Quiz 2 review

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1. General comments

Exam 2 covers all topics (1, 2, 3, 4 and 5) The topics covered are (very briefly summarized):

1. All of the topics from exam 1.

2. Orthogonal complements $S^{\perp}$ for subspaces S, especially (but not only) the four fundamental subspaces.

3. Dado el conjunto de soluciones, encontar un sistema de ecuaciones; es decir, pasar de las ecuaciones paramétricas a las implícitas y viceversa.

4. What happens to the four subspaces as we do matrix operations, especially elimination steps and more generally how the subspaces of $\mathbf{AB}$ compare to those of $\mathbf{A}$ and $\mathbf{B}$. The fact (important for projection and least-squares!) that $\mathbf{A}^{\mathsf{T}}\mathbf{A}$ has the same rank as $\mathbf{A}$, the same null space as $\mathbf{A}$, and the same column space as $\mathbf{A}^{\mathsf{T}}$, and why (we proved this in class and another way in homework).

5. Orthogonal projections: given a matrix $\mathbf{A}$, the projection of $\boldsymbol{b}$ onto $\mathcal{C}\left(\mathbf{A}\right)$ is $\boldsymbol{p} = \mathbf{A}\hat{\boldsymbol{x}}$ where $\hat{\boldsymbol{x}}$ solves $\mathbf{A}^{\mathsf{T}}\mathbf{A}\hat{\boldsymbol{x}} = \mathbf{A}^{\mathsf{T}}\boldsymbol{b}$ [always solvable since $\mathcal{C}\left(\mathbf{A}^{\mathsf{T}}\mathbf{A}\right) = \mathcal{C}\left(\mathbf{A}^{\mathsf{T}}\right)$]. If $\mathbf{A}$ has full column rank, then $\mathbf{A}^{\mathsf{T}}\mathbf{A}$ is invertible and we can write the projection matrix $\mathbf{P} = \mathbf{A}(\mathbf{A}^{\mathsf{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathsf{T}}$ (so that $\mathbf{A}\hat{\boldsymbol{x}} = \mathbf{P}\boldsymbol{b}$, but it is much quicker to solve $\mathbf{A}^{\mathsf{T}}\mathbf{A}\hat{\boldsymbol{x}} = \mathbf{A}^{\mathsf{T}}\boldsymbol{b}$ by elimination than to compute $\mathbf{P}$ in general). $\boldsymbol{e} = \boldsymbol{b} - \mathbf{A}\hat{\boldsymbol{x}}$ is in $\mathcal{N}\left(\mathbf{A}^{\mathsf{T}}\right)$, and $\mathbf{I} - \mathbf{P}$ is the projection matrix onto $\mathcal{N}\left(\mathbf{A}^{\mathsf{T}}\right)$.

6. Least-squares: $\hat{\boldsymbol{x}}$ minimizes $||\mathbf{A}\boldsymbol{x} - \boldsymbol{b}||^2$ over all $\boldsymbol{x}$, and is the least-squares solution. That is, $\boldsymbol{p} = \mathbf{A}\hat{\boldsymbol{x}}$ is the *closest* point to $\boldsymbol{b}$ in $\mathcal{C}\left(\mathbf{A}\right)$. Application to least-square curve fitting, minimizing the sum of the squares of the errors.

7. Orthonormal bases, forming the columns of a matrix $\mathbf{Q}$ with $\mathbf{Q}^{\mathsf{T}}\mathbf{Q} = \mathbf{I}$.

8. Determinants: their properties, how to compute them (simple formulas for 2×2 and 3×3, usually by elimination for matrices $> 3 \times 3$), their relationship to linear equations (zero determinant = singular), their use for eigenvalue problems.

9. Eigenvalues and eigenvectors: their definition $\boxed{\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}}$, their properties, the fact that for an eigenvector the matrix (or any function of the matrix) acts just like a number. Computing from the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I})$ and $\boldsymbol{x}$ from $\mathcal{N}\left(\mathbf{A} - \boldsymbol{\lambda}\mathbf{I}\right)$; zero eigenvalues $\lambda = 0$ just correspond to $\mathcal{N}\left(\mathbf{A}\right)$. Understand (from the definition) why, if $\mathbf{A}$ has an eigenvalue λ, then $\mathbf{A}^k$ has an eigenvalue λ^k, all with the *same* eigenvector.

10. Diagonalization $\mathbf{A} = \mathbf{SDS}^{-1}$: where it comes from, its use in understanding properties of matrices and eigenvalues. The basic idea that, to solve a problem involving A, you first expand your vector in the basis of the eigenvectors ($\mathbf{S}$), then for each eigenvector you treat $\mathbf{A}$ as just a number , then at the end you add up the solutions.

11. Using eigenvalues/eigenvectors to solve problems involving matrix powers.

12. If $\mathbf{A} = \mathbf{A}^{\mathsf{T}}$ (real-symmetric), then the eigenvalues are real and the eigenvectors are orthogonal (or can be chosen orthogonal), and $\mathbf{A}$ is diagonalizable as $\mathbf{A} = \mathbf{QDQ}^{\mathsf{T}}$ for an orthogonal $\mathbf{Q}$. If $\mathbf{A} = \mathbf{B}^{\mathsf{T}}\mathbf{B}$ where $\mathbf{B}$ has full column rank, then $\mathbf{A}$ is positive definite: all $\lambda > 0$ and all pivots > 0 and $\boldsymbol{y}\mathbf{A}\boldsymbol{y} > 0$ for any $\boldsymbol{y} \neq 0$; connection to minimization problems.

13. **This year we have not seen orthogonalization problems by the Gram-Schmidt method, nor the LU factorization**

As usual, the exam questions may turn these concepts around a bit, e.g. giving the answer and asking you to work backwards towards the question, or ask about the same concept in a slightly changed context. We want to know that you have really internalized these concepts, not just memorizing an algorithm but knowing why the method works and where it came from.

2. Past intermediate exams

Below you can find the intermediate examinations from the past years. Until the academic year 12/13 I explained gaussian elimination by rows (as in the book). Therefore some questions were designed assuming row elimination. For this reason here I have introduced some variations of the exercises in order to solve everything by column reduction.

Read the instructions carefully

Quiz 2 Review)

Grading

1.-

2.-

3.-

4.-

5.-

Name: ______________________________

INSTRUCTIONS

- Put your name in the blanks above (and also in all the paper provided).
- For each question, to receive full credit you must **show all work** (I have to be able to distinguish between a student who guesses the answers and one who understands the topic). Explain your answers fully and clearly. If you add false statements to a correct argument, you will lose points.
- **No electronic devices, books or notes of any form are allowed**.

2.1. Grupo C curso 23/24

EXERCISE 1. Fill in the blanks:

(a) (0.5^{pts}) $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is solvable if, and only if, $\boldsymbol{b}$ is orthogonal to every vector in the ________ space of $\mathbf{A}$.

(b) (0.5^{pts}) $\mathcal{C}\left((\mathbf{AB})^{\intercal}\right)$ must ________ (`contain` $\supseteq$ `/ be contained in` $\subseteq$ `/ be equal to` $=$) the ______ space of ____ ($\mathbf{A}$ `or` $\mathbf{B}$) for all 4×4 matrices $\mathbf{A}$ and $\mathbf{B}$.

(c) (0.5^{pts}) If $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are both solutions to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, then the vector $\boldsymbol{x}_1 - \boldsymbol{x}_2$ must be in the ________ space of $\mathbf{A}$.

Based on MIT Course 18.06 Exam 1, Fall, 2022

EXERCISE 2. Suppose the vectors $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$ form an orthonormal basis for $\mathbb{R}^3$ and the matrix $\mathbf{A}$ satisfies $\mathbf{A}\boldsymbol{q}_1 = (1, 0, 0,)$, $\mathbf{A}\boldsymbol{q}_2 = (0, 1, 0,)$, and $\mathbf{A}\boldsymbol{q}_3 = (0, 0, 1,)$.

(a) (0.5^{pts}) Describe the matrix $\mathbf{A}$ explicitly in terms of the vectors $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$.

(b) (1^{pts}) Write down all possibilities for $|\mathbf{A}|$.

MIT Course 18.06 Final Exam, Spring, 2023

EXERCISE 3. $\mathbf{A}$ is a square matrix such that $\mathcal{N}\left(\mathbf{A}-\mathbf{I}\right) = \mathcal{L}\left(\left[\begin{pmatrix}1\\2\end{pmatrix};\right]\right)$ and $\mathcal{N}\left(\mathbf{A}-5\mathbf{I}\right) = \mathcal{L}\left(\left[\begin{pmatrix}1\\-2\end{pmatrix};\right]\right)$.

(a) (1^{pts}) Without much calculation, explain why $\mathbf{A}$ (`is / is not`) (choose 1) symmetric.

(b) (1^{pts}) What is $\mathbf{A}$? You can leave your answer as a product of matrices and/or matrix inverses without multiplying/inverting them.

MIT Course 18.06 Exam 3, Spring, 2022

EXERCISE 4. Consider the quadratic form $q(x, y, z) = \boldsymbol{x}\mathbf{A}\boldsymbol{x} = 2x^2 + 8xy + 16yz - z^2$.

(a) (0.5^{pts}) Write the matrix $\mathbf{A}$ associated with the quadratic form $\boldsymbol{x}\mathbf{A}\boldsymbol{x}$.

(b) (0.5^{pts}) Is $\mathbf{A}$ diagonalizable? Is $\mathbf{A}$ invertible? (explain your answer)

(c) (1^{pts}) Classify the quadratic form.

(d) (1^{pts}) Express the quadratic form as a sum of squares.

EXERCISE 5.

(a) (0.5^{pts}) If $|\mathbf{A}| = 5$, where $\mathbf{A} = \begin{bmatrix} 1 & 8 & 3 \\ x & y & z \\ -3 & 7 & 2 \end{bmatrix}$, what is the determinant of $\mathbf{B} = \begin{bmatrix} x & y & z \\ 1 & 8 & 3 \\ -3-4x & 7-4y & 2-4z \end{bmatrix}$?

(b) (0.5^{pts}) Find an eigenvalue of $\mathbf{C} = \begin{bmatrix} 1 & 8 & 3 \\ u & v & w \\ u+1 & v+8 & w+3 \end{bmatrix}$.

EXERCISE 6.

(a) (0.5^{pts}) Find the parametric equations of the plane passing through the points $\boldsymbol{a} = (1, 1, 0,)$, $\boldsymbol{b} = (0, 0, 1,)$ and $\boldsymbol{c} = (1, 1, 1,)$

(b) (0.5^{pts}) Find a perpendicular vector to the plane in part (a).

2.2. Grupo E curso 23/24

EXERCISE 1. Fill in the blanks:

(a) (0.5$^{\text{pts}}$) If $\mathbf{A}$ is a 4×3 matrix and $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is not solvable for some $\boldsymbol{b}$ and the solutions are not unique when they exist, then possible values for the rank of $\mathbf{A}$ are ________ (list all possibilities).

(b) (0.5$^{\text{pts}}$) $\mathcal{C}(\mathbf{AB})$ must ________ (`contain` $\supseteq$ `/ be contained in` $\subseteq$ `/ be equal to` $=$) the ______ space of ____ ($\mathbf{A}$ `or` $\mathbf{B}$) for all 4×4 matrices $\mathbf{A}$ and $\mathbf{B}$.

(c) (0.5$^{\text{pts}}$) If $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ are both solutions to $\boldsymbol{y}\mathbf{A} = \boldsymbol{c}$, then the vector $\boldsymbol{y}_1 - \boldsymbol{y}_2$ must be in the ________ space of $\mathbf{A}$.

MIT Course 18.06 Exam 1, Fall, 2022

EXERCISE 2. Let $\mathbf{A} = \mathbf{L}\mathbf{U}\mathbf{L}^{-1}\mathbf{U}^{-1}$ for $\mathbf{L} = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ 0 & 3 & 1 & \\ 1 & 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{U} = \begin{bmatrix} 2 & 0 & 1 & 1 \\ & -1 & 0 & -1 \\ & & -2 & 1 \\ & & & 1 \end{bmatrix}$.

(a) (0.5$^{\text{pts}}$) Express $\mathbf{A}^{-1}$ in terms of $\mathbf{L}$, $\mathbf{U}$, $\mathbf{L}^{-1}$, and/or $\mathbf{U}^{-1}$ (but you don't need to actually multiply or invert the terms (or even write the full matrices)!).

(b) (1$^{\text{pts}}$) What is the determinant of $\mathbf{A}$?

MIT Course 18.06 Final Exam, Spring, 2022

EXERCISE 3. Each independent question refers to the matrix $\mathbf{A} = \begin{bmatrix} 4 & 1 \\ d & 2 \end{bmatrix}$.

(a) (0.5$^{\text{pts}}$) Give a value for d such that $(2, \quad 4,)$ is an eigenvector of $\mathbf{A}$.

(b) (0.5$^{\text{pts}}$) Give a value for d such that -2 is one of the eigenvalues of $\mathbf{A}$.

(c) (1$^{\text{pts}}$) Give a value for d such that $\mathbf{A}$ is a nondiagonalizable matrix.

EXERCISE 4. (1$^{\text{pts}}$) Can you find a matrix $\mathbf{A}$ such that $[\ (1, \quad 1, \quad 1,)\ ;\ ]$ is a basis for both $\mathcal{N}(\mathbf{A}^{\mathsf{T}})$ and $\mathcal{N}(\mathbf{A})$. If "yes", give a matrix $\mathbf{A}$. If "no", briefly explain why the matrix $\mathbf{A}$ cannot exist. *MIT Course 18.06 Quiz 2, Fall 1997*

EXERCISE 5. Let $\mathcal{V}$ be the following vector subspace of $\mathbb{R}^2$: $\mathcal{V} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ tal que } 3x + 4y = 0 \right\}$

(a) (1$^{\text{pts}}$) Find a basis for $\mathcal{W} = \mathcal{V}^{\perp}$ (in other words, $\mathcal{W}$ is the orthogonal complement of $\mathcal{V}$).

(b) (0.5$^{\text{pts}}$) Find some parametric equations for the line paralel to $\mathcal{W}$ that contains $(1, \quad 1,)$.

(c) (0.5$^{\text{pts}}$) Find some cartesian equations for the line paralel to $\mathcal{W}$ that contains $(1, \quad 1,)$.

Based on MIT Course 18.06 Exam 1, Spring, 2021

EXERCISE 6. Let $\mathbf{A} = \begin{bmatrix} -2 & 0 & -2 \\ 0 & 2 & -4 \\ -2 & -4 & 2 \end{bmatrix}$.

(a) (1$^{\text{pts}}$) How many eigenvalues of $\mathbf{A}$ are positive, how many are negative, and how many are zero?

(b) (1$^{\text{pts}}$) Express the quadratic form $\boldsymbol{x}\mathbf{A}\boldsymbol{x}$ as a sum of squares.

2.3. Grupo B curso 22/23

EXERCISE 1. Consider the following quadratic forms $\begin{cases} q_{\mathsf{A}}(x,y,z) & = 4x^2+4xz+y^2+z^2 \\ q_{\mathsf{B}}(\boldsymbol{x}) & = \begin{pmatrix} x, & y, & z, \end{pmatrix} \begin{bmatrix} -1 & -1 & 1 \\ -1 & a-1 & 1-a \\ 1 & 1-a & a \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{cases}$.

(a) (1^{pts}) Show that $q_{\mathsf{A}}(x,y,z)$ is positive semi-definite.

(b) (1^{pts}) Find, if it is possible, any value of a such that $q_{\mathsf{B}}(\boldsymbol{x})$ is negative definite.

EXERCISE 2. Consider the plane $\mathcal{P}$ in $\mathbb{R}^4$ spanned by $\boldsymbol{a} = \begin{pmatrix} 1, & 0, & 2, & 1, \end{pmatrix}$ y $\boldsymbol{b} = \begin{pmatrix} 1, & 1, & 0, & 1, \end{pmatrix}$.

(a) (1^{pts}) Write some parametric equations for the plane P^* that is perpendicular to $\mathcal{P}$ and contains the point $\boldsymbol{c} = \begin{pmatrix} 1, & 1, & 1, & 1, \end{pmatrix}$.

(b) (1^{pts}) Write some Cartesian equations for the mentioned plane P^*.

EXERCISE 3. Suppose that $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are orthonormal vectors in $\mathbb{R}^2$. Find all possible values for the following determinants of 2 by 2 matrices and explain your reasoning.
Hint: *Remember the geometric interpretation of the determinant.*

(a) (1^{pts}) $\det \left[\boldsymbol{q}_1;\ \boldsymbol{q}_2;\right]$.

(b) (1^{pts}) $\det \left[(\boldsymbol{q}_1-\boldsymbol{q}_2);\ (\boldsymbol{q}_1+\boldsymbol{q}_2);\right]$.

Based on MIT Course 18.06 Quiz 2, Fall 2006

EXERCISE 4. (1^{pts}) The equation $(\mathsf{A}^2-4\mathsf{I})\boldsymbol{x} = \boldsymbol{b}$ has no solution for some right-hand side $\boldsymbol{b}$. Give as much information as possible about the eigenvalues of the matrix A. *MIT 18.06 - Quiz 3, 2009*

EXERCISE 5. *Determine whether the following statements are true or false, and explain your reasoning (to receive full credit you must explain your answers in a clear and concise way).*

If R is a reduced echelon form (by columns) of A, then

(a) (0.5^{pts}) If $\boldsymbol{x}$ is a solution to $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ then $\boldsymbol{x}$ must be a solution to $\mathsf{R}\boldsymbol{x} = \boldsymbol{b}$.

(b) (0.5^{pts}) If $\boldsymbol{x}$ is a solution to $\mathsf{A}\boldsymbol{x} = \boldsymbol{0}$ then $\boldsymbol{x}$ must be a solution to $\mathsf{R}\boldsymbol{x} = \boldsymbol{0}$.

(c) (0.5^{pts}) If $\boldsymbol{x}$ is a solution to $\boldsymbol{x}\mathsf{A} = \boldsymbol{b}$ then $\boldsymbol{x}$ must be a solution to $\boldsymbol{x}\mathsf{R} = \boldsymbol{b}$.

(d) (0.5^{pts}) If $\boldsymbol{x}$ is a solution to $\boldsymbol{x}\mathsf{A} = \boldsymbol{0}$ then $\boldsymbol{x}$ must be a solution to $\boldsymbol{x}\mathsf{R} = \boldsymbol{0}$.

EXERCISE 6.

(a) (1^{pts}) Describe all vectors that are orthogonal to the nullspace of $\mathsf{A} = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 2 & 6 \\ 2 & 1 & 4 \end{bmatrix}$.

(*You can do this without computing the nullspace*).

MIT 18.06 - Final Exam, Monday May 16th, 2005

2.4. Grupo E curso 22/23

EXERCISE 1.

(a) (1^{pts}) Let $\mathbf{A}$ such that $\mathbf{A}\begin{pmatrix}1\\0\end{pmatrix}=2\begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{A}\begin{pmatrix}2\\1\end{pmatrix}=-2\begin{pmatrix}2\\1\end{pmatrix}$. Compute $(\mathbf{A}^3)\begin{pmatrix}1\\-1\end{pmatrix}$.

(b) (1^{pts}) Consider the following quadratic form $q(x,y,z)=az^2+2x^2+8xy+y^2$ Decide for which values a the quadratic form is positive definite, negative definite, semidefinite, or indefinite.

EXERCISE 2. Consider the matrix $\mathbf{A}=\begin{bmatrix}2&0&0&0\\3&-1&0&0\\-1&1&2&0\\-1&1&0&2\end{bmatrix}$.

(a) (1^{pts}) Compute the eigenvalues of $\mathbf{A}$ and bases of the corresponding eigenspaces.
(b) (0.5^{pts}) Is $\mathbf{A}$ diagonalizable?
(c) (0.5^{pts}) Is it possible to find a matrix $\mathbf{P}$ such as $\mathbf{A}=\mathbf{PDP}^{\mathsf{T}}$, where $\mathbf{D}$ is diagonal?
(d) (0.5^{pts}) Find $|\mathbf{A}^{-1}|$.

EXERCISE 3. Consider the matrix $\mathbf{A}=\begin{bmatrix}1&a&b\\c&1&d\\e&f&1\end{bmatrix}$

(a) (0.5^{pts}) $\mathbf{A}$ has a (double) eigenvalue $\lambda=2$. What is the other eigenvalue?
(b) (1^{pts}) Rank of $(\mathbf{A}-2\mathbf{I})$ is 1. ¿Is $\mathbf{A}$ diagonalizable?

Explain your answers.

EXERCISE 4. Consider 3 linearly independent vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ in $\mathbb{R}^5$ and also 3 orthonormal vectors $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$ in $\mathbb{R}^5$ such that $\mathcal{L}(\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3)=\mathcal{L}(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3)$. Put those vectors as columns of 5 by 3 matrices so that $\mathbf{A}=[\boldsymbol{a}_1;\boldsymbol{a}_2;\boldsymbol{a}_3;]$. and $\mathbf{B}=[\boldsymbol{q}_1;\boldsymbol{q}_2;\boldsymbol{q}_3;]$

(a) (0.5^{pts}) Give formulas using $\mathbf{B}$ and $\mathbf{A}$ for the projection matrices $\mathbf{P}_A$ and $\mathbf{P}_B$ onto $\mathcal{C}(\mathbf{A})$ and $\mathcal{C}(\mathbf{B})$.
(b) (0.5^{pts}) What is $\det(\mathbf{P}_B)$?
(c) (0.5^{pts}) What is $\mathbf{P}_B$ times $\mathbf{B}$? What is $\mathbf{P}_B$ times $\mathbf{A}$?
(d) (0.5^{pts}) What is $\mathbf{P}_B$ times $\mathbf{P}_A$? What is $\mathbf{P}_A$ times $\mathbf{P}_B$?

EXERCISE 5. Consider $\boldsymbol{x}=c_1\boldsymbol{q}_1+c_2\boldsymbol{q}_2+c_3\boldsymbol{q}_3$, where $\boldsymbol{q}_i$ are three orthonormal eigenvectors of 3 by 3 matrix $\mathbf{A}$ (so $\mathbf{A}\boldsymbol{q}_i=\lambda_i\boldsymbol{q}_i$).

(a) (1^{pts}) Compute $\boldsymbol{x}\cdot\boldsymbol{x}$ in terms of the c's.
(b) (1^{pts}) Compute $\boldsymbol{x}\mathbf{A}\boldsymbol{x}$ in terms of the c's and λ's.

MIT Course 18.06 Final, Fall 2006

2.5. Grupo D curso 21/22

EXERCISE 1. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$.

(a) (1^{pts}) Find an orthogonal basis for $\mathcal{C}(\mathbf{A})$.
(b) (1^{pts}) Find orthonormal vectors $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, and $\boldsymbol{q}_3$ so that $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ form a basis for $\mathcal{C}(\mathbf{A})$.
(c) (0.5^{pts}) Which of the four fundamental subspaces of $\mathbf{A}$ contains $\boldsymbol{q}_3$?
(d) (0.5^{pts}) Find the projection matrix $\mathbf{P}$ projecting onto the left nullspace (not the column space!) of $\mathbf{A}$.
(e) (1^{pts}) Find the projection $\boldsymbol{p}$ of $\boldsymbol{v} = (1, \quad 2, \quad 7,)$ onto $\mathcal{C}(\mathbf{A})$.
(f) (1^{pts}) Find the least squares solution to $\mathbf{A}\boldsymbol{x} = (1, \quad 2, \quad 7,)$.
(g) (1^{pts}) Describe $\mathcal{C}(\mathbf{A})$ with cartesian equations.

Based on MIT Course 18.06 Exam 2, April 12, 2000

EXERCISE 2. (1^{pts}) Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}$. For what (if any) values of d does $\mathbf{A}$ have all positive eigenvalues? (Hint: Do not try to compute the eigenvalues of $\mathbf{A}$). *MIT Course 18.06 Exam 3, May 3, 2000*

EXERCISE 3. Suppose $\mathbf{A}$ is a 3 by 3 matrix with eigenvalues 0, 1, 2. Find the following (and explain your answer):

(a) (0.5^{pts}) The rank of $\mathbf{A}$.
(b) (0.5^{pts}) The determinant of $\mathbf{A}^\intercal\mathbf{A}$.
(c) (0.5^{pts}) The determinant of $\mathbf{A} + \mathbf{I}$.
(d) (0.5^{pts}) The eigenvalues of $(\mathbf{A} + \mathbf{I})^{-1}$.

MIT Course 18.06 Exam 2, April 12, 2000

EXERCISE 4. (1^{pts}) Find an invertible matrix $\mathbf{S}$ that makes $\mathbf{S}^{-1}\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{S}$ a diagonal matrix. *MIT Course 18.06 Quiz 3, Fall 1997*

2.6. Grupo E curso 21/22

Exercise 1.

(a) (1^{pts}) Suppose $\mathbf{A}$ is a symmetric matrix. If you wish substract 3 times row 1 from row 3, after that you substract 3 times column 1 from column 3, is the resulting matrix $\mathbf{B}$ still symetric? Yes or not necessarily, with a reason!

(b) (1^{pts}) Suppose $\mathbf{A}$ is a symmetric matrix. If you wish substract 3 times row 1 from row 3, after that you add 3 times column 3 to column 1, has the resulting matrix $\mathbf{B}$ the same eigenvalues? Yes or not necesarily, with a reason!

(c) (0.5^{pts}) Create a non-symmetric matrix (if possible) with eigenvalues 1, 2, and 4.

(d) (0.5^{pts}) Create a rank-one matrix (if possible) with eigenvalues 1, 2, and 4.

(e) (1^{pts}) Create a symmetric positive definite matrix (but not diagonal) with eigenvalues 1, 2, and 4.

Based on MIT Course 18.06 Final, December 21, 2000

Exercise 2. Consider the following *projection matrix*: $\mathbf{P} = \frac{1}{9}\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$

(a) (1^{pts}) What subspace does $\mathbf{P}$ project onto? (describe the subspace with parametric equations).

(b) (1^{pts}) What is the distance from that subspace to $\boldsymbol{v} = \begin{pmatrix}1, & 2, & 1,\end{pmatrix}$.

(c) (1^{pts}) What are the three eigenvalues of $\mathbf{P}$? (*hint: it is better to think than to calculate*) Is $\mathbf{P}$ diagonalizable?

Exercise 3.

(a) (1^{pts}) Find a diagonalization $\mathbf{A} = \mathbf{SDS}^{-1}$ of $\mathbf{A} = \begin{bmatrix} \frac{1}{2} & 3 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) (1^{pts}) What is the limit of $\mathbf{A}^k$ as $k \to \infty$?

Based on MIT Course 18.06 Quiz 3, December 6, 2000

Exercise 4. (1^{pts}) Suppose $\mathbf{A}$ is similar to a matrix 3 by 3 matrix $\mathbf{B}$ that has eigenvalues 1, 1, 2. What can you say about

1. the eigenvalues of $\mathbf{A}$
2. diagonalizability of $\mathbf{A}$
3. symmetry of $\mathbf{A}$. ¿Is $\mathbf{A}$ positive definite?

MIT Course 18.06 Quiz 3, December 6, 2000

2.7. Grupo D curso 20/21

EXERCISE 1.

(a) (1^{pts}) If you transpose $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D}$ you learn that

- The eigenvalues of $\mathbf{A}^{\intercal}$ are ______________________________
- The eigenvectors of $\mathbf{A}^{\intercal}$ are ______________________________

(b) (1^{pts}) Complete the last row so that $\mathbf{B}$ is a singular matrix, with real eigenvalues, and orthogonal eigenvectors:

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ x & y & z \end{bmatrix}.$$

(c) (1^{pts}) $\mathbf{C}$ is a 3 by 3 matrix. I add colum 1 to column 2 to get $\mathbf{F} = \mathbf{C}_{\underset{[(1)\mathbf{1}+\mathbf{2}]}{\tau}} = \mathbf{C}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. This probably changes the eigenvalues. What should I do to the rows of $\mathbf{F}$ (your answer could be in words) to get back to the original eigenvalues of $\mathbf{C}$?

MIT Course 18.06 Quiz 3, Spring 1997

EXERCISE 2. The system of vectors $\left[\ (1,\ 1,\ 0,);\ (2,\ 0,\ 2,);\ \right]$ is a basis for $\mathcal{N}\left(\mathbf{A}^{\intercal}\right)$.

(a) (0.5^{pts}) Find a basis for $\mathcal{C}\left(\mathbf{A}\right)$.
(b) (0.5^{pts}) Find the projection matrix $\mathbf{P}$ onto $\mathcal{C}\left(\mathbf{A}\right)$.

MIT Course 18.06 Exam II, Fall 1996

EXERCISE 3. (1^{pts}) Given that $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$, find $|\mathbf{A}|$. *MIT Course 18.06 Quiz 2, Fall 1997*

EXERCISE 4. Consider the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & a \end{bmatrix}; \qquad \text{and} \qquad \boldsymbol{b} = \begin{pmatrix} 2 \\ 6 \\ c \end{pmatrix}.$$

(a) (1^{pts}) For which values a and c is this system solvable.
(b) (1^{pts}) Solve the system when $a = 5$ and $c = 10$ (a and c have fixed values *only* in this part b)).
(c) (1^{pts}) Classify the corresponding quadratic form $\boldsymbol{a}\mathbf{X}\boldsymbol{a}$.

EXERCISE 5.

(a) (1^{pts}) Find a parametric equations for the line L passing through the points $\boldsymbol{x}_p = \left(1,\ -3,\ 1,\right)$ and $\boldsymbol{x}_q = \left(-2,\ 2,\ -2,\right)$.
(b) (1^{pts}) Find a cartesian equations for the same line.

(Lang, 1986, Example 1 in Section 1.5)

2.8. Grupo E curso 20/21

EXERCISE 1. Sea $\mathbf{A} = \begin{bmatrix} 1 & -2 & -1 & 2 \\ 2 & -4 & -1 & 3 \end{bmatrix}$.

(a) (2^{pts}) Find the solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ that is closest to $(2, \quad -1, \quad 0, \quad 3,)$.
(b) (1^{pts}) Give an *orthonormal* basis for the nullspace of $\mathbf{A}$.

Based on MIT Course 18.06 Quiz 2, Fall 1997

EXERCISE 2.

(a) (0.5^{pts}) The linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution if and only if $\boldsymbol{b}$ is orthogonal to what subspace?
(b) (0.5^{pts}) Find the determinant of a 4 by 4 matrix $\mathbf{A}$ whose entries are ${}_{i|}\mathbf{A}_{|j} = \min(i^2,\ j^2)$.

Based on MIT Course 18.06 Quiz 2, Spring 1997

EXERCISE 3. Each independent question refers to the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ d & 1 \end{bmatrix}$.

(a) (0.5^{pts}) Give a value for d such that $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$.
(b) (0.5^{pts}) Give a value for d such that 2 is one of the eigenvalues of $\mathbf{A}$.
(c) (1^{pts}) Give a value for d such that $\mathbf{A}$ is a nondiagonalizable matrix.

EXERCISE 4. (1^{pts}) Give a vector $\boldsymbol{v}$ that makes $\left[\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix};\ \begin{pmatrix} 5 \\ 11 \\ -8 \end{pmatrix};\ \boldsymbol{v}; \right]$ an orthogonal basis for $\mathbb{R}^3$.

MIT Course 18.06 Quiz 2, Fall 1997

EXERCISE 5. Consider the linear system

$$\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}, \quad \text{with solution set} \quad \left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2,\ \boldsymbol{v} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}.$$

(a) (0.5^{pts}) Find the dimension of the row space of $\mathbf{A}$. Explain your answer.
(b) (1^{pts}) Construct the matrix $\mathbf{A}$. Explain your answer.
(c) (1^{pts}) For which right hand side vectors $\boldsymbol{b}$ the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is solvable?

EXERCISE 6. (0.5^{pts}) Consider $\mathbf{A}$ such that its *inverse* is $\begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Classify the quadratic form $\boldsymbol{x}\mathbf{A}\boldsymbol{x}$.

2.9. Grupo B curso 18/19

Exercise 1. Consider the following points in $\mathbb{R}^3$, $\boldsymbol{a} = (1, 0, 3,)$ and $\boldsymbol{b} = (-\frac{1}{3}, 0, -1,)$.

(a) (1^{pts}) Find a cartesian (or implicit) equations of the line that goes through $\boldsymbol{a}$ and $\boldsymbol{b}$.
(b) (0.5^{pts}) Is this line a subspace of $\mathbb{R}^3$? Explain your answer.
(c) (1^{pts}) Find the closest point of that line to $\boldsymbol{z} = (2, 2, 2,)$.
(d) ($0.5 + 0.5^{\text{pts}}$) Verify that the previous answer is correct and find the minimum distance between $\boldsymbol{z}$ and the line.

Exercise 2. Consider a 3×3 matrix $\mathbf{A}$ with eigenvalues $\lambda_1 = \lambda_2 = 1$ and the corresponding eigenvectors

$$\boldsymbol{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{y} \quad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}.$$

(a) (0.5^{pts}) Are $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ linearly independent? Are $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ orthogonal?
(b) (1^{pts}) Find a third eigenvector $\boldsymbol{v}_3$, corresponding to the third eigenvalue λ_3, such that matrix $\mathbf{A}$ is symmetric.
(c) (0.5^{pts}) If $\operatorname{tr}(\mathbf{A})$ is 2. What is λ_3? Is $\mathbf{A}$ positive definite?
(d) (1^{pts}) Find $\mathbf{A}$.

Exercise 3. You have a matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. (Look carefully at the columns of $\mathbf{A}$—very little calculation is needed!)

(a) (0.5^{pts}) Give the ranks of $\mathbf{A}$, $\mathbf{A}^\intercal$, and $\mathbf{A}^\intercal\mathbf{A}$,
(b) (1^{pts}) Give bases for $\mathcal{C}(\mathbf{A})$, $\mathcal{N}(\mathbf{A})$, and $\mathcal{N}(\mathbf{A}^\intercal\mathbf{A})$.
(c) (0.5^{pts}) Suppose we are looking for a least-square solution $\widehat{\boldsymbol{x}}$ that minimizes $\|\boldsymbol{b} - \mathbf{A}\boldsymbol{x}\|$ for $\boldsymbol{b} = \begin{pmatrix} 0 \\ 2 \\ 1 \\ -1 \end{pmatrix}$.

At that solution, $\boldsymbol{p} = \mathbf{A}\widehat{\boldsymbol{x}}$ will be the projection of $\boldsymbol{b}$ onto ____________________?
(d) (1^{pts}) Find that projection $\boldsymbol{p} = \mathbf{A}\widehat{\boldsymbol{x}}$. (Hint: your answer from (b) should help simplify the calculations.)

MIT Course 18.06 Exam 2, Problem 2. Fall 2018

Exercise 4. (0.5^{pts}) *Determine whether the following statements are true or false, and explain your reasoning (to receive full credit you must explain your answers in a clear and concise way).*

Consider $B = \{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$, a basis of the subespace $\mathcal{S}$ in $\mathbb{R}^4$. Then, the set

$$B^* = \{\boldsymbol{u} + \boldsymbol{v},\ \boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w},\ 2\boldsymbol{w}\}$$

is another basis of $\mathcal{S}$.

2.10. Grupo E curso 18/19

EXERCISE 1.

(a) (0.5^{pts}) Find the eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} .9 & .1 \\ .4 & .6 \end{bmatrix}$ (note the rows sum up one)

(b) (0.5^{pts}) Find the eigenspaces of $\mathbf{A}$.

(c) (0.5^{pts}) What is the limiting value of $\mathbf{A}^k \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ as the power k goes to infinity?

Based on MIT 18.06 - Quiz 3, December 5, 2005

EXERCISE 2. $\mathbf{A}$ has a $\mathcal{N}\left(\mathbf{A}\right)$ spanned by $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\mathcal{N}\left(\mathbf{A}^{\intercal}\right)$ spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$.

(a) (0.5^{pts}) What is the order of $\mathbf{A}$ and its rank?

(b) (1^{pts}) If we consider the vector $\boldsymbol{b} = \begin{pmatrix} -1 \\ \alpha \\ 0 \\ \beta \end{pmatrix}$, for what value(s) of α and β (if any) is $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ solvable? Will the solution (if any) be unique?

(c) ($1 + 1^{\text{pts}}$) Give the orthogonal projections of $\boldsymbol{y} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ onto two of the four fundamental subspaces of matrix $\mathbf{A}$.

MIT Course 18.06 Exam 2, Problem 1. Fall 2018

EXERCISE 3. Are the following matrices necessarily positive definite? Explain why or why not? ($\mathbf{D}$ is diagonal with $\begin{pmatrix}1, & 2, & 3, & 4\end{pmatrix}$ on the diagonal)

(a) (0.5^{pts}) $\mathbf{A} = \mathbf{QDQ}^{\intercal}$ where $\mathbf{Q}$ is some 4×4 orthogonal matrix.
(b) (0.5^{pts}) $\mathbf{A} = \mathbf{Q_1 D Q_1}^{\intercal} + \mathbf{Q_2 D Q_2}^{\intercal}$ where $\mathbf{Q_1}$ and $\mathbf{Q_2}$ are some 4x4 orthogonal matrices.
(c) (0.5^{pts}) $\mathbf{A} = \mathbf{XDX}^{\intercal}$ for some matrix $\mathbf{X}$ (Hint: Be careful.)
(d) (0.5^{pts}) $\mathbf{P}$ the projection matrix onto the spam of $\left(1, 2, 3, 4\right)$.
(e) (1^{pts}) $\mathbf{A}$ is the n by n tridiagonal matrix with 2 for each diagonal entry, and 1 for each superdiagonal and subdiagonal entry.

$$\mathbf{A}_n = \begin{bmatrix} 2 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 2 & 1 \\ & & & & 1 & 2 \end{bmatrix}$$

MIT Course 18.06 Quiz 3, Problem 3. May 4, 2018

EXERCISE 4. Consider $\boldsymbol{x} = (3, 2, 4,)$ and a square matrix $\mathbf{A}$ such that $\mathbf{A}\boldsymbol{x} = 2\boldsymbol{x}$ and such that the set of solutions to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ is spanned by $\boldsymbol{u} = (0, 1, 1,)$ and $\boldsymbol{v} = (1, 1, 0,)$.

(a) (1^{pts}) Is $\mathbf{A}$ diagonalizable?
(b) (1^{pts}) Is $\mathbf{A}$ symmetric?

2.11. Grupo E curso 17/18

EXERCISE 1. $\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -3 \\ 2 & 2 & -4 \end{bmatrix}$.

(a) (0.5^{pts}) Decide if $\mathbf{A}$ is singular or invertible.
(b) (1^{pts}) Find an orthonormal basis for its column space (if such a basis exists) (*hint*: $153 = 9 \times 17$).
(c) (1^{pts}) Why does $\mathbf{A}\left(\mathbf{A}^\intercal\mathbf{A}\right)^{-1}\mathbf{A}^\intercal$ not give the projection matrix onto the column space of $\mathbf{A}$? Somehow, it must be possible to get such projection matrix. How can we find it?
(d) (0.5^{pts}) Find that projection matrix.

EXERCISE 2. Consider the following points in $\mathbb{R}^3$, $\boldsymbol{a} = (1, 0, 3,)$ and $\boldsymbol{b} = (-\frac{1}{3}, 0, -1,)$.
(a) (1^{pts}) Find a cartesian (or implicit) equations of the line that goes through $\boldsymbol{a}$ and $\boldsymbol{b}$.

EXERCISE 3. Suppose that $\mathbf{A}$ is a positive definite matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & b & 0 \\ b & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

(a) (1^{pts}) What are the possible values of b?
(b) (1^{pts}) How do you know that the matrix $\mathbf{A}^2 + \mathbf{I}$ is positive definite for every b?
(c) (1^{pts}) Complete this sentence correctly for a general matrix $\mathbf{M}$, possibly rectangular: *The matrix* $\mathbf{M}^\intercal\mathbf{M}$ *is symmetric positive definite unless*____.

EXERCISE 4. $\mathbf{Q}$ is a 4×3 matrix with orthonormal columns $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, and $\boldsymbol{q}_3$. Assume that $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$, and $\boldsymbol{b}$ are linearly independent vectors in $\mathbb{R}^4$.
(a) (1^{pts}) What is the *row* space of $\mathbf{Q}$?
(b) (1^{pts}) What combination $\boldsymbol{p}$ of $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, and $\boldsymbol{q}_3$ is closest to $\boldsymbol{b}$?
(c) (1^{pts}) What combination of $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$, and $\boldsymbol{b}$ is in the nullspace of $\mathbf{Q}^\intercal$?

2.12. Grupo F curso 17/18

EXERCISE 1.

(a) (1^{pts}) Find the determinant of $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 4 & 5 \\ 0 & 0 & 5 & 6 \\ 2 & 0 & 0 & 7 \end{bmatrix}$.

(b) (1^{pts}) Suppose that $\mathbf{A}$ is a 3×2 matrix of rank 2. What is $\det(\mathbf{A}\left(\mathbf{A}^{\mathsf{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{T}})$?

EXERCISE 2. Let $\mathbf{A} = \begin{bmatrix} -15 & 8 \\ -28 & 15 \end{bmatrix}$.

(a) (1^{pts}) Find the eigenvalues and compute an eigenvector for each eigenvalue
(b) (1^{pts}) Find an invertible matrix $\mathbf{S}$ and diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$.
(c) (1^{pts}) Compute $\mathbf{A}^{37}$.

EXERCISE 3.

(a) (1^{pts}) Find the projection of $\boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ onto the plane spanned by $\boldsymbol{a}_1 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ and $\boldsymbol{a}_2 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$.

(b) (1^{pts}) Apply the Gram-Schmidt process to the vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{b}$ to find orthonormal vectors $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, and $\boldsymbol{q}_3$.

EXERCISE 4. Suppose that $\underset{3\times 3}{\mathbf{A}}$ has eigenvalues $\lambda = 0,\ 1,\ 2$, with respective eigenvectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.

(a) (1^{pts}) Describe the null space, column space, and row space of $\mathbf{A}$ in terms of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.
(b) (1^{pts}) Find all solutions to $\mathbf{A}\boldsymbol{x} = \boldsymbol{v} - \boldsymbol{w}$.
(c) (1^{pts}) Prove that $\mathbf{A}$ is not an orthogonal matrix.

2.13. Grupo B curso 16/17

Exercise 1. (1^{pts}) Solve the following linear system for x y and z

$$\begin{bmatrix}1 & 1 & 1\\ 2 & 2 & 1\\ 0 & 3 & -1\end{bmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}2\\ 3\\ 4\end{pmatrix}$$

Exercise 2.

(a) (0.5^{pts}) Write basis for the column space and the nullspace of $\mathbf{A}$

$$\begin{bmatrix}1 & 2 & 2 & 3\\ 1 & 2 & 4 & 3\\ 1 & 2 & 6 & 3\end{bmatrix}$$

(b) (1^{pts}) Write down all solutions to

$$\begin{bmatrix}1 & 2 & 2 & 3\\ 1 & 2 & 4 & 3\\ 1 & 2 & 6 & 3\end{bmatrix}\boldsymbol{x} = \begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$$

(c) (0.5^{pts}) If $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{A}^{-1}$ exists, then what is the column space and the nullspace of $\mathbf{A}$? Write down a basis for $\mathcal{C}\left(\mathbf{A}\right)$.

Exercise 3.

(a) (1^{pts}) Find the eigenvectors of $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix}2/4 & 1/4 & 1/4\\ 1/4 & 2/4 & 1/4\\ 1/4 & 1/4 & 2/4\end{bmatrix},$$

where the characteristic polynomial of the matrix $\mathbf{A}$ is $-(\lambda-1)(\lambda-\frac{1}{4})^2$.

(b) (1^{pts}) Find the limit of $\mathbf{A}^k$ as $k \to \infty$. (You may work with $\mathbf{A} = \mathbf{QDQ}^{-1}$ without computing every entry).

(c) (1^{pts}) Choose any positive numbers r, s, t so that

$$\begin{cases}\mathbf{A} - r\mathbf{I} & \text{is positive definite}\\ \mathbf{A} - s\mathbf{I} & \text{is neither positive nor negative definite}\\ \mathbf{A} - t\mathbf{I} & \text{is negative definite}\end{cases}$$

Basado en MIT 18.06 - Quiz 3, May 4, 2005

Exercise 4. Consider the plane $x + y + z = 0$ in $\mathbb{R}^3$.

