linearly dependent, there is a $\boldsymbol{x} \neq \mathbf{0}$ such that

$$x_1(\boldsymbol{v}_1) + x_2(\boldsymbol{v}_2) + x_3(\boldsymbol{v}_3) = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix} \boldsymbol{x} = \mathbf{V}\boldsymbol{x} = \mathbf{0},$$

where $\mathbf{V} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix}$. Hence, for that $\boldsymbol{x} \neq \mathbf{0}$ we have

$$\begin{bmatrix} \mathbf{A}\boldsymbol{v}_1 & \mathbf{A}\boldsymbol{v}_2 & \mathbf{A}\boldsymbol{v}_3 \end{bmatrix} \boldsymbol{x} = \mathbf{A}\mathbf{V}\boldsymbol{x} = \mathbf{A}\mathbf{0} = \mathbf{0}.$$

$\square$

(Grupo E curso 14/15) Exercise 1(b)
False. Take for example $\mathbf{A} = \underset{3\times 0}{\mathbf{4}}$ (the 4 by 3 zero matrix).

$\square$

(Grupo E curso 14/15) Exercise 1(c)
True. Since $\mathcal{N}\left(\mathbf{A}\right)$ is only 2 dimensional, the rank of $\mathbf{A}$ is 3. Therefore $\mathbf{A}$ is full row rank ($\mathcal{C}\left(\mathbf{A}\right) = \mathbb{R}^3$) and the system is solvable for any $\boldsymbol{b} \in \mathbb{R}^3$. Hence, it has infinitely many solutions since the columns of $\mathbf{A}$ are linearly dependent ($\dim \mathcal{N}\left(\mathbf{A}\right) = 2$).

$\square$

(Grupo E curso 14/15) Exercise 2.

The simplest example is

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

□

(Grupo E curso 14/15) Exercise 3(a)

$$\text{Basis} = \left[\begin{pmatrix}1\\4\\0\\2\end{pmatrix};\begin{pmatrix}0\\0\\1\\2\end{pmatrix};\right]. \qquad \mathcal{C}\left(\mathbf{A}\right) = \left\{\boldsymbol{x}\in\mathbb{R}^4 \,\middle|\, \exists\, a,b\in\mathbb{R} \text{ such that } \boldsymbol{x} = a\begin{pmatrix}1\\4\\0\\2\end{pmatrix} + b\begin{pmatrix}0\\0\\1\\2\end{pmatrix}\right\}.$$

□

(Grupo E curso 14/15) Exercise 3(b)

$$\text{Basis} = \left[\begin{pmatrix}3\\-2\\1\end{pmatrix};\right]. \qquad \mathcal{N}\left(\mathbf{A}\right) = \left\{\boldsymbol{x}\in\mathbb{R}^3 \,\middle|\, \exists a\in\mathbb{R} \text{ such that } \boldsymbol{x} = a\begin{pmatrix}3\\-2\\1\end{pmatrix}\right\}.$$

The basis cames from the last column of $\mathbf{E}$.

□

(Grupo E curso 14/15) Exercise 3(c)

The particular solution of 1's gives $\mathbf{A}\boldsymbol{x}$ = sum of the columns of $\mathbf{A}$.
Hence, the set of solutions is

$$\left\{\boldsymbol{x}\in\mathbb{R}^3 \,\middle|\, \exists\, a\in\mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix}1\\1\\1\end{pmatrix} + a\begin{pmatrix}3\\-2\\1\end{pmatrix}\right\}.$$

□

(Grupo E curso 14/15) Exercise 3(d)

The structure of $\mathbf{R}$ is

$$\begin{bmatrix}1&0&0\\4&0&0\\0&1&0\\2&2&0\end{bmatrix}, \quad \text{where} \quad \underset{2\times2}{\mathbf{I}} = \begin{bmatrix}1&0\\0&1\end{bmatrix}, \quad \text{and} \quad \underset{2\times2}{\mathbf{M}} = \begin{bmatrix}4&0\\2&2\end{bmatrix}.$$

Hence, writting the columns of $\mathbf{M}$ in the positions corresponding to the pivot row indexes, and the columns the identity matrix in the positions corresponding to the free row indexes, we get:

$$\begin{bmatrix}-4&1&0&0\\-2&0&-2&1\end{bmatrix}\begin{bmatrix}1&0&0\\4&0&0\\0&1&0\\2&2&0\end{bmatrix} = \begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}.$$

Therefore, a basis of $\mathcal{N}\left(\mathbf{A}^\intercal\right)$ consist of the two rows of the left matrix:

$$\text{Basis} = \left[\begin{pmatrix}-4\\1\\0\\0\end{pmatrix};\begin{pmatrix}-2\\0\\-2\\1\end{pmatrix};\right].$$

□

(Grupo E curso 14/15) Exercise 3(e)

$$\left[\begin{array}{ccc} 1 & -1 & 3 \\ 1 & 1 & -2 \\ 1 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(5)\mathbf{2}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{1}+\mathbf{2}] \\ [(-3)\mathbf{1}+\mathbf{3}] \\ [(2)\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \\ \hline 1 & 1 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{array}\right] \xrightarrow[{[(-1)\mathbf{3}+\mathbf{2}]}]{\substack{\boldsymbol{\tau} \\ [(2)\mathbf{1}] \\ [(-1)\mathbf{2}+\mathbf{1}] \\ [(-1)\mathbf{3}+\mathbf{1}]}} \left[\begin{array}{ccc} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ \hline 2 & 2 & -1 \\ -6 & -4 & 5 \\ -2 & -2 & 2 \end{array}\right] \xrightarrow[{[(\frac{1}{2})\mathbf{2}]}]{\substack{\boldsymbol{\tau} \\ [(\frac{1}{2})\mathbf{1}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 1 & -1 \\ -3 & -2 & 5 \\ -1 & -1 & 2 \end{array}\right] = \left[\begin{array}{c} \mathbf{I} \\ \hline \mathbf{E}^{-1} \end{array}\right]$$

So

$$\mathbf{E}^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -3 & -2 & 5 \\ -1 & -1 & 2 \end{bmatrix}$$

□

(Grupo E curso 14/15) Exercise 3(f)
Since $\mathbf{AE} = \mathbf{R}$, then $\mathbf{A} = \mathbf{RE}^{-1}$:

$$\mathbf{A} = \mathbf{RE}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ -3 & -2 & 5 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 4 & 4 & -4 \\ -3 & -2 & 5 \\ -4 & -2 & 8 \end{bmatrix}$$

□

(Grupo E curso 14/15) Exercise 3(g)
The pivot rows (rows 1 and 3) of $\mathbf{A}$ are a basis of $\mathcal{C}\left(\mathbf{A}^{\intercal}\right)$.

$$\text{Basis} = \left[\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}; \begin{pmatrix} -3 \\ -2 \\ 5 \end{pmatrix};\right]$$

□

(Grupo E curso 14/15) Exercise 3(h)

$$\left[\begin{array}{c} \mathbf{A} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{ccc} 1 & 1 & -1 \\ 4 & 4 & -4 \\ -3 & -2 & 5 \\ -4 & -2 & 8 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 4 & 0 & 0 \\ -3 & 1 & 2 \\ -4 & 2 & 4 \\ \hline 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 4 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 2 & 0 \\ \hline 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c} \mathbf{L} \\ \hline \dot{\mathbf{E}} \end{array}\right]$$

Hence, since $\dot{\mathbf{U}} = \dot{\mathbf{E}}^{-1}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{L}\dot{\mathbf{U}}.$$

□

(Grupo H curso 14/15) Exercise 1(a) **Yes.** As $\mathbf{A}$ is invertible, it's column space is the full space $\mathbb{R}^n$. The same is true of $\mathbf{A}^{-1}$.

□

(Grupo H curso 14/15) Exercise 1(b) **No**. Consider $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. It has nonzero column space, but its square is $\mathbf{0}$.

□

(Grupo H curso 14/15) Exercise 2(a)

Since the nullspace is spanned by the given three vectors, we may simply take $\mathbf{B}$ to consist of the three vectors as columns, i.e.,

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ -1 & 1 & 3 \\ 3 & 4 & 1 \end{bmatrix}$$

$\mathbf{B}$ need not be square.

$\square$

(Grupo H curso 14/15) Exercise 2(b)

For example, we may simply add a zero column to $\mathbf{B}$:

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 2 & 1 & -1 \\ 0 & -1 & 1 & 3 \\ 0 & 3 & 4 & 1 \end{bmatrix}$$

Or, we could interchange two columns. Or we could multiply one of the columns by -1. For example:

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -3 \\ 3 & 4 & -1 \end{bmatrix}$$

Or we could replace one of the columns by a linear combination of that column with the other two columns (any invertible column operation). Or we could replace $\mathbf{B}$ by $-\mathbf{B}$ or $2\mathbf{B}$. There are many possible solutions. In any case, the solution shouldn't require any significant calculation!

$\square$

(Grupo H curso 14/15) Exercise 2(c)

Since any solution $\boldsymbol{x}$ to the equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is of the form $\boldsymbol{x}_p + \boldsymbol{x}_n$ for some vector $\boldsymbol{x}_n$ in the nullspace, the vector $\boldsymbol{x} - \boldsymbol{x}_p$ must lie in the nullspace $\mathcal{N}\left(\mathbf{A}\right)$. Thus, we want to look at

$$\boldsymbol{x}_Z - \boldsymbol{x}_p = \begin{pmatrix} 0 \\ -1 \\ 0 \\ -4 \end{pmatrix}, \quad \boldsymbol{x}_H - \boldsymbol{x}_p \begin{pmatrix} 0 \\ -1 \\ 0 \\ -3 \end{pmatrix}.$$

To determine whether a vector $\boldsymbol{y}$ lies in the nullspace $\mathcal{N}\left(\mathbf{A}\right)$, we can just check whether it is in the column space of $\mathbf{B}$, i.e. check whether $\mathbf{B}\boldsymbol{z} = \boldsymbol{y}$ has a solution. As we learned in class, we can check this just by doing elimination: if elimination produces a zero vector in the last column of the augmented matrix $[\mathbf{B}| - \boldsymbol{y}]$, then $\boldsymbol{y}$ is a linear combination of the columns of $\mathbf{B}$:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 2 & 1 & -1 & 1 \\ -1 & 1 & 3 & 0 \\ 3 & 4 & 1 & -a \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{1}+\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 \\ -1 & 1 & 2 & 0 \\ 3 & 4 & 4 & -a \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{\substack{[(-1)\mathbf{2}+\mathbf{3}] \\ [(-1)\mathbf{2}+\mathbf{4}]}}]{\boldsymbol{\tau}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ 3 & 4 & 0 & -a\text{-}4 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{3}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 3 & 4 & 0 & -a\text{-}4 \\ \hline 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

We can get a solution if and only if $a = -4$. So Zarkon is correct.

$\square$

(Grupo H curso 14/15) Exercise 3.

