

## Mathematics II

Marcos Bujosa

Universidad Complutense de Madrid

14/01/2023

1 / 44

### 1 Highlights of Lesson 15

Always **squared** matrices in this topic

#### Highlights of Lesson 15

- **Eigenvalues, eigenvectors** (prefix eigen is the German word for innate, distinct, self)
- $|\mathbf{A} - \lambda \mathbf{I}| = 0$  *Characteristic equation*
- $\text{tr}(\mathbf{A})$ ,  $\det \mathbf{A}$  (demo in the next lesson)

2 / 44

– Marcos Bujosa. Copyright © 2008–2023

Algunos derechos reservados. Esta obra está bajo una licencia de Creative Commons Reconocimiento-CompartirIgual 4.0 Internacional. Para ver una copia de esta licencia, visite <http://creativecommons.org/licenses/by-sa/4.0/> o envíe una carta a Creative Commons, 559 Nathan Abbott Way, Stanford, California 94305, USA.

### 2 Eigenvalues and eigenvectors

Consider the equation

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \quad (\text{with } \mathbf{x} \neq \mathbf{0})$$

- *Eigenvalue* is any  $\lambda$  such that there are solutions.
- Such *non-null* solutions  $\mathbf{x}$  are called *eigenvectors*.  
 $\mathbf{x} \neq \mathbf{0}$  such that  $\mathbf{A}\mathbf{x}$  is *multiple*  $\mathbf{x}$

When  $\lambda$  is 0, What are the eigenvectors?

1 / 44

3 / 44

### 3 Example: projection matrix

- Orthogonal projection
- Which vectors are eigenvectors?  
What vectors are projected in the same starting direction?
- What are the eigenvalues of those eigenvectors?
- are there any other eigenvectors? with what eigenvalue?
- Two eigen-spaces

4 / 44

### 4 Another example: Interchange or swap matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- A vector that does not change after interchange?
- What is the eigenvalue?
- Is there an eigenvector corresponding to  $\lambda_2 = -1$ ?

$$\mathbf{A}\mathbf{x}_2 = -\mathbf{x}_2$$

$$\text{Note: } \text{tr}(\mathbf{A}) = 0 = \lambda_1 + \lambda_2; \quad \det \mathbf{A} = -1 = \lambda_1 \cdot \lambda_2.$$

5 / 44

### 5 how to find eigenvalues and eigenvectors?

How to solve

$$\mathbf{A}\mathbf{x} = \underbrace{\lambda}_{?} \underbrace{\mathbf{x}}_{?} ?$$

Here's the trick (simple idea). Bring the  $\mathbf{x}$  s onto the same side ...

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} =$$

idea If  $\mathbf{x} \neq \mathbf{0}$  what kind of matrix must be  $(\mathbf{A} - \lambda \mathbf{I})$ ?

and then its determinant must be?  $|\mathbf{A} - \lambda \mathbf{I}| =$

6 / 44

### 6 how to find eigenvalues and eigenvectors?

1. Eigenvalues are  $\lambda$ 's such that:  $|\mathbf{A} - \lambda \mathbf{I}| =$   
( Characteristic polynomial  $P_{\mathbf{A}}(\lambda)$  )
2. How to compute  $\mathbf{x}$  so that  $(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$ ?

Eigenspace (Set of eigenvectors +  $\mathbf{0}$ ):

$$\mathcal{E}_{\lambda}(\mathbf{A}) = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \lambda \mathbf{x} \right\}$$

Spectrum: set  $\{\lambda_1, \dots, \lambda_k\}$  of eigenvalues (roots of  $P_{\mathbf{A}}(\lambda)$ )

7 / 44

### 7 Example (we must compute the eigenvalues first!)

We are looking for a null determinant (Characteristic polynomial)

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}; \quad \det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{vmatrix} = (3 - \lambda)^2 - 1 = 0$$

Note:  $\text{tr}(\mathbf{A}) = 6 = \lambda_1 + \lambda_2; \quad \det \mathbf{A} = 8 = \lambda_1 \cdot \lambda_2.$

8 / 44

### 8 Example (...and then the eigenspaces)

And now we compute the null space  $\mathcal{N}(\mathbf{A} - \lambda \mathbf{I})$  ... for each  $\lambda$ .

For  $\lambda_1 = 4$

$$(\mathbf{A} - 4\mathbf{I}) = \begin{bmatrix} 3 - 4 & 1 \\ 1 & 3 - 4 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow$$

For  $\lambda_2 = 2$

$$(\mathbf{A} - 2\mathbf{I}) = \begin{bmatrix} 3 - 2 & 1 \\ 1 & 3 - 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow$$

Are they the only two eigenvectors?

$$\mathbf{A}\mathbf{x}_i = \lambda\mathbf{x}_i; \quad \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \mathbf{x}_i = \lambda\mathbf{x}_i.$$

9 / 44

### 9 Another example: 90° rotation matrix

$$\mathbf{Q} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- How much do the eigenvalues add up to?
- What is the determinant?

Difficulties

$$\lambda_1 + \lambda_2 = 0 \quad \text{and} \quad \lambda_1 \cdot \lambda_2 = 1 \quad (+) \cdot (-) = (+)?$$

What kind of vector can be parallel to itself after a 90° rotation?

$$\det(\mathbf{Q} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 =$$

10 / 44

### 10 There are even worse examples

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

- Eigenvalues

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 3 - \lambda & 1 \\ 0 & 3 - \lambda \end{vmatrix} = (3 - \lambda)(3 - \lambda) = 0 \quad \begin{cases} \lambda_1 = 3 \\ \lambda_2 = 3 \end{cases}$$

- Eigenvectors

- for  $\lambda_1$ :  $(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}_1; \quad \mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- for  $\lambda_2$ :

$\lambda = 3$  is repeated twice, but  $\dim \mathcal{E}_3(\mathbf{A}) = 1$

$$\mu(3) = 2 \neq 1 = \gamma(3)$$

11 / 44

**Summary:**

1. The eigenvalues are those numbers  $\lambda$  that makes the matrix  $(\mathbf{A} - \lambda \mathbf{I})$  singular. In other words, they are the roots of the Characteristic polynomial:  $\det(\mathbf{A} - \lambda \mathbf{I})$ .
2. Any  $n$  by  $n$  matrix has a characteristic polynomial of degree  $n$
3. A polynomial of degree  $n$  has  $n$  roots (perhaps some repeated roots).
4. The sum of eigenvalues of a matrix equals its trace
5. The product of eigenvalues of a matrix equals its determinant
6. The eigenvectors associated with  $\lambda$  are the **non-zero** vectors in  $\mathcal{N}(\mathbf{A} - \lambda \mathbf{I})$ .

12 / 44

**Questions of the Lecture 15**

(L-15) QUESTION 1. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} -3 & 4 & -4 \\ -3 & 5 & -3 \\ -1 & 2 & 0 \end{bmatrix}$$

(a) The three eigenvalues of  $\mathbf{A}$  are  $-1$ ,  $1$  and  $2$ ; and two of its eigenvectors are

$$\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}; \quad \mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Check that both vectors are eigenvectors of  $\mathbf{A}$ . What are the corresponding eigenvalues?

(b) Find a third linearly independent eigenvector.

(L-15) QUESTION 2. Find the eigenvalues and eigenvectors of

(a)

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

12 / 44

(b)

$$\mathbf{B} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

(Strang, 2006, exercise 12 from section 5.1.)

(L-15) QUESTION 3. If  $\mathbf{B}$  has eigenvalues 1, 2, 3,  $\mathbf{C}$  has eigenvalues 4, 5, 6, and  $\mathbf{D}$  has eigenvalues 7, 8, 9, what are the eigenvalues of the 6 by 6 matrix  $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}$ ? where  $\mathbf{B}$ ,  $\mathbf{C}$ ,  $\mathbf{D}$  are upper triangular matrices.  
(Strang, 2006, exercise 13 from section 5.1.)

(L-15) QUESTION 4. Find the eigenvalues and eigenvectors of

(a)

$$\mathbf{A} = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

(b)

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

(Strang, 2006, exercise 5 from section 5.1.)