(a) (1^{pts}) Find the parametric equations of the plane.

(b) (1^{pts}) Find the projection $\boldsymbol{p}$ of the vector $\boldsymbol{b} = (1, 2, 6,)$ onto the plane $x + y + z = 0$ in $\mathbb{R}^3$. (You may want to find a basis for this 2-dimensional subspace.)

Basado en MIT 18.06 - Quiz 2, April 1, 2005

Exercise 5.

(a) (0.5^{pts}) What are all possible values for the determinant of a projection matrix? (Please explain briefly.)

(b) (0.5^{pts}) What are all possible values for the determinant of a permutation matrix? (Please explain briefly.)

MIT 18.06 - Quiz 2, November 4, 2011

Exercise 6.

(a) (1^{pts}) In $\mathbb{R}^m$, suppose I gave you a vector $\boldsymbol{b}$ and a vector $\boldsymbol{p}$ and n linearly independent vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$. If I claim that $\boldsymbol{p}$ is the projection of $\boldsymbol{b}$ onto the subspace spanned by the $\boldsymbol{a}$'s, what tests would you make to see if this is true?

MIT 18.06 - Final Exam, Monday May 16th, 2005

2.14. Grupo E curso 16/17

Exercise 1. Suppose $\mathbf{A}$ is this 3 by 4 matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

(a) (0.5^{pts}) Describe the column space of $\mathbf{A}$

(b) (1^{pts}) For which vectors $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ does $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ have a solution? Give conditions on b_1, b_2, b_3.

(c) (0.5^{pts}) There is no 4 by 3 matrix $\mathbf{C}$ for which $\mathbf{AC} = \underset{3\times 3}{\mathbf{I}}$. Give a good reason (is this because $\mathbf{A}$ is rectangular?)

(d) (1^{pts}) Find the complete solution to $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

MIT 18.06 - Quiz 1, October 5, 2005

Exercise 2.

(a) (1^{pts}) Complete the matrix $\mathbf{A}$ (fill in the two blank entries) so that $\mathbf{A}$ has eigenvectors $\boldsymbol{x}_1 = (3, 1,)$ and $\boldsymbol{x}_2 = (2, 1,)$:

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ & \end{bmatrix}$$

(b) (1^{pts}) Find a different matrix $\mathbf{B}$ with those same eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, and with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 0$. What is $\mathbf{B}^{10}$?

MIT 18.06 - Final Exam, Monday May 16th, 2005

Exercise 3. (1^{pts}) Suppose $\mathbf{P}_1$ is the projection matrix onto the 1-dimensional subspace spanned by the first column of $\mathbf{A}$. Suppose $\mathbf{P}_2$ is the projection matrix onto the 2-dimensional column space of $\mathbf{A}$. After thinking a little, compute the product $\mathbf{P}_2\mathbf{P}_1$.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

MIT 18.06 - Quiz 2, April 1, 2005

Exercise 4. (1^{pts}) For which values of b does this matrix have 3 positive eigenvalues?

$$\mathbf{A} = \begin{bmatrix} 2 & b & 3 \\ b & 2 & b \\ 3 & b & 4 \end{bmatrix}$$

Exercise 5.

(a) (1^{pts}) I was looking for an m by n matrix $\mathbf{A}$ and vectors $\boldsymbol{b}$, $\boldsymbol{c}$ such that $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has no solution and $\mathbf{A}^{\mathsf{T}}\boldsymbol{y} = \boldsymbol{c}$ has exactly one solution. Why can I not find $\mathbf{A}$, $\boldsymbol{b}$, $\boldsymbol{c}$?

MIT 18.06 - Final Exam, Monday May 16th, 2005

Exercise 6. (1^{pts}) The matrix $\mathbf{A}$ has independent columns. The matrix $\mathbf{C}$ is square, diagonal, and has positive entries. Why is the matrix $\mathbf{K} = \mathbf{A}^{\mathsf{T}}\mathbf{CA}$ positive definite? You can use any of the basic tests for positive definiteness.
MIT 18.06 - Quiz 3, December 1, 2010

Exercise 7.

(a) (0.5$^{\text{pts}}$) Compute the determinant (as a function of x) of the 4×4 matrix

$$\mathbf{A} = \begin{bmatrix} x & x & x & x \\ x & x & 0 & 0 \\ x & 0 & x & x \\ x & 0 & x & 1 \end{bmatrix}$$

(b) (0.5$^{\text{pts}}$) Find all values of x for which $\mathbf{A}$ is singular.

MIT 18.06 - Quiz 3, December 1, 2010

2.15. Grupo E curso 15/16

EXERCISE 1.

(a) (0.5^{pts}) Find a 3 by 3 matrix $\mathbf{A}$ whose column space is the plane $x+y+z=0$ in $\mathbb{R}^3$. (This means: $\mathcal{C}(\mathbf{A})$ consists of all column vectors (x,y,z) with $x+y+z=0$.)

(b) (0.5^{pts}) How do you know that a 3 by 3 matrix $\mathbf{A}$ with that column space is not invertible?

EXERCISE 2. (1^{pts}) Let L be the intersection of the two planes

$$x+2y+3z=10 \quad \text{and} \quad 4x+5y+6z=28.$$

Find a parametric equation for L.

EXERCISE 3.

(a) (1^{pts}) Suppose $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $\mathbb{R}^n$ such that $\boldsymbol{u}+\boldsymbol{v}$ and $\boldsymbol{u}-\boldsymbol{v}$ are orthogonal (i.e., perpendicular) to each other. Show that $\|\boldsymbol{u}\|=\|\boldsymbol{v}\|$.

(b) (1^{pts}) Suppose $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are unit vectors in $\mathbb{R}^n$. (Recall that a unit vector is a vector whose length is 1.) Suppose each vector is orthogonal (i.e., perpendicular) to each of the other two. Show that the two vectors

$$(\boldsymbol{u}-3\boldsymbol{v}+2\boldsymbol{w}) \quad \text{and} \quad (\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w})$$

are orthogonal to each other.

EXERCISE 4. Let $\mathbf{A}=\begin{bmatrix}5&1\\1&3\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}2&-1&0\\2&1&1\\5&-7&6\end{bmatrix}$.

(a) (1^{pts}) Find the eigenvalues of $\mathbf{A}$.

(b) (1^{pts}) $\lambda=3$ is an eigenvalue of $\mathbf{B}$. (You do not need to check this.) Find all eigenvectors of $\mathbf{B}$ with eigenvalue 3.

EXERCISE 5. Suppose we measure $y=(0,0,0,1,0,0,0,)$ at times $x=(-3,-2,-1,0,1,2,3,)$.

(a) (0.5^{pts}) To fit these 7 measurements by a straight line $c+dx$, what system $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$, with 7 equations, would we want to solve? (note that $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$ could be an unsolvable system)

(b) (1^{pts}) Find the least squares solution $\widehat{\boldsymbol{\beta}}=\begin{pmatrix}\widehat{c}\\\widehat{d}\end{pmatrix}$.

(c) (1^{pts}) The projection of that vector $\boldsymbol{y}$ in $\mathbb{R}^7$ onto the column space of $\mathbf{A}$ is what vector $\boldsymbol{p}$?

MIT Course 18.06. Exam II. Professor Strang. April 10, 2015

EXERCISE 6. (1^{pts}) Suppose $\mathbf{A}$ is an $n\times n$ matrix and that $\boldsymbol{v}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ. Show that $\boldsymbol{v}$ is an eigenvector of $\mathbf{A}^2+\mathbf{A}$ with eigenvalue $\lambda^2+\lambda$.

EXERCISE 7. (0.5^{pts}) Compute the following determinant:

$$\begin{vmatrix}0&0&0&3&0\\-2&0&0&2&0\\8&-1&0&-7&2\\-1&2&2&3&2\\2&2&3&6&4\end{vmatrix}$$

2.16. Grupo H curso 15/16

Exercise 1. Consider the points $\boldsymbol{a} = (1, 1, 1,)$, $\boldsymbol{b} = (1, 3, 1,)$ and $\boldsymbol{c} = (1, 1, 4,)$ in $\mathbb{R}^3$.

(a) (1^{pts}) Find a parametric equation for the plane through the points $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

(b) (1^{pts}) Find an implicit (or cartesian) equation for the plane through the points $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

Exercise 2.

(a) (1^{pts}) Suppose $\boldsymbol{u}$ is a vector in $\mathbb{R}^4$. Let $\mathcal{V}$ be the set of all vectors in $\mathbb{R}^4$ which are orthogonal (i.e. perpendicular) to $\boldsymbol{u}$. That is,

$$\mathcal{V} = \left\{ \boldsymbol{x} \in \mathbb{R}^4 | \boldsymbol{x} \cdot \boldsymbol{u} = 0 \right\}.$$

Show that $\mathcal{V}$ is a subspace of $\mathbb{R}^4$.

(b) (1^{pts}) Suppose the vector $\boldsymbol{u}$ in part (a) is

$$\boldsymbol{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

Find a basis for $\mathcal{V}$.

(c) (0.5^{pts}) What is the dimension of the subspace $\mathcal{V}$ in part (b)?

Exercise 3.

(a) (0.5^{pts}) Find the eigenvalues of $\mathbf{A} = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 7 & 8 \\ 0 & 0 & 3 \end{bmatrix}$.

(b) (1^{pts}) Let $\mathbf{B} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & -1 \\ 2 & -4 & 1 \end{bmatrix}$; where $\lambda = 3$ is an eigenvalue of $\mathbf{B}$ (you do not need to verify this).

Find a basis for the eigenspace $\quad \mathcal{E}_3 = \left\{ \boldsymbol{v} \in \mathbb{R}^3 \,\middle|\, \mathbf{B}\boldsymbol{v} = 3\boldsymbol{v} \right\}$.

Exercise 4.

(a) (1^{pts}) What is the 3 by 3 projection matrix $\mathbf{P}_a$ onto the line through $\boldsymbol{a} = (2, 1, 2,)$?

(b) (1^{pts}) Suppose $\mathbf{P}_v$ is the 3 by 3 projection matrix onto the line through $\boldsymbol{v} = (1, 1, 1,)$. Find a basis for the column space of the matrix $\mathbf{A} = \mathbf{P}_a\mathbf{P}_v$ (product of 2 projections).

MIT Course 18.06. Final Exam. Professor Strang. May 18, 2015

Exercise 5. (1^{pts}) The equation $(\mathbf{A}^2 - 4\mathbf{I})\boldsymbol{x} = \boldsymbol{b}$ has no solution for some right-hand side $\boldsymbol{b}$. Give as much information as possible about the eigenvalues of the matrix $\mathbf{A}$ (the matrix $\mathbf{A}$ is diagonalizable).
MIT Course 18.06 Quiz 3. Spring, 2009

Exercise 6. (1^{pts}) Consider the following quadratic form

$$q(x, y, z) = x^2 + 6xy + y^2 + az^2;$$

Decide for which values a the quadratic form is positive definite, negative definite, semidefinite, or indefinite.

2.17. Grupo A curso 14/15

EXERCISE 1.
This 4 by 4 Hadmard matrix is an orthogonal matrix. Its columns are orthogonal unit vectors.

$$\mathbf{Q} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = [\boldsymbol{q}_1; \quad \boldsymbol{q}_2; \quad \boldsymbol{q}_3; \quad \boldsymbol{q}_4;]$$

(a) (0.5^{pts}) What projection matrix $\mathbf{P}_4$ (give numbers) will project every $\boldsymbol{y}$ in $\mathbb{R}^4$ onto the line through $\boldsymbol{q}_4$?
(b) (0.5^{pts}) What projection matrix $\mathbf{P}_{123}$ will project every $\boldsymbol{y}$ in $\mathbb{R}^4$ onto the subspace spanned by $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, and $\boldsymbol{q}_3$? Remember that those columns are orthogonal.
(c) (0.5^{pts}) Suppose $\mathbf{A}$ is the 4 by 3 matrix whose columns are $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$. Find the least-squares solution $\boldsymbol{\beta}$ to the four equations

$$\mathbf{A}\boldsymbol{\beta} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

(d) (0.5^{pts}) What is the error vector $\boldsymbol{e}$?

MIT Course 18.06 Quiz 2, 2013

EXERCISE 2.
(a) (1^{pts}) Suppose three matrices satisfy $\mathbf{AB} = \mathbf{C}$: If the columns of $\mathbf{B}$ are dependent, show that the columns of $\mathbf{C}$ are dependent.
(b) (0.5^{pts}) If $\mathbf{A}$ is 5 by 3 and $\mathbf{B}$ is 3 by 5, show using part (a) or otherwise that $\mathbf{AB} = \mathbf{I}$ is impossible.

MIT Course 18.06 Quiz 1, March 9, 2012

EXERCISE 3. (1^{pts}) Find the determinant of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \end{bmatrix}$. *MIT Course 18.06 Quiz 2, 2013*

EXERCISE 4. $\mathbf{A}$ is a 3 by 3 real-symmetric matrix. Two of its eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$ with eigenvectors $\boldsymbol{v}_1 = (1, 1, 1,)$ and $\boldsymbol{v}_2 = (1, -1, 0,)$, respectively. The third eigenvalue is $\lambda_3 = 0$.
(a) (0.5^{pts}) Give an eigenvector $\boldsymbol{v}_3$ for the eigenvalue λ_3. (*Hint*: what must be true of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$?)
(b) (0.5^{pts}) Write a squared orthonormal matrix whose columns are eigenvectors of $\mathbf{A}$.
(c) (0.5^{pts}) Find the eigenvalues and three linearly independent eigenvectors for $\mathbf{A}^4$.

Basado en MIT Course 18.06 Quiz 3. Spring, 2009

EXERCISE 5. *Determine whether the following statements are true or false, and explain your reasoning (to receive full credit you must explain your answers in a clear and concise way).*
(a) (0.5^{pts}) Matrix $\left[\begin{smallmatrix} 1 & 2 \\ 4 & 3 \end{smallmatrix}\right]$ is similar (same eigenvalues) to matrix $\left[\begin{smallmatrix} 4 & 1 \\ 3 & 2 \end{smallmatrix}\right]$.
(b) (0.5^{pts}) There is a matrix with column space $\mathcal{C}(\mathbf{A})$ is spanned by the vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, and with row space $\mathcal{C}(\mathbf{A}^\intercal)$ spanned by vectors $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

EXERCISE 6. (1^{pts}) Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 4 & 2 & 3 & 1 \\ 3 & 1 & 4 & 2 \end{bmatrix}$. Show $\boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector for $\mathbf{A}$.

EXERCISE 7. (1^{pts}) For which values a are the following vectors linearly independent? $\begin{pmatrix} 3 \\ 1 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -4 \\ a \end{pmatrix}$

.

EXERCISE 8. (1^{pts}) For wich numbers b does this matrix C have 3 positive eigenvalues?

$$\mathsf{C} = \begin{bmatrix} 2 & b & 3 \\ b & 2 & b \\ 3 & b & 4 \end{bmatrix}$$

2.18. Grupo C curso 14/15

Exercise 1.

$$\begin{cases} 2x_1 + x_2 + x_3 & = 2\alpha \\ 4x_1 + 2x_2 + 2x_3 & = 3\alpha \\ 6x_1 + 2x_2 + 3x_3 & = 2\beta \end{cases}$$

(a) (1^{pts}) What conditions on α and β make the system solvable?
(b) (1^{pts}) Solve the system in that case.

Exercise 2. (1^{pts}) Suppose the matrix $\mathbf{A}$ is m by n of rank r; and the matrix $\mathbf{B}$ is M by N of rank R: Suppose the column space $\mathcal{C}\left(\mathbf{A}\right)$ is contained in (possibly equal to) the column space $\mathcal{C}\left(\mathbf{B}\right)$: (This means that every vector in $\mathcal{C}\left(\mathbf{A}\right)$ is also in $\mathcal{C}\left(\mathbf{B}\right)$). What relations must hold between m and M; n and N; and r and R? It might be good to write down an example of $\mathbf{A}$ and $\mathbf{B}$ where all the columns are different. *MIT Course 18.06 Quiz 1, March 9, 2012*

Exercise 3.

(a) (0.5^{pts}) Give a 3×3 matrix $\mathbf{A}$ so that the homogeneous system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ has a nontrivial solution ($\boldsymbol{x} \neq \mathbf{0}$).
(b) (0.5^{pts}) If the characteristic polynomial of a matrix $\mathbf{A}$ is $p(\lambda) = \lambda^5 + 3\lambda^4 - 24\lambda^3 + 28\lambda^2 - 3\lambda + 10$. find the rank of $\mathbf{A}$.
(c) (1^{pts}) Suppose two of the eigenvalues of the 5×5 matrix $\mathbf{A}$ are -1 and 3, corresponding to the eigenvectors $\boldsymbol{u} = \begin{pmatrix}2, & -1, & 4, & 0, & 3,\end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix}3, & 1, & -2, & 1, & 2,\end{pmatrix}$, respectively. Compute $\mathbf{A}\boldsymbol{x}$ for $\boldsymbol{x} = \begin{pmatrix}12 & -1, & 8, & 2, & 13,\end{pmatrix}$. **Hint:** First write $\boldsymbol{x}$ as a linear combination of $\{\boldsymbol{u}, \boldsymbol{v}\}$. Note that

$$\begin{bmatrix} 2 & 3 & 12 \\ -1 & 1 & -1 \\ 4 & -2 & 8 \\ 0 & 1 & 2 \\ 3 & 2 & 13 \end{bmatrix} \begin{bmatrix} 3 & -3 & -3 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ -3 & 5 & 0 \\ 12 & -16 & 0 \\ 0 & 2 & 0 \\ 9 & -5 & 0 \end{bmatrix}.$$

Exercise 4. Consider a 4 by 4 real matrix

$$\mathbf{A} = \begin{bmatrix} 0 & x & y & z \\ x & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ z & 0 & 0 & 1 \end{bmatrix}$$

(a) (0.5^{pts}) Compute $|\mathbf{A}|$, the determinant of $\mathbf{A}$, in simplest form.
(b) (0.5^{pts}) For what values of x, y, z is $\mathbf{A}$ singular?

MIT Course 18.06 Quiz 2, November 7, 2012

Exercise 5. (1^{pts}) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & a \\ 1 & -2 & 1 \\ a & 1 & -2 \end{bmatrix}.$$

discuss whether the matrix is definite, semidefinite or not definite depending on the values of a.

Exercise 6. Sea la matriz

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

(a) (0.5^{pts}) Find the eigenvalues of $\mathbf{A}$
(b) (1^{pts}) Find the eigenvectors of $\mathbf{A}$

Exercise 7. (0.5^{pts}) If $\mathbf{A}^3 = 2\mathbf{A}^2 - \mathbf{A}$, what are the possible eigenvalues of $\mathbf{A}$?
MIT Course 18.06 Spring 2006 - Review Problems

Exercise 8.

(a) (0.5^{pts}) Find two eigenvalues and two linearly independent eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}.$$

(b) (0.5^{pts}) Express any vector $\boldsymbol{u}_0 = \begin{pmatrix} a \\ b \end{pmatrix}$ as a combination of the eigenvectors.

MIT Course 18.06 Quiz 3. May 6, 2011

2.19. Grupo E curso 14/15

Exercise 1. We look for the line $\boldsymbol{y} = c + d\boldsymbol{x}$ closest to 3 points $(x, y,) = (0, 1,)$ and $(1, 2,)$ and $(2, -1,)$.
(a) (0.5^{pts}) If the line went through those points (it doesn't), what three equations would be solved?
(b) (1^{pts}) Find the best c and d by the least squares method.
(c) (0.5^{pts}) Explain the result you get for c and d: how is the vector $\boldsymbol{y} = (1,\ 2,\ -1,)$ y el plano sobre el que se proyecta?
(d) (0.5^{pts}) What is the length of the error vector $\boldsymbol{e}$ (= distance to plane = $\|\boldsymbol{y} - \mathbf{A}\widehat{\boldsymbol{\beta}}\|$)?

MIT Course 18.06 Quiz 2, 1995

Exercise 2. In all of this problem, the 3 by 3 matrix $\mathbf{A}$ has eigenvalues λ_1, λ_2, λ_3, with *independent eigenvectors* $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$.
(a) (0.5^{pts}) What are the trace of $\mathbf{A}$ and the determinant of $\mathbf{A}$?
(b) (0.5^{pts}) Suppose: $\lambda_2 = \lambda_3$. Choose the true statement from 1), 2), 3):

1. $\mathbf{A}$ can be diagonalized. Why?
2. $\mathbf{A}$ can not be diagonalized. Why?
3. I need more information to decide. Why?

(c) (0.5^{pts}) From the eigenvalues and eigenvectors, how could you find the matrix $\mathbf{A}$? Give a formula for $\mathbf{A}$ and explain each part carefully.
(d) (1^{pts}) Suppose $\lambda_1 = 2$ and $\lambda_2 = 5$ and $\boldsymbol{x}_1 = (1, 1, 1,)$ and $\boldsymbol{x}_2 = (1, -2, 1,)$. Choose λ_3 and $\boldsymbol{x}_3$ so that $\mathbf{A}$ is *symmetric* positive *semidefinite* but not positive definite.

Exercise 3.
(a) (0.5^{pts}) Why is there no orthonormal matrix $\mathbf{Q}$ such that $\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \mathbf{D}$ if

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}?$$

(b) (0.5^{pts}) For what values of a and b is the quadratic form $ax^2 + 2xy + by^2 = \begin{pmatrix} x, & y, \end{pmatrix} \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ positive definite?

MIT Course 18.06 Final Exam, December 13, 1993

Exercise 4. Find all possible values for the determinant of the given type of 3×3 real matrix.
(a) (0.5^{pts}) A matrix with linearly independent columns.
(b) (0.5^{pts}) A matrix with $\mathbf{A}^2 = \mathbf{A}$.
(c) (0.5^{pts}) A matrix with pivots 1, 2 and 3.

MIT Course 18.06 Final Exam, December 13, 1993

Exercise 5.
(a) (0.5^{pts}) Give an example of a matrix with exactly two zero eigenvalues and no zero entries.
(b) (0.5^{pts}) What is the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$?

MIT Course 18.06 Final Exam, December 13, 1993

Exercise 6. The matrix $\mathbf{A}$ has a varing $1 - x$ in the (1,2) position:

$$\mathbf{A} = \begin{bmatrix} 2 & 1-x & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 9 \end{bmatrix}$$

(a) (1^{pts}) When $x = 1$ compute $\det \mathbf{A}$. What is the (1,1) element in the inverse when $x = 1$?
(b) (0.5^{pts}) When $x = 0$ compute $\det \mathbf{A}$.
(c) (0.5^{pts}) How do the properties of the determinant say that $\det \mathbf{A}$ is a linear function of x? For any x compute $\det \mathbf{A}$. For which x's is the matrix singular?

MIT Course 18.06 Quiz 2, November 2, 2005

2.20. Grupo H curso 14/15

Exercise 1.

(a) (0.5^{pts}) If $\mathbf{P}$ projects every vector $\boldsymbol{b}$ in $\mathbb{R}^5$ to the nearest point in the subspace spanned by $\boldsymbol{a}_1 = (1,0,1,0,4,)$ and $\boldsymbol{a}_2 = (2,0,0,0,4,)$, what is the rank of $\mathbf{P}$ and why?

(b) (0.5^{pts}) If these two vectors are the columns of the 5 by 2 matrix $\mathbf{A}$, which of the four fundamental subspaces for $\mathbf{A}$ is the nullspace of $\mathbf{P}$?

(c) (0.5^{pts}) If $\mathbf{P}$ is any (symmetric) projection matrix, show that $\mathbf{Q} = \mathbf{I} - 2\mathbf{P}$ is an orthogonal matrix.

MIT Course 18.06 Quiz 2, 2013

Exercise 2. (1^{pts}) Give an example of a vector $\boldsymbol{v}$ in $\mathbb{R}^4$ which is orthogonal to every solution $\boldsymbol{x}$ of the homogeneous linear system

$$\begin{cases} 3x_1 + x_2 + x_3 + 3x_4 & = 0 \\ \quad\;\; +x_2 \qquad\; + 2x_4 & = 0 \\ 2x_1 \qquad + x_3 & = 0 \end{cases}$$

Exercise 3.

(a) (1^{pts}) Apply row elimination to reduce this invertible matrix from $\mathbf{A}$ to $\mathbf{I}$.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

(b) (1^{pts}) Then write $\mathbf{A}^{-1}$ as a product of three (or more) simple matrices coming from that elimination. Multiply these matrices to find $\mathbf{A}^{-1}$.

MIT Course 18.06 Quiz 1, March 9, 2012

Exercise 4. (0.5^{pts}) Suppose $\mathbf{A}$ is a 5 by 3 matrix with orthonormal columns. Evaluate the following determinants: $\det \mathbf{A}^\mathsf{T}\mathbf{A}$

Exercise 5.

(a) (1^{pts}) Suppose the matrix $\mathbf{A}$ factors into $\mathbf{A} = \mathbf{PL\dot{U}}$ with a permutation matrix $\mathbf{P}$, and 1's on the diagonal of $\mathbf{\dot{U}}$ (upper triangular) and pivots $d_1, \dots, d_n$ on the diagonal of $\mathbf{L}$ (lower triangular). What is the determinant of $\mathbf{A}$? EXPLAIN WHAT RULES YOU ARE USING.

Based on MIT Course 18.06 Quiz 2, April 11, 2012

Exercise 6.

(a) (0.5^{pts}) Give an example of a square matrix (with all real eigenvalues) which is not diagonalizable.

(b) (0.5^{pts}) Find a unit vector with the same direction as $\boldsymbol{v} = (2,-1,0,4,-2,)$.

Exercise 7. (0.5^{pts}) Suppose $\mathbf{A}$ has eigenvalues 1, $\frac{1}{3}$, $\frac{1}{2}$ and its eigenvectors are the columns of $\mathbf{S}$:

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \text{ with } \mathbf{S}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

What are the eigenvalues and eigenvectors of $\mathbf{A}^{-1}$? *MIT Course 18.06 Quiz 3, 2013*

Exercise 8.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$$

(a) (0.5^{pts}) Find all eigenvalues of $\mathbf{A}$.

(b) (1^{pts}) Find a nonsingular matrix $\mathbf{S}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{D}$ (that it is $\mathbf{A} = \mathbf{SDS}^{-1}$).

(c) (1^{pts}) Find $\mathbf{A}^5$. (Si tiene que calcular la quinta potencia de un número a, puede dejarlo indicado como a^5.)

2.21. Grupo E curso 13/14

EXERCISE 1. An odd permutation matrix produces an odd number of "two-element swaps"; an even permutation matrix produces an even number of "two-element swaps".

(a) (0.5pts) When an odd permutation matrix $\mathbf{P}_1$ multiplies an even permutation matrix $\mathbf{P}_2$, the product $\mathbf{P}_1\mathbf{P}_2$ is ____________________ (EXPLAIN WHY).

(b) (0.5pts) If the columns of $\mathbf{B}$ are vectors in the nullspace of $\mathbf{A}$, then $\mathbf{AB}$ is ____________________ (EXPLAIN WHY)

(c) (1pts) If $c = 0$, factor this matrix into $\mathbf{A} = \mathbf{LU}$ (lower triangular times upper triangular):

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & c \end{bmatrix}.$$

(d) (0.5pts) That matrix $\mathbf{A}$ is invertible unless $c =$ ________.

MIT Course 18.06 Quiz 1, 2011

EXERCISE 2. (1pts) Is $\lambda = 3$ an eigenvalue of $\mathbf{A} = \begin{bmatrix} 3 & 0 & 1 \\ 6 & 5 & 7 \\ -3 & -1 & -2 \end{bmatrix}$? If yes, find one corresponding eigenvector.

EXERCISE 3.

(a) (0.5pts) For a really large number N, will this matrix be positive definite? Show why or why not.

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 3 \\ 4 & N & 1 \\ 3 & 1 & 4 \end{bmatrix}$$

(b) (1pts)

SUPPOSE:

$\mathbf{A}$ is positive definite symmetric

$\mathbf{Q}$ is orthogonal (same order as $\mathbf{A}$)

$\mathbf{B}$ is $\mathbf{Q}^\mathsf{T}\mathbf{AQ} = \mathbf{Q}^{-1}\mathbf{AQ}$

SHOW THAT:

1. $\mathbf{B}$ is also symmetric.
2. $\mathbf{B}$ is also positive definite.

MIT Course 18.06 Quiz 3, 2013

EXERCISE 4. Consider the following *unsolvable* linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

(a) (0.5pts) Find the projection matrix that projects any vector onto $\mathcal{C}(\mathbf{A})$.

(b) (0.5pts) Find the best solution to the *unsolvable* linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.

(c) (0.5pts) Find the error vector.

EXERCISE 5. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 \\ 1 & 2 & 0 & -1 & 6 \end{bmatrix}$

(a) (0.5pts) Find a basis of the column space $\mathcal{C}(\mathbf{A})$

(b) (0.5pts) Find a basis of the null space $\mathcal{N}(\mathbf{A})$

EXERCISE 6. True or false (to receive full credit you must explain your answers in a clear and concise way)

(a) (0.5pts) If $\boldsymbol{v}$ is an eigenvector of $\mathbf{A}$, then $\boldsymbol{v}$ must be an eigenvector of $\mathbf{A}^2$ as well.

(b) (0.5pts) If $\boldsymbol{v}$ is an eigenvector of $\mathbf{A}^2$, then $\boldsymbol{v}$ must be an eigenvector of $\mathbf{A}$ as well.

(c) (0.5^{pts}) If $\mathbf{A}$ is an invertible 3×3 matrix and $\boldsymbol{x}$ is a non-zero vector in $\mathbb{R}^3$, then the vectors $\boldsymbol{x}$, $\mathbf{A}\boldsymbol{x}$, and $\mathbf{A}^2\boldsymbol{x}$ must form a basis of $\mathbb{R}^3$.

Exercise 7.

(a) (0.5^{pts}) Is there a 2×2 matrix $\mathbf{A}$ with eigenvalues 4 and 6, such that all the four entries of $\mathbf{A}$ are positive (> 0)? Give an example of such a matrix $\mathbf{A}$, or explain why non exist.
(b) (0.5^{pts}) Is there a 2×2 matrix $\mathbf{B}$ that fails to be diagonalizable, such that all the four entries of $\mathbf{B}$ are positive (> 0)? Give an example of such a matrix $\mathbf{B}$, or explain why none exists.

2.22. Grupo G curso 13/14

EXERCISE 1. In all of this problem, the 3 by 3 matrix $\mathbf{A}$ has eigenvalues λ_1, λ_2, λ_3, with *independent eigenvectors* $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$.

(a) (0.5^{pts}) What are the trace of $\mathbf{A}$ and the determinant of $\mathbf{A}$?

(b) (0.5^{pts}) Suppose: $\lambda_2 = \lambda_3$. Choose the true statement from 1), 2), 3):

1. $\mathbf{A}$ can be diagonalized. Why?
2. $\mathbf{A}$ can not be diagonalized. Why?
3. I need more information to decide. Why?

(c) (0.5^{pts}) From the eigenvalues and eigenvectors, how could you find the matrix $\mathbf{A}$? Give a formula for $\mathbf{A}$ and explain each part carefully.

(d) (1^{pts}) Suppose $\lambda_1 = 2$ and $\lambda_2 = 5$ and $\boldsymbol{x}_1 = (1, 1, 1,)$ and $\boldsymbol{x}_2 = (1, -2, 1,)$. Choose λ_3 and $\boldsymbol{x}_3$ so that $\mathbf{A}$ is *symmetric* positive *semidefinite* but not positive definite.

EXERCISE 2.

(a) (0.5^{pts}) If an m by n matrix $\mathbf{Q}$ has orthonormal columns, is the matrix $\mathbf{Q}$ necessarily invertible? Give a reason or a counterexample.

(b) (0.5^{pts}) What is the nullspace of a matrix $\mathbf{Q}$ with orthonormal columns (and WHY)?

(c) (0.5^{pts}) What is the projection matrix onto the column space of $\mathbf{Q}$? Avoid inverses where possible.

MIT Course 18.06 Quiz 2, 1995

EXERCISE 3. We look for the line $\boldsymbol{y} = c + d\boldsymbol{x}$ closest to 3 points $(x, y,) = (0, 1,)$ and $(1, 2,)$ and $(2, -1,)$.

(a) (0.5^{pts}) If the line went through those points (it doesn't), what three equations would be solved?

(b) (1^{pts}) Find the best c and d by the least squares method.

(c) (0.5^{pts}) Explain the result you get for c and d: How is the vector $\boldsymbol{y} = (1,\ 2,\ -1,)$ y el plano sobre el que se proyecta?

(d) (0.5^{pts}) What is the length of the error vector $\boldsymbol{e}$ (= distance to plane = $\|\boldsymbol{y} - \mathbf{A}\widehat{\boldsymbol{\beta}}\|$)?

MIT Course 18.06 Quiz 2, 1995

EXERCISE 4. True or false (to receive full credit you must explain your answers in a clear and concise way)

(a) (0.5^{pts}) If 1 is the only eigenvalue of a $n \times n$ matrix $\mathbf{A}$, then $\mathbf{A}$ must be the identity matrix.

(b) (0.5^{pts}) If 1 is the only eigenvalue of a $n \times n$ *diagonalizable* matrix $\mathbf{A}$, then $\mathbf{A}$ must be the identity matrix.

(c) (0.5^{pts}) If the rank of a 9×10 matrix $\mathbf{A}$ is 5, then the $\mathcal{N}(\mathbf{A})$ is 4-dimensional.

(d) (0.5^{pts}) If $\mathbf{A}$ is similar to $\mathbf{B}$ (same eigenvalues and same Geometric multiplicities of eigenvectors), and $\mathbf{A}$ is invertible, then $\mathbf{B}$ must be invertible as well.

EXERCISE 5. (0.5^{pts}) Find the determinant of the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & 7 & 0 \\ 0 & 0 & 6 & 0 \\ 3 & 0 & 9 & 4 \\ 0 & 5 & 10 & 6 \end{bmatrix}$.

EXERCISE 6. Let $\mathcal{V}$ the space of all 2×2 matrices $\mathbf{A}$ such that the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is in $\mathcal{N}(\mathbf{A})$.

(a) (0.5^{pts}) Find a basis of $\mathcal{V}$ and thus determine $\dim \mathcal{V}$.

(b) (0.5^{pts}) Find the dimension of the space $\mathcal{W}$ of all 2×2 matrices $\mathbf{A}$ such that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ (Hint, using part (a), you can answer this cuestion without much computational work).

2.23. Grupo E curso 12/13

EXERCISE 1. The problem is to find the determinant of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} x & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

(a) (0.5^{pts}) Find $\det \mathbf{A}$ and give a reason.
(b) (0.5^{pts}) Find $\det \mathbf{B}$ using elimination.
(c) (0.5^{pts}) Find $\det \mathbf{C}$ for any value of x. For this you could use Multilinear Property of determinant function.

MIT Course 18.06 Quiz 2, 1995

EXERCISE 2.
(a) (0.5^{pts}) Why is there no orthonormal matrix $\mathbf{Q}$ such that $\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \mathbf{D}$ if

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}?$$

(b) (0.5^{pts}) For what values of a and b is the quadratic form $ax^2+2xy+by^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ positive definite?

MIT Course 18.06 Final Exam, December 13, 1993

EXERCISE 3. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in the Euclidean space $\mathbb{R}^n$, and let $\underset{n\times n}{\mathbf{A}}$ be the square matrix $[\boldsymbol{u}][\boldsymbol{v}]^{\intercal}$.
(a) (0.5^{pts}) Describe the row space and nullspace of $\mathbf{A}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$.
(b) (0.5^{pts}) Show that $\boldsymbol{u}$ is an eigenvector of $\mathbf{A}$, and find the corresponding eigenvalue.
(c) (0.5^{pts}) What condition must be satisfied by $\boldsymbol{u}$ and $\boldsymbol{v}$ for $\mathbf{A}$ to be skew-symmetric ($\mathbf{A} = -\mathbf{A}^{\intercal}$)?
(d) (0.5^{pts}) What condition must be satisfied by $\boldsymbol{u}$ and $\boldsymbol{v}$ so that $\mathbf{A}^2 = \mathbf{A}$?

MIT Course 18.06 Final Exam, December 13, 1993

EXERCISE 4. Suppose $\mathbf{A}$ is square matrix with eigenvalues $\lambda_1 = 0$, $\lambda_2 = c$ (real) and $\lambda_3 = 2$, and eigenvectors

$$\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{x}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \boldsymbol{x}_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

respectively. In each of the following questions, you must give a reason in order to get full credit.
(a) (0.5^{pts}) For wich values of c (if any) is $\mathbf{A}$ a diagonalizable matrix? Why?
(b) (0.5^{pts}) For wich values of c (if any) is $\mathbf{A}$ a symmetric matrix? Why?
(c) (0.5^{pts}) For wich values of c (if any) is $\mathbf{A}$ a positive definite matrix? Why?

basado en MIT Course 18.06 Quiz 3, November 22, 1993

EXERCISE 5. (1.5^{pts}) Diagonalize the matrix $\begin{bmatrix} 13 & 4 \\ 4 & 7 \end{bmatrix}$. *MIT Course 18.06 Final Exam, December 13, 1993*

EXERCISE 6. The left null space $\mathcal{N}\left(\mathbf{A}^{\intercal}\right)$ of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ is spanned by $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$.
(a) (0.5^{pts}) What is the rank of $\mathbf{A}$? What is the determinant of $\mathbf{A}$?
(b) (1^{pts}) Find a linear equation or equations for a, b and c whose solutions are those values for which

$$\mathbf{A}\boldsymbol{x} = \begin{pmatrix} a \\ b \\ c \\ 1 \end{pmatrix} \text{ can be solved.}$$

(c) (0.5^{pts}) The set of vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ where a, b and c satisfy the equation(s) of part (b), is (circle one):

the empty set, a point, a line, a plane, all of $\mathbb{R}^3$.

Explain why.

(d) (0.5^{pts}) The set of solutions of the equation $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$ is (circle one)

the empty set, a point, a line, a plane, a three-dimensional hyperplane in $\mathbb{R}^4$, all of $\mathbb{R}^4$.

Explain why.

MIT Course 18.06 Final Exam, December 13, 1993

Please answer the last two questions in this page

2.24. Grupo H curso 12/13

EXERCISE 1. Find all possible values for the determinant of the given type of 3×3 real matrix.

(a) (0.5^{pts}) A matrix with linearly independent columns.
(b) (0.5^{pts}) A matrix with $\mathbf{A}^2 = \mathbf{A}$.
(c) (0.5^{pts}) A matrix with pivots 1, 2 and 3.

MIT Course 18.06 Final Exam, December 13, 1993

EXERCISE 2. (1.5^{pts}) Diagonalize the matrix $\begin{bmatrix} 7 & 4 \\ 4 & 13 \end{bmatrix}$. *Basado en MIT Course 18.06 Final Exam, December 13, 1993*

EXERCISE 3.

(a) (0.5^{pts}) Give an example of a matrix with exactly two zero eigenvalues and no zero entries.
(b) (0.5^{pts}) What is the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$?

MIT Course 18.06 Final Exam, December 13, 1993

EXERCISE 4. Suppose that $\mathbf{A}$ has eigenvalues $\lambda = 0, 1, 2$, with respective eigenvectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$.

(a) (0.5^{pts}) Describe the null space of $\mathbf{A}$ in terms of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$.
(b) (0.5^{pts}) Describe the column space of $\mathbf{A}$ in terms of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$.
(c) (0.5^{pts}) Describe the row space of $\mathbf{A}$ in terms of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$.
(d) (0.5^{pts}) Find all solutions to $\mathbf{A}\boldsymbol{x} = (\boldsymbol{v} - \boldsymbol{w})$.

basado en MIT Course 18.06 Quiz 3, November 22, 1993

EXERCISE 5. Suppose that $\mathbf{A}$ is a positive definite matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & b & 0 \\ b & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

(a) (0.5^{pts}) What are the possible values of b?
(b) (0.5^{pts}) How do you know that the matrix $\mathbf{A}^2 + \mathbf{I}$ is positive definite for every b?
(c) (0.5^{pts}) Complete this sentence correctly for a general matrix $\mathbf{M}$, possibly rectangular:

The matrix $\mathbf{M}^{\mathsf{T}}\mathbf{M}$ is symmetric positive definite unless ____________________

____

MIT Course 18.06 Quiz 3, May 10, 1995

EXERCISE 6. Let $\mathbf{A}$ be the matrix $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

(a) (0.5^{pts}) Find a factorization $\mathbf{A} = \mathbf{L}\dot{\mathbf{U}}$, where $\mathbf{L}$ is a echelon form matrix, and $\dot{\mathbf{U}}$ is an unit upper triangular matrix.
(b) (1^{pts}) Find the general solution of $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.
(c) (1^{pts}) The vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is in the column space of $\mathbf{A}$ if a, b and c satisfy what linear conditions?

basado en MIT Course 18.06 Final Exam, December 13, 1993

2.25. Grupo E curso 11/12

EXERCISE 1.

(a) (1^{pts}) Find the determinant of $\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

(b) (2^{pts}) Let $\mathbf{A}$ be the 5 by 5 matrix $\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$. Find all five eigenvalues of $\mathbf{A}$ by noticing that $\mathbf{A} - \mathbf{I}$ has rank 1 and the trace of $\mathbf{A}$ is ____________. Find five linear independent eigenvectors of $\mathbf{A}$

(c) (0.5^{pts}) Find the (3, 1) and (1, 3) entries of $\mathbf{A}^{-1}$.

MIT Course 18.06. Final Exam. Professor Strang. May 16, 2005

EXERCISE 2. Suppose the 4 by 4 matrix $\mathbf{A}$ (with 2 by 2 blocks) is already reduced to its rref form

$$\underset{4\times 4}{\mathbf{A}} = \begin{bmatrix} \underset{2\times 2}{\mathbf{I}} & \underset{2\times 2}{3\mathbf{I}} \\ \underset{2\times 2}{\mathbf{0}} & \underset{2\times 2}{\mathbf{0}} \end{bmatrix}$$

(a) (0.5^{pts}) Find a basis for the column space $\mathcal{C}(\mathbf{A})$.

(b) (0.5^{pts}) Describe all possible bases for $\mathcal{C}(\mathbf{A})$

(c) (1^{pts}) Find a basis (special solutions are good) for the nullspace $\mathcal{N}(\mathbf{A})$.

(d) (0.5^{pts}) Find the complete solution $\boldsymbol{x}$ to the 4 by 4 system

$$\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 5 \\ 4 \\ 0 \\ 0 \end{pmatrix}.$$

MIT Course 18.06 Quiz 1, March 9, 2012

EXERCISE 3.

(a) (0.5^{pts}) Complete this 2 by 2 matrix A (depending on a) so that its eigenvalues are = 1 and = -1:

$$\mathbf{A} = \begin{bmatrix} a & 1 \\ & \end{bmatrix}$$

(b) (0.5^{pts}) How do you know that $\mathbf{A}$ has two independent eigenvectors?

(c) (0.5^{pts}) Which choices of a give orthogonal eigenvectors and which don't?

MIT Course 18.06 Quiz 3 2005 (spring)

EXERCISE 4. This matrix $\mathbf{Q}$ has orthonormal columns $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$:

$$\mathbf{Q} = \begin{bmatrix} .1 & .5 & a \\ .7 & .5 & b \\ .1 & -.5 & c \\ .7 & -.5 & d \end{bmatrix}$$

(a) (0.5^{pts}) What equations must be satisfied by numbers a, b, c, d?

(b) (0.5^{pts}) Is there a unique choice for those numbers, apart from multiplying them all by -1?

MIT Course 18.06 Quiz 2, November 2, 2005

EXERCISE 5.

(a) (0.5^{pts}) Find parametric equations of the plane that goes through the point (0,1,1,) and parallel to the vectors (0,1,2,) and (1,1,0,)

(b) (0.5^{pts}) Write the implicit equation of the same plane.

EXERCISE 6. (0.5^{pts}) Suppose $\mathbf{A}$ is a 5 by 3 matrix and $\mathbf{A}\boldsymbol{x}$ is never zero (except when $\boldsymbol{x}$ is the zero vector). What can you say about the columns of $\mathbf{A}$?