If $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has no solution, the column space of $\mathbf{A}$ must have dimension less than m. The rank is $r < m$. Since $\mathbf{A}^\intercal\boldsymbol{y} = \boldsymbol{c}$ has exactly one solution, the columns of $\mathbf{A}^\intercal$ are independent. This means that the rank of $\mathbf{A}^\intercal$ is $r = m$. This contradiction proves that we cannot find $\mathbf{A}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

$\square$

(Grupo H curso 14/15) Exercise 4(a)

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right]=\left[\begin{array}{ccc}-1&-1&1\\-6&-7&3\\1&1&\frac{-1}{2}\\ \hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow[\substack{[(-1)\mathbf{1}+\mathbf{2}]\\ [(1)\mathbf{1}+\mathbf{3}]}]{\boldsymbol{\tau}}\left[\begin{array}{ccc}-1&0&0\\-6&-1&-3\\1&0&\frac{1}{2}\\ \hline 1&-1&1\\0&1&0\\0&0&1\end{array}\right]\xrightarrow[{[(-3)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}}\left[\begin{array}{ccc}-1&0&0\\-6&-1&0\\1&0&\frac{1}{2}\\ \hline 1&-1&4\\0&1&-3\\0&0&1\end{array}\right]=\left[\begin{array}{c}\mathbf{L}\\ \hline \mathbf{E}\end{array}\right]$$

□

(Grupo H curso 14/15) Exercise 4(b)

Since $\dot{\mathbf{U}}=\dot{\mathbf{E}}^{-1}$:

$$\mathbf{A}=\mathbf{L}\dot{\mathbf{U}}=\begin{bmatrix}-1&0&0\\-6&-1&0\\1&0&1/2\end{bmatrix}\begin{bmatrix}1&1&-1\\0&1&3\\0&0&1\end{bmatrix}.$$

□

(Grupo H curso 14/15) Exercise 4(c)

If you change a_{33} from -1/2 to -1, the third pivot is reduced by 1/2 and $\mathbf{A}_{new}$ becomes singular. Its column space is the plane in $\mathbb{R}^3$ containing all combinations of the first columns (-1, -6, 1) and (0, -1, 0).

□

(Grupo H curso 14/15) Exercise 4(d)

$$\left[\begin{array}{c|c}\mathbf{A}&\text{-}\boldsymbol{b}\\ \hline \mathbf{I}&\mathbf{0}\\ \hline \mathbf{0}&1\end{array}\right]\xrightarrow[\substack{[(-1)\mathbf{1}+\mathbf{2}]\\ [(1)\mathbf{1}+\mathbf{3}]\\ [(-3)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}}\left[\begin{array}{ccc|c}-1&0&0&-b_1\\-6&-1&0&-b_2\\1&0&0&-b_3\\ \hline 1&-1&4&0\\0&1&-3&0\\0&0&1&0\\ \hline 0&0&0&1\end{array}\right]\xrightarrow[\substack{[(-6)\mathbf{2}+\mathbf{1}]\\ [(-1)\mathbf{1}]\\ [(-1)\mathbf{2}]}]{\boldsymbol{\tau}}\left[\begin{array}{ccc|c}1&0&0&-b_1\\0&1&0&-b_2\\-1&0&0&-b_3\\ \hline -7&1&4&0\\6&-1&-3&0\\0&0&1&0\\ \hline 0&0&0&1\end{array}\right]\xrightarrow[\substack{[(b_1)\mathbf{1}+\mathbf{4}]\\ [(b_2)\mathbf{2}+\mathbf{4}]}]{\boldsymbol{\tau}}\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\-1&0&0&-b_1-b_3\\ \hline -7&1&4&b_2-7b_1\\6&-1&-3&6b_1-b_2\\0&0&1&0\\ \hline 0&0&0&1\end{array}\right].$$

We need $b_1+b_3=0$ on the right side (since in that case, the last column in the augmented matrix is free, i.e., $\boldsymbol{b}$ is in $\mathcal{C}(\mathbf{A}_{new})$.

□

(Grupo H curso 14/15) Exercise 4(e)

$$\left\{\boldsymbol{x}\in\mathbb{R}^3\;\middle|\;\exists a\in\mathbb{R}\text{ tal que }\boldsymbol{x}=\begin{pmatrix}1\\0\\0\end{pmatrix}+a\begin{pmatrix}4\\-3\\1\end{pmatrix}\right\}$$

□

(Grupo E curso 13/14) Exercise 1(a)

The column space of $\mathbf{A}$ contains all linear combinations of the columns of $\mathbf{A}$, which are precisely vectors of the form $\mathbf{A}\boldsymbol{x}$ for an arbitrary vector $\boldsymbol{x}$. Thus,

$\boxed{\mathbf{A}\boldsymbol{x}=\boldsymbol{b}\text{ has a solution if and only if }\boldsymbol{b}\text{ is in the column space of }\mathbf{A}}$.

□

(Grupo E curso 13/14) Exercise 1(b)

$\boxed{\text{Yes}}$. Reducing the matrix combining $\mathbf{A}$ and $\boldsymbol{b}$ gives

$$\left[\begin{array}{c|c}\mathbf{A}&\text{-}\boldsymbol{b}\\ \hline \mathbf{I}&\mathbf{0}\\ \hline \mathbf{0}&1\end{array}\right]=\left[\begin{array}{cc|c}2&1&-8\\6&5&-28\\2&4&-14\\ \hline 1&0&0\\0&1&0\\ \hline 0&0&1\end{array}\right]\xrightarrow[\substack{[(-1/2)\mathbf{1}+\mathbf{2}]\\ [(4)\mathbf{1}+\mathbf{3}]}]{\boldsymbol{\tau}}\left[\begin{array}{cc|c}2&0&0\\6&2&-4\\2&3&-6\\ \hline 1&-1/2&4\\0&1&0\\ \hline 0&0&1\end{array}\right]\xrightarrow[{[(2)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}}\left[\begin{array}{cc|c}2&0&0\\6&2&0\\2&3&0\\ \hline 1&-1/2&3\\0&1&2\\ \hline 0&0&1\end{array}\right]$$

Thus, $\boldsymbol{x}=\begin{pmatrix}3\\2\end{pmatrix}$ is a solution to $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$, and therefore $\boldsymbol{b}$ is in the column space of $\mathbf{A}$.

□

(Grupo E curso 13/14) Exercise 2(a)

The four elementary operations in the Gauss-Jordan elimination process are

$$\mathbf{A}\left(\mathbf{I}_{\underset{[(-2)\mathbf{1}+\mathbf{2}]}{\tau}\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\tau}\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\tau}\underset{[(-3)\mathbf{2}+\mathbf{1}]}{\tau}}\right) = \mathbf{AE} = \mathbf{I}$$

or

$$\mathbf{A}\underbrace{\begin{bmatrix}1&-2&\\&1&\\&&1\end{bmatrix}\begin{bmatrix}1&&-3\\&1&\\&&1\end{bmatrix}}_{\begin{bmatrix}1&-2&-3\\&1&\\&&1\end{bmatrix}}\underbrace{\begin{bmatrix}1&&\\&1&\\&-1&1\end{bmatrix}\begin{bmatrix}1&&\\-3&1&\\&&1\end{bmatrix}}_{\begin{bmatrix}1&&\\-3&1&\\3&-1&1\end{bmatrix}} = \mathbf{A}\begin{bmatrix}-2&1&-3\\-3&1&0\\3&-1&1\end{bmatrix} = \mathbf{I}.$$

We can also write the same as: $\left[\begin{array}{c}\mathbf{A}\\\hline\mathbf{I}\end{array}\right] \xrightarrow[{[(-3)\mathbf{2}+\mathbf{1}]}]{\substack{\tau\\{[(-2)\mathbf{1}+\mathbf{2}]}\\{[(-3)\mathbf{1}+\mathbf{3}]}\\{[(-1)\mathbf{3}+\mathbf{2}]}}} \left[\begin{array}{c}\mathbf{A}\\\hline\mathbf{I}\end{array}\right]\mathbf{E} = \left[\begin{array}{c}\mathbf{I}\\\hline\mathbf{A}^{-1}\end{array}\right] = \left[\begin{array}{ccc}1&&\\&1&\\&&1\\\hline-2&1&-3\\-3&1&0\\3&-1&1\end{array}\right].$

Hence, $\mathbf{A}^{-1} = \begin{bmatrix}-2&1&-3\\-3&1&0\\3&-1&1\end{bmatrix}$.

□

(Grupo E curso 13/14) Exercise 2(b)

We can compute de inverse of $\mathbf{A}^{-1}$ in order to get $\mathbf{A}$, computing the inverse elementary operations on $\mathbf{I}$ (but in the reverse order).

$$\left[\begin{array}{c}\mathbf{A}^{-1}\\\hline\mathbf{I}\end{array}\right] = \left[\begin{array}{ccc}-2&-1&-3\\-3&1&0\\3&-1&1\\\hline1&&\\&1&\\&&1\end{array}\right] \xrightarrow[{[(2)\mathbf{1}+\mathbf{2}]}]{\substack{\tau\\{[(3)\mathbf{2}+\mathbf{1}]}\\{[(1)\mathbf{3}+\mathbf{2}]}\\{[(3)\mathbf{1}+\mathbf{3}]}}} \left[\begin{array}{ccc}1&&\\&1&\\&&1\\\hline1&2&3\\3&7&9\\0&1&1\end{array}\right] = \left[\begin{array}{c}\mathbf{I}\\\hline\mathbf{A}\end{array}\right],$$

computing the following product: $\left(\mathbf{I}_{\underset{[(3)\mathbf{2}+\mathbf{1}]}{\tau}}\right)\left(\mathbf{I}_{\underset{[(1)\mathbf{3}+\mathbf{2}]}{\tau}}\right)\left(\mathbf{I}_{\underset{[(3)\mathbf{1}+\mathbf{3}]}{\tau}}\right)\left(\mathbf{I}_{\underset{[(2)\mathbf{1}+\mathbf{2}]}{\tau}}\right) = \mathbf{A}.$

□

(Grupo E curso 13/14) Exercise 3(a)

The third column of $\mathbf{A}$ is two times the first.

$$\mathbf{A} = \begin{bmatrix}1&0&2\\2&0&4\\1&1&2\\2&2&4\\1&3&2\end{bmatrix}.$$

Since $\mathbf{L}$ has its first pivot in first row, first column; and the second pivot in third row, second column; then

A basis for $\mathcal{C}(\mathbf{A})$: $\left[\begin{pmatrix}1\\2\\1\\2\\1\end{pmatrix};\begin{pmatrix}0\\0\\1\\2\\3\end{pmatrix};\right]$. A basis for $\mathcal{C}(\mathbf{A})$: $\left[\begin{pmatrix}1, & 0, & 2,\end{pmatrix}; \begin{pmatrix}1, & 1, & 2,\end{pmatrix};\right]$.

□

(Grupo E curso 13/14) Exercise 3(b)

Since the third column of $\mathbf{L}$ is zero, the third column of $\mathbf{E} = \mathbf{U}^{-1}$ is a basis for $\mathcal{N}(\mathbf{A})$.

$$\mathbf{E} = \mathbf{U}^{-1} = \begin{bmatrix}1&0&-2\\0&1&0\\0&0&1\end{bmatrix}$$

Then, the set of solutions to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ is

$$\mathcal{N}\left(\mathbf{A}\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists a \in \mathbb{R} \;\text{ such that }\; \boldsymbol{x} = a \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

□

(Grupo E curso 13/14) Exercise 3(c)

$$\left[\begin{array}{c|c} \mathbf{A} & \textbf{-}\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 2 & 0 & 4 & -2 \\ 1 & 1 & 2 & -2 \\ 2 & 2 & 4 & -4 \\ 1 & 3 & 2 & -c \\ \hline 1 & & & \\ & 1 & & \\ & & 1 & \\ \hline & & & 1 \end{array}\right] \xrightarrow[\substack{[(-2)\mathbf{1}+\mathbf{3}] \\ [(1)\mathbf{1}+\mathbf{4}] \\ [(1)\mathbf{2}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 3 & 0 & 4-c \\ \hline 1 & & -2 & 1 \\ & 1 & & 1 \\ & & 1 & 0 \\ \hline & & & 1 \end{array}\right]$$

If $c \neq 4$ the system has no solution. When $c = 4$ the solution to the system is:

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists a \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

□

(Grupo E curso 13/14) Exercise 4(a)

There are three rows ($m = 3$). Since the first system has no solution, the rank is less than three, and since the second system has only one solution, all columns are pivot columns. Therefore $n = 1$ or $n = 2$, and so is the rank.

