Mathematics II

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You can find the last version of these course materials at

https://mbujosab.github.io/MatematicasII/

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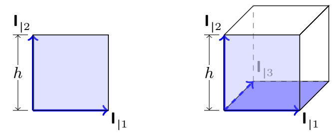
1 Highlights of **Lesson 14**

Highlights of Lesson 14

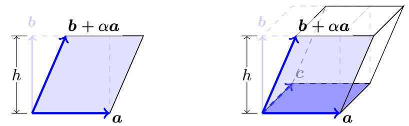
- Determinant: $\det(\mathbf{A}) \equiv |\mathbf{A}|$ $\quad [\det : \quad \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}]$
  - Volume vs determinant
  - Properties: **1, 2, 3**
- We will deduce properties: **4 – 9**

2 Area or volume

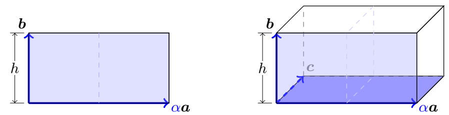
1. $\mathrm{Vol}(\underset{n\times n}{\mathbf{I}}) = 1.$



2. $\mathrm{Vol}(\mathbf{A}) \;=\; \mathrm{Vol}\Big(\mathbf{A}_{\underset{[(\alpha)\mathbf{k}+\mathbf{j}]}{\tau}}\Big)$ for $j \neq k$.



3. $|\alpha| \cdot \mathrm{Vol}(\mathbf{A}) \;=\; |\alpha| \cdot \mathrm{Vol}[\ldots;\mathbf{A}_{|k};\ldots] \;=\; \mathrm{Vol}[\ldots;\alpha\mathbf{A}_{|k};\ldots]$



3 Determinant: 3 properties that define the function

P-1 **Determinant of identity matrices:**

$$\det \underset{n\times n}{\mathbf{I}} = 1$$

P-2 **Type I elemen. transf. do not change the determinant**:

$$\det \mathbf{A} = \det \left(\mathbf{A}_{\underset{[(\alpha)\boldsymbol{k}+\boldsymbol{j}]}{\boldsymbol{\tau}}}\right)$$

P-3 **Multiplying a column by an scalar multiplies the det.**

$\alpha \cdot \det \mathbf{A} \;=\; \det\left[\ldots; \alpha\mathbf{A}_{|k}; \ldots\right]$ for any $k \in \{1:n\}$ and $\alpha \in \mathbb{R}$

$$\boxed{\text{Absolute value of } \det \mathbf{A} = \operatorname{Vol} \mathbf{A}}$$

Example

Then, we know that in $\mathbb{R}^3$:

$$\begin{vmatrix} a_1 & (b_1+\alpha c_1) & c_1 \\ a_2 & (b_2+\alpha c_2) & c_2 \\ a_3 & (b_3+\alpha c_3) & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix};$$

$$\det\left[\boldsymbol{a};\ (\boldsymbol{b}+\alpha\boldsymbol{c});\ \boldsymbol{c};\right] = \det\left[\boldsymbol{a};\ \boldsymbol{b};\ \boldsymbol{c};\right];$$

and also

$$\begin{vmatrix} a_1 & \alpha\, b_1 & c_1 \\ a_2 & \alpha\, b_2 & c_2 \\ a_3 & \alpha\, b_3 & c_3 \end{vmatrix}; = \alpha \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix};$$

$$\det\left[\boldsymbol{a};\ \alpha\boldsymbol{b};\ \boldsymbol{c};\right] = \alpha \det\left[\boldsymbol{a};\ \boldsymbol{b};\ \boldsymbol{c};\right];$$

