Mathematics II

Marcos Bujosa

Universidad Complutense de Madrid

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Highlights of Lesson 1

- Vector and matrix operations
  - Addition and scalar multiplication
  - Some properties of these operations

2 Vectors in $\mathbb{R}^n$

Vector in $\mathbb{R}^n$ is an ordered list of n real numbers

Example

$\boldsymbol{v} \in \mathbb{R}^3$: first component: 5, the second: 1 and the third: 10

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Example

$\boldsymbol{v} \in \mathbb{R}^3$: first component: 5, the second: 1 and the third: 10

$$\boldsymbol{v} = \begin{cases} v_1 = 5 \\ v_2 = 1 \\ v_3 = 10 \end{cases} \quad ; \qquad \boldsymbol{v} = \begin{pmatrix} 5 \\ 1 \\ 10 \end{pmatrix} = \begin{pmatrix} 5, & 1, & 10, \end{pmatrix}.$$

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$\boldsymbol{v} \in \mathbb{R}^3$: first component: 5, the second: 1 and the third: 10

$$\boldsymbol{v} = \begin{cases} v_1 = 5 \\ v_2 = 1 \\ v_3 = 10 \end{cases} ; \qquad \boldsymbol{v} = \begin{pmatrix} 5 \\ 1 \\ 10 \end{pmatrix} = (5, \quad 1, \quad 10,) \,.$$

Notation

- $\boldsymbol{a}$, $\boldsymbol{x}$, $\mathbf{0}$
- $\mathrm{elem}_3(\boldsymbol{v}) \equiv {}_{3|}\boldsymbol{v} \equiv \boldsymbol{v}_{|3} \equiv v_3 = 10$

A parenthesis around a list of numbers denotes a vector.

3 Basic operations with vectors

Vector addition: $\left(\boldsymbol{a}+\boldsymbol{b}\right)_{|i} = \boldsymbol{a}_{|i} + \boldsymbol{b}_{|i}$

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad \text{add to} \quad \boldsymbol{a}+\boldsymbol{b} = \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \end{pmatrix}.$$

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Scalar multiplication: $(\lambda\boldsymbol{a})_{|i} = \lambda(\boldsymbol{a}_{|i})$

$$2\boldsymbol{a} = \begin{pmatrix} 2a_1 \\ 2a_2 \end{pmatrix} \quad \text{and} \quad (-1)\boldsymbol{a} = \begin{pmatrix} -a_1 \\ -a_2 \end{pmatrix} \equiv -\boldsymbol{a}$$

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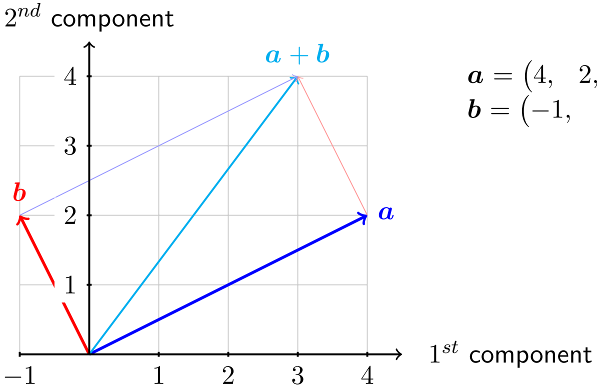
(Hence, the operator " $|i$ " is linear)

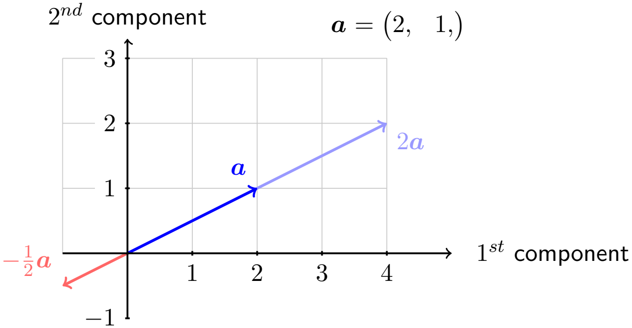
$\boldsymbol{a}$ and $\boldsymbol{b}$ (with n components) are equal when:
$\boldsymbol{a}_{|i} = \boldsymbol{b}_{|i}, \quad i = 1:n.$

4 Vector addition



$$\boldsymbol{a} = \big(4, \quad 2,\big)$$
$$\boldsymbol{b} = \big(-1, \quad 2,\big)$$

5 Scalar multiplication



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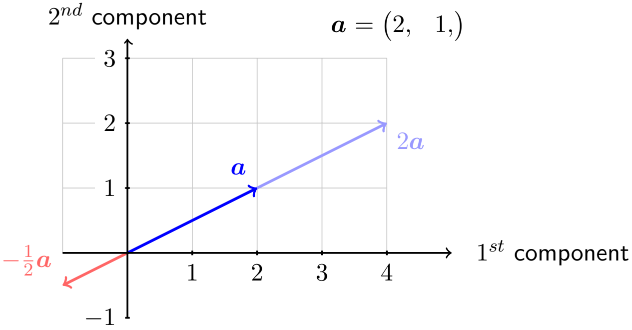

What is the picture of all multiples of $\boldsymbol{a}$?

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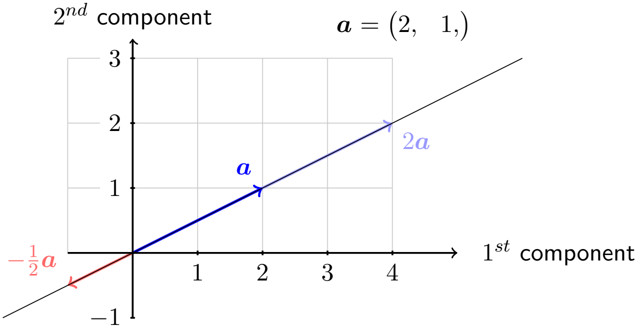

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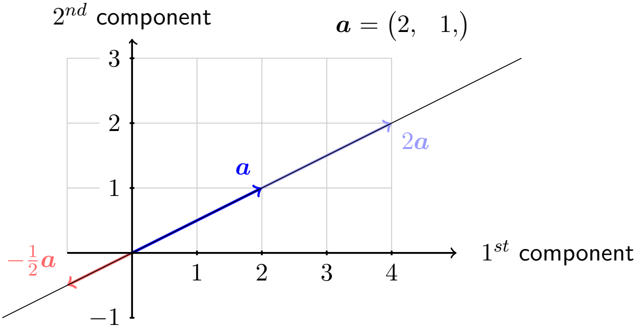

What is the picture of all multiples of $\boldsymbol{a}$?

Is $\mathbf{0}$ a multiple of $\boldsymbol{a}$?

6 Addition and scalar multiplication

$$\boxed{(\boldsymbol{a}+\boldsymbol{b})_{|i} = \boldsymbol{a}_{|i} + \boldsymbol{b}_{|i}} \qquad \boxed{(\lambda \boldsymbol{a})_{|i} = \lambda(\boldsymbol{a}_{|i})}$$

Let us recall some properties of scalars

Scalars

1. $a + b = b + a$
2. $a + (b + c) = (a + b) + c$
3. $a + 0 = a$
4. $a + (-a) = 0$
5. $ab = ba$
6. $a(b + c) = ab + ac$
7. $a(bc) = (ab)c$
8. $1a = a$

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5. $\lambda(\boldsymbol{a}+\boldsymbol{b})=\lambda\boldsymbol{a}+\lambda\boldsymbol{b}$
6. $(\lambda+\eta)\boldsymbol{a}=\lambda\boldsymbol{a}+\eta\boldsymbol{a}$
7. $\lambda(\eta\boldsymbol{a})=(\lambda\eta)\boldsymbol{a}$
8. $1\boldsymbol{a}=\boldsymbol{a}$

7 Matrices

Matrix in $\mathbb{R}^{m\times n}$ is an ordered list of n vectors in $\mathbb{R}^m$

Example

Three vectors in $\mathbb{R}^2$: $\boldsymbol{a} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 0 \\ 7 \end{pmatrix}$

$$\mathbf{A} = [\boldsymbol{a};\ \ \boldsymbol{b};\ \ \boldsymbol{c};] = \begin{bmatrix} 4 & -1 & 0 \\ 2 & 2 & 7 \end{bmatrix} \neq [\boldsymbol{c};\ \ \boldsymbol{b};\ \ \boldsymbol{a};]$$

A bracket around a vector list denotes a matrix.

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Two vectors in $\mathbb{R}^3$: $\boldsymbol{x} = (4, \quad -1, \quad 0,)$ and $\boldsymbol{y} = (2, \quad 2, \quad 7,)$

$$\mathbf{B} = [\boldsymbol{x}; \quad \boldsymbol{y};]$$

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$$\mathbf{B} = [\boldsymbol{x}; \ \boldsymbol{y};]$$

Notation

- $\mathbf{A}, \ \mathbf{B}, \ \mathbf{0}$
- $\underset{2\times 3}{\mathbf{A}}, \ \underset{3\times 2}{\mathbf{B}};$ $\qquad \underset{2\times 3}{\mathbf{A}} \neq \underset{3\times 2}{\mathbf{B}}$

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8 More notation

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 7 & 0 & 3 \end{bmatrix}$$

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Picking operators

- $\mathrm{elem}_{21}(\mathbf{A}) \;=\; {}_{2|}\mathbf{A}_{|1} \;=\; a_{21}: \qquad 7$

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Picking operators

- $\text{elem}_{21}(\mathbf{A}) \;=\; {}_{2|}\mathbf{A}_{|1} \;=\; a_{21} : \qquad 7$
- $\text{fila}_1\,(\mathbf{A}) \qquad = {}_{1|}\mathbf{A} \quad : \qquad (1, \quad 2, \quad 1,) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

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Picking operators

- $\text{elem}_{21}(\mathbf{A}) = {}_{2|}\mathbf{A}_{|1} = a_{21} : \quad 7$
- $\text{fila}_1(\mathbf{A}) = {}_{1|}\mathbf{A} : \quad (1, \; 2, \; 1,) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$
- $\text{col}_1(\mathbf{A}) = \mathbf{A}_{|1} : \quad \begin{pmatrix} 1 \\ 7 \end{pmatrix} = (1, \; 7,)$

9 Basic operations with matrices

Matrix addition: $\left(\mathbf{A}+\mathbf{B}\right)_{|j} = \mathbf{A}_{|j} + \mathbf{B}_{|j}$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \text{ add to } \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$$

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Scalar multiplication: $\left(\lambda\mathbf{A}\right)_{|j} = \lambda\left(\mathbf{A}_{|j}\right)$

$$7\mathbf{A} = \begin{bmatrix} 7a_{11} & 7a_{12} \\ 7a_{21} & 7a_{22} \end{bmatrix} \text{ and } (-1)\mathbf{A} = \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix} = -\mathbf{A}.$$

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(Hence, the operator " $|i$ " is linear)

A and **B** (with same order) are equal when: $\mathbf{A}_{|j} = \mathbf{B}_{|j}$

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Vectors

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11 Rewriting Rules

Distributive rules

$$(\boldsymbol{a}+\boldsymbol{b})_{|i} = \boldsymbol{a}_{|i} + \boldsymbol{b}_{|i} \qquad {}_{i|}(\boldsymbol{a}+\boldsymbol{b}) = {}_{i|}\boldsymbol{a} + {}_{i|}\boldsymbol{b}$$

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In addition, if we allow $\lambda\boldsymbol{a} = \boldsymbol{a}\lambda$ and $\lambda\mathbf{A} = \mathbf{A}\lambda$, then we get

Associative rules (moving parentheses)

$$(\lambda\boldsymbol{b})_{|i} = \lambda(\boldsymbol{b}_{|i}) \qquad {}_{i|}(\boldsymbol{b}\lambda) = ({}_{i|}\boldsymbol{b})\lambda$$

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Scalar and operator interchange

$$(\boldsymbol{b}\lambda)_{|i} = (\boldsymbol{b}_{|i})\lambda \qquad {}_{i|}(\lambda\boldsymbol{b}) = \lambda({}_{i|}\boldsymbol{b})$$

$$(\mathbf{A}\lambda)_{|j} = (\mathbf{A}_{|j})\lambda \qquad {}_{i|}(\lambda\mathbf{A}) = \lambda({}_{i|}\mathbf{A})$$

Questions of the Lecture 1

You should always complete the exercises in the theoretical sections previous to each lecture

(L-1) QUESTION 1. Give 3 by 3 examples (not just the zero matrix) of:

(a) A diagonal matrix: ${}_{i|}\mathbf{A}_{|j} = 0$ if $i \neq j$.

(b) A symmetric matrix: $\mathbf{A}_{|j} = {}_{j|}\mathbf{A}$.

(c) An upper triangular matrix: ${}_{i|}\mathbf{A}_{|j} = 0$ if $i > j$.

(d) A skew-symmetric matrix: ${}_{i|}\mathbf{A}_{|j} = -{}_{j|}\mathbf{A}_{|i}$.

(Strang, 1988, exercise 7 from section 1.4.)

Highlights of Lesson 2

- Dot product
- linear combinations
- The column picture of the Geometry of linear equations

2 Dot product

$$\boldsymbol{x} \cdot \boldsymbol{y} = x_1 y_1 + x_2 y_2 + x_3 y_3 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i.$$

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Symmetric

$$\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{y} \cdot \boldsymbol{x}$$

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$$\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{y} \cdot \boldsymbol{x}$$

Linear in the first argument

$$\begin{aligned} (a\boldsymbol{x}) \cdot \boldsymbol{y} &= a(\boldsymbol{x} \cdot \boldsymbol{y}) \\ (\boldsymbol{x} + \boldsymbol{y}) \cdot \boldsymbol{z} &= \boldsymbol{x} \cdot \boldsymbol{z} + \boldsymbol{y} \cdot \boldsymbol{z} \end{aligned}$$

$\boxed{2}$ Dot product

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Positive

$$\boldsymbol{x} \cdot \boldsymbol{x} \geq 0$$

2 Dot product

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Positive

$$\boldsymbol{x} \cdot \boldsymbol{x} \geq 0$$

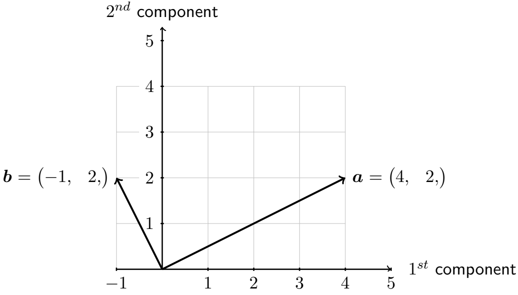
Definite

$$\boldsymbol{x} \cdot \boldsymbol{x} = 0 \Leftrightarrow \boldsymbol{x} = 0$$

3 Linear combinations

The sum of $x\boldsymbol{a}$ and $y\boldsymbol{b}$ is a linear combination of $\boldsymbol{a}$ and $\boldsymbol{b}$

$$x\boldsymbol{a}+y\boldsymbol{b} = x\begin{pmatrix}4\\2\end{pmatrix}+y\begin{pmatrix}-1\\2\end{pmatrix}$$



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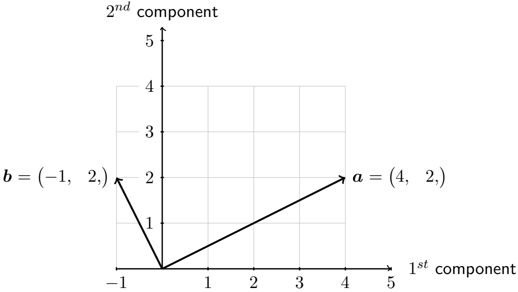
$$x\boldsymbol{a}+y\boldsymbol{b} = x\begin{pmatrix}4\\2\end{pmatrix}+y\begin{pmatrix}-1\\2\end{pmatrix} \;=\; [\boldsymbol{a};\ \boldsymbol{b};]\begin{pmatrix}x\\y\end{pmatrix} = \textbf{MATRIX}\times\boldsymbol{vector}$$



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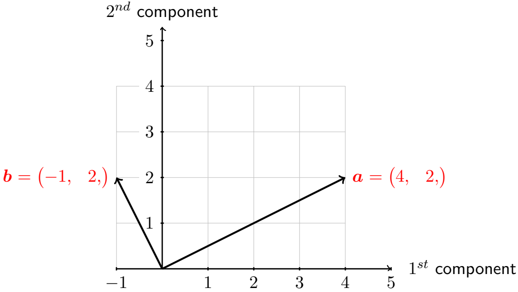

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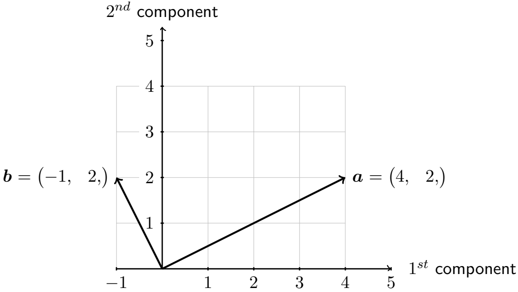

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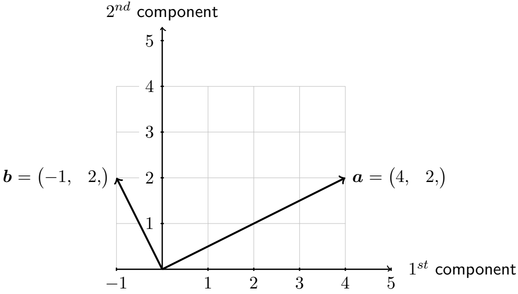
$$x\boldsymbol{a}+y\boldsymbol{b} = x\begin{pmatrix}4\\2\end{pmatrix}+y\begin{pmatrix}-1\\2\end{pmatrix} \;=\; [\boldsymbol{a};\ \boldsymbol{b};]\begin{pmatrix}x\\y\end{pmatrix} = \textbf{MATRIX}\times\boldsymbol{vector}$$



Is $\mathbf{0}$ a linear combination of $\boldsymbol{a}$ and $\boldsymbol{b}$?

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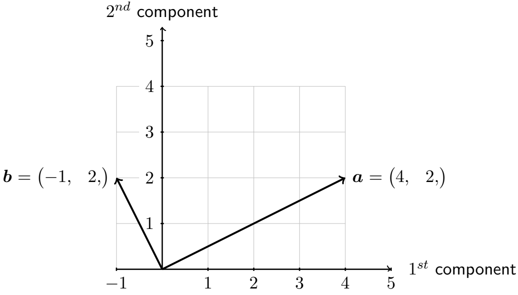
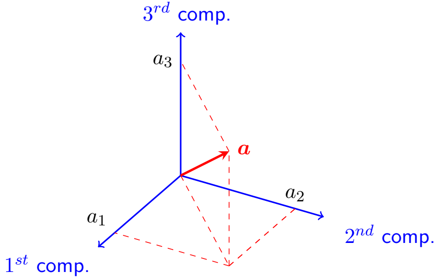


Is $\mathbf{0}$ a linear combination of $\boldsymbol{a}$ and $\boldsymbol{b}$? What is the picture of all linear combinations of $\boldsymbol{a}$ and $\boldsymbol{b}$?

4 Linear combinations in $\mathbb{R}^3$

$$\boldsymbol{a} = \big(a_1, \quad a_2, \quad a_3,\big) \ \in \ \mathbb{R}^3$$



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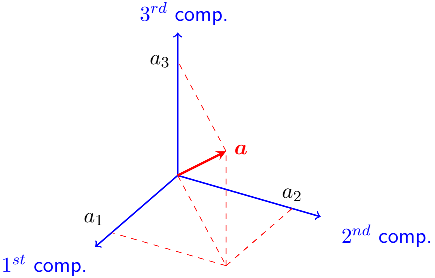

What is the picture of all multiples of $\boldsymbol{a}$?

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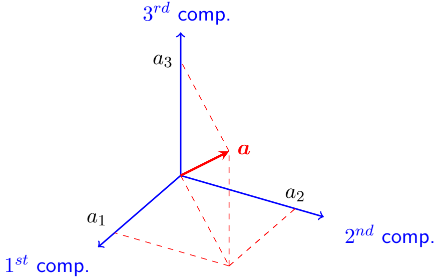

What is the picture of all multiples of $\boldsymbol{a}$?

What is the picture of all linear combinations of two vectors in $\mathbb{R}^3$?

(linear combination)

5 Matrix times vector

$$\mathbf{A}\boldsymbol{b} = b_1 \mathbf{A}_{|1} \quad + \quad b_2 \mathbf{A}_{|2} \quad + \cdots + \; b_n \mathbf{A}_{|n}$$

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Hence,

$$\boxed{{}_{i|}\left(\mathbf{A}\boldsymbol{b}\right) = \left({}_{i|}\mathbf{A}\right)\cdot\boldsymbol{b}}$$

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Hence,

$$\boxed{{}_{i|}\left(\mathbf{A}\boldsymbol{b}\right) = \left({}_{i|}\mathbf{A}\right)\cdot\boldsymbol{b}}$$

if we omit the period, we can simply write: ${}_{i|}\mathbf{A}\boldsymbol{b}$

6 Matrix times vector

If $\boldsymbol{b} \in \mathbb{R}^n$ then $\big(\underset{m\times n}{\mathbf{A}}\ \boldsymbol{b}\big) \in \mathbb{R}^m$; where ${}_{i|}\big(\mathbf{A}\boldsymbol{b}\big) = \big({}_{i|}\mathbf{A}\big) \cdot \boldsymbol{b}$

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Matrix times vector

1. $\mathbf{I}\boldsymbol{a} = \boldsymbol{a}$
2. $\mathbf{A}\big(\mathbf{I}_{|j}\big) = \mathbf{A}_{|j}$
3. $\mathbf{A}(\boldsymbol{b}+\boldsymbol{c}) = \mathbf{A}\boldsymbol{b} + \mathbf{A}\boldsymbol{c}$
4. $\mathbf{A}(\lambda\boldsymbol{b}) = \lambda(\mathbf{A}\boldsymbol{b})$
5. $\mathbf{A}(\lambda\boldsymbol{b}) = (\lambda\mathbf{A})\boldsymbol{b}$
6. $\mathbf{A}(\mathbf{B}\boldsymbol{c}) = \Big[\mathbf{A}(\mathbf{B}_{|1});\dots\quad \mathbf{A}(\mathbf{B}_{|n});\Big]\,\boldsymbol{c}$
7. $\big(\mathbf{A}+\mathbf{B}\big)\boldsymbol{c} = \mathbf{A}\boldsymbol{c} + \mathbf{B}\boldsymbol{c}$

Prove the above propositions
(follow the rewriting rules and properties of the dot product)

7 Example of linear system: 2 equations and 2 unknowns

$$\begin{cases} 2x - y = 0 \\ -x + 2y = 3 \end{cases}$$

$$=$$

$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$$

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$$\begin{cases} 2x - y = 0 \\ -x + 2y = 3 \end{cases}$$

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$$x\left(\mathbf{A}_{|1}\right) + y\left(\mathbf{A}_{|2}\right) \qquad = \; \boldsymbol{b}$$

8 Geometry of linear systems: Linear combination of columns

$$\begin{cases} 2x - y = 0 \\ -x + 2y = 3 \end{cases}$$

$$x\begin{pmatrix} \\ \\ \end{pmatrix} + y\begin{pmatrix} \\ \\ \end{pmatrix} = \begin{pmatrix} \\ \\ \end{pmatrix}$$

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Try: $x = 1; \quad y = 2.$

9 2 equations and 2 unknowns: Column picture

$$x\begin{pmatrix}2\\-1\end{pmatrix}+y\begin{pmatrix}-1\\2\end{pmatrix}=\begin{pmatrix}0\\3\end{pmatrix}$$

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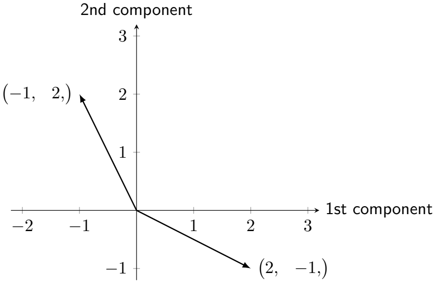
$$x\begin{pmatrix}2\\-1\end{pmatrix}+y\begin{pmatrix}-1\\2\end{pmatrix}=\begin{pmatrix}0\\3\end{pmatrix}$$

Which linear combination of $\begin{pmatrix}2\\-1\end{pmatrix}$ and $\begin{pmatrix}-1\\2\end{pmatrix}$ gives $\begin{pmatrix}0\\3\end{pmatrix}$?



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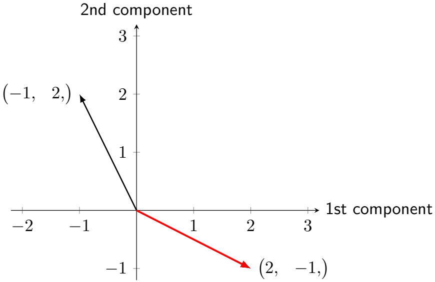

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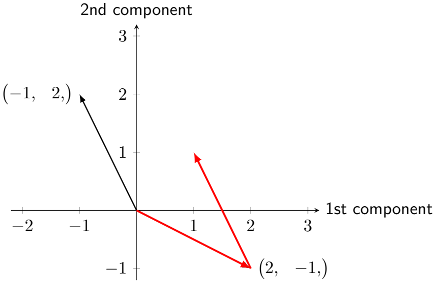

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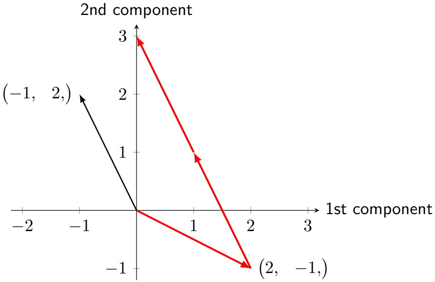

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$$1\begin{pmatrix}2\\-1\end{pmatrix}+2\begin{pmatrix}-1\\2\end{pmatrix}=\begin{pmatrix}0\\3\end{pmatrix}$$

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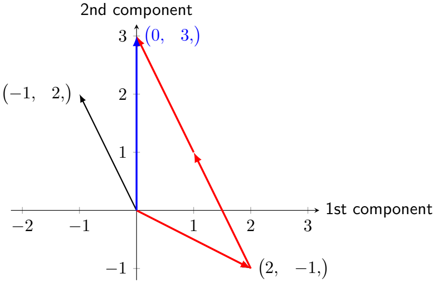

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What is the set of all possible combinations?

$\boxed{10}$ Example: 3 equations and 3 unknowns

$$\begin{cases} 2x - y \phantom{{}-4z} = 0 \\ -x + 2y - z = -1 \\ - 3y + 4z = 4 \end{cases}$$

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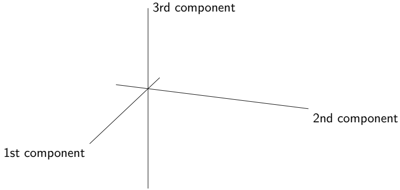
11 3 equations and 3 unknowns: Column picture

$$x \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + y \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} + z \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$$

Which linear combination of the columns gives $\boldsymbol{b}$?



$$\Big\{ x = ; \quad y = ; \quad z = \quad \Big\}$$

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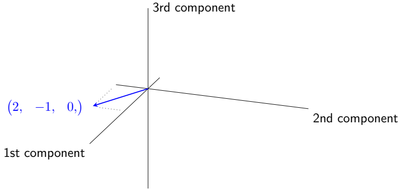

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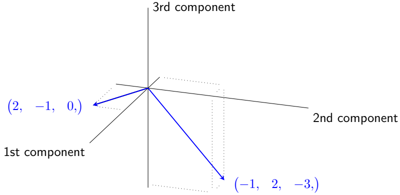

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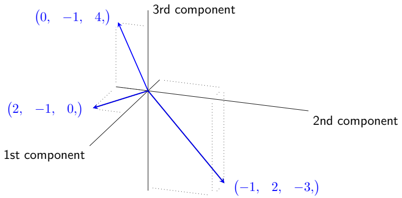

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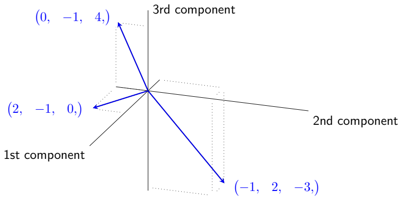

$$\left\{x =; \quad y =; \quad z = \quad \right\}$$

11 3 equations and 3 unknowns: Column picture

$$0\begin{pmatrix}2\\-1\\0\end{pmatrix}+0\begin{pmatrix}-1\\2\\-3\end{pmatrix}+1\begin{pmatrix}0\\-1\\4\end{pmatrix}=\begin{pmatrix}0\\-1\\4\end{pmatrix}$$

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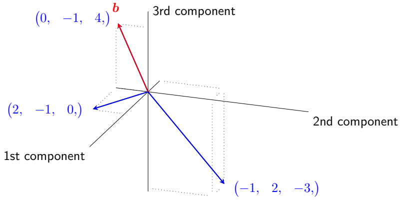


$$\left\{x=0; \quad y=0; \quad z=1 \quad \right\}$$ (Row picture)

11 3 equations and 3 unknowns: Column picture

$$x\begin{pmatrix}2\\-1\\0\end{pmatrix}+y\begin{pmatrix}-1\\2\\-3\end{pmatrix}+z\begin{pmatrix}0\\-1\\4\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$

Which linear combination of the columns gives $\boldsymbol{b}$?

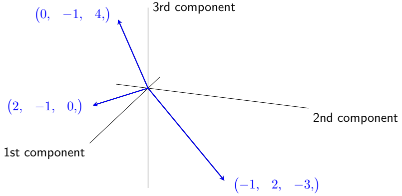
$$\left\{x=;\quad y=;\quad z=\quad\right\}$$

(Row picture)

What happens with a different $\boldsymbol{b}$?... let's see

12 3 equations and 3 unknowns: Column picture

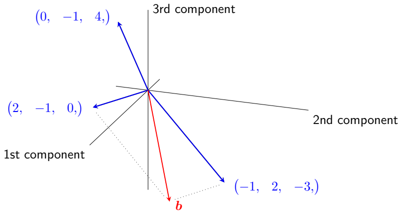
$$x\begin{pmatrix}2\\-1\\0\end{pmatrix}+y\begin{pmatrix}-1\\2\\-3\end{pmatrix}+z\begin{pmatrix}0\\-1\\4\end{pmatrix}=\begin{pmatrix} \\ \\ \end{pmatrix}$$



12 3 equations and 3 unknowns: Column picture

$$x\begin{pmatrix}2\\-1\\0\end{pmatrix}+y\begin{pmatrix}-1\\2\\-3\end{pmatrix}+z\begin{pmatrix}0\\-1\\4\end{pmatrix}=\begin{pmatrix}1\\1\\-3\end{pmatrix}$$

Which linear combination of the columns gives this new $\boldsymbol{b}$?



$$\left\{x= \quad ; \quad y= \quad ; \quad z= \quad \right\}$$

12 3 equations and 3 unknowns: Column picture

$$1\begin{pmatrix}2\\-1\\0\end{pmatrix}+1\begin{pmatrix}-1\\2\\-3\end{pmatrix}+0\begin{pmatrix}0\\-1\\4\end{pmatrix}=\begin{pmatrix}1\\1\\-3\end{pmatrix}$$

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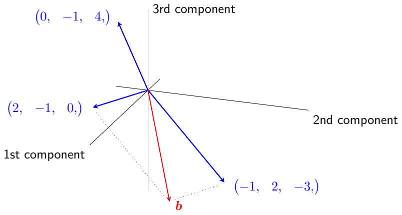

$$\left\{x=1;\quad y=1;\quad z=0\right\}$$

(rows picture)

13 What does Ax=b mean?

$\mathbf{A}x$ is a *linear combination* of columns of $\mathbf{A}$:

Example

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

13 What does Ax=b mean?

$\mathbf{A}x$ is a *linear combination* of columns of $\mathbf{A}$:

Example

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

13 What does Ax=b mean?

$\mathbf{A}x$ is a *linear combination* of columns of $\mathbf{A}$:

Example

$$\left[\begin{array}{cc} 2 & 5 \\ 1 & 3 \end{array}\right]\begin{pmatrix}1\\2\end{pmatrix} = 1\begin{pmatrix}2\\1\end{pmatrix} + 2\begin{pmatrix}5\\3\end{pmatrix} = \begin{pmatrix}12\\7\end{pmatrix}$$

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"$\mathbf{A}x = \boldsymbol{b}$" is asking for a particular linear combination:

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} ? \\ ? \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \end{pmatrix}$$

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"$\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$" is asking for a particular linear combination:

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} ? \\ ? \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \end{pmatrix}$$

To solve linear systems we will first learn to transform coeficient matrices by elimination (next lectures)

Questions of the Lecture 2

You must complete the exercises from the corresponding sections of the book

(L-2) QUESTION 1. Working a column at a time, compute the following products

(a)

$$\begin{bmatrix} 4 & 1 \\ 4 & 1 \\ 6 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

(b)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

(c)

$$\begin{bmatrix} 4 & 3 \\ 6 & 6 \\ 8 & 9 \end{bmatrix} \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix}$$

(Strang, 1988, exercise 2 from section 1.4.)

(L-2) Question 2. Can the three equations be solved simultaneously?

$$\begin{aligned} x + 2y &= 2 \\ x - y &= 2 \\ y &= 1. \end{aligned}$$

What happens if all right hand sides are zero? Is there any non-zero choice of right hand sides which allows the three equations to have a solution? How many non-zero choices have we?
(Strang, 1988, exercise 4 from section 1.2.)

(L-2) Question 3. Compute the product $\mathbf{A}\boldsymbol{x}$ with
$\mathbf{A} = \begin{bmatrix} 3 & -6 & 0 \\ 0 & 2 & -2 \\ 1 & -1 & -1 \end{bmatrix}$ and $\boldsymbol{x} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$. For this matrix $\mathbf{A}$, find a solution vector $\boldsymbol{x}$ to the system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$, with zeros on the right side of all three equations. Can you find more than one solution?
(Strang, 1988, exercise 5 from section 1.4.)