2.26. Grupo H curso 11/12

EXERCISE 1. (2^{pts}) Find the complete solution (also called the *general solution*) to

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

(Strang, 2003, exercise 4 from section 3.4.)

EXERCISE 2.

(a) (1^{pts}) Find a complete set of "special solutions" to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ by noticing the pivot variables and free variables (those have values 1 or 0).

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) (1^{pts}) and (c) (0.5^{pts}) Prove that those special solutions are a <u>basis</u> for the nullspace $\mathcal{N}(\mathbf{A})$. What two facts do you have to prove? Those are parts (b) and (c) of this problem.

MIT Course 18.06 Final Exam, May 16, 2005

EXERCISE 3. The matrix $\mathbf{A}$ has a varing $1 - x$ in the (1,2) position:

$$\mathbf{A} = \begin{bmatrix} 2 & 1-x & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 9 \end{bmatrix}$$

(a) (1^{pts}) When $x = 1$ compute $\det \mathbf{A}$. What is the (1,1) element in the inverse when $x = 1$?

(b) (0.5^{pts}) When $x = 0$ compute $\det \mathbf{A}$.

(c) (0.5^{pts}) How do the properties of the determinant say that $\det \mathbf{A}$ is a linear function of x? For any x compute $\det \mathbf{A}$. For which x's is the matrix singular?

MIT Course 18.06 Quiz 2, November 2, 2005

EXERCISE 4. Let

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 3 \end{bmatrix}$$

(a) (1^{pts}) Find the eigenvalues of $\mathbf{A}$.

(b) (1^{pts}) Find the eigenvectors of $\mathbf{A}$.

(c) (1^{pts}) Diagonalize $\mathbf{A}$, i.e., write $\mathbf{A} = \mathbf{P}\ \mathbf{D}\ \mathbf{P}^{-1}$ for some diagonal matrix $\mathbf{D}$ (escriba explícitamente las tres matrices).

EXERCISE 5. (0.5^{pts}) I was looking for an m by n matrix $\mathbf{A}$ and vectors $\boldsymbol{b}$, $\boldsymbol{c}$ such that $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has no solution and $\mathbf{A}^{\mathsf{T}}\boldsymbol{y} = \boldsymbol{c}$ has exactly one solution. Why can I not find $\mathbf{A}$, $\boldsymbol{b}$, $\boldsymbol{c}$? *MIT Course 18.06 Final Exam, May 16, 2005*

2.27. Grupo A curso 10/11

Exercise 1. Se pide

(a) (0.5^{pts}) Norma del vector $\boldsymbol{v} = (1, 2, 2,)$.
(b) (0.5^{pts}) Un vector ortogonal a $\boldsymbol{v} = (1, 2, 2,)$ con norma 2.
(c) (0.5^{pts}) Los valores de a y b tales que el vector $(1, 2, 1,)$ sea ortogonal al vector $(a, 0, b,)$.

Proporcionado por Javier Gavilanes

Exercise 2. (1^{pts}) Let $\mathbf{A}$ be a 3×3 matrix such that the equation

$$\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

has both $\boldsymbol{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ as solutions.

Find another solution to this equation. Explain.

Exercise 3. Sea la matriz

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix}$$

(a) (2^{pts}) Calcular los autovalores y autovectores de la matriz $\mathbf{A}$.
(b) (0.5^{pts}) ¿Es $\mathbf{A}$ diagonalizable? Justifique su respuesta (*sólo puntuará una respuesta correctamente justificada*)
(c) (0.5^{pts}) ¿Cómo emplearia usted lo que ya sabe de la matriz $\mathbf{A}$ si quisiera calcular su décima potencia $\left(\mathbf{A}^{10}\right)$, pero evitando multiplicar la matriz 10 veces (escriba cómo intervienen los elementos que usted usaría en el cómputo de la potencia de la matriz, pero sin llegar a realizar los cálculos).
(d) (0.5^{pts}) Obtenga $\mathbf{A}^4$ siguiendo de manera coherente a su respuesta al apartado anterior.
(e) (0.5^{pts}) Obtenga la forma cuadrática $f(x, y, z)$ asociada a la matriz $\mathbf{A}$, y clasifiquela.

Versión de un ejercicio proporcionado por Javier Gavilanes

Exercise 4. ¿Cuáles de los siguientes siguientes subconjuntos son subespacios vectoriales de $\mathbb{R}^3$? Justifique su respuesta (*sólo se puntuará si la respuesta está correctamente justificada*).

(a) (0.5^{pts}) $S_1 = \left\{\boldsymbol{x} \in \mathbb{R}^3 \text{ tales que } x_1 = x_3\right\}$.
(b) (0.5^{pts}) $S_2 = \left\{\boldsymbol{x} \in \mathbb{R}^3 \text{ tales que } x_1 = 2\right\}$.

Versión de un ejercicio proporcionado por Javier Gavilanes

Exercise 5. Consider the following matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 4 \\ 0 & 2 & -3 \end{bmatrix}$$

(a) (1^{pts}) Find $\det(\mathbf{A})$

Using the value of $\det(\mathbf{A})$ found above, and properties of determinants, find the following determinants.

(b) (0.5^{pts})

$$\begin{vmatrix} 2 & 1 & 4 \\ 1 & 2 & -3 \\ 0 & 2 & -3 \end{vmatrix}$$

(c) (0.5^{pts})

$$\begin{vmatrix} 3 & 6 & -9 \\ 2 & 1 & 4 \\ 0 & 2 & -3 \end{vmatrix}$$

(d) (0.5^{pts})

$$\begin{vmatrix} 2 & 4 & -6 \\ 4 & 2 & 8 \\ 0 & 4 & -6 \end{vmatrix}$$

(e) (0.5^{pts}) $\det \mathbf{A}^{-1}$

2.28. Grupo E curso 10/11

EXERCISE 1. Suppose $\mathbf{A}$ is a 2 by 2 matrix and $\mathbf{A}\boldsymbol{x} = \boldsymbol{x}$ and $\mathbf{A}\boldsymbol{y} = -\boldsymbol{y}$ (with $\boldsymbol{x} \neq \mathbf{0}$ and $\boldsymbol{y} \neq \mathbf{0}$).

(a) (0.5^{pts}) (Reverse engineering) What is the polynomial $p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$?

(b) (0.5^{pts}) If you know that the first column of $\mathbf{A}$ is $(2, \quad 1,)$, find the second column:

$$\mathbf{A} = \begin{bmatrix} 2 & a \\ 1 & b \end{bmatrix}.$$

(c) (1^{pts}) For that matrix in part (b), find an invertible $\mathbf{S}$ and a diagonal matrix $\mathbf{D}$ so that $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$.

(d) (1^{pts}) Compute $\mathbf{A}^{101}$. (If you don't solve parts (b)–(c), use the description of $\mathbf{A}$ at the start. In all questions show enough work so we can see your method and give due credit.)

(e) (1^{pts}) If $\mathbf{A}\boldsymbol{x} = \boldsymbol{x}$ and $\mathbf{A}\boldsymbol{y} = -\boldsymbol{y}$ (with $\boldsymbol{x} \neq \mathbf{0}$ and $\boldsymbol{y} \neq \mathbf{0}$) prove that $\boldsymbol{x}$ and $\boldsymbol{y}$ are *independent.*

Start of a proof: Suppose $\boldsymbol{z} = c\boldsymbol{x} + d\boldsymbol{y} = \mathbf{0}$. Then $\mathbf{A}\boldsymbol{z} = \dots$ (follow from here.)

MIT Course 18.06 Quiz 2. April 6, 2011

EXERCISE 2. Suppose the following information is known about a matrix $\mathbf{A}$:

$$i) \quad \mathbf{A}\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix}; \qquad ii) \quad \mathbf{A}\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 6 \end{pmatrix}; \qquad ii) \quad \mathbf{A} \text{ is symetric.}$$

Please note the right hand side vector in i) *is the opposite of the right hand side vector in* ii)*.*

(a) (1^{pts}) Is the nullspace of $\mathbf{A}$ zero?

(b) (1^{pts}) Is $\mathbf{A}$ invertible?

(c) (1^{pts}) Does $\mathbf{A}$ have linearly independent eigenvectors?

(d) (0.5^{pts}) Give a specific example of a matrix $\mathbf{A}$ satisfying the above three properties and whose eigenvalues add up to zero.

MIT Course 18.06 Spring 2006 - Review Problems

EXERCISE 3. Let $\mathbf{A}$ a 3×3 matrix with $\det \mathbf{A} = 0$. Determine if each of the following statements is true or false (to receive full credit you must explain your answers in a clear and concise way).

(a) (0.5^{pts}) $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ has a nontrivial solution ($\boldsymbol{x} \neq \mathbf{0}$).

(b) (0.5^{pts}) $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has at least one solution for every $\boldsymbol{b}$.

(c) (0.5^{pts}) For every 3×3 matrix $\mathbf{B}$, we have $\det(\mathbf{A} + \mathbf{B}) = \det(\mathbf{B})$.

(d) (0.5^{pts}) For every 3×3 matrix $\mathbf{B}$, we have $\det(\mathbf{A}\mathbf{B}) > 0$.

(e) (0.5^{pts}) There is a vector $\boldsymbol{b}$ in $\mathbb{R}^3$ such that for the augmented matrix $\operatorname{rg}\big([\mathbf{A}|\boldsymbol{b}]\big) > \operatorname{rg}\big(\mathbf{A}\big)$.

EXERCISE 4. Consider the following matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad \text{y} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) (1^{pts}) Compute the determinant of $\mathbf{A}$ and $\mathbf{B}$. Are these matrices invertible? Compute the inverse matrix when it is possible.

(b) (1^{pts}) Compute the following determinants when it is possible.

- $\det\left(\mathbf{A}\mathbf{A}^{\intercal}\right)$
- $\det\left(\mathbf{B}^{4}\mathbf{A}\right)$
- $\det\left(\mathbf{A}^{-1}\right)$

De un examen intermedio de Mercedes

2.29. Grupo G curso 10/11

Exercise 1. Dada la matriz

$$\mathbf{A} = \begin{bmatrix} 1 & a \\ 2 & b \end{bmatrix}$$

Responda a cada una de estas preguntas añadiendo una breve explicación en cada caso.

(a) (0.5^{pts}) Calcular los valores de los parámetros a y b para que el vector $\begin{pmatrix}-2, & 1,\end{pmatrix}$ sea un autovector asociado al autovalor $\lambda_1 = 5$ de $\mathbf{A}$.
(b) (0.5^{pts}) ¿Cuál es el otro autovalor λ_2?
(c) (0.5^{pts}) ¿Es diagonalizable?
(d) (0.5^{pts}) Considere la forma cuadrática $f(x, y)$ que resulta al multiplicar $\boldsymbol{x}\mathbf{A}\boldsymbol{x}$, donde $\boldsymbol{x} = \begin{pmatrix}x, & y,\end{pmatrix}$. ¿Es esta forma cuadrática definida? (*pista:* piense en su matriz simétrica asociada)
(e) (0.5^{pts}) ¿Es el punto $(0, 0,)$ un mínimo para dicha forma cuadrática $f(x, y)$?
(f) (0.5^{pts}) ¿Qué forma tiene la superficie definida por la forma cuadrática $f(x, y)$?. . . ¿un cuenco? ¿una silla de montar? ¿un valle?

Basado en un problema que me pasó Leonel Cerno

Exercise 2. This question is about the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 3 & 9 \end{bmatrix}$$

(a) (0.5^{pts}) Find the reduced echelon form $\mathbf{R}$. How many independent columns in $\mathbf{A}$?
(b) (1^{pts}) Find a basis for the nullspace of $\mathbf{A}$.
(c) (1^{pts}) If the vector $\boldsymbol{b}$ is the sum of the four columns of $\mathbf{A}$, write down the complete solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.

MIT Course 18.06 Final Exam. May 18, 2010

Exercise 3. (0.5^{pts}) I was looking for an m by n matrix $\mathbf{A}$ and vectors $\boldsymbol{b}$, $\boldsymbol{c}$ such that $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has no solution and $\mathbf{A}^\intercal\boldsymbol{y} = \boldsymbol{c}$ has exactly one solution. Why can I not find $\mathbf{A}$, $\boldsymbol{b}$, and $\boldsymbol{c}$?

Exercise 4. (1^{pts}) Let $\mathbf{A}$ be a 3×3 matrix such that the equation

$$\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

has both $\boldsymbol{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ as solutions.

Find another solution to this equation. Explain.

Exercise 5. Considere la matriz

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Calcule los siguientes determinantes

(a) (1^{pts}) $\det \mathbf{A}$
(b) (0.5^{pts}) $\det \mathbf{A}\mathbf{A}^\intercal$
(c) (0.5^{pts}) $\det \mathbf{A}^{-1}$
(d) (0.5^{pts}) $\det 2\mathbf{A}$
(e) (0.5^{pts})

$$\det\left(\begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 2 & -1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 1 \end{bmatrix}\right)$$

Nótese que hay dos modificaciones respecto a las columnas de $\mathbf{A}$.

2.30. Grupo F curso 09/10

Exercise 1. The determinant of the 1000 by 1000 matrix $\mathbf{A}$ is 12. What is the determinant of $-\mathbf{A}^\mathsf{T}$? (Careful: No credit for the wrong sign.)

Exercise 2. The matrix $\mathbf{A}$ has two special solutions:

$$\boldsymbol{x}_1 = \begin{pmatrix} c \\ 1 \\ 0 \end{pmatrix}; \qquad \boldsymbol{x}_2 = \begin{pmatrix} d \\ 0 \\ 1 \end{pmatrix}$$

(a) Describe all the possibilities for the number of columns of $\mathbf{A}$.
(b) Describe all the possibilities for the number of rows of $\mathbf{A}$.
(c) Describe all the possibilities for the rank of $\mathbf{A}$.

Briefly explain your answers.
(*MIT Course 18.06 Quiz 1, Fall, 2008*)

Exercise 3. Let $\mathbf{A}$ be any matrix and $\mathbf{R}$ its row reduced echelon form. Answer True or False to the statements below and briefly explain. (Note, if there are any counterexamples to a statement below you must choose false for that statement.)

(a) If $\boldsymbol{x}$ is a solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ then $\boldsymbol{x}$ must be a solution to $\mathbf{R}\boldsymbol{x} = \boldsymbol{b}$.
(b) If $\boldsymbol{x}$ is a solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ then $\boldsymbol{x}$ must be a solution to $\mathbf{R}\boldsymbol{x} = \mathbf{0}$.

Exercise 4. Consider the equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

$$\begin{bmatrix} 1 & 0 \\ 4 & 1 \\ 2 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

(a) Put the equation into echelon form $\mathbf{R}\boldsymbol{x} = \boldsymbol{d}$.
(b) For which $\boldsymbol{b}$ are there solutions?

Exercise 5. The equation $(\mathbf{A}^2 - 4\mathbf{I})\boldsymbol{x} = \boldsymbol{b}$ has no solution for some right-hand side $\boldsymbol{b}$. Give as much information as possible about the eigenvalues of the matrix $\mathbf{A}$ (the matrix $\mathbf{A}$ is diagonalizable).

Exercise 6. You are given the matrix

$$\mathbf{A} = \begin{bmatrix} 0.5 & 0.2 & 0.2 \\ 0.1 & 0.5 & 0.5 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$$

One of the eigenvalues is $\lambda = 1$. What are the eigenvalues of $\mathbf{A}$? [Hint: Very little calculation required! You should be able to see another eigenvalue by inspection of the form of $\mathbf{A}$, and the third by an easy calculation. You shouldn't need to compute $\det(\mathbf{A} - \lambda\mathbf{I})$ unless you really want to do it the hard way.]

2.31. Grupo H curso 09/10

EXERCISE 1. The determinant of the 1000 by 1000 matrix $\mathbf{A}$ is 12. What is the determinant of $-\mathbf{A}^\intercal$? (Careful: No credit for the wrong sign.)

EXERCISE 2. The matrix $\mathbf{A}$ has one special solution:

$$\boldsymbol{x}_1 = \begin{pmatrix} c \\ 1 \\ 0 \\ d \end{pmatrix}$$

(a) Describe all the possibilities for the number of columns of $\mathbf{A}$.
(b) Describe all the possibilities for the number of rows of $\mathbf{A}$.
(c) Describe all the possibilities for the rank of $\mathbf{A}$.

Briefly explain your answers.

EXERCISE 3. Your classmate, Nyarlathotep, performed the usual elimination steps to convert[1] $\mathbf{A}$ to echelon form $\mathbf{U}$, obtaining:

$$\mathbf{U} = \begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & 2 & 2 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a set of vectors spanning the nullspace $\mathcal{N}(\mathbf{A})$.

(b) If $\mathbf{U}\boldsymbol{y} = \begin{pmatrix} 9 \\ -12 \\ 0 \end{pmatrix}$, find the complete solution $\boldsymbol{y}$ (i.e. describe all possible solutions $\boldsymbol{y}$).

(c) If $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}$, then $\mathbf{U}\boldsymbol{x} = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$. (read the footnote to known the steps given by Nyarla).

EXERCISE 4. Let $\mathbf{A}$ be any matrix and $\mathbf{R}$ its row reduced echelon form. Answer True or False to the statements below and briefly explain. (Note, if there are any counterexamples to a statement below you must choose false for that statement.)

(a) If $\boldsymbol{x}$ is a solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ then $\boldsymbol{x}$ must be a solution to $\mathbf{R}\boldsymbol{x} = \boldsymbol{b}$.
(b) If $\boldsymbol{x}$ is a solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ then $\boldsymbol{x}$ must be a solution to $\mathbf{R}\boldsymbol{x} = \mathbf{0}$.

EXERCISE 5. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & a & b \\ c & 1 & d \\ e & f & 1 \end{bmatrix}$$

(a) $\mathbf{A}$ has a (double) eigenvalue $\lambda = 2$. What is the other eigenvalue?
(b) Rank of $(\mathbf{A} - 2\mathbf{I})$ is 1. ¿Is $\mathbf{A}$ diagonalizable?

Explain your answers.

References

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[1] Nyarla first subtracted 2 times the first row from the second row, then subtracted -1 time s the first row from the third row, then subtracted 3 times the second row from the third row.

Solutions to Exercises

(Grupo C curso 23/24) Exercise 1(a) *$\boldsymbol{b}$ must be orthogonal to every vector in the left nullspace of* $\mathbf{A}$ —since $\mathcal{N}\left(\mathbf{A}^\intercal\right)$ is the orthogonal complement of $\mathcal{C}\left(\mathbf{A}\right)$, being orthogonal to $\mathcal{N}\left(\mathbf{A}^\intercal\right)$ is equivalent to being in $\mathcal{C}\left(\mathbf{A}\right)$, and the condition for $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ to be solvable is $\boldsymbol{b} \in \mathcal{C}\left(\mathbf{A}\right)$.

□

(Grupo C curso 23/24) Exercise 1(b) Since ${}_{i|}(\mathbf{AB}) = ({}_{i|}\mathbf{A})\mathbf{B}$, each row of $\mathbf{AB}$ is a linear combination of the rows of $\mathbf{B}$. Therefore, *row space of* $\mathbf{AB}$ *must be contained in the row space of* $\mathbf{B}$, i.e. $\mathcal{C}\left((\mathbf{AB})^\intercal\right) \subseteq \mathcal{C}\left((\mathbf{B})^\intercal\right)$.

□

(Grupo C curso 23/24) Exercise 1(c) *$\boldsymbol{x}_1 - \boldsymbol{x}_2$ must be in the null space of* $\mathbf{A}$. The only way $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ can have multiple solutions is for them to differ by something in $\mathcal{N}\left(\mathbf{A}\right)$. We can see this explicitly from $\mathbf{A}(\boldsymbol{x}_1 - \boldsymbol{x}_2) = \mathbf{A}\boldsymbol{x}_1 - \mathbf{A}\boldsymbol{x}_2 = \boldsymbol{b} - \boldsymbol{b} = \boldsymbol{0}$.

□

(Grupo C curso 23/24) Exercise 2(a) The three equations tell us that $\mathbf{AQ} = \mathbf{I}$, where $\mathbf{Q} = \begin{bmatrix}\boldsymbol{q}_1; & \boldsymbol{q}_2; & \boldsymbol{q}_3;\end{bmatrix}$. So $\mathbf{A} = \mathbf{Q}^{-1}$, and since $\mathbf{Q}$ is an orthogonal matrix, we have $\mathbf{A} = \mathbf{Q}^{-1} = \mathbf{Q}^\intercal = \begin{bmatrix}\boldsymbol{q}_1; & \boldsymbol{q}_2; & \boldsymbol{q}_3;\end{bmatrix}^\intercal$. Therefore

$$\boxed{\mathbf{A} \text{ is the matrix whose rows are the vectors } \boldsymbol{q}_i\text{, that is, } \quad {}_{i|}\mathbf{A} = \boldsymbol{q}_i\ .}$$

□

(Grupo C curso 23/24) Exercise 2(b) Since $\mathbf{A} = \mathbf{Q}^\intercal$ then $\det \mathbf{A} = \det \mathbf{Q}$; and since $\mathbf{Q}^\intercal\mathbf{Q} = \mathbf{I}$

$$1 = \det \mathbf{I} = \det\left(\mathbf{Q}^\intercal\mathbf{Q}\right) = \det\left(\mathbf{Q}^\intercal\right)\det\left(\mathbf{Q}\right) = |\mathbf{Q}|^2 \quad \Rightarrow \quad \det \mathbf{A} = \pm 1.$$

□

(Grupo C curso 23/24) Exercise 3(a) The two nullspaces are eigenspaces of $\mathbf{A}$ for $\lambda = 1$ and 5 respectively; and they are clearly not orthogonal, so $\mathbf{A}$ *is not symmetric*.

□

(Grupo C curso 23/24) Exercise 3(b) From the vectors in the nullspaces, $\mathbf{A}$ must be a 2×2 matrix, and we are given two eigenvectors for two eigenvalues (1 and 5). Hence, $\mathbf{A}$ is diagonalizable and

$$\mathbf{A} = \begin{bmatrix}1 & 1\\ 2 & -2\end{bmatrix}\begin{bmatrix}1 & \\ & 5\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & -2\end{bmatrix}^{-1}.$$

You weren't required to simplify it further, but it turns out that $\mathbf{A} = \left[\begin{smallmatrix}3 & -1\\ -4 & 3\end{smallmatrix}\right]$ if you work it all out.

□

(Grupo C curso 23/24) Exercise 4(a) The matrix associated with the quadratic form is given by

$$\mathbf{A} = \begin{bmatrix}2 & 4 & 0\\ 4 & 0 & 8\\ 0 & 8 & -1\end{bmatrix}$$

□

(Grupo C curso 23/24) Exercise 4(b) Yes, since the matrix is symmetric. Yes, since $|\mathbf{A}| = -112 \neq 0$.

□

(Grupo C curso 23/24) Exercise 4(c) Diagonalizing by congruence we get

$$\left[\begin{array}{ccc}2 & 4 & 0\\ 4 & 0 & 8\\ 0 & 8 & -1\\ \hline 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right] \xrightarrow{\underset{[(-2)\mathbf{1}+\mathbf{2}]}{\tau}} \left[\begin{array}{ccc}2 & 0 & 0\\ 4 & -8 & 8\\ 0 & 8 & -1\\ \hline 1 & -2 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right] \xrightarrow[{[(-2)\mathbf{1}+\mathbf{2}]}]{\tau} \left[\begin{array}{ccc}2 & 0 & 0\\ 0 & -8 & 8\\ 0 & 8 & -1\\ \hline 1 & -2 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right] \xrightarrow{\underset{[(1)\mathbf{2}+\mathbf{3}]}{\tau}} \left[\begin{array}{ccc}2 & 0 & 0\\ 0 & -8 & 0\\ 0 & 8 & 7\\ \hline 1 & -2 & -2\\ 0 & 1 & 1\\ 0 & 0 & 1\end{array}\right] \xrightarrow[{[(1)\mathbf{2}+\mathbf{3}]}]{\tau} \left[\begin{array}{ccc}2 & 0 & 0\\ 0 & -8 & 0\\ 0 & 0 & 7\\ \hline 1 & -2 & -2\\ 0 & 1 & 1\\ 0 & 0 & 1\end{array}\right]$$

Therefore, $\mathbf{A}$ is neither positive nor negative definite.

□

(Grupo C curso 23/24) Exercise 4(d) Let $\mathbf{U} = \mathbf{I}_{\substack{\tau\\ [(-2)\mathbf{1}+\mathbf{2}]\\ [(1)\mathbf{2}+\mathbf{3}]}} = \begin{bmatrix}1 & -2 & -2\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$. We known that $\mathbf{D} = \mathbf{U}^\intercal\mathbf{A}\mathbf{U}$ and therefore

$$\mathbf{A} = (\mathbf{U}^{-1})^\intercal\mathbf{D}\mathbf{U}^{-1}, \quad \text{where} \quad \mathbf{U}^{-1} = \mathbf{I}_{\substack{\tau\\ [(-1)\mathbf{2}+\mathbf{3}]\\ [(2)\mathbf{1}+\mathbf{2}]}} = \begin{bmatrix}1 & 2 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}.$$

So,

$$\boldsymbol{x}\mathbf{A}\boldsymbol{x} = (x, y, z,) \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\,(x+2y)^2 - 8\,(y-z)^2 + 7z^2.$$

□

(Grupo C curso 23/24) Exercise 5(a) Since $\mathbf{A}_{\underset{[(-4)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}\underset{[\mathbf{1}\rightleftharpoons\mathbf{2}]}{\boldsymbol{\tau}}} = \mathbf{B}$, then $|\mathbf{B}| = -|\mathbf{A}| = -5$.

□

(Grupo C curso 23/24) Exercise 5(b) Since the last row is a linear combination of the other two, we know $|\mathbf{C}| = 0$ and therefore an eigenvalue of $\mathbf{C}$ is $\lambda = 0$.

□

(Grupo C curso 23/24) Exercise 6(a) If, for example, we take as direction vectors of the plane: $\boldsymbol{v} = \boldsymbol{b} - \boldsymbol{a} = (-1, -1, 1,)$ and $\boldsymbol{w} = \boldsymbol{c} - \boldsymbol{a} = (0, 0, 1,)$, and we choose the point $\boldsymbol{a}$ in the plane; we find the following parametric equation:

$$\left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2,\ \boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{bmatrix} -1 & 0 \\ -1 & 0 \\ 1 & 1 \end{bmatrix} \boldsymbol{p} \right\} = \left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2,\ \boldsymbol{v} = \begin{bmatrix} -1 & 0 \\ -1 & 0 \\ 1 & 1 \end{bmatrix} \boldsymbol{p} \right\}$$

□

(Grupo C curso 23/24) Exercise 6(b)

$$\left[\begin{array}{ccc} -1 & -1 & -1 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[\ ]{\substack{\underset{[(-1)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}} \\ (-1)\mathbf{1}+\mathbf{3}}} \left[\begin{array}{ccc} -1 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 1 & {\color{red}-1} & -1 \\ 0 & {\color{red}1} & 0 \\ 0 & {\color{red}0} & 1 \end{array}\right],$$

so the answer is the set of multiples of $\begin{pmatrix} -1 & 1 & 0 \end{pmatrix}$.

□

(Grupo E curso 23/24) Exercise 1(a) The rank of $\mathbf{A}$ has to be less than 4 for some vectors $\boldsymbol{b} \in \mathbb{R}^4$ not to belong to the $\mathcal{C}\left(\mathbf{A}\right)$, and it has to be less than 3 for the dimension of the $\mathcal{N}\left(\mathbf{A}\right)$ be greater than zero, so the rank of $\mathbf{A}$ can only be 0, 1, or 2..

□

(Grupo E curso 23/24) Exercise 1(b) Since $\left(\mathbf{AB}\right)_{|j} = \mathbf{A}\left(\mathbf{B}_{|j}\right)$, each column of $\mathbf{AB}$ is a linear combination of the columns of $\mathbf{A}$. Therefore, *column space of* $\mathbf{AB}$ *must be contained in the column space of* $\mathbf{A}$, i.e. $\mathcal{C}\left(\mathbf{AB}\right) \subseteq \mathcal{C}\left(\mathbf{A}\right)$.

□

(Grupo E curso 23/24) Exercise 1(c) $\boldsymbol{y}_1 - \boldsymbol{y}_2$ *must be in the left null space of* $\mathbf{A}$. The only way $\boldsymbol{y}\mathbf{A} = \boldsymbol{c}$ can have multiple solutions is for them to differ by something in $\mathcal{N}\left(\mathbf{A}^{\intercal}\right)$. We can see this explicitly from: $(\boldsymbol{y}_1 - \boldsymbol{y}_2)\mathbf{A} = \boldsymbol{y}_1\mathbf{A} - \boldsymbol{x}_2\mathbf{A} = \boldsymbol{c} - \boldsymbol{c} = \boldsymbol{0}$.

□

(Grupo E curso 23/24) Exercise 2(a) The inverse of a product of matrices is the product of its inverses in reverse order, so $\mathbf{A}^{-1} = \mathbf{U}\mathbf{L}\mathbf{U}^{-1}\mathbf{L}^{-1}$.

□

(Grupo E curso 23/24) Exercise 2(b)

$$\begin{aligned} \det \mathbf{A} &= \det\left(\mathbf{L}\mathbf{U}\mathbf{L}^{-1}\mathbf{U}^{-1}\right) \\ &= \det\left(\mathbf{L}\right)\det\left(\mathbf{U}\right)\det\left(\mathbf{L}^{-1}\right)\det\left(\mathbf{U}^{-1}\right) \\ &= \det\left(\mathbf{L}\right)\det\left(\mathbf{L}^{-1}\right)\det\left(\mathbf{U}\right)\det\left(\mathbf{U}^{-1}\right) \;=\; 1. \end{aligned}$$

□

(Grupo E curso 23/24) Exercise 3(a) $\begin{bmatrix} 4 & 1 \\ d & 2 \end{bmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 12 \\ 2d+8 \end{pmatrix} \implies \begin{cases} 12 = \lambda 2 \;\Rightarrow \lambda = 6 \\ 2d + 8 = \lambda 4 \end{cases} \implies$ $2d + 8 = 24 \implies \boxed{d = 8}$.

□

(Grupo E curso 23/24) Exercise 3(b) Matrix $(\mathbf{A} - (-2)\mathbf{I})$ must be singular. Hence

$$|\mathbf{A} - (-2)\mathbf{I}| = \det\begin{bmatrix} 6 & 1 \\ d & 4 \end{bmatrix} = 24 - d = 0 \implies \boxed{d = 24}.$$

□

(Grupo E curso 23/24) Exercise 3(c) The issue of nondiagonalizability only comes up for a matrix that has some repeated eigenvalues. So $\lambda_1 = \lambda_2 = \lambda$. Therefore $2\lambda = \operatorname{tr}(\mathbf{A}) = 6 \implies \lambda = 3$. Hence $|\mathbf{A}| = \lambda^2 = 9$

$$|\mathbf{A}| = \det\begin{bmatrix} 4 & 1 \\ d & 2 \end{bmatrix} = 8 - d = 9 \implies \boxed{d = -1}. \text{ With } \dim\mathcal{N}(\mathbf{A} - 3\mathbf{I}) = \dim\mathcal{N}\left(\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\right) = 1.$$

□

(Grupo E curso 23/24) Exercise 4. The components for each basis indicates that $\mathbf{A}$ has $m = 3$ rows, $n = 3$ columns, and rank $r = 2$. The left nullspace indicates that the sum of the rows of $\mathbf{A}$ is $\mathbf{0}$. Lets find a basis for its orthogonal complement of $\mathcal{N}(\mathbf{A}^\intercal)$, that is, a basis for $\mathcal{C}(\mathbf{A})$:

$$\left[\begin{array}{ccc} 1 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ \hline 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]; \qquad \text{the special solutions form a basis for } \mathcal{C}(\mathbf{A}).$$

So, for example, $\mathbf{A}_{|1} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{A}_{|2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. The nullspace indicates that the sum of the columns of $\mathbf{A}$ is $\mathbf{0}$. Therefore $\mathbf{A}_{|3} = -(\mathbf{A}_{|1} + \mathbf{A}_{|2}) = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$. Hence, one example is $\mathbf{A} = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$.

□

(Grupo E curso 23/24) Exercise 5(a) $\left[\begin{pmatrix} 3 \\ 4 \end{pmatrix};\right]$.

□

(Grupo E curso 23/24) Exercise 5(b) $\left\{\boldsymbol{v} \in \mathbb{R}^2 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}\boldsymbol{p}\right\}$

□

(Grupo E curso 23/24) Exercise 5(c)

$$\left[\begin{array}{cc} 3 & 4 \\ \hline x & y \\ \hline 1 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(3)\mathbf{2}] \\ [(-4)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc} 3 & 0 \\ \hline x & -4x+3y \\ \hline 1 & -1 \end{array}\right] \quad \Rightarrow \quad \left\{\boldsymbol{v} \in \mathbb{R}^2 \mid \begin{bmatrix} -4 & 3 \end{bmatrix}\boldsymbol{v} = (-1,)\right\}$$

□

(Grupo E curso 23/24) Exercise 6(a) Since $\left[\begin{array}{ccc} -2 & 0 & -2 \\ 0 & 2 & -4 \\ -2 & -4 & 2 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc} -2 & 0 & 0 \\ 0 & 2 & -4 \\ 0 & -4 & 4 \\ \hline 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{\boldsymbol{\tau} \\ [(2)\mathbf{2}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ [(2)\mathbf{2}+\mathbf{3}]}}$

$$\left[\begin{array}{ccc} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \\ \hline 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{array}\right];$$

$\mathbf{A}$ has two negative eigenvalues and one positive eigenvalue.

□

(Grupo E curso 23/24) Exercise 6(b) Since

$$\mathsf{U}^{\mathsf{T}}\mathsf{A}\mathsf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & -2 \\ 0 & 2 & -4 \\ -2 & -4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \mathsf{D}$$

we have that $\mathsf{A} = \left(\mathsf{U}^{\mathsf{T}}\right)^{-1}\mathsf{D}\mathsf{U}^{-1}$ and therefore $\boldsymbol{x}\mathsf{A}\boldsymbol{x} = \boldsymbol{x}(\mathsf{U}^{-1})^{\mathsf{T}}\mathsf{D}(\mathsf{U}^{-1})\boldsymbol{x}$, where

$$(\mathsf{U}^{-1})\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+z \\ y-2z \\ z \end{pmatrix}.$$

Hence $\boldsymbol{x}\mathsf{A}\boldsymbol{x} = -2\,(x+z)^2 + 2\,(y-2z)^2 - 4z^2$.

□

(Grupo B curso 22/23) Exercise 1(a) The corresponding matrix is $\mathsf{A} = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$. Its characteristic polynomial is:

$$\det(\mathsf{A} - \lambda\mathsf{I}) = \det \begin{bmatrix} 4-\lambda & 0 & 2 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & 1-\lambda \end{bmatrix} = \lambda\left(-\lambda^2 + 6\lambda - 5\right) = 0.$$

Hence the matrix is *semi*-definite positive, since its spectrum (the roots) is $\{0, 1, 5\}$. Therefore, the matrix is positive semidefinite since its spectrum (the roots) is 0, 1, 5. But it is much easier to check that there are no negative entries on the diagonal of the matrix obtained by diagonalizing by *congruence*:

$$\begin{bmatrix} 4 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{\tau \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(2)\mathbf{3}]}]{\substack{\tau \\ [(2)\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{3}]}} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

□

(Grupo B curso 22/23) Exercise 1(b) Lets use a method that cannot fail

$$\begin{bmatrix} -1 & -1 & 1 \\ -1 & a-1 & 1-a \\ 1 & 1-a & a \end{bmatrix} \xrightarrow[\substack{\tau \\ [(1)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{2}]}]{\substack{\tau \\ [(-1)\mathbf{1}+\mathbf{2}] \\ [(1)\mathbf{1}+\mathbf{3}]}} \begin{bmatrix} -1 & 0 & 0 \\ 0 & a & -a \\ 0 & -a & a+1 \end{bmatrix} \xrightarrow[\substack{\tau \\ [(1)\mathbf{2}+\mathbf{3}]}]{\substack{\tau \\ [(1)\mathbf{2}+\mathbf{3}]}} \begin{bmatrix} -1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is impossible, there is at least one negative and another positive eigenvalues regardless of the value of a.

□

(Grupo B curso 22/23) Exercise 2(a) $P^* = \left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2,\ \boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{bmatrix} -2 & -1 \\ 2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}$

since

$$\left[\begin{array}{cccc} 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-2)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ \hline 1 & 0 & -2 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(2)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

□

(Grupo B curso 22/23) Exercise 2(b) $P^* = \left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \begin{bmatrix} 0 & -1 & 2 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \boldsymbol{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$ since

$$\left[\begin{array}{cccc} -2 & 2 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 \\ \hline x & y & z & w \end{array}\right] \xrightarrow{\substack{\tau \\ [(1)\mathbf{1}+\mathbf{2}] \\ [(2)\mathbf{3}] \\ [(1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cccc} -2 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 \\ \hline 1 & 2 & 3 & 1 \\ \hline x & x+y & x+2z & w \end{array}\right] \xrightarrow{\substack{\tau \\ [(-1)\mathbf{2}+\mathbf{3}] \\ [(1)\mathbf{2}+\mathbf{4}]}} \left[\begin{array}{cccc} -2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ \hline 1 & 2 & 1 & 3 \\ \hline x & x+y & -y+2z & w+x+y \end{array}\right].$$

$\square$

(Grupo B curso 22/23) Exercise 3(a) Since both perpendicular vectors correspond to a square with sides of length 1, the enclosed area is 1. Therefore, $\boxed{\text{possible values for the determinant are 1 or } -1}$.

$\square$

(Grupo B curso 22/23) Exercise 3(b) Using elementary transformations we have that

$$\begin{aligned}
\det\left[(\boldsymbol{q}_1-\boldsymbol{q}_2);\ (\boldsymbol{q}_1+\boldsymbol{q}_2);\right] =& \det\left[(\boldsymbol{q}_1-\boldsymbol{q}_2);\ 2\boldsymbol{q}_1;\right] && \text{Adding the first column to the second}\\
=& 2\det\left[(\boldsymbol{q}_1-\boldsymbol{q}_2);\ \boldsymbol{q}_1;\right] &&\\
=& 2\det\left[-\boldsymbol{q}_2;\ \boldsymbol{q}_1;\right] && \text{Subtracting the second column from the first}\\
=& -2\det\left[\boldsymbol{q}_2;\ \boldsymbol{q}_1;\right] &&\\
=& 2\det\left[\boldsymbol{q}_1;\ \boldsymbol{q}_2;\right] && \text{swapping the columns}
\end{aligned}$$

That is, $\boxed{\text{this determinant can only be 2 or } -2}$.

$\square$

(Grupo B curso 22/23) Exercise 4. Since $(\mathbf{A}^2-4\mathbf{I})$ is singular, $\lambda=4$ is an eigenvalue of $\mathbf{A}^2$. Hence an eigenvalue of $\mathbf{A}$ is $\lambda=2$ or $\lambda=-2$ (or both).

$\square$

(Grupo B curso 22/23) Exercise 5(a) **False.** $(0,2)$ is a solution of $\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix}$, but not of $\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix}$.

$\square$

(Grupo B curso 22/23) Exercise 5(b) **False.** $(-1,1)$ is a solution of $\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$, but not of $\begin{bmatrix}1&0\\1&0\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$.

$\square$

(Grupo B curso 22/23) Exercise 5(c) **False.** $(1,1)$ is a solution of $\begin{pmatrix}x, & y,\end{pmatrix}\begin{bmatrix}1&1\\1&1\end{bmatrix}=\begin{pmatrix}2, & 2,\end{pmatrix}$, but not of $\begin{pmatrix}x, & y,\end{pmatrix}\begin{bmatrix}1&0\\1&0\end{bmatrix}=\begin{pmatrix}2, & 2,\end{pmatrix}$.

$\square$

(Grupo B curso 22/23) Exercise 5(d) **True.** Let $\mathbf{E}=\mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}$ such that $\mathbf{AE}=\mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}=\mathbf{R}$, then

$$\boldsymbol{x}\mathbf{A}=\mathbf{0} \quad\Longrightarrow\quad \boldsymbol{x}\mathbf{AE}=\mathbf{0E}=\mathbf{0}.$$

$\square$

(Grupo B curso 22/23) Exercise 6(a) The vectors orthogonal to the nullspace of $\mathbf{A}$ are the rows of $\mathbf{A}$. Since we know that the matrix $\mathbf{A}$ is singular and it is clearly not rank one, it follows that the rank of $\mathbf{A}$ is two. The first two rows are independent and therefore the orthogonal complement of the nullspace of $\mathbf{A}$ is spanned by the two first row vectors of $\mathbf{A}$:

$$\mathcal{N}\left(\mathbf{A}\right)^{\perp}=\mathcal{C}\left(\mathbf{A}^{\intercal}\right)=\left\{\boldsymbol{v}\in\mathbb{R}^3\ \middle|\ \exists\boldsymbol{p}\in\mathbb{R}^2,\ \boldsymbol{v}=\begin{bmatrix}1&2\\3&2\\7&6\end{bmatrix}\boldsymbol{p}\right\}=\mathcal{L}\left(\left[\begin{pmatrix}1\\3\\7\end{pmatrix};\ \begin{pmatrix}2\\2\\6\end{pmatrix};\right]\right).$$

$\square$

(Grupo E curso 22/23) Exercise 1(a) Como $\mathbf{A}^3=\mathbf{SD}^3\mathbf{S}^{-1}$,

$$\mathbf{A}^3\begin{pmatrix}1\\-1\end{pmatrix}=\begin{bmatrix}1&2\\0&1\end{bmatrix}\begin{bmatrix}(2)^3&\\&(-2)^3\end{bmatrix}\begin{bmatrix}1&-2\\0&1\end{bmatrix}\begin{pmatrix}1\\-1\end{pmatrix}=\begin{bmatrix}1&2\\0&1\end{bmatrix}\begin{bmatrix}8&-16\\0&-8\end{bmatrix}\begin{pmatrix}1\\-1\end{pmatrix}=$$
$$=\begin{bmatrix}8&-32\\0&-8\end{bmatrix}\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}40\\8\end{pmatrix}.$$

Otra forma de verlo: como $\left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{1}+\mathbf{3}]}]{\substack{\tau \\ [(-2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \\ \hline 1 & -2 & 1 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1 & -2 & 3 \\ 0 & 1 & -1 \\ \hline 0 & 0 & 1 \end{array}\right]$ entonces $\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-1)\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, Así $\mathbf{A}^3\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \mathbf{A}^3\begin{pmatrix} 3 \\ 0 \end{pmatrix} + \mathbf{A}^3\begin{pmatrix} -2 \\ -1 \end{pmatrix} = (2)^3\begin{pmatrix} 3 \\ 0 \end{pmatrix} + (-2)^3\begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 40 \\ 8 \end{pmatrix}$.

□

(Grupo E curso 22/23) Exercise 1(b) Since $q(x,y,z) = az^2+2x^2+8xy+y^2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}\begin{bmatrix} 2 & 4 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, diagonalizando por congruencia: $\begin{bmatrix} 2 & 4 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & a \end{bmatrix} \xrightarrow[\substack{\tau \\ [(-2)\mathbf{1}+\mathbf{2}]}]{\substack{\tau \\ [(-2)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & a \end{bmatrix}$, concluimos que $q(x,y,z)$ puede tomar valores tanto positivos como negativos (sea cual sea el valor de a).
Alternativamente: los menores principales son 2, -14 y $-14a$, por lo que la forma cuadrática puede tomar valores tanto positivos como negativos (sea cual sea el valor de a).

□

(Grupo E curso 22/23) Exercise 2(a) Since the matrix is triangular, the eigenvalues are the numbers on the main diagonal: $\lambda_1 = -1$ and $\lambda_2 = 2$.

For $\lambda = -1$. Como $(\mathbf{A} + 1\mathbf{I}) = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ -1 & 1 & 3 & 0 \\ -1 & 1 & 0 & 3 \end{bmatrix}$, a basis of eigespace $\mathcal{E}_{(\lambda=-1)}$ is $\left[\begin{pmatrix} 0 \\ -3 \\ 1 \\ 1 \end{pmatrix};\right]$.