4 Determinant of a matrix with a zero column

P-4 **Det. of a matrix A with a zero column**

If $\mathbf{A}$ *has a zero column* $\mathbf{0}$, *then*

$$\boxed{\det(\mathbf{A}) = 0}$$

prove **P-4**

5 Elementary matrices

We already know

$$\det\left(\mathbf{A}_{\underset{[(\alpha)\boldsymbol{k}+\boldsymbol{j}]}{\boldsymbol{\tau}}}\right) = |\mathbf{A}|; \qquad \det\left(\mathbf{A}_{\underset{[(\alpha)\boldsymbol{k}]}{\boldsymbol{\tau}}}\right) = \alpha|\mathbf{A}|.$$

Determinant of elementary matrices

$$\det\left(\mathbf{I}_{\underset{[(\alpha)\boldsymbol{k}+\boldsymbol{j}]}{\boldsymbol{\tau}}}\right) = 1 \qquad \text{and} \qquad \det\left(\mathbf{I}_{\underset{[(\alpha)\boldsymbol{j}]}{\boldsymbol{\tau}}}\right) = \alpha.$$

Hence, since $\mathbf{A}_{\boldsymbol{\tau}} = \mathbf{A}(\mathbf{I}_{\boldsymbol{\tau}})$, then

$$\left|\mathbf{A}(\mathbf{I}_{\boldsymbol{\tau}})\right| = |\mathbf{A}| \cdot |\mathbf{I}_{\boldsymbol{\tau}}| \qquad (1)$$

where $\mathbf{I}_{\boldsymbol{\tau}}$ is an elementary matrix

Exercise 1. Prove the following propositions

(a) $\det(\mathbf{A}_{\tau_1\cdots\tau_k}) = |\mathbf{A}|\cdot|\mathbf{I}_{\tau_1}|\cdots|\mathbf{I}_{\tau_k}|$.

(b) If $\mathbf{B}$ is a full rank matrix, i.e., if $\mathbf{B}=\mathbf{I}_{\tau_1\cdots\tau_k}$, then $|\mathbf{B}| = |\mathbf{I}_{\tau_1}|\cdots|\mathbf{I}_{\tau_k}|$, and therefore $|\mathbf{B}|\neq 0$.

(c) *If $\mathbf{A}$ and $\mathbf{B}$ have order n and $\mathbf{B}$ is full rank, then*

$$\det(\mathbf{AB}) = |\mathbf{A}|\cdot|\mathbf{B}| \tag{2}$$

6 Determinant after a sequence of elementary transformations

Example

a sequence $\tau_1\cdots\tau_k$ of *Type I* elementary transformations does not change the determinant.

$$|\mathbf{A}_{\tau_1\cdots\tau_k}| = |\mathbf{A}(\mathbf{I}_{\tau_1\cdots\tau_k})| = |\mathbf{A}|\cdot|\mathbf{I}_{\tau_1\cdots\tau_k}| = |\mathbf{A}|\cdot 1 = |\mathbf{A}|$$

Example

but a sequence of *Type II* can.

$$\begin{vmatrix} 2a & 3c \\ 2b & 3d \end{vmatrix} = \underline{\;?\;} \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

7 Antisymmetric property

P-5 **[Antisymmetric property]**
Column exchange changes the sign of the determinant.

Proof.
Column exchange is a sequence of *Type I* transformation and just only one *Type II* transformation that multiplies a column by -1 □

Therefore:

$$\begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix} = (-1)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

8 Singular matrices. Inverse of a matrix

P-6 **If $\mathbf{A}$ is singular then $|\mathbf{A}| = 0$.**

P-7 $\det(\mathbf{A}^{-1}) = \big(\det\mathbf{A}\big)^{-1}$.

Proof.
Let $\mathbf{A}_{\tau_1\cdots\tau_k} = \underset{n\times n}{\mathbf{R}}$ be a reduced equelon form (and $\mathbf{E}=\mathbf{I}_{\tau_1\cdots\tau_k}$).