□

(Grupo E curso 13/14) Exercise 4(b)

Since ther is no free column, the only solution is the zero vector: $\boldsymbol{x} = \mathbf{0}$.

□

(Grupo E curso 13/14) Exercise 4(c)

$$\begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix}, \qquad \text{or} \qquad \begin{bmatrix} 0 & c \\ a & 0 \\ 0 & c \end{bmatrix}; \quad \text{and } a, c \neq 0;$$

or any matrix obtained by elementary column operations from those two examples.

□

(Grupo E curso 13/14) Exercise 4(d)

The rank is the maximun number of column vectors of $\mathbf{A}$ that we can take, keeping a linearly independent the set; that is, in such way that the only linear combination of those vectors

$$a \cdot \mathbf{A}_{|i} + b \cdot \mathbf{A}_{|j} + \cdots + m \cdot \mathbf{A}_{|p}$$

that equals the zero vector is when all parameters are equal to zero. But in this definition the order in which the columns are added is irrelevant.

□

(Grupo E curso 12/13) Exercise 1.

$$\left[\begin{array}{c|c} \mathbf{A} & -\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 2 & 1 & -2 & -5 \\ 2 & 4 & 1 & 1 & -9 \\ 3 & 6 & 2 & -1 & -14 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-2)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(2)\mathbf{1}+\mathbf{4}] \\ [(5)\mathbf{1}+\mathbf{5}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & -1 & 5 & 1 \\ 3 & 0 & -1 & 5 & 1 \\ \hline 1 & -2 & -1 & 2 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(5)\mathbf{3}+\mathbf{4}] \\ [(1)\mathbf{3}+\mathbf{5}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 & 0 \\ \hline 1 & -2 & -1 & -3 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

We conclude that the general solutions to this system are given by

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 \text{ such that } \boldsymbol{x} = \begin{pmatrix} 4 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 5 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}.$$

□

(Grupo E curso 12/13) Exercise 2(a)

The "upwards" elimination procedure is

$$\mathbf{A} = \begin{bmatrix} 7 & 6 & 2 \\ 6 & 3 & 0 \\ 4 & 12 & 1 \end{bmatrix} \xrightarrow[{\underset{[(-2)\mathbf{3}+\mathbf{1}]}{\tau}}]{} \begin{bmatrix} -1 & -18 & 0 \\ 6 & 3 & 0 \\ 4 & 12 & 1 \end{bmatrix} \xrightarrow[{\underset{[(6)\mathbf{2}+\mathbf{1}]}{\tau}}]{} \begin{bmatrix} 35 & 0 & 0 \\ 6 & 3 & 0 \\ 4 & 12 & 1 \end{bmatrix} = \mathbf{L},$$

where the first step is $\underset{[(-2)\mathbf{3}+\mathbf{1}]}{\tau}\mathbf{A}$ and the second step is $\underset{[(6)\mathbf{2}+\mathbf{1}]}{\tau}\left(\underset{[(-2)\mathbf{3}+\mathbf{1}]}{\tau}\mathbf{A}\right)$. Since these operations are linear combinations of the rows, they correspond to multiplying on the *left* by elimination matrices:

$$\mathbf{L} = \begin{bmatrix} 35 & 0 & 0 \\ 6 & 3 & 0 \\ 4 & 12 & 1 \end{bmatrix} = \left(\underset{[(6)\mathbf{2}+\mathbf{1}]}{\tau}\mathbf{I}\right)\left(\underset{[(-2)\mathbf{3}+\mathbf{1}]}{\tau}\mathbf{I}\right)\mathbf{A} = \begin{bmatrix} 1 & 6 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{A}$$

(One way to get these elimination matrices, as usual, is to do the corresponding operation on the 3×3 identity matrix.)

□

(Grupo E curso 12/13) Exercise 2(b)

As above, $\mathbf{L} = \mathbf{EA}$ (the elimination matrices multiply on the *left*). It follows that the column space of $\mathbf{A}$ is different from the column space of $\mathbf{L}$, but the null spaces are the same:

For example

$$\mathbf{L} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \mathbf{EA} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

where

$$\mathcal{C}\left(\mathbf{L}\right) = \mathcal{L}\left(\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix};\right]\right) \quad \neq \quad \mathcal{C}\left(\mathbf{A}\right) = \mathcal{L}\left(\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix};\right]\right).$$

Whereas for all invertible matrix $\mathbf{E}$

$$\boldsymbol{x} \in \mathcal{N}\left(\mathbf{A}\right) \Rightarrow \mathbf{A}\boldsymbol{x} = \mathbf{0} \;\Rightarrow\; \mathbf{EA}\boldsymbol{x} = \mathbf{0} \;\Rightarrow\; \mathbf{L}\boldsymbol{x} = \mathbf{0} \;\Rightarrow\; \boldsymbol{x} \in \mathcal{N}\left(\mathbf{L}\right)$$
$$\boldsymbol{x} \in \mathcal{N}\left(\mathbf{L}\right) \Rightarrow \mathbf{L}\boldsymbol{x} = \mathbf{0} \;\Rightarrow\; \mathbf{E}^{-1}\mathbf{L}\boldsymbol{x} = \mathbf{0} \;\Rightarrow\; \mathbf{A}\boldsymbol{x} = \mathbf{0} \;\Rightarrow\; \boldsymbol{x} \in \mathcal{N}\left(\mathbf{A}\right)$$

so $\mathcal{N}\left(\mathbf{A}\right) = \mathcal{N}\left(\mathbf{L}\right)$.

□

(Grupo E curso 12/13) Exercise 2(c) The simplest way is to give any counterexample: e.g., apply rightwards elimination to $\mathbf{A}$ above and you will get a different result. For example, rightward elimination never changes the upper-left corner (7), but the upper-left corner was changed (to 35) by upwards elimination above.

Abstractly, we know from class that rightwards elimination always preserves the column space, whereas we just saw that upwards elimination does not. So, they cannot be the same.

It is *not* sufficient to simply say that up-elimination does different sorts of operations than right-elimination—there are lots of problems where you can do a different sequence of operations and still get the same result. (For example, we could use upwards elimination to find $\mathbf{A}^{-1}$, and of course there is only one possible $\mathbf{A}^{-1}$ if it exists at all).

□

(Grupo E curso 12/13) Exercise 3(a) <u>YES</u>

It is given by a linear equation equal to 0. You can also think about it as the nullspace of the matrix $\begin{pmatrix} 2, & -2, & 1, \end{pmatrix}$.

□

(Grupo E curso 12/13) Exercise 3(b) <u>**NO**</u>

The vector $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is in the set, but if you multiply by -1, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ is not.

□

(Grupo E curso 12/13) Exercise 3(c) <u>**NO**</u>

It is given by a linear equation <u>not</u> set equal to 0. In particular, it doesn't contain the **0** vector.

□

(Grupo E curso 12/13) Exercise 3(d) <u>**YES**</u>

It is the intersection of two subspaces! We can think about this set as the nullspace of the matrix $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -2 \end{bmatrix}$.

□

(Grupo E curso 12/13) Exercise 3(e) <u>**NO**</u>

It is the union of two planes! Take for example $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$; which is not in the set.

□

(Grupo E curso 12/13) Exercise 4(a) The columns are linearly independent, so the rank of the matrix is 3. Then $\dim \mathcal{C}\left(\mathbf{A}\right) = 3$, $\dim \mathcal{C}\left(\mathbf{A}^{\mathsf{T}}\right) = 3$, $\dim \mathcal{N}\left(\mathbf{A}\right) = 0$, $\dim \mathcal{N}\left(\mathbf{A}^{\mathsf{T}}\right) = 1$.

□

(Grupo E curso 12/13) Exercise 4(b) Since $\mathcal{C}\left(\mathbf{A}^{\mathsf{T}}\right)$ is a 3-dimensional subspace of $\mathbb{R}^3$, it is all of $\mathbb{R}^3$. Since $\mathcal{N}\left(\mathbf{A}\right)$ is a 0-dimensional subspace of $\mathbb{R}^3$ it is $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$.

□

(Grupo E curso 12/13) Exercise 4(c) **A** and **B** can be different. For example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

□

(Grupo E curso 12/13) Exercise 5(a)

No solution. The last row of **A** is all zeros. If $s = 1$, we get $0 = 1$.

□

(Grupo E curso 12/13) Exercise 5(b)

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^3 \text{ such that } \boldsymbol{x} = \begin{pmatrix} p \\ 0 \\ q \\ 0 \\ r \\ 0 \end{pmatrix} + \begin{bmatrix} -a & -d & -g \\ 1 & 0 & 0 \\ -b & -e & -h \\ 0 & 1 & 0 \\ -c & -f & -i \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}.$$

□

(Grupo H curso 12/13) Exercise 1(b)

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -b_1 \\ 1 & 2 & 3 & 5 & -b_2 \\ 1 & 3 & 5 & 9 & -b_3 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -b_1 \\ 1 & 1 & 2 & 4 & -b_2 \\ 1 & 2 & 4 & 8 & -b_3 \\ \hline 1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-1)\mathbf{2}+\mathbf{1}] \\ [(-2)\mathbf{2}+\mathbf{3}] \\ [(-4)\mathbf{2}+\mathbf{4}] \\ [(b_1)\mathbf{1}+\mathbf{5}] \\ [(b_2)\mathbf{2}+\mathbf{5}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & b_3 - 2b_2 + b_1 \\ \hline 2 & -1 & 1 & 3 & 2b_1 - b_2 \\ -1 & 1 & -2 & -4 & -b_1 + b_2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$\square$

(Grupo H curso 12/13) Exercise 1(a) Rank 2 (two pivots).

$\square$

(Grupo H curso 12/13) Exercise 1(b) We see from the last row of the reduced matrix that the condition for a vector to be in the column space is $b_1 - 2b_2 + b_3 = 0$. Thus $\mathcal{C}\left(\mathbf{A}\right)$ is $\mathcal{N}\left(\mathbf{B}\right)$ for

$$\mathbf{B} = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}.$$

$\square$

(Grupo H curso 12/13) Exercise 1(c) The last two vectors can't belong to the column space because they are in $\mathbb{R}^4$. From the condition mentioned in part (b), we see that $\begin{pmatrix} 2, & 0, & -2, \end{pmatrix}$ is in the column space $\mathcal{C}\left(\mathbf{A}\right)$, but $\begin{pmatrix} 1, & -2, & 1, \end{pmatrix}$ is not.

$\square$

(Grupo H curso 12/13) Exercise 2(c)

$$\left[\begin{array}{c|c} \mathbf{A} & -\boldsymbol{b} \\ \hline \mathbf{I} & \\ \hline & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 1 & 2 & 3 & -3 \\ 3 & 4 & k & -7 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(2)\mathbf{1}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & -1 \\ 3 & 1 & k-3 & -1 \\ \hline 1 & -1 & -1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-2)\mathbf{2}+\mathbf{3}] \\ [(1)\mathbf{2}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 1 & k-5 & 0 \\ \hline 1 & -1 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right].$$

$\square$

(Grupo H curso 12/13) Exercise 2(a) We see from this that no matter what k is there is always at least one solution. We could have seen that by inspection from the original matrix, since

$$\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}.$$

For $k \neq 5$, the matrix has rank 3, so there is a unique solution.

$\square$

(Grupo H curso 12/13) Exercise 2(b) For $k = 5$ the matrix has rank 2, so there are infinitely many solutions.