For $\lambda = 2$. Como $(\mathbf{A} - 2\mathbf{I}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & -3 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}$, a basis of $\mathcal{E}_{(\lambda=2)}$ is $\left[\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix};\right]$.

□

(Grupo E curso 22/23) Exercise 2(b) Yes, since there are 4 linearly independent eigenvectors

□

(Grupo E curso 22/23) Exercise 2(c) This factorization $\mathbf{A} = \mathbf{PDP}^\intercal$ implies that $\mathbf{A}$ must be symetric; but $\mathbf{A}$ is not. Therefore, it is not possible.

□

(Grupo E curso 22/23) Exercise 2(d) $|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|} = \frac{1}{\text{product of eigenvalues of }\mathbf{A}} = -\frac{1}{8}$.

□

(Grupo E curso 22/23) Exercise 3(a) $\lambda_1+\lambda_2+\lambda_3 = \operatorname{tr}(\mathbf{A}) \quad\Rightarrow\quad 2+2+\lambda_3 = 3 \quad\Rightarrow\quad \boxed{\lambda_3 = -1}$.

□

(Grupo E curso 22/23) Exercise 3(b) $\operatorname{rg}((\mathbf{A} - 2\mathbf{I})) = 1 \quad\Rightarrow\quad \dim\mathcal{N}((\mathbf{A} - 2\mathbf{I})) = 2$; therefore $\mathbf{A}$ *is diagonalizable* since the geometric multiplicity of $\lambda = 2$ is equal to its algebraic multiplicity.

□

(Grupo E curso 22/23) Exercise 4(a) $\mathbf{P}_A = \mathbf{A}(\mathbf{A}^\intercal\mathbf{A})^{-1}\mathbf{A}^\intercal$ and $\mathbf{P}_B = \mathbf{B}(\mathbf{B}^\intercal\mathbf{B})^{-1}\mathbf{B}^\intercal = \mathbf{BB}^\intercal$ (since columns of $\mathbf{B}$ are orthonormal).

□

(Grupo E curso 22/23) Exercise 4(b) Since $\mathcal{C}(\mathbf{B}) \subset \mathbb{R}^5$ has dimension 3, its orthogonal complement has dimension 2. So, there are non-zero vectors $\boldsymbol{y} \in \mathcal{C}(\mathbf{B})^\perp$, and for any $\boldsymbol{y} \in \mathcal{C}(\mathbf{B})^\perp$ the proyectiont $(\mathbf{P}_B)\boldsymbol{y} = \mathbf{0}$. Hence $\mathbf{P}_B$ is singular and $\boxed{\det(\mathbf{P}_B) = 0}$.

□

(Grupo E curso 22/23) Exercise 4(c) Since $\mathbf{P}_B$ projects any vector of $\mathbb{R}^5$ onto $\mathcal{C}(\mathbf{A}) = \mathcal{C}(\mathbf{B})$, and since de columns of both $\mathbf{A}$ and $\mathbf{B}$ belong to $\mathcal{C}(\mathbf{A}) = \mathcal{C}(\mathbf{B})$, then $(\mathbf{P}_B)\mathbf{B} = \mathbf{B}$ and $(\mathbf{P}_B)\mathbf{A} = \mathbf{A}$.

□

(Grupo E curso 22/23) Exercise 4(d) Since the columns of $(\mathbf{P}_B)$ are linear combinations of the columns of $\mathbf{B}$ and the columns of $(\mathbf{P}_A)$ are linear combinations of the columns of $\mathbf{A}$, using the same reasoning $(\mathbf{P}_B)(\mathbf{P}_A) = \mathbf{P}_A$ and $(\mathbf{P}_A)(\mathbf{P}_B) = \mathbf{P}_B$.

Another way: $\begin{cases} (\mathbf{P}_B)(\mathbf{P}_A) = (\mathbf{P}_B)\mathbf{A}\left(\mathbf{A}^\intercal\mathbf{A}\right)^{-1}\mathbf{A}^\intercal = \mathbf{A}\left(\mathbf{A}^\intercal\mathbf{A}\right)^{-1}\mathbf{A}^\intercal = (\mathbf{P}_A) \\ (\mathbf{P}_A)(\mathbf{P}_B) = (\mathbf{P}_B)\mathbf{B}\mathbf{B}^\intercal = \mathbf{B}\mathbf{B}^\intercal = (\mathbf{P}_B) \end{cases}$.

□

(Grupo E curso 22/23) Exercise 5(a)

$$\begin{aligned} \boldsymbol{x}\cdot\boldsymbol{x} &= (c_1\boldsymbol{q}_1 + c_2\boldsymbol{q}_2 + c_3\boldsymbol{q}_3)\cdot(c_1\boldsymbol{q}_1 + c_2\boldsymbol{q}_2 + c_3\boldsymbol{q}_3) \\ &= c_1^2(\boldsymbol{q}_1\cdot\boldsymbol{q}_1) + c_1c_2(\boldsymbol{q}_1\cdot\boldsymbol{q}_2) + \cdots + c_3c_2(\boldsymbol{q}_3\cdot\boldsymbol{q}_2) + c_3^2(\boldsymbol{q}_3\cdot\boldsymbol{q}_3) \\ &= c_1^2 + c_2^2 + c_3^2. \end{aligned}$$

□

(Grupo E curso 22/23) Exercise 5(b)

$$\begin{aligned} \boldsymbol{x}\mathbf{A}\boldsymbol{x} = \boldsymbol{x}\cdot\mathbf{A}\boldsymbol{x} &= (c_1\boldsymbol{q}_1 + c_2\boldsymbol{q}_2 + c_3\boldsymbol{q}_3)\cdot(c_1\mathbf{A}\boldsymbol{q}_1 + c_2\mathbf{A}\boldsymbol{q}_2 + c_3\mathbf{A}\boldsymbol{q}_3) \\ &= (c_1\boldsymbol{q}_1 + c_2\boldsymbol{q}_2 + c_3\boldsymbol{q}_3)\cdot(c_1\lambda_1\boldsymbol{q}_1 + c_2\lambda_2\boldsymbol{q}_2 + c_3\lambda_3\boldsymbol{q}_3) \\ &= c_1^2\lambda_1 + c_2^2\lambda_2 + c_3^2\lambda_3. \end{aligned}$$

□

(Grupo D curso 21/22) Exercise 1(a) Let's find a vector in $\mathcal{C}\left(\mathbf{A}\right)$ perpendicular to $\mathbf{A}_{|1}$. That is $(\mathbf{A}_{|1})\cdot\left(a\mathbf{A}_{|1} + b\mathbf{A}_{|2}\right) = 0$:

$$\begin{pmatrix}1\\2\\-2\end{pmatrix}\cdot\left(a\begin{pmatrix}1\\2\\-2\end{pmatrix} + b\begin{pmatrix}1\\-1\\4\end{pmatrix}\right) = \begin{pmatrix}1\\2\\-2\end{pmatrix}\cdot\begin{pmatrix}a+b\\2a-b\\-2\,(a-2b)\end{pmatrix} = 9\,(a-b) = 0;$$

so $a = b$. Then (for $a = 1$), $\mathbf{A}_{|1} + \mathbf{A}_{|2} = \begin{pmatrix}2, & 1, & 2,\end{pmatrix}$ belongs to $\mathcal{C}\left(\mathbf{A}\right)$ and it is perpendicular to $\mathbf{A}_{|1}$. Hence, an orthogonal basis for $\mathcal{C}\left(\mathbf{A}\right)$ is $\left[\begin{pmatrix}1\\2\\-2\end{pmatrix}; \begin{pmatrix}2\\1\\2\end{pmatrix};\right]$.

□

(Grupo D curso 21/22) Exercise 1(b) Applying elimination to find a vector perpendicular to the previous basis vectors:

$$\left[\begin{array}{ccc}1&2&-2\\2&1&2\\\hline 1&0&0\\0&1&0\\0&0&1\end{array}\right] \xrightarrow[{[(2)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau}\\{[(-2)\mathbf{1}+\mathbf{2}]}}} \left[\begin{array}{ccc}1&0&0\\2&-3&6\\\hline 1&-2&2\\0&1&0\\0&0&1\end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\{[(2)\mathbf{2}+\mathbf{3}]}}} \left[\begin{array}{ccc}1&0&0\\2&-3&0\\\hline 1&-2&-2\\0&1&2\\0&0&1\end{array}\right].$$

So the system $\left[\begin{pmatrix}1\\2\\-2\end{pmatrix}; \begin{pmatrix}2\\1\\2\end{pmatrix}; \begin{pmatrix}-2\\2\\1\end{pmatrix};\right]$ is orthogonal. As we want orthonormal vectors, we have to divide each vector by its length: $\boldsymbol{q}_1 = \frac{1}{3}\begin{pmatrix}1\\2\\-2\end{pmatrix}$; $\quad \boldsymbol{q}_2 = \frac{1}{3}\begin{pmatrix}2\\1\\2\end{pmatrix}$; $\quad \boldsymbol{q}_3 = \frac{1}{3}\begin{pmatrix}-2\\2\\1\end{pmatrix}$.

□

(Grupo D curso 21/22) Exercise 1(c) The orthogonal complement of $\mathcal{C}\left(\mathbf{A}\right)$, i.e. the left null space: $\mathcal{N}\left(\mathbf{A}^\intercal\right)$.

□

(Grupo D curso 21/22) Exercise 1(d) Since $[\boldsymbol{q}_3]^\intercal[\boldsymbol{q}_3] = [1] = \underset{1\times 1}{\mathbf{I}}$,

$$\mathbf{P} = [\boldsymbol{q}_3]\left([\boldsymbol{q}_3]^\intercal[\boldsymbol{q}_3]\right)^{-1}[\boldsymbol{q}_3]^\intercal = [\boldsymbol{q}_3][\boldsymbol{q}_3]^\intercal = \frac{1}{9}\begin{bmatrix}4&-4&-2\\-4&4&2\\-2&2&1\end{bmatrix}.$$

□

(Grupo D curso 21/22) Exercise 1(e) Using the projection matrix $(\mathbf{I}-\mathbf{P})$ we can compute the projection onto $\mathcal{C}\left(\mathbf{A}\right)$. Hence

$$\boldsymbol{p} = (\mathbf{I}-\mathbf{P})\boldsymbol{v} = \boldsymbol{v}-\mathbf{P}\boldsymbol{v} = \begin{pmatrix}1\\2\\7\end{pmatrix} - \frac{1}{9}\begin{bmatrix}4&-4&-2\\-4&4&2\\-2&2&1\end{bmatrix}\begin{pmatrix}1\\2\\7\end{pmatrix} = \begin{pmatrix}3\\0\\6\end{pmatrix}.$$

□

(Grupo D curso 21/22) Exercise 1(f) It is the vector $\boldsymbol{x}$ with the coefficients of the linear combination of the columns of $\mathbf{A}$ that is closest to $\boldsymbol{v}$. So we can solve the normal equations, but, since we already known the projection, $\boldsymbol{p}$, we can simply solve $\mathbf{A}\boldsymbol{x}=\boldsymbol{p}$.

$$\left[\begin{array}{cc|c}1&1&-3\\2&-1&0\\-2&4&-6\\\hline 1&0&0\\0&1&0\\\hline 0&0&1\end{array}\right] \xrightarrow[{[(3)\mathbf{1}+\mathbf{3}]}]{\overset{\tau}{[(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc|c}1&0&0\\2&-3&6\\-2&6&-12\\\hline 1&-1&3\\0&1&0\\\hline 0&0&1\end{array}\right] \xrightarrow{\overset{\tau}{[(2)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{cc|c}1&0&0\\2&-3&0\\-2&6&0\\\hline 1&-1&1\\0&1&2\\\hline 0&0&1\end{array}\right].$$

Hence, the least squares solution is $\boldsymbol{x}=\begin{pmatrix}1\\2\end{pmatrix}$.

□

(Grupo D curso 21/22) Exercise 1(g) We only need to find a matrix whose row is a nonzero vector orthogonal to $\mathcal{C}\left(\mathbf{A}\right)$ (and we have found such a vector in part b). Hence

$$\mathcal{C}\left(\mathbf{A}\right) = \left\{\boldsymbol{v}\in\mathbb{R}^3 \;\middle|\; \begin{bmatrix}-2&2&1\end{bmatrix}\boldsymbol{v}=(0,)\right\}.$$

□

(Grupo D curso 21/22) Exercise 2. Diagonalizing $\mathbf{A}$ by congruence we get: $\begin{bmatrix}1&2&3\\2&d&4\\3&4&5\end{bmatrix} \xrightarrow[{[(-3)\mathbf{1}+\mathbf{3}]}]{\overset{\tau}{[(-2)\mathbf{1}+\mathbf{2}]}}$

$$\begin{bmatrix}1&0&0\\2&d-4&-2\\3&-2&-4\end{bmatrix} \xrightarrow[\substack{\overset{\tau}{[(-3)\mathbf{1}+\mathbf{3}]}\\ [(-2)\mathbf{1}+\mathbf{2}]}]{} \begin{bmatrix}1&0&0\\0&d-4&-2\\0&-2&-4\end{bmatrix} \xrightarrow{\substack{\overset{\tau}{[(d-4)\mathbf{3}]}\\ [(2)\mathbf{2}+\mathbf{3}]}} \begin{bmatrix}1&0&0\\0&d-4&0\\0&-2&-4(d-3)\end{bmatrix} \xrightarrow[\substack{\overset{\tau}{[(2)\mathbf{2}+\mathbf{3}]}\\ [(d-4)\mathbf{3}]}]{} \begin{bmatrix}1&0&0\\0&d-4&0\\0&0&-4(d-4)(d-3)\end{bmatrix}.$$

If $d-4>0$ then $-4(d-4)(d-3)<0$. So, $\boxed{\text{there is no such value of } d}$.

Another way to see the same is computing the leading principal minors of $\mathbf{A}$: 1, $(d-4)$ and $(12-4d)$. These are never all positive.

□

(Grupo D curso 21/22) Exercise 3(a) For any $\mathbf{A}$ of order n we known that $\dim\mathcal{N}\left(\mathbf{A}\right)=n-\operatorname{rg}\left(\mathbf{A}\right)=$ the number of eigenvalues of $\mathbf{A}$ which are 0. So: $\operatorname{rg}\left(\mathbf{A}\right)=2$.

□

(Grupo D curso 21/22) Exercise 3(b) $\det(\mathbf{A}^{\mathsf{T}}\mathbf{A})=\det\mathbf{A}^{\mathsf{T}}\cdot\det\mathbf{A}=0\cdot 0=0$.

□

(Grupo D curso 21/22) Exercise 3(c) When we add $\mathbf{I}$ to a matrix, it increases the eigenvalues by 1 (since we need to subtract another unit from the main diagonal to get a singular matrix). So the eigenvalues of $\mathbf{A}+\mathbf{I}$ are 1, 2 and 3, and $|\mathbf{A}+\mathbf{I}|=1\cdot 2\cdot 3=6$.

□

(Grupo D curso 21/22) Exercise 3(d) The eigenvalues of $\left(\mathbf{A}+\mathbf{I}\right)^{-1}$ are $\frac{1}{1}$, $\frac{1}{2}$ and $\frac{1}{3}$.

□

(Grupo D curso 21/22) Exercise 4. Since $\begin{bmatrix}-2&1\\0&0\end{bmatrix}$ is triangular, its eigenvalues are the diagonal entries:

$$\begin{cases}\text{For } \lambda=-2: \quad \mathbf{A}+2\mathbf{I}=\begin{bmatrix}0&1\\0&2\end{bmatrix} \implies \mathcal{E}_{(-2)}(\mathbf{A})=\mathcal{L}\left(\left[\begin{pmatrix}1\\0\end{pmatrix};\right]\right)\\[2ex] \text{For } \lambda=0: \quad \mathbf{A}-0\mathbf{I}=\begin{bmatrix}-2&1\\0&0\end{bmatrix} \implies \mathcal{E}_{(0)}(\mathbf{A})=\mathcal{L}\left(\left[\begin{pmatrix}1\\2\end{pmatrix};\right]\right)\end{cases}; \quad \text{so } \boxed{\mathbf{S}=\begin{bmatrix}1&1\\0&2\end{bmatrix}}.$$

$\square$

(Grupo E curso 21/22) Exercise 1(a) For any symmetric A, the matrix $\mathsf{C}^\intercal\mathsf{AC}$ is also symmetric since $\left(\mathsf{C}^\intercal\mathsf{AC}\right)^\intercal = \mathsf{C}^\intercal\mathsf{AC}$. Hence $\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\mathsf{A}\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}} = \underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\mathsf{IAI}\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}$ is symetric since $\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\mathsf{I} = \Big(\mathsf{I}\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\Big)^\intercal$.

$\square$

(Grupo E curso 21/22) Exercise 1(b) For any symmetric A, the matrix $\mathsf{C}^{-1}\mathsf{AC}$ is similar to A. Hence $\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\mathsf{A}\underset{[(3)\mathbf{3}+\mathbf{1}]}{\boldsymbol{\tau}} = \underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\mathsf{IAI}\underset{[(3)\mathbf{3}+\mathbf{1}]}{\boldsymbol{\tau}}$ is similar to A since $\big(\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\mathsf{I}\big)\big(\mathsf{I}\underset{[(3)\mathbf{3}+\mathbf{1}]}{\boldsymbol{\tau}}\big) = \mathsf{I}$.

$\square$

(Grupo E curso 21/22) Exercise 1(c) Any triangular matrix with those values in the main diagonal. For example: $\begin{bmatrix} 4 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

$\square$

(Grupo E curso 21/22) Exercise 1(d) Impossible. To be a rank-one matrix only one non-zero eigenvalue is posible (with multiplicity 1).

$\square$

(Grupo E curso 21/22) Exercise 1(e) We need an orthogonal matrix. For example $\mathsf{Q} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Then $\mathsf{Q}^\intercal\mathsf{DQ} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has eigenvalues 1, 2, 4.

$\square$

(Grupo E curso 21/22) Exercise 2(a) The projection matrix P projects onto the column space of P which is the line

$$\mathcal{C}\left(\mathsf{P}\right) = \left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\boldsymbol{p} \right\}.$$

$\square$

(Grupo E curso 21/22) Exercise 2(b) The difference between $\boldsymbol{v}$ and its projection is

$$\boldsymbol{e} = \boldsymbol{v} - \mathsf{P}\boldsymbol{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \frac{1}{9}\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix};$$

so the distance is $\|\boldsymbol{e}\| = \sqrt{\boldsymbol{e}\cdot\boldsymbol{e}} = \sqrt{2}$.

$\square$

(Grupo E curso 21/22) Exercise 2(c) Since P projects onto a line, its three eigenvalues are 0, 0, 1. Since P is symmetric, it has a full set of (orthogonal) eigenvectors, and is then diagonalizable.

$\square$

(Grupo E curso 21/22) Exercise 3(a)

$$\begin{cases} \text{For } \lambda = \frac{1}{2}: \quad \mathsf{A} - \frac{1}{2}\mathsf{I} = \begin{bmatrix} 0 & 3 & 4 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \implies \mathcal{E}_{\left(\frac{1}{2}\right)}(\mathsf{A}) = \mathcal{L}\left(\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix};\right]\right). \\[2em] \text{For } \lambda = 1: \quad \mathsf{A} - 1\mathsf{I} = \begin{bmatrix} -\frac{1}{2} & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \mathcal{E}_{(1)}(\mathsf{A}) = \mathcal{L}\left(\left[\begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix};\ \begin{pmatrix} 8 \\ 0 \\ 1 \end{pmatrix};\right]\right). \end{cases}$$

So, $\quad \mathsf{A} = \mathsf{SDS}^{-1} = \begin{bmatrix} 1 & 6 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -6 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

$\square$

(Grupo E curso 21/22) Exercise 3(b) $\mathbf{A}^{\infty} = \begin{bmatrix} 1 & 6 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -6 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

$\square$

(Grupo E curso 21/22) Exercise 4.

1. The eigenvalues of $\mathbf{A}$ are: 1, 1, 2.
2. $\mathbf{A}$ might or might not be diagonalizable.
3. $\mathbf{A}$ might or might not be symmetric.

   $\mathbf{A}$ definetely (!) has positive eigenvalues. However it might not be symmetric, so $\mathbf{A}$ might or might not be positive definite.

$\square$

(Grupo D curso 20/21) Exercise 1(a) Since $\mathbf{D} = \mathbf{D}^{\mathsf{T}} = \mathbf{S}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\left(\mathbf{S}^{-1}\right)^{\mathsf{T}}$

- The eigenvalues of $\mathbf{A}^{\mathsf{T}}$ are $\underline{\text{the same as the eigenvalues of } \mathbf{A}}$
- The eigenvectors of $\mathbf{A}^{\mathsf{T}}$ are $\underline{\text{the columns of } \left(\mathbf{S}^{-1}\right)^{\mathsf{T}}}$

$\square$

(Grupo D curso 20/21) Exercise 1(b) $x = 3$ and $y = 4$ for symmetry, and $z = 5$ to be singular:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & z \end{bmatrix} \xrightarrow[{[(-3)\mathbf{1}+\mathbf{3}]}]{\boldsymbol{\tau}_{[(-2)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & -2 \\ 3 & -2 & z-9 \end{bmatrix} \xrightarrow{\boldsymbol{\tau}_{[(-2)\mathbf{2}+\mathbf{3}]}} \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 3 & -2 & z-5 \end{bmatrix} \Longrightarrow \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

$\square$

(Grupo D curso 20/21) Exercise 1(c) I need to multiply on the left by

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underset{[(-1)\mathbf{2}+\mathbf{1}]}{\boldsymbol{\tau}}\mathbf{I}.$$

So, $\boxed{\textbf{substract the second row from the first.}}$ Since $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{C} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underset{[(-1)\mathbf{2}+\mathbf{1}]}{\boldsymbol{\tau}}\mathbf{C}\underset{[(1)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}$ is similar to $\mathbf{C}$ and therefore it has the same eigenvalues.

$\square$

(Grupo D curso 20/21) Exercise 2(a) $\mathcal{C}\left(\mathbf{A}\right)$ is the orthogonal complement of $\mathcal{N}\left(\mathbf{A}^{\mathsf{T}}\right)$:

$$\left[\begin{array}{ccc} 1 & 1 & 0 \\ 2 & 0 & 2 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\boldsymbol{\tau}_{[(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 2 & -2 & 2 \\ \hline 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\boldsymbol{\tau}_{[(1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 2 & -2 & 0 \\ \hline 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right] \Longrightarrow \text{Basis: } \left[\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix};\right].$$

$\square$

(Grupo D curso 20/21) Exercise 2(b)

$$\mathbf{P} = \mathbf{A}\left(\mathbf{A}^{\mathsf{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{T}} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

$\square$

(Grupo D curso 20/21) Exercise 3. Since $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0\\ 2 & 5 & 3 & 0\\ 1 & 0 & 8 & 0\\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau\\ [(-2)\mathbf{1}+\mathbf{2}]\\ [(-3)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0\\ 2 & 1 & -3 & 0\\ 1 & -2 & 5 & 0\\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau\\ [(3)\mathbf{2}+\mathbf{3}]}}$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0\\ 2 & 1 & 0 & 0\\ 1 & -2 & -1 & 0\\ \hline 0 & 0 & 0 & 1 \end{array}\right]$; and since $\left[\begin{array}{ccc|c} 3 & 0 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 3 & 0 & 0\\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{\tau\\ [(-1)\mathbf{4}]}]{\substack{\tau\\ [\mathbf{2}\rightleftharpoons\mathbf{3}]}} \left[\begin{array}{ccc|c} 3 & 0 & 0 & 0\\ 0 & 3 & 0 & 0\\ 0 & 0 & 3 & 0\\ \hline 0 & 0 & 0 & -1 \end{array}\right]$; then, if we call the first matrix $\mathbf{B}$ and we call the last matrix $\mathbf{C}$ $\boxed{|\mathbf{A}| = \frac{|\mathbf{C}|}{|\mathbf{B}|} = \frac{-27}{-1} = 27}$.

□

(Grupo D curso 20/21) Exercise 4(a) It is solvable when $a \neq 5$ or $c = 10$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2\\ 1 & 2 & 3 & -6\\ 1 & 3 & a & -c\\ \hline 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau\\ [(-1)\mathbf{1}+\mathbf{2}]\\ [(-1)\mathbf{1}+\mathbf{3}]\\ [(2)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0\\ 1 & 1 & 2 & -4\\ 1 & 2 & a-1 & 2-c\\ \hline 1 & -1 & -1 & 2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau\\ [(-2)\mathbf{2}+\mathbf{3}]\\ [(4)\mathbf{2}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 1 & 2 & a-5 & 10-c\\ \hline 1 & -1 & 1 & -2\\ 0 & 1 & -2 & 4\\ 0 & 0 & 1 & 0\\ \hline 0 & 0 & 0 & 1 \end{array}\right].$$

□

(Grupo D curso 20/21) Exercise 4(b) The solution set is $\left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{pmatrix} -2\\ 4\\ 0 \end{pmatrix} + \begin{bmatrix} 1\\ -2\\ 1 \end{bmatrix} \boldsymbol{p} \right\}.$

□

(Grupo D curso 20/21) Exercise 4(c) We only need to check the signs of the pivots of any diagonal matrix that is congruent with $\mathbf{A}$:

$$\begin{bmatrix} 1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 3 & a \end{bmatrix} \xrightarrow{\substack{\tau\\ [(-1)\mathbf{1}+\mathbf{2}]\\ [(-1)\mathbf{1}+\mathbf{3}]}} \begin{bmatrix} 1 & 0 & 0\\ 1 & 1 & 2\\ 1 & 2 & a-1 \end{bmatrix} \xrightarrow[\substack{\tau\\ [(-1)\mathbf{1}+\mathbf{3}]\\ [(-1)\mathbf{1}+\mathbf{2}]}]{} \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 2\\ 0 & 2 & a-1 \end{bmatrix} \xrightarrow{\substack{\tau\\ [(-2)\mathbf{2}+\mathbf{3}]}} \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 2 & a-5 \end{bmatrix} \xrightarrow[\substack{\tau\\ [(-2)\mathbf{2}+\mathbf{3}]}]{} \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & a-5 \end{bmatrix}$$

- When $a > 5$ positive definite.
- When $a = 5$ positive *semi*definite.
- When $a < 5$ Neither positive nor negative definite.

□

(Grupo D curso 20/21) Exercise 5(a) We first have to find a vector in the direction of the line. We let

$$\boldsymbol{r} = \boldsymbol{x}_p - \boldsymbol{x}_q = \begin{pmatrix}1, & -3, & 1,\end{pmatrix} - \begin{pmatrix}-2, & 2, & -2,\end{pmatrix} = \begin{pmatrix}3, & -5, & 3,\end{pmatrix}.$$

A parametric representation of the line is therefore

$$L = \left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{pmatrix} 1\\ -3\\ 1 \end{pmatrix} + \begin{bmatrix} 3\\ -5\\ 3 \end{bmatrix} \boldsymbol{p} \right\}.$$

□

(Grupo D curso 20/21) Exercise 5(b)

$$\left[\begin{array}{ccc} 3 & -5 & 3\\ \hline v_1 & v_2 & v_3\\ \hline 1 & -3 & 1 \end{array}\right] \xrightarrow{\substack{\tau\\ [(3)\mathbf{2}]\\ [(5)\mathbf{1}+\mathbf{2}]\\ [(-1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc} 3 & 0 & 0\\ \hline v_1 & 5v_1+3v_2 & -v_1+v_3\\ \hline 1 & -4 & 0 \end{array}\right]$$

so

$$L = \left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} 5 & 3 & 0\\ -1 & 0 & 1 \end{bmatrix} \boldsymbol{v} = \begin{pmatrix} -4\\ 0 \end{pmatrix} \right\}.$$

□

(Grupo E curso 20/21) Exercise 1(a) Lets find a basis for $\mathcal{N}\left(\mathbf{A}\right)$:

$$\left[\begin{array}{cccc} 1 & -2 & -1 & 2 \\ 2 & -4 & -1 & 3 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-2)\mathbf{1}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(2)\mathbf{1}+\mathbf{2}] \\ [(1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & -1 \\ \hline 1 & 2 & 1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ \hline 1 & 2 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

The closest vector in $\mathcal{N}\left(\mathbf{A}\right)$ to $\begin{pmatrix} 2 \\ -1 \\ 0 \\ 3 \end{pmatrix}$ is the linear combination $\mathbf{N}\boldsymbol{c}$ where $\mathbf{N} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$, such that $\boldsymbol{c}$ satisfies $\mathbf{N}^\intercal\mathbf{N}\boldsymbol{c} = \mathbf{N}^\intercal\boldsymbol{b}$ (the normal equations). Since $\mathbf{N}^\intercal\mathbf{N} = \begin{bmatrix} 5 & -2 \\ -2 & 3 \end{bmatrix}$ and $\mathbf{N}^\intercal\boldsymbol{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$:

$$\left[\begin{array}{cc|c} 5 & -2 & -3 \\ -2 & 3 & -1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(5)\mathbf{2}] \\ [(2)\mathbf{1}+\mathbf{2}] \\ [(5)\mathbf{3}] \\ [(3)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cc|c} 5 & 0 & 0 \\ -2 & 11 & -11 \\ \hline 1 & 2 & 3 \\ 0 & 5 & 0 \\ \hline 0 & 0 & 5 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{cc|c} 5 & 0 & 0 \\ -2 & 11 & 0 \\ \hline 1 & 2 & 5 \\ 0 & 5 & 5 \\ \hline 0 & 0 & 5 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(\frac{1}{5})\mathbf{3}]}} \left[\begin{array}{cc|c} 5 & 0 & 0 \\ -2 & 11 & 0 \\ \hline 1 & 2 & 1 \\ 0 & 5 & 1 \\ \hline 0 & 0 & 1 \end{array}\right].$$

So, the solution $\boldsymbol{x}$ to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ that is closest to $\begin{pmatrix} 2 \\ -1 \\ 0 \\ 3 \end{pmatrix}$ is $\boldsymbol{x} = 1\mathbf{N}_{|1} + 1\mathbf{N}_{|2} = 1\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

□

(Grupo E curso 20/21) Exercise 1(b) If we choose the first column of $\mathbf{N}$ as one of the vectors of the basis, we need to find a linear combination of the columns of $\mathbf{N}$ that is orthogonal to $\mathbf{N}_{|1}$. So

$$(\mathbf{N}_{|1}) \cdot (a\mathbf{N}_{|1} + b\mathbf{N}_{|2}) = \begin{pmatrix} 2, & 1, & 0, & 0, \end{pmatrix} \cdot \left(a\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right) = 0 \quad \Rightarrow \quad a(5) + b(-2) = 0.$$

Hence, if $a = 2$ and $b = 5$ then $2\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 5\begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 5 \\ 5 \end{pmatrix} \in \mathcal{C}\left(\mathbf{N}\right)$ and it is orthogonal to $\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$. Dividing each vector by its length we get an othonormal basis:

$$\left[\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \; \frac{1}{\sqrt{55}}\begin{pmatrix} -1 \\ 2 \\ 5 \\ 5 \end{pmatrix}; \right]$$

□

(Grupo E curso 20/21) Exercise 2(a) $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution if and only if $\boldsymbol{b} \in \mathcal{C}\left(\mathbf{A}\right) = \left(\mathcal{N}\left(\mathbf{A}^\intercal\right)\right)^\perp \quad \Rightarrow \quad \boxed{\boldsymbol{b} \perp \mathcal{N}\left(\mathbf{A}^\intercal\right).}$

□

(Grupo E curso 20/21) Exercise 2(b) $\boxed{|\mathbf{A}| = (1)\cdot(3)\cdot(5)\cdot(7) = 105}$.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 4 & 4 & 4 & 0 \\ 1 & 4 & 9 & 9 & 0 \\ 1 & 4 & 9 & 16 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 3 & 0 \\ 1 & 3 & 8 & 8 & 0 \\ 1 & 3 & 8 & 15 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}+\mathbf{3}] \\ [(-1)\mathbf{2}+\mathbf{4}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 1 & 3 & 5 & 5 & 0 \\ 1 & 3 & 5 & 12 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 1 & 3 & 5 & 0 & 0 \\ 1 & 3 & 5 & 7 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

□

(Grupo E curso 20/21) Exercise 3(a) $\begin{bmatrix} 1 & 1 \\ d & 1 \end{bmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ d+3 \end{pmatrix} \implies \begin{cases} 4 = \lambda 1 \;\Rightarrow \lambda = 4 \\ d+3 = \lambda 3 \end{cases} \implies$ $d + 3 = 12 \implies \boxed{d = 9}$.

$\square$

(Grupo E curso 20/21) Exercise 3(b) Matrix $\big(\mathbf{A}-(2)\mathbf{I}\big)$ must be singular. Hence

$$|\mathbf{A}-(2)\mathbf{I}| = \det\begin{bmatrix} -1 & 1 \\ d & -1 \end{bmatrix} = 1-d = 0 \implies \boxed{d=1}.$$

$\square$

(Grupo E curso 20/21) Exercise 3(c) The issue of nondiagonalizability only comes up for a matrix that has some repeated eigenvalues. So $\lambda_1=\lambda_2=\lambda$. Therefore $2\lambda = \operatorname{tr}\big(\mathbf{A}\big) = 2 \implies \lambda = 1$. Hence $|\mathbf{A}| = \lambda^2 = \lambda = 1$

$$|\mathbf{A}| = \det\begin{bmatrix} 1 & 1 \\ d & 1 \end{bmatrix} = 1-d = 1 \implies \boxed{d=0}.$$

$\square$

(Grupo E curso 20/21) Exercise 4.

$$\left[\begin{array}{ccc} 1 & 1 & 2 \\ 5 & 11 & -8 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-2)\mathbf{1}+\mathbf{3}]}]{\overset{\tau}{[(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 5 & 6 & -18 \\ \hline 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\overset{\tau}{[(3)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 5 & 6 & 0 \\ \hline 1 & -1 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right] \Rightarrow \boldsymbol{v} = \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix}$$

$\square$

(Grupo E curso 20/21) Exercise 5(a)

Since the right hand side vector $\boldsymbol{b}$ belongs to $\mathbb{R}^3$, then $\mathbf{A}$ has three rows. In addition, $\boldsymbol{x}$ also belongs to $\mathbb{R}^3$, thus $\mathbf{A}$ has three columns.

Besides, there are two special solutions; therefore $\operatorname{rg}\big(\mathbf{A}\big) = 3 - \dim \mathcal{N}\big(\mathbf{A}\big) = 3-2 = 1$. It follows that there is only one pivot row, hence $\dim \mathcal{C}\big(\mathbf{A}^{\intercal}\big) = 1$.

$\square$

(Grupo E curso 20/21) Exercise 5(b)

From the particular solution, it follows that $2\mathbf{A}_{|1} = \begin{pmatrix}2, & 4, & 2,\end{pmatrix}$, hence, $\mathbf{A}_{|1} = \begin{pmatrix}1, & 2, & 1,\end{pmatrix}$. Because the rank is 1, the other columns are multiples of the first one. From the first special solution we know that the second column must be the opposite of the first one, since, $\mathbf{A}_{|1} + \mathbf{A}_{|2} = \mathbf{0}$. Finally, from the second special solution it follows that, $\mathbf{A}_{|3} = \mathbf{0}$. Consequently,

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}.$$

$\square$

(Grupo E curso 20/21) Exercise 5(c) For any $\boldsymbol{b} \in \mathcal{C}\big(\mathbf{A}\big)$; therefore, *only for any multiple of the first column.*

$\square$

(Grupo E curso 20/21) Exercise 6. Since λ and $\frac{1}{\lambda}$ have the same sing when $\lambda \neq 0$, we can answer checking the signs of the eigenvalues of $\mathbf{A}^{-1}$:

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{\overset{\tau}{[(1)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix} -1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow[{\overset{\tau}{[(1)\mathbf{1}+\mathbf{2}]}}]{} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Since $\mathbf{A}^{-1}$ have positive and negative eigenvalues, $\mathbf{A}$ is indefinite.

$\square$

(Grupo B curso 18/19) Exercise 1(a) A parametric representation of that line is: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boldsymbol{a} + \alpha(\boldsymbol{a} - \boldsymbol{b}) = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \alpha \begin{pmatrix} 4/3 \\ 0 \\ 4 \end{pmatrix}$. Hence, multiplying the parametric part by 4 (in order to avoid fractions), we can find an implicit equation of the same line.

$$\left[\begin{array}{ccc} x & y & z \\ \hline 1 & 0 & 3 \\ \hline 4 & 0 & 12 \end{array}\right] \rightarrow \left[\begin{array}{ccc} x & y & (z-3x) \\ \hline 1 & 0 & 0 \\ \hline 4 & 0 & 0 \end{array}\right] \Rightarrow \boxed{\begin{cases} -3x \quad + z = 0 \\ \qquad y \quad = 0 \end{cases}}$$

$\square$

(Grupo B curso 18/19) Exercise 1(b) Yes, it is the set of solutions to an homogeneous linear system. Note that it is the set of multiples of $\boldsymbol{a} = \begin{pmatrix}1, & 0, & 3,\end{pmatrix}$. In particular, $\boldsymbol{b}$ is $\frac{-1}{3}\boldsymbol{a}$.

$\square$

(Grupo B curso 18/19) Exercise 1(c) The cosest point $\boldsymbol{p}$ is the proyection of $\boldsymbol{z}$ onto the spam of $\begin{pmatrix}1, & 0, & 3,\end{pmatrix}$, that is

$$\boldsymbol{p} = \mathbf{P}\boldsymbol{z} = \begin{bmatrix}1\\0\\3\end{bmatrix}\left(\begin{bmatrix}1 & 0 & 3\end{bmatrix}\begin{bmatrix}1\\0\\3\end{bmatrix}\right)^{-1}\begin{bmatrix}1 & 0 & 3\end{bmatrix}\begin{pmatrix}2\\2\\2\end{pmatrix} = \frac{1}{10}\begin{bmatrix}1 & 0 & 3\\0 & 0 & 0\\3 & 0 & 9\end{bmatrix}\begin{pmatrix}2\\2\\2\end{pmatrix} = \frac{1}{10}\begin{pmatrix}8\\0\\24\end{pmatrix}.$$

$\square$

(Grupo B curso 18/19) Exercise 1(d) On the one hand, $\boldsymbol{p}$ is a point in the line since $\boldsymbol{p} = \frac{8}{10}\boldsymbol{a}$;

On the other hand $\boldsymbol{e} = \begin{pmatrix}2, & 2, & 2,\end{pmatrix} - \frac{8}{10}\begin{pmatrix}1, & 0, & 3,\end{pmatrix} = \frac{1}{10}\begin{pmatrix}20, & 20, & 20,\end{pmatrix} - \frac{8}{10}\begin{pmatrix}1, & 0, & 3,\end{pmatrix} = \frac{1}{10}\begin{pmatrix}12, & 20, & -4,\end{pmatrix}$ is perpendicular to the line, since $\boldsymbol{e}\cdot\boldsymbol{a} = 0$. Hence, $\boldsymbol{p}$ is multiple of $\boldsymbol{a}$ closest to $(2, 2, 2,)$.

The distance is $\|\boldsymbol{e}\| = \sqrt{\boldsymbol{e}\cdot\boldsymbol{e}} = \sqrt{\frac{144+400+16}{100}} = \frac{1}{10}\sqrt{560}$.

$\square$

(Grupo B curso 18/19) Exercise 2(a) Since the rank of $\begin{bmatrix}-1 & 0\\0 & 2\\1 & 0\end{bmatrix}$ is 2, these columns are linearly independent. The dot product $\boldsymbol{v}_1\cdot\boldsymbol{v}_2$ is zero, and therefore these vectors are orthogonal.

$\square$

(Grupo B curso 18/19) Exercise 2(b)

When $\mathbf{A}$ is symmetric, we can find three orthogonal eigenvectors. Using gaussian elimination we can find a third eigenvector $\boldsymbol{v}_3$ perpendicular to $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$

$$\left[\begin{array}{ccc}1 & 0 & -1\\0 & 2 & 0\\\hline 1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{array}\right] \xrightarrow{\underset{[(1)3+1]}{\tau}} \left[\begin{array}{ccc}1 & 0 & 0\\0 & 2 & 0\\\hline 1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{array}\right] \quad \rightarrow \quad \boldsymbol{v}_3 = \begin{pmatrix}1\\0\\1\end{pmatrix} \quad \text{belongs to that subspace.}$$

$\square$

(Grupo B curso 18/19) Exercise 2(c)

If $\operatorname{tr}\left(\mathbf{A}\right) = \lambda_1 + \lambda_2 + \lambda_3 = 2$ then $\lambda_3 = 0$. Matrix $\mathbf{A}$ is positive ***semi***definite.

$\square$

(Grupo B curso 18/19) Exercise 2(d)

Since $\mathbf{A}$ is symmetric, it is diagonalizable as $\mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}^{\intercal}$, where columns of the orthonormal matrix $\mathbf{Q}$ are eigenvectors of $\mathbf{A}$. Hence,

$$\begin{aligned}\mathbf{A} &= \frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 0 & 1\\0 & \sqrt{2} & 0\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}-1 & 0 & 1\\0 & \sqrt{2} & 0\\1 & 0 & 1\end{bmatrix}^{\intercal}\frac{1}{\sqrt{2}}\\ &= \frac{1}{2}\begin{bmatrix}-1 & 0 & 1\\0 & \sqrt{2} & 0\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}-1 & 0 & 1\\0 & \sqrt{2} & 0\\1 & 0 & 1\end{bmatrix}\\ &= \frac{1}{2}\begin{bmatrix}-1 & 0 & 0\\0 & \sqrt{2} & 0\\1 & 0 & 0\end{bmatrix}\begin{bmatrix}-1 & 0 & 1\\0 & \sqrt{2} & 0\\1 & 0 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 0 & -1\\0 & 2 & 0\\-1 & 0 & 1\end{bmatrix}\end{aligned}$$

$\square$

(Grupo B curso 18/19) Exercise 3(a) The first and third columns of $\mathbf{A}$ are the same, while the first and second columns are linearly independent. This means that the rank of $\mathbf{A}$ is $\boxed{2}$. The rank of $\mathbf{A}^{\intercal}$ and the rank of $\mathbf{A}^{\intercal}\mathbf{A}$ are equal to the rank of $\mathbf{A}$.

$\square$

(Grupo B curso 18/19) Exercise 3(b) A basis for $\mathcal{C}\left(\mathbf{A}\right)$ is then just the first two columns $\boxed{\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}2\\1\\1\\0\end{pmatrix}\right\}}$.

The nullspace of $\mathbf{A}$ is one dimensional, and since the first and third columns are the same, a basis for $\mathcal{N}\left(\mathbf{A}\right)$ is given by the vector $\boxed{\begin{pmatrix}1\\0\\-1\end{pmatrix}}$. Finally, $\boxed{\mathcal{N}\left(\mathbf{A}\right) = \mathcal{N}(\mathbf{A}^\intercal\mathbf{A})}$, and so our basis for $\mathcal{N}\left(\mathbf{A}\right)$ is also a basis for $\mathcal{N}(\mathbf{A}^\intercal\mathbf{A})$.

□

(Grupo B curso 18/19) Exercise 3(c) $\boldsymbol{p} = \mathbf{A}\widehat{\boldsymbol{x}}$ is the projection of $\boldsymbol{b}$ onto $\mathcal{C}\left(\mathbf{A}\right)$.