Since $\mathbf{AE}=\mathbf{R}$, then: $|\mathbf{A}|\cdot|\mathbf{E}| = |\mathbf{R}|$; with only two cases:

$$\begin{cases} \mathbf{A} \text{ singular } (\mathbf{R}_{|n} = \mathbf{0}\ ): & |\mathbf{A}|\cdot|\mathbf{E}| = 0 \;\Rightarrow\; |\mathbf{A}| = 0 \\ \mathbf{A} \text{ not singular } (\mathbf{R}=\mathbf{I}): & |\mathbf{A}|\cdot|\mathbf{E}| = 1 \;\Rightarrow\; |\mathbf{E}| = \left|\mathbf{A}^{-1}\right| = \big(|\mathbf{A}|\big)^{-1} \end{cases}.$$

□

Example

For $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$:

$$\left[\begin{array}{cc} 1 & 2 \\ 2 & 2 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] \xrightarrow[TypeI]{\overset{\tau}{[(-2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc} 1 & 0 \\ 2 & -2 \\ \hline 1 & -2 \\ 0 & 1 \end{array}\right] \xrightarrow[TypeII]{\overset{\tau}{[(-1/2)\mathbf{2}]}} \left[\begin{array}{cc} 1 & 0 \\ 2 & 1 \\ \hline 1 & 1 \\ 0 & -1/2 \end{array}\right] \xrightarrow[TypeI]{\overset{\tau}{[(-2)\mathbf{2}+\mathbf{1}]}} \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline -1 & 1 \\ 1 & -1/2 \end{array}\right]$$

So

$$\left|\mathbf{A}^{-1}\right| = \left|\mathbf{I}_{\underset{[(-2)\mathbf{1}+\mathbf{2}]}{\tau}}\right| \cdot \left|\mathbf{I}_{\underset{[(-1/2)\mathbf{2}]}{\tau}}\right| \cdot \left|\mathbf{I}_{\underset{[(-2)\mathbf{2}+\mathbf{1}]}{\tau}}\right| = 1 \cdot \frac{-1}{2} \cdot 1 = \frac{-1}{2};$$

that is $\boxed{|\mathbf{A}| = -2.}$

9 Determinant of a product

P-8 **[Determinant of a product of matrices]**

$$\boxed{\det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B}).} \qquad (3)$$

$$\begin{cases} \mathbf{B} \text{ singular, then so it is } \mathbf{AB} \Rightarrow & \det(\mathbf{AB}) = 0 = \det(\mathbf{A}) \cdot \det(\mathbf{B}) \\ \mathbf{B} = \mathbf{I}_{\tau_1 \cdots \tau_k} \Rightarrow & \det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B}) \end{cases}$$

Exercise 2. **[Transposed matrices]**

(a) What is the relation between the determinant of an elementary matrix $\mathbf{I}_{\tau}$ and the determinant of its transpose ${}_{\tau}\mathbf{I}$?

(b) Consider $\mathbf{B}$, a full rank matrix, proof that $|\mathbf{B}| = |\mathbf{B}^{\intercal}|$.

10 Determinant of a transpose

P-9 **Determinant of a transpose**

$$|\mathbf{A}| = |\mathbf{A}^{\intercal}|.$$

Proof.

$$\begin{cases} \text{if } \mathbf{A} \text{ singular:} & \mathbf{A}^{\intercal} \text{ singular } \Rightarrow \det \mathbf{A}^{\intercal} = \det \mathbf{A} = 0 \\ \text{if } \mathbf{A} \text{ NO singular:} & \mathbf{A} = \mathbf{I}_{\tau_1 \cdots \tau_k} \Rightarrow \det \mathbf{A}^{\intercal} = \det \mathbf{A} \end{cases}.$$

$\square$

Questions of the Lecture 14

(L-14) QUESTION 1. Complete the proofs of this lecture.