12 / 44

(L-15) QUESTION 5. The eigenvalues of  $\mathbf{A}$  equal the eigenvalues of  $\mathbf{A}^T$ . This is because  $\det(\mathbf{A} - \lambda \mathbf{I})$  equals  $\det(\mathbf{A}^T - \lambda \mathbf{I})$ .

(a) That is true because \_\_\_\_\_

(b) Show by an example that, nevertheless, the eigenvectors of  $\mathbf{A}$  and  $\mathbf{A}^T$  are not the same.

(Strang, 2006, exercise 11 from section 5.1.)

(L-15) QUESTION 6. Consider the matrix  $\mathbf{B}$  and its eigenvector  $\mathbf{x}$  associated to the eigenvalue  $\lambda$ , that is  $\mathbf{B}\mathbf{x} = \lambda\mathbf{x}$ ; and also consider the matrix  $\mathbf{A} = (\mathbf{B} + \alpha \mathbf{I})$ . Prove that  $\mathbf{x}$  is also an eigenvector of  $\mathbf{A}$  with eigenvalue  $(\lambda + \alpha)$ .

(L-15) QUESTION 7.

(a) Encuentre los autovalores y los auto-vectores de la matriz  $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$ .

Compruebe que la traza es igual a la suma de los autovalores, y que el determinante es igual a su producto.

(b) Si consideramos una nueva matriz, generada a partir de la anterior como

$$\mathbf{B} = (\mathbf{A} - 7\mathbf{I}) = \begin{bmatrix} -6 & -1 \\ 2 & -3 \end{bmatrix}.$$

¿Cuáles son los autovalores y auto-vectores de la nueva matriz, y como están relacionados con los de  $\mathbf{A}$ ?

(Strang, 2006, exercise 1 and 3 from section 5.1.)

12 / 44

(L-15) QUESTION 8. Suponga que  $\lambda$  es un auto-valor de  $\mathbf{A}$ , y que  $\mathbf{x}$  es un auto-vector tal que  $\mathbf{Ax} = \lambda\mathbf{x}$ .

- (a) Demuestre que ese mismo  $\mathbf{x}$  es un auto-vector de  $\mathbf{B} = \mathbf{A} - 7\mathbf{I}$ , y encuentre el correspondiente auto-valor de  $\mathbf{B}$ .
- (b) Suponga que  $\lambda \neq 0$  ( y que  $\mathbf{A}$  es invertible), demuestre que  $\mathbf{x}$  también es un auto-vector de  $\mathbf{A}^{-1}$ , y encuentre el correspondiente auto-valor. ¿Qué relación tiene con  $\lambda$ ?

(Strang, 2006, exercise 7 from section 5.1.)

(L-15) QUESTION 9. Suponga que  $\mathbf{A}$  es una matriz de dimensiones  $n \times n$ , y que  $\mathbf{A}^2 = \mathbf{A}$ . ¿Qué posibles valores pueden tomar los autovalores de  $\mathbf{A}$ ?

(L-15) QUESTION 10. Suponga la matriz  $\mathbf{A}$  con autovalores 1, 2 y 3. Si  $\mathbf{v}_1$  es un auto-vector asociado al auto-valor 1,  $\mathbf{v}_2$  al auto-valor 2 y  $\mathbf{v}_3$  al auto-valor 3; entonces ¿cuanto es  $\mathbf{A}(\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3)$ ?

(L-15) QUESTION 11. Proporcione un ejemplo que muestre que los auto-valores pueden cambiar cuando un múltiplo de una columna se resta de otra. ¿Por qué los pasos de eliminación no modifican los autovalores nulos?

(Strang, 2006, exercise 6 from section 5.1.)

(L-15) QUESTION 12. El polinomio característico de una matriz  $\mathbf{A}$  se puede factorizar como

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda).$$

Demuestre, partiendo de esta factorización, que el determinante de  $\mathbf{A}$  es igual al producto de sus valores propios (autovalores). Para ello haga una elección inteligente del valor de  $\lambda$ .

(Strang, 2006, exercise 8 from section 5.1.)

(L-15) QUESTION 13. Calcule los valores característicos (autovalores o valores propios) y los vectores característicos de  $\mathbf{A}$  y  $\mathbf{A}^2$ :

$$\mathbf{A} = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{y} \quad \mathbf{A}^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}$$

$\mathbf{A}^2$  tiene los mismos \_\_\_\_\_ que  $\mathbf{A}$ . Cuando los autovalores de  $\mathbf{A}$  son  $\lambda_1$  y  $\lambda_2$ , los autovalores de  $\mathbf{A}^2$  son \_\_\_\_\_.

(Strang, 2006, exercise 22 from section 5.1.)

(L-15) QUESTION 14. Suponga que los valores característicos de  $\mathbf{A}$  son 1, 2 y 4, ¿cuál es la traza de  $\mathbf{A}^2$ ? ¿Cuál es el determinante de  $(\mathbf{A}^{-1})^T$ ?

(Strang, 2006, exercise 10 from section 5.2.)

(L-15) QUESTION 15. The equation  $(\mathbf{A}^2 - 4\mathbf{I})\mathbf{x} = \mathbf{b}$  has no solution for some right-hand side  $\mathbf{b}$ . Give as much information as possible about the eigenvalues of the matrix  $\mathbf{A}$  (the matrix  $\mathbf{A}$  is diagonalizable).

(L-15) QUESTION 16. You are given the matrix

$$\mathbf{A} = \begin{bmatrix} 0.5 & 0.2 & 0.2 \\ 0.1 & 0.5 & 0.5 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$$

One of the eigenvalues is  $\lambda = 1$ . What are the eigenvalues of  $\mathbf{A}$ ? [Hint: Very little calculation required! You should be able to see another eigenvalue by inspection of the form of  $\mathbf{A}$ , and the third by an easy calculation. You shouldn't need to compute  $\det(\mathbf{A} - \lambda\mathbf{I})$  unless you really want to do it the hard way.]

## 1 Highlights of Lesson 16

### Highlights of Lesson 16

- Similar matrices:  $\mathbf{C} = \mathbf{S}^{-1}\mathbf{AS}$
- Triangular block diagonalizing a matrix

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{I} \end{bmatrix} \xrightarrow[\text{esp}(\tau_1^{-1} \dots \tau_p^{-1})]{\tau_1 \dots \tau_p} \begin{bmatrix} \mathbf{C} \\ \mathbf{S} \end{bmatrix} \quad \text{where} \quad \mathbf{S} = \mathbf{I}_{\tau_1 \dots \tau_p}.$$

- Diagonalizable matrices: when  $\mathbf{C}$  is diagonal.

## 2 Similar matrices

### Similarity

**A** and **C** are *similar* if there is an invertible **S** such that

$$\mathbf{C} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$$

If **A** and **C** are similar (see demos in the book):

- The same determinant:  $\det \mathbf{A} = \det \mathbf{C}$
- The same characteristic polynomial:  $|\mathbf{A} - \lambda \mathbf{I}| = |\mathbf{C} - \lambda \mathbf{I}|$
- The same eigenvalues (same *algebraic* and *geometric* multiplicities).
- The same trace.

Mirror inverse transf.:  $(\mathbf{I}_{(\tau_1 \dots \tau_k)})^{-1} = \text{esp}(\tau_k^{-1} \dots \tau_1^{-1}) \mathbf{I}$

$$\mathbf{I} = \tau_1 \dots \tau_k \begin{bmatrix} \tau \\ (-\alpha)j+i \end{bmatrix} \tau_{(k+1)} \dots \tau_p \begin{bmatrix} \tau \\ (\alpha)i+j \end{bmatrix} \mathbf{I} = \tau_1 \dots \tau_k \begin{bmatrix} \tau \\ (\frac{1}{\alpha})j \end{bmatrix} \tau_{(k+1)} \dots \tau_p \begin{bmatrix} \tau \\ (\alpha)j \end{bmatrix} \mathbf{I} \Rightarrow \mathbf{A} \text{ similar to } \tau_1 \dots \tau_p$$

14 / 44

## 3 Block diagonalizing a matrix (toothed matrix)

Consider  $\mathbf{A} = \left[ \begin{array}{c|c} \mathbf{C} & \\ \hline * & \mathbf{L} \end{array} \right] \in \mathbb{C}^{n \times n}$  where