□

(Grupo B curso 18/19) Exercise 3(d) To find $\widehat{\boldsymbol{x}}$ we must solve the normal equations $\mathbf{A}^\intercal\mathbf{A}\widehat{\boldsymbol{x}} = \mathbf{A}^\intercal\boldsymbol{b}$. However, since $\mathbf{A}$ only has two linearly independent columns we can simplify our calculations by instead using the full column rank matrix $\mathbf{B} = \begin{bmatrix}1&2\\0&1\\1&1\\1&0\end{bmatrix}$, with $\mathcal{C}\left(\mathbf{A}\right) = \mathcal{C}\left(\mathbf{B}\right)$; and solve the normal equations $\mathbf{B}^\intercal\mathbf{B}\widehat{\boldsymbol{x}} = \mathbf{B}^\intercal\boldsymbol{b}$ to find $\boldsymbol{p} = \mathbf{B}\widehat{\boldsymbol{x}}$. We can calculate

$$\mathbf{B}^\intercal\mathbf{B} = \begin{bmatrix}1&0&1&1\\2&1&1&0\end{bmatrix}\begin{bmatrix}1&2\\0&1\\1&1\\1&0\end{bmatrix} = \begin{bmatrix}3&3\\3&6\end{bmatrix}, \qquad \mathbf{B}^\intercal\boldsymbol{b} = \begin{bmatrix}1&0&1&1\\2&1&1&0\end{bmatrix}\begin{pmatrix}0\\2\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\3\end{pmatrix};$$

the normal equations are then: $\begin{bmatrix}3&3\\3&6\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\3\end{pmatrix} \quad \Rightarrow \quad \widehat{\boldsymbol{x}} = \begin{pmatrix}-1\\1\end{pmatrix}$. Finally, we can compute

$$\boldsymbol{p} = \mathbf{B}\widehat{\boldsymbol{x}} = \begin{bmatrix}1&2\\0&1\\1&1\\1&0\end{bmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\0\\-1\end{pmatrix}.$$

□

(Grupo B curso 18/19) Exercise 4. **False:** We can get a matrix whose columns are the vectors in B^* using the following product of matrices, where the columns of the first matrix are vectors in B.

$$\begin{bmatrix}\boldsymbol{u} & \boldsymbol{v} & \boldsymbol{w}\end{bmatrix}\begin{bmatrix}1&1&0\\1&1&0\\0&1&2\end{bmatrix} = \begin{bmatrix}(\boldsymbol{u}+\boldsymbol{v}), & (\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}), & 2\boldsymbol{w}\end{bmatrix}$$

Since the right hand side matrix is singular, vectors in B^* must be dependent. We can find the same result by gaussian elimination

$$\begin{bmatrix}(\boldsymbol{u}+\boldsymbol{v}), & (\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}), & 2\boldsymbol{w}\end{bmatrix} \xrightarrow{\overset{\tau}{[(-1)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix}(\boldsymbol{u}+\boldsymbol{v}), & \boldsymbol{w}, & 2\boldsymbol{w}\end{bmatrix} \xrightarrow{\overset{\tau}{[(-2)\mathbf{2}+\mathbf{3}]}} \begin{bmatrix}(\boldsymbol{u}+\boldsymbol{v}), & \boldsymbol{w}, & \mathbf{0}\end{bmatrix}.$$

□

(Grupo E curso 18/19) Exercise 1(a) Since $\mathbf{A}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$ one eigenvalue is $\lambda_1 = 1$. Since the trace is 1.5 the second eigenvalue is $\lambda_2 = 0.5$.

□

(Grupo E curso 18/19) Exercise 1(b) We already known that all non-zero multiples of $(1,1)$ are the eigenvectors corresponding to $\lambda_1 = 1$. To find the eigenvectors corresponding to $\lambda_2 = 0.5$, we look at $\mathbf{A} - \lambda_2\mathbf{I}$:

$$\left[\mathbf{A} - \lambda_2\mathbf{I}\right]\boldsymbol{x}_2 = \left[\mathbf{A} - 0.5\mathbf{I}\right]\boldsymbol{x}_2 = \begin{bmatrix}.4 & .1\\.4 & .1\end{bmatrix}\boldsymbol{x}_2 = \mathbf{0} \;\Rightarrow\; \boldsymbol{x}_1 = \begin{pmatrix}1\\-4\end{pmatrix};$$

$$\mathcal{E}_{\lambda=1} = \mathcal{L}\left\{(1,1)\right\}; \qquad \mathcal{E}_{\lambda=0.5} = \mathcal{L}\left\{(1,-4)\right\}.$$

□

(Grupo E curso 18/19) Exercise 1(c) We have

$$\begin{pmatrix}2\\-3\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}1\\-4\end{pmatrix} = \boldsymbol{x}_1 + \boldsymbol{x}_2$$

so

$$\mathbf{A}^k \begin{pmatrix}2\\-3\end{pmatrix} = \mathbf{A}^k(\boldsymbol{x}_1 + \boldsymbol{x}_2) = \mathbf{A}^k\boldsymbol{x}_1 + \mathbf{A}^k\boldsymbol{x}_2 = \boldsymbol{x}_1 + 0.5^k\boldsymbol{x}_2$$

Since $(0.5)^k$ goes to 0 as k goes to infinity, the limiting value of $\mathbf{A}^k \begin{pmatrix}2\\-3\end{pmatrix}$ is $\boldsymbol{x}_1 = \begin{pmatrix}1\\1\end{pmatrix}$.

□

(Grupo E curso 18/19) Exercise 2(a) Since $\mathcal{N}\left(\mathbf{A}\right) \subset \mathbb{R}^3$, matrix $\mathbf{A}$ has three columns. Since $\dim \mathcal{N}\left(\mathbf{A}\right) = 1$, the rank of $\mathbf{A}$ is 2. Since $\mathcal{N}\left(\mathbf{A}^\intercal\right) \subset \mathbb{R}^4$, matrix $\mathbf{A}$ has four rows.

□

(Grupo E curso 18/19) Exercise 2(b) If $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is solvable, then $\boldsymbol{b} \in \mathcal{C}\left(\mathbf{A}\right)$. Since $\mathcal{C}\left(\mathbf{A}\right)$ is the orthogonal complement of $\mathcal{N}\left(\mathbf{A}^\intercal\right)$, this means that an equivalent condition for $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ to be solvable is that $\boldsymbol{b}$ is orthogonal to $\mathcal{N}\left(\mathbf{A}^\intercal\right)$. This gives us two constraints on $\boldsymbol{b}$:

$$\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\end{bmatrix} \begin{pmatrix}-1\\ \alpha\\ 0\\ \beta\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \quad \Rightarrow \quad \begin{cases}\alpha + \beta = 1\\ \alpha - \beta = 1\end{cases} \quad \Rightarrow \quad \boxed{\alpha = 1; \quad \beta = 0}.$$

For these values of α and β, the solution of $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is not unique, since $\mathcal{N}\left(\mathbf{A}\right)$ has dimension 1: given any particuar solution of $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, we can add on any multiple of $\begin{pmatrix}1, & 0, & -1,\end{pmatrix}$ and the resulting vector would still be a solution.

□

(Grupo E curso 18/19) Exercise 2(c) The vector $\boldsymbol{y} = \begin{pmatrix}1, & 2 & ,-3,\end{pmatrix}$ is in $\mathbb{R}^3$, and so we can only project onto $\mathcal{N}\left(\mathbf{A}\right)$ and $\mathcal{C}\left(\mathbf{A}^\intercal\right)$. To project onto $\mathcal{N}\left(\mathbf{A}\right)$, we use the formula to project $\boldsymbol{y}$ onto the spam of $\begin{pmatrix}1, & 0 & ,-1,\end{pmatrix}$

$$\boldsymbol{p}_{\mathcal{N}\left(\mathbf{A}\right)} = \begin{bmatrix}1\\0\\-1\end{bmatrix} \left(\begin{bmatrix}1 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix}\right)^{-1} \begin{bmatrix}1 & 0 & -1\end{bmatrix}\begin{pmatrix}1\\2\\-3\end{pmatrix} = \frac{1}{2}\begin{bmatrix}1 & 0 & -1\\ 0 & 0 & 0\\ -1 & 0 & 1\end{bmatrix}\begin{pmatrix}1\\2\\-3\end{pmatrix} = \begin{pmatrix}2\\0\\-2\end{pmatrix}.$$

To compute the projection onto $\mathcal{C}\left(\mathbf{A}^\intercal\right)$, recall that if $\boldsymbol{p} = \mathbf{P}\boldsymbol{y}$ is the projection of $\boldsymbol{y}$ onto some subspace, then $(\mathbf{I} - \mathbf{P})\boldsymbol{y}$ will project $\boldsymbol{y}$ onto the orthogonal complement of this subspace. Since $\mathcal{C}\left(\mathbf{A}^\intercal\right)$ is orthogonal to $\mathcal{N}\left(\mathbf{A}\right)$, the projection of $\boldsymbol{y}$ onto $\mathcal{C}\left(\mathbf{A}^\intercal\right)$ is given by:

$$\boldsymbol{p}_{\mathcal{C}\left(\mathbf{A}^\intercal\right)} = \begin{pmatrix}1\\2\\-3\end{pmatrix} - \boldsymbol{p}_{\mathcal{N}\left(\mathbf{A}\right)} = \begin{pmatrix}1\\2\\-3\end{pmatrix} - \begin{pmatrix}2\\0\\-2\end{pmatrix} = \begin{pmatrix}-1\\2\\-1\end{pmatrix}.$$

□

(Grupo E curso 18/19) Exercise 3(a) First this matrix is clearly symmetric as $\mathbf{A}^\intercal = \mathbf{A}$. Since $\mathbf{Q}$ is orthogonal $\mathbf{Q}^\intercal = \mathbf{Q}^{-1}$ and so this gives an othogonal diagonalization (by similarity and congruence), so $\mathbf{A}$ has eigenvalues 1, 2, 3, 4 > 0 and is thus positive definite.

□

(Grupo E curso 18/19) Exercise 3(b) From part (a) we know that each of $\mathbf{A}_i = \mathbf{Q}_i\mathbf{D}\mathbf{Q}_i^\intercal$ are positive definite, so for $\boldsymbol{x} \neq \mathbf{0}$ $\boldsymbol{x}\mathbf{A}_i\boldsymbol{x} > 0$. So we get $\boldsymbol{x}\mathbf{A}\boldsymbol{x} = \boldsymbol{x}\mathbf{A}_1\boldsymbol{x} + \boldsymbol{x}\mathbf{A}_2\boldsymbol{x} > 0$ and it follows that $\mathbf{A}$ is positive definite as $\mathbf{A}$ is clearly symmetric as it is the sum of symmetric matrices.

□

(Grupo E curso 18/19) Exercise 3(c) We have $\mathbf{A}^\intercal = \mathbf{A}$, so $\mathbf{A}$ is symmetric. Also $\boldsymbol{v}\mathbf{A}\boldsymbol{v} = \boldsymbol{v}\mathbf{X}\mathbf{D}\mathbf{X}^\intercal\boldsymbol{v} = \left(\mathbf{X}^\intercal\boldsymbol{v}\right)\mathbf{D}\left(\mathbf{X}^\intercal\boldsymbol{v}\right) \geq 0$ as $\mathbf{D}$ is positive definite and the inequality is strict as long as $\mathbf{X}^\intercal\boldsymbol{v} \neq \mathbf{0}$. So we get that $\mathbf{A}$ is not positive definite as long as $\mathbf{X}^\intercal$ is not full column rank.

□

(Grupo E curso 18/19) Exercise 3(d) This is a projection matrix to a 1 dimensional space, so $\mathbf{P}$ has rank 1. It thus has a non-trivial nullspace and so has a 0 eigenvalue. So not all eigenvalues are positive and thus is not positive definite.

$\square$

(Grupo E curso 18/19) Exercise 3(e) Applying *Type I* elementary transformations we get

$$\begin{bmatrix} 2 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 2 & 1 \\ & & & & 1 & 2 \end{bmatrix} \xrightarrow{\substack{\overset{\tau}{[(-\frac{1}{2})\mathbf{2+1}]} \\ \overset{\tau}{[(-\frac{2}{3})\mathbf{3+2}]} \\ \overset{\tau}{[(-\frac{3}{4})\mathbf{2+3}]} \\ \vdots}} \begin{bmatrix} 2 & 0 & & & & \\ 1 & \frac{3}{2} & 0 & & & \\ & 1 & \frac{4}{3} & 0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & \frac{n}{n-1} & 0 \\ & & & & 1 & \frac{n+1}{n} \end{bmatrix} \xrightarrow[\substack{\overset{\tau}{[(-\frac{1}{2})\mathbf{2+1}]} \\ \overset{\tau}{[(-\frac{2}{3})\mathbf{3+2}]} \\ \overset{\tau}{[(-\frac{4}{3})\mathbf{4+3}]} \\ \vdots}]{} \begin{bmatrix} \frac{2}{1} & & & & \\ & \frac{3}{2} & & & \\ & & \frac{4}{3} & & \\ & & & \ddots & \\ & & & & \frac{n+1}{n} \end{bmatrix};$$

so it is positive definite.

Alternatively, we can use determinants: Lets denote by $T(n)$ the determinant of the $n \times n$ matrix $\mathbf{A}_n$. By expanding the determinant along the first column, we get the formula

$$T(n) = 2T(n-1) - \begin{vmatrix} 1 & 0 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 2 & 1 \\ & & & & 1 & 2 \end{vmatrix} = 2T(n-1) - T(n-2)$$

Were we expand the second determinant along the first row. Also $T(1) = 2$ and $T(2) = 3$, so we can check $T(n) = n + 1 > 0$. So we have $\mathbf{A}_n$ is symmetric and all top left corner determinants are positive so it is positive definite.

$\square$

(Grupo E curso 18/19) Exercise 4(a) $\mathbf{A}$ is 3 by 3. One eigenvalue is 2 and, since $\mathbf{A}$ is singular, another eigenvalue is 0. Since $\mathcal{N}(\mathbf{A})$ has dimension 2, the geometric multiplicity of $\lambda = 0$ is 2. Therefore, **$\mathbf{A}$ is diagonalizable** (we can find three linearly independent eigenvectors).

$\square$

(Grupo E curso 18/19) Exercise 4(b) Since the eigenspace corresponding to $\lambda = 0$ has dimension two, it is possible to find two perpendicular eigenvectors in that eigenspace. We need to verify if the eigenspace corresponding to $\lambda = 2$ is orthogonal to the eigenspace corresponding to $\lambda = 0$. Since

$$\begin{pmatrix} 3, & 2, & 4, \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 5,$$

these eigenspaces are not orthogonal. Hence, **$\mathbf{A}$ is not symmetric**.

$\square$

(Grupo E curso 17/18) Exercise 1(a) Since $\mathbf{A} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{0}$, it is singular.

$\square$

(Grupo E curso 17/18) Exercise 1(b) Since the rank is not 0, such a basis exist. We can apply Gram-Schmidt:

1. Elegimos un primer vector: $\boldsymbol{v}_1 = \boldsymbol{a}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

2. Proyectamos un segundo vector sobre el primero y nos quedamos con la diferencia ("la parte del segundo vector" que es ortogonal al primer vector):
$$\boldsymbol{v}_2 = \boldsymbol{a}_2 - \begin{bmatrix}\boldsymbol{a}_1\end{bmatrix}\left(\begin{bmatrix}\boldsymbol{a}_1\end{bmatrix}^\intercal\begin{bmatrix}\boldsymbol{a}_1\end{bmatrix}\right)^{-1}\begin{bmatrix}\boldsymbol{a}_1\end{bmatrix}^\intercal\boldsymbol{a}_2 = \begin{pmatrix}2\\1\\2\end{pmatrix} - \frac{8}{9}\begin{pmatrix}1\\2\\2\end{pmatrix} = \frac{1}{9}\begin{pmatrix}10\\-7\\2\end{pmatrix}.$$

3. . . . y comprobamos que son perpendiculares: $\frac{1}{9}\left(10, \quad -7, \quad 2,\right)\begin{pmatrix}1\\2\\2\end{pmatrix} = 0.$

4. Por último normalizamos los dos vectores: $\boldsymbol{q}_1 = \frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix}; \qquad \boldsymbol{q}_2 = \frac{1}{3\sqrt{17}}\begin{pmatrix}10\\-7\\2\end{pmatrix},$

   donde $\|\boldsymbol{v}_2\|^2 = \boldsymbol{v}_2\cdot\boldsymbol{v}_2 = \frac{(100+49+4)}{81} = \frac{153}{81} = \frac{17}{9}$ por lo que tenemos que $\|\boldsymbol{v}_2\| = \frac{\sqrt{17}}{3}$

An orthonormal basis:
$$\left\{\frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix};\quad \frac{1}{3\sqrt{17}}\begin{pmatrix}10\\-7\\2\end{pmatrix}\right\}.$$

□

(Grupo E curso 17/18) Exercise 1(c) Such product $\mathbf{A}\left(\mathbf{A}^\intercal\mathbf{A}\right)^{-1}\mathbf{A}^\intercal$ does not exist since $\mathbf{A}^\intercal\mathbf{A}$ is singular.

But we can use the two first columns of $\mathbf{A}$; hence, if $\mathbf{B} = \begin{bmatrix}\boldsymbol{a}_1 & \boldsymbol{a}_2\end{bmatrix}$, the projection matrix is $\mathbf{B}\left(\mathbf{B}^\intercal\mathbf{B}\right)^{-1}\mathbf{B}^\intercal$.

Or we can use an orthonormal basis of $\mathcal{C}\left(\mathbf{A}\right)$:
$$\mathbf{Q} = \frac{1}{3}\begin{bmatrix}1 & 10/\sqrt{17}\\2 & -7/\sqrt{17}\\2 & 2/\sqrt{17}\end{bmatrix} = \frac{1}{3}\begin{bmatrix}\sqrt{17}/\sqrt{17} & 10/\sqrt{17}\\2\sqrt{17}/\sqrt{17} & -7/\sqrt{17}\\2\sqrt{17}/\sqrt{17} & 2/\sqrt{17}\end{bmatrix} = \frac{1}{3\cdot\sqrt{17}}\begin{bmatrix}\sqrt{17} & 10\\2\sqrt{17} & -7\\2\sqrt{17} & 2\end{bmatrix},$$
so the projection matrix is $\mathbf{Q}\left(\mathbf{Q}^\intercal\mathbf{Q}\right)^{-1}\mathbf{Q}^\intercal = \mathbf{Q}\mathbf{Q}^\intercal$.

□

(Grupo E curso 17/18) Exercise 1(d) Hence, the projection matrix is
$$\mathbf{Q}\mathbf{Q}^\intercal = \frac{1}{(3\cdot\sqrt{17})^2}\begin{bmatrix}\sqrt{17} & 10\\2\sqrt{17} & -7\\2\sqrt{17} & 2\end{bmatrix}\begin{bmatrix}\sqrt{17} & 2\sqrt{17} & 2\sqrt{17}\\10 & -7 & 2\end{bmatrix} = \frac{1}{9\cdot 17}\begin{bmatrix}117 & -36 & 54\\-36 & 117 & 54\\54 & 54 & 72\end{bmatrix} = \frac{1}{17}\begin{bmatrix}13 & -4 & 6\\-4 & 13 & 6\\6 & 6 & 8\end{bmatrix}.$$

□

(Grupo E curso 17/18) Exercise 2(a)

A parametric equation for this line is:
$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \boldsymbol{a} + \alpha(\boldsymbol{a}-\boldsymbol{b}) = \begin{pmatrix}1\\0\\3\end{pmatrix} + \alpha\begin{pmatrix}4/3\\0\\4\end{pmatrix}$$
Since $(-3, 0, 1,)$ and $(0, 1, 0,)$ are orthogonal to $(\frac{4}{3}, 0, 4,)$, a cartesian equation for this line is
$$\begin{bmatrix}-3 & 0 & 1\\0 & 1 & 0\end{bmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{bmatrix}-3 & 0 & 1\\0 & 1 & 0\end{bmatrix}\begin{pmatrix}1\\0\\3\end{pmatrix} + \alpha\begin{bmatrix}-3 & 0 & 1\\0 & 1 & 0\end{bmatrix}\begin{pmatrix}4/3\\0\\4\end{pmatrix} \quad\Rightarrow\quad \begin{cases}-3x \quad + z = 0\\ \qquad y \quad = 0\end{cases}.$$

□

(Grupo E curso 17/18) Exercise 3(a) Using leading principal minors:
$$4 - b^2 > 0; \qquad 16 - 4 - 4b^2 = 12 - 4b^2 > 0;$$
or gaussian elimination
$$\begin{bmatrix}1 & b & 0\\b & 4 & 2\\0 & 2 & 4\end{bmatrix} \xrightarrow[{[(-b)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \begin{bmatrix}1 & 0 & 0\\b & 4-b^2 & 2\\0 & 2 & 4\end{bmatrix} \xrightarrow[{\left[\left(\frac{-2}{4-b^2}\right)\mathbf{2}+\mathbf{3}\right]}]{\boldsymbol{\tau}} \begin{bmatrix}1 & 0 & 0\\b & 4-b^2 & 0\\0 & 2 & 4-\frac{4}{4-b^2}\end{bmatrix} \Rightarrow \begin{cases}4-b^2 & > 0\\16-4b^2-4 = 12-4b^2 & > 0\end{cases}.$$

we get the same conclussion: $\begin{cases} 4-b^2>0 & \to \quad |b|<2 \\ 3-b^2>0 & \to \quad |b|<\sqrt{3} \end{cases}$; so $\boxed{-\sqrt{3}<b<\sqrt{3}.}$

□

(Grupo E curso 17/18) Exercise 3(b) Since

$$\boldsymbol{x}(\mathbf{A}^2+\mathbf{I})\boldsymbol{x} = \boldsymbol{x}(\mathbf{A}^2\boldsymbol{x}+\mathbf{I}\boldsymbol{x}) = \boldsymbol{x}\mathbf{A}^\intercal\mathbf{A}\boldsymbol{x}+\boldsymbol{x}\mathbf{I}\boldsymbol{x};$$

is a sum of squares, where $\boldsymbol{x}\mathbf{A}^\intercal\mathbf{A}\boldsymbol{x}\geq 0$ and $\boldsymbol{x}\mathbf{I}\boldsymbol{x}>0$.

□

(Grupo E curso 17/18) Exercise 3(c)
The matrix $\mathbf{M}^\intercal\mathbf{M}$ is symmetric positive definite unless not all columns are pivot columns ($rank<n$). In that case the matrix is symmetric positive *semi*definite.

□

(Grupo E curso 17/18) Exercise 4(a)
Since the rows of $\mathbf{Q}$ are vectors in $\mathbb{R}^3$, and since the rank of $\mathbf{Q}$ is 3, $\boxed{\mathcal{C}\left(\mathbf{Q}^\intercal\right)=\mathbb{R}^3}$.

□

(Grupo E curso 17/18) Exercise 4(b)
It is the proyection of $\boldsymbol{b}$ onto $\mathcal{C}\left(\mathbf{Q}\right)$. Hence, it is

$$\boldsymbol{p} = \mathbf{Q}\left[\left(\mathbf{Q}^\intercal\mathbf{Q}\right)^{-1}\mathbf{Q}^\intercal\boldsymbol{b}\right] = \mathbf{Q}\mathbf{Q}^\intercal\boldsymbol{b},$$

in other words: $\boldsymbol{p}=\begin{bmatrix}\boldsymbol{q}_1 & \boldsymbol{q}_2 & \boldsymbol{q}_3\end{bmatrix}\widehat{\boldsymbol{x}}=\mathbf{Q}\widehat{\boldsymbol{x}}$, where $\widehat{\boldsymbol{x}}=\left(\mathbf{Q}^\intercal\mathbf{Q}\right)^{-1}\mathbf{Q}^\intercal\boldsymbol{b}=\mathbf{Q}^\intercal\boldsymbol{b}$.

□

(Grupo E curso 17/18) Exercise 4(c)
The error vector $\boldsymbol{e}\equiv\boldsymbol{b}-\boldsymbol{p}\in\mathcal{N}\left(\mathbf{Q}^\intercal\right)$:

$$\boldsymbol{b}-\mathbf{Q}\mathbf{Q}^\intercal\boldsymbol{b} = \left[\mathbf{I}-\mathbf{Q}\mathbf{Q}^\intercal\right]\boldsymbol{b}.$$

Or, in a different way

$$\boldsymbol{b}-\boldsymbol{p} = \left[\boldsymbol{b}\,|\,\mathbf{Q}\right]\begin{pmatrix}1\\-\mathbf{Q}^\intercal\boldsymbol{b}\end{pmatrix} = \left[\boldsymbol{b}\,|\,\boldsymbol{q}_1 \quad \boldsymbol{q}_2 \quad \boldsymbol{q}_3\right]\begin{pmatrix}1\\-\widehat{\boldsymbol{x}}\end{pmatrix}, \qquad \text{where} \quad \widehat{\boldsymbol{x}}=\mathbf{Q}^\intercal\boldsymbol{b}.$$

The error vector $\boldsymbol{b}-\boldsymbol{p}$ belongs to $\mathcal{N}\left(\mathbf{Q}^\intercal\right)$ since: $\mathbf{Q}^\intercal\left(\boldsymbol{b}-\boldsymbol{p}\right)=\mathbf{Q}^\intercal\left[\mathbf{I}-\mathbf{Q}\mathbf{Q}^\intercal\right]\boldsymbol{b}=\left[\mathbf{Q}^\intercal-\mathbf{Q}^\intercal\right]\boldsymbol{b}=\mathbf{0}$.
□

(Grupo F curso 17/18) Exercise 1(a)

$$\begin{vmatrix}1&2&3&4\\0&3&4&5\\0&0&5&6\\2&0&0&7\end{vmatrix} = 1\cdot\begin{vmatrix}3&4&5\\0&5&6\\0&0&7\end{vmatrix} - 2\begin{vmatrix}2&3&4\\3&4&5\\0&5&6\end{vmatrix} = 1\cdot(3\cdot5\cdot7)-2\cdot(4)=97.$$

□

(Grupo F curso 17/18) Exercise 1(b)
$\boxed{\mathbf{A}\left(\mathbf{A}^\intercal\mathbf{A}\right)^{-1}\mathbf{A}^\intercal \text{ is singular and therefore its determinant is } 0}$. Let's see why it is singular:
Let $\mathbf{E}$ be a product of several elementary matrices such that $\mathbf{A}^\intercal\mathbf{E}=\underset{2\times3}{\mathbf{R}}$, is the reduced echelon form of $\mathbf{A}^\intercal$; since $\mathbf{A}$ has rank 2, the last column of $\mathbf{R}$ is full of zeros, therefore

$$\mathbf{A}\left(\mathbf{A}^\intercal\mathbf{A}\right)^{-1}\mathbf{A}^\intercal\mathbf{E}$$

has a column full of zeros; hence $\mathbf{A}(\mathbf{A}^\intercal\mathbf{A})^{-1}\mathbf{A}^\intercal$ is singular. In other words, since $\mathbf{A}\left(\mathbf{A}^\intercal\mathbf{A}\right)^{-1}\mathbf{A}^\intercal\mathbf{E}$ has a column full of zeros

$$0=\det(\mathbf{A}\left(\mathbf{A}^\intercal\mathbf{A}\right)^{-1}\mathbf{A}^\intercal\mathbf{E})=\det(\mathbf{A}\left(\mathbf{A}^\intercal\mathbf{A}\right)^{-1}\mathbf{A}^\intercal)\cdot\det(\mathbf{E}),$$

but since $\det(\mathbf{E})\neq0$ it follows that $\det(\mathbf{A}\left(\mathbf{A}^\intercal\mathbf{A}\right)^{-1}\mathbf{A}^\intercal)=0$.

□

(Grupo F curso 17/18) Exercise 2(a)
The eigenvalues are 1 and -1 since

$$\begin{cases} \lambda_1 + \lambda_2 = 0; & (\operatorname{tr}(\mathbf{A}) = 0) \\ \lambda_1 \lambda_2 = -1; & (\det \mathbf{A} = -1) \end{cases}$$

For $\lambda = 1$ we get

$$\mathbf{A} - \mathbf{I} = \begin{bmatrix} -16 & 8 \\ -28 & 14 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} -16 & 8 \\ -28 & 14 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \mathbf{0}.$$

Hence $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector associated to the eigenvalue $\lambda = 1$.

For $\lambda = -1$ we get $\mathbf{A} + \mathbf{I} = \begin{bmatrix} -14 & 8 \\ -28 & 16 \end{bmatrix}$

$$\left[\begin{array}{cc} -14 & 8 \\ -28 & 16 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] \xrightarrow[\substack{\tau \\ [(7)\mathbf{2}]}]{\substack{\tau \\ [(4)\mathbf{1}]}} \left[\begin{array}{cc} -56 & 56 \\ -112 & 112 \\ \hline 4 & 0 \\ 0 & 7 \end{array}\right] \xrightarrow{\substack{\tau \\ [(1)\mathbf{2}+\mathbf{1}]}} \left[\begin{array}{cc} -56 & 0 \\ -112 & 0 \\ \hline 4 & \mathbf{4} \\ 0 & \mathbf{7} \end{array}\right]$$

Hence $\begin{pmatrix} 4 \\ 7 \end{pmatrix}$ is an eigenvector associated to the eigenvalue $\lambda = -1$.

□

(Grupo F curso 17/18) Exercise 2(b)
Therefore

$$\mathbf{S} = \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}; \qquad \mathbf{D} = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}.$$

□

(Grupo F curso 17/18) Exercise 2(c)

$$\mathbf{A}^{37} = \mathbf{S}\mathbf{D}^{37}\mathbf{S}^{-1} = \mathbf{S} \begin{bmatrix} 1^{37} & \\ & -1^{37} \end{bmatrix} \mathbf{S}^{-1} = \mathbf{S} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \mathbf{S}^{-1} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1} = \mathbf{A}.$$

□

(Grupo F curso 17/18) Exercise 3(a)

Let $\mathbf{A} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \\ -1 & 2 \end{bmatrix}$, then

$$\boldsymbol{p} = \mathbf{A}\left(\mathbf{A}^\intercal\mathbf{A}\right)^{-1}\mathbf{A}^\intercal\boldsymbol{b} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 9 & \\ & 9 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{1}{9} \begin{bmatrix} 2 & 2 \\ 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

□

(Grupo F curso 17/18) Exercise 3(b)

1. Elegimos un primer vector: $\boldsymbol{v}_1 = \boldsymbol{a}_1$
2. Proyectamos un segundo vector sobre el primero y nos quedamos con la diferencia (el componente del segundo vector que es ortogonal al primer vector):

   pero en este caso, como $\boldsymbol{a}_1$ ya es perpendicular a $\boldsymbol{a}_2$, resulta que $\boldsymbol{v}_2 = \boldsymbol{a}_2$.

   (Si no nos damos cuenta y lo calculamos, obtenemos lo que acabamos de indicar:)

$$\boldsymbol{v}_2 = \boldsymbol{a}_2 - \left[\boldsymbol{a}_1\right]\left(\left[\boldsymbol{a}_1\right]^\intercal\left[\boldsymbol{a}_1\right]\right)^{-1}\left[\boldsymbol{a}_1\right]^\intercal\boldsymbol{a}_2 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}.$$

3. Proyectamos $\boldsymbol{b}$ sobre el espacio generado por los vectores anteriores y nos quedamos con la diferencia (el componente de $\boldsymbol{b}$ que es ortogonal a $\boldsymbol{v}_1$ y $\boldsymbol{v}_2$. Como ya hemos calculado dicha proyección en el apartado anterior, solo queda calcular la diferencia:

$$\boldsymbol{v}_3 = \boldsymbol{b} - \boldsymbol{p} = \begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\2\\2\end{pmatrix}$$

4. Por último normalizamos los tres vectores. Como todos ellos tienen norma 3 tenemos que

$$\boldsymbol{q}_1 = \frac{1}{3}\begin{pmatrix}2\\2\\-1\end{pmatrix}; \qquad \boldsymbol{q}_2 = \frac{1}{3}\begin{pmatrix}2\\-1\\2\end{pmatrix}; \qquad \boldsymbol{q}_3 = \frac{1}{3}\begin{pmatrix}-1\\2\\2\end{pmatrix}.$$

$\square$

(Grupo F curso 17/18) Exercise 4(a)

Los autovectores $\boldsymbol{u}$, $\boldsymbol{v}$ y $\boldsymbol{w}$ son linealmente independientes puesto que corresponden a autovalores distintos. Además, puesto que dos autovalores son distintos de cero, rg $\left(\mathbf{A}\right)$ =2, y dim$\mathcal{N}\left(\mathbf{A}\right) = 1$. Así,

- $\mathbf{A}\boldsymbol{u} = \mathbf{0} \Rightarrow \quad \mathcal{N}\left(\mathbf{A}\right)$ is the spam of $\boldsymbol{u}$
  (es la recta consistente en todos los múltiplos de $\boldsymbol{u}$)
- $\begin{cases}\mathbf{A}\boldsymbol{v} = \boldsymbol{v}\\ \frac{1}{2}\mathbf{A}\boldsymbol{w} = \boldsymbol{w}\end{cases} \quad \Rightarrow \; \boldsymbol{v}, \boldsymbol{w} \in \mathcal{C}\left(\mathbf{A}\right) \qquad \mathcal{C}\left(\mathbf{A}\right)$ is the spam of $\{\boldsymbol{v}, \boldsymbol{w}\}$
  (es el plano consistente en todas las combinaciones lineales de $\boldsymbol{v}$ y $\boldsymbol{w}$)
- $\mathcal{N}\left(\mathbf{A}\right) \perp \mathcal{C}\left(\mathbf{A}^{\intercal}\right) \Rightarrow \quad \mathcal{C}\left(\mathbf{A}^{\intercal}\right) = \{\boldsymbol{x} \mid \boldsymbol{x}\cdot\boldsymbol{u} = 0\}$
  (es el plano consistente en todos los vectores perpendiculares a $\boldsymbol{u}$)

$\square$

(Grupo F curso 17/18) Exercise 4(b)

Since $\mathcal{N}\left(\mathbf{A}\right)$ is the spam of $\boldsymbol{u}$, we only need to find a particular solution. Since $\mathbf{A}\boldsymbol{v} = \boldsymbol{v}$ and $\mathbf{A}\boldsymbol{w} = 2\boldsymbol{w}$ it follows that $\mathbf{A}(\boldsymbol{v} - \frac{1}{2}\boldsymbol{w}) = \boldsymbol{v} - \boldsymbol{w}$; hence the set of all solutions is

$$\left\{\boldsymbol{x} \;\middle|\; \boldsymbol{x} = \left(\boldsymbol{v} - \frac{1}{2}\boldsymbol{w}\right) + a\boldsymbol{u}; \quad \forall a \in \mathbb{R}\right\}.$$

$\square$

(Grupo F curso 17/18) Exercise 4(c)

For any orthogonal matrix $\mathbf{Q}$ we get $\mathbf{Q}^{\intercal}\mathbf{Q} = \mathbf{Q}\mathbf{Q}^{\intercal} = \mathbf{I}$. Hence $\det(\mathbf{Q})^2 = 1$, and therefore $\det(\mathbf{Q})$ is either 1 or -1. But $\det(\mathbf{A}) = 0$.

$\square$

(Grupo B curso 16/17) Exercise 1.

$$\left[\begin{array}{ccc|c}1&1&1&-2\\2&2&1&-3\\0&3&-1&-4\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right] \xrightarrow[{[(2)\boldsymbol{1}+\boldsymbol{4}]}]{\substack{\boldsymbol{\tau}\\{[(-1)\boldsymbol{1}+\boldsymbol{2}]}\\\boldsymbol{\tau}\\{[(-1)\boldsymbol{1}+\boldsymbol{3}]}\\\boldsymbol{\tau}}} \left[\begin{array}{ccc|c}1&0&0&0\\2&0&-1&1\\0&3&-1&-4\\\hline 1&-1&-1&2\\0&1&0&0\\0&0&1&0\end{array}\right] \xrightarrow{\overset{\boldsymbol{\tau}}{[(1)\boldsymbol{3}+\boldsymbol{4}]}} \left[\begin{array}{ccc|c}1&0&0&0\\2&0&-1&0\\0&3&-1&-5\\\hline 1&-1&-1&1\\0&1&0&0\\0&0&1&1\end{array}\right] \xrightarrow{\overset{\boldsymbol{\tau}}{[(5/3)\boldsymbol{2}+\boldsymbol{4}]}} \left[\begin{array}{ccc|c}1&0&0&0\\2&0&-1&0\\0&3&-1&0\\\hline 1&-1&-1&-2/3\\0&1&0&5/3\\0&0&1&1\end{array}\right]$$

So $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-2/3\\5/3\\1\end{pmatrix}$.

$\square$

(Grupo B curso 16/17) Exercise 2(a)

$$\left[\begin{array}{cccc}1&2&2&3\\1&2&4&3\\1&2&6&3\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow{\substack{\boldsymbol{\tau}\\[(-2)\mathbf{1}+\mathbf{2}]\\ \boldsymbol{\tau}\\[(-2)\mathbf{1}+\mathbf{3}]\\ \boldsymbol{\tau}\\[(-3)\mathbf{1}+\mathbf{4}]}}\left[\begin{array}{cccc}1&0&0&0\\1&0&2&0\\1&0&4&0\\\hline 1&-2&-2&-3\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]$$

So, a basis for $\mathcal{C}\left(\mathbf{A}\right)$: $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix};\begin{bmatrix}0\\2\\4\end{bmatrix}\right\}$; a basis for $\mathcal{N}\left(\mathbf{A}\right)$: $\left\{\begin{pmatrix}-2\\1\\0\\0\end{pmatrix};\begin{pmatrix}-3\\0\\0\\1\end{pmatrix}\right\}$.

□

(Grupo B curso 16/17) Exercise 2(b)

It is the set of vectors

$$\left\{\boldsymbol{x}\in\mathbb{R}^4\;\middle|\;\boldsymbol{x}=\begin{pmatrix}0\\0\\1/2\\0\end{pmatrix}+a\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}+b\begin{pmatrix}-3\\0\\0\\1\end{pmatrix},\qquad\forall a,b\in\mathbb{R}\right\}$$

□

(Grupo B curso 16/17) Exercise 2(c)

$\mathcal{C}\left(\mathbf{A}\right)=\mathbb{R}^n$; $\mathcal{N}\left(\mathbf{A}\right)=\{\mathbf{0}\}$; Basis for $\mathcal{C}\left(\mathbf{A}\right)$: the n columns of $\underset{n\times n}{\mathbf{I}}$; (Or the n columns of $\mathbf{A}$ or $\mathbf{A}^{-1}$).

□

(Grupo B curso 16/17) Exercise 3(a)

The eigenvalues of $\mathbf{A}$ are 1 with multiplicity one and $1/4$ with multiplicity two. Clearly the eigenvectors with eigenvalue 1 are the non-zero multiples of the vector $(1,1,1,)$. The remaining eigenvectors are all non-zero vector of the orthogonal complement of $\mathcal{L}([(1,1,1,);])$. Since

$$\left[\begin{array}{ccc}1&1&1\\\hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow{\substack{\boldsymbol{\tau}\\[(-1)\mathbf{1}+\mathbf{2}]\\ \boldsymbol{\tau}\\[(-1)\mathbf{1}+\mathbf{3}]}}\left[\begin{array}{ccc}1&0&0\\\hline 1&-1&-1\\0&1&0\\0&0&1\end{array}\right]$$

a basis for this orthogonal complement is $\left[\begin{pmatrix}-1, & 1, & 0,\end{pmatrix};\ \begin{pmatrix}-1, & 0, & 1,\end{pmatrix};\right]$, so they belong to the eigenspace $\mathcal{E}_{\frac{1}{4}}(\mathbf{A})$ corresponding to $\lambda=1/4$. Hence, they are the vectors $\begin{pmatrix}(-y-z), & y, & z,\end{pmatrix}$ for $y,z\in\mathbb{R}$.

□

(Grupo B curso 16/17) Exercise 3(b)

but they are not perpendicular. Lets find two perpendicular eigenvectors in $\mathcal{E}_{\frac{1}{4}}(\mathbf{A})$.

$$0=(-1,1,0,)\cdot\left(a\begin{pmatrix}-1\\1\\0\end{pmatrix}+b\begin{pmatrix}-1\\0\\1\end{pmatrix}\right)=2a+b\ \Rightarrow\ b=-2a.\ \Rightarrow\ \begin{pmatrix}-1\\1\\0\end{pmatrix}-2\begin{pmatrix}-1\\0\\1\end{pmatrix}=\begin{pmatrix}1\\1\\-2\end{pmatrix}\in\mathcal{E}_{\frac{1}{4}}(\mathbf{A}).$$

For the matrix $\mathbf{Q}$ we may choose the orthogonal matrix

$$\mathbf{Q}=\begin{bmatrix}1/\sqrt{3}&-1/\sqrt{2}&1/\sqrt{6}\\1/\sqrt{3}&1/\sqrt{2}&1/\sqrt{6}\\1/\sqrt{3}&0&-2/\sqrt{6}\end{bmatrix}$$

and the matrix $\mathbf{D}$ is then the diagonal matrix with entries 1, $1/4$, and $1/4$ along the diagonal. We have

$$\lim_{k\to\infty}\mathbf{A}^k=\mathbf{Q}\left(\lim_{k\to\infty}\mathbf{D}^k\right)\mathbf{Q}^{-1}=\mathbf{Q}\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\mathbf{Q}^{\intercal},$$

and therefore

$$\lim_{k\to\infty} \mathbf{A}^k = \begin{bmatrix} 1/\sqrt{3} & 0 & 0 \\ 1/\sqrt{3} & 0 & 0 \\ 1/\sqrt{3} & 0 & 0 \end{bmatrix} \mathbf{Q}^\mathsf{T} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

□

(Grupo B curso 16/17) Exercise 3(c)

Remember, is $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ then

$$(\mathbf{A} - b\mathbf{I})\boldsymbol{x} = \mathbf{A}\boldsymbol{x} - b\boldsymbol{x} = (\lambda - b)\boldsymbol{x}$$

- Any $r < 1/4$ is such that $\mathbf{A} - r\mathbf{I}$ is positive definite. Since we want r to be positive, we may choose $r = 1/8$.
- Any $1/4 < s < 1$ is such that $\mathbf{A} - s\mathbf{I}$ is neither positive nor negative definite. We may choose $s = 1/2$.
- Any $1 < t$ is such that $\mathbf{A} - t\mathbf{I}$ is negative definite. We may choose $t = 2$.

□

(Grupo B curso 16/17) Exercise 4(a)

The vectors $(-1, 1, 0)$ and $(-1, 0, 1)$ form a basis for the subspace $x + y + z = 0$. Hence, the plane consist of the points in the set

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} x \\ y \\ z \end{pmatrix} = a\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \forall a, b \in \mathbb{R} \right\}$$

□

(Grupo B curso 16/17) Exercise 4(b)

Let $\mathbf{A}$ be the matrix whose columns are the two vectors found above. Thus the projection matrix $\mathbf{P}$ onto the subspace $x + y + z = 0$ is

$$\begin{aligned} \mathbf{P} =& \mathbf{A}(\mathbf{A}^\mathsf{T}\mathbf{A})^{-1}\mathbf{A}^\mathsf{T} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \\ =& \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \\ =& \frac{1}{3}\begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \\ =& \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}. \end{aligned}$$

The projection of $(1, 2, 6,)$ onto the plane $x + y + z = 0$ is thus simply

$$\boldsymbol{p} = \mathbf{P}\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}.$$

□

(Grupo B curso 16/17) Exercise 5(a)

Since $\mathbf{P}^2 = \mathbf{P}$, $\det(\mathbf{P})^2 = \det(\mathbf{P})$, so that only 0 or 1 are possible.

□

(Grupo B curso 16/17) Exercise 5(b)

Starting with $\mathbf{I}$, a permutation matrix is obtained through column (or row) exchanges, therefore we can get only ± 1.

□

(Grupo B curso 16/17) Exercise 6(a)

We need to check two statements: the vector $(\boldsymbol{b}-\boldsymbol{p})$ is orthogonal to the space generated by $\mathcal{L}\left([\boldsymbol{a}_1;\ldots\boldsymbol{a}_n;]\right)$ and the vector $\boldsymbol{p}$ lies in that subspace. The first condition we check by seeing if the scalar products $\boldsymbol{a}_1\cdot(\boldsymbol{b}-\boldsymbol{p}),\ldots,\ \boldsymbol{a}_n\cdot(\boldsymbol{b}-\boldsymbol{p})$ are all zero.

The second condition we check by considering the $m\times(n+1)$ matrix whose first n columns are the coordinates of the $\boldsymbol{a}_i$'s and whose last column consists of the coordinates of $\boldsymbol{p}$. The vector $\boldsymbol{p}$ is in the span of the $\boldsymbol{a}_i$'s if and only if the last column of the augmented matriz becomes zero in elimination.