(L-2) QUESTION 4. Suppose $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has two solutions $\boldsymbol{v}$ and $\boldsymbol{w}$ (with $\boldsymbol{b} \neq \mathbf{0}$). Then show that $\frac{1}{2}(\boldsymbol{v} + \boldsymbol{w})$ is also a solution, although $\boldsymbol{v} + \boldsymbol{w}$ is not.

Hint

Use the following properties: $\mathbf{A}(\boldsymbol{b} + \boldsymbol{c}) = \mathbf{A}\boldsymbol{b} + \mathbf{A}\boldsymbol{c}$ and $\mathbf{A}(c\boldsymbol{b}) = c(\mathbf{A}\boldsymbol{b})$.

(L-2) QUESTION 5. *"It is impossible for a system of linear equations to have exactly two solutions"*. Explain why (answering the next question):

(a) If $\boldsymbol{v}$ y $\boldsymbol{w}$ are two solutions, what is another one?

(Strang, 2003, exercise 19 from section 2.2.)

(L-2) QUESTION 6. Draw $\boldsymbol{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, along with $\boldsymbol{v} + \boldsymbol{w}$, $2\boldsymbol{v} + \boldsymbol{w}$, and $\boldsymbol{v} - \boldsymbol{w}$ in a plane (first component on the horizontal axis and second component on the vertical axis).

(L-2) QUESTION 7. draw the column picture of the following system with solution $x = 3$ and $y = -1$.

$$\begin{cases} 2x + y = 5 \\ x - 3y = 6 \end{cases}$$

(L-2) QUESTION 8. draw the column picture of the following system.

$$\begin{cases} 2x - y = 3 \\ x + y = 1 \end{cases} \quad ; \qquad \left(\text{the solution is :} \qquad x = 1 + \frac{1}{3}, \quad y = -\frac{1}{3}\right).$$

No deje de hacer los ejercicios del libro.

Highlights of Lesson 3

- Matrix multiplication: $\left(\mathbf{AB}\right)_{|j} = \mathbf{A}\left(\mathbf{B}_{|j}\right)$
  - Properties
- Transpose of a matrix
- $\mathbf{A}x$ and $x\mathbf{A}$ (linear combinations)
- Other ways to compute the product
- Transpose of $\mathbf{AB}$

$\boxed{2}$ Matrix multiplication (by **columns**)

Column j of $\underset{m\times p}{\mathbf{A}}$ times $\underset{p\times n}{\mathbf{B}}$ is:

$$\boxed{(\mathbf{AB})_{|j} = \mathbf{A}(\mathbf{B}_{|j})}$$

Each column of **AB** is a linear combination of the p columns of **A**

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Each column of **AB** is a linear combination of the p columns of **A**

Example

$$\begin{bmatrix} 2 & 1 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{A}(\mathbf{B}_{|1}); & \mathbf{A}(\mathbf{B}_{|2}); \end{bmatrix}$$

$$= \left[\begin{bmatrix} 2 & 1 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \begin{bmatrix} 2 & 1 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{pmatrix} 6 \\ 0 \end{pmatrix}; \right] = \begin{bmatrix} 3 & 12 \\ 11 & 18 \\ 13 & 24 \end{bmatrix}$$

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Example

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3 Matrix multiplication properties

$$\textbf{MATRIX} \times \textbf{MATRIX} = \textbf{MATRIX}$$

1. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ remember $\mathbf{A}(\mathbf{B}\boldsymbol{c}) = \left[\mathbf{A}\left(\mathbf{B}_{|1}\right); \ldots\ \mathbf{A}\left(\mathbf{B}_{|n}\right);\right] \boldsymbol{c}$
2. $\left(\mathbf{A} + \mathbf{B}\right)\mathbf{C} = \mathbf{AC} + \mathbf{BC}$.
3. $\mathbf{A}\left(\mathbf{B} + \mathbf{C}\right) = \mathbf{AB} + \mathbf{AC}$.
4. $\mathbf{A}(\lambda\mathbf{B}) = (\lambda\mathbf{A})\mathbf{B} = \lambda(\mathbf{AB})$.
5. $\mathbf{IA} = \mathbf{A}$.
6. $\mathbf{AI} = \mathbf{A}$.

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MATRIX $\times$ MATRIX $=$ MATRIX

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4 Transposing a matrix

Transpose

$$(\text{column } i \text{ of } \mathbf{A}^{\mathsf{T}}) = (\text{row } i \text{ of } \mathbf{A}) \quad \leftrightarrow \quad (\mathbf{A}^{\mathsf{T}})_{|i} = {}_{i|}\mathbf{A}$$

4 Transposing a matrix

Transpose

$$(\text{column } i \text{ of } \mathbf{A}^\intercal) = (\text{row } i \text{ of } \mathbf{A}) \quad \leftrightarrow \quad (\mathbf{A}^\intercal)_{|i} = {}_{i|}\mathbf{A}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 4 & 1 \end{bmatrix}; \quad \mathbf{A}^\intercal = \begin{bmatrix} 1 & & \\ 3 & & \end{bmatrix}$$

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$${}_{i|}\mathbf{A}_{|j} = {}_{j|}(\mathbf{A}^\intercal)_{|i}; \qquad \left(\mathbf{A}^\intercal\right)^\intercal = \mathbf{A}; \qquad {}_{j|}(\mathbf{A}^\intercal) = \mathbf{A}_{|j}$$

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Symmetric matrices $\quad \mathbf{A}^\intercal = \mathbf{A}$

$$\begin{bmatrix} 3 & 1 & 7 \\ & 2 & 9 \\ & & 1 \end{bmatrix}$$

4 Transposing a matrix

Transpose

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Symmetric matrices $\quad \mathbf{A}^\intercal = \mathbf{A}$

$$\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 9 \\ & & 1 \end{bmatrix}$$

4 Transposing a matrix

Transpose

$$(\text{column } i \text{ of } \mathbf{A}^\top) = (\text{row } i \text{ of } \mathbf{A}) \quad \leftrightarrow \quad (\mathbf{A}^\top)_{|i} = {}_{i|}\mathbf{A}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 4 & 1 \end{bmatrix}; \quad \mathbf{A}^\top = \begin{bmatrix} 1 & 1 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

$${}_{i|}\mathbf{A}_{|j} = {}_{j|}(\mathbf{A}^\top)_{|i}; \qquad \left(\mathbf{A}^\top\right)^\top = \mathbf{A}; \qquad {}_{j|}(\mathbf{A}^\top) = \mathbf{A}_{|j}$$

Symmetric matrices $\quad \mathbf{A}^\top = \mathbf{A}$

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$\boxed{4}$ Transposing a matrix

Transpose

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Symmetric matrices $\quad \mathbf{A}^\intercal = \mathbf{A}$

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5 Vectors, row matrices, column matrices

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$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}; \quad {}_{2|}\mathbf{A} = (2,\quad 3,) = \begin{pmatrix} 2 \\ 3 \end{pmatrix};$$

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When writting vectors between “square brackets” we get a matrix whose columns are those vectors

$$\left[{}_{3|}\mathbf{A}; \quad {}_{1|}\mathbf{A};\right] = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix};$$

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$$\Big[{}_{3|}\mathbf{A};\quad {}_{1|}\mathbf{A};\Big] = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}; \qquad \mathbf{A}^{\intercal} = \Big[{}_{1|}\mathbf{A};\quad {}_{2|}\mathbf{A};\quad {}_{3|}\mathbf{A};\Big]$$

6 Linear combination of rows and columns

Linear combination of columns

$$\begin{bmatrix} \diamondsuit & \clubsuit \\ \heartsuit & \spadesuit \\ \diamondsuit & \clubsuit \end{bmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} =$$

6 Linear combination of rows and columns

Linear combination of columns

$$\begin{bmatrix} \diamondsuit & \clubsuit \\ \heartsuit & \spadesuit \\ \diamondsuit & \clubsuit \end{bmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3 \begin{pmatrix} \diamondsuit \\ \heartsuit \\ \diamondsuit \end{pmatrix} + 4 \begin{pmatrix} \clubsuit \\ \spadesuit \\ \clubsuit \end{pmatrix}$$

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MATRIX $\times$ $\boldsymbol{vector} = \boldsymbol{vector}$

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MATRIX $\times$ *vector* $=$ *vector*

Linear combination of rows

$$(1, \quad 2, \quad 7,) \begin{bmatrix} \diamondsuit & \clubsuit \\ \heartsuit & \spadesuit \\ \diamondsuit & \clubsuit \end{bmatrix} =$$

6 Linear combination of rows and columns

Linear combination of columns

$$\begin{bmatrix} \diamondsuit & \clubsuit \\ \heartsuit & \spadesuit \\ \diamondsuit & \clubsuit \end{bmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3 \begin{pmatrix} \diamondsuit \\ \heartsuit \\ \diamondsuit \end{pmatrix} + 4 \begin{pmatrix} \clubsuit \\ \spadesuit \\ \clubsuit \end{pmatrix}$$

$$\textbf{MATRIX} \times \textit{vector} = \textit{vector}$$

Linear combination of rows

$$\begin{pmatrix} 1, & 2, & 7, \end{pmatrix} \begin{bmatrix} \diamondsuit & \clubsuit \\ \heartsuit & \spadesuit \\ \diamondsuit & \clubsuit \end{bmatrix} = 1 \begin{pmatrix} \diamondsuit, & \clubsuit, \end{pmatrix} + 2 \begin{pmatrix} \heartsuit, & \spadesuit, \end{pmatrix} + 7 \begin{pmatrix} \diamondsuit, & \clubsuit, \end{pmatrix}$$

6 Linear combination of rows and columns

Linear combination of columns

$$\begin{bmatrix} \diamondsuit & \clubsuit \\ \heartsuit & \spadesuit \\ \diamondsuit & \clubsuit \end{bmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3 \begin{pmatrix} \diamondsuit \\ \heartsuit \\ \diamondsuit \end{pmatrix} + 4 \begin{pmatrix} \clubsuit \\ \spadesuit \\ \clubsuit \end{pmatrix}$$

MATRIX $\times$ $\boldsymbol{vector} = \boldsymbol{vector}$

Linear combination of rows

$$\begin{pmatrix} 1, & 2, & 7, \end{pmatrix} \begin{bmatrix} \diamondsuit & \clubsuit \\ \heartsuit & \spadesuit \\ \diamondsuit & \clubsuit \end{bmatrix} = 1 \begin{pmatrix} \diamondsuit, & \clubsuit, \end{pmatrix} + 2 \begin{pmatrix} \heartsuit, & \spadesuit, \end{pmatrix} + 7 \begin{pmatrix} \diamondsuit, & \clubsuit, \end{pmatrix}$$

$\boldsymbol{vector}$ $\times$ **MATRIX** $= \boldsymbol{vector}$

6 Linear combination of rows and columns

Linear combination of columns

$$\begin{bmatrix} \diamondsuit & \clubsuit \\ \heartsuit & \spadesuit \\ \diamondsuit & \clubsuit \end{bmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3 \begin{pmatrix} \diamondsuit \\ \heartsuit \\ \diamondsuit \end{pmatrix} + 4 \begin{pmatrix} \clubsuit \\ \spadesuit \\ \clubsuit \end{pmatrix}$$

MATRIX $\times$ $\boldsymbol{vector} = \boldsymbol{vector}$

Linear combination of rows

$$(1, \quad 2, \quad 7,) \begin{bmatrix} \diamondsuit & \clubsuit \\ \heartsuit & \spadesuit \\ \diamondsuit & \clubsuit \end{bmatrix} = 1\,(\diamondsuit, \quad \clubsuit,) + 2\,(\heartsuit, \quad \spadesuit,) + 7\,(\diamondsuit, \quad \clubsuit,)$$

$\boldsymbol{vector}$ $\times$ **MATRIX** $= \boldsymbol{vector}$

Linear combinations $\longrightarrow$ $\boxed{\boldsymbol{a}\mathbf{B} = (\mathbf{B}^{\intercal})\boldsymbol{a}}$

7 Vector times matrix

Remember that $\quad {}_{i|}(\mathbf{A}\boldsymbol{b}) = ({}_{i|}\mathbf{A}) \cdot \boldsymbol{b}; \quad$ hence

$$(\boldsymbol{a}\mathbf{B})_{|j} = {}_{j|}(\boldsymbol{a}\mathbf{B}) = {}_{j|}\Big((\mathbf{B}^\top)\boldsymbol{a}\Big) = \Big({}_{j|}(\mathbf{B}^\top)\Big) \cdot \boldsymbol{a} = (\mathbf{B}_{|j}) \cdot \boldsymbol{a} = \boldsymbol{a} \cdot (\mathbf{B}_{|j})$$

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Remember that $\quad {}_{i|}(\mathbf{A}\boldsymbol{b}) = ({}_{i|}\mathbf{A}) \cdot \boldsymbol{b}; \quad$ hence

$$(\boldsymbol{a}\mathbf{B})_{|j} = {}_{j|}(\boldsymbol{a}\mathbf{B}) = {}_{j|}\Big((\mathbf{B}^{\mathsf{T}})\boldsymbol{a}\Big) = \Big({}_{j|}(\mathbf{B}^{\mathsf{T}})\Big) \cdot \boldsymbol{a} = (\mathbf{B}_{|j}) \cdot \boldsymbol{a} = \boldsymbol{a} \cdot (\mathbf{B}_{|j})$$

Rewriting rules

$$\boxed{{}_{i|}(\mathbf{A}\boldsymbol{b}) = ({}_{i|}\mathbf{A}) \cdot \boldsymbol{b}} \qquad \text{and} \qquad \boxed{(\boldsymbol{a}\mathbf{B})_{|j} = \boldsymbol{a} \cdot (\mathbf{B}_{|j})}$$

7 Vector times matrix

Remember that $\quad {}_{i|}\big(\mathbf{A}b\big) = \big({}_{i|}\mathbf{A}\big)\cdot b; \quad$ hence

$$\big(a\mathbf{B}\big)_{|j} = {}_{j|}\big(a\mathbf{B}\big) = {}_{j|}\Big((\mathbf{B}^\top)a\Big) = \Big({}_{j|}(\mathbf{B}^\top)\Big)\cdot a = \big(\mathbf{B}_{|j}\big)\cdot a = a\cdot\big(\mathbf{B}_{|j}\big)$$

Rewriting rules

$$\boxed{{}_{i|}\big(\mathbf{A}b\big) = \big({}_{i|}\mathbf{A}\big)\cdot b} \qquad \text{and} \qquad \boxed{\big(a\mathbf{B}\big)_{|j} = a\cdot\big(\mathbf{B}_{|j}\big)}$$

Thus, if we omit the period, we can simply write:

$${}_{i|}\mathbf{A}b \qquad \text{and} \qquad a\mathbf{B}_{|j}$$

8 Matrix multiplication: **rows times columns**

Consider $\underset{m\times p}{\mathbf{A}}$ and $\underset{p\times n}{\mathbf{B}}$, then:

$$\boxed{{}_{i|}\big(\mathbf{AB}\big)_{|j} = \big({}_{i|}\mathbf{A}\big)\cdot\big(\mathbf{B}_{|j}\big)}$$

Proof.

Remember that $\quad {}_{i|}\big(\mathbf{A}\boldsymbol{b}\big) = \big({}_{i|}\mathbf{A}\big)\cdot\boldsymbol{b}, \quad$ hence

$${}_{i|}\big(\mathbf{AB}\big)_{|j} = {}_{i|}\Big(\big(\mathbf{AB}\big)_{|j}\Big) = {}_{i|}\Big(\mathbf{A}\big(\mathbf{B}_{|j}\big)\Big) = \big({}_{i|}\mathbf{A}\big)\cdot\big(\mathbf{B}_{|j}\big)$$

$\square$

8 Matrix multiplication: **rows times columns**

Consider $\underset{m\times p}{\mathbf{A}}$ and $\underset{p\times n}{\mathbf{B}}$, then:

$$\boxed{{}_{i|}(\mathbf{AB})_{|j} = ({}_{i|}\mathbf{A}) \cdot (\mathbf{B}_{|j})}$$

Proof.

Remember that $\quad {}_{i|}(\mathbf{A}\boldsymbol{b}) = ({}_{i|}\mathbf{A}) \cdot \boldsymbol{b}, \quad$ hence

$${}_{i|}(\mathbf{AB})_{|j} = {}_{i|}\Big((\mathbf{AB})_{|j}\Big) = {}_{i|}\Big(\mathbf{A}(\mathbf{B}_{|j})\Big) = ({}_{i|}\mathbf{A}) \cdot (\mathbf{B}_{|j})$$

□

Thus, if we omit the period, we can simply write:

$${}_{i|}\mathbf{AB}_{|j}$$

9 Matrix multiplication (**by rows**)

Consider $\underset{m\times p}{\mathbf{A}}$ and $\underset{p\times n}{\mathbf{B}}$, then:

$$\boxed{{}_{i|}\big(\mathbf{AB}\big) = \big({}_{i|}\mathbf{A}\big)\mathbf{B}}$$

Proof.

Let's see the jth components are equal:

$$\Big({}_{i|}(\mathbf{AB})\Big)_{|j} = {}_{i|}\Big((\mathbf{AB})_{|j}\Big) = \big({}_{i|}\mathbf{A}\big)\cdot\big(\mathbf{B}_{|j}\big) = \Big(({}_{i|}\mathbf{A})\mathbf{B}\Big)_{|j}$$

so ${}_{i|}\big(\mathbf{AB}\big) = \big({}_{i|}\mathbf{A}\big)\mathbf{B}$. $\square$

9 Matrix multiplication (**by rows**)

Consider $\underset{m\times p}{\mathbf{A}}$ and $\underset{p\times n}{\mathbf{B}}$, then:

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so ${}_{i|}\big(\mathbf{AB}\big) = \big({}_{i|}\mathbf{A}\big)\mathbf{B}$. □

Thus, we can simply write:

$${}_{i|}\mathbf{AB}$$

10 Matrix multiplication (**by rows**)

Each row of **AB** is a linear combination of the p rows of **B**

$$\begin{bmatrix} 2 & 1 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 12 \\ 11 & 18 \\ 13 & 24 \end{bmatrix} \text{ where } \begin{cases} \begin{pmatrix} 2, & 1, \end{pmatrix} \begin{bmatrix} 1 & 6 \\ 1 & 0 \end{bmatrix} = \begin{pmatrix} 3, & 12, \end{pmatrix} \\ \begin{pmatrix} 3, & 8, \end{pmatrix} \begin{bmatrix} 1 & 6 \\ 1 & 0 \end{bmatrix} = \begin{pmatrix} 11, & 18, \end{pmatrix} \\ \begin{pmatrix} 4, & 9, \end{pmatrix} \begin{bmatrix} 1 & 6 \\ 1 & 0 \end{bmatrix} = \begin{pmatrix} 13, & 24, \end{pmatrix} \end{cases}$$

11 Transposing a product of matrices

Since

- $(\mathbf{A}^\mathsf{T})_{|j} = {}_{j|}\mathbf{A}$
- $\boldsymbol{a}\mathbf{B} = (\mathbf{B}^\mathsf{T})\boldsymbol{a}$

11 Transposing a product of matrices

Since

- $(\mathbf{A}^\intercal)_{|j} = {}_{j|}\mathbf{A}$
- $\boldsymbol{a}\mathbf{B} = (\mathbf{B}^\intercal)\boldsymbol{a}$

it follows that:

$$\boxed{\left(\mathbf{AB}\right)^\intercal = (\mathbf{B}^\intercal)(\mathbf{A}^\intercal)}$$

11 Transposing a product of matrices

Since

- $(\mathbf{A}^\intercal)_{|j} = {}_{j|}\mathbf{A}$
- $\boldsymbol{a}\mathbf{B} = (\mathbf{B}^\intercal)\boldsymbol{a}$

it follows that:

$$\boxed{\left(\mathbf{AB}\right)^\intercal = (\mathbf{B}^\intercal)(\mathbf{A}^\intercal)}$$

Proof.

$$(\mathbf{AB})^\intercal{}_{|j} \;=\; {}_{j|}\mathbf{AB} \;=\; (\mathbf{B}^\intercal)\left({}_{j|}\mathbf{A}\right) \;=\; (\mathbf{B}^\intercal)(\mathbf{A}^\intercal)_{|j}.$$

□

11 Transposing a product of matrices

Since

- $(\mathbf{A}^\intercal)_{|j} = {}_{j|}\mathbf{A}$
- $\boldsymbol{a}\mathbf{B} = (\mathbf{B}^\intercal)\boldsymbol{a}$

it follows that:

$$\boxed{\left(\mathbf{AB}\right)^\intercal = (\mathbf{B}^\intercal)(\mathbf{A}^\intercal)}$$

Proof.

$$(\mathbf{AB})^\intercal{}_{|j} = {}_{j|}\mathbf{AB} = (\mathbf{B}^\intercal)\left({}_{j|}\mathbf{A}\right) = (\mathbf{B}^\intercal)(\mathbf{A}^\intercal)_{|j}.$$

□

Matrix times its transpose is always symmetric

Librería nacal para Python

Revise la implementación de las operaciones del álgebra matricial en la librería nacal para Python que acompaña al curso: Sección 1.3 de la documentación (o estudie directamente el código).

`https://github.com/mbujosab/nacallib`

Verá que el código es una traducción literal de las *definiciones* vistas aquí; pero que no hay ni una línea de código que describa las propiedades que hemos demostrado en estas tres lecciones. ¡No es necesario! Las definiciones implican las propiedades (como hemos comprobado teóricamente con las demostraciones de estas lecciones). Verifique con ejemplos que todas las propiedades se cumplen. Estudie los notebooks de Jupyter correspondientes a las tres primeras lecciones.

Questions of the Lecture 3

No deje de hacer los ejercicios del libro.

(L-3) QUESTION 1. Multiply these matrices in the orders **EF**, **FE** and $\mathbf{E}^2$

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}; \qquad \mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}.$$

(Strang, 1988, exercise 34 from section 1.4.)

(L-3) QUESTION 2. True or false; give a specific counterexample when false.

(a) If the first and third columns of **B** are the same, so are the first and third columns of **AB**.
(b) If the first and third rows of **B** are the same, so are the first and third rows of **AB**.
(c) If the first and third rows of **A** are the same, so are the first and third rows of **AB**.
(d) $(\mathbf{AB})^2 = \mathbf{A}^2\mathbf{B}^2$.

(Strang, 1988, exercise 10 from section 1.4.)

(L-3) QUESTION 3. Consider the vectors
$\boldsymbol{a} = \begin{pmatrix}1, & -2, & 7,\end{pmatrix}$ y $\boldsymbol{b} = \begin{pmatrix}3, & 5, & 1,\end{pmatrix}$. Compute the following products

(a) $\boldsymbol{a} \cdot \boldsymbol{a}$ (b) $\boldsymbol{a} \cdot \boldsymbol{b}$ (c) $[\boldsymbol{a}]\,[\boldsymbol{b}]^{\mathsf{T}}$

(Strang, 1988, exercise 3 from section 1.4.)

(L-3) QUESTION 4. Write down the 2 by 2 matrices **A** and **B** that have entries $a_{ij} = i + j$ and $b_{ij} = (-1)^{i+j}$. Multiply them to find **AB** and **BA**.
(Strang, 1988, exercise 6 from section 1.4.)

(L-3) QUESTION 5. The product of two lower triangular matrices is again lower triangular (all its entries above the main diagonal are zero). Confirm this with a 3 by 3 example, and then explain how it follows from the laws of matrix multiplication.
(Strang, 1988, exercise 12 from section 1.4.)

(L-3) Question 6. consider the matrices **A**, **B**, **C**, **D**, **E** and **F** .

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & 4 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

Compute (*in particular, ¡note that* $\mathbf{EF} \neq \mathbf{FE}$*!*)

(a) $\mathbf{B} + \mathbf{D}$
(b) $2\mathbf{E} - \mathbf{F}$
(c) $\mathbf{AC}$
(d) $\mathbf{BC}$
(e) $\mathbf{CB}$
(f) $\mathbf{ACD}$
(g) $\mathbf{EF}$
(h) $\mathbf{FE}$
(i) $\mathbf{CEF}$

Highlights of Lesson 4

- Elementary transformations
- Indentifying singular matrices by elimination
- Matrix multiplication of Elementary matrices

2 Elementary transformations of a matrix

Type *I*: $\mathbf{A}_{\underset{[(\lambda)\boldsymbol{i}+\boldsymbol{j}]}{\boldsymbol{\tau}}}$ (with $i \neq j$)

add λ times i-th column ($\lambda \mathbf{A}_{|i}$) to j-th column ($\mathbf{A}_{|j}$)

$$\begin{bmatrix} 1 & -3 & 0 \\ 1 & -6 & 3 \end{bmatrix}_{\underset{[(-2)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}} = \begin{bmatrix} 1 & -3 & -2 \\ 1 & -6 & 1 \end{bmatrix}$$

$\boxed{2}$ Elementary transformations of a matrix

Type *I*: $\mathbf{A}_{\underset{[(\lambda)\boldsymbol{i}+\boldsymbol{j}]}{\boldsymbol{\tau}}}$ (with $i \neq j$)

add λ times i-th column ($\lambda \mathbf{A}_{|i}$) to j-th column ($\mathbf{A}_{|j}$)

$$\begin{bmatrix} 1 & -3 & 0 \\ 1 & -6 & 3 \end{bmatrix}_{\underset{[(-2)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}} = \begin{bmatrix} 1 & -3 & -2 \\ 1 & -6 & 1 \end{bmatrix}$$

Type *II*: $\mathbf{A}_{\underset{[(\alpha)\boldsymbol{i}]}{\boldsymbol{\tau}}}$ (with $\alpha \neq 0$)

multiply by α the i-th column

$$\begin{bmatrix} 1 & -3 & 0 \\ 1 & -6 & 3 \end{bmatrix}_{\underset{[(10)\mathbf{2}]}{\boldsymbol{\tau}}} = \begin{bmatrix} 1 & -30 & 0 \\ 1 & -60 & 3 \end{bmatrix}$$

3 Elimination and pre-echelon form of a matrix

- *Pivot* is the first non-zero component of each column.
- *Elimination*: modifies a matrix until all components at the right-hand side of each pivot are zeros

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 8 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

3 Elimination and pre-echelon form of a matrix

- *Pivot* is the first non-zero component of each column.
- *Elimination*: modifies a matrix until all components at the right-hand side of each pivot are zeros

Subtract 3 times the 1^{st} column from the 2^{nd}

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 8 & 4 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[{[(-3)\mathbf{1}+\mathbf{2}]}]{\tau}$$

3 Elimination and pre-echelon form of a matrix

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Subtract 3 times the 1^{st} column from the 2^{nd}

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 8 & 4 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[{[(-3)\mathbf{1}+\mathbf{2}]}]{\tau} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix}$$

$\boxed{3}$ Elimination and pre-echelon form of a matrix

- *Pivot* is the first non-zero component of each column.
- *Elimination*: modifies a matrix until all components at the right-hand side of each pivot are zeros

Subtract 3 times the 1^{st} column from the 2^{nd}

Then subtract 2 times the 2^{nd} col. from the 3^{rd}

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 8 & 4 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[{[(-3)\mathbf{1}+\mathbf{2}]}]{\tau} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix} \xrightarrow[{[(-2)\mathbf{2}+\mathbf{3}]}]{\tau}$$

3 Elimination and pre-echelon form of a matrix

- *Pivot* is the first non-zero component of each column.
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$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 8 & 4 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[{[(-3)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix} \xrightarrow[{[(-2)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & -2 & 5 \end{bmatrix} = \mathbf{L}$$

3 Elimination and pre-echelon form of a matrix

- *Pivot* is the first non-zero component of each column.
- *Elimination*: modifies a matrix until all components at the right-hand side of each pivot are zeros

Subtract 3 times the 1^{st} column from the 2^{nd}
Then subtract 2 times the 2^{nd} col. from the 3^{rd}

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 8 & 4 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[{[(-3)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix} \xrightarrow[{[(-2)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & -2 & 5 \end{bmatrix} = \mathbf{L}$$

Elimination algorithm on **A**

modifies **A** using a sequence of *elementary transformations*

4 Elimination

Elimination algorithm on **A**

modifies **A** using a sequence of *elementary transformations*

Goal

to get a *(pre)echelon* form

$\boxed{4}$ Elimination

Elimination algorithm on $\mathbf{A}$

modifies $\mathbf{A}$ using a sequence of *elementary transformations*

Goal

to get a *(pre)echelon* form

- *pre-echelon*: all components on the right side of each pivot are zero.

4 Elimination

Elimination algorithm on $\mathbf{A}$

modifies $\mathbf{A}$ using a sequence of *elementary transformations*

Goal

to get a *(pre)echelon* form

- *pre-echelon*: all components on the right side of each pivot are zero.
- *echelon*: if any column before a non-null column $\mathbf{A}_{|j}$ is non-null column and its pivot is above the pivot of $\mathbf{A}_{|j}$.

4 Elimination

Elimination algorithm on $\mathbf{A}$

modifies $\mathbf{A}$ using a sequence of *elementary transformations*

Goal

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It is always possible to find a (pre)echelon form by elimination

Elimination algorithm on $\mathbf{A}$

modifies $\mathbf{A}$ using a sequence of *elementary transformations*

Goal

to get a *(pre)echelon* form

- *pre-echelon*: all components on the right side of each pivot are zero.
- *echelon*: if any column before a non-null column $\mathbf{A}_{|j}$ is non-null column and its pivot is above the pivot of $\mathbf{A}_{|j}$.

It is always possible to find a (pre)echelon form by elimination

Rank (rg): the number of pivots in any of its pre-echelon forms

4 Elimination

Elimination algorithm on **A**

modifies **A** using a sequence of *elementary transformations*

Goal
to get a *(pre)echelon* form

- *pre-echelon*: all components on the right side of each pivot are zero.
- *echelon*: if any column before a non-null column $\mathbf{A}_{|j}$ is non-null column and its pivot is above the pivot of $\mathbf{A}_{|j}$.

It is always possible to find a (pre)echelon form by elimination

Rank (rg): the number of pivots in any of its pre-echelon forms

$\underset{n\times n}{\mathbf{A}}$ is *singular* if its pre-echelon forms have null-columns $(\mathrm{rg} < n)$

5 Elimination: When can't we find n pivots?

$n \times n$ matrices are **singular** if we can't find n pivots $\rightarrow$ ☹

$$\begin{bmatrix} 0 & 1 & 3 \\ 4 & 2 & 8 \\ 1 & 1 & 1 \end{bmatrix}$$

5 Elimination: When can't we find n pivots?

$n \times n$ matrices are **singular** if we can't find n pivots $\rightarrow$ ☹

$$\begin{bmatrix} 0 & 1 & 3 \\ 4 & 2 & 8 \\ 1 & 1 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 0 & 1 & 3 \\ 4 & 2 & 8 \\ 1 & 1 & 1 \end{bmatrix}$$

☺

Has this matrix n pivots? $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ 1 & 1 & 1 \end{bmatrix}$

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$n \times n$ matrices are **singular** if we can't find n pivots $\rightarrow$

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Has this matrix n pivots? $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ 1 & 1 & 1 \end{bmatrix}$

$\boxed{5}$ Elimination: When can't we find n pivots?