$\square$

(Grupo H curso 12/13) Exercise 2(c) From the gausian elimination we known that $\mathbf{AE} = \mathbf{L}$, and since $\mathbf{E}$ is invertible, then $\mathbf{A} = \mathbf{LE}^{-1}$; lets find $\mathbf{E}^{-1}$:

$$\left[\begin{array}{c} \mathbf{E} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{ccc} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ \hline 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(2)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c} \mathbf{I} \\ \hline \mathbf{E}^{-1} \end{array}\right] = \left[\begin{array}{c} \mathbf{I} \\ \hline \mathbf{U} \end{array}\right]$$

hence

$$\mathbf{LU} = \mathbf{LE}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 4 & 4 \end{bmatrix} = \mathbf{A}$$

$\square$

(Grupo H curso 12/13) Exercise 2(d) As noted above, for $k \neq 5$ there is a unique solution, given by $\boldsymbol{x}_p = \begin{pmatrix} 1, & 1, & 0, \end{pmatrix}$. We can get this from the eliminated matrix, or as mentioned above, by inspection.

For $k = 5$, $\boldsymbol{x}_p = \begin{pmatrix} 1, & 1, & 0, \end{pmatrix}$ is a particular solution; the general solution is given by adding vectors in the nullspace:

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists a \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

□

(Grupo H curso 12/13) Exercise 3(d)

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 0 & 0 & -b_1\\ 1 & 2 & 0 & 2 & 2 & -b_2\\ 1 & 2 & -1 & 0 & 0 & -b_3\\ 2 & 4 & 0 & 4 & 4 & -b_4\\ \hline 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau}\\ [(-2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -b_1\\ 1 & 0 & 1 & 2 & 2 & -b_2\\ 1 & 0 & 0 & 0 & 0 & -b_3\\ 2 & 0 & 2 & 4 & 4 & -b_4\\ \hline 1 & -2 & 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(3)\mathbf{3}+\mathbf{5}]}]{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{3}+\mathbf{1}]\\ [(-2)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -b_1\\ 0 & 0 & 1 & 0 & 0 & -b_2\\ 1 & 0 & 0 & 0 & 0 & -b_3\\ 0 & 0 & 2 & 0 & 0 & -b_4\\ \hline 0 & -2 & 1 & -2 & -2 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ -1 & 0 & 1 & -2 & -2 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[{[(b_2)\mathbf{3}+\mathbf{6}]}]{\substack{\boldsymbol{\tau}\\ [(b_1)\mathbf{1}+\mathbf{6}]}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & b_1-b_3\\ 0 & 0 & 2 & 0 & 0 & 2b_2-b_4\\ \hline 0 & -2 & 1 & -2 & -2 & b_2\\ 0 & 1 & 0 & 0 & 0 & 0\\ -1 & 0 & 1 & -2 & -2 & b_2-b_1\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

□

(Grupo H curso 12/13) Exercise 3(a)

$$\text{For example:}\quad \text{basis } \mathcal{C}\left(\mathbf{A}\right) = \left[\begin{pmatrix}1\\0\\1\\0\end{pmatrix};\ \begin{pmatrix}0\\1\\0\\2;\end{pmatrix}\right];\quad \text{but also:}\quad \text{basis } \mathcal{C}\left(\mathbf{A}\right) = \left[\begin{pmatrix}1\\1\\1\\2\end{pmatrix};\ \begin{pmatrix}-1\\0\\-1\\0\end{pmatrix};\right].$$

□

(Grupo H curso 12/13) Exercise 3(b) For example: $\text{basis } \mathcal{N}\left(\mathbf{A}\right) = \left[\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix};\ \begin{pmatrix}-2\\0\\-2\\1\\0\end{pmatrix};\ \begin{pmatrix}-2\\0\\-2\\0\\1\end{pmatrix};\right].$

□

(Grupo H curso 12/13) Exercise 3(c) $\begin{cases} b_1 - b_3 = 0\\ 2b_2 - b_4 = 0 \end{cases}$

□

(Grupo H curso 12/13) Exercise 3(d) Since $\boldsymbol{x}_p = \begin{pmatrix} b_2\\ 0\\ b_2-b_1\\ 0\\ 0 \end{pmatrix} = \begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix}$, then

$$\left\{ \boldsymbol{x}\in\mathbb{R}^5 \;\middle|\; \exists\boldsymbol{p}\in\mathbb{R}^3 \text{ such that } \boldsymbol{x} = \begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix} + \begin{bmatrix} -2 & -2 & -2\\ 1 & 0 & 0\\ 0 & -2 & -2\\ 0 & 1 & 0\\ 0 & 0 & 0 \end{bmatrix}\boldsymbol{p} \right\}.$$

□

(Grupo H curso 12/13) Exercise 4(a) **False.** Counterexample: if $\mathbf{A} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, then $\mathbf{A}^2 = \mathbf{A}$ but $\mathbf{A} \neq \mathbf{I}$ and $\mathbf{A} \neq \mathbf{0}$.

Note that if we assume $\mathbf{A}$ is invertible, then the only solution is $\mathbf{A} = \mathbf{I}$ (multiply both sides of $\mathbf{A}^2 = \mathbf{A}$ by $\mathbf{A}^{-1}$), but this assumption is not warranted here.

□

(Grupo H curso 12/13) Exercise 4(b) **True.** We need to know that linear combinations of vectors stay in the vectors space. If $\mathbf{B}$ is a matrix where $\mathbf{AB} = \mathbf{BA}$, then clearly $\mathbf{A}(\mathbf{B}c) = (c\mathbf{B})\mathbf{A}$ for any c. If $\mathbf{B'}$ is another matrix where $\mathbf{AB'} = \mathbf{B'A}$, then $\mathbf{A}(\mathbf{B}+\mathbf{B'}) = \mathbf{AB} + \mathbf{AB'} = \mathbf{BA} + \mathbf{B'A} = (\mathbf{B}+\mathbf{B'})\mathbf{A}$. (The other properties of a vector space, associativity etcetera, need not be shown since they are automatic for the usual addition and multiplication operations).

□

(Grupo H curso 12/13) Exercise 4(c) **True.** Suppose the rank of $\mathbf{A}$ is r, then the dimension of column space is r, and the dimension of null space is $3-r$. Obviously no matter $r=0,1,2,3$, we always have $r \neq 3-r$. (Equivalently, $r=3-r$ would imply a fractional rank $r=3/2$!) This shows that the two spaces are not the same, since they must have different dimensions.

□

(Grupo H curso 12/13) Exercise 4(d) **True.**

$$\left(\mathbf{A}^2\right)^{\intercal} = \left(\mathbf{A}\mathbf{A}\right)^{\intercal} = \mathbf{A}^{\intercal}\mathbf{A}^{\intercal} = \mathbf{A}\mathbf{A} = \mathbf{A}^2$$

□

(Grupo H curso 12/13) Exercise 4(e) **False.** That would be true if you knew the matrix $\mathbf{A}$ was square..

□

(Grupo H curso 12/13) Exercise 5(a) None. Since the rank is four, there are no free columns.

□

(Grupo H curso 12/13) Exercise 5(b) The row space of $\mathbf{A}$ will be all of $\mathbb{R}^4$ (since the rank is 4). Then every vector $\boldsymbol{b}$ in $\mathbb{R}^4$ is a combination of the rows of $\mathbf{A}$, which means that $\mathbf{A}^{\intercal}\boldsymbol{y}=\boldsymbol{b}$ is solvable for every right side $\boldsymbol{b}$.

□

(Grupo E curso 11/12) Exercise 1(a)

$$\left[\begin{array}{ccc} 2 & 2 & 0 \\ 1 & 4 & 9 \\ 1 & 3 & 9 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc} 2 & 0 & 0 \\ 1 & 3 & 9 \\ 1 & 2 & 9 \\ \hline 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-3)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 2 & 3 \\ \hline 1 & -1 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{array}\right]$$

Therefore

$$\mathbf{E} = \left(\mathbf{I}_{\underset{[(-1)\mathbf{1}+\mathbf{2}]}{\tau}}\right)\left(\mathbf{I}_{\underset{[(-3)\mathbf{2}+\mathbf{3}]}{\tau}}\right) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

(Grupo E curso 11/12) Exercise 1(b) Since $\mathbf{AE}=\mathbf{L}$, then $\mathbf{A}=\mathbf{LE}^{-1}$; hence $\mathbf{U}=\mathbf{E}^{-1}$.

$$\mathbf{U} = \mathbf{E}^{-1} = \left(\mathbf{I}_{\underset{[(-1)\mathbf{1}+\mathbf{2}]}{\tau}\underset{[(-3)\mathbf{2}+\mathbf{3}]}{\tau}}\right)^{-1} = \left(\mathbf{I}_{\underset{[(3)\mathbf{2}+\mathbf{3}]}{\tau}}\right)\left(\mathbf{I}_{\underset{[(1)\mathbf{1}+\mathbf{2}]}{\tau}}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

(Grupo E curso 11/12) Exercise 2(a)

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & -3 \\ 1 & 1 & 5 & 4 & -6 \\ 3 & 3 & 9 & 6 & -c \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-1)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{4}] \\ [(3)\mathbf{1}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 3 & -3 \\ 3 & 0 & 3 & 3 & -c+9 \\ \hline 1 & -1 & -2 & -1 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-1)\mathbf{3}+\mathbf{4}] \\ [(1)\mathbf{3}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 3 & 0 & 3 & 0 & -c+12 \\ \hline 1 & -1 & -2 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Thus

$$\text{A basis of } \mathcal{N}\left(\mathbf{A}\right) = \left[\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix};\right]$$

□

(Grupo E curso 11/12) Exercise 2(b) Only for $c = 12$.

□

(Grupo E curso 11/12) Exercise 2(c) The complete solution is

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 \text{ tal que } \boldsymbol{x} = \begin{pmatrix}1\\0\\1\\0\end{pmatrix} + \begin{bmatrix}-1 & 1\\ 1 & 0\\ 0 & -1\\ 0 & 1\end{bmatrix}\boldsymbol{p} \right\}$$

□

(Grupo E curso 11/12) Exercise 3(a) The space of all vectors in $\mathbb{R}^4$ that are orthogonal (perpendicular) to both of these vectors is the set of solutions to the system:

$$\begin{bmatrix}1 & 2 & 3 & 1\\ 0 & 0 & 1 & 2\end{bmatrix}\boldsymbol{x} = \boldsymbol{0}$$

Hence,

$$\left[\begin{array}{cccc}1 & 2 & 3 & 1\\ 0 & 0 & 1 & 2\\ \hline 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{4}]}]{\substack{\boldsymbol{\tau}\\ [(-2)\mathbf{1}+\mathbf{2}]\\ [(-3)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cccc}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 2\\ \hline 1 & -2 & -3 & -1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right] \xrightarrow[{[(-2)\mathbf{3}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ \hline 1 & -2 & -3 & 5\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 1\end{array}\right]$$

Therefore, a basis is

$$\left[\begin{pmatrix}-2\\1\\0\\0\end{pmatrix};\begin{pmatrix}5\\0\\-2\\1\end{pmatrix};\right].$$

□

(Grupo E curso 11/12) Exercise 3(b) It depends on the rank of the matrix (the number of pivots) $\begin{bmatrix}\boldsymbol{u}; & \boldsymbol{v}; & \boldsymbol{w};\end{bmatrix}$. Since there are three non-zero columns, then the matrix is not de zero matrix and then, the dimensión of its column space could be 1, 2, or 3.