**C** (of order  $m$ ) is *singular* and **L** is *full rank lower triangular*; then there exists an invertible **R** such that

$$\mathbf{R}^{-1}\mathbf{A}\mathbf{R} = \left[ \begin{array}{c|c} * & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline * & \begin{matrix} d_{m+1} & \beta_{m+1} \\ d_{m+2} & * & \beta_{m+2} \\ \vdots & * & * & \ddots \\ d_n & * & * & \dots & \beta_n \end{matrix} \end{array} \right]$$

$$\tau_1 \dots \tau_k \begin{pmatrix} \dots & [(-\alpha_j) \tau_{m+j}] \dots \end{pmatrix} \tau_{(k+1)} \dots \tau_p \begin{pmatrix} \dots & [(\alpha_j) \tau_{j+m}] \dots \end{pmatrix} \mathbf{A}; \quad j = 1, \dots, m-1.$$

15 / 44

## 4 Block diagonalizing a matrix (toothed matrix)

Consider  $\mathbf{A} = \left[ \begin{array}{c|c} \mathbf{C} & \\ \hline * & \mathbf{L} \end{array} \right] \in \mathbb{C}^{n \times n}$  where

**C** (of order  $m$ ) is singular and **L** is full rank lower triangular, then there exists **S** = **RP** (invertible) such that

$$\mathbf{P}^{-1}\mathbf{R}^{-1}\mathbf{A}\mathbf{R}\mathbf{P} = \left[ \begin{array}{c|c} * & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline * & \begin{matrix} 0 & \beta_{m+1} \\ 0 & * & \beta_{m+2} \\ \vdots & * & * & \ddots \\ 0 & * & * & \dots & \beta_n \end{matrix} \end{array} \right]$$

$$\tau_1 \dots \tau_k \begin{pmatrix} \dots & [(-\alpha_j) \tau_{m+j}] \dots \end{pmatrix} \tau_{(k+1)} \dots \tau_p \begin{pmatrix} \dots & [(\alpha_j) \tau_{j+m}] \dots \end{pmatrix} \mathbf{R}^{-1}\mathbf{A}\mathbf{R}; \quad j = m+1, \dots, n.$$

16 / 44

## 5 A very simple example

### Example

Consider  $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}$  with eigenvalues 0, 1 and 1.

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{I} \end{bmatrix} \xrightarrow[\text{OI}]{(-)} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\begin{matrix} \tau \\ [(1)1+2] \\ [(2)3+2] \\ [2 \rightleftharpoons 3] \end{matrix}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\begin{matrix} \tau \\ [2 \rightleftharpoons 3] \\ [(-2)2+3] \\ [(-1)2+1] \end{matrix}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow[\text{OI}]{(+)} \begin{bmatrix} \mathbf{C} \\ \mathbf{S} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{diagonal}}$$

17 / 44

### 6 A not so simple example

#### Example

Consider  $\mathbf{A} = \begin{bmatrix} -2 & 0 & 3 \\ 3 & -2 & -9 \\ -1 & 2 & 6 \end{bmatrix}$  with eigenvalues 1, 1 and 0.

$$\begin{array}{l}
 \xrightarrow[11]{(-)} \begin{bmatrix} -3 & 0 & 3 \\ 3 & -3 & -9 \\ -1 & 2 & 5 \end{bmatrix} \xrightarrow[\tau]{[(1)1+3], [(-2)2+3]} \begin{bmatrix} -3 & 0 & 0 \\ 3 & -3 & 0 \\ -1 & 2 & 0 \end{bmatrix} \xrightarrow[\tau]{[(2)3+2], [(-1)3+1]} \begin{bmatrix} -2 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow[11]{(+)} \begin{bmatrix} -2 & -2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \\
 \xrightarrow[11]{(-)} \begin{bmatrix} -2 & -2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow[\tau]{[(-1)1+2]} \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 3 & 0 \end{bmatrix} \xrightarrow[\tau]{[(1)2+1]} \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow[11]{(+)} \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \\
 \xrightarrow[01]{(-)} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \xrightarrow[\tau]{[(-1)2+1], [(4)3+1]} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow[\tau]{[(-4)1+3], [(1)1+2]} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & -1 & 1 \end{bmatrix} \xrightarrow[01]{(+)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -9 & 1 & -2 \end{bmatrix} \xrightarrow[\tau]{[(-4)1+3], [(1)1+2]} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & -1 & 1 \end{bmatrix}
 \end{array}$$

18 / 44

### 7 Every matrix is similar to a toothed matrix

For every  $\mathbf{A}$  there exists  $\mathbf{S}$  such that

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{C} \Rightarrow \mathbf{A}\mathbf{S} = \mathbf{S}\mathbf{C}$$

where  $\mathbf{C}$ , toothed, has the eigenvalues on the diagonal

#### Example

$$\begin{bmatrix} 6 & -1 & 1 \\ -9 & 1 & -2 \\ 4 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -2 & 0 & 3 \\ 3 & -2 & -9 \\ -1 & 2 & 6 \end{bmatrix} \begin{bmatrix} 6 & -1 & 1 \\ -9 & 1 & -2 \\ 4 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}}_{\text{toothed}}$$

#### Consequences

- $\sum \lambda_i = \text{tr}(\mathbf{A})$  and  $\prod \lambda_i = \det \mathbf{A}$
- $\mathbf{A}\mathbf{S}_{|j} = \mathbf{S}\mathbf{C}_{|j} \Rightarrow$  for  $j$  such that  $\mathbf{C}_{|j} = \lambda_i \mathbf{I}_{|j}$ :  
 $\mathbf{A}(\mathbf{S}_{|j}) = \lambda_i(\mathbf{S}_{|j}) \Rightarrow \mathbf{S}_{|j}$  is an eigenvector.

19 / 44

### 8 Back to the simple, "toothless" example

Consider  $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}$  with eigenvalues 0, 1 and 1.

$$\begin{array}{l}
 \xrightarrow[01]{(-)} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow[\tau]{[(1)1+2], [(2)3+2], [2 \leftrightarrow 3]} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[\tau]{[2 \leftrightarrow 3], [(-2)2+3], [(-1)2+1]} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[01]{(+)} \begin{bmatrix} \mathbf{C} \\ \mathbf{S} \end{bmatrix}
 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}(\mathbf{S}_{|j}) = \lambda_i(\mathbf{S}_{|j}) \Rightarrow \mathbf{S}_{|j} \text{ is an eigenvector.}$$

20 / 44

### 9 Diagonalizable matrices

- A matrix is diagonalizable **if and only if** algebraic and geometric multiplicities are equal for each eigenvalue
- If there are no repeated eigenvalues, there are no "teeth" either
- When there are no repeated eigenvalues  $\mathbf{A}$  is diagonalizable  
 $n \times n$   
(is sure to have  $n$  independent eigenvectors)

21 / 44

## 10 Diagonalizing a matrix

- Find the spectrum:  $\{\lambda_1, \lambda_2, \dots\}$
- Find the *algebraic multiplicity* of each eigenvalue:  $\mu(\lambda_i)$

then choose one of these alternatives:

- teething the matrix (implemented in NAcAL)
- ... or for every  $\lambda_i$ 
  - find the eigenspace

$$\mathcal{E}_{\lambda_i}(\mathbf{A}) = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \lambda_i \mathbf{x} \right\} = \mathcal{N}(\mathbf{A} - \lambda_i \mathbf{I}).$$

- check  $\mu(\lambda_i) = \dim \mathcal{E}_{\lambda_i}(\mathbf{A})$  (algebraic and geometric multiplicities are equal)

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix}; \quad \mathbf{S} = [\text{Basis for } \mathcal{E}_{\lambda_1}(\mathbf{A}) \# \dots \# \text{Basis for } \mathcal{E}_{\lambda_k}(\mathbf{A})]$$

$$\mathbf{S}^{-1} \mathbf{A} \mathbf{S} = \mathbf{D} \quad \Leftrightarrow \quad \mathbf{A} = \mathbf{S} \mathbf{D} \mathbf{S}^{-1}$$

22 / 44

## 11 Matrix powers

If  $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$  then  $\mathbf{A}^2\mathbf{x} = \mathbf{A}\mathbf{A}\mathbf{x} = \mathbf{A}(\lambda\mathbf{x}) = \lambda\mathbf{A}\mathbf{x} =$

- What can I say about the eigenvectors?
- What is the relationship between the eigenvalues of  $\mathbf{A}$  and those of  $\mathbf{A}^2$

In a matrix form (if  $\mathbf{A}$  is diagonalizable,  $\mathbf{A} = \mathbf{S} \mathbf{D} \mathbf{S}^{-1}$ ):

$$\mathbf{A}^2 = \mathbf{S} \mathbf{D} \mathbf{S}^{-1} \mathbf{S} \mathbf{D} \mathbf{S}^{-1} = \mathbf{S} \mathbf{D}^2 \mathbf{S}^{-1}$$

In general, for,  $n \in \mathbb{Z}$ ,  $n \geq 0$ ...  $\mathbf{A}^n =$   
what about  $\mathbf{A}$  both diagonalizable and invertible?