$n \times n$ matrices are **singular** if we can't find n pivots $\rightarrow$ ☹

$$\begin{bmatrix} 0 & 1 & 3 \\ 4 & 2 & 8 \\ 1 & 1 & 1 \end{bmatrix}$$ ☺

Has this matrix n pivots? $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ 1 & 1 & 1 \end{bmatrix}$ ☺

and this one? $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

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$n \times n$ matrices are **singular** if we can't find n pivots $\rightarrow$ ☹

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and this one? $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ ☹

and this one? $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & -4 \end{bmatrix}$

5 Elimination: When can't we find n pivots?

$n \times n$ matrices are **singular** if we can't find n pivots $\rightarrow$ ☹

$$\begin{bmatrix} 0 & 1 & 3 \\ 4 & 2 & 8 \\ 1 & 1 & 1 \end{bmatrix}$$ ☺

Has this matrix n pivots? $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ 1 & 1 & 1 \end{bmatrix}$ ☺

and this one? $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ ☹

and this one? $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & -4 \end{bmatrix}$ ☹

6 Matrix multiplication: elementary matrices

$$\underbrace{\begin{bmatrix} 1 & 3 & 0 \\ 2 & 8 & 4 \\ 1 & 1 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}_{\mathbf{I}_\tau} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix}}_{\mathbf{A}_\tau}$$

6 Matrix multiplication: elementary matrices

$$\underbrace{\begin{bmatrix} 1 & 3 & 0 \\ 2 & 8 & 4 \\ 1 & 1 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} ? & & \\ ? & & \\ ? & & \end{bmatrix}}_{\mathbf{I}_\tau} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix}}_{\mathbf{A}_\tau}$$

6 Matrix multiplication: elementary matrices

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6 Matrix multiplication: elementary matrices

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6 Matrix multiplication: elementary matrices

$$\underbrace{\begin{bmatrix} 1 & 3 & 0 \\ 2 & 8 & 4 \\ 1 & 1 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} 1 & & 0 \\ 0 & & 0 \\ 0 & & 1 \end{bmatrix}}_{\mathbf{I}_\tau} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix}}_{\mathbf{A}_\tau}$$

6 Matrix multiplication: elementary matrices

$$\underbrace{\begin{bmatrix}1 & 3 & 0\\ 2 & 8 & 4\\ 1 & 1 & 1\end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{bmatrix}1 & ? & 0\\ 0 & ? & 0\\ 0 & ? & 1\end{bmatrix}}_{\mathbf{I}_\tau} = \underbrace{\begin{bmatrix}1 & 0 & 0\\ 2 & 2 & 4\\ 1 & -2 & 1\end{bmatrix}}_{\mathbf{A}_\tau}$$

$\boxed{6}$ Matrix multiplication: elementary matrices

Subtract 3 times column one from column two

$$\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}$$

$$\underbrace{\begin{bmatrix} 1 & 3 & 0 \\ 2 & 8 & 4 \\ 1 & 1 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{I}_{\boldsymbol{\tau}}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix}}_{\mathbf{A}_{\boldsymbol{\tau}}}$$

We call $\mathbf{I}_{\boldsymbol{\tau}}$ "Elementary matrix":

$$\mathbf{A}\left(\mathbf{I}_{\boldsymbol{\tau}}\right) = \mathbf{A}_{\boldsymbol{\tau}}$$

6 Matrix multiplication: elementary matrices

Subtract 3 times column one from column two $\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}$

$$\underbrace{\begin{bmatrix}1 & 3 & 0\\ 2 & 8 & 4\\ 1 & 1 & 1\end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{bmatrix}1 & -3 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}}_{\mathbf{I}_{\boldsymbol{\tau}}}=\underbrace{\begin{bmatrix}1 & 0 & 0\\ 2 & 2 & 4\\ 1 & -2 & 1\end{bmatrix}}_{\mathbf{A}_{\boldsymbol{\tau}}}$$

We call $\mathbf{I}_{\boldsymbol{\tau}}$ "Elementary matrix":

$$\mathbf{A}\left(\mathbf{I}_{\boldsymbol{\tau}}\right)=\mathbf{A}_{\boldsymbol{\tau}}$$

This specific elementary matrix $\mathbf{I}_{\boldsymbol{\tau}}$ is written as $\mathbf{I}_{\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}$

6 Matrix multiplication: elementary matrices

Subtract 3 times column one from column two $\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}$

$$\underbrace{\begin{bmatrix}1 & 3 & 0\\2 & 8 & 4\\1 & 1 & 1\end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{bmatrix}1 & -3 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}}_{\mathbf{I}_{\boldsymbol{\tau}}}=\underbrace{\begin{bmatrix}1 & 0 & 0\\2 & 2 & 4\\1 & -2 & 1\end{bmatrix}}_{\mathbf{A}_{\boldsymbol{\tau}}}$$

We call $\mathbf{I}_{\boldsymbol{\tau}}$ "Elementary matrix":

$$\mathbf{A}\left(\mathbf{I}_{\boldsymbol{\tau}}\right)=\mathbf{A}_{\boldsymbol{\tau}}$$

This specific elementary matrix $\mathbf{I}_{\boldsymbol{\tau}}$ is written as $\mathbf{I}_{\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}$

6 Matrix multiplication: elementary matrices

Subtract 3 times column one from column two $\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}$

$$\underbrace{\begin{bmatrix}1 & 3 & 0\\ 2 & 8 & 4\\ 1 & 1 & 1\end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{bmatrix}1 & -3 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}}_{\mathbf{I}_{\boldsymbol{\tau}}}=\underbrace{\begin{bmatrix}1 & 0 & 0\\ 2 & 2 & 4\\ 1 & -2 & 1\end{bmatrix}}_{\mathbf{A}_{\boldsymbol{\tau}}}$$

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7 Matrix multiplication: elementary matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & -2 & 5 \end{bmatrix}$$

7 Matrix multiplication: elementary matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} ? & & \\ ? & & \\ ? & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & -2 & 5 \end{bmatrix}$$

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7 Matrix multiplication: elementary matrices

Subtract 2 times column two from column three

$$\begin{bmatrix}1 & 0 & 0\\ 2 & 2 & 4\\ 1 & -2 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 2 & 2 & 0\\ 1 & -2 & 5\end{bmatrix}$$

7 Matrix multiplication: elementary matrices

Subtract 2 times column two from column three

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7 Matrix multiplication: elementary matrices

Subtract 2 times column two from column three

$$\underset{[(-2)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}$$

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This specific elementary matrix $\mathbf{I}_{\boldsymbol{\tau}}$ is written as $\mathbf{I}_{\underset{[(-2)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}}$

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$$\mathbf{A}\left(\mathbf{I}_{\underset{[(-2)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}}\right)=\mathbf{A}_{\underset{[(-2)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}}$$

8 Elimination by elementary matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 8 & 4 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[{[(-3)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix} \xrightarrow[{[(-2)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & -2 & 5 \end{bmatrix} = \mathbf{L}$$

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$$\mathbf{A}_{\overset{\boldsymbol{\tau}}{[(-3)\mathbf{1}+\mathbf{2}]}} = \mathbf{A}_{\overset{\boldsymbol{\tau}}{[(-3)\mathbf{1}+\mathbf{2}]}} \qquad = \left(\mathbf{A}\left(\mathbf{I}_{\overset{\boldsymbol{\tau}}{[(-3)\mathbf{1}+\mathbf{2}]}}\right)\right)$$

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$$\mathbf{A}_{\substack{\boldsymbol{\tau} \\ [(-3)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{2}+\mathbf{3}]}} = \mathbf{A}_{\overset{\boldsymbol{\tau}}{[(-3)\mathbf{1}+\mathbf{2}]}\overset{\boldsymbol{\tau}}{[(-2)\mathbf{2}+\mathbf{3}]}} = \left(\mathbf{A}\left(\mathbf{I}_{\overset{\boldsymbol{\tau}}{[(-3)\mathbf{1}+\mathbf{2}]}}\right)\right)\left(\mathbf{I}_{\overset{\boldsymbol{\tau}}{[(-2)\mathbf{2}+\mathbf{3}]}}\right) = \mathbf{L}$$

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$$\mathbf{A}_{\substack{\tau \\ [(-3)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{2}+\mathbf{3}]}} = \mathbf{A}_{\overset{\tau}{[(-3)\mathbf{1}+\mathbf{2}]}\overset{\tau}{[(-2)\mathbf{2}+\mathbf{3}]}} = \left(\mathbf{A}\left(\mathbf{I}_{\overset{\tau}{[(-3)\mathbf{1}+\mathbf{2}]}}\right)\right)\left(\mathbf{I}_{\overset{\tau}{[(-2)\mathbf{2}+\mathbf{3}]}}\right) = \mathbf{L}$$

there is a matrix that does the whole job at once

$$\mathbf{A}_{\substack{\tau \\ [(-3)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{2}+\mathbf{3}]}} = \mathbf{A}\left(\left(\mathbf{I}_{\overset{\tau}{[(-3)\mathbf{1}+\mathbf{2}]}}\right)\left(\mathbf{I}_{\overset{\tau}{[(-2)\mathbf{2}+\mathbf{3}]}}\right)\right) = \mathbf{A}\mathbf{I}_{\substack{\tau \\ [(-3)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{2}+\mathbf{3}]}} = \mathbf{L}$$

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there is a matrix that does the whole job at once

$$\mathbf{A}_{\substack{\boldsymbol{\tau} \\ [(-3)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{2}+\mathbf{3}]}} = \mathbf{A}\left(\left(\mathbf{I}_{\overset{\boldsymbol{\tau}}{[(-3)\mathbf{1}+\mathbf{2}]}}\right)\left(\mathbf{I}_{\overset{\boldsymbol{\tau}}{[(-2)\mathbf{2}+\mathbf{3}]}}\right)\right) = \mathbf{A}\mathbf{I}_{\substack{\boldsymbol{\tau} \\ [(-3)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{2}+\mathbf{3}]}} = \mathbf{L}$$

$$\boxed{\mathbf{A}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k} = \mathbf{A}\left(\mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k}\right)}$$

9 how do I get from **L** back to **A**? Inverses

How do I reverse the first step? (it was subtract 3 times $\mathbf{A}_{|1}$ from $\mathbf{A}_{|2}$)

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

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$\mathbf{I}_{\overset{\boldsymbol{\tau}}{[(-\lambda)\boldsymbol{i}+\boldsymbol{j}]}}$ "undo" $\mathbf{I}_{\overset{\boldsymbol{\tau}}{[(\lambda)\boldsymbol{i}+\boldsymbol{j}]}}$

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$$\mathbf{I}_{[(-\lambda)\boldsymbol{i}+\boldsymbol{j}]}^{\boldsymbol{\tau}} \quad \text{“undo”} \quad \mathbf{I}_{[(\lambda)\boldsymbol{i}+\boldsymbol{j}]}^{\boldsymbol{\tau}}$$

How to undo $\mathbf{I}_{[(\alpha)\boldsymbol{i}]}^{\boldsymbol{\tau}}$?

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 2\end{bmatrix}\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}.$$

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$\mathbf{I}_{[(-\lambda)\boldsymbol{i}+\boldsymbol{j}]}^{\boldsymbol{\tau}}$ "undo" $\mathbf{I}_{[(\lambda)\boldsymbol{i}+\boldsymbol{j}]}^{\boldsymbol{\tau}}$

How to undo $\mathbf{I}_{[(\alpha)\boldsymbol{i}]}^{\boldsymbol{\tau}}$?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

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$\mathbf{I}_{\boldsymbol{\tau}_{[(-\lambda)\boldsymbol{i}+\boldsymbol{j}]}}$ "undo" $\mathbf{I}_{\boldsymbol{\tau}_{[(\lambda)\boldsymbol{i}+\boldsymbol{j}]}}$

How to undo $\mathbf{I}_{\boldsymbol{\tau}_{[(\alpha)\boldsymbol{i}]}}$? $\mathbf{I}_{\boldsymbol{\tau}_{\left[\left(\frac{1}{\alpha}\right)\boldsymbol{i}\right]}}$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}1 & & \\ & 1 & \\ & & 1/2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

$\boxed{\mathbf{10}}$ Interchange or swap matrices

Which matrix exchanges the columns?

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} = \begin{bmatrix} c & a \\ d & b \end{bmatrix}$$

$\boxed{10}$ Interchange or swap matrices

Which matrix exchanges the columns?

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & \quad \\ 1 & \quad \end{bmatrix} = \begin{bmatrix} c & a \\ d & b \end{bmatrix}$$

10 Interchange or swap matrices

Which matrix exchanges the columns?

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} c & a \\ d & b \end{bmatrix}$$

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Which matrix exchanges the rows? where do we put that matrix?

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \qquad\qquad = \begin{bmatrix} b & d \\ a & c \end{bmatrix}$$

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$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \qquad = \begin{bmatrix} b & d \\ a & c \end{bmatrix}$$

Matrix multiplication is not commutative!

$\boxed{11}$ Interchange of columns

Interchange of columns:

$\mathbf{A}_{\underset{[i\rightleftharpoons j]}{\tau}} \rightarrow$ swicht columns i and j of $\mathbf{A}$

$$\begin{bmatrix} 1 & -3 & 0 \\ 1 & -6 & 3 \end{bmatrix}_{\underset{[\mathbf{2}\rightleftharpoons\mathbf{3}]}{\tau}} = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 3 & -6 \end{bmatrix}$$

Interchange of columns:

$\mathbf{A}_{\underset{[i \rightleftharpoons j]}{\boldsymbol{\tau}}}$ $\rightarrow$ swicht columns i and j of **A**

$$\begin{bmatrix} 1 & -3 & 0 \\ 1 & -6 & 3 \end{bmatrix}_{\underset{[\mathbf{2} \rightleftharpoons \mathbf{3}]}{\boldsymbol{\tau}}} = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 3 & -6 \end{bmatrix}$$

We can switch two columns by a sequence of elementary transformations

11 Interchange of columns

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$\mathbf{A}_{\underset{[i\rightleftharpoons j]}{\tau}} \rightarrow$ swicht columns i and j of $\mathbf{A}$

$$\begin{bmatrix} 1 & -3 & 0 \\ 1 & -6 & 3 \end{bmatrix}_{\underset{[2\rightleftharpoons 3]}{\tau}} = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 3 & -6 \end{bmatrix}$$

We can switch two columns by a sequence of elementary transformations

Matrix $\mathbf{I}_{\underset{[i\rightleftharpoons j]}{\tau}}$ is call a exchange matrix

12 Permutation matrices

Product between exchange matrices $\mathbf{I}_{\underset{[\text{-}\rightleftharpoons\text{-}]}{\boldsymbol{\tau}}}$ is a permutation matrix $\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}$.

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$$\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}} = \text{Identity matrix } \mathbf{I} \text{ with rearranged columns}$$

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$$\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}} = \text{Identity matrix } \mathbf{I} \text{ with rearranged columns}$$

Let's see the 3×3 case

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad \mathbf{I}_{\underset{[\mathbf{1}\rightleftharpoons\mathbf{2}]}{\boldsymbol{\tau}}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

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How many 3×3 pemutations can we find?

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How many 3×3 pemutations can we find?

$3 \times 2 \times 1$

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$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad \mathbf{I}_{\underset{[\mathbf{1}\rightleftharpoons\mathbf{2}]}{\tau}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

How many 3×3 pemutations can we find?

$3 \times 2 \times 1 \qquad n!$ for $\mathbb{R}^{n \times n}$

12 Permutation matrices

Product between exchange matrices $\mathbf{I}_{\underset{[-\rightleftharpoons-]}{\tau}}$ is a permutation matrix $\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}}$.

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Let's see the 3×3 case

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad \mathbf{I}_{\underset{[1\rightleftharpoons 2]}{\tau}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

How many 3×3 pemutations can we find?

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what happens if I multiply two permutation matrices?

12 Permutation matrices

Product between exchange matrices $\mathbf{I}_{\underset{[\cdot\rightleftharpoons\cdot]}{\boldsymbol{\tau}}}$ is a permutation matrix $\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}$.

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Let's see the 3×3 case

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad \mathbf{I}_{\underset{[1\rightleftharpoons 2]}{\boldsymbol{\tau}}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

How many 3×3 pemutations can we find?

$3 \times 2 \times 1$ $\quad$ $n!$ for $\mathbb{R}^{n \times n}$

what happens if I multiply two permutation matrices?
we get another permutation matrix

Questions of the Lecture 4

(L-4) Question 1.

(a) Which three matrices $\mathbf{I}_{\underset{[(x)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}$, $\mathbf{I}_{\underset{[(y)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}}$ and $\mathbf{I}_{\underset{[(z)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}}$ put $\mathbf{A} = \begin{bmatrix} 1 & 4 & -2 \\ 1 & 6 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ into an echelon form?

(b) Multiply those $\mathbf{I}_{\boldsymbol{\tau}_i}$ to get one matrix $\mathbf{E}$ that does elimination: $\mathbf{AE} = \mathbf{K}$.

Based on (Strang, 1988, exercise 24 from section 1.4.)

(L-4) Question 2. Consider the matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ -1 & -3 & -2 \\ 0 & 1 & c \end{bmatrix}$$

For what value(s) of c the matrix is singular (we can't find three pivots)?

(L-4) QUESTION 3. Consider the following 3 by 3 matrices.

(a) $(\mathbf{I}_{[(-1)\mathbf{1}+\mathbf{2}]}^{\boldsymbol{\tau}})$ subtracts column 1 from column 2 and then $(\mathbf{I}_{[\mathbf{2}\rightleftharpoons\mathbf{3}]}^{\boldsymbol{\tau}})$ exchanges columns 2 and 3. What matrix **E** does both steps at once?

(b) $(\mathbf{I}_{[\mathbf{2}\rightleftharpoons\mathbf{3}]}^{\boldsymbol{\tau}})$ exchanges columns 2 and 3 and then $\mathbf{I}_{[(-1)\mathbf{1}+\mathbf{3}]}^{\boldsymbol{\tau}}$ subtracts column 1 from column 3. What matrix $\mathbf{N} = (\mathbf{I}_{[\mathbf{2}\rightleftharpoons\mathbf{3}]}^{\boldsymbol{\tau}})(\mathbf{I}_{[(-1)\mathbf{1}+\mathbf{3}]}^{\boldsymbol{\tau}})$ does both steps at once? Explain why **M** and **N** are the same but the $\mathbf{I}_{\boldsymbol{\tau}}$'s are different.

Based on (Strang, 1988, exercise 28 from section 1.4.)

(L-4) QUESTION 4. Elimination matrices $\mathbf{I}_{[(\text{-})\mathbf{1}+\mathbf{2}]}^{\boldsymbol{\tau}}$ and $\mathbf{I}_{[(\text{-})\mathbf{2}+\mathbf{3}]}^{\boldsymbol{\tau}}$ will reduce **A** to triangular form. Find **E** so that $\mathbf{AE} = \mathbf{L}$ is lower triangular (echelon), if **A** is

$$\begin{bmatrix} 2 & 2 & 0 \\ 1 & 4 & 9 \\ 1 & 3 & 9 \end{bmatrix}$$

(L-4) QUESTION 5. Although we will only consider as elementary the *Type I* and *II* transformations, in most of the Linear Algebra books appears a third type: the *exchange* of columns

$$\mathbf{A}_{\substack{\boldsymbol{\tau}\\[\boldsymbol{p}\rightleftharpoons\boldsymbol{s}]}} \rightarrow \text{Exchanges columns } p \text{ and } s \text{ of } \mathbf{A}.$$

Prove that a column exchange is, in fact, a sequence of *Type I* and *II* elementary transformations. Try transforming $\underset{2\times 2}{\mathbf{I}}$ in $\begin{bmatrix}0&1\\1&0\end{bmatrix}$ by elementary transformations of the columns.

(L-4) QUESTION 6. Write down the 3 by 3 matrices that produce these elimination steps:

(a) $\mathbf{I}_{\substack{\boldsymbol{\tau}\\[(-5)\mathbf{1}+\mathbf{2}]}}$ substracts 5 times column 1 from column 2,

(b) $\mathbf{I}_{\substack{\boldsymbol{\tau}\\[(-7)\mathbf{2}+\mathbf{3}]}}$ substracts 7 times column 2 from column 3,

(c) $\mathbf{I}_{\substack{\boldsymbol{\tau}\\[\mathfrak{S}]}}$ exchanges columns 1 and 2, and then columns 2 and 3.

(Strang, 2003, exercise 1 from section 2.3.)

(L-4) QUESTION 7. Consider the matrices of QUESTION 6:

(a) when multiplying by $\mathbf{I}_{\underset{[(-5)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}$ and then by $\mathbf{I}_{\underset{[(-7)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}}$ the matrix $\mathbf{A} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}$

we get $\mathbf{A}_{\substack{\boldsymbol{\tau} \\ [(-5)\mathbf{1}+\mathbf{2}] \\ [(-7)\mathbf{2}+\mathbf{3}]}} = [\quad ; \quad ; \quad].$

(b) But, when multiplying by $\mathbf{I}_{\underset{[(-5)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}$ before and then by $\mathbf{I}_{\underset{[(-7)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}}$ we get

$\mathbf{A}_{\substack{\boldsymbol{\tau} \\ [(-7)\mathbf{2}+\mathbf{3}] \\ [(-5)\mathbf{1}+\mathbf{2}]}} = [\quad ; \quad ; \quad].$

(c) When $\underset{[(-7)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}$ comes first, the column ____ feels no effect from column ____.

This property will become very important in the LU factorization!

(Strang, 2003, exercise 2 from section 2.3.)

(L-4) QUESTION 8. What matrix $\mathbf{M}$ sends $\boldsymbol{v} = \begin{pmatrix}1, & 0,\end{pmatrix}$ to $\begin{pmatrix}0, & 1,\end{pmatrix}$, es decir $\boldsymbol{v}\mathbf{M} = \begin{pmatrix}0, & 1,\end{pmatrix}$; and also sends $\boldsymbol{w} = \begin{pmatrix}0, & 1,\end{pmatrix}$ to $\begin{pmatrix}1, & 0,\end{pmatrix}$, es decir $\boldsymbol{w}\mathbf{M} = \begin{pmatrix}1, & 0,\end{pmatrix}$?

(L-4) QUESTION 9. Consider a permutation (interchange) matrix $\mathbf{I}_{\underset{[i\rightleftharpoons j]}{\boldsymbol{\tau}}}$, if we compute the product $\mathbf{A}\left(\mathbf{I}_{\underset{[i\rightleftharpoons j]}{\boldsymbol{\tau}}}\right)$, we get a new matrix like $\mathbf{A}$, but with exchanged columns. What happen if we compute the product $\left(\mathbf{I}_{\underset{[i\rightleftharpoons j]}{\boldsymbol{\tau}}}\right)\mathbf{A}$? Check your answer with a 2 by 2 example.

(L-4) QUESTION 10. If every column of **A** is a multiple of $\begin{pmatrix}1, & 1, & 1,\end{pmatrix}$, then $\mathbf{A}\boldsymbol{x}$ is always a multiple of $\begin{pmatrix}1, & 1, & 1,\end{pmatrix}$. Do a 3 by 3 example. How many pivots are produced by elimination?
(Strang, 1988, exercise 26 from section 1.4.)

Highlights of Lesson 5

- Inverse of $\mathbf{A}$
- Gauss-Jordan elimination / finding $\mathbf{A}^{-1}$
- Inverse of $\mathbf{AB}$, $\mathbf{A}^{\mathsf{T}}$

2 Inverse of a matrix (square matrices)

A squared of order n has inverse (is *invertible*) if exists **B** such that

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$$\mathbf{B} = \mathbf{A}^{-1} \qquad \text{and} \qquad \mathbf{A} = \mathbf{B}^{-1}.$$

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$$\mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A} = \mathbf{I}.$$

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Not all matrices have inverse

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Then

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Not all matrices have inverse

Squared matrices with no inverse are called *singular* matrices

3 Singular case (no inverse)

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

$\boxed{3}$ Singular case (no inverse)

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

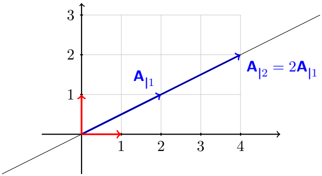
Is it possible to find a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I}$?

3 Singular case (no inverse)

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

Is it possible to find a matrix **B** such that $\mathbf{AB} = \mathbf{I}$?

. . . columns of **I** should be linear combinations of columns of **A**. . . but both columns lie on the same line.



So

$$\boxed{\mathbf{A} \text{ is singular}}$$

$\boxed{4}$ Singular case (no inverse)

Can we find $\boldsymbol{x} \neq \mathbf{0}$ such that $\mathbf{A}\boldsymbol{x} = \mathbf{0}$?

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{pmatrix} \quad \\ \quad \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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Can we find $x \neq \mathbf{0}$ such that $\mathbf{A}x = \mathbf{0}$?

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If $\mathbf{A}x = \mathbf{0}$ and $x \neq \mathbf{0}$ $\Rightarrow$ there is no $\mathbf{A}^{-1}$

$\boxed{\text{The existence of } \mathbf{A}^{-1} \text{ leads to a contradiction}}$

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$\boxed{\text{The existence of } \mathbf{A}^{-1} \text{ leads to a contradiction}}$

If $\mathbf{A}x = \mathbf{0}$ and $x \neq \mathbf{0}$ $\Rightarrow$ $\mathbf{A}^{-1}\mathbf{A}x = \mathbf{A}^{-1}\mathbf{0}$ $\Rightarrow$ $x = \mathbf{0}$.

When $\mathbf{A}^{-1}$ does exist

the only solution to $\mathbf{A}x = \mathbf{0}$ is $x = \mathbf{0}$.

5 Calculating the inverse matrix

$$\mathbf{A}(\mathbf{A}^{-1}) = \mathbf{I}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

5 Calculating the inverse matrix

$$\mathbf{A}(\mathbf{A}^{-1}) = \mathbf{I}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

5 Calculating the inverse matrix

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$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a & \\ b & \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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So. . . we are solving m systems (of m equations each)

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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6 Gauss-Jordan: solving two linear systems at once

Gauss-Jordan elimination (obtaining a reduced echelon form $\mathbf{R}$)

apply elementary transformations until a echelon matrix with only zeros to the left of each pivot (and all pivots equal to 1) is achieved

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applying Gauss-Jordan elimination on **A** stacked with **I**

$$\left[\begin{array}{c} \mathbf{A} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{cc} 1 & 3 \\ 2 & 7 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right]$$

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If $\mathbf{R}=\mathbf{I}$, we have found $\mathbf{A}^{-1}$

7 Gauss-Jordan: Why does it work?

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{cc}1 & 3\\ 2 & 7\\ \hline 1 & 0\\ 0 & 1\end{array}\right]$$

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$$\vdots$$

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that is, since $\mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k} = \mathbf{A}\left(\mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right)$:

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$$\left[\begin{array}{c} \mathbf{A} \\ \hline \mathbf{I} \end{array}\right]_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k} = \left[\begin{array}{c} \mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k} \\ \hline \mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k} \end{array}\right] = \left[\begin{array}{c} \mathbf{A}\left(\mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right) \\ \hline \mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k} \end{array}\right]$$

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$$\boxed{\text{who is } \mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\text{?}}$$

7 Gauss-Jordan: Why does it work?

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{cc}1 & 3\\ 2 & 7\\ \hline 1 & 0\\ 0 & 1\end{array}\right] \xrightarrow{\overset{\boldsymbol{\tau}}{[(-3)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc}1 & 0\\ 2 & 1\\ \hline 1 & -3\\ 0 & 1\end{array}\right] \xrightarrow{\overset{\boldsymbol{\tau}}{[(-2)\mathbf{2}+\mathbf{1}]}} \left[\begin{array}{cc}1 & 0\\ 0 & 1\\ \hline 7 & -3\\ -2 & 1\end{array}\right]$$

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therefore

$$\left(\mathbf{A}^{\mathsf{T}}\right)^{-1} = (\mathbf{A}^{-1})^{\mathsf{T}}$$

10 Interchanges and permutations

Are interchange matrices $\mathbf{I}_{\underset{[i \rightleftharpoons j]}{\boldsymbol{\tau}}}$, invertible?

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$\boxed{10}$ Interchanges and permutations

Are interchange matrices $\mathbf{I}_{\underset{[i\rightleftharpoons j]}{\boldsymbol{\tau}}}$, invertible? if I invert, then I'm just doing column exchanges to get back again!

It is easy to check that

$$\left(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}\right)^{\mathsf{T}}\left(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}\right) = \mathbf{I} \quad \Longrightarrow \quad \left(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}\right)^{\mathsf{T}} = \left(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}\right)^{-1}$$

11 Caracterización of invertible matrices

Given **A** of order n, the following statements are equivalent

1. No zero columns in $\mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_p} = \mathbf{K}$ (pre-echelon matrix).
2. **A** has inverse.
3. **A** is product of elementary matrices.

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$$\mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k} = \mathbf{A}\left(\mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right) = \mathbf{I} \qquad \Rightarrow \qquad \mathbf{A} = \left(\mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right)^{-1}$$

where

$$\left(\mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right)^{-1} = \Big((\mathbf{I}_{\boldsymbol{\tau}_1})\cdots(\mathbf{I}_{\boldsymbol{\tau}_k})\Big)^{-1} = (\mathbf{I}_{\boldsymbol{\tau}_k^{-1}})\cdots(\mathbf{I}_{\boldsymbol{\tau}_1^{-1}}) = \mathbf{I}_{\boldsymbol{\tau}_k^{-1}\cdots\boldsymbol{\tau}_1^{-1}}$$

Questions of the Lecture 5

(L-5) QUESTION 1. Use the Gauss-Jordan method to invert

(a) $\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) $\mathbf{A}_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

(c) $\mathbf{A}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

(Strang, 1988, exercise 6 from section 1.6.)

(L-5) QUESTION 2.

(a) If $\mathbf{A}$ is invertible and $\mathbf{AB} = \mathbf{AC}$, prove quickly that $\mathbf{B} = \mathbf{C}$.
(b) If $\mathbf{A} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, find an example with $\mathbf{AB} = \mathbf{AC}$, but $\mathbf{B} \neq \mathbf{C}$.
(Strang, 1988, exercise 4 from section 1.6.)

(L-5) QUESTION 3. Use the Gauss-Jordan method to invert the generic matrix 2×2

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The matrix is invertible (not singular) only when ...

(L-5) QUESTION 4. Use the Gauss-Jordan method to invert the following matrices.

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 6 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 4 & -2 \\ 1 & 3 & 1 \end{bmatrix}$$

(L-5) QUESTION 5. If the 3 by 3 matrix $\mathbf{A}$ has $\mathbf{A}_{|1} + \mathbf{A}_{|2} = \mathbf{A}_{|3}$, show that $\mathbf{A}$ is not invertible, by two different methods:

(a) Find a nonzero solution $\boldsymbol{x}$ to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.

(b) Elimination keeps $column\,1 + column\,2 = column\,3$. Explain why there is no third pivot.

(Strang, 1988, exercise 26 from section 1.6.)

(L-5) QUESTION 6. Find the inverses of

(a) $$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}.$$

(b) $$\mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}.$$

(c) $$\mathbf{A}_3 = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}.$$

(Strang, 1988, exercise 10 from section 1.6.)

(L-5) QUESTION 7. Find the inverse of

$$\mathbf{A} = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}$$

What values of a and b make the matrix singular?
(Strang, 1988, exercise 42 from section 1.6.)

(L-5) QUESTION 8. Find $\mathbf{E}^2$, $\mathbf{E}^8$ and $\mathbf{E}^{-1}$ if $\mathbf{E} = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$
(Strang, 1988, exercise 6 from section 1.5.)

(L-5) QUESTION 9. Consider the following permutation matrix:

$$\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Find $\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}}{}^{-1}$. Can you say something else about the relationship between $\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}}$ and $\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}}{}^{-1}$?

(L-5) QUESTION 10. The 3 by 3 matrix **A** reduces to the identity matrix **I** by the following three column operations (in order):

$\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}$: Subtract 4 times column 1 from column 2.

$\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}$: Subtract 3 times column 1 from column 3.

$\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}$: Subtract column 3 from column 2.

(a) Write $\mathbf{A}^{-1}$ in terms of elementary matrices $\mathbf{I}_{\boldsymbol{\tau}}$. **Then compute $\mathbf{A}^{-1}$.**

(b) What is the original matrix **A**?

(Based on *MIT Course 18.06 Quiz 1, October 4, 2006*)

(L-5) QUESTION 11. The 3 by 3 matrix **A** reduces to the identity matrix **I** by the following three **row** operations (in order):

$\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}$: Subtract 4 times row 1 from row 2.

$\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}$: Subtract 3 times row 1 from row 3.

$\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}$: Subtract row 3 from row 2.

(a) Write $\mathbf{A}^{-1}$ in terms of the **E**'s. **Then compute $\mathbf{A}^{-1}$.**
(b) What is the original matrix **A** ?

(*MIT Course 18.06 Quiz 1, October 4, 2006*)

(L-5) QUESTION 12.

(a) Find the inverse of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(b) Find the inverse of the following matrix **using the Gauss-Jordan method**

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & d \end{bmatrix}$$

(Poole, 2004, exercise 36, 38 and 59 from section 3.3.)

(L-5) QUESTION 13. Consider the squared matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. True or false?

(a) If $\mathbf{AB} = \mathbf{I}$ and $\mathbf{CA} = \mathbf{I}$ then $\mathbf{B} = \mathbf{C}$.

(b) $(\mathbf{AB})^2 = \mathbf{A}^2\mathbf{B}^2$.

(L-5) QUESTION 14. Consider the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & a & 0 & 2a \\ a & 0 & 1 & 0 \\ 1 & 0 & a & 1 \end{bmatrix}$

(a) Prove that **A** is invertible for any value of a.
(b) Compute $\mathbf{A}^{-1}$ when $a = 0$.

(L-5) QUESTION 15. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$. Find $\mathbf{A}^{-1}$.

(L-5) QUESTION 16. Find (if it is possible) the inverse of the following inverses

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}.$$

(L-5) QUESTION 17. There is a finite number ($n!$) of $n \times n$ permutation matrices. In addition, any power of a permutation matrix is a another permutation matrix. Use these facts to prove that $\left(\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}}\right)^r = \mathbf{I}$ for some integer numbers r.

Highlights of Lesson 6

- Introduction to vector spaces and sub-spaces

2 Introduction

What are the main operations that we do with vectors?

- We add them: $\boldsymbol{v} + \boldsymbol{w}$
- We multiply them by numbers, usually called scalars: $\lambda \boldsymbol{v}$

3 Vector space: definition

A *vector space* is a set $\mathcal{V}$ together with two operations

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Addition $(\vec{x} + \vec{y})$: $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$

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satisfying:

- $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
- $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$
- There exists a unique $\vec{0}$ such that $\vec{x} + \vec{0} = \vec{x}$
- For each $\vec{x}$ there is a unique $-\vec{x}$ such that $\vec{x} + (-\vec{x}) = \vec{0}$
- $\alpha(\vec{x} + \vec{y}) = \alpha \vec{x} + \alpha \vec{y}$
- $(\alpha + \beta)\vec{x} = \alpha \vec{x} + \beta \vec{x}$
- $(\alpha \cdot \beta)\vec{x} = \alpha(\beta \vec{x})$
- $1\vec{x} = \vec{x}$

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  (they could be numbers, lists of numbers, matrices, functions, etc. . .)

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For us, scalars will be always real numbers ($\mathbb{R}$).

5 Examples: $\mathbb{R}^2$

$\mathbb{R}^2$: The space of all vectors with two components

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix}; \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \quad \begin{pmatrix} \pi \\ e \end{pmatrix};$$

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$\mathbb{R}^2$ is represented by the usual xy plane

6 More examples

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$\mathbb{R}^n$: the space af all vectors with n components

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A subspace of $\mathcal{V}$ is a vector space inside the vector space $\mathcal{V}$.

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-
-
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Which ones of the following subsets of $\mathbb{R}^2$ are subspaces?

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Every *subspace* has its own zero vector $\mathbf{0}$

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and the subspaces of $\mathbb{R}^3$? gráfico 3D

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  Is the intersection a subspace? (proof?) ☺

Questions of the Lecture 6

(L-6) Question 1.