(L-14) QUESTION 2. Knowing that $|\mathbf{BC}| = |\mathbf{B}||\mathbf{C}|$; prove that for any invertible matrix $\mathbf{A}$ (so $\det \mathbf{A} \neq 0$)

$$\det(\mathbf{A}^{-1}) = \Big(\det(\mathbf{A})\Big)^{-1}.$$

(L-14) QUESTION 3. Consider $\underset{3\times 3}{\mathbf{A}}$ and $\underset{3\times 3}{\mathbf{B}}$ such that $\det(\mathbf{A}) = 2$ and $\det(\mathbf{B}) = -2$

(a) (0.5^{pts}) Compute the determinants of $\mathbf{A}(\mathbf{B})^2$ and $(\mathbf{AB})^{-1}$
(b) (0.5^{pts}) Is it possible to compute the rank of $\mathbf{A}+\mathbf{B}$? and the rank of $\mathbf{AB}$?

(L-14) QUESTION 4. Use the Gauss-Jordan method to compute the determinant

(a) $\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\mathbf{A}_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

(c) $\mathbf{A}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(L-14) QUESTION 5. The 3 by 3 matrix $\mathbf{A}$ reduces to the identity matrix $\mathbf{I}$ by the following three column operations (in order):

$\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}$: Subtract 4 times column 1 from column 2.

$\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}$: Subtract 3 times column 1 from column 3.

$\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}$: Subtract column 3 from column 2.

Find the determinant of $\mathbf{A}$.

(L-14) QUESTION 6.

(a) Find the determinant of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(b) Find the determinant of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & d \end{bmatrix}$ **using Gauss-Jordan**.

$\boxed{1}$ Highlights of **Lesson 15**

Highlights of Lesson 15

- Computing $|\mathbf{A}|$ by gaussian elimination
- **P-10** — Multilinear property
- Expansion of $\det \mathbf{A}$ in Cofactors (Laplace expansion).
- Application of determinants
  - Cramer's rule for solving linear equations
  - Computing the inverse of $\mathbf{A}$

$\boxed{2}$ Extended matrix

Extended matrix of $\mathbf{B}$: $\begin{bmatrix} \mathbf{B} & \\ & 1 \end{bmatrix}$

1. Given $\boldsymbol{\tau}$: $\begin{bmatrix} \mathbf{B}_{\boldsymbol{\tau}} & \\ & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \\ & 1 \end{bmatrix}_{\boldsymbol{\tau}}.$

2. Since $\begin{bmatrix} \mathbf{I} & \\ & 1 \end{bmatrix}_{\boldsymbol{\tau}}$ and $\mathbf{I}_{\boldsymbol{\tau}}$ same *type* Elem. Mat. $\Rightarrow$ same det.

Applying 1. k times, and then 2.

$$\left|\begin{bmatrix} \mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k} & \\ & 1 \end{bmatrix}\right| = \left|\begin{bmatrix} \mathbf{I} & \\ & 1 \end{bmatrix}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right| = \left|\begin{bmatrix} \mathbf{I} & \\ & 1 \end{bmatrix}_{\boldsymbol{\tau}_1} \cdots \begin{bmatrix} \mathbf{I} & \\ & 1 \end{bmatrix}_{\boldsymbol{\tau}_k}\right|$$
$$= \left|\mathbf{I}_{\boldsymbol{\tau}_1}\right| \cdots \left|\mathbf{I}_{\boldsymbol{\tau}_k}\right| = \left|\mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right|.$$

If $\mathbf{A}$ is the extended matrix of $\mathbf{B}$ $\begin{cases} \text{If } \mathbf{B} \text{ singular} & |\mathbf{B}| = 0 = |\mathbf{A}| \\ \text{If } \mathbf{B} \text{ invertible} & |\mathbf{B}| = |\mathbf{A}| \end{cases}$

EXERCISE 7. **[Triangular matrices]**

(a) Find the determinant of a full rank lower triangular matrix **L**
(b) Find the determinant of a triangular matrix with a zero entry in the main diagonal
(c) Find the determinant of an upper triangular matrix **U**

In addition $\begin{vmatrix} \mathbf{A} & \underset{m\times n}{\mathbf{0}} \\ \underset{n\times m}{\mathbf{C}} & \mathbf{B} \end{vmatrix} = |\mathbf{A}|\cdot|\mathbf{B}|$.