□

(Grupo E curso 11/12) Exercise 4(a) Checking that the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{c}$ is solvable, for example:

- Checking if $\mathbf{A}$ has the same number of pivots as the augmented matrix $[\mathbf{A}|\boldsymbol{c}]$ $\Big($that is, checking if $\operatorname{rg}(\mathbf{A}) = \operatorname{rg}([\mathbf{A}|\boldsymbol{c}])\Big)$.
- In different words, checking if the last column of the augmented matrix $[\mathbf{A}|\boldsymbol{c}]$ is a free column after Gaussian elimination by rows.
- or doing Gaussian elimination by columns on the augmented matrix $[\mathbf{A}|\boldsymbol{c}]$ and cheking the last column becomes a zero-vector: $[\mathbf{A}|\boldsymbol{c}] \xrightarrow{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_n} [\mathbf{L}|\mathbf{0}]$.

□

(Grupo E curso 11/12) Exercise 4(b) Checking if $\boldsymbol{r}$ becomes a row of zeros after gaussian row elimination:

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \boldsymbol{r}\end{array}\right] \xrightarrow[\boldsymbol{\tau}_n\cdots\boldsymbol{\tau}_1]{} \left[\begin{array}{c}\mathbf{U}\\ \hline \mathbf{0}\end{array}\right]$$

□

(Grupo E curso 11/12) Exercise 4(c) Since $\boldsymbol{c} \in \mathcal{C}(\mathbf{A})$ $\Big($or, since $\boldsymbol{r} \in \mathcal{C}(\mathbf{A}^{\intercal})$;$\Big)$ the rank of $\mathbf{A}$ is at least one.

□

(Grupo E curso 11/12) Exercise 4(d) At most the rank of $\mathbf{A}$ is two. Not all vector of $\mathbb{R}^3$ are in the row space $\mathcal{C}\left(\mathbf{A}^{\mathsf{T}}\right)$, since $\boldsymbol{r} \notin \mathcal{C}\left(\mathbf{A}^{\mathsf{T}}\right)$. This means the dimensión of $\mathcal{C}\left(\mathbf{A}^{\mathsf{T}}\right)$ is less than three (there are less than three pivots).

□

(Grupo H curso 11/12) Exercise 1(a)

The four elementary operations in the Gauss-Jordan elimination process are

$$\mathbf{A}\left(\mathbf{I}_{\underset{[(-2)\mathbf{1}+\mathbf{2}]}{\tau}\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\tau}\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\tau}\underset{[(-3)\mathbf{2}+\mathbf{1}]}{\tau}}\right) = \mathbf{AE} = \mathbf{I}$$

or

$$\mathbf{A}\underbrace{\begin{bmatrix}1&-2&\\&1&\\&&1\end{bmatrix}\begin{bmatrix}1&&-3\\&1&\\&&1\end{bmatrix}}_{\begin{bmatrix}1&-2&-3\\&1&\\&&1\end{bmatrix}}\underbrace{\begin{bmatrix}1&&\\&1&\\&-1&1\end{bmatrix}\begin{bmatrix}1&&\\-3&1&\\&&1\end{bmatrix}}_{\begin{bmatrix}1&&\\-3&1&\\3&-1&1\end{bmatrix}} = \mathbf{A}\begin{bmatrix}-2&1&-3\\-3&1&0\\3&-1&1\end{bmatrix} = \mathbf{I}.$$

We can also write the same as: $\left[\begin{array}{c}\mathbf{A}\\\hline\mathbf{I}\end{array}\right] \xrightarrow[{[(-3)\mathbf{2}+\mathbf{1}]}]{\substack{\tau\\{[(-2)\mathbf{1}+\mathbf{2}]}\\{[(-3)\mathbf{1}+\mathbf{3}]}\\{[(-1)\mathbf{3}+\mathbf{2}]}}} \left[\begin{array}{c}\mathbf{A}\\\hline\mathbf{I}\end{array}\right]\mathbf{E} = \left[\begin{array}{c}\mathbf{I}\\\hline\mathbf{A}^{-1}\end{array}\right] = \left[\begin{array}{ccc}1&&\\&1&\\&&1\\\hline-2&1&-3\\-3&1&0\\3&-1&1\end{array}\right].$

Hence, $\mathbf{A}^{-1} = \begin{bmatrix}-2&1&-3\\-3&1&0\\3&-1&1\end{bmatrix}.$

□

(Grupo H curso 11/12) Exercise 1(b)

We can compute de inverse of $\mathbf{A}^{-1}$ in order to get $\mathbf{A}$, computing the inverse elementary operations on $\mathbf{I}$ (but in the reverse order).

$$\left[\begin{array}{c}\mathbf{A}^{-1}\\\hline\mathbf{I}\end{array}\right] = \left[\begin{array}{ccc}-2&-1&-3\\-3&1&0\\3&-1&1\\\hline1&&\\&1&\\&&1\end{array}\right] \xrightarrow{\substack{\tau\\{[(3)\mathbf{2}+\mathbf{1}]}\\{[(1)\mathbf{3}+\mathbf{2}]}\\{[(3)\mathbf{1}+\mathbf{3}]}\\{[(2)\mathbf{1}+\mathbf{2}]}}} \left[\begin{array}{ccc}1&&\\&1&\\&&1\\\hline1&2&3\\3&7&9\\0&1&1\end{array}\right] = \left[\begin{array}{c}\mathbf{I}\\\hline\mathbf{A}\end{array}\right],$$

computing the following product: $\left(\mathbf{I}_{\underset{[(3)\mathbf{2}+\mathbf{1}]}{\tau}}\right)\left(\mathbf{I}_{\underset{[(1)\mathbf{3}+\mathbf{2}]}{\tau}}\right)\left(\mathbf{I}_{\underset{[(3)\mathbf{1}+\mathbf{3}]}{\tau}}\right)\left(\mathbf{I}_{\underset{[(2)\mathbf{1}+\mathbf{2}]}{\tau}}\right) = \mathbf{A}.$

□

(Grupo H curso 11/12) Exercise 2(a) The third column of $\mathbf{A}$ is two times the first.

$$\mathbf{A} = \begin{bmatrix}1&0&2\\2&0&4\\1&1&2\\2&2&4\\1&3&2\end{bmatrix}.$$

Since $\mathbf{L}$ has its first pivot in first row, first column; and the second pivot in third row, second column; then

A basis for $\mathcal{C}(\mathbf{A})$: $\left[\begin{pmatrix}1\\2\\1\\2\\1\end{pmatrix};\ \begin{pmatrix}0\\0\\1\\2\\3\end{pmatrix};\right]$. A basis for $\mathcal{C}\left(\mathbf{A}^{\mathsf{T}}\right)$: $\left[(1,\ \ 0,\ \ 2,);\ (1,\ \ 1,\ \ 2,);\right]$.

□

(Grupo H curso 11/12) Exercise 2(b) Since the third column of $\mathbf{L}$ is zero, the third column of $\mathbf{E}$ $=\mathbf{U}^{-1}$ is a basis for $\mathcal{N}(\mathbf{A})$.

$$\mathbf{E} = \mathbf{U}^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, the set of solutions to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ is

$$\mathcal{N}(\mathbf{A}) = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists a \in \mathbb{R} \text{ such that } \boldsymbol{x} = a \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

□

(Grupo H curso 11/12) Exercise 2(c)

$$\begin{bmatrix} \mathbf{A} & \text{-}\boldsymbol{b} \\ \hline \mathbf{I} & \\ \hline & 1 \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 2 & 0 & 4 & -2 \\ 1 & 1 & 2 & -2 \\ 2 & 2 & 4 & -4 \\ 1 & 3 & 2 & -c \\ \hline 1 & & & \\ & 1 & & \\ & & 1 & \\ \hline & & & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{2}+\mathbf{4}]}]{\substack{\tau \\ [(-2)\mathbf{1}+\mathbf{3}] \\ [(1)\mathbf{1}+\mathbf{4}]}} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 3 & 0 & 4-c \\ \hline 1 & & -2 & 1 \\ & 1 & & 1 \\ & & 1 & 0 \\ \hline & & & 1 \end{array}\right]$$

If $c \neq 4$ the system has no solution. When $c = 4$ the solution to the system is:

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists a \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

□

(Grupo H curso 11/12) Exercise 3(a) There are three rows ($m = 3$). Since the first system has no solution, the rank is less than three, and since the second system has only one solution, all columns are pivot columns. Therefore $n = 1$ or $n = 2$, and so is the rank.

□

(Grupo H curso 11/12) Exercise 3(b) Since ther is no free column, the only solution is the zero vector: $\boldsymbol{x} = \mathbf{0}$.

□

(Grupo H curso 11/12) Exercise 3(c)

$$\begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix}, \qquad \text{or} \qquad \begin{bmatrix} 0 & c \\ a & 0 \\ 0 & c \end{bmatrix}; \quad \text{and } a, c \neq 0;$$

or any matrix obtained by elementary column operations from those two examples.

□

(Grupo H curso 11/12) Exercise 3(d) The rank is the maximun number of column vectors of $\mathbf{A}$ that we can take, keeping a linearly independent the set; that is, in such way that the only linear combination of those vectors

$$a \cdot \boldsymbol{a}i + b \cdot \boldsymbol{a}j + \cdots + m \cdot \boldsymbol{a}p$$

that equals the zero vector is when all parameters are equal to zero. But in this definition the order on which the columns are added is irrelevant.

□

(Grupo H curso 11/12) Exercise 4(a) The rank is 4, so there will be 7 - 4 = 3 columns of zeros in $\mathbf{L}$ and $\mathbf{R}$.

□

(Grupo H curso 11/12) Exercise 4(b) The column space of $\mathbf{A}$ contains $\mathbb{R}^4$ (since the rank is 4). Then, every vector $\boldsymbol{c}$ in $\mathbb{R}^4$ is a combination of the rows of $\mathbf{A}$, which means that $\mathbf{A}\boldsymbol{y} = \boldsymbol{c}$ is solvable for every right hand side vector $\boldsymbol{c}$.

□

(Grupo A curso 10/11) Exercise 1(a)

$$\left[\begin{array}{cccc} 0&1&2&2\\ 0&3&8&7\\ 0&0&4&2\\ \hline 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{array}\right] \xrightarrow[\substack{[(-2)\mathbf{2}+\mathbf{3}]\\ [(-2)\mathbf{2}+\mathbf{4}]}]{\tau} \left[\begin{array}{cccc} 0&1&0&0\\ 0&3&2&1\\ 0&0&4&2\\ \hline 1&0&0&0\\ 0&1&-2&-2\\ 0&0&1&0\\ 0&0&0&1 \end{array}\right] \xrightarrow[\substack{[(2)\mathbf{4}]\\ [(-1)\mathbf{3}+\mathbf{4}]}]{\tau} \left[\begin{array}{cccc} 0&1&0&0\\ 0&3&2&0\\ 0&0&4&0\\ \hline 1&0&0&0\\ 0&1&-2&-2\\ 0&0&1&-1\\ 0&0&0&2 \end{array}\right]$$

so the special solutions are $\boxed{\boldsymbol{s}_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}; \quad \boldsymbol{s}_2 = \begin{pmatrix}0\\-2\\-1\\2\end{pmatrix}}$ Thus, $\mathcal{N}\left(\mathbf{A}\right)$ is **a plane in** $\mathbb{R}^4$ given by all linear combinations of the two special solutions.

$$\left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 \text{ such that } \boldsymbol{v} = \begin{bmatrix}1&0\\0&-2\\0&-1\\0&2\end{bmatrix} \boldsymbol{p} \right\} = \mathcal{L}\left(\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}; \begin{pmatrix}0\\-2\\-1\\2\end{pmatrix};\right]\right).$$

□

(Grupo A curso 10/11) Exercise 1(b) $\mathcal{C}\left(\mathbf{A}\right)$ is a plane in $\mathbb{R}^3$ given by all combinations of the pivot columns of $\mathbf{A}$, namely

$$\left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 \text{ such that } \boldsymbol{v} = \begin{bmatrix}1&2\\3&8\\0&4\end{bmatrix} \boldsymbol{p} \right\} = \mathcal{L}\left(\left[\begin{pmatrix}1\\3\\0\end{pmatrix}; \begin{pmatrix}2\\8\\4\end{pmatrix};\right]\right).$$

Or $\mathcal{C}\left(\mathbf{A}\right)$ is a plane in $\mathbb{R}^3$ given by all combinations of the pivot columns of $\mathbf{L}$, namely

$$\left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 \text{ such that } \boldsymbol{v} = \begin{bmatrix}1&0\\3&2\\0&4\end{bmatrix} \boldsymbol{p} \right\} = \mathcal{L}\left(\left[\begin{pmatrix}1\\3\\0\end{pmatrix}; \begin{pmatrix}0\\2\\4\end{pmatrix};\right]\right).$$

□

(Grupo A curso 10/11) Exercise 1(c) Note that $\mathbf{B}$ immediately reduces to

$$\mathbf{B} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{A} & \mathbf{0} \end{bmatrix}$$

We reduced $\mathbf{A}$ above: the row reduced echelon form of of $\mathbf{B}$ is thus

$$\boxed{\mathbf{B} = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{R} & \mathbf{0} \end{bmatrix}; \quad \text{where} \quad \mathbf{R} = \begin{bmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&-6&2&0 \end{bmatrix}.}$$

□

(Grupo A curso 10/11) Exercise 2(a) First of all, $m = 3$ since $\mathbf{A}\boldsymbol{x} \in \mathbb{R}^3$. In addition, $\mathbf{A}\boldsymbol{x} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ has *one* solution $\implies \mathcal{N}\left(\mathbf{A}\right) = \{0\}$, so $r = n$ (where r is the rank of $\mathbf{A}$).