(a) Find a subset W in $\mathbb{R}^2$ ($W \subseteq \mathbb{R}^2$) closed under vector addition (if $\boldsymbol{v}, \boldsymbol{w} \in W$, then $\boldsymbol{v} + \boldsymbol{w} \in W$), but not under scalar multiplication ($c\boldsymbol{v}$ is not necessarily in W).
(b) Find a subset W in $\mathbb{R}^2$ ($W \subseteq \mathbb{R}^2$) closed under scalar multiplication (if $\boldsymbol{v}, \boldsymbol{w} \in W$, then $c\boldsymbol{v} \in W$) but not under vector addition ($\boldsymbol{v} + \boldsymbol{w} \in W$ is not necessarily in W).

(Strang, 2006, exercise 1 from section 2.1.)

(L-6) Question 2. Consider $\mathbb{R}^2$ as a vector space. Which of the following are subspaces and which are not? If not, why not?

(a) $\{(a,\ a^2,) \mid a \in \mathbb{R}\}$
(b) $\{(b,\ 0,) \mid b \in \mathbb{R}\}$
(c) $\{(0,\ c,) \mid c \in \mathbb{R}\}$
(d) $\{(m,\ n,) \mid m, n \in \mathbb{Z}\}$ where $\mathbb{Z}$ is the set of integer numbers.
(e) $\{(d,\ e,) \mid d, e \in \mathbb{R},\ d \cdot e = 0\}$
(f) $\{(f,\ f,) \mid f \in \mathbb{R}\}$

(L-6) Question 3. Why isn't $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?
(Strang, 2006, exercise 31 from section 2.1.)

(L-6) QUESTION 4. Let P be the plane in $\mathbb{R}^3$ defined by the equation

$$x - y - z = 3.$$

Find two vectors in P and show that their sum is not in P.

(L-6) QUESTION 5. Show that for any $\boldsymbol{b} \neq \mathbf{0}$, the solution set $\{\boldsymbol{x} \mid \mathbf{A}\boldsymbol{x} = \boldsymbol{b}\}$ does not form a subspace.

(L-6) QUESTION 6. Consider the set $\mathbb{R}^{2\times 2}$ as a vector space. Let

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix};$$

(a) Name a subspace containing $\mathbf{A}$ but not $\mathbf{B}$.
(b) Name a subspace containing $\mathbf{B}$ but not $\mathbf{A}$.
(c) Is there a subspace containing $\mathbf{A}$ and $\mathbf{B}$ but not the 2×2 identity matrix $\underset{n\times n}{\mathbf{I}}$?

(L-6) QUESTION 7. Consider the set $\mathbb{R}^{n\times n}$ as a vector space. Which of the following are subspaces?

(a) The symmetric matrices, $\mathcal{S} = \{\mathbf{A} \in \mathbb{R}^{n\times n} \mid {}_{j|}\mathbf{A} = \mathbf{A}_{|j}\}$
(b) The non-symmetric matrices, $\mathcal{NS} = \{\mathbf{A} \in \mathbb{R}^{n\times n} \mid \mathbf{A}^{\mathsf{T}} \neq \mathbf{A}\}$
(c) The skew-symmetric matrices, $\mathcal{AS} = \{\mathbf{A} \in \mathbb{R}^{n\times n} \mid \mathbf{A}^{\mathsf{T}} = -\mathbf{A}\}$

(L-6) QUESTION 8.

(a) The intersection of two planes through $(0, 0, 0,)$ is probably a____________, but it could be a ____________.
(b) The intersection of a plane through $(0, 0, 0,)$ with a line through $(0, 0, 0,)$ is probably a ____________, but it could be a ____________.
(c) If $\mathcal{S}$ and $\mathcal{T}$ are subspaces of $\mathbb{R}^5$, their intersection $\mathcal{S} \cap \mathcal{T}$ (vectors in both sub-spaces) is a subspace of $\mathbb{R}^5$. Check the requirements on $\boldsymbol{x} + \boldsymbol{y}$ and $c\boldsymbol{x}$.

(Strang, 2006, exercise 18 from section 2.1.)

(L-6) QUESTION 9. Which of the following subsets of $\mathbb{R}^3$ are actually subspaces?

(a) The plane of vectors $\boldsymbol{b} = (b_1,\ b_2,\ b_3,)$ with first component $b_1 = 0$.
(b) The plane of vectors $\boldsymbol{b} = (b_1,\ b_2,\ b_3,)$ with first component $b_1 = 1$.
(c) The vectors $\boldsymbol{b}$ with $b_2 b_3 = 0$ (this is the union of two subspaces. the plane $b_2 = 0$ and the plane $b_3 = 0$).
(d) The solitary vector $\boldsymbol{b} = \mathbf{0}$.
(e) All combinations of two given vectors $(1,\ 1,\ 0,)$ and $(2,\ 0,\ 1,)$.
(f) The vectors $(b_1,\ b_2,\ b_3,)$ that satisfies $b_3 - b_2 + 3b_1 = 0$.

(Strang, 2006, exercise 2 from section 2.1.)

One more. . . not so easy

(L-6) QUESTION 10. Addition and scalar multiplication are required to satisfy these eight rules:

1. $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$.
2. $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$.

3. There is a unique $\mathbf{0}$ ("zero vector") such that $\boldsymbol{x} + \mathbf{0} = \boldsymbol{x}$ for all $\boldsymbol{x}$.
4. For each $\boldsymbol{x}$ there is a unique vector $-\boldsymbol{x}$ such that $\boldsymbol{x} + (-\boldsymbol{x}) = \mathbf{0}$.
5. $1\boldsymbol{x} = \boldsymbol{x}$.
6. $(a \cdot b)\boldsymbol{x} = a(b\boldsymbol{x})$.
7. $a(\boldsymbol{x} + \boldsymbol{y}) = a\boldsymbol{x} + a\boldsymbol{y}$.
8. $(a + b)\boldsymbol{x} = a\boldsymbol{x} + b\boldsymbol{x}$.

(a) Suppose addition in $\mathbb{R}^2$ adds an extra 1 to each component, so that $(3, 1,) + (5, 0,) = (9, 2,)$ instead of $(8, 1,)$. With scalar multiplication unchanged, which rules are broken?

(b) Show that the set of all positive real numbers is a vector space, when the addition and multiplication are redefined to be as follows:

$$\bullet\ \boldsymbol{x} + \boldsymbol{y} = xy \qquad\qquad \bullet\ \boldsymbol{x} = x^c$$

What is the "zero vector" $\mathbf{0}$?:

(c) Suppose $(x_1,\ x_2,) + (y_1,\ y_2,)$ is defined to be $\Big((x_1 + y_2),\ (x_2 + y_1), \Big)$; With the usual $c\boldsymbol{x} = (cx_1,\ cx_2,)$. which of the eight conditions are not satisfied?

(Strang, 1988, exercise 5 from section 2.1.)

Highlights of Lesson 7

- Null space of a matrix **A**: solving $\mathbf{A}x = \mathbf{0}$

A natural algorithm for solving $\mathbf{A}x = 0$?

by elimination (**column** reduction)

- **Column** (pre)echelon form
- pivot (*or endogenous*) variables and free (*or exogenous*) variables
- Special solutions

2 The subspaces of a matrix: the null space $\mathcal{N}(\mathbf{A})$

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\mathcal{N}(\mathbf{A})$ is the set of vectors $\boldsymbol{x}$ that solve $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.

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Find some solutions.

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And what does it look like? (graph) A line in $\mathbb{R}^3$

$\boxed{3}$ Is the null space $\mathcal{N}(\mathbf{A})$ a subspace?

We should check that the set of all solutions to $\mathbf{A}v = \mathbf{0}$ is a subspace.
We should check that for any $a, b \in \mathbb{R}$

If $\mathbf{A}v = \mathbf{0}$ and $\mathbf{A}w = \mathbf{0}$, then

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Therefore $\mathcal{N}\left(\mathbf{A}\right)$ is a subspace

4 Solutions of a general linear system

Let me change the right-hand side to (one, two, three, four).

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

That is a very special right-hand side. And we know that there are some solutions

$$\left\{ \qquad \qquad \qquad \qquad \right\}$$

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Do they form a subspace?

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Do they form a subspace?
Is the zero vector $\mathbf{0}$ a solution?

$\boxed{4}$ Solutions of a general linear system

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Do they form a subspace? NO;

Is the zero vector $\mathbf{0}$ a solution? NO;

It's a line that doesn't go through the origin.

5 A natural algorithm for finding $\mathcal{N}(\mathbf{A})$?

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

- Which columns are a linear combination of the other columns?

5 A natural algorithm for finding $\mathcal{N}(\mathbf{A})$?

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- Which columns are a linear combination of the other columns?
- Elimination will tell us...

6 Which columns are a linear combination of the other columns?

$$\left[\begin{array}{c} \mathbf{A} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{cccc} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \\ \hline 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right]$$

$\boxed{6}$ Which columns are a linear combination of the other columns?

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{cccc} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \\ \hline 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right] \xrightarrow[\ ]{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{3}] \\ [(-2)\mathbf{1}+\mathbf{4}]}}$$

6 Which columns are a linear combination of the other columns?

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{cccc} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \\ \hline 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right] \xrightarrow[\;]{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{3}] \\ [(-2)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 4 \\ 3 & 0 & 2 & 4 \\ \hline 1 & -2 & -2 & -2 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right]$$

6 Which columns are a linear combination of the other columns?

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{cccc} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \\ \hline 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right] \xrightarrow[\substack{[(-2)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{3}] \\ [(-2)\mathbf{1}+\mathbf{4}]}]{\tau} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 4 \\ 3 & 0 & 2 & 4 \\ \hline 1 & -2 & -2 & -2 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right] \xrightarrow[{[(-2)\mathbf{3}+\mathbf{4}]}]{\tau}$$

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$$\left[\frac{A}{I}\right] = \left[\begin{array}{cccc} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \\ \hline 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right] \xrightarrow[{[(-2)\mathbf{1}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ {[(-2)\mathbf{1}+\mathbf{2}]} \\ {[(-2)\mathbf{1}+\mathbf{3}]}}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 4 \\ 3 & 0 & 2 & 4 \\ \hline 1 & -2 & -2 & -2 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right] \xrightarrow[{[(-2)\mathbf{3}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc} 1 & \mathbf{0} & 0 & \mathbf{0} \\ 2 & \mathbf{0} & 2 & \mathbf{0} \\ 3 & \mathbf{0} & 2 & \mathbf{0} \\ \hline 1 & -2 & -2 & 2 \\ & 1 & & 0 \\ & 0 & 1 & -2 \\ & 0 & & 1 \end{array}\right]$$

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Then $\mathbf{A}\begin{pmatrix}-2\\ 1\\ 0\\ 0\end{pmatrix} = \mathbf{0} \qquad \Longrightarrow \qquad \mathbf{A}_{|2} = 2\mathbf{A}_{|1}$

6 Which columns are a linear combination of the other columns?

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Then $\quad \mathbf{A}\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{A}_{|2} = 2\mathbf{A}_{|1}$

and $\quad \mathbf{A}\begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{A}_{|4} = 2\mathbf{A}_{|3} - 2\mathbf{A}_{|1}$

7 How to compute $\mathcal{N}(\mathbf{A})$: elimination and "special solutions"

$$\boxed{\mathbf{A}\left(\mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right)_{|j} = \left(\mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right)_{|j}}$$

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{cccc} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \\ \hline 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right] \xrightarrow[{[(-2)\mathbf{1}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 4 \\ 3 & 0 & 2 & 4 \\ \hline 1 & -2 & -2 & -2 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{cccc} 1 & \mathbf{0} & 0 & \mathbf{0} \\ 2 & \mathbf{0} & 2 & \mathbf{0} \\ 3 & \mathbf{0} & 2 & \mathbf{0} \\ \hline 1 & -2 & -2 & 2 \\ & 1 & & 0 \\ & 0 & 1 & -2 \\ & 0 & & 1 \end{array}\right] = \left[\frac{\mathbf{K}}{\mathbf{E}}\right]$$

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If $\mathbf{A}\left(\mathbf{E}_{|j}\right) = \mathbf{0}$ them $\mathbf{E}_{|j}$ is a solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$

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$$\boxed{\text{The red vectors are (\textit{special}) solutions to } \mathbf{A}\boldsymbol{x} = \mathbf{0}}$$

7 How to compute $\mathcal{N}(\mathbf{A})$: elimination and "special solutions"

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The number of pivots of $\mathbf{K}$ is the *rank* of a matrix

8 How to compute $\mathcal{N}(\mathbf{A})$: general solution

The general solution: $\mathcal{N}(\mathbf{A})$

What is the set of ALL solutions?

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The general solution: $\mathcal{N}(\mathbf{A})$

What is the set of ALL solutions?

- $\mathcal{N}(\mathbf{A}) = \left\{ \text{all linear combinations of the special solutions} \right\}$

8 How to compute $\mathcal{N}(\mathbf{A})$: general solution

The general solution: $\mathcal{N}(\mathbf{A})$

What is the set of ALL solutions?

- $\mathcal{N}(\mathbf{A}) = \left\{ \text{all linear combinations of the special solutions} \right\}$
- $\mathcal{N}(\mathbf{A}) = \left\{ \boldsymbol{x} \in \phantom{\begin{matrix} a \\ b \\ c \end{matrix}} \right\}$

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The general solution: $\mathcal{N}(\mathbf{A})$

What is the set of ALL solutions?

- $\mathcal{N}(\mathbf{A}) = \left\{ \text{all linear combinations of the special solutions} \right\}$
- $\mathcal{N}(\mathbf{A}) = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \,\middle|\, \text{exists } \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \text{ such that } \boldsymbol{x} = \begin{bmatrix} -2 & 2 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right\}$

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How many special solutions are there?

One for every *null column* in **K**

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How many null columns are there?

$\boxed{8}$ How to compute $\mathcal{N}(\mathbf{A})$: general solution

The general solution: $\mathcal{N}(\mathbf{A})$

What is the set of ALL solutions?

- $\mathcal{N}(\mathbf{A}) = \left\{ \text{all linear combinations of the special solutions} \right\}$
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How many special solutions are there?
One for every *null column* in **K**

How many null columns are there?
n columns minus num. pivots (*rank*) in **K**

9 Why aren't there more solutions?

Consider $\mathbf{A}x = \mathbf{0}$ and $\mathbf{AE} = \mathbf{K}$ $\quad$ ($\mathbf{E} = \mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k}$ fullrank)

$\boxed{9}$ Why aren't there more solutions?

Consider $\mathbf{A}x = \mathbf{0}$ and $\mathbf{AE} = \mathbf{K}$ $(\mathbf{E} = \mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k}$ fullrank$)$

Is x a combination of cols. of $\mathbf{E}$? $(x = \mathbf{E}y)$

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Using $y = \mathbf{E}^{-1}x$, we have that $x = \mathbf{E}y$

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Is x a combination of cols. of $\mathbf{E}$? ($x = \mathbf{E}y$)

Using $y = \mathbf{E}^{-1}x$, we have that $x = \mathbf{E}y$

Do we need all columns of $\mathbf{E}$ to get x?

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Consider $\mathbf{A}x = \mathbf{0}$ and $\mathbf{AE} = \mathbf{K}$ $\qquad$ ($\mathbf{E} = \mathbf{I}_{\tau_1 \cdots \tau_k}$ fullrank)

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$$\begin{bmatrix} \mathbf{A} \\ \hline \mathbf{I} \end{bmatrix} \to \begin{bmatrix} \mathbf{K} \\ \hline \mathbf{E} \end{bmatrix} = \left[\begin{array}{ccccc} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ \hline \mathbf{E}_{|1} & \mathbf{E}_{|2} & \mathbf{E}_{|3} & \mathbf{E}_{|4} & \mathbf{E}_{|5} \end{array}\right]$$

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$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] \to \left[\begin{array}{c}\mathbf{K}\\ \hline \mathbf{E}\end{array}\right] = \left[\begin{array}{ccccc} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ \hline \mathbf{E}_{|1} & \mathbf{E}_{|2} & \mathbf{E}_{|3} & \mathbf{E}_{|4} & \mathbf{E}_{|5} \end{array}\right]$$

$$\mathbf{A}x = \mathbf{AE}y = \mathbf{K}y = \mathbf{0} \;\Rightarrow\; (y_j = ?\ \text{for pivot columns})$$

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Consider $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ and $\mathbf{AE} = \mathbf{K}$ $\qquad$ ($\mathbf{E} = \mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k}$ fullrank)

Is $\boldsymbol{x}$ a combination of cols. of $\mathbf{E}$? $\qquad (\boldsymbol{x} = \mathbf{E}\boldsymbol{y})$

Using $\boldsymbol{y} = \mathbf{E}^{-1}\boldsymbol{x}$, we have that $\boldsymbol{x} = \mathbf{E}\boldsymbol{y}$
Do we need all columns of $\mathbf{E}$ to get $\boldsymbol{x}$?

$$\begin{bmatrix} \mathbf{A} \\ \hline \mathbf{I} \end{bmatrix} \to \begin{bmatrix} \mathbf{K} \\ \hline \mathbf{E} \end{bmatrix} = \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ \hline \mathbf{E}_{|1} & \mathbf{E}_{|2} & \mathbf{E}_{|3} & \mathbf{E}_{|4} & \mathbf{E}_{|5} \end{bmatrix}$$

$$\mathbf{A}\boldsymbol{x} = \mathbf{AE}\boldsymbol{y} = \mathbf{K}\boldsymbol{y} = \mathbf{0} \;\Rightarrow\; (y_j = 0 \text{ for pivot columns})$$

9 Why aren't there more solutions?

Consider $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ and $\mathbf{AE} = \mathbf{K}$ $\quad$ ($\mathbf{E} = \mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k}$ fullrank)

Is $\boldsymbol{x}$ a combination of cols. of $\mathbf{E}$? $\quad (\boldsymbol{x} = \mathbf{E}\boldsymbol{y})$

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$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] \to \left[\begin{array}{c}\mathbf{K}\\ \hline \mathbf{E}\end{array}\right] = \left[\begin{array}{ccccc} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ \hline \mathbf{E}_{|1} & \mathbf{E}_{|2} & \mathbf{E}_{|3} & \mathbf{E}_{|4} & \mathbf{E}_{|5} \end{array}\right]$$

$$\mathbf{A}\boldsymbol{x} = \mathbf{AE}\boldsymbol{y} = \mathbf{K}\boldsymbol{y} = \mathbf{0} \;\Rightarrow\; (y_j = 0 \text{ for pivot columns})$$

$\forall \boldsymbol{x} \in \mathcal{N}(\mathbf{A})$, $\boldsymbol{x}$ is a combination of the *special solutions*

An algorithm for solving $\mathbf{A}x = 0$

1. Find a pre-echelon form: $\left[\frac{\mathbf{A}}{\mathbf{I}}\right] \xrightarrow{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k} \left[\frac{\mathbf{K}}{\mathbf{E}}\right]$

$\boxed{10}$ Computing $\mathcal{N}\left(\mathbf{A}\right)$: complete algorithm for solving $\mathbf{A}x = \mathbf{0}$

An algorithm for solving $\mathbf{A}x = 0$

1. Find a pre-echelon form: $\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] \xrightarrow{\tau_1 \cdots \tau_k} \left[\begin{array}{c}\mathbf{K}\\ \hline \mathbf{E}\end{array}\right]$

2. If there are *special solutions:*
   - Complete solution

$$\mathcal{N}\left(\mathbf{A}\right) = \{\text{linear combinations of special solutions }\}$$

10 Computing $\mathcal{N}\left(\mathbf{A}\right)$: complete algorithm for solving $\mathbf{A}x = \mathbf{0}$

An algorithm for solving $\mathbf{A}x = \mathbf{0}$

1. Find a pre-echelon form: $\left[\begin{array}{c} \mathbf{A} \\ \hline \mathbf{I} \end{array}\right] \xrightarrow{\tau_1 \cdots \tau_k} \left[\begin{array}{c} \mathbf{K} \\ \hline \mathbf{E} \end{array}\right]$
2. If there are *special solutions:*
   - Complete solution

$$\mathcal{N}\left(\mathbf{A}\right) = \{\text{linear combinations of special solutions }\}$$

3. If no *special solutions*
   - Complete solution:

$$\mathcal{N}\left(\mathbf{A}\right) = \{\mathbf{0}\}$$

11 Another example: $\mathcal{N}\left(\mathbf{A}^{\mathsf{T}}\right)$

$$\left[\begin{array}{c}\mathbf{A}^{\mathsf{T}} \\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

11 Another example: $\mathcal{N}\left(\mathbf{A}^\intercal\right)$

$$\left[\begin{array}{c}\mathbf{A}^\intercal\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{ccc}1 & 2 & 3\\ 2 & 4 & 6\\ 2 & 6 & 8\\ 2 & 8 & 10\\ \hline 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right] \xrightarrow[]{\substack{\boldsymbol{\tau}\\ [(-2)\mathbf{1}+\mathbf{2}]\\ [(-3)\mathbf{1}+\mathbf{3}]\\ [(-1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc}1 & 0 & 0\\ 2 & 0 & 0\\ 2 & 2 & 0\\ 2 & 4 & 0\\ \hline 1 & -2 & -1\\ 0 & 1 & -1\\ 0 & 0 & 1\end{array}\right] = \left[\begin{array}{c}\mathbf{L}\\ \hline \mathbf{E}\end{array}\right]$$

11 Another example: $\mathcal{N}\left(\mathbf{A}^{\intercal}\right)$

$$\left[\begin{array}{c}\mathbf{A}^{\intercal}\\ \hline \mathbf{I}\end{array}\right]=\left[\begin{array}{ccc}1&2&3\\2&4&6\\2&6&8\\2&8&10\\ \hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow[{[(-1)\mathbf{2}+\mathbf{3}]}]{\substack{\boldsymbol{\tau}\\ {[(-2)\mathbf{1}+\mathbf{2}]}\\ {[(-3)\mathbf{1}+\mathbf{3}]}}}\left[\begin{array}{ccc}1&0&0\\2&0&0\\2&2&0\\2&4&0\\ \hline 1&-2&-1\\0&1&-1\\0&0&1\end{array}\right]=\left[\begin{array}{c}\mathbf{L}\\ \hline \mathbf{E}\end{array}\right]$$

How many pivots are there?

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$$\left[\begin{array}{c}\mathbf{A}^\intercal \\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{2}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ {[(-2)\mathbf{1}+\mathbf{2}]} \\ {[(-3)\mathbf{1}+\mathbf{3}]}}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 4 & 0 \\ \hline 1 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c}\mathbf{L} \\ \hline \mathbf{E}\end{array}\right]$$

How many pivots are there? 2;

$\boxed{11}$ Another example: $\mathcal{N}\left(\mathbf{A}^\mathsf{T}\right)$

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$$\mathcal{N}\left(\mathbf{A}^\intercal\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \,\middle|\, \text{exists } a \in \mathbb{R} \text{ such that } \boldsymbol{x} = c \begin{pmatrix}-1 \\ -1 \\ 1\end{pmatrix} \right\}$$

Questions of the Lecture 7

(L-7) QUESTION 1. Reduce the matrices to a pre-echelon form, to find their ranks. Describe the nullspace with parametric equations.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 5 \\ 2 & 3 & 1 & 4 \\ -1 & -1 & -1 & 1 \end{bmatrix}.$

(b) $\mathbf{F} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 4 \\ 2 & -1 & -3 \end{bmatrix}$

(c) $\mathbf{G} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ -2 & -1 & 4 \end{bmatrix}.$

(d) $\mathbf{H} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ -1 & -3 \end{bmatrix}.$

(L-7) QUESTION 2. Describe the nullspace of the matrices with parametric equations

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 0 \end{bmatrix}$.

(b) $\mathbf{F} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$.

(c) $\mathbf{G} = \begin{bmatrix} 1 & 2 & -4 \\ -1 & 1 & 3 \\ 1 & 5 & -5 \end{bmatrix}$.

(L-7) QUESTION 3. Reduce $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$ to pre-echelon form, to find their ranks. Find the special solutions to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$. Find all solutions.
(Strang, 2006, exercise 2 from section 2.2.)

(L-7) QUESTION 4. Find a pre-echelon form and the rank of these matrices and the complete solution to the systems $\mathbf{A}\boldsymbol{x} = \mathbf{0}$:

(a) The 3 by 4 matrix of all ones.
(b) The 4 by 4 matrix with $a_{ij} = (-1)^{ij}$.
(c) The 3 by 4 matrix with $a_{ij} = (-1)^{j}$.

(Strang, 2006, exercise 13 from section 2.2.)

(L-7) QUESTION 5. The matrix **A** has two special solutions:

$$\boldsymbol{x}_1 = \begin{pmatrix} c \\ 1 \\ 0 \end{pmatrix}; \qquad \boldsymbol{x}_2 = \begin{pmatrix} d \\ 0 \\ 1 \end{pmatrix}$$

(a) Describe all the possibilities for the number of columns of **A**.
(b) Describe all the possibilities for the number of rows of **A**.
(c) Describe all the possibilities for the rank of **A**.

Briefly explain your answers.
(*MIT Course 18.06 Quiz 1, Fall, 2008*)

(L-7) QUESTION 6. Suppose **A** has column reduced echelon form **R**

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & \clubsuit \\ 2 & a & \clubsuit \\ 1 & 1 & \clubsuit \\ b & 8 & \clubsuit \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix}.$$

(a) What can you say about column 3 of **A**?
(b) What are the numbers a and b?
(c) Describe the nullspace of **A** if: $\quad \mathbf{A} \begin{bmatrix} -1 & 2 & 1 \\ 1 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{R}.$

Based on MIT Course 18.06 Quiz 1, March 1, 2004

(L-7) Question 7. Find the reduced column echelon form of these matrices

(a) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$.

(b) $\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix}$.

(c) $\mathbf{C} = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.

(d) $\mathbf{D} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$.

(L-7) QUESTION 8. Consider the *invertible* matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(a) Knowing $\mathbf{A}$ is invertible, and without any calculation ¿what is its reduced echelon form?
(b) Compute $\mathbf{A}^{-1}$.

(L-7) QUESTION 9. Consider the invertible matrix $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(a) Without any computation, say what is its reduced echelon form.
(b) Find the inverse of $\mathbf{A}$.

Highlights of Lesson 8

- The column space of a matrix $\mathbf{A}$: solving $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$
- We will completely solve the *general* linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, ...

Highlights of Lesson 8

- The column space of a matrix $\mathbf{A}$: solving $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$
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Highlights of Lesson 8

- The column space of a matrix **A**: solving $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$
- We will completely solve the *general* linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, . . . if it has a solution
  - is there only one solution?
  - or is there a whole family of solutions?

$$\left\{ \boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n \;\middle|\; \begin{array}{l} \mathbf{A}(\boldsymbol{x}_p) = \boldsymbol{b} \\ \\ \mathbf{A}(\boldsymbol{x}_n) = \mathbf{0} \end{array} \right\}$$

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3D graph

$\boxed{3}$ The subspaces of a matrix: The column space $\mathcal{C}(\mathbf{A})$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

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Let's connect this question with linear equations. . .

4 Link between $\mathcal{C}(\mathbf{A})$ and $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

does $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ have a solution for every $\boldsymbol{b}$?

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

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Can I find a solution for $\boldsymbol{b}_1 = \begin{pmatrix} 1, & 2, & 3, \end{pmatrix}$? and $\boldsymbol{b}_2 = \begin{pmatrix} 2, & 6, & 8, \end{pmatrix}$?

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I can solve $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ if and only if $\boldsymbol{b} \in \mathcal{C}(\mathbf{A})$

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Can I throw away any column and keep the same? $\mathcal{C}(\mathbf{A})$?

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$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Can I throw away any column and keep the same? $\mathcal{C}(\mathbf{A})$? Yes

5 Link between $\mathcal{C}(\mathbf{A})$ and $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Can I throw away any column and keep the same? $\mathcal{C}(\mathbf{A})$? Yes

and elimination will show which columns are linear combination of those to the left of it.

$\boxed{5}$ Link between $\mathcal{C}\left(\mathbf{A}\right)$ and $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Can I throw away any column and keep the same? $\mathcal{C}\left(\mathbf{A}\right)$? Yes

and elimination will show which columns are linear combination of those to the left of it.

But how does Gaussian elimination affect $\mathcal{N}\left(\mathbf{A}\right)$ and $\mathcal{C}\left(\mathbf{A}\right)$?

6 In the next example we will use the reduced echelon form

elimination:

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right]=\left[\begin{array}{cccc}1&2&2&2\\2&4&6&8\\3&6&8&10\\ \hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[\;]{\substack{[(-2)\mathbf{1}+\mathbf{2}]\\ [(-2)\mathbf{1}+\mathbf{3}]\\ [(-2)\mathbf{1}+\mathbf{4}]\\ [(-2)\mathbf{3}+\mathbf{4}]\\ [\mathbf{2}\rightleftharpoons\mathbf{3}]}}\left[\begin{array}{cccc}1&0&0&0\\2&2&0&0\\3&2&0&0\\ \hline 1&-2&-2&2\\0&0&1&0\\0&1&0&-2\\0&0&0&1\end{array}\right]$$

6 In the next example we will use the reduced echelon form

Gauss-Jordan elimination: all pivots are 1, with zeros at the left

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{cccc} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{[(-2)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{3}] \\ [(-2)\mathbf{1}+\mathbf{4}] \\ [(-2)\mathbf{3}+\mathbf{4}] \\ [\mathbf{2}\rightleftharpoons\mathbf{3}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ \hline 1 & -2 & -2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$\boxed{6}$ In the next example we will use the reduced echelon form

Gauss-Jordan elimination: all pivots are 1, with zeros at the left

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right]=\left[\begin{array}{cccc}1&2&2&2\\2&4&6&8\\3&6&8&10\\ \hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[{[\mathbf{2}\rightleftharpoons\mathbf{3}]}]{\substack{\boldsymbol{\tau}\\ [(-2)\mathbf{1}+\mathbf{2}]\\ [(-2)\mathbf{1}+\mathbf{3}]\\ [(-2)\mathbf{1}+\mathbf{4}]\\ [(-2)\mathbf{3}+\mathbf{4}]}}\left[\begin{array}{cccc}1&0&0&0\\2&2&0&0\\3&2&0&0\\ \hline 1&-2&-2&2\\0&0&1&0\\0&1&0&-2\\0&0&0&1\end{array}\right]\xrightarrow{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{2}+\mathbf{1}]\\ [\left(\frac{1}{2}\right)\mathbf{2}]}}\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\1&1&0&0\\ \hline 3&-1&-2&2\\0&0&1&0\\-1&\frac{1}{2}&0&-2\\0&0&0&1\end{array}\right]=\left[\begin{array}{c}\mathbf{R}\\ \hline \mathbf{E}\end{array}\right]$$

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$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right]=\left[\begin{array}{cccc}1&2&2&2\\2&4&6&8\\3&6&8&10\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[\substack{[(-2)\mathbf{1}+\mathbf{2}]\\[(-2)\mathbf{1}+\mathbf{3}]\\[(-2)\mathbf{1}+\mathbf{4}]\\[(-2)\mathbf{3}+\mathbf{4}]\\[\mathbf{2}\rightleftharpoons\mathbf{3}]}]{\boldsymbol{\tau}}\left[\begin{array}{cccc}1&0&0&0\\2&2&0&0\\3&2&0&0\\\hline 1&-2&-2&2\\0&0&1&0\\0&1&0&-2\\0&0&0&1\end{array}\right]\xrightarrow[\substack{[(-1)\mathbf{2}+\mathbf{1}]\\[(\frac{1}{2})\mathbf{2}]}]{\boldsymbol{\tau}}\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\1&1&0&0\\\hline 3&-1&-2&2\\0&0&1&0\\-1&\frac{1}{2}&0&-2\\0&0&0&1\end{array}\right]=\left[\frac{\mathbf{R}}{\mathbf{E}}\right]$$

$$\boxed{\left(\mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right)_{|j}=\mathbf{A}\left(\mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right)_{|j}}$$

.

6 In the next example we will use the reduced echelon form

Gauss-Jordan elimination: all pivots are 1, with zeros at the left

$$\left[\begin{array}{c}\mathbf{A}\\\hline\mathbf{I}\end{array}\right]=\left[\begin{array}{cccc}1&2&2&2\\2&4&6&8\\3&6&8&10\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[\substack{[(-2)\mathbf{1}+\mathbf{2}]\\ [(-2)\mathbf{1}+\mathbf{3}]\\ [(-2)\mathbf{1}+\mathbf{4}]\\ [(-2)\mathbf{3}+\mathbf{4}]\\ [\mathbf{2}\rightleftharpoons\mathbf{3}]}]{\boldsymbol{\tau}}\left[\begin{array}{cccc}1&0&0&0\\2&2&0&0\\3&2&0&0\\\hline 1&-2&-2&2\\0&0&1&0\\0&1&0&-2\\0&0&0&1\end{array}\right]\xrightarrow[\substack{[(-1)\mathbf{2}+\mathbf{1}]\\ [(\frac{1}{2})\mathbf{2}]}]{\boldsymbol{\tau}}\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\1&1&0&0\\\hline 3&-1&-2&2\\0&0&1&0\\-1&\frac{1}{2}&0&-2\\0&0&0&1\end{array}\right]=\left[\begin{array}{c}\mathbf{R}\\\hline\mathbf{E}\end{array}\right]$$

$$\boxed{\begin{array}{c}\left(\mathbf{A}_{\tau_1\cdots\tau_k}\right)_{|j}=\mathbf{A}\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)_{|j}\\[2em]\left(\mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right)\left(\mathbf{I}_{\boldsymbol{\tau}_k^{-1}\cdots\boldsymbol{\tau}_1^{-1}}\right)_{|j}=\mathbf{A}_{|j}\end{array}}$$

.