23 / 44

## Questions of the Lecture 16

(L-16) QUESTION 1. Factor these two matrices into  $\mathbf{S} \mathbf{D} \mathbf{S}^{-1}$ ;

(a)  $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

(b)  $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

(Strang, 2006, exercise 15 from section 5.2.)

(L-16) QUESTION 2. Which of these matrices cannot be diagonalized?

(a)

$$\mathbf{A}_1 = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$$

(b)

$$\mathbf{A}_2 = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}$$

(c)

$$\mathbf{A}_3 = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

(Strang, 2006, exercise 5 from section 5.2.)

23 / 44

(L-16) QUESTION 3. If  $\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$  find  $\mathbf{A}^{100}$  by diagonalizing  $\mathbf{A}$ .

(Strang, 2006, exercise 7 from section 5.2.)

(L-16) QUESTION 4. If the eigenvalues of  $\mathbf{A}$  are 1, 1 and 2, which of the following are certain to be true? Give a reason if true or a counterexample if false:

- (a)  $\mathbf{A}$  is invertible.  
 (b)  $\mathbf{A}$  is diagonalizable.  
 (c)  $\mathbf{A}$  is not diagonalizable

(Strang, 2006, exercise 11 from section 5.2.)

(L-16) QUESTION 5. Let  $\mathbf{A}$  be the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

- (a) (1<sup>pts</sup>) Determine if  $\mathbf{A}$  is diagonalizable, and if so, diagonalize it.  
 (b) (0.5<sup>pts</sup>) Compute  $(\mathbf{A}^6)\mathbf{v}$ , where  $\mathbf{v} = (0, 0, 0, 1)^T$ .  
 (c) (0.5<sup>pts</sup>) Using the the eigenvalues found in part (a) justify that  $\mathbf{A}$  is invertible.  
 (d) (0.5<sup>pts</sup>) What is the relation between the eigenvalues of  $\mathbf{A}$  and the eigenvalues of  $\mathbf{A}^{-1}$ ?

23 / 44



(L-16) QUESTION 6. Si  $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$ ; entonces  $\mathbf{A}^3 = ( \quad )( \quad )( \quad )$  y  $\mathbf{A}^{-1} = ( \quad )( \quad )( \quad )$ .  
(Strang, 2006, exercise 16 from section 5.2.)

(L-16) QUESTION 7. Considere la matriz

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

- (a) Encuentre los autovalores de  $\mathbf{A}$
- (b) Encuentre los auto-vectores de  $\mathbf{A}$
- (c) Diagonalice  $\mathbf{A}$ : escríbalo como  $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$ .

(L-16) QUESTION 8. ¿Falso o verdadero? Si los autovalores de  $\mathbf{A}$  son 2, 2 y 3 entonces sabemos que la matriz es

- (a) Invertible
- (b) Diagonalizable
- (c) No diagonalizable.

(L-16) QUESTION 9. Sean las matrices

$$\mathbf{A}_1 = \begin{bmatrix} 8 & \\ & 2 \end{bmatrix}; \quad \mathbf{A}_2 = \begin{bmatrix} 9 & 4 \\ & 1 \end{bmatrix}; \quad \mathbf{A}_3 = \begin{bmatrix} 10 & 5 \\ -5 & \end{bmatrix}$$

(a) Encuentre los autovalores y auto-vectores de la matriz  $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ .

(b) Explique por qué (o por qué no) la matriz  $\mathbf{A}$  es diagonalizable.

(L-16) QUESTION 14. Sea  $\mathbf{A}$  una matriz  $3 \times 3$ . Asuma que sus autovalores son 1 y 0, que una base de los autovectores asociados a  $\lambda = 1$  son  $[1, 0, 1]$  y  $[0, 0, 1]$ ; mientras que los asociados a  $\lambda = 0$  son paralelos a  $[1, 1, 2]$ .

- (a) ¿Es  $\mathbf{A}$  diagonalizable? En caso afirmativo escriba la matriz diagonal  $\mathbf{D}$  y la matriz  $\mathbf{S}$  tales que  $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$ .
- (b) Encuentre  $\mathbf{A}$ .

(L-16) QUESTION 15. Let  $\mathbf{A}$  be a  $2 \times 2$  matrix such that  $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$  is an eigenvector for  $\mathbf{A}$  with eigenvalue 2, and  $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$  is another eigenvector for  $\mathbf{A}$  with eigenvalue -2. If  $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ , compute  $(\mathbf{A}^3)\mathbf{v}$ .

- (a) Complete dichas matrices de modo que en los tres casos  $\det \mathbf{A}_i = 25$ . Así, la traza es en todos los casos igual a 10, y por tanto para las tres matrices el único auto-valor  $\lambda = 5$  está repetido dos veces ( $\lambda^2 = 25$  y  $\lambda + \lambda = 10$  implica  $\lambda = 5$ ).
- (b) Encuentre un vector característico con  $\mathbf{A}\mathbf{x} = 5\mathbf{x}$ . Estas tres matrices no son diagonalizable porque no hay un segundo auto-vector linealmente independiente del primero.

(Strang, 2006, exercise 27 from section 5.2.)

(L-16) QUESTION 10. Factorice las siguientes matrices en  $\mathbf{S}\mathbf{D}\mathbf{S}^{-1}$

$$(a) \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$(b) \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

(Strang, 2006, exercise 1 from section 5.2.)

(L-16) QUESTION 11. Encuentre la matriz  $\mathbf{A}$  cuyos autovalores son 1 y 4, cuyos autovectores son  $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$  y  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$  respectivamente.

(Strang, 2006, exercise 2 from section 5.2.)

(L-16) QUESTION 12. Si los elementos diagonales de una matriz triangular superior de orden  $3 \times 3$  son 1, 2 y 7, ¿puede saber si la matriz es diagonalizable? ¿Quién es  $\mathbf{D}$ ?  
(Strang, 2006, exercise 4 from section 5.2.)

(L-16) QUESTION 13.

## 1 Highlights of Lesson 17

### Highlights of Lesson 17

- Symetric matrices  $\mathbf{A} = \mathbf{A}^T$ 
  - Eigenvalues and eigenvectors
- Introd. positive Definiteness matrices

## 2 Symmetric matrices $\mathbf{A} = \mathbf{A}^T$

what's special about  $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$  when  $\mathbf{A}$  is symmetric?  
 $n \times n$

1. A symmetric matrix has only **REAL EIGENVALUES**
2.  $n$  **EIGENVECTORS** can be chosen **ORTHOGONAL**  
 (always diagonalizable)

The usual diagonalizable case:

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D} \iff \mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$$

Symmetric case:

I can choose perpendicular unit eigenvectors (**orthonormal** columns of  $\mathbf{S} = \mathbf{Q}$ )

$$(\text{if } \mathbf{A} = \mathbf{A}^T) \quad \mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}^{-1} = \mathbf{Q}\mathbf{D}\mathbf{Q}^T \quad \text{Spectral thm.}$$

Orthogonally diagonalizable.

25 / 44

## 3 Eigenspaces are orthogonal for symmetric matrices

Eigenvectors (corresponding to different eigenvalues) of a symmetric matrix are orthogonal.

Proof.