□

(Grupo E curso 16/17) Exercise 1(a)

Since the third row is the sum the two first ones, the rank is 2; so it is a plane in $\mathbb{R}^3$ formed by all linear combinations of the two first columns:

$$\mathcal{C}\left(\mathbf{A}\right)=\left\{\boldsymbol{x}\in\mathbb{R}^3\quad\text{such that}\quad\boldsymbol{x}=a\begin{pmatrix}1\\2\\3\end{pmatrix}+b\begin{pmatrix}2\\3\\4\end{pmatrix};\qquad\forall a,b\in\mathbb{R}\right\}$$

□

(Grupo E curso 16/17) Exercise 1(b)

By gaussian elimination we get...

$$\left[\begin{array}{cccc|c}1&2&3&4&-b_1\\2&3&4&5&-b_2\\3&4&5&6&-b_3\\\hline 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{array}\right]\xrightarrow{\begin{subarray}{c}\boldsymbol{\tau}_{[(-2)\mathbf{1}+\mathbf{2}]}\\\boldsymbol{\tau}_{[(-3)\mathbf{1}+\mathbf{3}]}\\\boldsymbol{\tau}_{[(-4)\mathbf{1}+\mathbf{4}]}\end{subarray}}\left[\begin{array}{cccc|c}1&0&0&0&-b_1\\2&-1&-2&-3&-b_2\\3&-2&-4&-6&-b_3\\\hline 1&-2&-3&-4&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{array}\right]\xrightarrow{\begin{subarray}{c}\boldsymbol{\tau}_{[(-2)\mathbf{2}+\mathbf{3}]}\\\boldsymbol{\tau}_{[(-3)\mathbf{2}+\mathbf{4}]}\end{subarray}}\left[\begin{array}{cccc|c}1&0&0&0&-b_1\\2&-1&0&0&-b_2\\3&-2&0&0&-b_3\\\hline 1&-2&1&2&0\\0&1&-2&-3&0\\0&0&1&0&0\\0&0&0&1&0\end{array}\right]$$

$$\xrightarrow{\begin{subarray}{c}\boldsymbol{\tau}_{[(2)\mathbf{1}+\mathbf{2}]}\\\boldsymbol{\tau}_{[(-1)\mathbf{2}]}\\\boldsymbol{\tau}_{[(b_1)\mathbf{1}+\mathbf{5}]}\end{subarray}}\left[\begin{array}{cccc|c}1&0&0&0&0\\0&1&0&0&-b_2\\-1&2&0&0&-b_3-b_1\\\hline -3&2&1&2&-3b_1\\2&-1&-2&-3&2b_1\\0&0&1&0&0\\0&0&0&1&0\end{array}\right]\xrightarrow{\boldsymbol{\tau}_{[(b_2)\mathbf{2}+\mathbf{5}]}}\left[\begin{array}{cccc|c}1&0&0&0&0\\0&1&0&0&0\\-1&2&0&0&2b_2-b_3-b_1\\\hline -3&2&1&2&2b_2-3b_1\\2&-1&-2&-3&2b_1-b_2\\0&0&1&0&0\\0&0&0&1&0\end{array}\right]$$

$\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$ have a solution if and only if $\boldsymbol{b}$ is a linear combination of vectors $\begin{pmatrix}1\\0\\-1\end{pmatrix}$ and $\begin{pmatrix}0\\1\\2\end{pmatrix}$:

$$\boldsymbol{b}=b_1\begin{pmatrix}1\\0\\-1\end{pmatrix}+b_2\begin{pmatrix}0\\1\\2\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\2b_2-b_1\end{pmatrix}.$$

The condition on b_1, b_2, b_3 is: $\qquad 2b_2-b_3-b_1=0.$

□

(Grupo E curso 16/17) Exercise 1(c) This is because $\mathbf{A}$ is not full row-rank, as shown in part (b).

If $\mathbf{AC}=\mathbf{I}$, then we could solve every equation $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$. Actually the solution would be $\boldsymbol{x}=\mathbf{C}\boldsymbol{b}$, since $\mathbf{AC}\boldsymbol{b}=\mathbf{I}\boldsymbol{b}$. But in part (b) we saw that $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$ has no solution for some $\boldsymbol{b}$.

□

(Grupo E curso 16/17) Exercise 1(d) In this case $b_1=1$, $b_2=0$, and $b_3=-1$, so the set of solutions is

$$\left\{\boldsymbol{x}\in\mathbb{R}^4;\text{ such that; }\boldsymbol{x}=\begin{pmatrix}-3\\2\\0\\0\end{pmatrix}+a\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}+b\begin{pmatrix}2\\-3\\0\\1\end{pmatrix};\quad a,b\in\mathbb{R}\right\}$$

□

(Grupo E curso 16/17) Exercise 2(a)

The answer is

$$\mathbf{A}=\begin{bmatrix}2&6\\-1&7\end{bmatrix}.$$

Reason:

$$\begin{bmatrix} 2 & 6 \\ a & b \end{bmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 12 \\ 3a+b \end{pmatrix} = \lambda_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

We deduce that $\lambda_1 = 4$ and $3a + b = 4$. Similarly, since $\boldsymbol{x}_2$ is an eigenvector we have

$$\begin{bmatrix} 2 & 6 \\ a & b \end{bmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 2a+b \end{pmatrix} = \lambda_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

We deduce that $\lambda_2 = 5$ and therefore that $2a + b = 5$. Solving $\left[\begin{smallmatrix} 3 & 1 \\ 2 & 1 \end{smallmatrix}\right] \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = \left(\begin{smallmatrix} 4 \\ 5 \end{smallmatrix}\right)$

$$\left[\begin{array}{cc|c} 3 & 1 & -4 \\ 2 & 1 & -5 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(4)\mathbf{1}+\mathbf{3}]}]{\substack{\tau \\ [(3)\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{2}] \\ [(3)\mathbf{3}]}} \left[\begin{array}{cc|c} 3 & 0 & 0 \\ 2 & 1 & -7 \\ \hline 1 & -1 & 4 \\ 0 & 3 & 0 \\ \hline 0 & 0 & 3 \end{array}\right] \xrightarrow{\overset{\tau}{[(7)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{cc|c} 3 & 0 & 0 \\ 2 & 1 & 0 \\ \hline 1 & -1 & -3 \\ 0 & 3 & 21 \\ \hline 0 & 0 & 3 \end{array}\right] \xrightarrow{\overset{\tau}{[(\frac{1}{3})\mathbf{3}]}} \left[\begin{array}{cc|c} 3 & 0 & 0 \\ 2 & 1 & 0 \\ \hline 1 & -1 & -1 \\ 0 & 3 & 7 \\ \hline 0 & 0 & 1 \end{array}\right],$$

we conclude that $a = -1$ and $b = 7$.

□

(Grupo E curso 16/17) Exercise 2(b)

$\mathbf{B} = \mathbf{S}\ \mathbf{D}\ \mathbf{S}^{-1}$, where the columns of $\mathbf{S}$ are the vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, and $\mathbf{D}$ y a diagonal matrix with entries 1 and 0:

$$\mathbf{B} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}.$$

Then $\mathbf{D}^{10} = \mathbf{D}$ and therefore $\mathbf{B}^{10} = \mathbf{S}\mathbf{D}^{10}\mathbf{S}^{-1} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1} = \mathbf{B}$.

□

(Grupo E curso 16/17) Exercise 3.

The product $\mathbf{P}_2\mathbf{P}_1$ is projection onto the column space of $\mathbf{P}_1$, followed by the projection onto the column space of $\mathbf{P}_2$. Since the column space of $\mathbf{P}_2$ contains the column space of $\mathbf{P}_1$, the second projection does not change the vectors anymore. Thus

$$\mathbf{P}_2\mathbf{P}_1 = \mathbf{P}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 & 0 & 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}.$$

□

(Grupo E curso 16/17) Exercise 4.

$$\begin{bmatrix} 2 & b & 3 \\ b & 2 & b \\ 3 & b & 4 \end{bmatrix} \xrightarrow{\substack{\overset{\tau}{[(-\frac{b}{2})\mathbf{2}+\mathbf{1}]} \\ \overset{\tau}{[(-\frac{3}{2})\mathbf{3}+\mathbf{1}]}}} \begin{bmatrix} 2 & 0 & 0 \\ b & 2-b & -b/2 \\ 3 & -b/2 & -1/2 \end{bmatrix} \xrightarrow[\substack{[(-\frac{3}{2})\mathbf{3}+\mathbf{1}] \\ \tau \\ [(-\frac{b}{2})\mathbf{2}+\mathbf{1}]}]{} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2-b & -b/2 \\ 0 & -b/2 & -1/2 \end{bmatrix},$$

since there is a negative number on the main diagonal, $\mathbf{A}$ can't be positive definite

Another way to solve the problem:

$\mathbf{A}$ has 3 positive eigenvalues if and only if it is positive- definite. To test for positive-definiteness, we check the three upper-left determinants to see when they are positive. The 1 by 1 upper-left determinant is 2, which is positive. The 2 by 2 upper-left determinant is $4 - b^2$, which is positive whenever $-2 < b - 2$. Finally we compute de 3 by 3 upper-left determinant, of $\det \mathbf{A}$:

$$\begin{aligned} \det \mathbf{A} =& 2 \det \begin{bmatrix} 2 & b \\ b & 4 \end{bmatrix} - b \det \begin{bmatrix} b & b \\ 3 & 4 \end{bmatrix} + 3 \det \begin{bmatrix} b & 2 \\ 3 & b \end{bmatrix} \\ =& 2(8 - b^2) - b(4b - 3b) + 3(b^2 - 6) = -2. \end{aligned}$$

which is always negative. Since $\det \mathbf{A} < 0$ regardless of the value of b, we conclude that $\mathbf{A}$ cannot have 3 positive eigenvalues.

□

(Grupo E curso 16/17) Exercise 5(a)

If $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ has no solution, the column space of A must have dimension less than m. The rank is $r < m$. Since $\mathsf{A}^\mathsf{T}\boldsymbol{y} = \boldsymbol{c}$ has exactly one solution, the columns of A^T are independent. This means that the rank of A^T is $r = m$. This contradiction proves that we cannot find A, $\boldsymbol{b}$ and $\boldsymbol{c}$.

□

(Grupo E curso 16/17) Exercise 6. Again, there are many ways to do this. First way: rememaber that a symmetric matrix is positive definite if and only if it can be written as $\mathsf{R}^\mathsf{T}\mathsf{R}$ where R has linearly independent columns. In our case, let $c_1, \ldots, c_n$ be the diagonal entries of C and let B be the diagonal matrix with diagonal entries $\sqrt{c_1}, \ldots, \sqrt{c_n}$ (take the positive square roots). Then $\mathsf{C} = \mathsf{B}^\mathsf{T}\mathsf{B}$ and so $\mathsf{K} = \mathsf{A}^\mathsf{T}\mathsf{B}^\mathsf{T}\mathsf{B}\mathsf{A} = (\mathsf{B}\mathsf{A})^\mathsf{T}(\mathsf{B}\mathsf{A})$ so we take $\mathsf{R} = \mathsf{B}\mathsf{A}$. Since A has linearly independent columns and B is invertible (because the c_i are nonzero numbers), we conclude that $\mathsf{B}\mathsf{A}$ also has linearly independent columns.

Second way: Let $\boldsymbol{x}$ be a nonzero vector. We have to show that $\boldsymbol{x}\mathsf{K}\boldsymbol{x} > 0$. First, since A has linearly independent columns, this means that its null space is $\{\mathbf{0}\}$, so $\mathsf{A}\boldsymbol{x} \neq \mathbf{0}$. Set $\boldsymbol{y} = \mathsf{A}\boldsymbol{x}$. Since C is diagonal and has positive diagonal entries, it is positive definite (this follows from the eigenvalue definition, or the submatrices definition, for example). So $\boldsymbol{y}\mathsf{C}\boldsymbol{y} > 0$, but $\boldsymbol{y}\mathsf{C}\boldsymbol{y} = \boldsymbol{x}\mathsf{K}\boldsymbol{x}$, so we're done.

□

(Grupo E curso 16/17) Exercise 7(a) There are several ways to do this. One way is to use the cofactor formula on the second row, which gives

$$-x\begin{vmatrix} x & x & x \\ 0 & x & x \\ 0 & x & 1 \end{vmatrix} + x\begin{vmatrix} x & x & x \\ x & x & x \\ x & x & 1 \end{vmatrix}.$$

The second determinant is zero because the first two columns are dependent, so we just need to expand the first. The first column is easy since there is only one nonzero element. Expanding gives $-x \cdot x(x - x^2) = -x^3(1-x)$

□

(Grupo E curso 16/17) Exercise 7(b) A is singular exactly when the determinant is 0. This means $-x \cdot x(x - x^2) = -x^3(1-x) = 0$, which means x is 0 or 1.

□

(Grupo E curso 15/16) Exercise 1(a) The column space is a plane, hence it has dimension 2. That means that any two independent columns vectors in the plane plus the zero vector will do. For example,

$$\mathsf{A} = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

□

(Grupo E curso 15/16) Exercise 1(b) The column space is a plane (a plane is 2-dimensional), so the rank of the matrix is 2, which is less than the order of the matrix. In particular the sum of the rows is the zero vector (since the sum of each column is zero), therefore, its rows are linearly dependent.

□

(Grupo E curso 15/16) Exercise 2.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & -10 \\ 4 & 5 & 6 & -28 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\tau}]{\underset{[(-2)\mathbf{1}+\mathbf{2}]}{\tau}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -10 \\ 4 & -3 & -6 & -28 \\ \hline 1 & -2 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[\underset{[(-2)\mathbf{2}+\mathbf{3}]}{\tau}]{\underset{[(10)\mathbf{1}+\mathbf{4}]}{\tau}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 4 & -3 & 0 & 12 \\ \hline 1 & -2 & 1 & 10 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\underset{[(4)\mathbf{2}+\mathbf{4}]}{\tau}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 4 & -3 & 0 & 0 \\ \hline 1 & -2 & 1 & 2 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

So, a parametric equation is

$$\boldsymbol{x} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + a\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

□

(Grupo E curso 15/16) Exercise 3(a) Two vectors are orthogonal if and only if their scalar product (dot product) is zero. So

$$(\boldsymbol{u} + \boldsymbol{v}) \cdot (\boldsymbol{u} - \boldsymbol{v}) = 0$$

The left hand side expands to

$$\boldsymbol{u}\cdot\boldsymbol{u}-\boldsymbol{u}\cdot\boldsymbol{v}+\boldsymbol{v}\cdot\boldsymbol{u}-\boldsymbol{v}\cdot\boldsymbol{v}=\boldsymbol{u}\cdot\boldsymbol{u}-\boldsymbol{v}\cdot\boldsymbol{v}=\|\boldsymbol{u}\|^2-\|\boldsymbol{v}\|^2=0$$

Thus $\|\boldsymbol{u}\|^2=\|\boldsymbol{v}\|^2$ so $\|\boldsymbol{u}\|=\|\boldsymbol{v}\|$.

□

(Grupo E curso 15/16) Exercise 3(b) Since $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are unit vectors, their lengths (and hence their lengths squared) are all equal to 1. So $\boldsymbol{u}\cdot\boldsymbol{u}=\boldsymbol{v}\cdot\boldsymbol{v}=\boldsymbol{w}\cdot\boldsymbol{w}=1$. Since each vector is perpendicular to the others, $\boldsymbol{u}\cdot\boldsymbol{v}=\boldsymbol{u}\cdot\boldsymbol{w}=\boldsymbol{v}\cdot\boldsymbol{w}=0$. So the dot product of the two vectors given is

$$\begin{aligned}(\boldsymbol{u}-3\boldsymbol{v}+2\boldsymbol{w})\cdot(\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w})= &\quad \boldsymbol{u}\cdot\boldsymbol{u}+\boldsymbol{u}\cdot\boldsymbol{v}+\boldsymbol{u}\cdot\boldsymbol{w}\\ &-3\boldsymbol{u}\cdot\boldsymbol{u}-3\boldsymbol{u}\cdot\boldsymbol{v}-3\boldsymbol{u}\cdot\boldsymbol{w}\\ &+2\boldsymbol{u}\cdot\boldsymbol{u}+2\boldsymbol{u}\cdot\boldsymbol{v}+2\boldsymbol{u}\cdot\boldsymbol{w}\\ =&1-3+2=0.\end{aligned}$$

so they are orthogonal.

□

(Grupo E curso 15/16) Exercise 4(a) The characteristic polynomial is

$$\det(\mathbf{A}-\lambda\mathbf{I})=(\lambda-5)(\lambda-3)-1=\lambda^2-8\lambda+14$$

The eigenvalues are its roots,

$$\lambda=\frac{8\pm\sqrt{64-56}}{2}=4\pm\sqrt{2}.$$

□

(Grupo E curso 15/16) Exercise 4(b) The eigenvectors with eigenvalue 3 are nonzero solutions of $\mathbf{B}\boldsymbol{v}=3\boldsymbol{v}$, or equivalently, nonzero solutions of $(\mathbf{B}-3\mathbf{I})\boldsymbol{v}=\mathbf{0}$. By column reduction we get

$$\left[\begin{array}{rrr}-1&-1&0\\2&-2&1\\5&-7&3\\\hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow[{[(-1)\mathbf{1}+\mathbf{2}]}]{\tau}\left[\begin{array}{rrr}-1&0&0\\2&-4&1\\5&-12&3\\\hline 1&-1&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow[{[(4)\mathbf{3}+\mathbf{2}]}]{\tau}\left[\begin{array}{rrr}-1&0&0\\2&0&1\\5&0&3\\\hline 1&-1&0\\0&1&0\\0&4&1\end{array}\right].$$

Thus the set of eigenvectors with eigenvalue 3 can be written as

$$\left\{\boldsymbol{v}\in\mathbb{R}^3\;\middle|\;\boldsymbol{v}=c\begin{pmatrix}-1\\1\\4\end{pmatrix},\forall c\in\mathbb{R}\right\}$$

□

(Grupo E curso 15/16) Exercise 5(a) We want to solve the following 7 equations: $\begin{cases}c-3d=0\\c-2d=0\\c-\;\;d=0\\c\qquad=1\\c+\;\;d=0\\c+2d=0\\c+3d=0\end{cases}.$

□

(Grupo E curso 15/16) Exercise 5(b) First we need to find the projection of $\boldsymbol{y}$ onto the plane generated by two vectors: $(1,1,1,1,1,1,1,)$ and $(-3,-2,-1,0,1,2,3,)$. As $\boldsymbol{y}$ is perpendicular to the second vector, we only need to find the projection of $\boldsymbol{y}$ on the line generated by the first vector, which is $(1/7,1/7,1/7,1/7,1/7,1/7,1/7,)$. Now we need to solve the seven equations: $\begin{cases}c-3d=1/7\\c-2d=1/7\\c-\;\;d=1/7\\c\qquad=1/7\\c+\;\;d=1/7\\c+2d=1/7\\c+3d=1/7\end{cases};$ so $c=1/7$ and $d=0$.

Alternatively, we can denote by A the matrix that has these two vectors as its two columns, then $\mathsf{A}^\intercal\mathsf{A} = \begin{bmatrix} 7 & \\ & 28 \end{bmatrix}$ and $\mathsf{A}^\intercal\boldsymbol{y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The two equations corresponding $\mathsf{A}^\intercal\mathsf{A}\widehat{\boldsymbol{\beta}} = \mathsf{A}^\intercal\boldsymbol{y}$ are $\begin{cases} 7c & = 1 \\ & 28d = 0 \end{cases}$, resulting in the same solution $c = 1/7$ and $d = 0$.

□

(Grupo E curso 15/16) Exercise 5(c) If we used the first method above, we already calculated the projection as $(1/7, 1/7, 1/7, 1/7, 1/7, 1/7, 1/7,)$. If we used the second method, the projection is $\mathsf{A}\boldsymbol{x} = (1/7, 1/7, 1/7, 1/7, 1/7, 1/7, 1/7,)$.

□

(Grupo E curso 15/16) Exercise 6.

$$(\mathsf{A}^2 + \mathsf{A})\boldsymbol{v} = \mathsf{A}^2\boldsymbol{v} + \mathsf{A}\boldsymbol{v} = \lambda^2\boldsymbol{v} + \lambda\boldsymbol{v} = (\lambda^2 + \lambda)\boldsymbol{v}.$$

□

(Grupo E curso 15/16) Exercise 7.

$$\begin{vmatrix} 0 & 0 & 0 & 3 & 0 \\ -2 & 0 & 0 & 2 & 0 \\ 8 & -1 & 0 & -7 & 2 \\ -1 & 2 & 2 & 3 & 2 \\ 2 & 2 & 3 & 6 & 4 \end{vmatrix} = -3 \begin{vmatrix} -2 & 0 & 0 & 0 \\ 8 & -1 & 0 & 2 \\ -1 & 2 & 2 & 2 \\ 2 & 2 & 3 & 4 \end{vmatrix} = (-3)\cdot(-2) \begin{vmatrix} -1 & 0 & 2 \\ 2 & 2 & 2 \\ 2 & 3 & 4 \end{vmatrix} = (-3)\cdot(-2)\cdot(2) = 12.$$

□

(Grupo H curso 15/16) Exercise 1(a) Two (linearly independent) vectors parallel to the plane are $\boldsymbol{v} = \boldsymbol{b} - \boldsymbol{a} = (0, 2, 0,)$ and $\boldsymbol{w} = \boldsymbol{c} - \boldsymbol{a} = (0, 0, 3,)$. Hence, since $\boldsymbol{a}$ is in the plane,

$$\boldsymbol{x} = \boldsymbol{a} + p\,\boldsymbol{v} + q\,\boldsymbol{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + p \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + q \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

is a parametric representation of the plane. There are many others.

□

(Grupo H curso 15/16) Exercise 1(b) Since $(1, 0, 0,)$ is ortogonal to $\boldsymbol{v}$ and $\boldsymbol{w}$, then

$$\left[\begin{array}{ccc} 0 & 2 & 0 \\ 0 & 0 & 3 \\ \hline x & y & z \\ \hline 1 & 1 & 1 \end{array}\right].$$

So, $\boxed{\{x = 1}$ is an implicit (or cartesian) representation of the plane. There are many others.

□

(Grupo H curso 15/16) Exercise 2(a) Since $\mathbf{0} \cdot \boldsymbol{u} = 0$, vector $\mathbf{0}$ is in $\mathcal{V}$, so $\mathcal{V}$ is nonempty. Now we need to verify the two subspace properties.

1. Let $\boldsymbol{x}$, $\boldsymbol{y}$ be vectors in $\mathcal{V}$. Then $\boldsymbol{x} \cdot \boldsymbol{u} = 0$ and $\boldsymbol{y} \cdot \boldsymbol{u} = 0$. So $(\boldsymbol{x} + \boldsymbol{y}) \cdot \boldsymbol{u} = \boldsymbol{x} \cdot \boldsymbol{u} + \boldsymbol{y} \cdot \boldsymbol{u} = 0 + 0 = 0$ which means $\boldsymbol{x} + \boldsymbol{y}$ is in $\mathcal{V}$. So $\mathcal{V}$ is closed under addition.
2. Let $\boldsymbol{x}$ be a vector in $\mathcal{V}$ and c any scalar. Since $\boldsymbol{x} \cdot \boldsymbol{u} = 0$, $(c\,\boldsymbol{x}) \cdot \boldsymbol{u} = c(\boldsymbol{x} \cdot \boldsymbol{u}) = 0$ so $c\,\boldsymbol{x}$ is in $\mathcal{V}$. Thus $\mathcal{V}$ is closed under scalar multiplication.

□

(Grupo H curso 15/16) Exercise 2(b)

$$\left[\begin{array}{c} [\boldsymbol{u}]^\intercal \\ \hline \mathsf{I} \end{array}\right] = \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[]{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ \boldsymbol{\tau} \\ [(-3)\mathbf{1}+\mathbf{3}] \\ \boldsymbol{\tau} \\ [(-4)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ \hline 1 & -2 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

So
$$\left\{ \begin{pmatrix} -2\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} -3\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} -4\\0\\0\\1 \end{pmatrix} \right\}$$
is a basis for $\mathcal{V}$.

□

(Grupo H curso 15/16) Exercise 2(c) The basis for $\mathcal{V}$ found in part (b) had 3 elements, so the dimension of $\mathcal{V}$ is $\boxed{3}$.

Aunque no hubieramos calculado una base, sabemos que la dimensión de $\mathcal{V}$ es 3 ya que $\mathcal{V}$ es el complemento ortogonal del espacio vectorial generado por $\boldsymbol{v}$
$$\mathcal{V} = \mathcal{L}\{\boldsymbol{u}\}^{\perp} \subset \mathbb{R}^4;$$
y puesto que $\dim(\mathcal{L}\{\boldsymbol{u}\}) = 1$, entonces necesariamente $\dim(\mathcal{V}) = 3$.

□

(Grupo H curso 15/16) Exercise 3(a)
$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det \begin{bmatrix} 2-\lambda & 4 & 6\\ 0 & 7-\lambda & 8\\ 0 & 0 & 3-\lambda \end{bmatrix} = (2-\lambda)(7-\lambda)(3-\lambda)$$
so the eigenvalues of $\mathbf{A}$ are 2, 7 and 3.

□

(Grupo H curso 15/16) Exercise 3(b)
$$\left[\begin{array}{c} \mathbf{A}-3\mathbf{I} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{ccc} -1 & 2 & 1\\ 1 & -2 & -1\\ 2 & -4 & -2\\ \hline 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{1}+\mathbf{3}]}]{\substack{\tau \\ [(2)\mathbf{1}+\mathbf{2}] \\ \tau}} \left[\begin{array}{ccc} -1 & 0 & 0\\ 1 & 0 & 0\\ 2 & 0 & 0\\ \hline 1 & 2 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{array}\right].$$
So a basis is
$$\left\{ \begin{pmatrix} 2\\1\\0 \end{pmatrix}; \begin{pmatrix} 1\\0\\1 \end{pmatrix} \right\}.$$
Hence,
$$\mathcal{E}_3 = \mathcal{N}\left(\mathbf{A} - 3\mathbf{I}\right) = \mathcal{L}\left(\left[\begin{pmatrix} 2\\1\\0 \end{pmatrix}; \begin{pmatrix} 1\\0\\1 \end{pmatrix};\right]\right).$$

□

(Grupo H curso 15/16) Exercise 4(a)
$$\mathbf{P}_a = \begin{bmatrix} 2\\1\\2 \end{bmatrix} \left(\begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2\\1\\2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 2\\1\\2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 4 & 2 & 4\\ 2 & 1 & 2\\ 4 & 2 & 4 \end{bmatrix}.$$

□

(Grupo H curso 15/16) Exercise 4(b)
$$(\mathbf{P}_a\mathbf{P}_v)\boldsymbol{v} = \mathbf{P}_a\boldsymbol{v} = \frac{1}{9}\begin{bmatrix} 10\\5\\10 \end{bmatrix} = \frac{5}{9}\boldsymbol{a}$$
and so $\boldsymbol{a} \in \mathcal{C}\left(\mathbf{P}_a\mathbf{P}_v\right) \subset \mathcal{C}\left(\mathbf{P}_a\right)$. Since $\mathcal{C}\left(\mathbf{P}_a\right)$ is spanned by $\boldsymbol{a}$, a basis for $\mathcal{C}\left(\mathbf{P}_a\mathbf{P}_v\right)$ is given by $\{\boldsymbol{a}\}$.

□

(Grupo H curso 15/16) Exercise 5.

The condition says that $\mathbf{A}^2 - 4\mathbf{I}$ is singular. But we know that, if $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n$ are eigenvalues of $\mathbf{A}$, then the eigenvalues of $\mathbf{A}^2 - 4\mathbf{I}$ are $\lambda_1^2 - 4, \ldots, \lambda_n^2 - 4$. The condition $\mathbf{A}^2 - 4\mathbf{I}$ being singular says that one of $(\lambda_i^2 - 4)$ is zero, and hence $\lambda_i = 2$ or $\lambda_i = -2$. That is to say $\mathbf{A}$ has an eigenvalue 2 or -2.

□

(Grupo H curso 15/16) Exercise 6. $q(x,y,z) = x^2 + 6xy + y^2 + az^2 = \begin{pmatrix} x, & y, & z, \end{pmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & a \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & a \end{bmatrix} \xrightarrow[\underset{[(-3)\mathbf{2}+\mathbf{1}]}{\tau}]{\underset{[(-3)\mathbf{2}+\mathbf{1}]}{\tau}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & a \end{bmatrix}$$

La forma cuadrática es *in*definida (sea cual sea el valor de a).

□

(Grupo A curso 14/15) Exercise 1(a) Letting $\mathbf{A} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$, the projection matrix that projects every $\boldsymbol{y} \in \mathbb{R}^4$ onto the column space of $\mathbf{A}$ (which is the line through $\boldsymbol{q}_4$) is given by the formula

$$\mathbf{A}\left(\mathbf{A}^\intercal\mathbf{A}\right)^{-1}\mathbf{A}^\intercal = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}$$
$$= \frac{1}{4}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

□

(Grupo A curso 14/15) Exercise 1(b) Letting $\mathbf{A} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$, the projection matrix that projects every $\boldsymbol{y} \in \mathbb{R}^4$ onto the column space of $\mathbf{A}$ (which is the subspace spanned by $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, and $\boldsymbol{q}_3$) is given by the formula

$$\mathbf{A}\left(\mathbf{A}^\intercal\mathbf{A}\right)^{-1}\mathbf{A}^\intercal = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & 1 & 1 & -1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ -1 & 1 & 1 & 3 \end{bmatrix}.$$

□

(Grupo A curso 14/15) Exercise 1(c) We must solve the new system $\mathbf{A}^\intercal\mathbf{A}\widehat{\boldsymbol{\beta}} = \mathbf{A}^\intercal\boldsymbol{y}$. Since $\mathbf{A}^\intercal\mathbf{A} = \mathbf{I}$, we have $\widehat{\boldsymbol{\beta}} = \mathbf{A}^\intercal\boldsymbol{y} = \begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix}$. Then $\mathbf{A}\widehat{\boldsymbol{\beta}} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$,

□

(Grupo A curso 14/15) Exercise 1(d) $\boldsymbol{e} = \boldsymbol{y} - \mathbf{A}\widehat{\boldsymbol{\beta}} = \mathbf{0}$.

□

(Grupo A curso 14/15) Exercise 2(a) The columns of $\mathbf{B}$ being dependent means by definition that there is a vector $\boldsymbol{x} \neq \mathbf{0}$ such that $\mathbf{B}\boldsymbol{x} = \mathbf{0}$. But then we also have $\mathbf{C}\boldsymbol{x} = (\mathbf{AB})\boldsymbol{x} = \mathbf{A}(\mathbf{B}\boldsymbol{x}) = \mathbf{A}(\mathbf{0}) = \mathbf{0}$; which means that the same $\boldsymbol{x} \neq \mathbf{0}$ works to show that the columns of $\mathbf{C}$ are dependent.

□

(Grupo A curso 14/15) Exercise 2(b) The columns of $\mathbf{B}$ are dependent, since these are five vectors in $\mathbb{R}^3$, and $5 > 3$. Thus, by part (a), the columns of $\mathbf{AB}$ must be dependent. However, columns of $\mathbf{I}$ are independent, so $\mathbf{AB}$ can never equal $\mathbf{I}$. [Note: Switching the order matters here. One can indeed find a

3×5 matrix A, and a 5×3 matrix B such that $\mathsf{AB} = \underset{3\times 3}{\mathsf{I}}$ — hence any "proof" that is insensitive to the order of A and B must be awed].

□

(Grupo A curso 14/15) Exercise 3.

$$\mathsf{A} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \end{bmatrix} \xrightarrow[TypeII]{\substack{\boldsymbol{\tau}\\[(1/2)\mathbf{2}] \\ \boldsymbol{\tau}\\[(1/3)\mathbf{3}] \\ \boldsymbol{\tau}\\[(1/4)\mathbf{4}]}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[TypeI]{\substack{\boldsymbol{\tau}\\[(-1)\mathbf{1}+\mathbf{2}] \\ \boldsymbol{\tau}\\[(-1)\mathbf{3}+\mathbf{4}] \\ \boldsymbol{\tau}\\[(1)\mathbf{4}+\mathbf{3}] \\ \boldsymbol{\tau}\\[(1)\mathbf{4}+\mathbf{1}] \\ \boldsymbol{\tau}\\[(-1)\mathbf{2}+\mathbf{3}]}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[Permutations]{\substack{\boldsymbol{\tau}\\[\mathbf{2}\rightleftharpoons\mathbf{3}] \\ \boldsymbol{\tau}\\[\mathbf{2}\rightleftharpoons\mathbf{4}]}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathsf{B}.$$

Since $\mathsf{AE} = \mathsf{B}$ then $|\mathsf{A}||\mathsf{E}| = |\mathsf{B}|$, with determinant equal to negative one $\Rightarrow |\mathsf{A}| = \frac{-1}{|\mathsf{E}|}$.

From the permutations and Type II elementary operations that we have taken we can see that, $|\mathsf{E}| = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot (-1) \cdot (-1) = \frac{1}{24}$, so $|\mathsf{A}| = -24$.

□

(Grupo A curso 14/15) Exercise 4(a) For a real-symmetric matrix, its eigenvectors are orthogonal to each other. So, by inspection, in order for $\boldsymbol{v}_3$ to be perpendicular to $\boldsymbol{v}_2$, we need its first two components same. Hence, we should take $\boldsymbol{v}_3$ to be $(1, 1, -2,)$.

□

(Grupo A curso 14/15) Exercise 4(b)

$$\begin{aligned} \boldsymbol{q}_1 =& \boldsymbol{v}_1/\|\boldsymbol{v}_1\| = (1, 1, 1,)/\sqrt{3} \\ \boldsymbol{q}_2 =& \boldsymbol{v}_2/\|\boldsymbol{v}_2\| = (1, -1, 0,)/\sqrt{2} \\ \boldsymbol{q}_2 =& \boldsymbol{v}_2/\|\boldsymbol{v}_2\| = (1, 1, -2,)/\sqrt{6} \end{aligned}$$

Hence $\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0/\sqrt{2} & -2/\sqrt{6} \end{bmatrix}.$

□

(Grupo A curso 14/15) Exercise 4(c) $\lambda_1 = 1$, $\lambda_2 = 1$ and $\lambda_3 = 0$. The eigenvectors of A^4 and A are the same.

□

(Grupo A curso 14/15) Exercise 5(a) **False.** If those matrices are similar, they have the same eigenvales, and therefore they have the same trace and determinant. But $\det \mathsf{B} = 5 = -\det \mathsf{A}$ and $\operatorname{tr}\big(\mathsf{A}\big) = 4$ and $\operatorname{tr}\big(\mathsf{B}\big) = 6$.

□

(Grupo A curso 14/15) Exercise 5(b) **False.** If the column space is spanned by $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, then, the matrix has rank one (the second vector is twice the first one). But the row vector $\big(2, \quad 2,\big)$ is not a multiple of $\big(1, \quad 4,\big)$, so the matrix must has rank 2. Hence both conditions are incompatible.

□

(Grupo A curso 14/15) Exercise 6. $\mathsf{A}\boldsymbol{v} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 4 & 2 & 3 & 1 \\ 3 & 1 & 4 & 2 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \\ 10 \\ 10 \end{pmatrix} = 10\boldsymbol{v}.$

□

(Grupo A curso 14/15) Exercise 7.

$$\left[\begin{array}{ccc} 3 & 1 & 1 \\ 1 & 1 & 0 \\ -4 & 4 & -4 \\ 6 & 4 & a \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\[\mathbf{1}\rightleftharpoons\mathbf{2}] \\ \boldsymbol{\tau}\\[(-3)\mathbf{1}+\mathbf{2}] \\ \boldsymbol{\tau}\\[(-1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & -2 & -1 \\ 4 & -16 & -8 \\ 4 & -6 & a-4 \\ \hline 0 & 1 & 0 \\ 1 & -3 & -1 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\[(-1/2)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 4 & -16 & 0 \\ 4 & -6 & a-1 \\ \hline 0 & 1 & -1/2 \\ 1 & -3 & 1/2 \\ 0 & 0 & 1 \end{array}\right]$$

They are linearly dependent when $a = 1$ (rank 2).

$\square$

(Grupo A curso 14/15) Exercise 8.

Puesto que el determinante es negativo independientemente del valor de b:

$$\begin{vmatrix} 2 & b & 3 \\ b & 2 & b \\ 3 & b & 4 \end{vmatrix} = 16 + 6b^2 - 18 - 6b^2 = -2 \quad < 0;$$

esta matriz nunca puede tener sus tres autovalores positivos.

$\square$

(Grupo C curso 14/15) Exercise 1(a)

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & -2\alpha \\ 4 & 2 & 2 & -3\alpha \\ 6 & 2 & 3 & -2\beta \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[{\substack{\boldsymbol{\tau} \\ [\mathbf{1}\rightleftharpoons\mathbf{3}] \\ \boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{2}] \\ \boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{3}]}}]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2\alpha \\ 2 & 0 & 0 & -3\alpha \\ 3 & -1 & 0 & -2\beta \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right] \xrightarrow[{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}] \\ \boldsymbol{\tau} \\ [(-3)\mathbf{2}+\mathbf{1}]}}]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2\alpha \\ 2 & 0 & 0 & -3\alpha \\ 0 & 1 & 0 & -2\beta \\ \hline 0 & 0 & 1 & 0 \\ 3 & -1 & 0 & 0 \\ -2 & 1 & -2 & 0 \end{array}\right] \xrightarrow[{\substack{\boldsymbol{\tau} \\ [(2\alpha)\mathbf{1}+\mathbf{4}] \\ \boldsymbol{\tau} \\ [(2\beta)\mathbf{2}+\mathbf{4}]}}]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 3 & -1 & 0 & 6\alpha - 2\beta \\ -2 & 1 & -2 & -4\alpha + 2\beta \end{array}\right]$$

The system is solvable if $\alpha = 0$.

$\square$

(Grupo C curso 14/15) Exercise 1(b)

$$\vec{x} = \begin{pmatrix} 0 \\ -2\beta \\ 2\beta \end{pmatrix} + a \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad \text{for all } a \in \mathbb{R}.$$

$\square$

(Grupo C curso 14/15) Exercise 2.

The column space of $\mathbf{A}$ is contained in $\mathbb{R}^m$, and the column space of $\mathbf{B}$ is contained in $\mathbb{R}^M$. If $\mathcal{C}\left(\mathbf{A}\right) \subseteq \mathcal{C}\left(\mathbf{B}\right)$, this means they are contained in the same Euclidean space, so $M = m$. The dimension of the column space is the rank of the matrix, so if $\mathcal{C}\left(\mathbf{A}\right) \subseteq \mathcal{C}\left(\mathbf{B}\right)$, this means $\dim \mathcal{C}\left(\mathbf{A}\right) \leq \dim \mathcal{C}\left(\mathbf{B}\right)$, hence $r \leq R$. There are no relations between N and n; for example $n = N$ if $\mathbf{A} = \mathbf{B}$, $N \leq n$ if $\mathbf{B} = [\mathbf{A}|\mathbf{A}]$, and $n \leq N$ if $\mathbf{A} = [\mathbf{B}|\mathbf{B}]$.

$\square$

(Grupo C curso 14/15) Exercise 3(a)

For example $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

$\square$

(Grupo C curso 14/15) Exercise 3(b)

Since $\mathbf{A}$ has a characteristic polynomial of degree 5, we know that $\mathbf{A}$ is a 5×5 matrix. Since 0 is not a root of $p(\cdot)$ and so is not an eigenvalue, we know $\mathbf{A}$ is invertible so $rank(A) = 5$.

$\square$

(Grupo C curso 14/15) Exercise 3(c)

Following the Hint, we get $-3\boldsymbol{u} - 2\boldsymbol{v} + \boldsymbol{x} = \mathbf{0} \;\Rightarrow\; \boldsymbol{x} = 3\boldsymbol{u} + 2\boldsymbol{v}$. Hence

$$\mathbf{A}\boldsymbol{x} = 3(\mathbf{A}\boldsymbol{u}) + 2(\mathbf{A}\boldsymbol{v}) = 3(-\boldsymbol{u}) + 2(3\boldsymbol{v}) = 3\begin{pmatrix} -2 \\ 1 \\ -4 \\ 0 \\ -3 \end{pmatrix} + 2\begin{pmatrix} 9 \\ 3 \\ -6 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 12 \\ 9 \\ -24 \\ 6 \\ 3 \end{pmatrix}$$

where we have used the given fact that $\boldsymbol{u}$, $\boldsymbol{v}$ are eigenvectors to get $\mathbf{A}\boldsymbol{u} = -\boldsymbol{u}$ and $\mathbf{A}\boldsymbol{v} = 3\boldsymbol{v}$.

$\square$

(Grupo C curso 14/15) Exercise 4(a)

The answer is $\det \mathbf{A} = -x^2 - y^2 - z^2$. But before we discuss how to get this answer, I'd like to call your attention to that fact that the expression $-x^2 - y^2 - z^2$ is symmetric in the three variables x; y; z. That is to say, if we swap the roles of any two of these variables, the expression as a whole is unchanged. Why might we have predicted that $\det \mathbf{A}$ has this property? Well, if we swap rows 2 and 3 of $\mathbf{A}$, and then swap columns 2 and 3 of the result, we end up with

$$\mathbf{A}' = \begin{bmatrix} 0 & y & x & z \\ y & 1 & 0 & 0 \\ x & 0 & 1 & 0 \\ z & 0 & 0 & 1 \end{bmatrix},$$

which is the same as $\mathbf{A}$, but with the roles of x and y swapped. In performing one row swap and one column swap, we have multiplied the determinant by $(-1)^2 = 1$, so $\mathbf{A}'$ has the same determinant as $\mathbf{A}$. From this we conclude that $\det \mathbf{A}$, whatever it is, must be an expression that's symmetric in x and y. Similar considerations show that it's symmetric in all three variables x, y, z. Anyway, let's actually compute $\det \mathbf{A}$.

By Type I elementary operations we reach the echelon form. From column 1 of $\mathbf{A}$, we subtract x times column 2, y times column 3, and z times column 4. These operations do not change the determinant, so

$$|\mathbf{A}| = \begin{vmatrix} -x^2 - y^2 - z^2 & y & x & y \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -x^2 - y^2 - z^2.$$

□

(Grupo C curso 14/15) Exercise 4(b)

A square matrix is singular if and only if its determinant equals zero. So we are asked to find all triples $(x, y, z,)$ such that

$$\det \mathbf{A} = -x^2 - y^2 - z^2 = 0,$$

or in other words

$$x^2 + y^2 + z^2 = 0.$$

So far, we have been talking about real numbers x, y, z in this course, so the left-hand side is just the square of the distance from $(x, y, z,)$ to the origin in $\mathbb{R}^3$. Since only the origin is at a distance 0 from the origin, the matrix $\mathbf{A}$ is singular if and only if $x = y = z = 0$.

□

(Grupo C curso 14/15) Exercise 5.

$\left|-2\right| < 0$; $\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} > 0$; $\begin{vmatrix} -2 & 1 & a \\ 1 & -2 & 1 \\ a & 1 & -2 \end{vmatrix} = -2a^2+2a+4$; parabolic function that cross the x axis in 1 and -2.

- If $-2 < a < 1$ Negative definite
- If $a = -1$ or $a = 2$ negative semi-defined
- Not definite in other cases.

□

(Grupo C curso 14/15) Exercise 6(a)

The eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

are the solutions to the characteristic equation

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = (3-\lambda)(-3-\lambda) - 16 = 0; \quad \Rightarrow \quad \lambda_1 = 5;\ \lambda_2 = -5.$$

We can also use

$$\begin{cases} \lambda_1 \cdot \lambda_2 = & \det \mathbf{A} = 25 \\ \lambda_1 + \lambda_2 = & \operatorname{tr} \mathbf{A} = 0 \end{cases}$$

with the same result.