$\boxed{3}$ Computing by Gaussian elimination

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} : \left[\begin{array}{cc|c} 1 & 5 & \\ 2 & 3 & \\ \hline & & 1 \end{array}\right] \xrightarrow[{[(-5)\mathbf{1}+\mathbf{2}]}]{\tau} \left[\begin{array}{cc|c} 1 & 0 & \\ 2 & -7 & \\ \hline & & 1 \end{array}\right] \quad \boxed{|\mathbf{A}| = -7}$$

Example

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 9 & 6 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(\frac{1}{2})\mathbf{4}]}]{\substack{\tau \\ [(2)\mathbf{3}] \\ [(-1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc|c} 0 & 2 & 0 & 0 \\ 9 & 6 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{2} \end{array}\right] \xrightarrow[{[(-1)\mathbf{4}]}]{\substack{\tau \\ [\mathbf{1}\rightleftharpoons\mathbf{2}]}} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 6 & 9 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & -\frac{1}{2} \end{array}\right]$$

$$\begin{vmatrix} 0 & 2 & 1 \\ 9 & 6 & 3 \\ 0 & 1 & 1 \end{vmatrix} = -9,$$

Matrices of order 1, $\mathbf{A} = [a]$:

$$\left[\begin{array}{c|c} a & 0 \\ \hline 0 & 1 \end{array}\right] \Rightarrow |\mathbf{A}| = a.$$

Matrices of order 2:

$$\left[\begin{array}{cc|c} a & b & 0 \\ c & d & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-\frac{b}{a})\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc|c} a & 0 & 0 \\ c & d-\frac{bc}{a} & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

$$|\mathbf{A}| = ad - bc = a\det[d] - b\det[c].$$

Matrices of order 3:

$$\left[\begin{array}{ccc|c} a & b & c & 0 \\ d & e & f & 0 \\ g & h & i & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-\frac{b}{a})\mathbf{1}+\mathbf{2}] \\ [(-\frac{c}{a})\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc|c} a & 0 & 0 & 0 \\ d & e-\frac{bd}{a} & f-\frac{cd}{a} & 0 \\ g & h-\frac{bg}{a} & i-\frac{cg}{a} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(\frac{-af+cd}{ae-bd})\mathbf{2}+\mathbf{3}]}}$$

$$\left[\begin{array}{ccc|c} a & 0 & 0 & 0 \\ d & e-\frac{bd}{a} & 0 & 0 \\ g & h-\frac{bg}{a} & \frac{aei-afh-bdi+bfg+cdh-ceg}{ae-bd} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$|\mathbf{A}| = \underbrace{aei - afh - bdi + bfg + cdh - ceg}_{\text{(Rule of Sarrus)}} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

Matrices of order 4:

$$\left[\begin{array}{cccc|c} a & b & c & d & 0 \\ e & f & g & h & 0 \\ i & j & k & l & 0 \\ m & n & o & p & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-\frac{b}{a})\mathbf{1}+\mathbf{2}] \\ [(-\frac{c}{a})\mathbf{1}+\mathbf{3}] \\ [(-\frac{d}{a})\mathbf{1}+\mathbf{4}] \\ \left[\left(\frac{-ag+ce}{af-be}\right)\mathbf{2}+\mathbf{3}\right] \\ \left[\left(\frac{-ah+de}{af-be}\right)\mathbf{2}+\mathbf{4}\right] \\ \left[\left(\frac{-afl+ahj+bel-bhi-dej+dfi}{afk-agj-bek+bgi+cej-cfi}\right)\mathbf{3}+\mathbf{4}\right]}}$$