$\boxed{6}$ In the next example we will use the reduced echelon form

Gauss-Jordan elimination: all pivots are 1, with zeros at the left

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right]=\left[\begin{array}{cccc}1&2&2&2\\2&4&6&8\\3&6&8&10\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[\begin{subarray}{c}[(-2)\mathbf{1}+\mathbf{2}]\\ [(-2)\mathbf{1}+\mathbf{3}]\\ [(-2)\mathbf{1}+\mathbf{4}]\\ [(-2)\mathbf{3}+\mathbf{4}]\\ [\mathbf{2}\rightleftharpoons\mathbf{3}]\end{subarray}]{\boldsymbol{\tau}}\left[\begin{array}{cccc}1&0&0&0\\2&2&0&0\\3&2&0&0\\\hline 1&-2&-2&2\\0&0&1&0\\0&1&0&-2\\0&0&0&1\end{array}\right]\xrightarrow[\begin{subarray}{c}[(-1)\mathbf{2}+\mathbf{1}]\\ [\left(\frac{1}{2}\right)\mathbf{2}]\end{subarray}]{\boldsymbol{\tau}}\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\1&1&0&0\\\hline 3&-1&-2&2\\0&0&1&0\\-1&\frac{1}{2}&0&-2\\0&0&0&1\end{array}\right]=\left[\frac{\mathbf{R}}{\mathbf{E}}\right]$$

$$\boxed{\begin{array}{c}\left(\mathbf{A}_{\tau_1\cdots\tau_k}\right)_{|j}=\mathbf{A}\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)_{|j}\\[2ex]\left(\mathbf{A}_{\tau_1\cdots\tau_k}\right)\left(\mathbf{I}_{\tau_k^{-1}\cdots\tau_1^{-1}}\right)_{|j}=\mathbf{A}_{|j}\end{array}}\quad\Longrightarrow\quad\mathcal{C}\left(\mathbf{A}\right)=\mathcal{C}\left(\mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right).$$

6 In the next example we will use the reduced echelon form

Gauss-Jordan elimination: all pivots are 1, with zeros at the left

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right]=\left[\begin{array}{cccc}1&2&2&2\\2&4&6&8\\3&6&8&10\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[\substack{[(-2)\mathbf{1}+\mathbf{2}]\\[(-2)\mathbf{1}+\mathbf{3}]\\[(-2)\mathbf{1}+\mathbf{4}]\\[(-2)\mathbf{3}+\mathbf{4}]\\[\mathbf{2}\rightleftharpoons\mathbf{3}]}]{\boldsymbol{\tau}}\left[\begin{array}{cccc}1&0&0&0\\2&2&0&0\\3&2&0&0\\\hline 1&-2&-2&2\\0&0&1&0\\0&1&0&-2\\0&0&0&1\end{array}\right]\xrightarrow[\substack{[(-1)\mathbf{2}+\mathbf{1}]\\[(\frac{1}{2})\mathbf{2}]}]{\boldsymbol{\tau}}\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\1&1&0&0\\\hline 3&-1&-2&2\\0&0&1&0\\-1&\frac{1}{2}&0&-2\\0&0&0&1\end{array}\right]=\left[\frac{\mathbf{R}}{\mathbf{E}}\right]$$

$$\boxed{\begin{array}{c}\left(\mathbf{A}_{\tau_1\cdots\tau_k}\right)_{|j}=\mathbf{A}\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)_{|j}\\[2ex]\left(\mathbf{A}_{\tau_1\cdots\tau_k}\right)\left(\mathbf{I}_{\tau_k^{-1}\cdots\tau_1^{-1}}\right)_{|j}=\mathbf{A}_{|j}\end{array}}\quad\Longrightarrow\quad\mathcal{C}\left(\mathbf{A}\right)=\mathcal{C}\left(\mathbf{A}_{\tau_1\cdots\tau_k}\right).$$

But in general $\quad\mathcal{N}\left(\mathbf{A}\right)\neq\mathcal{N}\left(\mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right).$

7 Linear system of equations example

$$\begin{cases} x_1 + 2x_2 + 2x_3 + \ \ 2x_4 = b_1 \\ 2x_1 + 4x_2 + 6x_3 + \ \ 8x_4 = b_2 \\ 3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3 \end{cases}$$

$\boxed{7}$ Linear system of equations example

$$\begin{cases} x_1 + 2x_2 + 2x_3 + \ \ 2x_4 = b_1 \\ 2x_1 + 4x_2 + 6x_3 + \ \ 8x_4 = b_2 \\ 3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3 \end{cases}$$

What is going to discover elimination about the columns?

7 Linear system of equations example

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Columns that become $\mathbf{0}$ are linear combinations of those to its left

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If $b_1 = 1$ and $b_2 = 5$, what is the only b_3 that would be OK?

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$$\begin{cases} x_1 + 2x_2 + 2x_3 + \ \ 2x_4 = b_1 \\ 2x_1 + 4x_2 + 6x_3 + \ \ 8x_4 = b_2 \\ 3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3 \end{cases}$$

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Let's see!

$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$$

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$$\begin{cases} x_1 + 2x_2 + 2x_3 + 2x_4 = b_1 \\ 2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2 \\ 3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3 \end{cases}$$

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$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b} \quad \Leftrightarrow \quad \mathbf{A}\boldsymbol{x} - 1\boldsymbol{b} = \mathbf{0}$$

7 Linear system of equations example

$$\begin{cases} x_1 + 2x_2 + 2x_3 + \ \ 2x_4 = b_1 \\ 2x_1 + 4x_2 + 6x_3 + \ \ 8x_4 = b_2 \\ 3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3 \end{cases}$$

What is going to discover elimination about the columns?
Columns that become $\mathbf{0}$ are linear combinations of those to its left

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If $b_1 = 1$ and $b_2 = 5$, what is the only b_3 that would be OK?
Let's see!

$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b} \quad \Leftrightarrow \quad \mathbf{A}\boldsymbol{x} - 1\boldsymbol{b} = \mathbf{0} \quad \Leftrightarrow \quad \left[\begin{array}{c|c} \mathbf{A} & -\boldsymbol{b} \end{array}\right] \begin{pmatrix} \boldsymbol{x} \\ \hline 1 \end{pmatrix} = \mathbf{0}$$

8 Linear system of equations: condition for solvability

$$\left[\ \mathbf{A} \;\middle|\; -\boldsymbol{b}\ \right] \left(\boldsymbol{x}, \quad 1,\right) = \mathbf{0}$$

$$\left[\begin{array}{c|c} \mathbf{A} & -b \\ \hline \mathbf{I} & \mathbf{0} \\ \hline & 1 \end{array}\right] \rightarrow$$

8 Linear system of equations: condition for solvability

$$\left[\ \mathsf{A} \ \middle|\ -b\ \right](x, \quad 1,) = \mathbf{0}$$

$$\left[\begin{array}{c|c} \mathsf{A} & \text{-}b \\ \hline \mathsf{I} & \mathbf{0} \\ \hline & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \text{-}b_1 \\ 0 & 1 & 0 & 0 & \text{-}b_2 \\ 1 & 1 & 0 & 0 & \text{-}b_3 \\ \hline 3 & \text{-}1 & \text{-}2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \text{-}1 & \frac{1}{2} & 0 & \text{-}2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

8 Linear system of equations: condition for solvability

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$$\left[\begin{array}{c|c} \mathbf{A} & \text{-}b \\ \hline \mathbf{I} & \mathbf{0} \\ \hline & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \text{-}b_1 \\ 0 & 1 & 0 & 0 & \text{-}b_2 \\ 1 & 1 & 0 & 0 & \text{-}b_3 \\ \hline 3 & \text{-}1 & \text{-}2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \text{-}1 & \frac{1}{2} & 0 & \text{-}2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\,]{\substack{\boldsymbol{\tau} \\ [(b_1)\mathbf{1}+\mathbf{5}] \\ [(b_2)\mathbf{2}+\mathbf{5}]}}$$

8 Linear system of equations: condition for solvability

$$\left[\ \mathbf{A}\ \middle|\ -\boldsymbol{b}\ \right](\boldsymbol{x},\ \ 1,) = \mathbf{0}$$

$$\left[\begin{array}{c|c} \mathbf{A} & \text{-}\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \text{-}b_1 \\ 0 & 1 & 0 & 0 & \text{-}b_2 \\ 1 & 1 & 0 & 0 & \text{-}b_3 \\ \hline 3 & \text{-}1 & \text{-}2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \text{-}1 & \frac{1}{2} & 0 & \text{-}2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(b_2)\mathbf{2}+\mathbf{5}]}]{\substack{\boldsymbol{\tau} \\ [(b_1)\mathbf{1}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & b_1+b_2\text{-}b_3 \\ \hline 3 & \text{-}1 & \text{-}2 & 2 & 3b_1\text{-}b_2 \\ 0 & 0 & 1 & 0 & 0 \\ \text{-}1 & \frac{1}{2} & 0 & \text{-}2 & \text{-}b_1+\frac{1}{2}b_2 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

8 Linear system of equations: condition for solvability

$$\left[\ \mathbf{A}\ \middle|\ -b\ \right](x,\quad 1,) = \mathbf{0}$$

$$\left[\begin{array}{c|c} \mathbf{A} & \text{-}b \\ \hline \mathbf{I} & \mathbf{0} \\ \hline & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \text{-}b_1 \\ 0 & 1 & 0 & 0 & \text{-}b_2 \\ 1 & 1 & 0 & 0 & \text{-}b_3 \\ \hline 3 & \text{-}1 & \text{-}2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \text{-}1 & \frac{1}{2} & 0 & \text{-}2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(b_2)\mathbf{2}+\mathbf{5}]}]{\substack{\boldsymbol{\tau} \\ [(b_1)\mathbf{1}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & b_1+b_2\text{-}b_3 \\ \hline 3 & \text{-}1 & \text{-}2 & 2 & 3b_1\text{-}b_2 \\ 0 & 0 & 1 & 0 & 0 \\ \text{-}1 & \frac{1}{2} & 0 & \text{-}2 & \text{-}b_1+\frac{1}{2}b_2 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Then the condition for solvability is:

8 Linear system of equations: condition for solvability

$$\left[\,\mathbf{A}\;\middle|\;-b\,\right](x,\;1,) = \mathbf{0}$$

$$\left[\begin{array}{c|c} \mathbf{A} & -b \\ \hline \mathbf{I} & \mathbf{0} \\ \hline & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -b_1 \\ 0 & 1 & 0 & 0 & -b_2 \\ 1 & 1 & 0 & 0 & -b_3 \\ \hline 3 & -1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(b_2)\mathbf{2}+\mathbf{5}]}]{\substack{\boldsymbol{\tau} \\ [(b_1)\mathbf{1}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & b_1+b_2-b_3 \\ \hline 3 & -1 & -2 & 2 & 3b_1-b_2 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & -2 & -b_1+\frac{1}{2}b_2 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Then the condition for solvability is: $b_1 + b_2 - b_3 = 0$;

8 Linear system of equations: condition for solvability

$$\left[\,\mathbf{A} \mid -\boldsymbol{b}\,\right](\boldsymbol{x},\ 1,) = \mathbf{0}$$

$$\left[\begin{array}{c|c} \mathbf{A} & -\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -b_1 \\ 0 & 1 & 0 & 0 & -b_2 \\ 1 & 1 & 0 & 0 & -b_3 \\ \hline 3 & -1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(b_2)\mathbf{2}+\mathbf{5}]}]{\substack{\boldsymbol{\tau} \\ [(b_1)\mathbf{1}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & b_1+b_2-b_3 \\ \hline 3 & -1 & -2 & 2 & 3b_1-b_2 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & -2 & -b_1+\frac{1}{2}b_2 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Then the condition for solvability is: $b_1 + b_2 - b_3 = 0$;

$$\mathcal{C}\left(\mathbf{A}\right) = \left\{\boldsymbol{b} \in \mathbb{R}^3 \mid b_1 + b_2 - b_3 = 0\right\}$$

8 Linear system of equations: condition for solvability

$$\left[\ \mathbf{A}\ \middle|\ -\boldsymbol{b}\ \right](\boldsymbol{x},\ \ 1,) = \mathbf{0}$$

$$\left[\begin{array}{c|c} \mathbf{A} & -\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -b_1 \\ 0 & 1 & 0 & 0 & -b_2 \\ 1 & 1 & 0 & 0 & -b_3 \\ \hline 3 & -1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(b_2)\mathbf{2}+\mathbf{5}]}]{\substack{\boldsymbol{\tau} \\ [(b_1)\mathbf{1}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & b_1+b_2-b_3 \\ \hline 3 & -1 & -2 & 2 & 3b_1-b_2 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & -2 & -b_1+\frac{1}{2}b_2 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Then the condition for solvability is: $b_1 + b_2 - b_3 = 0$;

$$\mathcal{C}\left(\mathbf{A}\right) = \left\{\boldsymbol{b} \in \mathbb{R}^3 \mid b_1 + b_2 - b_3 = 0\right\}$$

If $b_1 = 1$ and $b_2 = 5$ then $b_3 = 6$

8 Linear system of equations: condition for solvability

$$\left[\ \mathbf{A}\ \middle|\ -\boldsymbol{b}\ \right](\boldsymbol{x},\quad 1,) = \mathbf{0}$$

$$\left[\begin{array}{c|c} \mathbf{A} & \text{-}\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \text{-}b_1 \\ 0 & 1 & 0 & 0 & \text{-}b_2 \\ 1 & 1 & 0 & 0 & \text{-}b_3 \\ \hline 3 & \text{-}1 & \text{-}2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \text{-}1 & \frac{1}{2} & 0 & \text{-}2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(b_2)\mathbf{2}+\mathbf{5}]}]{\substack{\boldsymbol{\tau} \\ {[(b_1)\mathbf{1}+\mathbf{5}]}}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & b_1+b_2\text{-}b_3 \\ \hline 3 & \text{-}1 & \text{-}2 & 2 & 3b_1\text{-}b_2 \\ 0 & 0 & 1 & 0 & 0 \\ \text{-}1 & \frac{1}{2} & 0 & \text{-}2 & \text{-}b_1+\frac{1}{2}b_2 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Then the condition for solvability is: $b_1 + b_2 - b_3 = 0$;

$$\mathcal{C}\left(\mathbf{A}\right) = \left\{\boldsymbol{b} \in \mathbb{R}^3 \mid b_1 + b_2 - b_3 = 0\right\}$$

If $b_1 = 1$ and $b_2 = 5$ then $b_3 = 6$

If $\boldsymbol{b} = (1, \quad 5, \quad 6,)$ what is the last column when the system is solvable?

8 Linear system of equations: condition for solvability

$$\left[\,\mathbf{A} \mid -\boldsymbol{b}\,\right](\boldsymbol{x},\ 1,) = \mathbf{0}$$

$$\left[\begin{array}{c|c} \mathbf{A} & \textbf{-}\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -b_1 \\ 0 & 1 & 0 & 0 & -b_2 \\ 1 & 1 & 0 & 0 & -b_3 \\ \hline 3 & -1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(b_2)\mathbf{2}+\mathbf{5}]}]{\substack{\boldsymbol{\tau} \\ [(b_1)\mathbf{1}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \hline 3 & -1 & -2 & 2 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & -2 & 3/2 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{R} & \mathbf{0} \\ \hline \mathbf{E} & \boldsymbol{x}_p \\ \hline & 1 \end{array}\right]$$

Then the condition for solvability is: $b_1 + b_2 - b_3 = 0$;

$$\mathcal{C}\left(\mathbf{A}\right) = \left\{\boldsymbol{b} \in \mathbb{R}^3 \mid b_1 + b_2 - b_3 = 0\right\}$$

If $b_1 = 1$ and $b_2 = 5$ then $b_3 = 6$

If $\boldsymbol{b} = (1,\ 5,\ 6,)$ what is the last column when the system is solvable?

8 Linear system of equations: condition for solvability

$$\left[\ \mathbf{A}\ \middle|\ -\boldsymbol{b}\ \right]\left(\boldsymbol{x},\quad 1,\right) = \mathbf{0}$$

$$\left[\begin{array}{c|c} \mathbf{A} & \text{-}\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -b_1 \\ 0 & 1 & 0 & 0 & -b_2 \\ 1 & 1 & 0 & 0 & -b_3 \\ \hline 3 & -1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(b_2)\mathbf{2}+\mathbf{5}]}]{\substack{\boldsymbol{\tau} \\ {[(b_1)\mathbf{1}+\mathbf{5}]}}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \hline 3 & -1 & -2 & 2 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & -2 & 3/2 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{R} & \mathbf{0} \\ \hline \mathbf{E} & \boldsymbol{x}_p \\ \hline & 1 \end{array}\right]$$

Then the condition for solvability is: $b_1 + b_2 - b_3 = 0$;

$$\mathcal{C}\left(\mathbf{A}\right) = \left\{\boldsymbol{b} \in \mathbb{R}^3 \mid b_1 + b_2 - b_3 = 0\right\}$$

If $b_1 = 1$ and $b_2 = 5$ then $b_3 = 6$

If $\boldsymbol{b} = \left(1, \quad 5, \quad 6,\right)$ what is the last column when the system is solvable?

Solve for $\boldsymbol{b} = \left(2, \quad 2, \quad 4,\right)$

9 An algorithm to solve the linear system $\mathbf{A}x = \boldsymbol{b}$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \; ; \quad \boldsymbol{b} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}; \qquad \mathbf{A}x = \boldsymbol{b}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & -2 \\ 2 & 4 & 6 & 8 & -2 \\ 3 & 6 & 8 & 10 & -4 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

9 An algorithm to solve the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \;;\quad \boldsymbol{b} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix};\qquad \mathbf{A}\boldsymbol{x} = \boldsymbol{b}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & \text{-}2 \\ 2 & 4 & 6 & 8 & \text{-}2 \\ 3 & 6 & 8 & 10 & \text{-}4 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 4 & 2 \\ 3 & 0 & 2 & 4 & 2 \\ \hline 1 & \text{-}2 & \text{-}2 & \text{-}2 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

9 An algorithm to solve the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \; ; \quad \boldsymbol{b} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} ; \qquad \mathbf{A}\boldsymbol{x} = \boldsymbol{b}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & -2 \\ 2 & 4 & 6 & 8 & -2 \\ 3 & 6 & 8 & 10 & -4 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 4 & 2 \\ 3 & 0 & 2 & 4 & 2 \\ \hline 1 & -2 & -2 & -2 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{3}+\mathbf{5}]}]{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 \\ 3 & 0 & 2 & 0 & 0 \\ \hline 1 & -2 & -2 & 2 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

9 An algorithm to solve the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \; ; \quad \boldsymbol{b} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} ; \qquad \mathbf{A}\boldsymbol{x} = \boldsymbol{b}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & -2 \\ 2 & 4 & 6 & 8 & -2 \\ 3 & 6 & 8 & 10 & -4 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 4 & 2 \\ 3 & 0 & 2 & 4 & 2 \\ \hline 1 & -2 & -2 & -2 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{3}+\mathbf{5}]}]{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 \\ 3 & 0 & 2 & 0 & 0 \\ \hline 1 & -2 & -2 & 2 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \text{exists } \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \text{ such that } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -1 \\ 0 \end{pmatrix} + a \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$$

10 Complete algorithm to solve the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

Apply elimination to solve $\left[\ \mathbf{A} \ \middle|\ -\boldsymbol{b}\ \right] \begin{pmatrix} \boldsymbol{x} \\ \hline 1 \end{pmatrix} = \mathbf{0}$

$$\left[\begin{array}{c|c} \mathbf{A} & \text{-}\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline 0 \cdots 0 & 1 \end{array}\right] \xrightarrow{\text{Elimination}} \left[\begin{array}{c|c} \mathbf{K} & \boldsymbol{c} \\ \hline \mathbf{E} & \boldsymbol{x}_p \\ \hline 0 \cdots 0 & 1 \end{array}\right], \quad \text{where} \quad \mathbf{K} = \mathbf{A}\mathbf{E}.$$

10 Complete algorithm to solve the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

Apply elimination to solve $\left[\ \mathbf{A} \ \middle|\ -\boldsymbol{b}\ \right] \begin{pmatrix} \boldsymbol{x} \\ \hline 1 \end{pmatrix} = \mathbf{0}$

$$\left[\begin{array}{c|c} \mathbf{A} & -\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline 0 \cdots 0 & 1 \end{array}\right] \xrightarrow{\text{Elimination}} \left[\begin{array}{c|c} \mathbf{K} & \boldsymbol{c} \\ \hline \mathbf{E} & \boldsymbol{x}_p \\ \hline 0 \cdots 0 & 1 \end{array}\right], \quad \text{where} \quad \mathbf{K} = \mathbf{AE}.$$

- If $\boldsymbol{c} \neq \mathbf{0}$, the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is not solvable.

10 Complete algorithm to solve the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

Apply elimination to solve $\left[\begin{array}{c|c} \mathbf{A} & -\boldsymbol{b} \end{array}\right] \begin{pmatrix} \boldsymbol{x} \\ \hline 1 \end{pmatrix} = \mathbf{0}$

$$\left[\begin{array}{c|c} \mathbf{A} & -\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline 0 \cdots 0 & 1 \end{array}\right] \xrightarrow{\text{Elimination}} \left[\begin{array}{c|c} \mathbf{K} & \boldsymbol{c} \\ \hline \mathbf{E} & \boldsymbol{x}_p \\ \hline 0 \cdots 0 & 1 \end{array}\right], \quad \text{where} \quad \mathbf{K} = \mathbf{AE}.$$

- If $\boldsymbol{c} \neq \mathbf{0}$, the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is not solvable.
- If $\boldsymbol{c} = \mathbf{0}$ then $\boldsymbol{b} \in \mathcal{C}\left(\mathbf{A}\right)$ and the set of solutions is

$$\left\{\boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \text{exists } \boldsymbol{y} \in \mathcal{N}\left(\mathbf{A}\right) \text{ such that } \boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{y}\right\}.$$

$\boxed{10}$ Complete algorithm to solve the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

Apply elimination to solve $\left[\ \mathbf{A} \ \middle|\ -\boldsymbol{b}\ \right] \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} = \mathbf{0}$

$$\left[\begin{array}{c|c} \mathbf{A} & \text{-}\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline 0 \cdots 0 & 1 \end{array}\right] \xrightarrow{\text{Elimination}} \left[\begin{array}{c|c} \mathbf{K} & \boldsymbol{c} \\ \hline \mathbf{E} & \boldsymbol{x}_p \\ \hline 0 \cdots 0 & 1 \end{array}\right], \quad \text{where} \quad \mathbf{K} = \mathbf{AE}.$$

- If $\boldsymbol{c} \neq \mathbf{0}$, the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is not solvable.
- If $\boldsymbol{c} = \mathbf{0}$ then $\boldsymbol{b} \in \mathcal{C}\left(\mathbf{A}\right)$ and the set of solutions is

$$\left\{\boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \text{exists } \boldsymbol{y} \in \mathcal{N}\left(\mathbf{A}\right) \text{ such that } \boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{y}\right\}.$$

If $\mathcal{N}\left(\mathbf{A}\right) = \{\mathbf{0}\}$, then $\boldsymbol{x}_p$ is the unique solution.

11 Rouché-Frobenius theorem

$$\left[\begin{array}{c|c} \mathbf{A} & \text{-}\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] \xrightarrow{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k} \left[\begin{array}{c|c} \mathbf{R} & \text{-}\boldsymbol{b} \\ \hline \mathbf{E} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{ccccccc|c} \mathbf{1} & 0 & \cdots & 0 & \cdots & 0 & 0 & \text{-}b_1 \\ 0 & \mathbf{1} & \cdots & 0 & \cdots & 0 & 0 & \text{-}b_2 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{1} & \cdots & 0 & 0 & \text{-}b_h \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \vdots & 0 & \vdots \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \mathbf{1} & \text{-}b_m \\ \hline \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 \end{array}\right]$$

where **A** (with order $m \times n$) has rank r; and where "**1**" are pivots.

11 Rouché-Frobenius theorem

$$\left[\begin{array}{c|c} \mathbf{A} & -\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] \xrightarrow{\tau_1\cdots\tau_k} \left[\begin{array}{c|c} \mathbf{R} & -\boldsymbol{b} \\ \hline \mathbf{E} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{ccccccc|c} \mathbf{1} & 0 & \cdots & 0 & \cdots & 0 & 0 & -b_1 \\ 0 & \mathbf{1} & \cdots & 0 & \cdots & 0 & 0 & -b_2 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{1} & \cdots & 0 & 0 & -b_h \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \vdots & 0 & \vdots \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \mathbf{1} & -b_m \\ \hline \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 \end{array}\right]$$

where **A** (with order $m \times n$) has rank r; and where "**1**" are pivots.

11 Rouché-Frobenius theorem

$$
\left[\begin{array}{c|c}
\mathbf{A} & \text{-}\boldsymbol{b} \\ \hline
\mathbf{I} & \mathbf{0} \\ \hline
\mathbf{0} & 1
\end{array}\right]
\xrightarrow{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}
\left[\begin{array}{c|c}
\mathbf{R} & \text{-}\boldsymbol{b} \\ \hline
\mathbf{E} & \mathbf{0} \\ \hline
\mathbf{0} & 1
\end{array}\right]
=
\left[\begin{array}{ccccccc|c}
\mathbf{1} & 0 & \cdots & 0 & \cdots & 0 & 0 & \text{-}b_1 \\
0 & \color{blue}{\mathbf{1}} & \cdots & 0 & \cdots & 0 & 0 & \color{blue}{\text{-}b_2} \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \mathbf{1} & \cdots & 0 & 0 & \text{-}b_h \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & \vdots & 0 & \vdots \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & \mathbf{1} & \text{-}b_m \\ \hline
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1
\end{array}\right]
$$

where **A** (with order $m \times n$) has rank r; and where "**1**" are pivots.

11 Rouché-Frobenius theorem

$$\left[\begin{array}{c|c} \mathbf{A} & \text{-}\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] \xrightarrow{\tau_1\cdots\tau_k} \left[\begin{array}{c|c} \mathbf{R} & \text{-}\boldsymbol{b} \\ \hline \mathbf{E} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{ccccccc|c} \mathbf{1} & 0 & \cdots & 0 & \cdots & 0 & 0 & \text{-}b_1 \\ 0 & \mathbf{1} & \cdots & 0 & \cdots & 0 & 0 & \text{-}b_2 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \color{blue}{\mathbf{1}} & \cdots & 0 & 0 & \color{blue}{\text{-}b_h} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \vdots & 0 & \vdots \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \mathbf{1} & \text{-}b_m \\ \hline \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 \end{array}\right]$$

where **A** (with order $m \times n$) has rank r; and where "**1**" are pivots.

11 Rouché-Frobenius theorem

Sist. $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$	$r = m = n$	$r = n < m$	$r = m < n$	$r < m; \quad r < n$
solutions				

$$\left[\begin{array}{c|c} \mathbf{A} & -\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] \xrightarrow{\tau_1 \cdots \tau_k} \left[\begin{array}{c|c} \mathbf{R} & -\boldsymbol{b} \\ \hline \mathbf{E} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{ccccccc|c} \mathbf{1} & 0 & \cdots & 0 & \cdots & 0 & 0 & -b_1 \\ 0 & \mathbf{1} & \cdots & 0 & \cdots & 0 & 0 & -b_2 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{1} & \cdots & 0 & 0 & -b_h \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \vdots & 0 & \vdots \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \mathbf{1} & -b_m \\ \hline \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 \end{array}\right]$$

where $\mathbf{A}$ (with order $m \times n$) has rank r; and where "$\mathbf{1}$" are pivots.

11 Rouché-Frobenius theorem

Sist. $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$	$r = m = n$	$r = n < m$	$r = m < n$	$r < m;\quad r < n$
solutions	1			

$$\left[\begin{array}{c|c} \mathbf{A} & -\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] \xrightarrow{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k} \left[\begin{array}{c|c} \mathbf{R} & -\boldsymbol{b} \\ \hline \mathbf{E} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{ccccccc|c} \mathbf{1} & 0 & \cdots & 0 & \cdots & 0 & 0 & -b_1 \\ 0 & \mathbf{1} & \cdots & 0 & \cdots & 0 & 0 & -b_2 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{1} & \cdots & 0 & 0 & -b_h \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \vdots & 0 & \vdots \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \mathbf{1} & -b_m \\ \hline \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 \end{array}\right]$$

where **A** (with order $m \times n$) has rank r; and where "**1**" are pivots.

11 Rouché-Frobenius theorem

Sist. $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$	$r = m = n$	$r = n < m$	$r = m < n$	$r < m;\quad r < n$
solutions	1			

$$\left[\begin{array}{c|c} \mathbf{A} & -\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] \xrightarrow{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k} \left[\begin{array}{c|c} \mathbf{R} & -\boldsymbol{b} \\ \hline \mathbf{E} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{ccccccc|c} \mathbf{1} & 0 & \cdots & 0 & \cdots & 0 & 0 & -b_1 \\ 0 & \mathbf{1} & \cdots & 0 & \cdots & 0 & 0 & -b_2 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{1} & \cdots & 0 & 0 & -b_h \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \vdots & \color{blue}{\mathbf{1}} & \color{blue}{-b_k} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \color{red}{\vdots} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \color{red}{-b_m} \\ \hline \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 \end{array}\right]$$

where **A** (with order $m \times n$) has rank r; and where "**1**" are pivots.

11 Rouché-Frobenius theorem

Sist. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$	$r = m = n$	$r = n < m$	$r = m < n$	$r < m;\quad r < n$
solutions	1	0 or 1		

$$\left[\begin{array}{c|c} \mathbf{A} & \text{-}\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] \xrightarrow{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k} \left[\begin{array}{c|c} \mathbf{R} & \text{-}\boldsymbol{b} \\ \hline \mathbf{E} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{ccccccc|c} \mathbf{1} & 0 & \cdots & 0 & \cdots & 0 & 0 & \text{-}b_1 \\ 0 & \mathbf{1} & \cdots & 0 & \cdots & 0 & 0 & \text{-}b_2 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{1} & \cdots & 0 & 0 & \text{-}b_h \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \vdots & \mathbf{1} & \text{-}b_k \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \text{-}b_m \\ \hline \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 \end{array}\right]$$

where **A** (with order $m \times n$) has rank r; and where "**1**" are pivots.

11 Rouché-Frobenius theorem

Sist. $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$	$r = m = n$	$r = n < m$	$r = m < n$	$r < m; \quad r < n$
solutions	1	0 or 1		

$$\left[\begin{array}{c|c} \mathbf{A} & \text{-}\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] \xrightarrow{\tau_1 \cdots \tau_k} \left[\begin{array}{c|c} \mathbf{R} & \text{-}\boldsymbol{b} \\ \hline \mathbf{E} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{ccccccc|c} \mathbf{1} & 0 & \cdots & 0 & \cdots & 0 & 0 & \text{-}b_1 \\ 0 & \mathbf{1} & \cdots & 0 & \cdots & 0 & 0 & \text{-}b_2 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{1} & \cdots & 0 & 0 & \text{-}b_h \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \vdots & 0 & \vdots \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \mathbf{1} & 0 & \text{-}b_m \\ \hline \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 \end{array}\right]$$

where **A** (with order $m \times n$) has rank r; and where "**1**" are pivots.

11 Rouché-Frobenius theorem

Sist. $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$	$r = m = n$	$r = n < m$	$r = m < n$	$r < m; \quad r < n$
solutions	1	0 or 1	∞	

$$\left[\begin{array}{c|c} \mathbf{A} & \text{-}\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] \xrightarrow{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k} \left[\begin{array}{c|c} \mathbf{R} & \text{-}\boldsymbol{b} \\ \hline \mathbf{E} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{ccccccc|c} \mathbf{1} & 0 & \cdots & 0 & \cdots & 0 & 0 & \text{-}b_1 \\ 0 & \mathbf{1} & \cdots & 0 & \cdots & 0 & 0 & \text{-}b_2 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{1} & \cdots & 0 & 0 & \text{-}b_h \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \vdots & 0 & \vdots \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \mathbf{1} & 0 & \text{-}b_m \\ \hline \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 \end{array}\right]$$

where **A** (with order $m \times n$) has rank r; and where "**1**" are pivots.

11 Rouché-Frobenius theorem

Sist. $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$	$r = m = n$	$r = n < m$	$r = m < n$	$r < m;\quad r < n$
solutions	1	0 or 1	∞	

$$\left[\begin{array}{c|c} \mathbf{A} & \text{-}\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] \xrightarrow{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k} \left[\begin{array}{c|c} \mathbf{R} & \text{-}\boldsymbol{b} \\ \hline \mathbf{E} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{ccccccc|c} \mathbf{1} & 0 & \cdots & 0 & \cdots & 0 & 0 & \text{-}b_1 \\ 0 & \mathbf{1} & \cdots & 0 & \cdots & 0 & 0 & \text{-}b_2 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{1} & \cdots & 0 & 0 & \text{-}b_h \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \mathbf{1} & 0 & \text{-}b_k \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & 0 & \text{-}b_m \\ \hline \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 \end{array}\right]$$

where **A** (with order $m \times n$) has rank r; and where "**1**" are pivots.

11 Rouché-Frobenius theorem

Sist. $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$	$r = m = n$	$r = n < m$	$r = m < n$	$r < m;\quad r < n$
solutions	1	0 or 1	∞	0 or ∞

$$\left[\begin{array}{c|c} \mathbf{A} & -\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] \xrightarrow{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k} \left[\begin{array}{c|c} \mathbf{R} & -\boldsymbol{b} \\ \hline \mathbf{E} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{ccccccc|c} \mathbf{1} & 0 & \cdots & 0 & \cdots & 0 & 0 & -b_1 \\ 0 & \mathbf{1} & \cdots & 0 & \cdots & 0 & 0 & -b_2 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{1} & \cdots & 0 & 0 & -b_h \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \mathbf{1} & 0 & -b_k \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & 0 & -b_m \\ \hline \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 \end{array}\right]$$

where **A** (with order $m \times n$) has rank r; and where "**1**" are pivots.

Questions of the Lecture 8

(L-8) Question 1. Which of these rules give a correct definition of the rank of **A**?

(a) The number of non-zero columns in **R** (reduced column echelon form).
(b) The number of columns minus the total number of rows.
(c) The number of columns minus the number of free columns.
(d) The number of ones in **R**.