But $\mathbf{A}\boldsymbol{x} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ has no solution $\implies \mathcal{C}\left(\mathbf{A}\right) \neq \mathbb{R}^3$, so $r < m = 3$.

There are two possibilities : $\begin{matrix} m=3 \\ r=n=1 \end{matrix}$ and $\begin{matrix} m=3 \\ r=n=2 \end{matrix}$.

□

(Grupo A curso 10/11) Exercise 2(b) Since $\mathcal{N}(\mathbf{A}) = \{0\}$ (because $\mathbf{A}\boldsymbol{x} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ has 1 solution), there is a unique solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$, which is clearly $\boldsymbol{x} = \mathbf{0}$. (Can be either $\boldsymbol{x} = (0,)$ or $\boldsymbol{x} = \begin{pmatrix}0\\0\end{pmatrix}$ depending on if $n = 1$ or $n = 2$.)

□

(Grupo A curso 10/11) Exercise 2(c) $\mathbf{A}$ could be $\begin{bmatrix}0\\a\\0\end{bmatrix}$, for $a \neq 0$; or $\begin{bmatrix}0&c\\0&0\\0&c\end{bmatrix}$, or $\begin{bmatrix}c&0\\0&a\\c&0\end{bmatrix}$ and both columns linearly independent.

□

(Grupo A curso 10/11) Exercise 3(a) the equation $Ax = b$ $\boxed{\text{always}}$ has $\boxed{\text{many solutions}}$.

□

(Grupo A curso 10/11) Exercise 3(b) $\boxed{\text{The column space is a 3-dimensional space inside a 3-dimensional space}}$, i.e. it contains all the vectors in $\mathbb{R}^3$, and $\boxed{\text{the nullspace has dimension } 5-3=2>0 \text{ inside } \mathbb{R}^5}$, a plane in $\mathbb{R}^5$ through the origin.

□

(Grupo A curso 10/11) Exercise 4(a) We first have to find a vector in the direction of the line. We let

$$\boldsymbol{v} = \boldsymbol{p} - \boldsymbol{q} = \begin{pmatrix}2\\2\end{pmatrix} - \begin{pmatrix}-1\\4\end{pmatrix} = \begin{pmatrix}3\\-2\end{pmatrix}.$$

A parametric representation of the line is therefore

$$\boldsymbol{x} = \boldsymbol{p} + a\boldsymbol{v} \quad \Rightarrow \quad \begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix} + a\begin{pmatrix}3\\-2\end{pmatrix} \quad \text{or} \quad \begin{cases}x_1 = 2 + 3a\\ x_2 = 2 - 2a\end{cases}.$$

□

(Grupo A curso 10/11) Exercise 4(b) We need to eliminate the "parametric" part $(a\boldsymbol{v})$, since

$$\begin{pmatrix}2, & 3,\end{pmatrix} \cdot \boldsymbol{v} = \begin{pmatrix}2, & 3,\end{pmatrix} \cdot \begin{pmatrix}3\\-2\end{pmatrix} = \begin{pmatrix}0\end{pmatrix}$$

then it is easy to eliminate the parametric part:

$$\begin{pmatrix}2, & 3,\end{pmatrix}\boldsymbol{x} = \begin{pmatrix}2, & 3,\end{pmatrix} \cdot \boldsymbol{p} + a\begin{pmatrix}2, & 3,\end{pmatrix} \cdot \boldsymbol{v} \longrightarrow \begin{pmatrix}2, & 3,\end{pmatrix} \cdot \boldsymbol{x} = \begin{pmatrix}2, & 3,\end{pmatrix} \cdot \boldsymbol{p} + a \cdot 0$$

Therefore, in this case

$$\begin{pmatrix}2, & 3,\end{pmatrix} \cdot \begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}2, & 3,\end{pmatrix} \cdot \begin{pmatrix}2\\2\end{pmatrix} + a\begin{pmatrix}2, & 3,\end{pmatrix} \cdot \begin{pmatrix}3\\-2\end{pmatrix} \longrightarrow 2x + 3y = 10 + a \cdot 0$$

y therefore the line is $\{2x + 3y = 10$.

An easy way to compute all this calculations is by...Gaussian Elimination!

$$\left[\begin{array}{cc} x & y \\ \hline 2 & 2 \\ \hline\hline 3 & -2 \end{array}\right] \xrightarrow[{[(3)\mathbf{2}]}]{\overset{\tau}{[(2)\mathbf{1}]}} \left[\begin{array}{cc} 2x & 3y \\ \hline 4 & 6 \\ \hline\hline 6 & -6 \end{array}\right] \xrightarrow{\overset{\tau}{[(1)\mathbf{1+2}]}} \left[\begin{array}{cc} 2x & 2x+3y \\ \hline 4 & 10 \\ \hline\hline 6 & 0 \end{array}\right] \Longrightarrow \{2x + 3y = 10.$$

□

(Grupo E curso 10/11) Exercise 1(a) First, since $\mathbf{R}$ is in reduced row echelon form, we must have $\boldsymbol{d} = \begin{pmatrix}4, & 0, & 0,\end{pmatrix}$. The other two vectors provide special solutions for $\mathbf{R}$, showing that $\mathbf{R}$ has rank 1: again, since it is in reduced row echelon form, the bottom two rows must be all 0, and the top row is $\begin{pmatrix}1, & -2, & -5,\end{pmatrix}$, i.e.

$$\boxed{\mathbf{R} = \begin{bmatrix}1 & -2 & -5\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}}.$$

□

(Grupo E curso 10/11) Exercise 1(b) The matrix connecting $\mathbf{R}$ and $\boldsymbol{d}$ to the original $\mathbf{A}$ and $\boldsymbol{b}$ is

$$\mathbf{E} = \mathbf{I}_{\underset{[(-5)\mathbf{3}+\mathbf{1}]}{\tau}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}}$$

That is, $\mathbf{R} = \mathbf{E}\mathbf{A}$ and $\mathbf{E}\boldsymbol{b} = \boldsymbol{d}$. Thus, $\mathbf{A} = \mathbf{E}^{-1}\mathbf{R}$ and $\boldsymbol{b} = \mathbf{E}^{-1}\boldsymbol{d}$, giving

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & -2 & -5 \\ 3 & -6 & -15 \\ 5 & -10 & -25 \end{bmatrix}}$$

$$\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}}.$$

□

(Grupo E curso 10/11) Exercise 2(a)

- $\text{rg}\left(\mathbf{A}\right) =$ number of pivots $= 3$.
- $\dim(\mathcal{C}\left(\mathbf{A}\right)) = \dim(\mathcal{C}\left(\mathbf{A}^\intercal\right)) = \text{rg}\left(\mathbf{A}\right) = 3$.
- $\dim(\mathcal{N}\left(\mathbf{A}\right)) =$ number of free columns $= 0$
- $\dim(\mathcal{N}\left(\mathbf{A}^\intercal\right)) =$ number of free rows $= 5 - 3 = 2$.

□

(Grupo E curso 10/11) Exercise 2(b) The three pivot columns of $\mathbf{R}$. Also $\mathbf{A}_{|1}$ $\mathbf{A}_{|2}$ and $\mathbf{A}_{|4}$.

□

(Grupo E curso 10/11) Exercise 2(c) The first, second and fourth rows of $\mathbf{A}$

□

(Grupo E curso 10/11) Exercise 2(d) Since there are two free rows, we need to compute two linearly independent solutions for $\boldsymbol{x}\mathbf{A} = \mathbf{0}$. Since

$$\mathbf{N}\mathbf{R} = \begin{bmatrix} \mathbf{-3} & \mathbf{2} & 1 & 0 & \mathbf{0} \\ \mathbf{-1} & \mathbf{0} & 0 & 1 & \mathbf{0} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \mathbf{3} & \mathbf{-2} & 0 & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & 0 & \mathbf{0} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \underset{2\times 4}{\mathbf{0}}$$

The system that consist of rows $\boldsymbol{x}_a = \begin{pmatrix}-3, & 2, & 1, & 0, & 0,\end{pmatrix}$ and $\boldsymbol{x}_b = \begin{pmatrix}-1, & 0, & 0, & 1, & 0,\end{pmatrix}$ of $\mathbf{N}$ is a basis of $\mathcal{N}\left(\mathbf{A}^\intercal\right)$.

And since any linear combination of both solutions is another solution, the left nullspace $\mathcal{N}\left(\mathbf{A}^\intercal\right)$ is

$$\mathcal{N}\left(\mathbf{A}^\intercal\right) = \left\{\boldsymbol{w} \in \mathbb{R}^5 \,\middle|\, \exists \alpha, \beta \in \mathbb{R} \text{ such that } \boldsymbol{w} = \alpha\boldsymbol{x}_a + \beta\boldsymbol{x}_b \right\}.$$

□

(Grupo E curso 10/11) Exercise 2(e) $3\mathbf{A}_{|1} - 2\mathbf{A}_{|2} + 0\mathbf{A}_{|4} + 0\mathbf{A}_{|5} = \mathbf{A}_{|3}$. Note that the third row of $\mathbf{R}$ is telling us a linear combination. We can find more, but that is the only possibility if we only use the first and second rows.

□

(Grupo E curso 10/11) Exercise 3(a) the equation $Ax = b$ $\boxed{\text{always}}$ has $\boxed{\text{many solutions}}$.

□

(Grupo E curso 10/11) Exercise 3(b) $\boxed{\text{The column space is a 3-dimensional space inside a 3-dimensional space}}$, i.e. it contains all the vectors in $\mathbb{R}^3$, and $\boxed{\text{the nullspace has dimension } 5-3=2>0 \text{ inside } \mathbb{R}^5}$, a plane in $\mathbb{R}^5$ through the origin.