$$\left[\begin{array}{cccc|c} a & 0 & 0 & 0 & 0 \\ e & f-\frac{be}{a} & 0 & 0 & 0 \\ i & j-\frac{bi}{a} & \frac{afk-agj-bek+bgi+cej-cfi}{af-be} & 0 & 0 \\ m & n-\frac{bm}{a} & \frac{afo-agn-beo+bgm+cen-cfm}{af-be} & p+\frac{\left(-l+\frac{\left(h-\frac{de}{a}\right)\left(j-\frac{bi}{a}\right)}{f-\frac{be}{a}}+\frac{di}{a}\right)\left(-o+\frac{\left(g-\frac{ce}{a}\right)\left(n-\frac{bm}{a}\right)}{f-\frac{be}{a}}+\frac{cm}{a}\right)}{-k+\frac{\left(g-\frac{ce}{a}\right)\left(j-\frac{bi}{a}\right)}{f-\frac{be}{a}}+\frac{ci}{a}}-\frac{\left(h-\frac{de}{a}\right)\left(n-\frac{bm}{a}\right)}{f-\frac{be}{a}}-\frac{dm}{a} & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$|\mathbf{A}| =$

$afkp-aflo-agjp+agln+ahjo-ahkn-bekp+belo+bgip-bglm-bhio+bhkm+$
$cejp-celn-cfip+cflm+chin-chjm-dejo+dekn+dfio-dfkm-dgin+dgjm$

$$= a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c\begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

4 Multilinear property

P-10 Multilinear property

$$\det\left[\ldots;(\beta \boldsymbol{b}+\psi \boldsymbol{c});\ldots\right]=\beta\det\left[\ldots;\boldsymbol{b};\ldots\right]+\psi\det\left[\ldots;\boldsymbol{c};\ldots\right]$$

Example

Then, in the 2 dimensional case $\mathbb{R}^2$

$$\begin{vmatrix} a+\alpha & c \\ b+\beta & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} + \begin{vmatrix} \alpha & c \\ \beta & d \end{vmatrix};$$

therefore

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = \begin{vmatrix} a & c \\ 0 & d \end{vmatrix} + \begin{vmatrix} & c \\ & d \end{vmatrix}.$$

5 minors and cofactors

Definition minors and cofactors

We denote a submatrix of $\mathbf{A}$ *obtained by deleting row* i *and column* j *of* $\mathbf{A}$ *by*

$${}^{i\upharpoonleft}\mathbf{A}^{\upharpoonright j};$$

Its determinant is called the minor of a_{ij}. *And*

$$\mathrm{cof}_{ij}\left(\mathbf{A}\right)=(-1)^{i+j}\det\left({}^{i\upharpoonleft}\mathbf{A}^{\upharpoonright j}\right)$$

is called the cofactor of a_{ij}.

Example

For $\mathbf{A}=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, we have

$${}^{1\upharpoonleft}\mathbf{A}^{\upharpoonright 2}=\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}, \qquad {}^{3\upharpoonleft}\mathbf{A}^{\upharpoonright 3}=\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

hence

$$\mathrm{cof}_{12}\left(\mathbf{A}\right)=(-1)^{1+2}\det\left({}^{1\upharpoonleft}\mathbf{A}^{\upharpoonright 2}\right)=(-1)\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}.$$

and

$$\mathrm{cof}_{33}\left(\mathbf{A}\right)=(-1)^{3+3}\det\left({}^{3\upharpoonleft}\mathbf{A}^{\upharpoonright 3}\right)=\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}.$$

6 Expansion by cofactors

Theorem [Laplace expansion]

For $\mathbf{A}$ n *by* n, $\det(\mathbf{A})$ *may be computed as the sum of the products of the elements of any column (row) of* $\mathbf{A}$ *by their cofactors:*

$$\det(\mathbf{A})=\sum_{i=1}^{n} a_{ij}\,\mathrm{cof}_{ij}\left(\mathbf{A}\right), \qquad \textit{the expansion by the } j\textit{th column}$$

or

$$\det(\mathbf{A})=\sum_{j=1}^{n} a_{ij}\,\mathrm{cof}_{ij}\left(\mathbf{A}\right), \qquad \textit{the expansion by the } i\textit{th row}$$