Consider  $\mathbf{A}\mathbf{x} = \lambda_1\mathbf{x}$  and  $\mathbf{A}\mathbf{y} = \lambda_2\mathbf{y}$  (with  $\lambda_1 \neq \lambda_2$ ). then

$$\lambda_1\mathbf{x} \cdot \mathbf{y} = \mathbf{A}\mathbf{x} \cdot \mathbf{y} = \mathbf{x}(\mathbf{A}^T)\mathbf{y} = \mathbf{x}\mathbf{A}\mathbf{y} = (\mathbf{x} \cdot \mathbf{y})\lambda_2.$$

Since  $\lambda_1 \neq \lambda_2$  then:

$$\lambda_1(\mathbf{x} \cdot \mathbf{y}) - \lambda_2(\mathbf{x} \cdot \mathbf{y}) = 0 \implies (\lambda_1 - \lambda_2)\mathbf{x} \cdot \mathbf{y} = 0 \implies \mathbf{x} \cdot \mathbf{y} = 0.$$

□

26 / 44

## 4 Quadratic forms

Quadratic form:

$$\mathbf{x}\mathbf{A}\mathbf{x}; \quad \text{with } \mathbf{A}^T = \mathbf{A}$$

Since  $\mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}^T$  (with  $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{I}$ ), then

$$\mathbf{x}\mathbf{A}\mathbf{x} = \mathbf{x}\mathbf{Q}\mathbf{D}\mathbf{Q}^T\mathbf{x} = (\mathbf{Q}^T\mathbf{x})\mathbf{D}(\mathbf{Q}^T\mathbf{x}) \quad (\text{weighted sum of squares})$$

Positive definite quadratic form:

$$\mathbf{x}\mathbf{A}\mathbf{x} > 0 \quad \forall \mathbf{x} \neq \mathbf{0} \iff \lambda_i > 0, \quad i = 1 : n.$$

then we also say  $\mathbf{A}$  is positive definite.

27 / 44

## 5 Positive definite matrices

Meaning:

$$\mathbf{x}\mathbf{A}\mathbf{x} > 0 \quad (\text{except for } \mathbf{x} = \mathbf{0})$$

Some properties

Consider a positive definite symmetric  $\mathbf{A}$ : What about  $\mathbf{A}^{-1}$ ?

$$\mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}^{-1} = \mathbf{Q}\mathbf{D}\mathbf{Q}^T$$

Consider two positive definite symmetric matrices  $\mathbf{A}$ ,  $\mathbf{B}$ : What about  $\mathbf{A} + \mathbf{B}$ ?

the answer must be...

28 / 44

## 6 The matrix product $\mathbf{A}^T \mathbf{A}$

Consider the rectangular matrix  $\mathbf{A}$ . Is  $\mathbf{A}^T \mathbf{A}$  positive definite?  
 $m \times n$

$$x(\mathbf{A}^T \mathbf{A})x =$$

It can only be 0 when  $\mathbf{A}x$  is 0

How can we guarantee that  $\mathbf{A}x \neq 0$  when  $x \neq 0$ ?

29 / 44

## 8 Positive definite symmetric matrices

- All eigenvalues are:
- All pivots are:

$$\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$$

**Pivots:**

What is the sign of each eigenvalue?

$$\lambda^2 - 8\lambda + 11 = 0 \rightarrow \lambda = 4 \pm \sqrt{5} > 0$$

31 / 44

## 7 Symmetric matrices: signs of eigenvalues

are all  $\lambda_i$  positive? are they negative?

Computing eigenvalues of  $\mathbf{A}$  is impossible in general! (5th degree polynomial)  
 $5 \times 5$

**Good news:** The signs of the pivots of echelon form are the same as the signs of the eigenvalues  $\lambda_i$   
 (if we do not change the sign of the determinant with *Type II* elementary transformations)

num. of positive pivots = num. of positive eigenvalues

30 / 44

**Summary** (for symmetric matrices):

1. Symmetric matrices have *real eigenvalues* and *perpendicular eigenvectors* can be chosen
2.  $\mathbf{A} = \mathbf{Q} \mathbf{D} \mathbf{Q}^T$  where  $\mathbf{Q}$  is orthogonal
3.  $\mathbf{A}$  is symmetric if and only if it is *orthogonally* diagonalizable
4. The signs of the pivots in the echelon form are same as the signs of the eigenvalues  $\lambda_i$  (only if we do not change the sign of the determinant with *Type II* elementary transformations)

32 / 44

## Questions of the Lecture 17

(L-17) QUESTION 1. Write  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{C}$  in the form  $\mathbf{QDQ}^T$  of the spectral theorem:

(a)  $\mathbf{A} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

(b)  $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c)  $\mathbf{C} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

(Strang, 2006, exercise 11 from section 5.5.)

(L-17) QUESTION 2. Find the eigenvalues and the unit eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(Strang, 2003, exercise 3 from section 6.4.)

(L-17) QUESTION 3. Find an orthonormal  $\mathbf{Q}$  that diagonalizes this symmetric matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$$

32 / 44

(L-17) QUESTION 6. Sean

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{B} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

- (a) Encuentre los valores característicos de  $\mathbf{A}$  (recuerde que  $i^2 = -1$ ).
- (b) Encuentre los valores característicos de  $\mathbf{B}$  (en este caso quizá le resulte más sencillo encontrar primero los autovectores, y deducir entonces los autovalores).
- (c) De los siguientes tipos de matrices: ortogonales, invertibles, permutación, hermíticas, de rango 1. diagonalizables, de Markov ¿a qué tipos pertenece  $\mathbf{A}$ ?
- (d) ¿y  $\mathbf{B}$ ?

(Strang, 2006, exercise 14 from section 5.5.)

(L-17) QUESTION 7. Si  $\mathbf{A}^3 = \mathbf{0}$  entonces los autovalores de  $\mathbf{A}$  deben ser \_\_\_\_\_. De un ejemplo tal que  $\mathbf{A} \neq \mathbf{0}$ . Ahora bien, si  $\mathbf{A}$  es además simétrica, demuestre que entonces  $\mathbf{A}^3$  es necesariamente  $\mathbf{0}$ .

32 / 44

(Strang, 2003, exercise 5 from section 6.4.)

(L-17) QUESTION 4. Suppose  $\mathbf{A}$  is a symmetric 3 by 3 matrix with eigenvalues 0, 1, 2.

- (a) What properties can be guaranteed for the corresponding unit eigenvectors  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$
- (b) In terms of  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{w}$ , describe the nullspace, left nullspace, row space, and column space of  $\mathbf{A}$ .
- (c) Find a vector  $\mathbf{x}$  that satisfies  $\mathbf{Ax} = \mathbf{v} + \mathbf{w}$ . Is  $\mathbf{x}$  unique?
- (d) Under what conditions on  $\mathbf{b}$  does  $\mathbf{Ax} = \mathbf{b}$  have a solution?
- (e) If  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{w}$  are the columns of  $\mathbf{S}$ , what are  $\mathbf{S}^{-1}$  and  $\mathbf{S}^{-1}\mathbf{AS}$ .

(Strang, 2006, exercise 13 from section 5.5.)

(L-17) QUESTION 5. Escriba un hecho destacado sobre los valores característicos de cada uno de estos tipos de matrices:

- (a) Una matriz simétrica real.
- (b) Una matriz diagonalizable tal que  $\mathbf{A}^n \rightarrow \mathbf{0}$  cuando  $n \rightarrow \infty$ .
- (c) Una matriz no diagonalizable
- (d) Una matriz singular

(Strang, 2006, exercise 16 from section 5.5.)

32 / 44

(L-17) QUESTION 8. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} a & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

- (a) Prove that  $\mathbf{A}$  is not diagonalizable when  $a = 3$ .
- (b) Is  $\mathbf{A}$  diagonalizable when  $a = 2$ ? (explain). If it is diagonalizable, find an eigenvalue diagonal matrix  $\mathbf{D}$  and an eigenvector matrix  $\mathbf{S}$  such as  $\mathbf{A} = \mathbf{SDS}^{-1}$ .
- (c) Is  $\mathbf{A}^T\mathbf{A}$  diagonalizable for any value  $a$ ? Is it possible to find a full set of orthonormal eigenvectors of  $\mathbf{A}^T\mathbf{A}$ ?
- (d) Find all posible values  $a$  such as  $\mathbf{A}$  is invertible and diagonalizable.