□

(Grupo E curso 10/11) Exercise 4(a) We first have to find a vector in the direction of the line. We let

$$\boldsymbol{v} = \boldsymbol{p} - \boldsymbol{q} = \begin{pmatrix}2\\4\end{pmatrix} - \begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}.$$

A parametric representation of the line is therefore

$$\boldsymbol{x} = \boldsymbol{p} + a\boldsymbol{v} \quad \Rightarrow \quad \begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix} + a\begin{pmatrix}1\\1\end{pmatrix} \quad \text{or} \quad \begin{cases}x_1 = 1 + a\\ x_2 = 4 + a\end{cases}.$$

□

(Grupo E curso 10/11) Exercise 4(b) We need to eliminate the "parametric" part ($a\boldsymbol{v}$), since

$$\begin{pmatrix}-1, & 1,\end{pmatrix}\boldsymbol{v} = \begin{pmatrix}-1, & 1,\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\end{pmatrix}$$

then it is easy to eliminate the parametric part:

$$\begin{pmatrix}-1, & 1,\end{pmatrix}\boldsymbol{x} = \begin{pmatrix}-1, & 1,\end{pmatrix}\boldsymbol{p} + a\begin{pmatrix}-1, & 1,\end{pmatrix}\boldsymbol{v} \longrightarrow \begin{pmatrix}-1, & 1,\end{pmatrix}\boldsymbol{x} = \begin{pmatrix}-1, & 1,\end{pmatrix}\boldsymbol{p} + a\cdot 0$$

Therefore, in this case

$$\begin{pmatrix}-1, & 1,\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-1, & 1,\end{pmatrix}\begin{pmatrix}2\\4\end{pmatrix} + a\begin{pmatrix}-1, & 1,\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} \longrightarrow -x + y = 2 + a\cdot 0$$

y therefore the line is

$$\begin{cases}-x + y = 2.\end{cases}$$

An easy way to compute all this calculations is by...Gaussian Elimination!

$$\left[\begin{array}{cc}x & y\\ \hline 2 & 4\\ \hline\hline 1 & 1\end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \left[\begin{array}{cc}x & y-x\\ \hline 2 & 2\\ \hline\hline 1 & 0\end{array}\right] \Longrightarrow \begin{cases}-x + y = 2\,.\end{cases}$$

□

(Grupo G curso 10/11) Exercise 1(a)

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{cccc}1&2&1&4\\3&6&3&9\\2&4&2&9\\ \hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right] \xrightarrow[\substack{[(-2)\mathbf{1}+\mathbf{2}]\\ [(-1)\mathbf{1}+\mathbf{3}]\\ [(-4)\mathbf{1}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc}1&0&0&0\\3&0&0&-3\\2&0&0&1\\ \hline 1&-2&-1&-4\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right] = \left[\begin{array}{c}\mathbf{K}\\ \hline \mathbf{E}\end{array}\right]$$

$r = \operatorname{rg}\left(\mathbf{A}\right) = 2$, pivot columns are $\begin{pmatrix}1\\3\\2\end{pmatrix}$ and $\begin{pmatrix}4\\9\\9\end{pmatrix}$.

The column space $\mathcal{C}\left(\mathbf{A}\right)$ is **a plane in** $\mathbb{R}^3$ spaned by the two pivot columns of $\mathbf{A}$ (or $\mathbf{L}$).

□

(Grupo G curso 10/11) Exercise 1(b) Special solutions: $\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}$, and $\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}$.

$$\mathcal{N}\left(\mathbf{A}\right) = \left\{\boldsymbol{v}\in\mathbb{R}^4 \;\middle|\; \exists\boldsymbol{p}\in\mathbb{R}^2 \text{ such that } \boldsymbol{v} = \begin{bmatrix}-2&0\\1&-1\\0&2\\0&0\end{bmatrix}\boldsymbol{p}\right\}.$$

□

(Grupo G curso 10/11) Exercise 1(c)

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & -3 \\ 3 & 6 & 3 & 9 & -b_3 \\ 2 & 4 & 2 & 9 & -100 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(-4)\mathbf{1}+\mathbf{4}] \\ [(3)\mathbf{1}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & -3 & 0 \\ 2 & 0 & 0 & 1 & 6-b_3 \\ \hline 1 & -2 & -1 & -4 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Hence to have a solution we need $\boxed{b_3 = 6}$

For this value of b_3, a particular solution is given by $\boldsymbol{x}_p = \begin{pmatrix} 3, & 0, & 0, & 0, \end{pmatrix}$.

Complete solution: $\left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 \text{ such that } \boldsymbol{x} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{p} \right\}.$

$\square$

(Grupo G curso 10/11) Exercise 2(a) Since $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is solvable for any $\boldsymbol{b} \in \mathbb{R}^3$, then any vector $\boldsymbol{b}$ in $\mathbb{R}^3$ is a linear combination of the columns of $\mathbf{A}$: $\boldsymbol{b} = x_1\mathbf{A}_{|1} + \cdots + x_n\mathbf{A}_{|n}$. Then, every $\boldsymbol{b}$ in $\mathbb{R}^3$ is in $\mathcal{C}\left(\mathbf{A}\right)$; so **the column space $\mathcal{C}\left(\mathbf{A}\right)$ is the whole three dimensional space** $\mathbb{R}^3$.

$\square$

(Grupo G curso 10/11) Exercise 2(b) Since $\mathcal{C}\left(\mathbf{A}\right) = \mathbb{R}^3$, we have $r = \text{rg}\left(\mathbf{A}\right) = 3$ and therefore num. free variables = num. special solutions $= n - r = 5 - 3 = 2$.

Hence $\mathcal{N}\left(\mathbf{A}\right)$ is a plane in $\mathbb{R}^5$.

$\square$

(Grupo G curso 10/11) Exercise 2(c) By (a) we know $\mathcal{C}\left(\mathbf{A}\right) = \mathbb{R}^3$ therefore $r = \text{rg}\left(\mathbf{A}\right) = 3$.

$\square$

(Grupo G curso 10/11) Exercise 2(d) We can use elimination to obtain

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{A} \end{bmatrix} \mathbf{E} = \begin{bmatrix} \mathbf{AE} \\ \mathbf{AE} \end{bmatrix} = \begin{bmatrix} \mathbf{R} \\ \mathbf{R} \end{bmatrix}.$$

But $\mathbf{R}$ has rank 3. Therefore $\text{rg}\left(\mathbf{B}\right) = 3$.

$\square$

(Grupo G curso 10/11) Exercise 3(a) the equation $Ax = b$ $\boxed{\text{always}}$ has $\boxed{\text{many solutions}}$.

$\square$

(Grupo G curso 10/11) Exercise 3(b) $\boxed{\text{The column space is a 3-dimensional space inside a 3-dimensional space}}$, i.e. it contains all the vectors in $\mathbb{R}^3$, and $\boxed{\text{the nullspace has dimension } 5 - 3 = 2 > 0 \text{ inside } \mathbb{R}^5}$, a plane in $\mathbb{R}^5$ through the origin.

$\square$

(Grupo G curso 10/11) Exercise 4(a) We first have to find a vector in the direction of the line. We let

$$\boldsymbol{v} = \boldsymbol{p} - \boldsymbol{q} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

A parametric representation of the line is therefore

$$\boldsymbol{x} = \boldsymbol{p} + a\boldsymbol{v} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + a \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{or} \quad \begin{cases} x = a \\ y = 1 - 2a \end{cases}.$$

$\square$

(Grupo G curso 10/11) Exercise 4(b) We need to eliminate the "parametric" part ($a\boldsymbol{v}$), and repeat the same operations on $\boldsymbol{x}$ and $\boldsymbol{x}_p$

$$\left[\begin{array}{cc} x & y \\ \hline 0 & 1 \\ \hline\hline 1 & -2 \end{array}\right] \xrightarrow{\substack{\tau \\ [(2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc} 2x & 2x+y \\ \hline 1 & 1 \\ \hline\hline 1 & 0 \end{array}\right] \Longrightarrow \left\{ 2x + y = 1 \right..$$

$\square$

(Grupo F curso 09/10) Exercise 1(a) The first one is an elementary matrix, its inverse is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}.$$

The second one is a permutation matrix, its inverse is its transpose.

$\square$

(Grupo F curso 09/10) Exercise 1(b)

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & d \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1/d)\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{d} \end{array}\right] \xrightarrow[{\substack{[(-c)\mathbf{4}+\mathbf{3}] \\ [(-b)\mathbf{4}+\mathbf{2}] \\ [(-a)\mathbf{4}+\mathbf{1}]}}]{\boldsymbol{\tau}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{a}{d} & -\frac{b}{d} & -\frac{c}{d} & \frac{1}{d} \end{array}\right] = \left[\frac{\mathbf{I}}{\mathbf{A}^{-1}}\right]$$

Then, the inverse of the matrix is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{a}{d} & -\frac{b}{d} & -\frac{c}{d} & \frac{1}{d} \end{bmatrix}$

$\square$

(Grupo F curso 09/10) Exercise 2(b)

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 0 & 0 & -b_1 \\ 1 & 2 & 0 & 2 & 2 & -b_2 \\ 1 & 2 & -1 & 0 & 0 & -b_3 \\ 2 & 4 & 0 & 4 & 4 & -b_4 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{\substack{[(-2)\mathbf{1}+\mathbf{2}] \\ [(1)\mathbf{1}+\mathbf{3}]}}]{\boldsymbol{\tau}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -b_1 \\ 1 & 0 & 1 & 2 & 2 & -b_2 \\ 1 & 0 & 0 & 0 & 0 & -b_3 \\ 2 & 0 & 2 & 4 & 4 & -b_4 \\ \hline 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{\substack{[(-1)\mathbf{3}+\mathbf{1}] \\ [(-2)\mathbf{3}+\mathbf{4}] \\ [(3)\mathbf{3}+\mathbf{5}]}}]{\boldsymbol{\tau}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -b_1 \\ 0 & 0 & 1 & 0 & 0 & -b_2 \\ 1 & 0 & 0 & 0 & 0 & -b_3 \\ 0 & 0 & 2 & 0 & 0 & -b_4 \\ \hline 0 & -2 & 1 & -2 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -2 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[{\substack{[(b_1)\mathbf{1}+\mathbf{6}] \\ [(b_2)\mathbf{3}+\mathbf{6}]}}]{\boldsymbol{\tau}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & b_1 - b_3 \\ 0 & 0 & 2 & 0 & 0 & 2b_2 - b_4 \\ \hline 0 & -2 & 1 & -2 & -2 & b_2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -2 & -2 & b_2 - b_1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$\square$

(Grupo F curso 09/10) Exercise 2(a)

For example: basis $\mathcal{C}\left(\mathbf{A}\right) = \left[\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix};\right]$; but also: basis $\mathcal{C}\left(\mathbf{A}\right) = \left[\begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}; \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix};\right]$.

$\square$

(Grupo F curso 09/10) Exercise 2(b)

For example: basis $\mathcal{N}\left(\mathbf{A}\right) = \left[\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} -2 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix};\right]$.

$\square$

(Grupo F curso 09/10) Exercise 2(c) $\begin{cases} b_1 - b_3 = 0 \\ 2b_2 - b_4 = 0 \end{cases}$

□

(Grupo F curso 09/10) Exercise 2(d) Since $\boldsymbol{x}_p = \begin{pmatrix} b_2 \\ 0 \\ b_2 - b_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$; then

$$\left\{ \boldsymbol{x} \in \mathbb{R}^5 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^3 \text{ such that } \boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{bmatrix} -2 & -2 & -2 \\ 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{p} \right\}.$$

□

(Grupo F curso 09/10) Exercise 3(a) **False.** Consider for example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix on the right hand side is invertible, but the two on the left hand side are not.