Exercise 8. Compute $\det \mathbf{A} = \begin{vmatrix} 2 & 0 & 3 & 2 \\ 5 & 1 & 2 & 4 \\ 3 & 0 & 1 & 2 \\ 5 & 3 & 2 & 1 \end{vmatrix}$

7 Cramer's Rule

$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b}; \qquad |\mathbf{A}| \neq 0 \quad \text{then}$$

$$\boldsymbol{b} = (\mathbf{A}_{|1})x_1 + \cdots + (\mathbf{A}_{|j})x_j + \cdots + (\mathbf{A}_{|n})x_n.$$

$$\det\left[\mathbf{A}_{|1}; \ \ldots \ \overbrace{\boldsymbol{b}}^{\text{pos. } j}; \ \ldots \ \mathbf{A}_{|n}\right] = x_j \cdot \det(\mathbf{A}).$$

$$x_j = \frac{\det\left[\mathbf{A}_{|1}; \ \ldots \ \overbrace{\boldsymbol{b}}^{\text{pos. } j}; \ \ldots \ \mathbf{A}_{|n}\right]}{\det(\mathbf{A})}.$$

Computational issues when $\det \mathbf{A} \simeq 0$ (tiny angle between vectors)

8 The inverse of a matrix

$$[\mathbf{Adj}(\mathbf{A})] \cdot \mathbf{A} =$$

$$\begin{bmatrix} \operatorname{cof}_{11}(\mathbf{A}) & \operatorname{cof}_{21}(\mathbf{A}) & \cdots & \operatorname{cof}_{n1}(\mathbf{A}) \\ \operatorname{cof}_{12}(\mathbf{A}) & \operatorname{cof}_{22}(\mathbf{A}) & \cdots & \operatorname{cof}_{n2}(\mathbf{A}) \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{cof}_{1n}(\mathbf{A}) & \operatorname{cof}_{2n}(\mathbf{A}) & \cdots & \operatorname{cof}_{nn}(\mathbf{A}) \end{bmatrix} \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{\mathbf{A}}$$

Questions of the Lecture 15

(L-15) **Question 1.** Complete the proofs of the exercises of this lecture.

(L-15) **Question 2.** Consider $\mathbf{A} = \left[\mathbf{A}_{|1}; \quad \mathbf{A}_{|2}; \quad \mathbf{A}_{|3};\right]$ with $\det \mathbf{A} = 2$.

(a) What are $\det(2\mathbf{A})$ and $\det \mathbf{A}^{-1}$?

(b) What is $\det\left[(3\mathbf{A}_{|1} + 2\mathbf{A}_{|2}); \quad \mathbf{A}_{|3}; \quad \mathbf{A}_{|2};\right]$

(L-15) **Question 3.** The determinant of the 1000 by 1000 matrix $\mathbf{A}$ is 12. What is the determinant of $-\mathbf{A}^{\intercal}$? (Careful: No credit for the wrong sign.) (*MIT Course 18.06 Quiz 2, Fall, 2008*)

(L-15) **Question 4.** Consider the squared matrix $\mathbf{A}$. True or false? (to receive full credit you must explain your answer in a clear and concise way) $|\mathbf{A}\mathbf{A}^{\intercal}| = |\mathbf{A}|^2$.

(L-15) **Question 5.** We have a 3×3 matrix $\mathbf{A} = \begin{bmatrix} a & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$ with $\det \mathbf{A} = 3$.

Compute the determinant of the following matrices:

(a) (0.5 pts) $\begin{bmatrix} a-2 & 1 & 2 \\ b-4 & 3 & 4 \\ c-6 & 5 & 6 \end{bmatrix}$

(b) (0.5 pts) $\begin{bmatrix} 7a & 7 & 14 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$

(c) (1 pts) $(2\mathbf{A})^{-1}\mathbf{A}^{\intercal}$

(d) (0.5 pts) $\begin{bmatrix} a-2 & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$

(L-15) QUESTION 6.