(Strang, 2006, exercise 12 from section 2.2.)

(L-8) Question 2. Find the complete solution (also called the *general solution*) to

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

(Strang, 2003, exercise 4 from section 3.4.)

(L-8) QUESTION 3. Find the complete solution (also called the *general solution*) to

$$\begin{bmatrix} 1 & 2 & -1 & -2 & 1 \\ 1 & 2 & 0 & 0 & 3 \\ 2 & 4 & 1 & 2 & 9 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -4 \end{pmatrix}$$

(L-8) QUESTION 4. Resuelva el siguiente sistema de ecuaciones

$$\begin{aligned} x_1 + x_3 + x_5 &= 1 \\ x_2 + x_4 &= 1 \\ x_1 + x_2 + x_3 + x_4 &= 2 \\ x_3 + x_4 &= 2 \end{aligned}$$

(L-8) QUESTION 5.

Consider

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 6 \end{bmatrix}; \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

(a) Find the column echelon form
(b) Find the free variables
(c) Find the special solutions:
(d) $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is consistent (has a solution) when $\boldsymbol{b}$ satisfies $b_2 =$ ____.
(e) Find the complete solution to the system when b_2 satisfies the consistency condition.

(Strang, 2006, exercise 3 from section 2.2.)

(L-8) QUESTION 6. Carry out the same steps as in the previous problem to find the complete solution of $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 3 & 6 \end{bmatrix}; \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

(Strang, 2006, exercise 4 from section 2.2.)

(L-8) QUESTION 7. Describe the set of attainable right-hand sides $\boldsymbol{b}$? (the column space $\mathcal{C}(\mathbf{A})$) for

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

by finding the constraints on $\boldsymbol{b}$ (after elimination). What is the rank? give a particular solution to the system?
(Strang, 2006, exercise 6 from section 2.2.)

(L-8) QUESTION 8. Suppose a paint company paints automobiles, trains and planes.
Each automobile takes 10 man hours to prepare, 30 man hours to paint, and 12 man hours to add finishing touches (the painters are quite meticulous).
Each train takes 20 man hours to prepare, 75 man hours to paint, and 36 man hours to add finishing touches.
Each plane takes 40 man hours to prepare, 135 man hours to paint, and 64 man hours to add finishing touches.
If the paint company decides to use 760 man hours towards preparation, 2595 man hours towards painting, and 1224 man hours towards finishing touches each week, how many planes, trains and automobiles do they paint each week?

(L-8) QUESTION 9. Para el sistema $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ dado por

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 10 \\ 3 & 1 & c \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \\ 20 \end{pmatrix}$$

(a) Encuentre el valor de c que hace a la matriz **A** no invertible. Use dicho valor en los apartados siguientes.
(b) Encuentre la solución completa al sistema $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.
(c) Describa el sistema de ecuaciones mediante la visión por columnas (columnas de **A** y el vector $\boldsymbol{b}$), o bien mediante la visión por filas (las tres ecuaciones del sistema).

(L-8) QUESTION 10. Construct a matrix whose column space contains $(1, 1, 5)$ and $(0, 3, 1)$ and whose nullspace contains $(1, 1, 2)$.
(Strang, 2006, exercise 62 from section 2.2.)

(L-8) QUESTION 11. For which vectors $\boldsymbol{b}$ do these systems have a solution?

(a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

(Strang, 2006, exercise 24 from section 2.1.)

(L-8) QUESTION 12. Under what conditions on b_1 and b_2 (if any) does $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ have a solution?

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & 7 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Find two vectors in the nullspace of $\mathbf{A}$, and the complete solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$. (Strang, 2006, exercise 8 from section 2.2.)

(L-8) QUESTION 13. Sea la matriz

$$\underset{3\times 4}{\mathbf{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Sin realizar la multiplicación, diga una base de $\mathcal{N}\left(\mathbf{B}\right)$, y el rango de $\mathbf{B}$. Explique su respuesta.

(b) ¿Cuál es la solución completa a $\mathbf{B}\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$?

(L-8) QUESTION 14. For which right-hand sides (find a condition on $\boldsymbol{b}$) are these systems solvable?

(a)

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Is the column space $\mathcal{C}(\mathbf{A})$ the whole 3 dimensional space $\mathbb{R}^3$? , or is it only a plane? a line? a point?

(b)

$$\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Is the column space $\mathcal{C}(\mathbf{A})$ the whole 3 dimensional space $\mathbb{R}^3$? , or is it only a plane? a line? a point? *Based on (Strang, 2006, exercise 22 from section 2.1.)*

(L-8) QUESTION 15. The complete solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b} \in \mathbb{R}^m$ is:

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists\, c_1, c_2 \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

What is $\mathbf{A}$?

(L-8) QUESTION 16. Mike, Shai, and Tara all decide that they are unhappy with the color scheme at Math II classroom, and they do something about it. They go down to the paint store, and each buy some paint. Mike buys one gallon of red paint, six gallons of blue paint, and one gallon of yellow paint. He spends 44 euros. Shai, on the other hand, buys no red paint, two gallons of blue paint and three gallons of yellow paint. He spends 24 euros. Tara finally buys one gallon of red paint and five gallons of blue paint, and spends 33 euros.

(a) How much does each color of paint cost?
(b) What is wrong with your answer to the previous problem?
(c) When Mike, Shai and Tara compare receipts, they realize that one of them was charged 4 too little. Who was it?
(d) Tras intentar dar respuesta a la pregunta anterior, se habrá dado cuenta de que es un tanto "trabajoso" dar con el resultado. Intente lo siguiente: genere la matriz ampliada $\left[\mathbf{A}|\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{c}\right]$ donde $\mathbf{A}$ es la matriz de coeficientes del sistema de ecuaciones, y $\boldsymbol{a}$ es el vector de precios suponiendo que a Ana deberían haberle cobrado 4 euros más (es decir 48 en lugar de 44), $\boldsymbol{b}$ el vector de precios suponiendo que sólo a Belén deberían haberle cobrado 4 euros más, y $\boldsymbol{c}$ lo mismo para Carlos. Calcule la forma escalonada reducida de la matriz ampliada. A la vista de lo obtenido ¿cuanto vale cada bote de pintura? y ¿a quien han cobrado 4 euros de menos?

(L-8) QUESTION 17. Suponga que el sistema de ecuaciones $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ es consistente (que tiene solución), donde $\underset{m \times n}{\mathbf{A}}$ y $\boldsymbol{x} = (x_1, ..., x_n)$. **Demuestre** las siguientes afirmaciones:

(a) $\boldsymbol{b} \in \mathcal{C}\left(\mathbf{A}\right)$.
(b) Si $\boldsymbol{x}_0$ es una solución particular del sistema, entonces cualquier vector de la forma $\boldsymbol{x}_0 + \boldsymbol{z}$, donde $\boldsymbol{z} \in \mathcal{N}\left(\mathbf{A}\right)$, es también solución del sistema.
(c) Demuestre que si hay dependencia lineal entre las columnas de $\mathbf{A}$, entonces hay más de una solución.

(L-8) QUESTION 18. Solve the following system of equations using Gaussian elimination.

$$\begin{cases} 3x & + & y & + & z & = & 6 \\ x & - & y & - & z & = & -2 \\ & & 4y & + & z & = & 3 \end{cases}$$

(L-8) QUESTION 19. Write these ancient problems in a 2 by 2 matrix form $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, and solve them:

(a) X is twice as old as Y and their ages add to 39.
(b) $(x, y,) = (2, 5,)$ and $(x, y,) = (3, 7,)$ lie on the line $y = mx + c$. Find m and c.

(Strang, 1988, exercise 32 from section 1.4.)

(L-8) QUESTION 20. The parabola $y = a + bx + cx^2$ goes through the points $(x, y,) = (1, 4,)$, $(2, 8,)$ and $(3, 14,)$. Find and solve a matrix equation for the unknowns $\boldsymbol{x} = \left(a, \quad b, \quad c,\right)$.

(Strang, 1988, exercise 33 from section 1.4.)

(L-8) QUESTION 21. Explain why the system

$$\begin{cases} u + v + w = 2 \\ u + 2v + 3w = 1 \\ v + 2w = 0 \end{cases}$$

is singular and has no solution.
What value should replace the last zero on the right hand side, to allow the the equations to have solutions— and what is one of the solutions?
(Strang, 1988, exercise 8 from section 1.2.)

(L-8) QUESTION 22. Choose a coefficient b that makes this system singular. Then choose a value for g that makes it solvable. Find two solutions in that singular case.

$$\begin{cases} 2x + by &= 16 \\ 4x + 8y &= g \end{cases}$$

Based on (Strang, 2003, exercise 6 from section 2.2.)

(L-8) QUESTION 23. Solve the following nonsingular triangular system. Show that your solution gives the linear combination of the columns that equals the column of the right $\boldsymbol{b} = \begin{pmatrix} b_1, & b_2, & b_3, \end{pmatrix}$.

$$\begin{aligned} u - v + w &= b_1 \\ v + w &= b_2 \\ w &= b_3. \end{aligned}$$

Check your answer multiplying **A** by your solution vector.
(Strang, 1988, exercise 2 from section 1.2.)

(L-8) QUESTION 24. Find **A** and **B** with the given property or explain why you can't.

(a) The only solution to $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is $\boldsymbol{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(b) The only solution to $\mathbf{B}\boldsymbol{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is $\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

(Strang, 2006, exercise 49 from section 2.2.)

(L-8) QUESTION 25. The complete solution to $\mathbf{A}x = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is $\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Find **A**.
(Strang, 2006, exercise 50 from section 2.2.)

(L-8) QUESTION 26. Suppose the fifth column of **L** has no pivot. Then x_5 is a ____________ variable. The zero vector (is) (is not) the only solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$. If $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution, then it has____________ solutions.
(Strang, 2006, exercise 40 from section 2.2.)

(L-8) QUESTION 27. Consider a linear system of algebraic equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$. Here the matrix **A** has three rows and four columns.

(a) Does such a linear system always have at least one solution? If not provide an example for which no solution exists.
(b) Can such a linear system have a unique solution? If so, provide and example of a problem with this property.
(c) Formulate, if possible, necessary and sufficient conditions on **A** and $\boldsymbol{b}$ which guarantee that at least one solution exists.
(d) Formulate, if possible, necessary and sufficient conditions on **A** which guarantee that at least one solution exists for any choice of $\boldsymbol{b}$.

(L-8) QUESTION 28. By performing column eliminations on the 4×7 matrix **A**

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 2 & 0 & -1 & 0 \\ 2 & -2 & 1 & 5 & 0 & -1 & 0 \\ -3 & 3 & -1 & -7 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

we got the following matrix $\mathbf{B} = \mathbf{A}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k}$:

$$\mathbf{B} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}; \quad \text{where } \mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k} = \begin{bmatrix} 2 & 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 2 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

(a) What is the rank of **A**? Find the complete solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.
(b) Write, if it is possible, the general solution as a function of x_2, x_4, and x_6.
(c) Is it possible to find a vector $\boldsymbol{b}$ in $\mathbb{R}^4$ that is not in the column space of **A** ($\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has no solution)? If it is, give an example.
(d) Give a vector $\boldsymbol{b}$ such that the vector $\boldsymbol{x} = \mathbf{I}_{|1}$ is a solution to the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.
(e) If $\boldsymbol{b}$ is the sum of columns of **A**, find, if it is possible, the full solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.

A modified version of MIT Course 18.06 Quiz 1, October 4, 2004

(L-8) QUESTION 29. $\underset{m\times n}{\mathbf{A}}$ es una matriz de rango r. Suponga que $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$ no tiene solución para algunos vectores $\boldsymbol{b}$, pero infinitas soluciones para otros vectores $\boldsymbol{b}$.

(a) Decida si el espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$ contiene sólo el vector cero, y explique porqué.
(b) Decida si el espacio columna $\mathcal{C}\left(\mathbf{A}\right)$ es todo $\mathbb{R}^m$ y explique porqué.
(c) Para esta matriz $\mathbf{A}$, encuentre las relaciones entre los números r, m; y entre r y n.

(d) ¿Puede existir un lado derecho $\boldsymbol{b}$ para el que $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$ tenga una y sólo una solución? ¿Porqué es posible o porqué no?

(L-8) QUESTION 30. Sea la matriz

$$\mathbf{A}=\begin{bmatrix} 2 & 3 & 1 & -1 \\ 6 & 9 & 3 & -2 \end{bmatrix}$$

(a) Encuentre un conjunto de soluciones del sistema $\mathbf{A}\boldsymbol{x}=\mathbf{0}$ y describa con él el espacio nulo de $\mathbf{A}$.
(b) Encuentre la solución completa— es decir todas las soluciones (x_1,x_2,x_3,x_4) — de

$$\mathbf{A}\boldsymbol{x}=\begin{pmatrix}1\\2\end{pmatrix}.$$

(c) Cuando una matriz $\underset{m\times n}{\mathbf{A}}$ tiene rango $r=m$ ¿para qué vectores $\boldsymbol{b}$ el sistema $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$ puede resolverse? ¿Cuantas soluciones especiales tiene $\mathbf{A}\boldsymbol{x}=\mathbf{0}$ (dimensión del espacio nulo)?

(L-8) QUESTION 31. consider the system of linear equations,

$$\begin{cases} x + y + 2z = 1 \\ 2x + 2y - z = 1 \\ \quad\; y + cz = 2 \end{cases}$$

For which number c the system has no solution? Only one? and infinite solutions?

(L-8) QUESTION 32. Consider the following system of linear equations

$$\begin{cases} x - y + 2z & = 1 \\ 2x - 3y + mz & = 3 \\ -x + 2y + 3z & = 2m \end{cases}$$

(a) Show that the system has solution for any value m
(b) Find the solution when $m = -1$.
(c) Is the set of solutions to the system in the last question $(m = -1)$ a line in $\mathbb{R}^3$? Is there any m such as the set of solutions to the system is a plane in $\mathbb{R}^3$?... and a point in $\mathbb{R}^3$?
(d) Find the solution to the system when $m = 1$.

(L-8) QUESTION 33. Which descriptions are correct? The solutions $\boldsymbol{x}$ of

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

form:

(a) A plane
(b) A line (c) A point (d) A subspace
(e) The nullspace of **A**. (f) The column space of **A**.

(Strang, 2006, exercise 8 from section 2.1.)

(L-8) QUESTION 34. Consider the equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

$$\begin{bmatrix} 1 & 0 \\ 4 & 1 \\ 2 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

For which $\boldsymbol{b}$ are there solutions?
Based in *MIT Course 18.06 Quiz 1, Fall 2008*

(L-8) QUESTION 35. En un teatro de barrio, tres grupos están haciendo cola. Hay cuatro tipos de tarifas; tercera edad (t), adulto (a), infantil (i) y tarifa con descuento para empleados del teatro y familiares (d).
El primer grupo compra tres entradas de adulto y tres infantiles por 39 euros.
El segundo grupo compra tres entradas de adulto y cuatro de la tercera edad por 44 euros
El tercer grupo compra dos entradas con descuento y dos entradas infantiles por 22 euros

(a) Si intenta descubrir el precio de cada entrada ¿cuantas soluciones puede encontrar? Ninguna, una, o infinitas
(b) Si las entradas de la tercera edad valen lo mismo que las infantiles. ¿Cuánto vale cada tipo de entrada?

(L-8) QUESTION 36. Consider the following system of linear equations

$$\begin{cases} x_1 + 2x_2 - x_3 + x_4 = -1 \\ -x_1 - 2x_2 + 3x_3 + 5x_4 = -5 \\ -x_1 - 2x_2 - x_3 - 7x_4 = 7 \end{cases}$$

(a) (0.5 pts) What is the rank of the coeficient matrix?
(b) (1.5 pts) Find all solutions to the system of linear equations
(c) (0.5 pts) Describe the geometric shape of the collection of all solutions to the above equations considered as a subset of $\mathbb{R}^4$.

(L-8) QUESTION 37. Consider the following linear system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ a & 1 & 1 & 2 \end{bmatrix}.$$

(a) (0.5^{pts}) Find the values of a such as the set of solutions of the linear system is a line.
(b) (0.5^{pts}) Find the values of a such as the set of solutions of the linear system is a plane?

(L-8) QUESTION 38. Find the complete solution to the system

$$\begin{bmatrix} 1 & 3 & 2 & 4 & -3 \\ 2 & 6 & 0 & -1 & -2 \\ 0 & 0 & 6 & 2 & -1 \\ 1 & 3 & -1 & 4 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -7 \\ 0 \\ 12 \\ -6 \end{pmatrix}$$

(L-8) QUESTION 39. Sea la matriz $\underset{3\times 4}{\mathbf{A}}$ y el vector columna $\boldsymbol{b}$ de $\mathbb{R}^3$:

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 & 2 \\ 2 & 7 & 6 & 8 \\ 3 & 9 & 6 & 7 \end{bmatrix}; \qquad \boldsymbol{b} = \begin{pmatrix} 2 \\ 7 \\ 7 \end{pmatrix}$$

(a) Encuentre todas la soluciones al sistema $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ (si es que existen soluciones). Describa el conjunto de soluciones geométricamente. ¿Es dicho conjunto un sub-espacio vectorial?

(b) ¿Quién es el espacio columna $\mathcal{C}\left(\mathbf{A}\right)$? Cambie el 7 de la esquina inferior derecha por un número que conduzca a un espacio columna más pequeño de la nueva matriz (digamos $\mathbf{M}$). Dicho número es ____.

(c) Encuentre un lado derecho $\boldsymbol{b}$ tal que, para la nueva matriz, el sistema $\mathbf{M}\boldsymbol{x} = \boldsymbol{b}$ tenga solución; y otro lado derecho $\boldsymbol{b}$ tal que $\mathbf{M}\boldsymbol{x} = \boldsymbol{b}$ no tenga solución.

Highlights of Lesson 9

- Linear *independence*
- vectors *spanning* a space
- **BASIS** and dimension

$\boxed{2}$ Homogeneous equations: our starting point

Suppose I have a matrix $\underset{m\times n}{\mathbf{A}}$ with $m < n$ and I look at $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.

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(more unknowns than equations ($m < n$), *free columns*)

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Then, there's something in $\mathcal{N}\left(\mathbf{A}\right)$, other than just the zero vector.

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Then, there's something in $\mathcal{N}\left(\mathbf{A}\right)$, other than just the zero vector.

There are non-trivial linear combinations $\mathbf{A}\boldsymbol{x}$ that give $\mathbf{0}$

3 Linear independence

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if no combination gives the zero vector $\vec{0}$

$$(\vec{v}_1)p_1 + (\vec{v}_2)p_2 + \cdots + (\vec{v}_n)p_n \neq \vec{0}$$

except the zero combination (all $p_i = 0$).

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$[\vec{v}_1; \ \ldots \ \vec{v}_n;]\boldsymbol{p} = \vec{0}$ if and only if $\boldsymbol{p} = \mathbf{0}$

4 linear independence: examples in $\mathbb{R}^2$

Can you find numbers a and b such that $a\boldsymbol{v} + b\boldsymbol{w} = \mathbf{0}$?

- $\boldsymbol{v}$ and $\boldsymbol{w} = 2\boldsymbol{v}$

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- $\boldsymbol{v}$ and $\boldsymbol{w} = \mathbf{0}$ — $a = 0;\ b = 100$

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$$\underset{\times}{\mathbf{A}}\ \boldsymbol{x} = [\boldsymbol{v}_1; \quad \boldsymbol{v}_2; \quad \boldsymbol{v}_3;] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

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dependent vectors if: $\quad \mathcal{N}\left(\mathbf{A}\right)$ bigger than $\{\mathbf{0}\}$

$\boxed{5}$ linear independence and rank of a matrix

Columns of $\underset{m\times n}{\mathbf{A}}$ are:

- independent:

  If the null space $\mathcal{N}\left(\mathbf{A}\right)$ *is*

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Columns of $\underset{m \times n}{\mathbf{A}}$ are:

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5 linear independence and rank of a matrix

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6 Space spanned by a system of vectors: Generating system

Generating system

The system $\mathsf{Z} = \left[\vec{z}_1; \ldots \vec{z}_j;\right]$ **spans** subspace $\mathcal{W}$ if their linear combinations fill $\mathcal{W}$

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Example

- The column space:

$$\mathcal{C}\left(\mathbf{A}\right) = \{\boldsymbol{b} \mid \exists \boldsymbol{x} \text{ such that } \boldsymbol{b} = \mathbf{A}\boldsymbol{x}\} = \mathcal{L}\Big(\text{columns of } \mathbf{A}\Big).$$

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7 A Basis for a Vector Space

Basis for a subspace $\mathcal{W}$

is a system of vectors $\left[\vec{z}_1; \ldots \vec{z}_d;\right]$ such that;

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examples

$\mathbb{R}^3$:

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All bases for a given subspace $\mathcal{W}$ contain the same *number* of vectors

8 Dimension

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That number indicates the “size” of the space

9 Examples: $\mathcal{C}\left(\mathbf{A}\right)$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix};$$

- do the columns span $\mathcal{C}\left(\mathbf{A}\right)$?

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- are the columns a basis for $\mathcal{C}(\mathbf{A})$? No
- What is the rank of $\mathbf{A}$?

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$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix};$$

- do the columns span $\mathcal{C}\left(\mathbf{A}\right)$? Yes
- are the columns a basis for $\mathcal{C}\left(\mathbf{A}\right)$? No
- What is the rank of $\mathbf{A}$? r=2

9 Examples: $\mathcal{C}\left(\mathbf{A}\right)$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix};$$

- do the columns span $\mathcal{C}\left(\mathbf{A}\right)$? Yes
- are the columns a basis for $\mathcal{C}\left(\mathbf{A}\right)$? No
- What is the rank of $\mathbf{A}$? r=2

$$\boxed{\mathrm{rg}\left(\mathbf{A}\right) = \text{num. pivots} = \text{dimension of } \mathcal{C}\left(\mathbf{A}\right)}$$

9 Examples: $\mathcal{C}(\mathbf{A})$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix};$$

- do the columns span $\mathcal{C}(\mathbf{A})$? Yes
- are the columns a basis for $\mathcal{C}(\mathbf{A})$? No
- What is the rank of $\mathbf{A}$? r=2
- write down some bases for $\mathcal{C}(\mathbf{A})$

$$\boxed{\mathrm{rg}(\mathbf{A}) = \text{num. pivots} = \text{dimension of } \mathcal{C}(\mathbf{A})}$$

$\boxed{10}$ Examples: $\mathcal{N}(\mathbf{A})$

$$\underset{m\times n}{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}; \qquad \boldsymbol{v} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

- Is $\boldsymbol{v}$ in $\mathcal{N}(\mathbf{A})$?

10 Examples: $\mathcal{N}(\mathbf{A})$

$$\underset{m\times n}{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}; \qquad \boldsymbol{v} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

- Is $\boldsymbol{v}$ in $\mathcal{N}(\mathbf{A})$? Yes

$\boxed{10}$ Examples: $\mathcal{N}(\mathbf{A})$

$$\underset{m \times n}{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}; \qquad \boldsymbol{v} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

- Is v in $\mathcal{N}(\mathbf{A})$? Yes
- Does $\boldsymbol{v}$ span $\mathcal{N}(\mathbf{A})$?

$\boxed{10}$ Examples: $\mathcal{N}(\mathbf{A})$

$$\underset{m\times n}{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}; \qquad \boldsymbol{v} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

- Is v in $\mathcal{N}(\mathbf{A})$? Yes
- Does $\boldsymbol{v}$ span $\mathcal{N}(\mathbf{A})$? NO

$\boxed{10}$ Examples: $\mathcal{N}(\mathbf{A})$

$$\underset{m\times n}{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}; \qquad \boldsymbol{v} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

- Is $\boldsymbol{v}$ in $\mathcal{N}(\mathbf{A})$? Yes
- Does $\boldsymbol{v}$ span $\mathcal{N}(\mathbf{A})$? NO
- write down another vector of $\mathcal{N}(\mathbf{A})$ independent of $\boldsymbol{v}$.

$\boxed{10}$ Examples: $\mathcal{N}(\mathbf{A})$

$$\underset{m\times n}{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}; \qquad \boldsymbol{v} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{w} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- Is $\boldsymbol{v}$ in $\mathcal{N}(\mathbf{A})$? Yes
- Does $\boldsymbol{v}$ span $\mathcal{N}(\mathbf{A})$? NO
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$\boxed{\mathbf{10}}$ Examples: $\mathcal{N}(\mathbf{A})$

$$\underset{m\times n}{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}; \qquad \boldsymbol{v} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{w} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- Is $\boldsymbol{v}$ in $\mathcal{N}(\mathbf{A})$? Yes
- Does $\boldsymbol{v}$ span $\mathcal{N}(\mathbf{A})$? NO
- write down another vector of $\mathcal{N}(\mathbf{A})$ independent of $\boldsymbol{v}$.
- do $\boldsymbol{v}$ and $\boldsymbol{w}$ span $\mathcal{N}(\mathbf{A})$?

$\boxed{10}$ Examples: $\mathcal{N}\left(\mathbf{A}\right)$

$$\underset{m\times n}{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}; \qquad \boldsymbol{v} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{w} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- Is v in $\mathcal{N}\left(\mathbf{A}\right)$? Yes
- Does v span $\mathcal{N}\left(\mathbf{A}\right)$? NO
- write down another vector of $\mathcal{N}\left(\mathbf{A}\right)$ independent of v.
- do $\boldsymbol{v}$ and $\boldsymbol{w}$ span $\mathcal{N}\left(\mathbf{A}\right)$? Yes

10 Examples: $\mathcal{N}(\mathbf{A})$

$$\underset{m\times n}{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}; \qquad \boldsymbol{v} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{w} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- Is $\boldsymbol{v}$ in $\mathcal{N}(\mathbf{A})$? Yes
- Does $\boldsymbol{v}$ span $\mathcal{N}(\mathbf{A})$? NO
- write down another vector of $\mathcal{N}(\mathbf{A})$ independent of $\boldsymbol{v}$.
- do $\boldsymbol{v}$ and $\boldsymbol{w}$ span $\mathcal{N}(\mathbf{A})$? Yes
- are $\boldsymbol{v}$ and $\boldsymbol{w}$ a basis for $\mathcal{N}(\mathbf{A})$?

$\boxed{10}$ Examples: $\mathcal{N}(\mathbf{A})$

$$\underset{m\times n}{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}; \qquad \boldsymbol{v} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{w} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- Is $\boldsymbol{v}$ in $\mathcal{N}(\mathbf{A})$? Yes
- Does $\boldsymbol{v}$ span $\mathcal{N}(\mathbf{A})$? NO
- write down another vector of $\mathcal{N}(\mathbf{A})$ independent of $\boldsymbol{v}$.
- do $\boldsymbol{v}$ and $\boldsymbol{w}$ span $\mathcal{N}(\mathbf{A})$? Yes
- are $\boldsymbol{v}$ and $\boldsymbol{w}$ a basis for $\mathcal{N}(\mathbf{A})$? Yes

10 Examples: $\mathcal{N}(\mathbf{A})$

$$\underset{m\times n}{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}; \qquad \boldsymbol{v} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{w} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- Is $\boldsymbol{v}$ in $\mathcal{N}(\mathbf{A})$? Yes
- Does $\boldsymbol{v}$ span $\mathcal{N}(\mathbf{A})$? NO
- write down another vector of $\mathcal{N}(\mathbf{A})$ independent of $\boldsymbol{v}$.
- do $\boldsymbol{v}$ and $\boldsymbol{w}$ span $\mathcal{N}(\mathbf{A})$? Yes
- are $\boldsymbol{v}$ and $\boldsymbol{w}$ a basis for $\mathcal{N}(\mathbf{A})$? Yes
- What is the dimension of $\mathcal{N}(\mathbf{A})$?

10 Examples: $\mathcal{N}(\mathbf{A})$

$$\underset{m\times n}{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}; \qquad \boldsymbol{v} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{w} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- Is $\boldsymbol{v}$ in $\mathcal{N}(\mathbf{A})$? Yes
- Does $\boldsymbol{v}$ span $\mathcal{N}(\mathbf{A})$? NO
- write down another vector of $\mathcal{N}(\mathbf{A})$ independent of $\boldsymbol{v}$.
- do $\boldsymbol{v}$ and $\boldsymbol{w}$ span $\mathcal{N}(\mathbf{A})$? Yes
- are $\boldsymbol{v}$ and $\boldsymbol{w}$ a basis for $\mathcal{N}(\mathbf{A})$? Yes
- What is the dimension of $\mathcal{N}(\mathbf{A})$? $n-r$

10 Examples: $\mathcal{N}(\mathbf{A})$

$$\underset{m\times n}{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}; \qquad \boldsymbol{v} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{w} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- Is $\boldsymbol{v}$ in $\mathcal{N}(\mathbf{A})$? Yes
- Does $\boldsymbol{v}$ span $\mathcal{N}(\mathbf{A})$? NO
- write down another vector of $\mathcal{N}(\mathbf{A})$ independent of $\boldsymbol{v}$.
- do $\boldsymbol{v}$ and $\boldsymbol{w}$ span $\mathcal{N}(\mathbf{A})$? Yes
- are $\boldsymbol{v}$ and $\boldsymbol{w}$ a basis for $\mathcal{N}(\mathbf{A})$? Yes
- What is the dimension of $\mathcal{N}(\mathbf{A})$? $n-r$

$$\boxed{n - \operatorname{rg}(\mathbf{A}) = \text{ num. free variables} = \dim \mathcal{N}(\mathbf{A})}$$

Questions of the Lecture 9

(L-9) QUESTION 1. Decide whether or not the following vectors are linearly independent, by solving $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 + c_4\boldsymbol{v}_4 = \mathbf{0}$:

$$\boldsymbol{v}_1 = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \qquad \boldsymbol{v}_2 = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \qquad \boldsymbol{v}_3 = \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \qquad \boldsymbol{v}_4 = \begin{pmatrix}0\\1\\0\\1\end{pmatrix}.$$

Decide also if they span $\mathbb{R}^4$, by trying to solve
$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 + c_4\boldsymbol{v}_4 = \begin{pmatrix}0, & 0, & 0, & 1,\end{pmatrix}$.
(Strang, 2006, exercise 16 from section 2.3.)

(L-9) QUESTION 2. Suppose $\boldsymbol{v}_1 \ldots \boldsymbol{v}_6$ are six vectors in $\mathbb{R}^4$.

(a) Those vectors (do)(do not)(might not) span $\mathbb{R}^4$.
(b) Those vectors (are)(are not)(might be) linearly independent.
(c) If those vectors are the columns of $\mathbf{A}$, then $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ (has) (does not have) (might not have) a solution.
(d) If those vectors are the columns of $\mathbf{A}$, then $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ (has) (does not have) (might not have) a sole solution.

(Strang, 2006, exercise 22 from section 2.3.)

(L-9) QUESTION 3. Find **B** and **C** with the given property or explain why you can't.

(a) The complete solution to $\mathbf{B}\boldsymbol{x} = \begin{pmatrix}1\\2\\4\end{pmatrix}$ is $\boldsymbol{x} = \begin{pmatrix}1\\0\end{pmatrix}$. Find **B** or explain why you can't.

(b) The complete solution to $\mathbf{C}\boldsymbol{x} = \begin{pmatrix}5\\1\end{pmatrix}$ is $\boldsymbol{x} = \begin{pmatrix}1\\4\\-2\end{pmatrix}$. Find **C** or explain why you can't.

(L-9) QUESTION 4. Show that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are independent but $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ are dependent:

$$\boldsymbol{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}; \quad \boldsymbol{v}_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}; \quad \boldsymbol{v}_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}; \quad \boldsymbol{v}_4 = \begin{pmatrix}2\\3\\4\end{pmatrix}.$$

Solve $\mathbf{A}\boldsymbol{c} = \mathbf{0}$ (where the $\boldsymbol{v}$'s go in the columns of **A**).
(Strang, 2006, exercise 1 from section 2.3.)

(L-9) QUESTION 5. True or false?

If $\mathbf{A}^\intercal = 2\mathbf{A}$, *then the rows of* **A** *are linearly dependent.*

(L-9) Question 6. Which of the following sets of vectors span $\mathbb{R}^3$?

(a) $(1, 2, 0,)$ and $(0, -1, 1,)$.
(b) $(1, 1, 0,)$, $(0, 1, -2,)$, and $(1, 3, 1,)$.
(c) $(-1, 2, 3,)$, $(2, 1, -1,)$, and $(4, 7, 3,)$.
(d) $(1, 0, 2,)$, $(0, 1, 0,)$, $(-1, 3, 0,)$, and $(1, -4, 1,)$.

(L-9) Question 7. Which of the following systems of vectors are linearly independent? In case of linear dependence, write one of the vectors as a linear combination of the others.

(a) $(-1, 2, 3,)$, $(2, 1, -1,)$, and $(4, 7, 3,)$ in $\mathbb{R}^3$.
(b) $(1, 2, 0,)$ and $(0, -1, 1,)$ in $\mathbb{R}^3$.
(c) $(1, 2,)$, $(2, 3,)$, and $(8, -2,)$ in $\mathbb{R}^2$.
(d) $t^2 + 2t + 1$, $t^3 - t^2$, $t^3 + 1$, and $t^3 + t + 1$ in P_3.

(L-9) Question 8. Suppose the only solution to $\underset{m\times n}{\mathbf{A}}\ \boldsymbol{x} = \mathbf{0}$ (m equations in n unknowns) is $\boldsymbol{x} = \mathbf{0}$. What is the rank and why? The columns of A are linearly ____________.
(Strang, 2006, exercise 8 from section 2.4.)

(L-9) Question 9. **[Important]** If **A** has order 4×6, prove that the columns of **A** are linearly dependent.

(L-9) QUESTION 10. **A** is such that $\mathcal{N}(\mathbf{A}) = \mathcal{L}\left(\left[\begin{pmatrix}1\\2\\-1\\3\end{pmatrix}; \quad \begin{pmatrix}0\\1\\1\\4\end{pmatrix}; \quad \begin{pmatrix}-1\\-1\\3\\1\end{pmatrix};\right]\right)$.