(L-17) QUESTION 9. Sea la matriz

$$\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};$$

- (a) Expresé  $\mathbf{B}$  en la forma  $\mathbf{B} = \mathbf{A} = \mathbf{QDQ}^T$  del teorema espectral.
- (b) ¿Es  $\mathbf{B}$  diagonalizable? Si no lo es, diga las razones; y en caso contrario genere una matriz  $\mathbf{S}$  que diagonalice a  $\mathbf{B}$ .

32 / 44

## 1 Highlights of Lesson 18

### Highlights of Lesson 18

- Positive and Negative (semi)definite matrices
- Completing the squares
- Diagonalization by congruence

33 / 44

### Example

What number do I have to put there for the matrix  $\mathbf{A}$  to be singular?

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & ? \end{bmatrix}$$

- Eigenvalues:
- Leading principal minors:
- For the following quadratic form

$$q_{\mathbf{A}}(x) = x\mathbf{A}x = (x, y) \begin{bmatrix} 2 & 6 \\ 6 & ? \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 12xy + ?y^2$$

Is there a  $x \neq 0$  such that  $x\mathbf{A}x = 0$ ?

35 / 44

## 2 Quadratic forms

- Positive definite:  $\forall x \neq 0 \Rightarrow x\mathbf{A}x > 0$ .
- Positive semi-definite:  $\forall x \neq 0 \Rightarrow x\mathbf{A}x \geq 0$ .
- Negative definite:  $\forall x \neq 0 \Rightarrow x\mathbf{A}x < 0$ .
- Negative semi-definite:  $\forall x \neq 0 \Rightarrow x\mathbf{A}x \leq 0$ .
- Indefinite: neither positive semi-definite, nor negative semi-definite.

34 / 44

### Example

If  $\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & ? \end{bmatrix}$  then  $(x, y) \begin{bmatrix} 2 & 6 \\ 6 & ? \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 12xy + ?y^2$

- Are there numbers  $x$  and  $y$  that make  $x\mathbf{A}x$  negative?
- Does the function go through the origin?
- When  $y = 0$  and  $x = 1$ , is it positive? (and when  $x = -1$ ?)
- When  $x = 0$  and  $y = 1$ , is it positive? (and when  $y = -1$ ?)
- Is it always positive?

$(0, 0)$  **saddle point**: minimum in some directions, maximum in others.

$$\lambda_1 = -2, \quad \begin{pmatrix} -6 \\ 4 \end{pmatrix}; \quad \lambda_2 = 11, \quad \begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

36 / 44

### Example

If  $\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$  then  $(x, y) \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 12xy + 20y^2$

Positive definite.

Does it pass the tests?

- Are the leading principal minors positive?
- Are the eigenvalues positive?

$$q_{\mathbf{A}}(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x} > 0 \quad \text{for all } \mathbf{x} \neq \mathbf{0}$$

37 / 44

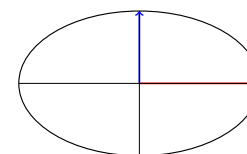
### 3 Completing the squares

If we could express  $q(\mathbf{x})$  as a sum of squares, we would know whether  $q(\mathbf{x})$  is positive definite.

Let's complete the square!

- $q(x, y) = 2x^2 + 12xy + 20y^2 = 2(x + ?y)^2 + ?$
- $q(x, y) = 2x^2 + 12xy + 7y^2$
- $q(x, y) = 2x^2 + 12xy + 18y^2$
- $q(x, y) = 2x^2 + 12xy + 20y^2$  (graph)

If positive definite:  $q(x, y) = a; \quad a > 0$ : ellipse



is

$$q(x, y, z, w, t) = 2t^2 - 2tx - 2tz + w^2 - 2wy + 2x^2 - 2xy + 2y^2 + z^2$$

positive definite? 😞😞😞😞😞!!!!???

38 / 44

### 4 Congruent matrices

$\mathbf{A}$  and  $\mathbf{C}$  are congruent if there exists an invertible  $\mathbf{B}$  such that

$$\mathbf{C} = \mathbf{B}^T \mathbf{A} \mathbf{B}$$

Diagonalization by congruence

For each  $\mathbf{A}$  (symmetric) exists  $\mathbf{B} = \mathbf{I}_{\tau_1 \dots \tau_k}$  (invertible) such that

$$\mathbf{D} = \mathbf{B}^T \mathbf{A} \mathbf{B} \quad \text{is diagonal} \quad (\mathbf{B}^T = \tau_k \dots \tau_1 \mathbf{I})$$

Spectral Theorem: |Diagonalization by similarity and congruence!

$$\mathbf{D} = \mathbf{Q}^{-1} \mathbf{A} \mathbf{Q} = \mathbf{Q}^T \mathbf{A} \mathbf{Q}.$$

Hence, every quadratic form can be written as a sum of squares

$$\mathbf{x}^T \mathbf{A} \mathbf{x} = \mathbf{x} (\mathbf{B}^{-1})^T \mathbf{D} \mathbf{B}^{-1} \mathbf{x} = \mathbf{y}^T \mathbf{D} \mathbf{y}; \quad \text{where} \quad \mathbf{y} = \mathbf{B}^{-1} \mathbf{x}.$$

39 / 44

### 5 Completing the squares

$$2x^2 + 12xy + 20y^2$$

$$\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \xrightarrow{[(-3)1+2]} \begin{bmatrix} 2 & 0 \\ 6 & 2 \end{bmatrix} \xrightarrow{[(-3)1+2]} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix};$$

therefore, we get:

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \mathbf{D} = \mathbf{E}^T \mathbf{A} \mathbf{E} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

hence  $\mathbf{A} = (\mathbf{E}^T)^{-1} \mathbf{D} \mathbf{E}^{-1}$  so

$$\begin{aligned} \mathbf{x}^T \mathbf{A} \mathbf{x} &= (x, y) \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (\mathbf{x} (\mathbf{E}^{-1})^T) \mathbf{D} (\mathbf{E}^{-1} \mathbf{x}) \\ &= ((x + 3y), y) \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{pmatrix} x + 3y \\ y \end{pmatrix} = 2(x + 3y)^2 + 2y^2 \end{aligned}$$

40 / 44

### 6 example 3 by 3

Is  $\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$  positive definite?

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{\begin{bmatrix} \tau \\ (\frac{1}{2})1+2 \\ (\frac{1}{2})1+2 \end{bmatrix}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{\begin{bmatrix} \tau \\ (\frac{2}{3})2+3 \\ (\frac{2}{3})2+3 \end{bmatrix}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

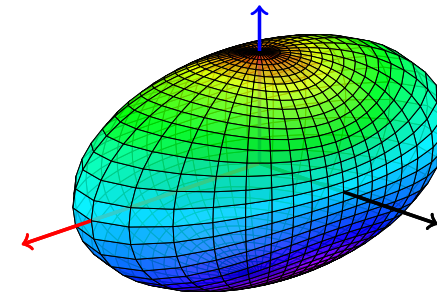
$$\mathbf{xAx} = 2x^2 + 2y^2 + 2z^2 - 2xy - 2yz > 0$$

$$\mathbf{xAx} = 1 : (\text{ellipsoid}) \text{ axes are eigenvectors } \mathbf{A} = \mathbf{Q}^T \boldsymbol{\Lambda} \mathbf{Q}$$

41 / 44

### 7 Positive definite matrices and ellipsoids: example 3 by 3

- The region  $(\mathbf{xAx} = a)$  is an (ellipsoid).
- The eigenvectors of  $\mathbf{Q}$  are in the direction of the three principal axes.
- Lengths of axes determined by the eigenvalues



42 / 44

### 8 Another example 3 by 3

Is  $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$  positive definite?