(a) Escalone la matriz $\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 5 & 2 \\ 4 & 6 & 0 \end{bmatrix}$.

(b) ¿Es $\mathbf{A}$ invertible?

(c) En caso afirmativo calcule $|\mathbf{A}^{-1}|$; en caso contrario calcule $|\mathbf{A}|$

(d) La matriz $\mathbf{C}$ es igual al producto de $\mathbf{A}$ con la *traspuesta* de la matriz $\mathbf{B}$, es decir

$$\mathbf{C} = \mathbf{A}\mathbf{B}^{\intercal} \qquad \text{donde} \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

¿Cuánto vale el determinante de $\mathbf{C}$? ¿Es $\mathbf{C}$ invertible?

(L-15) QUESTION 7. What is the determinant of the following matrices using Laplace expansions.

(a) $\begin{bmatrix} 1 & 2 \\ -4 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & -2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 1 & -2 \end{bmatrix}$

(L-15) QUESTION 8. Compute the following determinant using Laplace expansions:

$$\begin{vmatrix} 0 & 0 & 0 & 3 & 0 \\ -2 & 0 & 0 & 2 & 0 \\ 8 & -1 & 0 & -7 & 2 \\ -1 & 2 & 2 & 3 & 2 \\ 2 & 2 & 3 & 6 & 4 \end{vmatrix}$$

(L-15) QUESTION 9. Compute $\det \mathbf{A} = \begin{vmatrix} 2 & 2 & 0 & 0 \\ 5 & 5 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{vmatrix}$

(L-15) QUESTION 10. Compute the value of $\det \mathbf{A}$ using Laplace expansion

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 3 & \cdots & 3 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & n \end{bmatrix}$$

(L-15) QUESTION 11. Consider a n by n matrix $\mathbf{A}_n$ full of 3s in its diagonal, and twos just below the diagonal, and another 2 at the position $(1, n)$; for example, for $n = 4$:

$$\mathbf{A}_4 = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 2 & 3 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix}.$$

(a) Find, using the cofactors of the first row, the determinant of $\mathbf{A}_4$.

(b) Find the determinant of $\mathbf{A}_n$ for $n > 4$.

(L-15) QUESTION 12. Consider the following block matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}$$

Prove $|\mathbf{A}| = |\mathbf{B}||\mathbf{C}|$.

Hint

$$\begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}$$

(L-15) QUESTION 13. Solve the following linear systems using Cramer's Rule

(a) $\begin{cases} 2x + 5y = 1 \\ x + 4y = 2 \end{cases}$.

(b) $\begin{cases} 2x + y = 1 \\ x + 2y + z = 0 \\ y + 2z = 0 \end{cases}$.

(exercise 13 from section 4.4 of Strang (2006))

(L-15) QUESTION 14. Find the inverse of the following matrices using the *adjoint matrix*

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

(exercise 18 from section 4.4 of Strang (2006))

(L-15) QUESTION 15. Consider the matrices

$\mathbf{A} = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 2 & 3 & 3 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & a \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$; and the vector $\boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

(a) (0.5^{pts}) For wich values of a the matrix $\mathbf{A}$ is invertible?
(b) (1^{pts}) Consider $a = 5$. Using the Cramer's rule, compute the fourth coordinate x_4 of $\boldsymbol{x}$ for linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.
(c) (1^{pts}) Compute $\mathbf{B}^{-1}$. Use the matrix $\mathbf{B}^{-1}$ to solve $\mathbf{B}\boldsymbol{x} = \boldsymbol{b}$.

Strang, G. (2006). *Linear algebra and its applications*. Thomson Learning, Inc., fourth ed. ISBN 0-03-010567-6.