□

(Grupo F curso 09/10) Exercise 3(b) **True.** For $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ to have no solution $\boldsymbol{b}$ must be linearly independent of the columns of $\mathbf{A}$. Therefore, in the augmented matrix $[\mathbf{A}| - \boldsymbol{b}]$, the last column must be a pivot column; so we must have a free row in the reduced echelon form of $\mathbf{A}$. Hence, the number of pivots in $\mathbf{A}$ will be less than the number of rows; thus, the matrix $\mathbf{A}$ does not have full rank.

□

(Grupo F curso 09/10) Exercise 3(c) **False.** Suppose $\mathbf{AB}$ is invertible, and consider $\mathbf{M} = (\mathbf{AB})^{-1}$ $\mathbf{A}$. Then $\mathbf{MB} = (\mathbf{AB})^{-1}\mathbf{AB} = \mathbf{I}$, so $\mathbf{M}$ is an inverse for $\mathbf{B}$.

Another way to see the same thing, if $\mathbf{B}$ is singular, it means that its echelon form has at least one free column:

$$\mathbf{BE} = \mathbf{L} = \left[\mathbf{C}|\mathbf{0}\right]$$

Hence, applying the same elimination steps on $(\mathbf{AB})$ we get

$$(\mathbf{AB})\mathbf{E} = \mathbf{AL} = \mathbf{A}\left[\mathbf{C}|\mathbf{0}\right] = \left[\mathbf{AC}|\mathbf{0}\right]$$

so $(\mathbf{AB})$ has at least one free column, and there fore is also singular.

□

(Grupo F curso 09/10) Exercise 3(d) **False.** Consider the exchange matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Then

$$\mathbf{P}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \neq \mathbf{I}.$$

□

(Grupo F curso 09/10) Exercise 4(a) Since the nullspace is spanned by the given three vectors, we may simply take $\mathbf{B}$ to consist of the three vectors as columns, i.e.,

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ -1 & 1 & 3 \\ 3 & 4 & 1 \end{bmatrix}$$

$\mathbf{B}$ need not be square.

□

(Grupo F curso 09/10) Exercise 4(b) For example, we may simply add a zero column to $\mathbf{B}$:

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 2 & 1 & -1 \\ 0 & -1 & 1 & 3 \\ 0 & 3 & 4 & 1 \end{bmatrix}$$

Or, we could interchange two columns. Or we could multiply one of the columns by -1. For example:

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -3 \\ 3 & 4 & -1 \end{bmatrix}$$

Or we could replace one of the columns by a linear combination of that column with the other two columns (any invertible column operation). Or we could replace $\mathbf{B}$ by $-\mathbf{B}$ or $2\mathbf{B}$. There are many possible solutions. In any case, the solution shouldn't require any significant calculation!

□

(Grupo F curso 09/10) Exercise 4(c) Since any solution $\boldsymbol{x}$ to the equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is of the form $\boldsymbol{x}_p + \boldsymbol{x}_n$ for some vector $\boldsymbol{x}_n$ in the nullspace, the vector $\boldsymbol{x} - \boldsymbol{x}_p$ must lie in the nullspace $\mathcal{N}\left(\mathbf{A}\right)$. Thus, we want to look at

$$\boldsymbol{x}_Z - \boldsymbol{x}_p = \begin{pmatrix} 0 \\ -1 \\ 0 \\ -4 \end{pmatrix}, \quad \boldsymbol{x}_H - \boldsymbol{x}_p \begin{pmatrix} 0 \\ -1 \\ 0 \\ -3 \end{pmatrix}.$$

To determine whether a vector $\boldsymbol{y}$ lies in the nullspace $\mathcal{N}\left(\mathbf{A}\right)$, we can just check whether it is in the column space of $\mathbf{B}$, i.e. check whether $\mathbf{B}\boldsymbol{z} = \boldsymbol{y}$ has a solution. As we learned in class, we can check this just by doing elimination: if elimination produces a zero vector in the last column of the augmented matrix $[\mathbf{B}| - \boldsymbol{y}]$, then $\boldsymbol{y}$ is a linear combination of the columns of $\mathbf{B}$:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 2 & 1 & -1 & 1 \\ -1 & 1 & 3 & 0 \\ 3 & 4 & 1 & -a \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{1}+\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 \\ -1 & 1 & 2 & 0 \\ 3 & 4 & 4 & -a \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{\substack{[(-1)\mathbf{2}+\mathbf{3}] \\ [(-1)\mathbf{2}+\mathbf{4}]}}]{\boldsymbol{\tau}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ 3 & 4 & 0 & -a-4 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{3}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 3 & 4 & 0 & -a-4 \\ \hline 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

We can get a solution if and only if $a = -4$. So Zarkon is correct.

□

(Grupo H curso 09/10) Exercise 1(a) We know that $\left(\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}\mathbf{I}\right)\mathbf{A} = \mathbf{I}$, hence

$$\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}\mathbf{I} = \underset{[(-1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underset{[(-1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ -3 & 0 & 1 \end{bmatrix} = \mathbf{A}^{-1}$$

□

(Grupo H curso 09/10) Exercise 1(b) Apply the inverse operations in reverse order to $\mathbf{I}$, i.e. $\mathbf{A} = \underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\underset{[(1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}\mathbf{I}$:

$$\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\underset{[(1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}\mathbf{I} = \underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix} = \mathbf{A}$$

$$\text{Check:} \quad \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

□

(Grupo H curso 09/10) Exercise 2(a) No. This is not a vector space because $\mathbf{0}$ is not in this subspace.

□

(Grupo H curso 09/10) Exercise 2(b) Yes. (This is actually just the left nullspace of the matrix whose columns are $\boldsymbol{y}$ and $\boldsymbol{z}$.)

$$\boldsymbol{x}\left[\boldsymbol{z};\quad \boldsymbol{y};\right]=\mathbf{0}.$$

□

(Grupo H curso 09/10) Exercise 2(c) No. For example, the zero matrix $\mathbf{0}$ is not in this subset.

□

(Grupo H curso 09/10) Exercise 2(d) Yes. If the nullspaces of A_1 and A_2 contain $\begin{pmatrix}1\\2\\3\end{pmatrix}$ then any linear combination of these matrices, $(a\mathsf{A}_1+b\mathsf{A}_1)$, does too:

$$(a\mathsf{A}_1+b\mathsf{A}_1)\begin{pmatrix}1\\2\\3\end{pmatrix}=a\mathsf{A}_1\begin{pmatrix}1\\2\\3\end{pmatrix}+b\mathsf{A}_2\begin{pmatrix}1\\2\\3\end{pmatrix}=a\mathbf{0}+b\mathbf{0}=\mathbf{0}.$$

□

(Grupo H curso 09/10) Exercise 3.

$$\left[\begin{array}{cccc|c}1&3&1&2&-1\\2&6&4&8&-3\\0&0&2&4&-1\\\hline 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\\hline 0&0&0&0&1\end{array}\right]\xrightarrow[\substack{[(-3)\mathbf{1}+\mathbf{2}]\\ [(-1)\mathbf{1}+\mathbf{3}]\\ [(-2)\mathbf{1}+\mathbf{4}]\\ [(1)\mathbf{1}+\mathbf{5}]}]{\tau}\left[\begin{array}{cccc|c}1&0&0&0&0\\2&0&2&4&-1\\0&0&2&4&-1\\\hline 1&-3&-1&-2&1\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\\hline 0&0&0&0&1\end{array}\right]\xrightarrow[\substack{[(-2)\mathbf{3}+\mathbf{4}]\\ [(2)\mathbf{5}]\\ [(1)\mathbf{3}+\mathbf{5}]\\ [(\frac{1}{2})\mathbf{5}]}]{\tau}\left[\begin{array}{cccc|c}1&0&0&0&0\\2&0&2&0&0\\0&0&2&0&0\\\hline 1&-3&-1&0&1/2\\0&1&0&0&0\\0&0&1&-2&1/2\\0&0&0&1&0\\\hline 0&0&0&0&1\end{array}\right].$$

$$\left\{\boldsymbol{x}\in\mathbb{R}^4\;\middle|\;\exists\boldsymbol{p}\in\mathbb{R}^2 \text{ tal que } \boldsymbol{x}=\begin{pmatrix}1/2\\0\\1/2\\0\end{pmatrix}+\begin{bmatrix}-3&0\\1&0\\0&-2\\0&1\end{bmatrix}\boldsymbol{p}\right\}.$$

□

(Grupo H curso 09/10) Exercise 4(a) 4. There are 4 pivots in the reduced echelon form of A.

□

(Grupo H curso 09/10) Exercise 4(b)

- $\dim(\mathcal{C}\left(\mathsf{A}\right))=\dim(\mathcal{C}\left(\mathsf{A}^{\intercal}\right))=\operatorname{rg}\left(\mathsf{A}\right)=4$
- $\dim(\mathcal{N}\left(\mathsf{A}\right))=n-\operatorname{rg}\left(\mathsf{A}\right)=4-4=0$
- $\dim(\mathcal{N}\left(\mathsf{A}^{\intercal}\right))=m-\operatorname{rg}\left(\mathsf{A}\right)=8-4=4$

□

(Grupo H curso 09/10) Exercise 4(c) Since A is not full row rank, $\mathsf{A}\boldsymbol{x}=\boldsymbol{b}$ will no solutions if $\boldsymbol{b}\notin\mathcal{C}\left(\mathsf{A}\right)$, and only one solution otherwise, since there are no free columns (A is a full column rank).

□

(Grupo H curso 09/10) Exercise 4(d) Yes. The reduced echelon form of A has linearly independent columns, and column operations preserve the column space.

□

(Grupo H curso 09/10) Exercise 4(e) No. Rows 4, 5, 6 and 7 in B are dependent, and column operations preserve linear dependence and independence of rows. Hence, rows 4, 5, 6 and 7 of A are dependent.

□

(Grupo H curso 09/10) Exercise 4(f) We saw that $\dim(\mathcal{N}\left(\mathbf{A}\right)) = 0$. Hence, $\mathcal{N}\left(\mathbf{A}\right)$ contains only the zero vector; and the isn't any basis for this space.

□

(Grupo H curso 09/10) Exercise 4(g) since

$$\begin{bmatrix} -2 & \mathbf{1} & 0 & & & 0 & & 0 \\ -3 & & -1 & \mathbf{1} & & 0 & & 0 \\ 1 & & -2 & & \mathbf{1} & 0 & & 0 \\ -2 & & 0 & & & 1 & \mathbf{1} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mathbf{2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 0 & 0 \\ \mathbf{3} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{-1} & \mathbf{2} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 1 & 0 \\ \mathbf{2} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From these we get the 4 special solutions corresponding to the 4 free variables x_2, x_4, x_5 and x_7 . The special solutions are a basis for the nullspace. Hence, our basis is

$$\text{A basis for } \mathcal{N}\left(\mathbf{A}^{\mathsf{T}}\right) : \left[\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; \begin{pmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} ; \right].$$

□

(Grupo H curso 09/10) Exercise 4(h) Yes; $\mathbf{E}$ is just the product of the elimination matrices (including possibly permutation matrices) which are applied to $\mathbf{A}$ to get $\mathbf{B}$. Consider pivot rows 1, 3, 6 and 8 of $\mathbf{A}$ — those which become the pivot rows of $\mathbf{B}$. The matrix $\mathbf{E}$ is what performs this change on the columns. Hence

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & -2 & 2 & 0 \\ -2 & 1 & -2 & 2 \\ 1 & -2 & 1 & 2 \\ -5 & 3 & -5 & 4 \\ 0 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 \\ -3 & 0 & 1 & 2 \end{bmatrix} \mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which of course means that this matrix is $\mathbf{E}^{-1}$:

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ -2 & 1 & -2 & 2 \\ 0 & 0 & 1 & 0 \\ -3 & 0 & 1 & 2 \end{bmatrix} \mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

□