(a) Find a matrix **B** such that its column space $\mathcal{C}(\mathbf{B}) = \mathcal{N}(\mathbf{A})$. [Thus, any vector $\boldsymbol{y} \in \mathcal{N}(A)$ satisfies $\mathbf{B}\boldsymbol{u} = \boldsymbol{y}$ for some $\boldsymbol{u}$.]

(b) Give a different possible answer to (a): another **B** with $\mathcal{C}(\mathbf{B}) = \mathcal{N}(\mathbf{A})$.

(c) For some vector $\boldsymbol{b}$, you are told that a particular solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is

$$\boldsymbol{x}_p = (1, \quad 2, \quad 3, \quad 4,)$$

Now, your classmate Zarkon tells you that a second solution is:

$$\boldsymbol{x}_Z = (1, \quad 1, \quad 3, \quad 0,)$$

while your other classmate Hastur tells you "No, Zarkon's solution can't be right, but here's a second solution that is correct:"

$$\boldsymbol{x}_H = (1, \quad 1, \quad 3, \quad 1,)$$

Is Zarkon's solution correct, or Hastur's solution, or are both correct? (Hint: what should be true of $\boldsymbol{x}$ - $\boldsymbol{x}_p$ if $\boldsymbol{x}$ is a valid solution?)

MIT Course 18.06 Quiz 1, Spring, 2009

(L-9) QUESTION 11. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 & 0 & 0 \\ 1 & 2 & 0 & 2 & 2 \\ 1 & 2 & -1 & 0 & 0 \\ 2 & 4 & 0 & 4 & 4 \end{bmatrix}$$

(a) Find a basis of the column space $\mathcal{C}\left(\mathbf{A}\right)$.
(b) Find a basis of the nullspace $\mathcal{N}\left(\mathbf{A}\right)$.
(c) Find linear conditions on a, b, c, d that guarantee that the system $\mathbf{A}\boldsymbol{x} = (a, b, c, d,)$ has a solution.
(d) Find the complete solution for the system $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}$.

MIT Course 18.06 Quiz 1, March 5, 2007

(L-9) QUESTION 12. Si a una matriz $\mathbf{A}$ se le “añade” una nueva columna extra $\boldsymbol{b}$, entonces el espacio columna se vuelve más grande, a no ser que ____________. Proporcione un ejemplo en el que espacio columna se haga más grande, y uno en el que no. ¿Por qué $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ es resoluble cuando el espacio columna no crece al añadir $\boldsymbol{b}$?

(L-9) QUESTION 13. If the 9 by 12 system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is solvable for every $\boldsymbol{b}$, then $\mathcal{C}(\mathbf{A}) = \_\_\_\_$
(Strang, 2006, exercise 30 from section 2.1.)

(L-9) QUESTION 14. **[Importante]**[1] Suponga que el sistema $\left[\boldsymbol{v}_1; \;\ldots\; \boldsymbol{v}_n\right]$ de vectores de $\mathbb{R}^m$ genera el subespacio $\mathcal{V}$, y suponga que $\boldsymbol{v}_n$ es una combinación lineal de los vectores $\boldsymbol{v}_1, \ldots \boldsymbol{v}_{n-1}$. Demuestre que el sistema $\left[\boldsymbol{v}_1; \;\ldots\; \boldsymbol{v}_{n-1}\right]$ también genera el subespacio $\mathcal{V}$.

(L-9) QUESTION 15.

(a) Find the general (complete) solution to this equation

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

(b) Find a basis for the column of the 3 by 9 block matrix $\left[\mathbf{A}; \quad 2\mathbf{A}; \quad \mathbf{A}^2;\right]$.
MIT Course 18.06 Final, May 18, 1998

[1] *pista:* Piense si el espacio $\mathcal{V}$ se puede expresar como el espacio columna de una matriz $\mathbf{V}$ cuyas columnas son los vectores $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$. Una vez expresado de esa manera, recuerde que las operaciones entre columnas no alteran el espacio columna de la matriz. Por último, transforme $\mathbf{V}$ de manera que transforme una de las columnas en un vector de ceros.

(L-9) QUESTION 16. ¿Cuáles de los siguientes vectores generan el espacio de polinomios de, a lo sumo, grado 4; es decir, el conjunto de polinomios $P_3 = \{at^3 + bt^2 + ct + d\}$?

(a) $t + 1,\ \ t^2 - t,$ y $t^3.$
(b) $t^3 + t$ y $t^2 + 1.$
(c) $t^2 + t + 1,\ \ t + 1,\ \ 1,$ y $t^3.$
(d) $t^3 + t^2,\ \ t^2 - t,\ \ 2t + 4,$ y $t^3 + 2t^2 + t + 4.$

(L-9) QUESTION 17. Considere los vectores $\boldsymbol{u}_1 = (1, 0, 1,)$ y $\boldsymbol{u}_2 = (1, -1, 1,).$

(a) **Demuestre** que $\boldsymbol{u}_1$ y $\boldsymbol{u}_2$ son linealmente independientes.
(b) ¿Pertenece $\boldsymbol{v} = (2, 1, 2,)$ al espacio generado por $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$? Explique las razones de su respuesta.
(c) Encuentre una base de $\mathbb{R}^3$ que contenga a $\boldsymbol{u}_1$ y a $\boldsymbol{u}_2$. Explique su respuesta.

(L-9) QUESTION 18.

(a) ¿Son linealmente independientes los siguientes vectores? Explique su respuesta.

$$\boldsymbol{v}_1 = \begin{pmatrix} -2 \\ -1 \\ 3 \\ 4 \end{pmatrix}; \qquad \boldsymbol{v}_2 = \begin{pmatrix} -8 \\ 2 \\ -2 \\ 1 \end{pmatrix}$$

(b) ¿Son los siguientes vectores una base de $\mathbb{R}^4$? Explique su respuesta.

$$\boldsymbol{v}_1 = \begin{pmatrix} -2 \\ -1 \\ 3 \\ 4 \end{pmatrix}; \qquad \boldsymbol{v}_2 = \begin{pmatrix} 8 \\ 2 \\ 2 \\ 1 \end{pmatrix}; \qquad \boldsymbol{v}_3 = \begin{pmatrix} 10 \\ 1 \\ 1 \\ 6 \end{pmatrix}; \qquad \boldsymbol{v}_4 = \begin{pmatrix} -2 \\ -1 \\ 3 \\ 4 \end{pmatrix}$$

(c) ¿Son los siguientes vectores una base del subespacio descrito por el plano tridimensional $x_1 + 2x_2 + 3x_3 + 6x_4 = 0$? Explique su respuesta.

$$\boldsymbol{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \qquad \boldsymbol{v}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}; \qquad \boldsymbol{v}_3 = \begin{pmatrix} -4 \\ -2 \\ 2 \\ 1 \end{pmatrix}$$

(d) Encuentre el valor de q para el que los siguientes vectores no generan $\mathbb{R}^3$.

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}; \qquad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}; \qquad \boldsymbol{v}_3 = \begin{pmatrix} -1 \\ 12 \\ 10 \end{pmatrix}; \qquad \boldsymbol{v}_4 = \begin{pmatrix} q \\ 3 \\ 1 \end{pmatrix}$$

(L-9) QUESTION 19. Suponga que tiene 4 vectores columna $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ y $\boldsymbol{z}$ en el espacio tridimensional $\mathbb{R}^3$.

(a) Dé un ejemplo donde el espacio columna de **A** contenga $\boldsymbol{u}$, $\boldsymbol{v}$ y $\boldsymbol{w}$, pero no a $\boldsymbol{z}$. (escriba unos vectores $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ y $\boldsymbol{z}$; y una matriz **A** que cumplan lo anterior).

(b) ¿Cuáles son las dimensiones del espacio columna y del espacio nulo de su matriz ejemplo **A** del apartado anterior?

Highlights of Lesson 10

- The Four Fundamental Subspaces of a matrix $\mathbf{A}$
  - Column space $\mathcal{C}\left(\mathbf{A}\right)$
  - Nullspace $\mathcal{N}\left(\mathbf{A}\right)$
  - Row space $\mathcal{C}\left(\mathbf{A}^\intercal\right)$
  - Left nullspace $\mathcal{N}\left(\mathbf{A}^\intercal\right)$

2 The Four Fundamental Subspaces of a matrix $\mathbf{A}$

- Column space $\mathcal{C}\left(\mathbf{A}\right)$

$\boxed{2}$ The Four Fundamental Subspaces of a matrix $\mathbf{A}$

- Column space $\mathcal{C}\left(\mathbf{A}\right)$
- Nullspace $\mathcal{N}\left(\mathbf{A}\right)$

2 The Four Fundamental Subspaces of a matrix $\mathbf{A}$

- Column space $\mathcal{C}(\mathbf{A})$
- Nullspace $\mathcal{N}(\mathbf{A})$
- Row space

$\boxed{2}$ The Four Fundamental Subspaces of a matrix $\mathbf{A}$

- Column space $\mathcal{C}\left(\mathbf{A}\right)$
- Nullspace $\mathcal{N}\left(\mathbf{A}\right)$
- Row space
  Linear combinations of the rows

$\boxed{2}$ The Four Fundamental Subspaces of a matrix $\mathbf{A}$

- Column space $\mathcal{C}\left(\mathbf{A}\right)$
- Nullspace $\mathcal{N}\left(\mathbf{A}\right)$
- Row space
  Linear combinations of the rows
  Linear combinations of the columns of $\mathbf{A}^\intercal$

2 The Four Fundamental Subspaces of a matrix $\mathbf{A}$

- Column space $\mathcal{C}\left(\mathbf{A}\right)$
- Nullspace $\mathcal{N}\left(\mathbf{A}\right)$
- Row space
  Linear combinations of the rows
  Linear combinations of the columns of $\mathbf{A}^\intercal = \mathcal{C}\left(\mathbf{A}^\intercal\right)$

2 The Four Fundamental Subspaces of a matrix $\mathbf{A}$

- Column space $\mathcal{C}\left(\mathbf{A}\right)$
- Nullspace $\mathcal{N}\left(\mathbf{A}\right)$
- Row space
  Linear combinations of the rows
  Linear combinations of the columns of $\mathbf{A}^\intercal = \mathcal{C}\left(\mathbf{A}^\intercal\right)$
- Left nullspace of $\mathbf{A}$,

2 The Four Fundamental Subspaces of a matrix $\mathbf{A}$

- Column space $\mathcal{C}\left(\mathbf{A}\right)$
- Nullspace $\mathcal{N}\left(\mathbf{A}\right)$
- Row space
  Linear combinations of the rows
  Linear combinations of the columns of $\mathbf{A}^\intercal = \mathcal{C}\left(\mathbf{A}^\intercal\right)$
- Left nullspace of $\mathbf{A}$, $\quad \mathcal{N}\left(\mathbf{A}^\intercal\right)$

2 The Four Fundamental Subspaces of a matrix **A**

Where are those subspaces if $\underset{m\times n}{\mathbf{A}}$?

- Column space $\mathcal{C}\left(\mathbf{A}\right)$
- Nullspace $\mathcal{N}\left(\mathbf{A}\right)$
- Row space
  Linear combinations of the rows
  Linear combinations of the columns of $\mathbf{A}^\intercal = \mathcal{C}\left(\mathbf{A}^\intercal\right)$
- Left nullspace of **A**, $\quad \mathcal{N}\left(\mathbf{A}^\intercal\right)$

2 The Four Fundamental Subspaces of a matrix $\mathbf{A}$

Where are those subspaces if $\underset{m\times n}{\mathbf{A}}$?

- Column space $\mathcal{C}\left(\mathbf{A}\right)$ $\mathbb{R}^m$
- Nullspace $\mathcal{N}\left(\mathbf{A}\right)$
- Row space
  Linear combinations of the rows
  Linear combinations of the columns of $\mathbf{A}^\intercal = \mathcal{C}\left(\mathbf{A}^\intercal\right)$
- Left nullspace of $\mathbf{A}$, $\mathcal{N}\left(\mathbf{A}^\intercal\right)$

Where are those subspaces if $\underset{m\times n}{\mathbf{A}}$?

- Column space $\mathcal{C}\left(\mathbf{A}\right)$ $\mathbb{R}^m$
- Nullspace $\mathcal{N}\left(\mathbf{A}\right)$ $\mathbb{R}^n$
- Row space
  Linear combinations of the rows
  Linear combinations of the columns of $\mathbf{A}^\intercal = \mathcal{C}\left(\mathbf{A}^\intercal\right)$
- Left nullspace of **A**, $\mathcal{N}\left(\mathbf{A}^\intercal\right)$

Where are those subspaces if $\underset{m\times n}{\mathbf{A}}$?

- Column space $\mathcal{C}\left(\mathbf{A}\right)$ $\mathbb{R}^m$
- Nullspace $\mathcal{N}\left(\mathbf{A}\right)$ $\mathbb{R}^n$
- Row space $\mathbb{R}^n$
  Linear combinations of the rows
  Linear combinations of the columns of $\mathbf{A}^\intercal = \mathcal{C}\left(\mathbf{A}^\intercal\right)$ $\mathbb{R}^n$
- Left nullspace of **A**, $\mathcal{N}\left(\mathbf{A}^\intercal\right)$

$\boxed{2}$ The Four Fundamental Subspaces of a matrix **A**

Where are those subspaces if $\underset{m\times n}{\mathbf{A}}$?

- Column space $\mathcal{C}\left(\mathbf{A}\right)$ $\mathbb{R}^m$
- Nullspace $\mathcal{N}\left(\mathbf{A}\right)$ $\mathbb{R}^n$
- Row space $\mathbb{R}^n$
  Linear combinations of the rows
  Linear combinations of the columns of $\mathbf{A}^\intercal = \mathcal{C}\left(\mathbf{A}^\intercal\right)$ $\mathbb{R}^n$
- Left nullspace of **A**, $\mathcal{N}\left(\mathbf{A}^\intercal\right)$ $\mathbb{R}^m$

$\boxed{3}$ Bases for the 4 subspaces: row space

$$\left[\begin{array}{c} \mathbf{A} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{cccc} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$\boxed{3}$ Bases for the 4 subspaces: row space

$$\left[\begin{array}{c} \mathbf{A} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{cccc} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(-3)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 1 & -2 & -3 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

3 Bases for the 4 subspaces: row space

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right]=\left[\begin{array}{cccc}1&2&3&1\\1&1&2&1\\1&2&3&1\\ \hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[{[(-1)\mathbf{1}+\mathbf{4}]}]{\substack{\tau\\ {[(-2)\mathbf{1}+\mathbf{2}]}\\ {[(-3)\mathbf{1}+\mathbf{3}]}}}\left[\begin{array}{cccc}1&0&0&0\\1&-1&-1&0\\1&0&0&0\\ \hline 1&-2&-3&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[{[(1)\mathbf{2}+\mathbf{3}]}]{\substack{\tau\\ {[(-1)\mathbf{2}]}\\ {[(-1)\mathbf{2}+\mathbf{1}]}}}\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\1&0&0&0\\ \hline -1&2&-1&-1\\1&-1&-1&0\\0&0&1&0\\0&0&0&1\end{array}\right]=\left[\begin{array}{c}\mathbf{R}\\ \hline \mathbf{E}\end{array}\right]$$

3 Bases for the 4 subspaces: row space

$$\left[\begin{array}{c}\mathbf{A}\\\hline\mathbf{I}\end{array}\right]=\left[\begin{array}{cccc}1&2&3&1\\1&1&2&1\\1&2&3&1\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[{[(\text{-}1)\mathbf{1}+\mathbf{4}]}]{\substack{\tau\\{[(\text{-}2)\mathbf{1}+\mathbf{2}]}\\{[(\text{-}3)\mathbf{1}+\mathbf{3}]}}}\left[\begin{array}{cccc}1&0&0&0\\1&\text{-}1&\text{-}1&0\\1&0&0&0\\\hline 1&\text{-}2&\text{-}3&\text{-}1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[{[(1)\mathbf{2}+\mathbf{3}]}]{\substack{\tau\\{[(\text{-}1)\mathbf{2}]}\\{[(\text{-}1)\mathbf{2}+\mathbf{1}]}}}\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\1&0&0&0\\\hline \text{-}1&2&\text{-}1&\text{-}1\\1&\text{-}1&\text{-}1&0\\0&0&1&0\\0&0&0&1\end{array}\right]=\left[\begin{array}{c}\mathbf{R}\\\hline\mathbf{E}\end{array}\right]$$

column operations preserve $\mathcal{C}\left(\mathbf{A}\right)$ (but not the row space)

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What's the dimension of the row space $\mathcal{C}\left(\mathbf{A}^\intercal\right)$?

3 Bases for the 4 subspaces: row space

$$\left[\begin{array}{c}\mathbf{A}\\\hline\mathbf{I}\end{array}\right]=\left[\begin{array}{cccc}1&2&3&1\\1&1&2&1\\1&2&3&1\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[{[(-1)\mathbf{1}+\mathbf{4}]}]{\substack{\tau\\{[(-2)\mathbf{1}+\mathbf{2}]}\\{[(-3)\mathbf{1}+\mathbf{3}]}}}\left[\begin{array}{cccc}1&0&0&0\\1&-1&-1&0\\1&0&0&0\\\hline 1&-2&-3&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[{[(1)\mathbf{2}+\mathbf{3}]}]{\substack{\tau\\{[(-1)\mathbf{2}]}\\{[(-1)\mathbf{2}+\mathbf{1}]}}}\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\1&0&0&0\\\hline -1&2&-1&-1\\1&-1&-1&0\\0&0&1&0\\0&0&0&1\end{array}\right]=\left[\begin{array}{c}\mathbf{R}\\\hline\mathbf{E}\end{array}\right]$$

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What's the dimension of the row space $\mathcal{C}\left(\mathbf{A}^\intercal\right)$? r

3 Bases for the 4 subspaces: row space

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{cccc} 1 & 2 & 3 & 1\\ 1 & 1 & 2 & 1\\ 1 & 2 & 3 & 1\\ \hline 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(\text{-}1)\mathbf{1}+\mathbf{4}]}]{\substack{\tau\\ {[(\text{-}2)\mathbf{1}+\mathbf{2}]}\\ {[(\text{-}3)\mathbf{1}+\mathbf{3}]}}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0\\ 1 & \text{-}1 & \text{-}1 & 0\\ 1 & 0 & 0 & 0\\ \hline 1 & \text{-}2 & \text{-}3 & \text{-}1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{2}+\mathbf{3}]}]{\substack{\tau\\ {[(\text{-}1)\mathbf{2}]}\\ {[(\text{-}1)\mathbf{2}+\mathbf{1}]}}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ \hline \text{-}1 & 2 & \text{-}1 & \text{-}1\\ 1 & \text{-}1 & \text{-}1 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c}\mathbf{R}\\ \hline \mathbf{E}\end{array}\right]$$

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What's the dimension of the row space $\mathcal{C}(\mathbf{A}^\intercal)$? $\quad r$

What's a basis for the row space of $\mathbf{A}$?

3 Bases for the 4 subspaces: row space

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{cccc} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(\text{-}1)\mathbf{1}+\mathbf{4}]}]{\substack{\tau \\ [(\text{-}2)\mathbf{1}+\mathbf{2}] \\ [(\text{-}3)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & \text{-}1 & \text{-}1 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 1 & \text{-}2 & \text{-}3 & \text{-}1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{2}+\mathbf{3}]}]{\substack{\tau \\ [(\text{-}1)\mathbf{2}] \\ [(\text{-}1)\mathbf{2}+\mathbf{1}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline \text{-}1 & 2 & \text{-}1 & \text{-}1 \\ 1 & \text{-}1 & \text{-}1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\frac{\mathbf{R}}{\mathbf{E}}\right]$$

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What's the dimension of the row space $\mathcal{C}(\mathbf{A}^\intercal)$? r

What's a basis for the row space of **A**? the r pivot rows of **A**

3 Bases for the 4 subspaces: row space

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What's the dimension of the row space $\mathcal{C}\left(\mathbf{A}^\intercal\right)$? r

What's a basis for the row space of **A**? the r pivot rows of **A**

A basis for the column space of **A**?

$\boxed{4}$ Left null space: why that name?

$\mathcal{N}\left(\mathbf{A}^\intercal\right)$

$$(\mathbf{A}^\intercal)\boldsymbol{y} = \mathbf{0}$$

$\boxed{4}$ Left null space: why that name?

$\mathcal{N}\left(\mathbf{A}^\intercal\right)$

$$(\mathbf{A}^\intercal)\boldsymbol{y} = \mathbf{0}$$

$$\begin{bmatrix} & & \\ & \mathbf{A}^\intercal & \\ & & \end{bmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

4 Left null space: why that name?

$\mathcal{N}\left(\mathbf{A}^\mathsf{T}\right)$

$$(\mathbf{A}^\mathsf{T})\boldsymbol{y} = \mathbf{0}$$

$$\begin{bmatrix} & \mathbf{A}^\mathsf{T} & \end{bmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

so. . .

$$\boldsymbol{y}\mathbf{A} = \mathbf{0}$$

4 Left null space: why that name?

$\mathcal{N}\left(\mathbf{A}^\intercal\right)$

$$(\mathbf{A}^\intercal)\boldsymbol{y} = \mathbf{0}$$

$$\begin{bmatrix} & & \\ & \mathbf{A}^\intercal & \\ & & \end{bmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

so. . .

$$\boldsymbol{y}\mathbf{A} = \mathbf{0}$$

$$\begin{pmatrix} y_1, & \dots & y_m, \end{pmatrix} \begin{bmatrix} & & \\ & \mathbf{A} & \\ & & \end{bmatrix} = \begin{pmatrix} 0, & \dots & 0, \end{pmatrix}$$

$\boxed{5}$ Column elimination preserves the left null space

Let $\underset{n\times n}{\mathbf{E}} = \mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}$ be invertible

5 Column elimination preserves the left null space

Let $\underset{n\times n}{\mathbf{E}} = \mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}$ be invertible then

- If $\boldsymbol{x} \in \mathcal{N}\left(\mathbf{A}^\mathsf{T}\right)$

$$\boldsymbol{x}\mathbf{A} = \mathbf{0} \quad \text{and} \quad \boldsymbol{x}\mathbf{AE} = \mathbf{0E} = \mathbf{0} \quad \Rightarrow \boldsymbol{x} \in \mathcal{N}\left((\mathbf{AE})^\mathsf{T}\right);$$

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- If $\boldsymbol{x} \in \mathcal{N}\left((\mathbf{AE})^\mathsf{T}\right)$

$$\boldsymbol{x}\mathbf{AE} = \mathbf{0} \quad \text{and} \quad \boldsymbol{x}\mathbf{A} = \mathbf{0E}^{-1} = \mathbf{0} \quad \Rightarrow \boldsymbol{x} \in \mathcal{N}\left(\mathbf{A}^\mathsf{T}\right).$$

$\boxed{5}$ Column elimination preserves the left null space

Let $\underset{n\times n}{\mathbf{E}} = \mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}$ be invertible then

- If $x \in \mathcal{N}\left(\mathbf{A}^\mathsf{T}\right)$

$$x\mathbf{A} = 0 \quad \text{and} \quad x\mathbf{AE} = 0\mathbf{E} = 0 \quad \Rightarrow x \in \mathcal{N}\left((\mathbf{AE})^\mathsf{T}\right);$$

- If $\boldsymbol{x} \in \mathcal{N}\left((\mathbf{AE})^\mathsf{T}\right)$

$$\boldsymbol{x}\mathbf{AE} = \mathbf{0} \quad \text{and} \quad \boldsymbol{x}\mathbf{A} = \mathbf{0}\mathbf{E}^{-1} = \mathbf{0} \quad \Rightarrow \boldsymbol{x} \in \mathcal{N}\left(\mathbf{A}^\mathsf{T}\right).$$

5 Column elimination preserves the left null space

Let $\underset{n\times n}{\mathbf{E}} = \mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}$ be invertible then

- If $x \in \mathcal{N}\left(\mathbf{A}^\top\right)$

$$x\mathbf{A} = 0 \quad \text{and} \quad x\mathbf{AE} = 0\mathbf{E} = 0 \quad \Rightarrow x \in \mathcal{N}\left((\mathbf{AE})^\top\right);$$

- If $x \in \mathcal{N}\left((\mathbf{AE})^\top\right)$

$$x\mathbf{AE} = 0 \quad \text{and} \quad x\mathbf{A} = 0\mathbf{E}^{-1} = 0 \quad \Rightarrow x \in \mathcal{N}\left(\mathbf{A}^\top\right).$$

Therefore,

$$\mathcal{N}\left(\mathbf{A}^\top\right) = \mathcal{N}\left((\mathbf{AE})^\top\right) = \mathcal{N}\left(\left(\mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\right)^\top\right).$$

$\boxed{6}$ Finding a basis of $\mathcal{N}\left(\mathbf{A}^\intercal\right)$ by column reduction

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(-3)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{4}]}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}] \\ [(-1)\mathbf{2}+\mathbf{1}] \\ [(1)\mathbf{2}+\mathbf{3}]}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \mathbf{R}$$

Basis for $\mathcal{N}\left(\mathbf{A}^\intercal\right)$?

6 Finding a basis of $\mathcal{N}\left(\mathbf{A}^{\intercal}\right)$ by column reduction

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \xrightarrow[]{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(-3)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{4}]}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \text{-1} & \text{-1} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}] \\ [(-1)\mathbf{2}+\mathbf{1}] \\ [(1)\mathbf{2}+\mathbf{3}]}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \mathbf{R}$$

Basis for $\mathcal{N}\left(\mathbf{A}^{\intercal}\right)$?

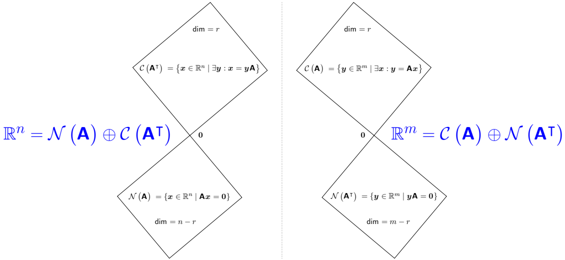
$$\left(-1, \quad 0, \quad 1,\right)$$

$\boxed{7}$ Finding a basis of $\mathcal{N}\left(\mathbf{A}^\intercal\right)$ by column reduction

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \\ a & b & c \\ d & e & f \end{bmatrix} \xrightarrow[{[(1)\mathbf{1}+\mathbf{3}]}]{\boldsymbol{\tau}} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & a+c \\ d & e & d+f \end{bmatrix} \xrightarrow[{[(1)\mathbf{2}+\mathbf{1}]}]{\boldsymbol{\tau}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ a+b & b & a+c \\ d+e & e & d+f \end{bmatrix}$$

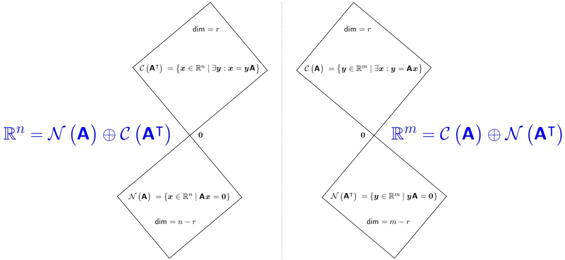
Basis for $\mathcal{N}\left(\mathbf{A}^\intercal\right)$?

8 The Four Fundamental Subspaces



$\underset{m \times n}{\mathbf{A}}$ dimensions?

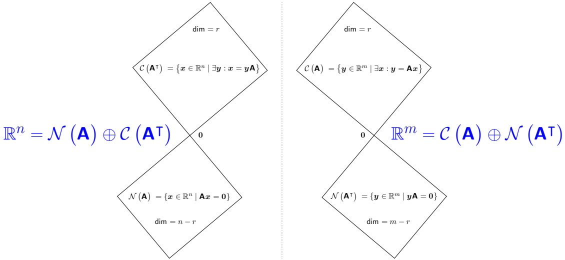
8 The Four Fundamental Subspaces



$\underset{m\times n}{\mathbf{A}}$ dimensions?

- $\dim(\mathcal{C}(\mathbf{A})) =$

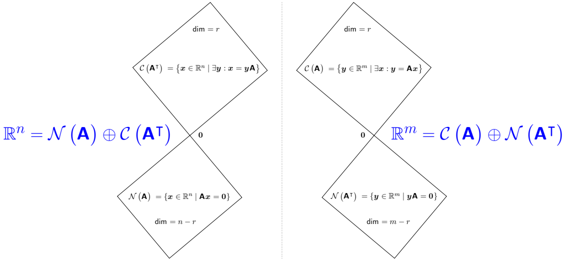
8 The Four Fundamental Subspaces



$\underset{m\times n}{\mathbf{A}}$ dimensions?

- $\dim\left(\mathcal{C}(\mathbf{A})\right) = \mathrm{rg}(\mathbf{A}) = r$

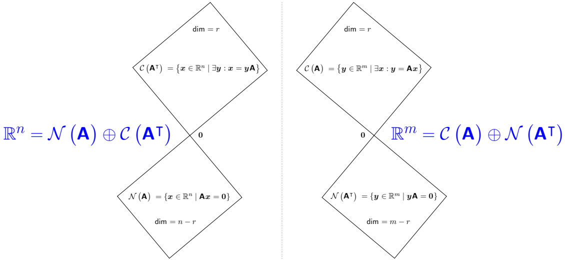
8 The Four Fundamental Subspaces



$\underset{m\times n}{\mathbf{A}}$ dimensions?

- $\dim\left(\mathcal{C}(\mathbf{A})\right) = \operatorname{rg}(\mathbf{A}) = r = \dim\left(\mathcal{C}(\mathbf{A}^\intercal)\right)$

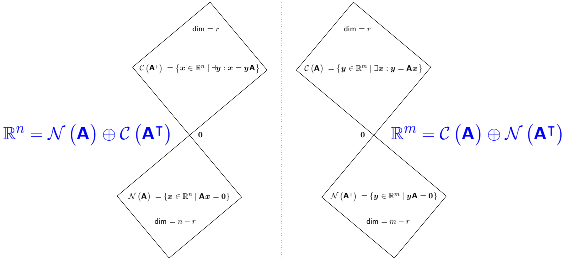
8 The Four Fundamental Subspaces



$\underset{m \times n}{\mathbf{A}}$ dimensions?

- $\dim\left(\mathcal{C}(\mathbf{A})\right) = \mathrm{rg}(\mathbf{A}) = r = \dim\left(\mathcal{C}(\mathbf{A}^\intercal)\right)$
- $\dim\left(\mathcal{N}(\mathbf{A})\right) =$

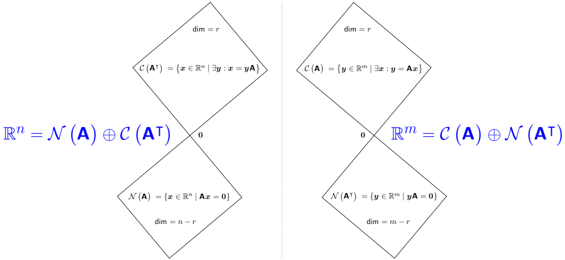
8 The Four Fundamental Subspaces



$\underset{m \times n}{\mathbf{A}}$ dimensions?

- $\dim\left(\mathcal{C}(\mathbf{A})\right) = \operatorname{rg}(\mathbf{A}) = r = \dim\left(\mathcal{C}(\mathbf{A}^\top)\right)$
- $\dim\left(\mathcal{N}(\mathbf{A})\right) = n - r$

8 The Four Fundamental Subspaces



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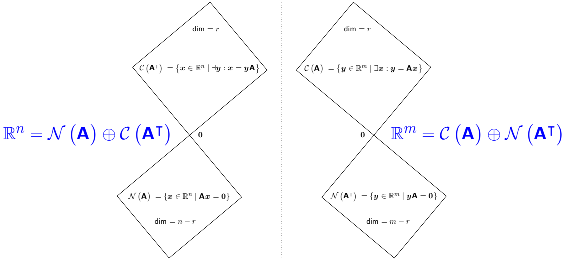
8 The Four Fundamental Subspaces



$\underset{m\times n}{\mathbf{A}}$ dimensions?

- $\dim\left(\mathcal{C}(\mathbf{A})\right) = \mathrm{rg}(\mathbf{A}) = r = \dim\left(\mathcal{C}(\mathbf{A}^\mathsf{T})\right)$
- $\dim\left(\mathcal{N}(\mathbf{A})\right) = n - r$
- $\dim\left(\mathcal{N}(\mathbf{A}^\mathsf{T})\right) = m - r$

Questions of the Lecture 10

(L-10) QUESTION 1. Find the dimension and construct a basis for the four subspaces associated with each of the matrices

(a) $\mathbf{A} = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix}$

(b) What is the sum $\dim \mathcal{C}\left(\mathbf{A}\right) + \dim \mathcal{N}\left(\mathbf{A}^\intercal\right)$? and $\dim \mathcal{C}\left(\mathbf{A}^\intercal\right) + \dim \mathcal{N}\left(\mathbf{A}\right)$?

(c) $\mathbf{U} = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(d) What is the sum $\dim \mathcal{C}\left(\mathbf{U}\right) + \dim \mathcal{N}\left(\mathbf{U}^\intercal\right)$? and $\dim \mathcal{C}\left(\mathbf{U}^\intercal\right) + \dim \mathcal{N}\left(\mathbf{U}\right)$?

(Strang, 2006, exercise 2 from section 2.4.)

(L-10) QUESTION 2. Describe the four subspaces in three-dimensional space associated with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(Strang, 2006, exercise 4 from section 2.4.)

(L-10) QUESTION 3.

(a) If a 7 by 9 matrix has rank 5, what are the dimensions of the four subspaces? What is the sum of all four dimensions?

(b) If a 3 by 4 matrix has rank 3, what are its column space $\mathcal{C}\left(\mathbf{A}\right)$ and left nullspace $\mathcal{N}\left(\mathbf{A}^\intercal\right)$?