□

(Grupo C curso 14/15) Exercise 6(b)

- for $\lambda_1 = 5$, the null space of
$$(\mathbf{A} - \lambda_1 \mathbf{I}) = \begin{bmatrix} 3-5 & 4 \\ 4 & -3-5 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix}$$
consists of all multiples of the eigenvector $\boldsymbol{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

- for $\lambda_2 = -5$, the null space of
$$(\mathbf{A} - \lambda_2 \mathbf{I}) = \begin{bmatrix} 3+5 & 4 \\ 4 & -3+5 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}$$
consists of all multiples of the eigenvector $\boldsymbol{x}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

□

(Grupo C curso 14/15) Exercise 7.

If $\boldsymbol{v}$ is an eigenvector with corresponding eigenvalue λ, then we have that $\mathbf{A}^2\boldsymbol{v} = \mathbf{A}(\mathbf{A}\boldsymbol{v}) = \lambda\mathbf{A}\boldsymbol{v} = \lambda^2\boldsymbol{v}$. Similarly $\mathbf{A}^3 = \lambda^3\boldsymbol{v}$. Thus λ must satisfy $\lambda^3 = 2\lambda^2 - \lambda$, which means that is either 0 or 1.

□

(Grupo C curso 14/15) Exercise 8(a)

$\lambda_1 = 2$ and $\lambda_2 = 5$

$$\mathcal{N}\left(\mathbf{A} - 2\mathbf{I}\right) = \mathcal{N}\left(\begin{bmatrix} 0 & 3 \\ 0 & 3 \end{bmatrix}\right) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} \to \boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\mathcal{N}\left(\mathbf{A} - 5\mathbf{I}\right) = \mathcal{N}\left(\begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix}\right) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} \to \boldsymbol{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

□

(Grupo C curso 14/15) Exercise 8(b)

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbf{S}^{-1}\boldsymbol{u}_0 = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}^{-1} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ & 1 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a-b \\ b \end{pmatrix}$$

So $\boldsymbol{u}_0 = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 = (a-b)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

□

(Grupo E curso 14/15) Exercise 1(a)

$$\begin{cases} c + 0\cdot d &= 1 \\ c + 1\cdot d &= 2 \\ c + 2\cdot d &= -1 \end{cases} \qquad \Rightarrow \qquad \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

□

(Grupo E curso 14/15) Exercise 1(b)

The normal equations $\mathbf{A}^{\intercal}\mathbf{A}\widehat{\boldsymbol{\beta}} = \mathbf{A}^{\intercal}\boldsymbol{y}$ are

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} \widehat{c} \\ \widehat{d} \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

or

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{pmatrix} \widehat{c} \\ \widehat{d} \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Cuya solución es

$$\left[\begin{array}{cc|c} 3 & 3 & -2 \\ 3 & 5 & -0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow[{[(1/3)\mathbf{1}]}]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{2}] \\ \boldsymbol{\tau}}} \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 1 & 2 & 0 \\ \hline 1/3 & -1 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow[{[(-1)\mathbf{2}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ [(2)\mathbf{1}+\mathbf{3}] \\ \boldsymbol{\tau}}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 2 & 0 \\ \hline 1/3 & -1 & 5/3 \\ 0 & 1 & -1 \end{array}\right] \quad \Rightarrow \quad \begin{pmatrix} \widehat{c} \\ \widehat{d} \end{pmatrix} = \begin{pmatrix} 5/3 \\ -1 \end{pmatrix}.$$

$\square$

(Grupo E curso 14/15) Exercise 1(c)

$\boldsymbol{y} \notin \mathcal{C}\left(\mathbf{A}\right)$ so $y = \frac{5}{3} - x$ is the best fit, and

$$\boldsymbol{p} = \mathbf{A}\widehat{\boldsymbol{\beta}} = \frac{5}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}0\\1\\2\end{pmatrix} = \frac{1}{3}\begin{pmatrix}5\\2\\-1\end{pmatrix}$$

is the closest point in $\mathcal{C}\left(\mathbf{A}\right)$ to $\boldsymbol{y}$.

$\square$

(Grupo E curso 14/15) Exercise 1(d)

$$\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{p} = \begin{pmatrix}1\\2\\-1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}5\\2\\-1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}-2\\4\\-2\end{pmatrix}$$

so

$$\|\boldsymbol{e}\|^2 = \boldsymbol{e}\cdot\boldsymbol{e} = \frac{1}{9}\left(-2, \quad 4, \quad -2,\right)\begin{pmatrix}-2\\4\\-2\end{pmatrix} = \frac{24}{9}; \quad \Rightarrow \quad \|\boldsymbol{e}\| = \frac{\sqrt{24}}{3}.$$

$\square$

(Grupo E curso 14/15) Exercise 2(a)

Trace of $\mathbf{A}$ is $\lambda_1 + \lambda_2 + \lambda_3$; determinant of $\mathbf{A}$ is $\lambda_1 \cdot \lambda_2 \cdot \lambda_3$.

$\square$

(Grupo E curso 14/15) Exercise 2(b)

$\mathbf{A}$ can be diagonalized, since the eigenvectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ are linearly independent.

$\square$

(Grupo E curso 14/15) Exercise 2(c)

We can recover $\mathbf{A}$ using $\mathbf{SDS}^{-1}$ where $\mathbf{S}$ is a matrix whose columns are $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$,

$$\mathbf{S} = \left[\boldsymbol{x}_1;\ \boldsymbol{x}_2;\ \boldsymbol{x}_3;\right]$$

and $\mathbf{D}$ is a diagonal matrix whose diagonal entries are λ_1, λ_2, λ_3

$$\mathbf{D} = \begin{bmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3\end{bmatrix}.$$

$\square$

(Grupo E curso 14/15) Exercise 2(d)

If we want $\mathbf{A}$ to be symmetric, the third eigenvector $\boldsymbol{x}_3$ had better be orthogonal to the other two.

$$\left[\begin{array}{rrr}1 & 1 & 1\\ 1 & -2 & 1\\ \hline 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right] \xrightarrow{\substack{\tau_{[(-1)\mathbf{1}+\mathbf{2}]}\\ \tau_{[(-1)\mathbf{1}+\mathbf{3}]}}} \left[\begin{array}{rrr}1 & 0 & 0\\ 1 & -3 & 0\\ \hline 1 & -1 & \mathbf{-1}\\ 0 & 1 & \mathbf{0}\\ 0 & 0 & \mathbf{1}\end{array}\right]$$

You could also just notice the first and last entries match and guess the answer from that. Either way, $\boldsymbol{x}_3$ should be a multiple of $\begin{bmatrix}1 & 0 & -1\end{bmatrix}$. As for the eigenvalue, to get a matrix that's positive semidefinite but not positive definite, we need to use $\lambda_3 = 0$.

It doesn't actually ask you to compute $\mathbf{A}$, but here's one that works:

$$\mathbf{A} = \begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 0\\ 1 & 1 & -1\end{bmatrix}\begin{bmatrix}2 & & \\ & 5 & \\ & & 0\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 0\\ 1 & 1 & -1\end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix}3 & -2 & 3\\ -2 & 8 & -2\\ 3 & -2 & 3\end{bmatrix}.$$

$\square$

(Grupo E curso 14/15) Exercise 3(a)

Because $\mathbf{A}$ is not symmetric.

□

(Grupo E curso 14/15) Exercise 3(b)
When $a > 0$ and $\begin{vmatrix} a & 1 \\ 1 & b \end{vmatrix} = ab - 1 > 0$; that is, when $a > 0$ and $b > \frac{1}{a}$.

□

(Grupo E curso 14/15) Exercise 4(a)
Any value except zero.

□

(Grupo E curso 14/15) Exercise 4(b)
Since the eigenvalues of $\mathbf{A}^2$ are the square of the eigenvalues of $\mathbf{A}$, then the only posible eigenvalues are zero or one. Threfore the determinant is either zero or one.

□

(Grupo E curso 14/15) Exercise 4(c)
Determinant equal to 6.

□

(Grupo E curso 14/15) Exercise 5(a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

□

(Grupo E curso 14/15) Exercise 5(b)
Since the determiant is not zero, the matrix has full rank, that is, the rank is 3.

□

(Grupo E curso 14/15) Exercise 6(a)

$$\begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 9 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 2 \cdot 2 = 4.$$

El elemento (1,1) de la inversa de $\mathbf{A}$ es el primer elemento de la matriz adjunta **Adj(A)** dividido por el determinante.

$$\frac{\operatorname{cof}(\mathbf{A})_{11}}{\det \mathbf{A}} = \frac{2}{4} = \frac{1}{2}.$$

□

(Grupo E curso 14/15) Exercise 6(b)

$$\begin{vmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 9 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 4 - 2 = 2.$$

□

(Grupo E curso 14/15) Exercise 6(c)

$$\det \mathbf{A} = \begin{vmatrix} 2 & 1-x & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 9 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 9 \end{vmatrix} + \begin{vmatrix} 2 & -x & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 2 & 4 \\ 1 & 0 & 3 & 9 \end{vmatrix} = 2 + x \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 2 + 2x.$$

Por tanto, $\det \mathbf{A} = 2x + 2$; y cuando $x = -1$ la matriz es singular ($\det \mathbf{A} = 0$).

□

(Grupo H curso 14/15) Exercise 1(a) The rank of $\mathbf{P}$ is 2. Any vector perpendicular to the subspace spanned by $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ is in the nullspace of $\mathbf{P}$, and the orthogonal complement of the subspace spanned by $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ is 3-dimensional (that is, there are three independent vectors that project to $\mathbf{0}$ by $\mathbf{P}$). This is exactly the nullspace of $\mathbf{P}$, and since $\operatorname{rg}(\mathbf{P}) = \dim \mathcal{C}(\mathbf{P}) = 5 - \dim \mathcal{N}(\mathbf{P})$, the rank of $\mathbf{P}$ is $5 - 3 = 2$.

□

(Grupo H curso 14/15) Exercise 1(b) The nullspace of $\mathbf{P}$ is the left nullspace of $\mathbf{A}$. Indeed, we have

$$\begin{aligned}\mathbf{P}\boldsymbol{v} = \mathbf{0} &\Leftrightarrow \boldsymbol{a}_1 \cdot \boldsymbol{v} = 0 \text{ and } \boldsymbol{a}_2 \cdot \boldsymbol{v} = 0\\ &\Leftrightarrow \boldsymbol{v} \cdot \boldsymbol{a}_1 = 0 \text{ and } \boldsymbol{v} \cdot \boldsymbol{a}_2 = 0\\ &\Leftrightarrow \boldsymbol{v}\mathbf{A} = \mathbf{0}.\end{aligned}$$

□

(Grupo H curso 14/15) Exercise 1(c) Since $\mathbf{P}$ is a projection matrix, we have $\mathbf{P} = \mathbf{P}^\intercal$. To show that $\mathbf{Q}$ is an orthogonal matrix, we need to check that $\mathbf{Q}\mathbf{Q}^\intercal = \mathbf{I}$. We have

$$\begin{aligned}\mathbf{Q}\mathbf{Q}^\intercal &= (\mathbf{I} - 2\mathbf{P})(\mathbf{I} - 2\mathbf{P})^\intercal\\ &= (\mathbf{I} - 2\mathbf{P})(\mathbf{I}^\intercal - 2\mathbf{P}^\intercal)\\ &= (\mathbf{I} - 2\mathbf{P})(\mathbf{I} - 2\mathbf{P}) && \mathbf{I} \text{ and } \mathbf{P} \text{ are symmetric}\\ &= \mathbf{I}(\mathbf{I} - 2\mathbf{P}) - 2\mathbf{P}(\mathbf{I} - 2\mathbf{P})\\ &= \mathbf{I} - 2\mathbf{P} - 2\mathbf{P} + 4\mathbf{P}^2\\ &= \mathbf{I} && \text{Since for a projection matrix we have } \mathbf{P}^2 = \mathbf{P}\end{aligned}$$

□

(Grupo H curso 14/15) Exercise 2. Any row in the coeficient matrix is Ok. But the easiest way to do this is to realize that $\boldsymbol{v} = \mathbf{0} = (0,0,0,0)^\intercal$ is orthogonal to every vector in $\mathbb{R}^4$, including all solutions.

□

(Grupo H curso 14/15) Exercise 3(a) $\mathbf{A} = \begin{bmatrix}0&1&1\\1&0&0\\4&0&1\end{bmatrix} \xrightarrow{\overset{\boldsymbol{\tau}}{[\mathbf{1}\rightleftharpoons\mathbf{2}]}} \begin{bmatrix}1&0&1\\0&1&0\\0&4&1\end{bmatrix} \xrightarrow{\overset{\boldsymbol{\tau}}{[(-1)\mathbf{1}+\mathbf{3}]}} \begin{bmatrix}1&0&0\\0&1&0\\0&4&1\end{bmatrix} \xrightarrow{\overset{\boldsymbol{\tau}}{[(-4)\mathbf{3}+\mathbf{2}]}}$
$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \mathbf{I}.$

□

(Grupo H curso 14/15) Exercise 3(b) Swapping columns 1 and 2 corresponds to

$$\mathbf{I}_{\underset{[\mathbf{1}\rightleftharpoons\mathbf{2}]}{\boldsymbol{\tau}}} = \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}.$$

Subtracting column 1 from column 3 corresponds to

$$\mathbf{I}_{\underset{[(-1)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}} := \begin{bmatrix}1&0&-1\\0&1&0\\0&0&1\end{bmatrix}.$$

Subtracting 4 times column 3 from column 2 corresponds to

$$\mathbf{I}_{\underset{[(-4)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}} = \begin{bmatrix}1&0&0\\0&1&0\\0&-4&1\end{bmatrix}.$$

Putting them together, we get

$$\mathbf{A}\cdot\left(\mathbf{I}_{\underset{[\mathbf{1}\rightleftharpoons\mathbf{2}]}{\boldsymbol{\tau}}}\right)\cdot\left(\mathbf{I}_{\underset{[(-1)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}}\right)\cdot\left(\mathbf{I}_{\underset{[(-4)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}}\right) = \mathbf{A}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-4&1\end{bmatrix} = \mathbf{I}.$$

Hence, $\mathbf{A}^{-1} = \left(\mathbf{I}_{\underset{[\mathbf{1}\rightleftharpoons\mathbf{2}]}{\boldsymbol{\tau}}}\right)\cdot\left(\mathbf{I}_{\underset{[(-1)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}}\right)\cdot\left(\mathbf{I}_{\underset{[(-4)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}}\right) = \begin{bmatrix}0&1&0\\1&4&-1\\0&-4&1\end{bmatrix}.$

□

(Grupo H curso 14/15) Exercise 4. $\det \mathbf{A}^\intercal\mathbf{A} = \det \underset{3\times 3}{\mathbf{I}} = 1$

□

(Grupo H curso 14/15) Exercise 5(a) Use

$$\det(\mathsf{A}) = \det(\mathsf{P}) \cdot \det(\mathsf{L}) \cdot \det(\mathsf{U}),$$

where we make two uses of the rule $\det(\mathsf{MN}) = \det(\mathsf{M})\det(\mathsf{N})$, for any two $n \times n$ matrices M and N. We will compute each of the determinants on the right-hand side. The determinant of a triangular matrix is the product of its diagonal entries; this is true whether the matrix is upper or lower triangular. Thus $\det(\mathsf{U}) = 1$ and $\det(\mathsf{L}) = d_1 \cdot d_2 \cdot \ldots \cdot d_n$. The determinant changes sign whenever two columns or rows are swapped. Thus

$$\det(\mathsf{P}) = \begin{cases} +1 \text{ if } \mathsf{P} \text{ is even (even number of colunm exchanges)} \\ -1 \text{ if } \mathsf{P} \text{ is odd (odd number of colunm exchanges)}; \end{cases}$$

and so

$$\det(\mathsf{A}) = \pm d_1 \cdot d_2 \cdot \ldots \cdot d_n$$

where the sign depends on the parity of P.

□

(Grupo H curso 14/15) Exercise 6(a) $\mathsf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ with characteristic polynomial λ^2 (so the only eigenvalue is $\lambda = 0$) and noted that all eigenvectors are the multiples of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

□

(Grupo H curso 14/15) Exercise 6(b) $\|\boldsymbol{v}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v} = 4 + 1 + 0 + 16 + 4 = 25$ so we take $\boldsymbol{u} = \boldsymbol{v}/\|\boldsymbol{v}\| = (2/5, -1/5, 0.4/5, -2/5)$.

□

(Grupo H curso 14/15) Exercise 7. The eigenvectors of A^{-1} are the same as those of A. Its eigenvalues are the inverses of those of A: 1, 3, and 2.

□

(Grupo H curso 14/15) Exercise 8(a) since it is a triangular matrix, the numbers on the main diagonal are the eigenvalues (Note that this is only true when the matrix is triangular!)

$$\lambda_1 = 1; \quad \lambda_2 = 3.$$

□

(Grupo H curso 14/15) Exercise 8(b) First we need to find an eigenvector for each eigenvalue (there are no repeated ones).

For $\lambda_1 = 1$ $(\mathsf{A} - \mathsf{I})\boldsymbol{x} = \boldsymbol{0}$.

$$(\mathsf{A} - \mathsf{I})\boldsymbol{x} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \boldsymbol{x} = \boldsymbol{0}.$$

Hence $\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

For $\lambda_2 = 3$ $(\mathsf{A} - 3\mathsf{I})\boldsymbol{x} = \boldsymbol{0}$.

$$(\mathsf{A} - \mathsf{I})\boldsymbol{x} = \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x} = \boldsymbol{0}.$$

Hence $\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

So, $\mathsf{D} = \begin{bmatrix} 1 & \\ & 3 \end{bmatrix}$; $\mathsf{S} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

□

(Grupo H curso 14/15) Exercise 8(c) $\mathsf{A}^5 = (\mathsf{SDS}^{-1})^5 = \mathsf{SD}^5\mathsf{S}^{-1}$ First we need S^{-1}

$$\left[\begin{array}{c} \mathsf{S} \\ \hline \mathsf{I} \end{array}\right] = \left[\begin{array}{cc} 1 & 1 \\ 0 & 2 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 2 \\ \hline 1 & -1 \\ 0 & 1 \end{array}\right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 1 & -1/2 \\ 0 & 1/2 \end{array}\right] = \left[\begin{array}{c} \mathsf{I} \\ \hline \mathsf{S}^{-1} \end{array}\right].$$

And now $\mathbf{D}^5$

$$\mathbf{D}^5 = \begin{bmatrix} 1 & \\ & 3 \end{bmatrix}^5 = \begin{bmatrix} 1^5 & \\ & 3^5 \end{bmatrix} = \begin{bmatrix} 1 & \\ & 243 \end{bmatrix}.$$

Hence,

$$\mathbf{A}^5 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & \\ & 243 \end{bmatrix} \begin{bmatrix} 1 & -0.5 \\ 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 1 & 121 \\ 0 & 243 \end{bmatrix}.$$

□

(Grupo E curso 13/14) Exercise 1(a) *When an odd permutation matrix $\mathbf{P}_1$ multiplies an even permutation matrix $\mathbf{P}_2$, the product $\mathbf{P}_1\mathbf{P}_2$ is **odd**.*

$\mathbf{P}_1$ applies an odd number of column exchanges to $\mathbf{I}$ and $\mathbf{P}_2$ applies an even number of column exchanges to $\mathbf{I}$. Hence the permutation matrix $\mathbf{P}_1\mathbf{P}_2$ applies an (even+odd)= odd number of column exchanges.

□

(Grupo E curso 13/14) Exercise 1(b) $\mathbf{AB}$ *is **the zero matrix**.*

Let $\mathbf{B} = \begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_n \end{bmatrix}$, where $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ are the columns of $\mathbf{B}$. Since each $\boldsymbol{b}_i$ is in $\mathcal{N}(\mathbf{A})$ we have $\mathbf{A}\boldsymbol{b}_i = \mathbf{0}$. Then $\mathbf{AB} = \begin{bmatrix} \mathbf{A}\boldsymbol{b}_1 & \cdots & \mathbf{A}\boldsymbol{b}_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} = \mathbf{0}$.

□

(Grupo E curso 13/14) Exercise 1(c)

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & c \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-3)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ \boldsymbol{\tau}}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 2 & 6 \\ 1 & 6 & c-3 \\ \hline 1 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-3)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 6 & c-21 \\ \hline 1 & -2 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{array}\right] = \left[\frac{\mathbf{L}}{\mathbf{E}}\right]$$

When $c = 0$, $\quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 6 & -21 \end{bmatrix}, \qquad \dot{\mathbf{U}} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}; \qquad \mathbf{A} = \mathbf{L}\dot{\mathbf{U}}.$

□

(Grupo E curso 13/14) Exercise 1(d) *That matrix $\mathbf{A}$ is invertible unless $c = 21$.*

□

(Grupo E curso 13/14) Exercise 2.

$$\mathbf{A} - 3\mathbf{I} = \begin{bmatrix} 0 & 0 & 1 \\ 6 & 2 & 7 \\ -3 & -1 & -5 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-3)\mathbf{2}+\mathbf{1}]}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 7 \\ 0 & -1 & -5 \end{bmatrix}$$

So $(\mathbf{A} - 3\mathbf{I})$ is a singular matrix, and therefore 3 is an eigenvalue of $\mathbf{A}$. The vectors in $\mathcal{N}(\mathbf{A} - 3\mathbf{I})$ are the corresponding eigenvectors:

$$\left[\frac{\mathbf{A} - 3\mathbf{I}}{\mathbf{I}}\right] = \left[\begin{array}{ccc} 0 & 0 & 1 \\ 6 & 2 & 7 \\ -3 & -1 & -5 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-3)\mathbf{2}+\mathbf{1}]}} \left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 2 & 7 \\ 0 & -1 & -5 \\ \hline \mathbf{1} & 0 & 0 \\ \mathbf{-3} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{array}\right],$$

Hence, $\begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$ (or any multiple of this vector) is a corresponding eigenvector.

□

(Grupo E curso 13/14) Exercise 3(a) By Type I elementary *column* operations we get

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 3 \\ 4 & N & 1 \\ 3 & 1 & 4 \end{bmatrix} \xrightarrow[{[(-3/2)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ \boldsymbol{\tau}}} \begin{bmatrix} 2 & 0 & 0 \\ 4 & N-8 & -5 \\ 3 & -5 & -1/2 \end{bmatrix}$$

And now, by Type I elementary *row* operations we get

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & N-8 & -5 \\ 3 & -5 & -1/2 \end{bmatrix} \xrightarrow{\underset{[(-10)\mathbf{2}+\mathbf{3}]}{\tau}} \begin{bmatrix} 2 & 0 & 0 \\ -26 & N+42 & 0 \\ 3 & -5 & -1/2 \end{bmatrix}$$

Since we have used only Type I elementary *row* operations, and since we can find positive and negative pivots, this matrix is not definite.

We can also check whether the upper-left determinants are positive:

- 1×1: This is 2, which is always greater than 0.
- 2×2: This is $2N - 16$ which is greater than 0 if N is really large (in particular if $N > 8$).
- 3×3: Use the method of your choice to compute the determinant of $\mathbf{A}$, in terms of N.

$$\det \mathbf{A} = -N - 42.$$

This is going to be very negative if N is really large. So the matrix will not be positive definite.

□

(Grupo E curso 13/14) Exercise 3(b)

1. $\mathbf{B}^\intercal = (\mathbf{Q}^\intercal\mathbf{A}\mathbf{Q})^\intercal = \mathbf{Q}^\intercal\mathbf{A}^\intercal(\mathbf{Q}^\intercal)^\intercal = \mathbf{Q}^\intercal\mathbf{A}\mathbf{Q} = \mathbf{B}$, since $\mathbf{A}$ is symmetric.
2. Since $\mathbf{A}$ is positive definite, $\boldsymbol{x}\mathbf{B}\boldsymbol{x} = \boldsymbol{x}\mathbf{Q}^\intercal\mathbf{A}\mathbf{Q}\boldsymbol{x} = \boldsymbol{y}\mathbf{A}\boldsymbol{y} > 0$; where $\mathbf{Q}\boldsymbol{x}$ is a vector $\boldsymbol{y}$.

□

(Grupo E curso 13/14) Exercise 4(a)

$$\mathbf{P} = \mathbf{A}(\mathbf{A}^\intercal\mathbf{A})^{-1}\mathbf{A}^\intercal = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

□

(Grupo E curso 13/14) Exercise 4(b) $\widehat{\boldsymbol{x}} = (\mathbf{A}^\intercal\mathbf{A})^{-1}\mathbf{A}^\intercal\boldsymbol{b} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

□

(Grupo E curso 13/14) Exercise 4(c) $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \boldsymbol{b} - \mathbf{A}\widehat{\boldsymbol{x}} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 2/3 \\ -2/3 \end{pmatrix}.$

□

(Grupo E curso 13/14) Exercise 5(a)

$$\left[\begin{array}{c} \mathbf{A} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{ccccc} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 \\ 1 & 2 & 0 & -1 & 6 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\underset{[(-3)\mathbf{1}+\mathbf{5}]}{\tau}]{\underset{[(-2)\mathbf{1}+\mathbf{2}]}{\tau}} \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 \\ 1 & 0 & 0 & -1 & 3 \\ \hline 1 & -2 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\underset{[(3)\mathbf{4}+\mathbf{5}]}{\tau}]{\underset{[(-4)\mathbf{3}+\mathbf{5}]}{\tau}} \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ \hline 1 & -2 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c} \mathbf{L} \\ \hline \mathbf{E} \end{array}\right]$$

Hence, a basis for $\mathcal{C}(\mathbf{A})$ is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\}$

□

(Grupo E curso 13/14) Exercise 5(b) $\left\{\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix};\begin{pmatrix}-3\\0\\-4\\3\\1\end{pmatrix}\right\}$

□

(Grupo E curso 13/14) Exercise 6(a) **True.** Matrices $\mathbf{A}^n$ for $n \in \mathbb{N}$, share the same eigenvectors, since

$$\mathbf{A}\boldsymbol{v} = \lambda\boldsymbol{v} \quad \Rightarrow \quad \mathbf{A}\mathbf{A}\boldsymbol{v} = \lambda\mathbf{A}\boldsymbol{v} = \lambda^2\boldsymbol{v}.$$

□

(Grupo E curso 13/14) Exercise 6(b) **False.** Any vector in $\mathbb{R}^2$ is an eigenvector of the 2 by 2 identity matrix $\mathbf{I}$. For example

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix};$$

but that vector is not an eigenvector of the permutation matrix $\mathbf{P}$

$$\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix};$$

although it is an eigenvector of $\mathbf{P}^2 = \mathbf{I}$. Another example is $\mathbf{A} = \begin{bmatrix}0&1\\0&0\end{bmatrix}$, whose eigenvectors are the multiples of $\begin{pmatrix}0\\1\end{pmatrix}$, nevertheless, note that $\mathbf{A}^2 = \mathbf{0}$ so any non-zero vector in $\mathbb{R}^2$ is an eigenvector of $\mathbf{A}^2$ with eigenvalue $\lambda = 0$.

□

(Grupo E curso 13/14) Exercise 6(c) **False.** If $\boldsymbol{x}$ is an egenvector of $\mathbf{A}$, then $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$, and $\mathbf{A}^2\boldsymbol{x} = \lambda^2\boldsymbol{x}$, so in this case $\mathbf{A}\boldsymbol{x}$ and $\mathbf{A}^2\boldsymbol{x}$ are linearly dependent vectors (not a basis).

□

(Grupo E curso 13/14) Exercise 7(a) We are looking for matrices $\mathbf{A} = \begin{bmatrix}a&b\\c&b\end{bmatrix}$ such that $\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix}a-\lambda & b\\ c & d-\lambda\end{vmatrix} = \lambda^2 - (a+d)\lambda + (ad-bc) = (\lambda-4)(\lambda-6) = \lambda^2 - 10\lambda + 24.$ It is required that $\operatorname{tr}\left(\mathbf{A}\right) = a + d = 10$ and $\det\mathbf{A} = ad - bc = 24$ (compare coeficients!). One possible solution is $a = d = 5$, $b = c = 1$. Thus $\mathbf{A} = \begin{bmatrix}5&1\\1&5\end{bmatrix}$ does the job.

□

(Grupo E curso 13/14) Exercise 7(b) We are looking for matrices $\mathbf{A} = \begin{bmatrix}a&b\\c&b\end{bmatrix}$ such that fails to be diagonalizable, so $\lambda_1 = \lambda_2$ (repeated eigenvalues, otherwise the matrix must be diagonalizable). Hence $2\lambda = \operatorname{tr}\left(\mathbf{A}\right) = a + b$, or

$$\lambda_1 = \lambda_2 = \frac{a+b}{2} > 0,$$

since both, a and b are positive. But also $\lambda^2 = ad - bc$, or

$$\lambda = \pm\sqrt{ad-bc} \quad \Rightarrow \quad \lambda_1 = -\lambda_2.$$

Therefore, there is no such matrix, since conditions $\lambda_1 = \lambda_2 > 0$, and $\lambda_1 = -\lambda_2$ are incompatible.

□

(Grupo G curso 13/14) Exercise 1(a) Trace of $\mathbf{A}$ is $\lambda_1 + \lambda_2 + \lambda_3$; determinant of $\mathbf{A}$ is $\lambda_1 \cdot \lambda_2 \cdot \lambda_3$.

□

(Grupo G curso 13/14) Exercise 1(b) $\mathbf{A}$ can be diagonalized, since the eigenvectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ are linearly independent.

□

(Grupo G curso 13/14) Exercise 1(c) We can recover $\mathbf{A}$ using $\mathbf{S}\mathbf{D}\mathbf{S}^{-1}$ where $\mathbf{S}$ is a matrix whose columns are $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$,

$$\mathbf{S} = \left[\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\right]$$

and $\mathbf{D}$ is a diagonal matrix whose diagonal entries are λ_1, λ_2, λ_3

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}.$$

$\square$

(Grupo G curso 13/14) Exercise 1(d) If we want $\mathbf{A}$ to be symmetric, the third eigenvector $\boldsymbol{x}_3$ had better be orthogonal to the other two.

$$\left[\begin{array}{rrr} 1 & 1 & 1 \\ 1 & -2 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau_{[(-1)\mathbf{1}+\mathbf{2}]} \\ \tau_{[(-1)\mathbf{1}+\mathbf{3}]}}} \left[\begin{array}{rrr} 1 & 0 & 0 \\ 1 & -3 & 0 \\ \hline 1 & -1 & \mathbf{-1} \\ 0 & 1 & \mathbf{0} \\ 0 & 0 & \mathbf{1} \end{array}\right]$$

You could also just notice the first and last entries match and guess the answer from that. Either way, $\boldsymbol{x}_3$ should be a multiple of $\begin{pmatrix} 1 & 0 & -1 \end{pmatrix}$. As for the eigenvalue, to get a matrix that's positive semidefinite but not positive definite, we need to use $\lambda_3 = 0$.

It doesn't actually ask you to compute $\mathbf{A}$, but here's one that works:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & 5 & \\ & & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 3 & -2 & 3 \\ -2 & 8 & -2 \\ 3 & -2 & 3 \end{bmatrix}.$$

$\square$

(Grupo G curso 13/14) Exercise 2(a) NO. Since the columns are linearly independent, $\mathbf{Q}$ is invertible only if $m = n$. Otherwise, it is a non-square matrix.

$\square$

(Grupo G curso 13/14) Exercise 2(b) Since the columns are linearly independent, the only solution to $\mathbf{Q}\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$; hence $\mathcal{N}\left(\mathbf{Q}\right) = \{\mathbf{0}\}$.

$\square$

(Grupo G curso 13/14) Exercise 2(c) Since $\operatorname{rg}\left(\underset{m\times n}{\mathbf{Q}}\right) = n$, then $\mathbf{Q}^\intercal\mathbf{Q} = \underset{n\times n}{\mathbf{I}}$ and

$$\underset{m\times m}{\mathbf{P}} = \underset{m\times n}{\mathbf{Q}}\,(\mathbf{Q}^\intercal\mathbf{Q})^{-1}\,\mathbf{Q}^\intercal = \underset{m\times n}{\mathbf{Q}}\left(\underset{n\times n}{\mathbf{I}}\right)^{-1}\mathbf{Q}^\intercal = \mathbf{Q}\mathbf{Q}^\intercal.$$

$\square$

(Grupo G curso 13/14) Exercise 3(a) $\begin{cases} c + 0\cdot d & = 1 \\ c + 1\cdot d & = 2 \\ c + 2\cdot d & = -1 \end{cases} \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$

$\square$

(Grupo G curso 13/14) Exercise 3(b) The normal equations $\mathbf{A}^\intercal\mathbf{A}\widehat{\boldsymbol{\beta}} = \mathbf{A}^\intercal\boldsymbol{y}$ are

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} \widehat{c} \\ \widehat{d} \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

or

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{pmatrix} \widehat{c} \\ \widehat{d} \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Cuya solución es

$$\left[\begin{array}{rr|r} 3 & 3 & -2 \\ 3 & 5 & -0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{\substack{\tau_{[(-1)\mathbf{1}+\mathbf{2}]} \\ \tau_{[(1/3)\mathbf{1}]}}} \left[\begin{array}{rr|r} 1 & 0 & -2 \\ 1 & 2 & 0 \\ \hline 1/3 & -1 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{\substack{\tau_{[(2)\mathbf{1}+\mathbf{3}]} \\ \tau_{[(-1)\mathbf{2}+\mathbf{3}]}}} \left[\begin{array}{rr|r} 1 & 0 & 0 \\ 1 & 2 & 0 \\ \hline 1/3 & -1 & 5/3 \\ 0 & 1 & -1 \end{array}\right] \Rightarrow \begin{pmatrix} \widehat{c} \\ \widehat{d} \end{pmatrix} = \begin{pmatrix} 5/3 \\ -1 \end{pmatrix}.$$

$\square$

(Grupo G curso 13/14) Exercise 3(c) $\boldsymbol{y} \notin \mathcal{C}\left(\mathbf{A}\right)$ so $y = \frac{5}{3} - x$ is the best fit, and

$$\boldsymbol{p} = \mathbf{A}\widehat{\boldsymbol{\beta}} = \frac{5}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}0\\1\\2\end{pmatrix} = \frac{1}{3}\begin{pmatrix}5\\2\\-1\end{pmatrix}$$

is the closest point in $\mathcal{C}\left(\mathbf{A}\right)$ to $\boldsymbol{y}$.

□

(Grupo G curso 13/14) Exercise 3(d)

$$\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{p} = \begin{pmatrix}1\\2\\-1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}5\\2\\-1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}-2\\4\\-2\end{pmatrix}$$

so, $\quad \|\boldsymbol{e}\|^2 = \boldsymbol{e}\cdot\boldsymbol{e} = \frac{1}{9}\begin{pmatrix}-2 & 4 & -2\end{pmatrix}\begin{pmatrix}-2\\4\\-2\end{pmatrix} = \frac{24}{9}; \quad \Rightarrow \quad \|\boldsymbol{e}\| = \frac{\sqrt{24}}{3}.$

□

(Grupo G curso 13/14) Exercise 4(a) **False.** Example: $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$.

□

(Grupo G curso 13/14) Exercise 4(b) **True.** Since $\mathbf{A}$ is diagonalizable, $\dim \mathcal{N}\left(\mathbf{A}-\mathbf{I}\right) = n$. Hence the n by n matrix $(\mathbf{A}-\mathbf{I})$ has n zero columns, so $\mathbf{A}-\mathbf{I}$ is the zero matrix $\mathbf{0}$, and therefore $\mathbf{A} = \mathbf{I}$.

□

(Grupo G curso 13/14) Exercise 4(c) **False.** If the rank is 5, there are 5 pivot columns, and 5 free columns, so the dimension of $\mathcal{N}\left(\mathbf{A}\right)$ is also 5.

□

(Grupo G curso 13/14) Exercise 4(d) **True**. If $\mathbf{A}$ is invertible, then is eigenvalues are not zeros. Since $\mathbf{B}$ has the same eigenvalues, it is also invertible.

□

(Grupo G curso 13/14) Exercise 5(d) $\begin{vmatrix}1 & -1 & 7 & 0\\0 & 0 & 6 & 0\\3 & 0 & 9 & 4\\0 & 5 & 10 & 6\end{vmatrix} = -6\begin{vmatrix}1 & -1 & 0\\3 & 0 & 4\\0 & 5 & 6\end{vmatrix} = -6\left(1\begin{vmatrix}0 & 4\\5 & 6\end{vmatrix} + 1\begin{vmatrix}3 & 4\\0 & 6\end{vmatrix}\right) =$
$-6(-20+18) = 12.$

□

(Grupo G curso 13/14) Exercise 6(a) We are looking for matrices $\mathbf{A} = \begin{bmatrix}a & b\\c & d\end{bmatrix}$ such that

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{bmatrix}a+2b\\c+2d\end{bmatrix} = \begin{pmatrix}0\\0\end{pmatrix},$$

or $a = -2b$ and $c = -2d$. The general element of $\mathcal{V}$ is

$$\begin{bmatrix}-2b & b\\-2d & d\end{bmatrix} = b\begin{bmatrix}-2 & 1\\0 & 0\end{bmatrix} + d\begin{bmatrix}0 & 0\\-2 & 1\end{bmatrix},$$

so that

$$\left\{\begin{bmatrix}-2 & 1\\0 & 0\end{bmatrix}; \begin{bmatrix}0 & 0\\-2 & 1\end{bmatrix}\right\}$$

is a basis of $\mathcal{V}$, and $\dim \mathcal{V} = 2$.

□

(Grupo G curso 13/14) Exercise 6(b) The space $\mathcal{V}$ of part (a) is a subspace of $\mathcal{W}$ (if $(1,2,)$ is in $\mathcal{N}\left(\mathbf{A}\right)$, then $(1,2,)$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda = 0$). Since not all 2×2 matrices are in $\mathcal{W}$, then $\dim \mathcal{W} < 4$; since there are matrices in $\mathcal{W}$ that do not belong to $\mathcal{V}$ (for example the identity matrix), then $\dim \mathcal{W} > 2$. Therefore $\dim \mathcal{W}$ must be 3.

□

(Grupo E curso 12/13) Exercise 1(a) Since the two first columns are equal, $\det \mathbf{A} = 0$.

□

(Grupo E curso 12/13) Exercise 1(b) $\det \mathbf{B} = -\begin{vmatrix}1&0&1&1\\1&1&1&1\\1&1&1&0\\1&1&0&0\end{vmatrix} = -\begin{vmatrix}1&0&0&0\\1&1&0&0\\1&1&0&-1\\1&1&-1&-2\end{vmatrix} = \begin{vmatrix}1&0&0&0\\1&1&0&0\\1&1&-1&0\\1&1&-2&-1\end{vmatrix} =$ 1.

□

(Grupo E curso 12/13) Exercise 1(c) $\det \mathbf{C} = \begin{vmatrix}0&1&1&1\\1&1&1&1\\1&1&1&0\\1&1&0&0\end{vmatrix} + \begin{vmatrix}x&1&1&1\\0&1&1&1\\0&1&1&0\\0&1&0&0\end{vmatrix} = 1 + x\begin{vmatrix}1&1&1\\1&1&0\\1&0&0\end{vmatrix} =$ $1 - x\begin{vmatrix}1&1&1\\0&1&1\\0&0&1\end{vmatrix} = 1 - x.$

□

(Grupo E curso 12/13) Exercise 2(a) Because $\mathbf{A}$ is not symmetric.

□

(Grupo E curso 12/13) Exercise 2(b) When $a > 0$ and $\begin{vmatrix}a&1\\1&b\end{vmatrix} = ab - 1 > 0$; that is, when $a > 0$ and $b > \frac{1}{a}$.

Or diagonalizing by congruence

$$\begin{bmatrix}a&1\\1&b\end{bmatrix} \xrightarrow{\underset{[(-\frac{1}{a})\mathbf{2}+\mathbf{1}]}{\tau}} \begin{bmatrix}a&0\\1&b-\frac{1}{a}\end{bmatrix} \xrightarrow[\underset{[(-\frac{1}{a})\mathbf{2}+\mathbf{1}]}{\tau}]{} \begin{bmatrix}a&0\\0&b-\frac{1}{a}\end{bmatrix} \qquad \Rightarrow \quad b - \frac{1}{a} > 0.$$

□

(Grupo E curso 12/13) Exercise 3(a)

$$\mathcal{C}\left(\mathbf{A}^{\intercal}\right) = \{\boldsymbol{x} \in \mathbb{R}^n | \, \boldsymbol{x} = a\,\boldsymbol{v} \text{ for all } a \in \mathbb{R}\}.$$

And, since $\mathcal{N}\left(\mathbf{A}\right)$ is orthogonal to the rows of $\mathbf{A}$,

$$\mathcal{N}\left(\mathbf{A}\right) = \{\boldsymbol{x} \in \mathbb{R}^n \ | \ \boldsymbol{v}\cdot\boldsymbol{x} = 0\}.$$

Note that when $\boldsymbol{x}$ is orthogonal to the rows of $\mathbf{A}$ (when $\boldsymbol{v}\cdot\boldsymbol{x} = \mathbf{0}$), then $\mathbf{A}\boldsymbol{x} = [\boldsymbol{u}][\boldsymbol{v}]^{\intercal}\boldsymbol{x} = [\boldsymbol{u}]^{\intercal}\mathbf{0} = \mathbf{0}$.

□

(Grupo E curso 12/13) Exercise 3(b) Puesto que $[\boldsymbol{v}]^{\intercal}\boldsymbol{u} = \sum u_i([\boldsymbol{v}]^{\intercal})_{|i} = \sum u_i(v_i,) = \left(\sum u_i\,v_i,\right) = (\boldsymbol{u}\cdot\boldsymbol{v},)$ es un vector de $\mathbb{R}^1$, tenemos

$$\mathbf{A}\boldsymbol{u} = [\boldsymbol{u}][\boldsymbol{v}]^{\intercal}\boldsymbol{u} = [\boldsymbol{u}]\left(\sum u_i\,v_i,\right) = [\boldsymbol{u}](\lambda,) = \lambda\boldsymbol{u}, \qquad \text{where} \qquad \lambda = \boldsymbol{v}\cdot\boldsymbol{u}.$$

□

(Grupo E curso 12/13) Exercise 3(c) Since all columns are multiples of $\boldsymbol{u}$, and skew-symmetric matrix has zeros on the main diagonal, then all the entries must be zero. Hence, the zero vector is either $\boldsymbol{u}$ or $\boldsymbol{v}$, or both.

□

(Grupo E curso 12/13) Exercise 3(d) Since $[\boldsymbol{v}]^{\intercal}[\boldsymbol{u}] = [\boldsymbol{v}\cdot\boldsymbol{u}] = \lambda \underset{1\times 1}{\mathbf{I}}$ then

$$\mathbf{A}^2 = [\boldsymbol{u}][\boldsymbol{v}]^{\intercal}[\boldsymbol{u}][\boldsymbol{v}]^{\intercal} = [\boldsymbol{u}]\Big([\boldsymbol{v}]^{\intercal}[\boldsymbol{u}]\Big)[\boldsymbol{v}]^{\intercal} = \lambda[\boldsymbol{u}]\,[\boldsymbol{v}]^{\intercal} = \lambda\,\mathbf{A}.$$

Hence, if $\mathbf{A}^2 = \mathbf{A}$; then $\boldsymbol{v}\cdot\boldsymbol{u} = \lambda = 1$.

□

(Grupo E curso 12/13) Exercise 4(a) For all c, since $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ are linearly independent.

□

(Grupo E curso 12/13) Exercise 4(b) For all c, since $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ are perpendiclar one to each other.

$\square$

(Grupo E curso 12/13) Exercise 4(c) The matrix $\mathbf{A}$ can't be positive definite since one eigenvalue is zero, $\lambda_1 = 0$.

$\square$

(Grupo E curso 12/13) Exercise 5.
The eigenvalues are 5 and 15. For $\lambda = 5$ we get

$$\mathbf{A} - 5\mathbf{I} = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \mathbf{0}.$$

For $\lambda = 15$ we get

$$\mathbf{A} - 15\mathbf{I} = \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \mathbf{0}.$$

Hence $\quad \mathbf{A} = \dfrac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 & \\ & 15 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \dfrac{1}{\sqrt{5}}.$

$\square$

(Grupo E curso 12/13) Exercise 6(a) Since the dimensión of $\mathcal{N}\left(\mathbf{A}^\intercal\right)$ is one, the dimension of $\mathcal{C}\left(\mathbf{A}^\intercal\right)$ is three; so rank three. Since the matrix is singular (not full rank), then $\det \mathbf{A} = 0$.