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{\begin{bmatrix} \tau \\ (1)3+1 \\ (1)3+1 \end{bmatrix}} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{\begin{bmatrix} \tau \\ (-\frac{1}{2})1+3 \\ (-\frac{1}{2})1+3 \end{bmatrix}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \xrightarrow{\begin{bmatrix} \tau \\ [2 \leftrightarrow 3] \\ [2 \leftrightarrow 3] \end{bmatrix}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Indefinite matrix

43 / 44

### 9 "Classification" of quadratic

$$\mathbf{xAx} \leq 0; \text{ for all } \mathbf{x} \neq \mathbf{0}$$

#### Methods

Check the signs of

- Elem. diag.:  $\mathbf{D} = \mathbf{B}^T \mathbf{A} \mathbf{B}$  (Diagonalization by congruence) 😊
- Computing eigenvalues: (Roots of a polynomial) 😊
- Leading principal minors: (Sylvester's criterion) 😊

#### Law of inertia

the number of positive, negative and zero entries of the diagonal of  $\mathbf{D}$  is an invariant of  $\mathbf{A}$ , i.e. it does not depend on  $\mathbf{B}$

(Orthogonal diagonalization  $\mathbf{D} = \mathbf{Q}^T \mathbf{A} \mathbf{Q}$  is a special case)

44 / 44

## Questions of the Lecture 18

(L-18) QUESTION 1. Decide for or against the positive definiteness of these matrices, and write out the corresponding quadratic form  $f = \mathbf{x} \mathbf{A} \mathbf{x}$  :

- (a)  $\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$   
 (b)  $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$   
 (c)  $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$   
 (d)  $\begin{bmatrix} -1 & 2 \\ 2 & -8 \end{bmatrix}$

(e) The determinant in (b) is zero; along what line is  $f(x, y) = 0$ ?

(Strang, 2006, exercise 2 from section 6.1.)

(L-18) QUESTION 2. What is the quadratic  $f = ax^2 + 2bxy + cy^2$  for each of these matrices? Complete the square to write  $f$  as a sum of one or two squares  $d_1(\quad)^2 + d_2(\quad)^2$ .

- (a)  $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix}$   
 (b)  $\mathbf{B} = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$

(Strang, 2006, exercise 15 from section 6.1.)

44 / 44

(L-18) QUESTION 3. Which one of the following matrices has two positive eigenvalues? Test  $a > 0$  and  $ac > b^2$ , don't compute the eigenvalues.  $\mathbf{x} \mathbf{A} \mathbf{x} < 0$ .

- (a)  $\mathbf{A} = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix}$   
 (b)  $\mathbf{B} = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix}$   
 (c)  $\mathbf{C} = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix}$   
 (d)  $\mathbf{D} = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}$

(Strang, 2006, exercise 14 from section 6.1.)

(L-18) QUESTION 4. Show that  $f(x, y) = x^2 + 4xy + 3y^2$  does not have a minimum at  $(0, 0)$  even though it has positive coefficients. Write  $f(x, y)$  as a difference of squares and find a point  $(x, y)$  where  $f(x, y)$  is negative.  
 (Strang, 2006, exercise 16 from section 6.1.)

(L-18) QUESTION 5. Show from the eigenvalues that if  $\mathbf{A}$  is positive definite, so is  $\mathbf{A}^2$  and so is  $\mathbf{A}^{-1}$ .

(Strang, 2006, exercise 4 from section 6.2.)

44 / 44

(L-18) QUESTION 6. Consider the following quadratic forms

$$\begin{aligned} q_1(x, y, z) &= x^2 + 4y^2 + 5z^2 - 4xy. \\ q_2(x, y, z) &= -x^2 + 4y^2 + z^2 + 2xy - 2axz. \end{aligned}$$

- (a) Show that  $q_1(x, y, z)$  is positive semi-definite.  
 (b) Find, if it is possible, any value of  $a$  such as  $q_2(x, y, z)$  is negative definite.

(L-18) QUESTION 7. Decide for or against the positive definiteness of

- (a)  $\mathbf{A} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$   
 (b)  $\mathbf{B} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$   
 (c)  $\mathbf{C} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}^2$

(Strang, 2006, exercise 2 from section 6.2.)

44 / 44

(L-18) QUESTION 8. Consider the following quadratic form

$$q(x, y, z) = x^2 + 6xy + y^2 + az^2;$$

Decide for which values  $a$  the quadratic form is positive definite, negative definite, semidefinite, or indefinite.

(L-18) QUESTION 9. Si  $\mathbf{A} = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$  es definida positiva, pruebe que  $\mathbf{A}^{-1}$  es definida positiva.

(Strang, 2006, exercise 8 from section 6.1.)

(L-18) QUESTION 10. Si una matriz simétrica de 2 por 2 satisface  $a > 0$ , y  $ac > b^2$ , demuestre que sus autovalores son reales y positivos (definida positiva). Emplee la ecuación característica y el hecho de que el producto de los autovalores es igual al determinante.  
 (Strang, 2006, exercise 3 from section 6.1.)

44 / 44



(L-18) QUESTION 11. Decida si las siguientes matrices son definidas positivas, definidas negativas, semi-definidas, o indefinidas.

$$(a) \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{bmatrix}$$

$$(b) \mathbf{B} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 6 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 3 \end{bmatrix}$$

$$(c) \mathbf{C} = -\mathbf{B}$$

$$(d) \mathbf{D} = \mathbf{A}^{-1}$$

(L-18) QUESTION 12. Una matriz definida positiva no puede tener un cero (o incluso peor; un número negativo) en su diagonal principal. Demuestre que esta matriz no cumple  $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0$ , para todo  $\mathbf{x} \neq \mathbf{0}$ :

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{no es positiva cuando} \quad \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$$

(Strang, 2006, exercise 21 from section 6.2.)

(L-18) QUESTION 13. Demuestre que si  $\mathbf{A}$  y  $\mathbf{B}$  son definidas positivas entonces  $\mathbf{A} + \mathbf{B}$  también es definida positiva. Para esta demostración los pivotes y los valores característicos no son convenientes. Es mejor emplear  $\mathbf{x}^T (\mathbf{A} + \mathbf{B}) \mathbf{x} > 0$  (Strang, 2006, exercise 5 from section 6.2.)

(L-18) QUESTION 14. Find the  $\mathbf{LDL}^T$  factorization for the following symmetric matrices.

(a)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

(L-18) QUESTION 15. La forma cuadrática  $f(x, y) = 3(x + 2y)^2 + 4y^2$  es definida positiva. Encuentre la matriz  $\mathbf{A}$ , factorícela en  $\mathbf{LDL}^T$ , y relacione los elementos en  $\mathbf{D}$  y  $\mathbf{L}$  con 3, 2 y 4 en  $f$ . (Strang, 2006, exercise 9 from section 6.1.)

(L-18) QUESTION 16. Consider the following matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & a & a \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- (a) (0.5pts) Compute the eigenvalues of  $\mathbf{A}$ .
- (b) (0.5pts) Prove that when  $a = 2$  the matrix  $\mathbf{A}$  is not diagonalisable.
- (c) (1pts) For matrix  $\mathbf{B}$ , find a diagonal matrix  $\mathbf{D}$  and an orthonormal matrix  $\mathbf{P}$  such as  $\mathbf{B} = \mathbf{PDP}^T$ .
- (d) (0.5pts) Find the quadratic form  $f(x, y, z)$  associated to  $\mathbf{B}$ , and prove it is positive defined.

(L-18) QUESTION 17. Given the matrix  $\mathbf{A} = \begin{pmatrix} a & 3/5 \\ b & 4/5 \end{pmatrix}$ , compute the values (if they exist) of  $a$  and  $b$  such as

- (a) (0.5pts)  $\mathbf{A}$  is ortho-normal.
- (b) (0.5pts) Columns of  $\mathbf{A}$  are linearly independent.
- (c) (0.5pts)  $\lambda = 0$  is an eigenvalue of  $\mathbf{A}$ .
- (d) (0.5pts)  $\mathbf{A}$  is a symmetric definite negative matrix.

(L-18) QUESTION 18.

- (a) Consider the quadratic form  $q(x, y, z) = x^2 + 2xy + ay^2 + 8z^2$  and find its corresponding symmetric matrix  $\mathbf{Q}$ ; determine if  $\mathbf{Q}$  is positive-definite, positive-semidefinite, negative-definite, negative-semidefinite or indefinite when the parameter  $a$  is equal to one ( $a = 1$ ).
- (b) If  $a \neq 1$ , determine whether the matrix is positive-definite, positive-semidefinite, negative-definite, negative-semidefinite or indefinite.

## Questions of the Optional Lecture 2

(L-OPT-2) QUESTION 1. Consider the following matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- (a) (0.5pts) Prove  $\mathbf{A}$  is invertible if and only if  $a \neq 0$ .
- (b) (0.5pts) Is  $\mathbf{A}$  positive definite when  $a = 1$ ? Explain your answer.
- (c) (1pts) Compute  $\mathbf{A}^{-1}$  when  $a = 2$ .
- (d) (0.5pts) How many variables can be chosen as pivot (or exogenous) variables in the system  $\mathbf{A}\mathbf{x} = \mathbf{o}$  when  $a = 0$ ? Which ones?

(L-OPT-2) QUESTION 2. True or false (to receive full credit you must explain your answers in a clear and concise way)

- (a) If  $\mathbf{A}$  is symmetric, then so it is  $\mathbf{A}^2$ .
- (b) If  $\mathbf{A}^2 = \mathbf{A}$  then  $(\mathbf{I} - \mathbf{A})^2 = (\mathbf{I} - \mathbf{A})$  where  $\mathbf{I}$  is the identity matrix.
- (c) If  $\lambda = 0$  is an eigenvalue of the squared matrix  $\mathbf{A}$ , then the linear system  $\mathbf{A}\mathbf{x} = \mathbf{0}$  is always solvable and has only one solution.
- (d) If  $\lambda = 0$  is an eigenvalue of the squared matrix  $\mathbf{A}$ , then the linear system  $\mathbf{A}\mathbf{x} = \mathbf{b}$  could be unsolvable.
- (e) If a matrix is orthogonal (perpendicular columns of norm one), then so it is the inverse of that matrix.

44 / 44

(L-OPT-2) QUESTION 4. En las preguntas siguientes  $\mathbf{A}$  y  $\mathbf{B}$  son matrices  $n \times n$ . Indique si las siguientes afirmaciones son verdaderas o falsas (incluya una breve explicación, o un contra ejemplo que justifique su respuesta):

- (a) Si  $\mathbf{A}$  no es cero entonces  $\det(\mathbf{A}) \neq 0$
- (b) Si  $\det(\mathbf{AB}) \neq 0$  entonces  $\mathbf{A}$  es invertible.
- (c) Si intercambio las dos primeras filas de  $\mathbf{A}$  sus autovalores cambian.
- (d) Si  $\mathbf{A}$  es real y simétrica, entonces sus autovalores son reales (**aquí no es necesaria una justificación**).
- (e) Si la forma reducida de echelon de  $(\mathbf{A} - 5\mathbf{I})$  es la matriz identidad, entonces 5 no es un autovalor de  $\mathbf{A}$ .
- (f) Sea  $\mathbf{b}$  un vector columna de  $\mathbb{R}^n$ . Si el sistema  $\mathbf{A}\mathbf{x} = \mathbf{b}$  no tiene solución, entonces  $\det(\mathbf{A}) \neq 0$
- (g) Sea  $\mathbf{C}$  de orden  $3 \times 5$ . El rango de  $\mathbf{C}$  puede ser 4.
- (h) Sea  $\mathbf{C}$  de orden  $n \times m$ , y  $\mathbf{b}$  un vector columna de  $\mathbb{R}^n$ . Si  $\mathbf{C}\mathbf{x} = \mathbf{b}$  no tiene solución, entonces  $\text{rg}(\mathbf{C}) < n$ .
- (i) Toda matriz diagonalizable es invertible.
- (j) Si  $\mathbf{A}$  es invertible, entonces su forma reducida de echelon es la matriz identidad.

44 / 44

- (f) If 1 is the only eigenvalue of a  $2 \times 2$  matrix  $\mathbf{A}$ , then  $\mathbf{A}$  must be the identity matrix  $\mathbf{I}$ .

(L-OPT-2) QUESTION 3. complete los blancos, o responda Verdadero/Falso.

- (a) Cualquier sistema generador de un espacio vectorial contiene una base del espacio (V/F)

- (b) Que los vectores  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  sean linealmente independientes significa que

- (c) El conjunto que sólo contiene el vector  $\mathbf{0}$  es un conjunto linealmente independiente. (V/F)

- (d) Una matriz cuadrada de orden  $n$  por  $n$  es diagonalizable cuando:

- (e) Si  $\mathbf{u} = (1, 2, -1, 1)$ , entonces  $\|\mathbf{u}\| =$

- (f) Si  $\mathbf{u} = (1, 2, -1, 1)$  y  $\mathbf{v} = (-2, 1, 0, 0)$ , entonces  $\mathbf{u} \cdot \mathbf{v} =$

44 / 44

(L-OPT-2) QUESTION 5. Sean

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & 5 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 0 & 4 \\ 0 & 0 & 6 \end{bmatrix}; \quad \mathbf{D} = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Los autovalores de  $\mathbf{B}$  son 0 y 2. Use esta información para responder a las siguientes cuestiones. Para cada matriz debe dar una explicación. Puede haber más de una matriz que cumpla la condición:

- (a) ¿Qué matrices son invertibles?
- (b) ¿Qué matrices tienen un autovalor repetido?
- (c) ¿Qué matrices tienen rango menor a tres?
- (d) ¿Qué matrices son diagonalizables?
- (e) ¿Para qué matrices diagonalizables podemos encontrar tres autovectores ortogonales entre sí?

(L-OPT-2) QUESTION 6. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

- (a) Find the eigenvalues and eigenvectors of  $\mathbf{A}$ .
- (b) Is  $\mathbf{A}$  diagonalizable?

44 / 44

- (c) Is it possible to find a matrix  $\mathbf{P}$  such as  $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T$ , where  $\mathbf{D}$  is diagonal?  
 (d) Find  $|\mathbf{A}^{-1}|$ .

(L-OPT-2) QUESTION 7. Consider a 3 by 3 matrix  $\mathbf{A}$  with eigenvalues  $\lambda_1 = 1$ ,  $\lambda_2 = 2$ , and  $\lambda_3 = -1$ ; and let  $\mathbf{v}_1 = (1, 0, 1)^T$  and  $\mathbf{v}_2 = (1, 1, 1)^T$  be the corresponding eigenvectors to  $\lambda_1$  and  $\lambda_2$ .

- (a) Is  $\mathbf{A}$  diagonalizable?  
 (b) Is  $\mathbf{v}_3 = (-1, 0, -1)^T$  an eigenvector associated to the eigenvalue  $\lambda_3 = -1$ ?  
 (c) Compute  $\mathbf{A}(\mathbf{v}_1 - \mathbf{v}_2)$ .

(L-OPT-2) QUESTION 8.

- (a) (0.5pts) Find an homogeneous system  $\mathbf{A}\mathbf{x} = \mathbf{0}$  such as its solutions set is

$$\left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \mid \exists \alpha, \beta, \gamma \in \mathbb{R} \text{ such that } \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

- (b) (0.5pts) If the characteristic polynomial of a matrix  $\mathbf{A}$  is  $p(\lambda) = \lambda^5 + 3\lambda^4 - 24\lambda^3 + 28\lambda^2 - 3\lambda + 10$ , find the rank of  $\mathbf{A}$ .

(L-OPT-2) QUESTION 9. Suponga una matriz cuadrada e invertible  $\mathbf{A}_{n \times n}$ .

- (a) ¿Cuáles son sus espacios columna  $\mathcal{C}(\mathbf{A})$  y espacio nulo  $\mathcal{N}(\mathbf{A})$ ? (no responda con la definición, diga qué conjunto de vectores compone cada espacio).  
 (b) Suponga que  $\mathbf{A}$  puede ser factorizada en  $\mathbf{A} = \mathbf{LU}$ :

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 7 & 3 & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Describe el primer paso de eliminación en la reducción de  $\mathbf{A}$  a  $\mathbf{U}$ . ¿porqué sabe que  $\mathbf{U}$  es también una matriz invertible? ¿Cuanto vale el determinante de  $\mathbf{A}$ ?

- (c) Encuentre una matriz particular de dimensiones  $3 \times 3$  e invertible  $\mathbf{A}$  que no pueda ser factorizada en la forma  $\mathbf{LU}$  (sin permutar previamente las filas). ¿Qué factorización es todavía posible en su ejemplo? (no es necesario que realice la factorización). ¿Cómo sabe que su matriz  $\mathbf{A}$  es invertible?

Strang, G. (2003). *Introduction to Linear Algebra*.

Wellesley-Cambridge Press, Wellesley, Massachusetts. USA, third ed. ISBN 0-9614088-9-8.

Strang, G. (2006). *Linear algebra and its applications*. Thomson Learning, Inc., fourth ed. ISBN 0-03-010567-6.