$\square$

(Grupo E curso 12/13) Exercise 6(b) Since the system is solvable only if the right hand side vector is in $\mathcal{C}\left(\mathbf{A}\right)$, and since $\mathcal{C}\left(\mathbf{A}\right)$ is orthogonal to $\mathcal{N}\left(\mathbf{A}^\intercal\right)$, we just only need to find a subspace in $\mathbb{R}^4$ orthogonal to $\begin{pmatrix} -1 & -1 & 1 & 1 \end{pmatrix}$; By gaussian elimination we get,

$$\left[\begin{array}{rrrr} -1 & -1 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-1)\mathbf{1}+\mathbf{2}] \\ \tau \\ [(1)\mathbf{1}+\mathbf{3}] \\ \tau \\ [(1)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{rrrr} -1 & 0 & 0 & 0 \\ \hline 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

Therefore $\begin{pmatrix} a \\ b \\ c \\ 1 \end{pmatrix}$ must be a linear combination of $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Thus

$$\begin{pmatrix} a \\ b \\ c \\ 1 \end{pmatrix} = b \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -b+c+1 \\ b \\ c \\ 1 \end{pmatrix},$$

or $\boxed{a = -b+c+1}$.

$\square$

(Grupo E curso 12/13) Exercise 6(c) Since there is only one linear restriction in $\mathbb{R}^3$ $(a+b-c=1)$, it is **a plane** (two free columns in the system $a+b-c=1$).

$\square$

(Grupo E curso 12/13) Exercise 6(d) Since $\mathcal{C}\left(\mathbf{A}\right)$ is orthogonal to $\mathcal{N}\left(\mathbf{A}^\intercal\right)$, and $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \in \mathcal{N}\left(\mathbf{A}^\intercal\right)$, this right hand side vector is not in the column space. Therefore **the set of solutions is the empty set**.

$\square$

(Grupo H curso 12/13) Exercise 1(a) Any value except zero.

$\square$

(Grupo H curso 12/13) Exercise 1(b) Since the eigenvalues of $\mathbf{A}^2$ are the square of the eigenvalues of $\mathbf{A}$, then the only posible eigenvalues are zero or one. Threfore the determinant is either zero or one.

$\square$

(Grupo H curso 12/13) Exercise 1(c) Determinant equal to 6.

□

(Grupo H curso 12/13) Exercise 2. The eigenvalues are 5 and 15. For $\lambda = 5$ we get

$$\mathbf{A} - 5\mathbf{I} = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \quad \to \quad \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \mathbf{0}.$$

For $\lambda = 15$ we get

$$\mathbf{A} - 15\mathbf{I} = \begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix} \quad \to \quad \begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \mathbf{0}.$$

Hence $\quad \mathbf{A} = \dfrac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 & \\ & 15 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \dfrac{1}{\sqrt{5}}.$

□

(Grupo H curso 12/13) Exercise 3(a) $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \quad a \neq 0.$

□

(Grupo H curso 12/13) Exercise 3(b) Since the determiant is not zero, the matrix has full rank, that is, the rank is 3.

□

(Grupo H curso 12/13) Exercise 4(a) $\mathcal{N}\left(\mathbf{A}\right) = \left\{\boldsymbol{x} \in \mathbb{R}^3 \text{ such that } \boldsymbol{x} = a\boldsymbol{u} \text{ for all } a \in \mathbb{R}\right\}.$

□

(Grupo H curso 12/13) Exercise 4(b) Since there are 3 eigenvalues, $\mathbf{A}$ is 3×3. Since there are no repeated eigenvalues, $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are linearly independent. Since only one eigenvalue is zero, the rank of $\mathbf{A}$ is 2, and since $\boldsymbol{v}$ and $\boldsymbol{w}$ are eigenvectors of $\mathbf{A}$ with eigenvalue 1 and 2, then

$$\mathbf{A}\boldsymbol{v} = \boldsymbol{v}, \qquad \mathbf{A}\left(\boldsymbol{w}/2\right) = \boldsymbol{w}$$

so $\boldsymbol{v}$ and $\boldsymbol{w}$ belong to $\mathcal{C}\left(\mathbf{A}\right)$; therefore

$$\mathcal{C}\left(\mathbf{A}\right) = \left\{\boldsymbol{x} \in \mathbb{R}^3 \text{ such that } \boldsymbol{x} = a\boldsymbol{v} + b\boldsymbol{w} \text{ for all } a, b \in \mathbb{R}\right\}.$$

□

(Grupo H curso 12/13) Exercise 4(c) Since $\mathcal{N}\left(\mathbf{A}\right)$ is perpendicular to $\mathcal{C}\left(\mathbf{A}^{\intercal}\right)$,

$$\mathcal{C}\left(\mathbf{A}^{\intercal}\right) = \left\{\boldsymbol{x} \in \mathbb{R}^3 \text{ such that } \boldsymbol{x} \cdot \boldsymbol{u} = 0\right\}.$$

□

(Grupo H curso 12/13) Exercise 4(d) Since

$$\mathbf{A}\boldsymbol{v} = \boldsymbol{v}, \qquad \mathbf{A}\boldsymbol{w} = 2\boldsymbol{w}$$

then

$$\boldsymbol{v} - \boldsymbol{w} = \mathbf{A}\boldsymbol{v} - \frac{1}{2}\mathbf{A}\boldsymbol{w} = \mathbf{A}\left(\boldsymbol{v} - \boldsymbol{w}/2\right).$$

Hence, a particular solution is $(\boldsymbol{v} - \boldsymbol{w}/2)$, and the complete solution is

$$\boldsymbol{x} = (\boldsymbol{v} - \boldsymbol{w}/2) + a\boldsymbol{u}, \qquad \text{for all } a \in \mathbb{R}.$$

□

(Grupo H curso 12/13) Exercise 5(a)

$$\begin{vmatrix} 1 & b \\ b & 4 \end{vmatrix} = 4 - b^2 > 0 \Rightarrow \quad -\sqrt{4} < b < \sqrt{4}$$

and

$$\begin{vmatrix} 1 & b & 0 \\ b & 4 & 2 \\ 0 & 2 & 4 \end{vmatrix} = 16 - 4 - 4b^2 = 12 - 4b^2 = 4(3 - b^2) > 0 \Rightarrow \quad -\sqrt{3} < b < \sqrt{3}.$$

Since $\sqrt{3} < \sqrt{4}$, $\mathbf{A}$ is positive definite only if $-\sqrt{3} < b < \sqrt{3}$.

□

(Grupo H curso 12/13) Exercise 5(b) The eigenvalues λ_i of A^2 are the square of the eigenvalues of A, therefore $\lambda_i \geq 0$. Hence $\boldsymbol{x}\mathsf{A}^2\boldsymbol{x} \geq 0$ for all $\boldsymbol{x} \neq \mathbf{0}$. On the other hand I is positive definite, and then

$$\boldsymbol{x}(\mathsf{A}^2 + \mathsf{I})\boldsymbol{x} = \underbrace{\boldsymbol{x}\mathsf{A}\boldsymbol{x}}_{\geq 0} + \underbrace{\boldsymbol{x}\mathsf{I}\boldsymbol{x}}_{>0} > 0 \quad \text{for every } b.$$

$\square$

(Grupo H curso 12/13) Exercise 5(c) The matrix $\mathsf{M}^\intercal\mathsf{M}$ is symmetric positive definite unless M *is not full column rank.*

If M has linearly dependent columns, then $\mathsf{M}^\intercal\mathsf{M}$ is symmetric positive **semi**-definite, since $\boldsymbol{x}\mathsf{M}^\intercal\mathsf{M}\boldsymbol{x} = 0$ when $\boldsymbol{x} \in \mathcal{N}\left(\mathsf{M}\right)$.

$\square$

(Grupo H curso 12/13) Exercise 6(c)

$$\left[\begin{array}{cccc|c|c} 1 & 2 & 1 & 2 & -2 & -a \\ 2 & 1 & 2 & 1 & -1 & -b \\ 1 & 1 & 1 & 1 & -1 & -c \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ (-1)\mathbf{1}+\mathbf{3} \\ (-2)\mathbf{1}+\mathbf{4} \\ (2)\mathbf{1}+\mathbf{5} \\ (a)\mathbf{1}+\mathbf{6}}} \left[\begin{array}{cccc|c|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & -3 & 0 & -3 & 3 & 2a-b \\ 1 & -1 & 0 & -1 & 1 & a-c \\ \hline 1 & -2 & -1 & -2 & 2 & a \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}+\mathbf{4}] \\ (1)\mathbf{2}+\mathbf{5} \\ (\frac{2a-b}{3})\mathbf{2}+\mathbf{6}}} \left[\begin{array}{cccc|c|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & -3 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & (a+b-3c)/3 \\ \hline 1 & -2 & -1 & 0 & 0 & (2b-a)/3 \\ 0 & 1 & 0 & -1 & 1 & (2a-b)/3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$\square$

(Grupo H curso 12/13) Exercise 6(a) We have seen that

$$\left[\begin{array}{c} \mathsf{A} \\ \hline \mathsf{I} \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ (-1)\mathbf{1}+\mathbf{3} \\ (-2)\mathbf{1}+\mathbf{4} \\ (-1)\mathbf{2}+\mathbf{4}}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 1 & -2 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c} \mathsf{L} \\ \hline \dot{\mathsf{E}} \end{array}\right].$$

Since $\quad \dot{\mathsf{U}} = \dot{\mathsf{E}}^{-1} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad$ we get $\quad \mathsf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$

$\square$

(Grupo H curso 12/13) Exercise 6(b) $\boldsymbol{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + d\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + e\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ for all $d, e \in \mathbb{R}$.

$\square$

(Grupo H curso 12/13) Exercise 6(c) The vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is in $\mathcal{C}\left(\mathsf{A}\right)$ if and only if $a + b - 3c = 0$.

$\square$

(Grupo E curso 11/12) Exercise 1(a) Using column elementary operations:

$$\begin{vmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -|\mathsf{I}| = -1.$$

$\square$

(Grupo E curso 11/12) Exercise 1(b) Puesto que la matriz es simétrica, sabemos que es invertible, es decir, que es posible encontrar 5 autovectores linealmente independientes.

Para el autovalor $\lambda = 1$, cuatro autovectores linealmente independientes son:

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Puesto que la traza es 10, el quinto autovalor es $\lambda = 6$. En tal caso

$$\mathbf{A} - 6\mathbf{I} = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix};$$

y puesto que la suma de las columnas de dicha matriz es $\mathbf{0}$, un quinto autovector es $\begin{pmatrix} 1, & 1, & 1, & 1, & 1, \end{pmatrix}$. Nótese que los cinco autovectores son ortogonales.

$\square$

(Grupo E curso 11/12) Exercise 1(c) El elemento $(3,1)$ de $\mathbf{A}^{-1}$ es $\dfrac{\operatorname{cof}(\mathbf{A})_{1,3}}{\det \mathbf{A}}$; es decir

$$\frac{\operatorname{cof}(\mathbf{A})_{1,3}}{\det \mathbf{A}} = \frac{\begin{vmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix}}{\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \lambda_4 \cdot \lambda_5 \cdot \lambda_6} = \frac{-1}{6}.$$

Y como el menor (3,1) de la matriz del enunciado, $M_{31}(\mathbf{A})$, es igual a la traspuesta del menor (1,3); entonces los cofactores $\operatorname{cof}(\mathbf{A})_{1,3}$ y $\operatorname{cof}(\mathbf{A})_{1,3}$ son iguales, y por tanto también son iguales los elementos (3,1) y (1,3) de $\mathbf{A}^{-1}$; ambos iguales a $\frac{-1}{6}$.

$\square$

(Grupo E curso 11/12) Exercise 2(a) The column space is spanned by the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}.$$

We then put them in a matrix and do a Gaussian elimination to find independent vectors. This tells us that a basis for the column space is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

$\square$

(Grupo E curso 11/12) Exercise 2(b) The column space can be described by

$$\mathcal{C}\left(\mathbf{A}\right) = \left\{ \begin{pmatrix} x \\ y \\ 0 \\ 0 \end{pmatrix} \middle| \, x; y \in \mathbb{R} \right\};$$

so the basis of $\mathcal{C}\left(\mathbf{A}\right)$ is the set of any two independent vectors $\begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} x_3 \\ x_4 \\ 0 \\ 0 \end{pmatrix}$. This means that the matrix

$$\mathbf{A} = \begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix}$$

has full rank (in other words $x_4 - x_2x_3/x_1 \neq 0 \quad$ or $\quad x_1x_4 - x_2x_3 \neq 0$must hold).

□

(Grupo E curso 11/12) Exercise 2(c) We observe that $(-3, 0, 1, 0,)$ and $(0, -3, 0, 1,)$ are two independent vectors belonging to the null space. Since the column space has dimension 2, the null space has dimension $4 - 2 = 2$, so any basis of $\mathcal{N}\left(\mathbf{A}\right)$ has two elements. Hence, $\{(-3, 0, 1, 0,);\ (0, -3, 0, 1,)\}$ is a basis for $\mathcal{N}\left(\mathbf{A}\right)$.

□

(Grupo E curso 11/12) Exercise 2(d) We start by looking for $\boldsymbol{x}_{particular}$ via elimination. Note that the matrix is already in a reduced row echelon form:

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 5 \\ 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So $\boldsymbol{x}_{particular} = (5; 4; 0; 0)$. Then the complete solution is given by

$$\boldsymbol{x} = \boldsymbol{x}_{particular} + \boldsymbol{x}_{nullspace} = \begin{pmatrix} 5 \\ 4 \\ 0 \\ 0 \end{pmatrix} + a \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 - 3a \\ 4 - 3b \\ a \\ b \end{pmatrix}$$

for any $a; b \in \mathbb{R}$.

□

(Grupo E curso 11/12) Exercise 3(a) La traza debe valer 0; por tanto $\mathbf{A} = \begin{bmatrix} a & 1 \\ x & -a \end{bmatrix}$ y el determinante -1; por tanto

$$-a^2 - x = -1 \quad \Rightarrow \quad x = 1 - a^2;$$

es decir

$$\mathbf{A} = \begin{bmatrix} a & 1 \\ 1 - a^2 & -a \end{bmatrix}.$$

□

(Grupo E curso 11/12) Exercise 3(b) Porque los autovalores son distintos.

□

(Grupo E curso 11/12) Exercise 3(c) Los que hacen la matriz simétrica, es decir, aquellos para los que $1 - a^2 = 1$, por tanto, sólo para $a = 0$. En tal caso

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \text{que tiene los autovectores ortogonales} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

□

(Grupo E curso 11/12) Exercise 4(a) $\begin{bmatrix} .1 & .7 & .1 & .7 \\ .5 & .5 & .5 & .5 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \text{y} \qquad \sqrt{a^2 + b^2 + c^2 + d^2} = 1.$

□

(Grupo E curso 11/12) Exercise 4(b) Puesto que la matriz de coeficientes tiene rango 2 (los vectores fila apuntan en direcciones distintas), el conjunto de soluciones al primer sistema de ecuaciones es un espacio vectorial de dimensión 2 (hay todo un plano de puntos posibles, es decir, hay infinitas posibilidades para la elección de estos números). Así pues, hay infinitos vectores de longitud uno en el plano, que son los situados en la circunferencia de radio uno, centrada en el origen.

□

(Grupo E curso 11/12) Exercise 5(a) $\boldsymbol{x} = \boldsymbol{p} + a\boldsymbol{v} + b\boldsymbol{w}; \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + a \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$

□

(Grupo E curso 11/12) Exercise 5(b) We just need an orthogonal vector to $\boldsymbol{v}$ and $\boldsymbol{w}$.

$$\left[\begin{array}{ccc} x & y & z \\ \hline 0 & 1 & 1 \\ \hline 1 & 1 & 0 \\ 0 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc} x & y-x & z \\ \hline 0 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc} x & y-x & z-2y+2x \\ \hline 0 & 1 & -1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]; \quad \text{hence} \quad 2x-2y+z=-1.$$

□

(Grupo E curso 11/12) Exercise 6. $\mathcal{N}\left(\mathbf{A}\right)=\{\mathbf{0}\}$ so $\mathbf{A}$ has full column rank $r=n=3$: the columns are linearly independent.

□

(Grupo H curso 11/12) Exercise 1.

$$[\mathbf{A}|\boldsymbol{b}] = \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = [\mathbf{R}|\boldsymbol{d}]$$

$$\boldsymbol{x} = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix} + a\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

□

(Grupo H curso 11/12) Exercise 2(a)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\underset{[(-1)\mathbf{2}+\mathbf{1}]}{\boldsymbol{\tau}}} \begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

por tanto, el conjunto de "soluciones especiales" está compuesto por los siguientes tres vectores:

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

□

(Grupo H curso 11/12) Exercise 2(b)

Empezaremos por la pregunta:

- **c)** Las dos cosas a probar son que el conjunto de vectores genera el conjunto de todas las soluciones del sistema $\mathbf{A}\boldsymbol{x}=\mathbf{0}$ (que es un sistema generador); y que los vectores del sistema generador son linealmente independiestes.
- **b) Primera parte de la demo** (*El conjunto genera todo el espacio de soluciones*). Lo que hay que demostrar es que todo vector $\boldsymbol{x}$ combinación de las soluciones especiales es una solución a $\mathbf{A}\boldsymbol{x}=\mathbf{0}$ (pertenece a $\mathcal{N}\left(\mathbf{A}\right)$); es decir:

$$\text{Si} \quad \boldsymbol{x} = \begin{bmatrix} -2 & -3 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad \text{entonces} \qquad \mathbf{A}\boldsymbol{x}=\mathbf{0}.$$

Veamoslo

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -2 & -3 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{0}$$

Segunda parte de la demo (*los vectores son linealmente independientes*). Por eliminación Gaussiana es inmediato ver que las cuatro columnas de

$$\mathbf{N} = \begin{bmatrix} -2 & -3 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

son columnas pivote, y por tanto que en el sistema $\mathbf{N}\boldsymbol{x} = \mathbf{0}$ no hay columnas libres; así pues, la única combinación de dichas columnas que es igual al vector cero $\mathbf{0}$ es la solución trivial ($\boldsymbol{x} = \mathbf{0}$), es decir, los vectores columna de $\mathbf{N}$ son linealmente independientes.

□

(Grupo H curso 11/12) Exercise 3(a)

$$\begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 9 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 2 \cdot 2 = 4.$$

El elemento (1,1) de la inversa de $\mathbf{A}$ es el primer elemento de la matriz adjunta $\mathbf{Adj(A)}$ dividido por el determinante.

$$\frac{\operatorname{cof}(\mathbf{A})_{11}}{\det \mathbf{A}} = \frac{2}{4} = \frac{1}{2}.$$

□

(Grupo H curso 11/12) Exercise 3(b)

$$\begin{vmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 9 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 4 - 2 = 2.$$

□

(Grupo H curso 11/12) Exercise 3(c)

$$\det \mathbf{A} = \begin{vmatrix} 2 & 1-x & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 9 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 9 \end{vmatrix} + \begin{vmatrix} 2 & -x & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 2 & 4 \\ 1 & 0 & 3 & 9 \end{vmatrix} = 2 + x \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 2 + 2x.$$

Por tanto, $\det \mathbf{A} = 2x + 2$; y cuando $x = -1$ la matriz es singular ($\det \mathbf{A} = 0$).

□

(Grupo H curso 11/12) Exercise 4(a)

Debemos resolver la ecuación característica:

$$\begin{vmatrix} 0-\lambda & 0 & -2 \\ 0 & -2-\lambda & 0 \\ -2 & 0 & 3-\lambda \end{vmatrix} = (-\lambda)(-2-\lambda)(3-\lambda) - 4(-2-\lambda) = 0$$

Evidentemente una raiz es $\lambda = -2$; y dividiendo el polinomio por $(-2-\lambda)$ obtenemos las otras dos

$$0 = (-\lambda)(3-\lambda) - 4 = \lambda^2 - 3\lambda - 4 \Rightarrow \begin{cases} \lambda = 4 \\ \lambda = -1 \end{cases}.$$

□

(Grupo H curso 11/12) Exercise 4(b)

Y ahora calculamos un autovector para cada autovalor:

- Para $\lambda = -2$

$$[\mathbf{A} + 2\mathbf{I}] = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 5 \end{bmatrix} \xrightarrow[{[(1)3+1]}]{\boldsymbol{\tau}} \begin{bmatrix} 2 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow \boldsymbol{x}_{(-2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

- Para $\lambda = 4$

$$[\mathbf{A} - 4\mathbf{I}] = \begin{bmatrix} -4 & 0 & -2 \\ 0 & -6 & 0 \\ -2 & 0 & -1 \end{bmatrix} \xrightarrow[{[(1/2)\mathbf{3}+\mathbf{1}]}]{\boldsymbol{\tau}} \begin{bmatrix} -4 & 0 & -2 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{x}_{(4)} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

- Para $\lambda = -1$

$$[\mathbf{A} - 4\mathbf{I}] = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \\ -2 & 0 & 4 \end{bmatrix} \xrightarrow[{[(2)\mathbf{3}+\mathbf{1}]}]{\boldsymbol{\tau}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{x}_{(-1)} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

□

(Grupo H curso 11/12) Exercise 4(c)
Por ejemplo:

$$\mathbf{P} = \begin{bmatrix} \boldsymbol{x}_{(4)}, & \boldsymbol{x}_{(-2)}, & \boldsymbol{x}_{(-1)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}; \quad \mathbf{D} = \begin{bmatrix} 4 & & \\ & -2 & \\ & & -1 \end{bmatrix}$$

de manera que

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 3 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & & \\ & -2 & \\ & & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & & \\ & -2 & \\ & & -1 \end{bmatrix} \begin{bmatrix} 1/5 & 0 & -2/4 \\ 0 & 1 & 0 \\ 2/5 & 0 & 1/5 \end{bmatrix}$$

ya que

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(2)\mathbf{1}+\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 2 & 0 & 1 \end{array}\right] \xrightarrow[{[(1/5)\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2/5 & 0 & 1/5 \end{array}\right]$$

$$\xrightarrow[{[(-2)\mathbf{3}+\mathbf{1}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 0 & -2/4 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2/5 & 0 & 1/5 \end{array}\right]$$

□

(Grupo H curso 11/12) Exercise 5. Por una parte

$$\text{Si} \quad \underset{m\times n}{\mathbf{A}}\ \boldsymbol{x} = \boldsymbol{b} \quad \text{no tiene solución} \quad \Rightarrow \text{rg}\left(\mathbf{A}\right) < m.$$

Por otra

$$\text{Si} \quad \underset{m\times n}{\mathbf{A}^{\intercal}}\ \boldsymbol{x} = \boldsymbol{c} \quad \text{tiene sólo una solución} \quad \Rightarrow \text{rg}\left(\mathbf{A}\right) = m.$$

Y por tanto ambas condiciones son incompatibles.

□

(Grupo A curso 10/11) Exercise 1(a) Norma es $\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v}\cdot\boldsymbol{v}} = \sqrt{1^2+2^2+2^2} = \sqrt{9} = 3.$

□

(Grupo A curso 10/11) Exercise 1(b) *Sólo se le pide encontrar un vector ortogonal de norma dos, pero aquí vamos a desarrollar una respuesta un poco más extensa (en el enunciado no se le pide tanto. . .)*
Mediante la eliminación de Gauss podemos calcular el espacio nulo por la izquierda de $\begin{bmatrix}\boldsymbol{v}\end{bmatrix}$

$$\begin{bmatrix}\mathbf{I}|\boldsymbol{v}\end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \end{array}\right]$$

Por tanto cualquier combinación lineal de los vectores $\begin{pmatrix}-2, & 1, & 0,\end{pmatrix}$ y $\begin{pmatrix}-2, & 0, & 1,\end{pmatrix}$ es perpendicular al vector dado; y puesto que ambos tienen norma 5, tomando por ejemplo el doble de la versión normalizada del primero, tenemos $2 \times \frac{1}{\sqrt{5}}\begin{pmatrix}-2, & 1, & 0,\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}-4, & 2, & 0,\end{pmatrix}$ que tiene norma 2 y es perpendicular al vector del enunciado.

Pero ésta no es la única solución posible. Sabemos que cualquier vector de la forma

$$a\begin{pmatrix}-2, & 1, & 0,\end{pmatrix} + b\begin{pmatrix}-2, & 0, & 1,\end{pmatrix} = \begin{pmatrix}-2(a+b), & a, & b,\end{pmatrix}$$

es perpendicular; y que sólo queremos vectores de norma 2. Es decir

$$\Big(-2(a+b)\Big)^2 + a^2 + b^2 = 4;$$

por tanto

$$5a^2 + 5b^2 + 8ab = 4$$

es la condición que deben cumplir los valores de a y b para que el vector perpendicular $\begin{pmatrix}-2(a+b), & a, & b,\end{pmatrix}$ tenga norma 2.

$\square$

(Grupo A curso 10/11) Exercise 1(c)

Es sencillo ver que la respuesta es $a = -b$.

$\square$

(Grupo A curso 10/11) Exercise 2.

Entonces

$$\mathbf{A}(\boldsymbol{v} - \boldsymbol{w}) = \mathbf{A}\boldsymbol{v} - \mathbf{A}\boldsymbol{w} = \mathbf{0}$$

y por tanto el vector diferencia $(\boldsymbol{v} - \boldsymbol{w})$

$$\boldsymbol{v} - \boldsymbol{w} = \begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}3\\1\\4\end{pmatrix} = \begin{pmatrix}-2\\1\\-1\end{pmatrix}$$

es una solución al sistema homogéneo $\mathbf{A}\boldsymbol{x} = \mathbf{0}$. Así pues

$$\begin{pmatrix}1\\2\\3\end{pmatrix} + (\boldsymbol{v} - \boldsymbol{w}) = \begin{pmatrix}1\\2\\3\end{pmatrix} + \begin{pmatrix}-2\\1\\-1\end{pmatrix} = \begin{pmatrix}-1\\3\\2\end{pmatrix}$$

Es otra solución.

$\square$

(Grupo A curso 10/11) Exercise 3(a)

Ecuación característica:

$$\begin{vmatrix}4-\lambda & 0 & -1\\ 0 & 3-\lambda & 0\\ -1 & 0 & 4-\lambda\end{vmatrix} = (4-\lambda)(3-\lambda)(4-\lambda) - (3-\lambda) = (3-\lambda)\Big((4-\lambda)^2 - 1\Big) = 0$$

Por tanto una raiz es $\lambda_1 = 3$.

Las otras dos raices las obtenemos de

$$0 = (4-\lambda)^2 - 1 = \lambda^2 - 8\lambda + 15 \Rightarrow \begin{cases}\lambda_2 = & 3\\ \lambda_5 = & 5\end{cases}$$

Para $\lambda = 5$

$$\mathbf{A} - 5\mathbf{I} = \begin{bmatrix}-1 & 0 & -1\\ 0 & -2 & 0\\ -1 & 0 & -1\end{bmatrix}$$

Por lo que

$$\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

es un autovector para $\lambda = 5$.

Para $\lambda = 3$

$$\mathbf{A} - 5\mathbf{I} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Por lo que

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{y} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

son dos autovectores linealmente independientes para $\lambda = 3$.

□

(Grupo A curso 10/11) Exercise 3(b)

La matriz $\mathbf{A}$ es diagonalizable, ya que es posible encontrar un número suficiente (en este caso 3) de autovectores linealmente independientes (algo que ya sabíamos antes responder al primer apartado, ya que $\mathbf{A}$ es simétrica).

□

(Grupo A curso 10/11) Exercise 3(c)

$$\mathbf{A}^{10} = \mathbf{S}\mathbf{D}^{10}\mathbf{S}^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5^{10} & & \\ & 3^{10} & \\ & & 3^{10} \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1}.$$

donde $\mathbf{S}$ es una matriz cuyas columnas son los autovectores, y $\mathbf{D}$ es una matriz diagonal con los correspondientes autovalores.

□

(Grupo A curso 10/11) Exercise 3(d)

$$\begin{aligned}\mathbf{A}^{4} = \mathbf{S}\mathbf{D}^{4}\mathbf{S}^{-1} = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix}^4 &= \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5^{4} & & \\ & 3^{4} & \\ & & 3^{4} \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 625 & & \\ & 81 & \\ & & 81 \end{bmatrix} \begin{bmatrix} -1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 353 & 0 & -272 \\ 0 & 81 & 0 \\ -272 & 0 & 353 \end{bmatrix};\end{aligned}$$

donde $\mathbf{S}$ es una matriz cuyas columnas son los autovectores, y $\mathbf{D}$ es una matriz diagonal con los correspondientes autovalores.

□

(Grupo A curso 10/11) Exercise 3(e)

$$f(x,y,z) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{bmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4x^2 + 3y^2 + 4z^2 - 2xz.$$

Y sabemos que es definida positiva, ya que los autovalores de $\mathbf{A}$ son mayores que cero (3, 3 y 5).

□

(Grupo A curso 10/11) Exercise 4(a)

Es subespacio vectorial, ya que el conjunto es cerrado para la suma

$$\begin{pmatrix} a, & b, & a, \end{pmatrix} + \begin{pmatrix} c, & d, & c, \end{pmatrix} = \begin{pmatrix} a+c, & b+d, & a+c, \end{pmatrix}$$

y también es cerrado para el producto por un escalar

$$a \begin{pmatrix} b, & c, & d, \end{pmatrix} = \begin{pmatrix} ab, & ac, & ab, \end{pmatrix}$$

en concreto S_1 es un plano en $\mathbb{R}^3$ que pasa por el origen, y constituye el conjunto de soluciones del sistema homogéneo

$$\mathbf{A}\boldsymbol{x} = \mathbf{0}; \qquad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0};$$

es decir, que $S_1 = \mathcal{N}\left(\mathbf{A}\right)$.

□

(Grupo A curso 10/11) Exercise 4(b)
No es un subespacio. Por ejemplo el vector $\boldsymbol{x} = \begin{pmatrix}2, & 0, & 0,\end{pmatrix}$ pertenece a S_2, pero $2\boldsymbol{x}$ no. Así pues, el conjunto S_2 no es cerrado para el producto por un escalar (es fácil comprobar que tampoco lo es para la suma).

□

(Grupo A curso 10/11) Exercise 5(a)
$\det \mathbf{A} = -11.$

□

(Grupo A curso 10/11) Exercise 5(b)
Se han intercambiado las dos primeras filas, por tanto

$$\begin{vmatrix}2 & 1 & 4\\ 1 & 2 & -3\\ 0 & 2 & -3\end{vmatrix} = -\det \mathbf{A} = 11.$$

□

(Grupo A curso 10/11) Exercise 5(c)
Se ha multiplicado la primera fila por 3, por tanto

$$\begin{vmatrix}3 & 6 & -9\\ 2 & 1 & 4\\ 0 & 2 & -3\end{vmatrix} = 3\det \mathbf{A} = -33.$$

□

(Grupo A curso 10/11) Exercise 5(d)
Se han multiplicado todas las filas por 2, por tanto

$$\begin{vmatrix}2 & 4 & -6\\ 4 & 2 & 8\\ 0 & 4 & -6\end{vmatrix} = 2^3\det \mathbf{A} = -88.$$

□

(Grupo A curso 10/11) Exercise 5(e)
$\det \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} = \frac{-1}{11}.$

□

(Grupo E curso 10/11) Exercise 1(a)
Since $\lambda_1 = 1$ and $\lambda_2 = -1$:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (-1-\lambda)(1-\lambda) = \lambda^2 - 1.$$

□

(Grupo E curso 10/11) Exercise 1(b)
Trace $(\lambda_1 + \lambda_2)$ must be equal to 0; therefore $b = -2$. In addition $\det \mathbf{A} = \lambda_1 \cdot \lambda_2 = -1$, so $-4 - a = -1$, or $a = -3$. Then

$$\mathbf{A} = \begin{bmatrix}2 & -3\\ 1 & -2\end{bmatrix}$$

□

(Grupo E curso 10/11) Exercise 1(c)
For $\lambda_1 = 1$

$$\mathbf{A} - \mathbf{I} = \begin{bmatrix}1 & -3\\ 1 & -3\end{bmatrix} \qquad \text{with eigenvector} \qquad \boldsymbol{x}_1 = \begin{pmatrix}3\\ 1\end{pmatrix}$$

For $\lambda_2 = -1$

$$\mathbf{A} - \mathbf{I} = \begin{bmatrix}3 & -3\\ 1 & -1\end{bmatrix} \qquad \text{with eigenvector} \qquad \boldsymbol{x}_2 = \begin{pmatrix}1\\ 1\end{pmatrix}$$

Therefore

$$\mathbf{S} = \begin{bmatrix}3 & 1\\ 1 & 1\end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix}1 & \\ & -1\end{bmatrix}.$$

□

(Grupo E curso 10/11) Exercise 1(d)
Since, in this case, $\mathbf{D}^{101} = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}^{101} = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}^{\text{odd number}} = \mathbf{D}$

$$\mathbf{A}^{101} = \mathbf{S}\mathbf{D}^{101}\mathbf{S}^{-1} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1} = \mathbf{A}.$$

□

(Grupo E curso 10/11) Exercise 1(e)
Suppose $\boldsymbol{z} = c\boldsymbol{x} + d\boldsymbol{y} = \mathbf{0}$. Then $\mathbf{A}\boldsymbol{z} = c\mathbf{A}\boldsymbol{x} + d\mathbf{A}\boldsymbol{y} = c\boldsymbol{x} - d\boldsymbol{y} = \mathbf{0}$. Since $\mathbf{A}\boldsymbol{z} = \mathbf{A}\mathbf{0} = \mathbf{0}$. Therefore

$$\begin{cases} c\boldsymbol{x} + d\boldsymbol{y} = & \mathbf{0} \\ c\boldsymbol{x} - d\boldsymbol{y} = & \mathbf{0}. \end{cases} \quad \text{but, since} \quad \boldsymbol{x} \neq \mathbf{0} \text{ and } \boldsymbol{y} \neq \mathbf{0} \Longrightarrow \quad \text{the only possibility is} \quad c = d = 0.$$

□

(Grupo E curso 10/11) Exercise 2(a)
No. $\mathbf{A}\left[\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}\right] = \mathbf{0}$. So $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ is in the nullspace of $\mathbf{A}$.

□

(Grupo E curso 10/11) Exercise 2(b)
No. From part (a), $\dim(\mathcal{N}\left(\mathbf{A}\right)) > 0$.

□

(Grupo E curso 10/11) Exercise 2(c)
Yes because the eigenvectors of a symmetric matrix are linearly independent (¡all symmetric matrices are diagonalizable!).

□

(Grupo E curso 10/11) Exercise 2(d)
$\mathbf{A}\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ gives 2 times the first column and $\mathbf{A}\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ gives -1 times the second column of $\mathbf{A}$.
By the symmetry condition (iii), we get $a_{13} = a_{31}$ and $a_{23} = a_{32}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -3 & -6 & a_{33} \end{bmatrix}$$

For a_{33} , we know that $\operatorname{tr}\left(A\right) = 1 + 4 + a_{33} = 0$ so $a_{33} = -5$.

□

(Grupo E curso 10/11) Exercise 3(a)
True. Since the matrix is not full rank, $\dim \mathcal{N}\left(\mathbf{A}\right) > 0$.

□

(Grupo E curso 10/11) Exercise 3(b)
False. Since the matrix is not full rank, $\mathcal{C}\left(\mathbf{A}\right)$ is smaller than $\mathbb{R}^3$, that is, there are some $\boldsymbol{b}$ in $\mathbb{R}^3$ that do not belong to $\mathcal{C}\left(\mathbf{A}\right)$.

□

(Grupo E curso 10/11) Exercise 3(c)
False. For example

$$\mathbf{A} = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 0 \end{bmatrix}; \qquad \mathbf{B} = \mathbf{I}; \quad \text{and then} \quad \det\left(\mathbf{A} + \mathbf{B}\right) = \begin{vmatrix} 0 & & \\ & 0 & \\ & & 1 \end{vmatrix} = 0 \neq \det \mathbf{B} = 1.$$

□

(Grupo E curso 10/11) Exercise 3(d)
False. $\det(\mathbf{A}\mathbf{B}) = \det \mathbf{A} \cdot \det \mathbf{B} = 0 \cdot \det \mathbf{B} = 0$.

□

(Grupo E curso 10/11) Exercise 3(e)

True. Since the matrix is not full rank, there are some $\boldsymbol{b}$ in $\mathbb{R}^3$ that do not belong to $\mathcal{C}\left(\mathbf{A}\right)$; therefore there are linearly independent vectors $\boldsymbol{b}$ in $\mathbb{R}^3$, such as $\operatorname{rg}\left([\mathbf{A}|\boldsymbol{b}]\right) = 3$.

□

(Grupo E curso 10/11) Exercise 4(a)

$$\begin{vmatrix}1 & 0 & 1\\0 & 0 & -1\\0 & -1 & 1\end{vmatrix} = 1\cdot\begin{vmatrix}0 & -1\\-1 & 1\end{vmatrix} = -1$$

$$\begin{vmatrix}1&0&0&-1&0\\0&1&0&1&-1\\0&1&2&-1&1\\0&2&2&0&0\\0&0&0&0&1\end{vmatrix} = 1\begin{vmatrix}1&0&1&-1\\1&2&-1&1\\2&2&0&0\\0&0&0&1\end{vmatrix} + 1\begin{vmatrix}0&1&0&-1\\0&1&2&1\\0&2&2&0\\0&0&0&1\end{vmatrix} = 1\cdot 1\cdot\begin{vmatrix}1&0&1\\1&2&-1\\2&2&0\end{vmatrix} + 0 = 0$$

$$\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&0&-1&0&1&0\\0&-1&1&0&0&1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&-1&1&0&0&1\\0&0&-1&0&1&0\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&-1&0&0&-1\\0&0&1&0&-1&0\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&0&0&1&1&0\\0&1&0&0&-1&-1\\0&0&1&0&-1&0\end{array}\right]$$

□

(Grupo E curso 10/11) Exercise 4(b)

$\det\left(\mathbf{A}\mathbf{A}^{\intercal}\right) = \det\mathbf{A}\cdot\det\mathbf{A}^{\intercal} = \det\mathbf{A}\cdot\det\mathbf{A} = 1$

□

(Grupo E curso 10/11) Exercise 4(b)

$\mathbf{B}^4\mathbf{A}$ is not defined.

□

(Grupo E curso 10/11) Exercise 4(b)

$\det\left(\mathbf{A}^{-1}\right) = \frac{1}{\det\mathbf{A}} = -1$

□

(Grupo G curso 10/11) Exercise 1(a)

$$\begin{bmatrix}1 & a\\2 & b\end{bmatrix}\begin{pmatrix}-2\\1\end{pmatrix} = \begin{pmatrix}-10\\5\end{pmatrix} \Longrightarrow \begin{cases}-2+a &= -10\\-4+b &= 5\end{cases} \Longrightarrow \begin{cases}a &= -8\\b &= 9\end{cases} \Longrightarrow \quad \mathbf{A} = \begin{bmatrix}1 & -8\\2 & 9\end{bmatrix}$$

□

(Grupo G curso 10/11) Exercise 1(b)

La suma de los autovalores ($\lambda_1 + \lambda_2$) debe ser igual a la traza de la matriz (10), por tanto $\lambda_2 = 5$.

□

(Grupo G curso 10/11) Exercise 1(c)

La matriz no es simétrica, veamos si es diagoonalizable;

$$\mathbf{A} - 5\mathbf{I} = \begin{bmatrix}-4 & -8\\2 & 4\end{bmatrix}$$

que es de rango 1. Así pues $\dim\mathcal{N}(\mathbf{A} - 5\mathbf{I}) = 1$, y entonces sólo podemos encontrar un autovector linealmente independiente para el autovalor $\lambda = 5$ (de multiplicidad 2): por tanto la matriz no es diagonalizable.

□

(Grupo G curso 10/11) Exercise 1(d)

$$f(x,y) = (x, \quad y,)\begin{bmatrix}1 & -8\\2 & 9\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} = (x, \quad y,)\begin{bmatrix}x-8y\\2x+9y\end{bmatrix} = x^2 - 8yx + 2xy + 9y^2 = x^2 - 6xy + 9y^2.$$

La matriz asociada a esta forma cuadrática es

$$\mathbf{B} = \begin{bmatrix}1 & -3\\-3 & 9\end{bmatrix};$$

que es singular, y por tanto sus autovalores son $\lambda_1 = 0$ y $\lambda_2 = 10$ (la suma debe ser igual a la traza). Así pues, es *semi*-definida positiva.

□

(Grupo G curso 10/11) Exercise 1(e)
No, sólo prodría serlo si la forma cuadrática fuera definida positiva.

□

(Grupo G curso 10/11) Exercise 1(f)
Un valle. En ciertas direcciones la función crece, pero en la dirección del autovector correspondiente al autovalor cero ($\boldsymbol{x} = \begin{pmatrix}3, & 1,\end{pmatrix}$), la función es siempre cero.

□

(Grupo G curso 10/11) Exercise 2(a)

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 3 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{R}$$

Por tanto el rango es dos, y dos es el máximo número de columnas linealmente independientes.

□

(Grupo G curso 10/11) Exercise 2(b)
La dimensión es dos (el número de columnas libres). Las dos soluciones especiales constituyen una base de $\mathcal{N}\left(\mathbf{A}\right)$:

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad \begin{pmatrix} -1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$$

□

(Grupo G curso 10/11) Exercise 2(c)

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + a \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} -1 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

□

(Grupo G curso 10/11) Exercise 3.
If $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has no solution, the column space of $\mathbf{A}$ must have dimension less than m. The rank is $r < m$. Since $\mathbf{A}^\intercal\boldsymbol{y} = \boldsymbol{c}$ has exactly one solution, the columns of $\mathbf{A}^\intercal$ are independent. This means that the rank of $\mathbf{A}^\intercal$ is $r = m$. This contradiction proves that we cannot find $\mathbf{A}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

□

(Grupo G curso 10/11) Exercise 4.
Entonces

$$\mathbf{A}(\boldsymbol{v} - \boldsymbol{w}) = \mathbf{A}\boldsymbol{v} - \mathbf{A}\boldsymbol{w} = \mathbf{0}$$

y por tanto el vector diferencia $(\boldsymbol{v} - \boldsymbol{w})$

$$\boldsymbol{v} - \boldsymbol{w} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$$

es una solución al sistema homogéneo $\mathbf{A}\boldsymbol{x} = \mathbf{0}$. Así pues

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + (\boldsymbol{v} - \boldsymbol{w}) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$$

Es otra solución.

□

(Grupo G curso 10/11) Exercise 5(a)

$\det \mathbf{A} = -4$

□

(Grupo G curso 10/11) Exercise 5(b)
$\det \mathbf{A}\mathbf{A}^{\mathsf{T}} = \det \mathbf{A} \cdot \det \mathbf{A}^{\mathsf{T}} = \det \mathbf{A} \cdot \det \mathbf{A} = (\det \mathbf{A})^2 = (-4)^2 = 16$

□

(Grupo G curso 10/11) Exercise 5(c)
$\det \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} = \frac{-1}{4}$

□

(Grupo G curso 10/11) Exercise 5(d)
$\det 2\mathbf{A} = 2^n \det \mathbf{A} = 2^5 \det \mathbf{A} = 32 * (-4) = -128$

□

(Grupo G curso 10/11) Exercise 5(e)
$2(-\det \mathbf{A}) = 8.$

□

(Grupo F curso 09/10) Exercise 1.

$$\det(-\mathbf{A}^{\mathsf{T}}) = \det(-\mathbf{A}) = (-1)^n \mathbf{A} = \det(\mathbf{A})$$

since n is an even number.
(*MIT Course 18.06 Quiz 2, Fall, 2008*)

□

(Grupo F curso 09/10) Exercise 2(a)
Since $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^3$, $\mathbf{A}$ has 3 columns.

□

(Grupo F curso 09/10) Exercise 2(b)
Any number greater (or equal) than one.

□

(Grupo F curso 09/10) Exercise 2(c)
Three columns and two special solutions (2 free columns) means rank 1 (only one pivot column).

□