(Strang, 2006, exercise 20 from section 2.4.)

(L-10) QUESTION 4. If $\mathbf{A}$ has the same four fundamental subspaces as $\mathbf{B}$, does $\mathbf{A} = c\mathbf{B}$?
(Strang, 2006, exercise 19 from section 2.4.)

(L-10) QUESTION 5. Find the dimension and a basis for the four fundamental subspaces for

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}; \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \mathbf{AE}; \quad \text{where} \quad \mathbf{E} = \begin{bmatrix} 1 & -2 & 2 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Basado en (Strang, 2006, exercise 3 from section 2.4.)

(L-10) QUESTION 6. Find the dimensions of these vector spaces:

(a) The space of all vectors in $\mathbb{R}^4$ whose components add to zero.
(b) The nullspace of the 4 by 4 identity matrix.
(c) The space of all 4 by 4 matrices

(Strang, 2006, exercise 32 from section 2.3.)

(L-10) QUESTION 7. Without multiplying matrices, find bases for the row and column spaces of **A**:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

How do you know from these shapes that **A** is not invertible?
(Strang, 2006, exercise 36 from section 2.4.)

(L-10) QUESTION 8. Which of the following (if any) are subspaces? For any that are not a subspace, give an example of how they violate a property of subspaces.

(a) Given 3×5 matrix **A** with full row rank, the set of all solutions to

$$\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(b) All vectors $\boldsymbol{x}$ with $\langle \vec{x}, \vec{y} \rangle = \mathbf{0}$ and $\langle \vec{x}, \vec{z} \rangle = \mathbf{0}$ for some given vectors $\boldsymbol{y}$ and $\boldsymbol{z}$.

(c) All 3×5 matrices with $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. in their column space.

(d) All 5×3 matrices with $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in their nullspace.

MIT Course 18.06 Quiz 1, Spring, 2009

(L-10) QUESTION 9. ¿Cuál es el espacio columna $\mathcal{C}\left(\mathbf{A}\right)$ y el espacio fila $\mathcal{C}\left(\mathbf{A}^\intercal\right)$ de la matriz

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 5 & -2 \\ 2 & -1 & 3 & -4 \\ -1 & 4 & 2 & 2 \end{bmatrix}$$

MIT Course 18.06 Final. May 18, 1998

(L-10) QUESTION 10. If $\underset{5\times 4}{\mathbf{A}}$ is a matrix with linearly independent columns, find each of these explicitily:

(a) The nulls space of $\mathbf{A}$.
(b) The dimension of the left null space $\mathcal{N}\left(\mathbf{A}^\intercal\right)$.
(c) One particular solution $\boldsymbol{x}_p$ to the system $\mathbf{A}\boldsymbol{x} = \mathbf{A}_{|2}$.
(d) The general (complete) solution to $\mathbf{A}\boldsymbol{x} = \mathbf{A}_{|2}$.
(e) The reduced echelon form $\mathbf{R}$ of $\mathbf{A}$.

(L-10) QUESTION 11. Verdadero o falso

(a) Si una matriz es cuadrada $(m = n)$, entonces el espacio columna es igual al espacio fila.

(b) La matriz $\mathbf{A}$ y la matriz $(-\mathbf{A})$ comparten los mismos cuatro sub-espacios fundamentales.

(c) Si $\mathbf{A}$ y $\mathbf{B}$ comparten los mismos cuatro sub-espacios fundamentales, entonces $\mathbf{A}$ es un múltiplo de $\mathbf{B}$.

(d) Indique si la siguiente aseveración es verdadera o falsa. Si es verdadera explique el motivo, si es falsa encuentre un contraejemplo: *"Un sistema con n ecuaciones y n incógnitas es resoluble cuando las columnas de la matriz de coeficientes son independientes."*

(L-10) QUESTION 12. Se conoce la siguiente información sobre **A**:

$$\mathbf{A}\boldsymbol{v} = \mathbf{A}\begin{pmatrix}1\\-2\\3\\1\end{pmatrix} = \begin{pmatrix}-6\\3\end{pmatrix}; \qquad \text{y que} \qquad \mathbf{A}\boldsymbol{w} = \mathbf{A}\begin{pmatrix}3\\-1\\1\\2\end{pmatrix} = \begin{pmatrix}-18\\9\end{pmatrix}.$$

De hecho, $\mathbf{A}\boldsymbol{x}$ es siempre algún múltiplo del vector $\begin{pmatrix}-2, & 1,\end{pmatrix}$ sea cual sea el vector $\boldsymbol{x} \in \mathbb{R}^4$.

(a) ¿Cuál es el orden y el rango de **A**?
(b) ¿Cuál es la dimensión del espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$?
(c) ¿Cuál es la dimensión del espacio fila $\mathcal{C}\left(\mathbf{A}^\intercal\right)$?
(d) ¿Cuál es la dimensión del espacio nulo por la izquierda $\mathcal{N}\left(\mathbf{A}^\intercal\right)$?
(e) Encuentre una solución $\boldsymbol{x}$ no nula al sistema $\mathbf{A}\boldsymbol{x} = \begin{pmatrix}0\\0\end{pmatrix}$.

(L-10) QUESTION 13. Consider the matrix $\mathbf{A}$ with its column reduced echelon form $\mathbf{R}$ computed by gaussian elimination without permutations:

$$\begin{bmatrix} \mathbf{A} \\ \hline \mathbf{I} \end{bmatrix} = \left[\begin{array}{ccccc} 1 & -1 & 5 & 1 & 0 \\ 2 & 1 & 4 & 2 & 1 \\ 3 & 0 & 9 & 3 & 1 \\ -1 & -1 & -1 & -1 & 2 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 3/8 & 2/8 & -3 & -1 & -1/8 \\ -5/8 & 2/8 & 2 & 0 & -1/8 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1/8 & 2/8 & 0 & 0 & 3/8 \end{array}\right] = \begin{bmatrix} \mathbf{R} \\ \hline \mathbf{E} \end{bmatrix}$$

(a) What is the rank of $\mathbf{A}$? What are the dimensions of the column space $\mathcal{C}(\mathbf{A})$, the row space $\mathcal{C}(\mathbf{A}^\intercal)$ and the nullspace $\mathcal{N}(\mathbf{A})$?

(b) Find a basis of the row space $\mathcal{C}(\mathbf{A}^\intercal)$.

(c) Find a basis for the column space $\mathcal{C}(\mathbf{A})$.

(d) Find a basis for the nullspace $\mathcal{N}(\mathbf{A})$.

(e) Write down $\mathbf{A}_{|3}$ as a linear combination of $\mathbf{A}_{|1}$, $\mathbf{A}_{|2}$, $\mathbf{A}_{|4}$ and $\mathbf{A}_{|5}$.

(L-10) Question 14. Construct a matrix whose nullspace consists of all combinations of $(2, 2, 1, 0,)$ and $(3, 1, 0, 1,)$.
(Strang, 2006, exercise 60 from section 2.2.)

(L-10) Question 15. Construct a matrix whose nullspace consists of all multiples of $(4, 3, 2, 1)^\intercal$
(Strang, 2006, exercise 61 from section 2.2.)

(L-10) Question 16.

(a) Suponga que el producto de **A** y **B** es la matriz nula: $\mathbf{AB} = \mathbf{0}$. Entonces el espacio (I)________________ de la matriz **A** contiene el espacio (II)________________ de la matriz **B**. También el espacio (III)________________ de la matriz **B** contiene el espacio (IV)________________ de la matriz **A**. (incluya los nombres de los cuatro espacios fundamentales en los lugares apropiados)

(I)________________, (II)________________, (III)________________, (IV)________________

(b) Suponga que la matriz **A** es de dimensiones 5 por 7 con rango r, y **B** es de dimensiones 7 por 9 de rango s. ¿Cuáles son las dimensiones de los espacios (I) y (II)? Del hecho de que el espacio (I) contiene el espacio (II), ¿qué sabe acerca de $r + s$?

(L-10) QUESTION 17. By performing column eliminations (and possibly permutations) on the following 4×8 matrix $\mathbf{A}$

$$\begin{bmatrix} \mathbf{A} \\ \hline \mathbf{I} \end{bmatrix} = \left[\begin{array}{rrrrrrrr} 1 & 2 & -2 & 1 & -5 & 0 & 2 & -3 \\ -1 & -2 & 1 & -2 & 3 & 0 & -2 & 0 \\ 1 & 2 & -2 & 1 & -5 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 4 & 0 & 0 & 2 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{rrrrrrrr} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 2/4 & -2 & -2/4 & -3 & 1 & 0 & -2 & 3/4 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2/4 & 0 & 2/4 & -1 & -2 & 0 & 0 & 3/4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -2/4 & 0 & -2/4 & 0 & 0 & 0 & 0 & -1/4 \end{array}\right] = \begin{bmatrix} \mathbf{R} \\ \hline \mathbf{E} \end{bmatrix}$$

(a) What is the rank of $\mathbf{A}$?
(b) What are the dimensions of the 4 fundamental subspaces?

(c) How many solutions does $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ have? Does it depend on $\boldsymbol{b}$? Justify.
(d) Are the rows of $\mathbf{A}$ linearly independent? Why?
(e) Give a basis of $\mathcal{N}\left(\mathbf{A}\right)$.
(f) Give a basis of $\mathcal{N}\left(\mathbf{A}^{\mathsf{T}}\right)$.
(g) Give, if possible, matrix $\left[\mathbf{A}_{|1};\ \mathbf{A}_{|3};\ \mathbf{A}_{|6};\ \mathbf{A}_{|7};\right]^{-1}$
(h) Give, if possible, matrix $\left[\mathbf{A}_{|1};\ \mathbf{A}_{|3};\ \mathbf{A}_{|6};\ \mathbf{A}_{|8};\right]^{-1}$

Based on *MIT Course 18.06 Quiz 1, October 4, 2004*

(L-10) QUESTION 18. Consider the 5 by 3 matrix $\mathbf{R}$ (in its column reduced echelon form) with three pivots ($r = 3$).

(a) What is its null space $\mathcal{N}\left(\mathbf{R}\right)$?
(b) Consider the 10 by 3 block matrix; $\mathbf{B} = \begin{bmatrix}\mathbf{R} \\ 2\mathbf{R}\end{bmatrix}$. What is its column reduced echelon form? What is its rank?
(c) Consider the 10 by 6 block matrix; $\mathbf{C} = \begin{bmatrix}\mathbf{R} & \mathbf{R} \\ \mathbf{R} & \mathbf{0}\end{bmatrix}$. What is its column reduced echelon form?
(d) What is the rank of $\mathbf{C}$?
(e) What is the dimension of the null space of $\mathbf{C}^{\mathsf{T}}$; $\dim\mathcal{N}\left(\mathbf{C}^{\mathsf{T}}\right)$?

Based on MIT Course 18.06 Quiz 1, Fall 1993

(L-10) QUESTION 19. Consider the linear system $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$

with solution $\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists\, c, d \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$

(a) (1^{pts}) Find the dimension of the row space of **A**. Explain your answer.
(b) (1^{pts}) Construct the matrix **A**. Explain your answer.
(c) (0.5^{pts}) For which right hand side vectors $\boldsymbol{b}$ the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is solvable?

(L-10) QUESTION 20. True or false (give a good reason)?

(a) If the columns of a matrix are dependent, so are the rows.
(b) The column space of a 2 by 2 matrix is the same as its row space.
(c) The column space of a 2 by 2 matrix has the same dimension as its row space.
(d) The columns of a matrix are a basis for the column space.

(Strang, 2006, exercise 28 from section 2.3.)

(L-10) QUESTION 21. Let **A** be any matrix and **R** its **row reduced** echelon form. Answer True or False to the statements below and briefly explain. (Note, if there are any counterexamples to a statement below you must choose false for that statement.)

(a) If $\boldsymbol{x}$ is a solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ then $\boldsymbol{x}$ must be a solution to $\mathbf{R}\boldsymbol{x} = \boldsymbol{b}$.
(b) If $\boldsymbol{x}$ is a solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ then $\boldsymbol{x}$ must be a solution to $\mathbf{R}\boldsymbol{x} = \mathbf{0}$.
(c) What would be your anwsers if **R** is the **column** reduced form of **A**?

basado en MIT Course 18.06 Quiz 1, Fall 2008

Highlights of Lesson

- Bases of new vector spaces
- Rank one matrices
- Free variables

$\boxed{2}$ A new vector space

$\mathbb{R}^{3\times 3}$: All matrices of order 3×3! $\quad \mathbf{A}+\mathbf{B}$; $\quad c\mathbf{A}$; $\quad \underset{3\times 3}{\mathbf{0}}$

2 A new vector space

$\mathbb{R}^{3\times 3}$: All matrices of order 3×3! $\mathbf{A} + \mathbf{B}$; $c\mathbf{A}$; $\underset{3\times 3}{\mathbf{0}}$

subspaces of $\mathbb{R}^{3\times 3}$

- $\mathcal{U}$: Upper triangular matrices

$\boxed{2}$ A new vector space

$\mathbb{R}^{3\times 3}$: All matrices of order 3×3! $\quad \mathbf{A} + \mathbf{B}$; $\quad c\mathbf{A}$; $\quad \underset{3\times 3}{\mathbf{0}}$

subspaces of $\mathbb{R}^{3\times 3}$

- $\mathcal{U}$: Upper triangular matrices
- $\mathcal{S}$: Symmetric matrices

2 A new vector space

$\mathbb{R}^{3\times 3}$: All matrices of order 3×3! $\quad \mathbf{A} + \mathbf{B}$; $\quad c\mathbf{A}$; $\quad \underset{3\times 3}{\mathbf{0}}$

subspaces of $\mathbb{R}^{3\times 3}$

- $\mathcal{U}$: Upper triangular matrices
- $\mathcal{S}$: Symmetric matrices
- $\mathcal{U} \cap \mathcal{S}$: The intersection (the vectors that are in both $\mathcal{S}$ and $\mathcal{U}$):

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What are the dimensions of these subspaces?

2 A new vector space

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What are the dimensions of these subspaces? 6, 6 and 3

2 A new vector space

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Is $\mathcal{U} \cup \mathcal{S}$ a subspace?

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What are the dimensions of these subspaces? 6, 6 and 3

Is $\mathcal{U} \cup \mathcal{S}$ a subspace? NO

$\boxed{2}$ A new vector space

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subspaces of $\mathbb{R}^{3\times 3}$

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What are the dimensions of these subspaces? 6, 6 and 3

Is $\mathcal{U} \cup \mathcal{S}$ a subspace? NO

Let $\mathcal{U} + \mathcal{S}$ be the set of all sums of vectors in $\mathcal{U}$ plus vectors in $\mathcal{S}$; then $\mathcal{U} + \mathcal{S} =$

2 A new vector space

$\mathbb{R}^{3\times 3}$: All matrices of order 3×3! $\quad \mathbf{A} + \mathbf{B}$; $\quad c\mathbf{A}$; $\quad \underset{3\times 3}{\mathbf{0}}$

subspaces of $\mathbb{R}^{3\times 3}$

- $\mathcal{U}$: Upper triangular matrices
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Is $\mathcal{U} \cup \mathcal{S}$ a subspace? NO

Let $\mathcal{U} + \mathcal{S}$ be the set of all sums of vectors in $\mathcal{U}$ plus vectors in $\mathcal{S}$; then $\mathcal{U} + \mathcal{S} = \mathbb{R}^{3\times 3}$

3 Rank one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ & & \end{bmatrix}$$

3 Rank one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$$

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- tell me a basis for the row space:

3 Rank one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$$

- tell me a basis for the row space: Any of the rows

$\boxed{3}$ Rank one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$$

- tell me a basis for the row space: Any of the rows
- tell me a basis for the column space

3 Rank one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$$

- tell me a basis for the row space: Any of the rows
- tell me a basis for the column space Any of the columns

3 Rank one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$$

- tell me a basis for the row space: Any of the rows
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What's the dimension of $\mathcal{C}\left(\mathbf{A}\right)$, $\mathcal{C}\left(\mathbf{A}^{\intercal}\right)$ and $\mathrm{rg}\left(\mathbf{A}\right)$?

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- tell me a basis for the row space: Any of the rows
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What's the dimension of $\mathcal{C}\left(\mathbf{A}\right)$, $\mathcal{C}\left(\mathbf{A}^\intercal\right)$ and $\mathrm{rg}\left(\mathbf{A}\right)$? 1, 1 and 1

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What's the dimension of $\mathcal{C}\left(\mathbf{A}\right)$, $\mathcal{C}\left(\mathbf{A}^\intercal\right)$ and $\mathrm{rg}\left(\mathbf{A}\right)$? 1, 1 and 1

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \end{bmatrix}$$

3 Rank one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$$

- tell me a basis for the row space: Any of the rows
- tell me a basis for the column space Any of the columns

What's the dimension of $\mathcal{C}\left(\mathbf{A}\right)$, $\mathcal{C}\left(\mathbf{A}^\intercal\right)$ and $\mathrm{rg}\left(\mathbf{A}\right)$? 1, 1 and 1

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \end{bmatrix}$$

Every rank one matrix has the form: a column times a row.

$$\mathbf{A} = \left[\mathbf{A}_{|1}\right]\left[{}_{1|}\mathbf{A}\right]^\intercal = \text{column matrix times row matrix}$$

4 Rank one matrices

Think about the following subset of $\mathbb{R}^4$:

$$\mathcal{S} = \left\{ \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; v_1 + v_2 + v_3 + v_4 = 0 \right\}$$

4 Rank one matrices

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$$\mathcal{S} = \left\{ \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; v_1 + v_2 + v_3 + v_4 = 0 \right\}$$

Is $\mathcal{S}$ a subspace?

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Is $\mathcal{S}$ a subspace? Yes

$\boxed{4}$ Rank one matrices

Think about the following subset of $\mathbb{R}^4$:

$$\mathcal{S} = \left\{ \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; v_1 + v_2 + v_3 + v_4 = 0 \right\}$$

Is $\mathcal{S}$ a subspace? Yes

What's the dimension? can you tell me a basis?

4 Rank one matrices

Think about the following subset of $\mathbb{R}^4$:

$$\mathcal{S} = \left\{ \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; v_1 + v_2 + v_3 + v_4 = 0 \right\}$$

Is $\mathcal{S}$ a subspace? Yes

What's the dimension? can you tell me a basis?

$\mathcal{S}$ is the null space of a matrix **A** $(\mathbf{A}\boldsymbol{v} = 0)\ldots$ Which matrix?

4 Rank one matrices

Think about the following subset of $\mathbb{R}^4$:

$$\mathcal{S} = \left\{ \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; v_1 + v_2 + v_3 + v_4 = 0 \right\}$$

Is $\mathcal{S}$ a subspace? Yes

What's the dimension? can you tell me a basis?

$\mathcal{S}$ is the null space of a matrix **A** $(\mathbf{A}\boldsymbol{v} = 0)$... Which matrix?

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

5 Rank one matrices

$$\underset{1\times 4}{\mathbf{A}} = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}; \qquad \operatorname{rg}\left(\mathbf{A}\right) = \qquad\qquad \mathcal{S} = \mathcal{N}\left(\mathbf{A}\right)$$

5 Rank one matrices

$$\underset{1\times 4}{\mathbf{A}} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}; \qquad \operatorname{rg}\left(\mathbf{A}\right) = 1. \qquad\qquad \mathcal{S} = \mathcal{N}\left(\mathbf{A}\right)$$

- $\dim \mathcal{S} = \dim \mathcal{N}\left(\mathbf{A}\right) =$

5 Rank one matrices

$$\underset{1\times 4}{\mathbf{A}} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}; \qquad \operatorname{rg}(\mathbf{A}) = 1. \qquad \mathcal{S} = \mathcal{N}(\mathbf{A})$$

- $\dim \mathcal{S} = \dim \mathcal{N}(\mathbf{A}) = n - r = 3.$

5 Rank one matrices

$$\underset{1\times 4}{\mathbf{A}} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}; \qquad \operatorname{rg}\left(\mathbf{A}\right) = 1. \qquad \mathcal{S} = \mathcal{N}\left(\mathbf{A}\right)$$

- $\dim \mathcal{S} = \dim \mathcal{N}\left(\mathbf{A}\right) = n - r = 3.$
- basis of $\mathcal{S} = \mathcal{N}\left(\mathbf{A}\right)$?

5 Rank one matrices

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- $\dim \mathcal{S} = \dim \mathcal{N}\left(\mathbf{A}\right) = n - r = 3.$
- basis of $\mathcal{S} = \mathcal{N}\left(\mathbf{A}\right)$? which ones are pivot variables, which ones are free?

$$\left[\quad ; \quad ; \quad ; \right]$$

5 Rank one matrices

$$\underset{1\times 4}{\mathbf{A}} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}; \qquad \operatorname{rg}\left(\mathbf{A}\right) = 1. \qquad \mathcal{S} = \mathcal{N}\left(\mathbf{A}\right)$$

- $\dim \mathcal{S} = \dim \mathcal{N}\left(\mathbf{A}\right) = n - r = 3.$
- basis of $\mathcal{S}$ $=\mathcal{N}\left(\mathbf{A}\right)$? which ones are pivot variables, which ones are free?

$$\left[\begin{pmatrix} \\ 1 \\ 0 \\ 0 \end{pmatrix}; \qquad ; \qquad ;\right]$$

5 Rank one matrices

$$\underset{1\times 4}{\mathbf{A}} = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}; \qquad \operatorname{rg}\left(\mathbf{A}\right) = 1. \qquad \mathcal{S} = \mathcal{N}\left(\mathbf{A}\right)$$

- $\dim \mathcal{S} = \dim \mathcal{N}\left(\mathbf{A}\right) = n - r = 3.$
- basis of $\mathcal{S} = \mathcal{N}\left(\mathbf{A}\right)$? which ones are pivot variables, which ones are free?

$$\left[\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}; \qquad ; \qquad ;\right]$$

5 Rank one matrices

$$\underset{1\times 4}{\mathbf{A}} = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}; \qquad \operatorname{rg}(\mathbf{A}) = 1. \qquad \mathcal{S} = \mathcal{N}(\mathbf{A})$$

- $\dim \mathcal{S} = \dim \mathcal{N}(\mathbf{A}) = n - r = 3.$
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$$\left[\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}; \quad \begin{pmatrix}\\0\\1\\0\end{pmatrix}; \qquad ; \right]$$

5 Rank one matrices

$$\underset{1\times 4}{\mathbf{A}} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}; \qquad \mathrm{rg}\left(\mathbf{A}\right) = 1. \qquad \mathcal{S} = \mathcal{N}\left(\mathbf{A}\right)$$

- $\dim \mathcal{S} = \dim \mathcal{N}\left(\mathbf{A}\right) = n - r = 3.$
- basis of $\mathcal{S}$ =$\mathcal{N}\left(\mathbf{A}\right)$? which ones are pivot variables, which ones are free?

$$\left[\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \qquad ; \right]$$

5 Rank one matrices

$$\underset{1\times 4}{\mathbf{A}} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}; \qquad \operatorname{rg}(\mathbf{A}) = 1. \qquad \mathcal{S} = \mathcal{N}(\mathbf{A})$$

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$$\left[\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} \\ 0 \\ 0 \\ 1 \end{pmatrix};\right]$$

5 Rank one matrices

$$\underset{1\times 4}{\mathbf{A}} = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}; \qquad \operatorname{rg}\left(\mathbf{A}\right) = 1. \qquad \mathcal{S} = \mathcal{N}\left(\mathbf{A}\right)$$

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$$\left[\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}; \quad \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}; \quad \begin{pmatrix}-1\\0\\0\\1\end{pmatrix};\right]$$

5 Rank one matrices

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$$\left[\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}; \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}; \begin{pmatrix}-1\\0\\0\\1\end{pmatrix};\right]$$

- $\dim \mathcal{C}\left(\mathbf{A}^{\intercal}\right) =$

5 Rank one matrices

$$\underset{1\times 4}{\mathbf{A}} = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}; \qquad \operatorname{rg}\left(\mathbf{A}\right) = 1. \qquad \mathcal{S} = \mathcal{N}\left(\mathbf{A}\right)$$

- $\dim \mathcal{S} = \dim \mathcal{N}\left(\mathbf{A}\right) = n - r = 3.$
- basis of $\mathcal{S} = \mathcal{N}\left(\mathbf{A}\right)$? which ones are pivot variables, which ones are free?

$$\left[\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}; \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}; \begin{pmatrix}-1\\0\\0\\1\end{pmatrix};\right]$$

- $\dim \mathcal{C}\left(\mathbf{A}^{\intercal}\right) = 1.$

5 Rank one matrices

$$\underset{1\times 4}{\mathbf{A}} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}; \qquad \operatorname{rg}(\mathbf{A}) = 1. \qquad \mathcal{S} = \mathcal{N}(\mathbf{A})$$

- $\dim \mathcal{S} = \dim \mathcal{N}(\mathbf{A}) = n - r = 3.$
- basis of $\mathcal{S} = \mathcal{N}(\mathbf{A})$? which ones are pivot variables, which ones are free?

$$\left[\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix};\right]$$

- $\dim \mathcal{C}(\mathbf{A}^\intercal) = 1.$ $\boxed{\dim \mathcal{N}(\mathbf{A}) + \dim \mathcal{C}(\mathbf{A}^\intercal) = 3 + 1 = n}$

5 Rank one matrices

$$\underset{1\times 4}{\mathbf{A}} = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}; \qquad \operatorname{rg}\left(\mathbf{A}\right) = 1. \qquad \mathcal{S} = \mathcal{N}\left(\mathbf{A}\right)$$

- $\dim \mathcal{S} = \dim \mathcal{N}\left(\mathbf{A}\right) = n - r = 3.$
- basis of $\mathcal{S} = \mathcal{N}\left(\mathbf{A}\right)$? which ones are pivot variables, which ones are free?

$$\left[\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}; \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}; \begin{pmatrix}-1\\0\\0\\1\end{pmatrix};\right]$$

- $\dim \mathcal{C}\left(\mathbf{A}^{\intercal}\right) = 1.$ $\boxed{\dim \mathcal{N}\left(\mathbf{A}\right) + \dim \mathcal{C}\left(\mathbf{A}^{\intercal}\right) = 3 + 1 = n}$
- $\dim \mathcal{N}\left(\mathbf{A}^{\intercal}\right) =$

5 Rank one matrices

$$\underset{1\times 4}{\mathbf{A}} = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}; \qquad \operatorname{rg}\left(\mathbf{A}\right) = 1. \qquad \mathcal{S} = \mathcal{N}\left(\mathbf{A}\right)$$

- $\dim \mathcal{S} = \dim \mathcal{N}\left(\mathbf{A}\right) = n - r = 3.$
- basis of $\mathcal{S} = \mathcal{N}\left(\mathbf{A}\right)$? which ones are pivot variables, which ones are free?

$$\left[\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}; \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}; \begin{pmatrix}-1\\0\\0\\1\end{pmatrix};\right]$$

- $\dim \mathcal{C}\left(\mathbf{A}^{\intercal}\right) = 1.$ $\boxed{\dim \mathcal{N}\left(\mathbf{A}\right) + \dim \mathcal{C}\left(\mathbf{A}^{\intercal}\right) = 3 + 1 = n}$
- $\dim \mathcal{N}\left(\mathbf{A}^{\intercal}\right) = 0.$ basis $\mathcal{N}\left(\mathbf{A}^{\intercal}\right)$?

5 Rank one matrices

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$$\left[\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}; \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}; \begin{pmatrix}-1\\0\\0\\1\end{pmatrix};\right]$$

- $\dim \mathcal{C}\left(\mathbf{A}^{\intercal}\right) = 1.$ $\boxed{\dim \mathcal{N}\left(\mathbf{A}\right) + \dim \mathcal{C}\left(\mathbf{A}^{\intercal}\right) = 3 + 1 = n}$
- $\dim \mathcal{N}\left(\mathbf{A}^{\intercal}\right) = 0.$ basis $\mathcal{N}\left(\mathbf{A}^{\intercal}\right)$? $\left[\;\right]$

5 Rank one matrices

$$\underset{1\times 4}{\mathbf{A}} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}; \qquad \operatorname{rg}\left(\mathbf{A}\right) = 1. \qquad \mathcal{S} = \mathcal{N}\left(\mathbf{A}\right)$$

- $\dim \mathcal{S} = \dim \mathcal{N}\left(\mathbf{A}\right) = n - r = 3.$
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$$\left[\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix};\right]$$

- $\dim \mathcal{C}\left(\mathbf{A}^\intercal\right) = 1.$ $\boxed{\dim \mathcal{N}\left(\mathbf{A}\right) + \dim \mathcal{C}\left(\mathbf{A}^\intercal\right) = 3 + 1 = n}$
- $\dim \mathcal{N}\left(\mathbf{A}^\intercal\right) = 0.$ basis $\mathcal{N}\left(\mathbf{A}^\intercal\right)$? $\left[\ \right]$
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5 Rank one matrices

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$$\left[\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}; \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}; \begin{pmatrix}-1\\0\\0\\1\end{pmatrix};\right]$$

- $\dim \mathcal{C}\left(\mathbf{A}^{\intercal}\right) = 1.$ $\boxed{\dim \mathcal{N}\left(\mathbf{A}\right) + \dim \mathcal{C}\left(\mathbf{A}^{\intercal}\right) = 3 + 1 = n}$
- $\dim \mathcal{N}\left(\mathbf{A}^{\intercal}\right) = 0.$ basis $\mathcal{N}\left(\mathbf{A}^{\intercal}\right)$? $\left[\ \right]$
- $\dim \mathcal{C}\left(\mathbf{A}\right) = 1.$

5 Rank one matrices

$$\underset{1\times 4}{\mathbf{A}} = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}; \qquad \operatorname{rg}\left(\mathbf{A}\right) = 1. \qquad \mathcal{S} = \mathcal{N}\left(\mathbf{A}\right)$$

- $\dim \mathcal{S} = \dim \mathcal{N}\left(\mathbf{A}\right) = n - r = 3.$
- basis of $\mathcal{S}$ $=\mathcal{N}\left(\mathbf{A}\right)$? which ones are pivot variables, which ones are free?

$$\left[\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}; \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}; \begin{pmatrix}-1\\0\\0\\1\end{pmatrix};\right]$$

- $\dim \mathcal{C}\left(\mathbf{A}^\intercal\right) = 1.$ $\qquad \boxed{\dim \mathcal{N}\left(\mathbf{A}\right) + \dim \mathcal{C}\left(\mathbf{A}^\intercal\right) = 3 + 1 = n}$
- $\dim \mathcal{N}\left(\mathbf{A}^\intercal\right) = 0.$ $\qquad$ basis $\mathcal{N}\left(\mathbf{A}^\intercal\right)$? $\qquad \begin{bmatrix}\ \end{bmatrix}$
- $\dim \mathcal{C}\left(\mathbf{A}\right) = 1.$ $\qquad \boxed{\dim \mathcal{N}\left(\mathbf{A}^\intercal\right) + \dim \mathcal{C}\left(\mathbf{A}\right) = 0 + 1 = m}$

6 A problem from Microeconomics

Solve Y in terms of X to get PPF

$$\begin{cases} X & & & = 4L_x \\ & Y & & = 3L_y \\ & & L_x + L_y & = 80 \end{cases}$$

6 A problem from Microeconomics

Solve Y in terms of X to get PPF

$$\begin{cases} X & = 4L_x \\ \quad Y & = 3L_y \\ \qquad L_x + L_y & = 80 \end{cases} \rightarrow \begin{cases} X - 4L_x & = 0 \\ Y - 3L_y & = 0 \\ L_x + L_y & = 80 \end{cases}$$

6 A problem from Microeconomics

Solve Y in terms of X to get PPF

$$\begin{cases} X & & & = 4L_x \\ & Y & & = 3L_y \\ & & L_x + L_y & = 80 \end{cases} \quad \to \quad \begin{cases} X & & -4L_x & & = 0 \\ & Y & & -3L_y & = 0 \\ & & L_x + & L_y & = 80 \end{cases}$$

("in terms of" X means X free)

$\boxed{6}$ A problem from Microeconomics

Solve Y in terms of X to get PPF

$$\begin{cases} X & & & = 4L_x \\ & Y & & = 3L_y \\ & & L_x + L_y & = 80 \end{cases} \rightarrow \begin{cases} X & & - 4L_x & & = 0 \\ & Y & & - 3L_y & = 0 \\ & & L_x + & L_y & = 80 \end{cases}$$

("in terms of" X means X free)

$$\left[\begin{array}{cccc|c} 1 & 0 & -4 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 1 & -80 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(3)\mathbf{2}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ {[(4)\mathbf{1}+\mathbf{3}]}}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -80 \\ \hline 1 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(80)\mathbf{3}+\mathbf{5}]}]{\substack{\boldsymbol{\tau} \\ {[(-1)\mathbf{3}+\mathbf{4}]}}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 4 & -4 & 320 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$\boxed{6}$ A problem from Microeconomics

Solve Y in terms of X to get PPF

$$\begin{cases} X & & = 4L_x \\ & Y & = 3L_y \\ & & L_x + L_y = 80 \end{cases} \quad \to \quad \begin{cases} X & -4L_x & = 0 \\ Y & -3L_y & = 0 \\ & L_x + \ L_y & = 80 \end{cases}$$

("in terms of" X means X free)

$$\left[\begin{array}{cccc|c} 1 & 0 & -4 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 1 & -80 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(3)\mathbf{2}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(4)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -80 \\ \hline 1 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(80)\mathbf{3}+\mathbf{5}]}]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 4 & -4 & 320 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{pmatrix} X \\ Y \\ L_x \\ L_y \end{pmatrix} = \begin{pmatrix} 320 \\ 0 \\ 80 \\ 0 \end{pmatrix} + a \begin{pmatrix} -4 \\ 3 \\ -1 \\ 1 \end{pmatrix}$$

$\boxed{6}$ A problem from Microeconomics

Solve Y in terms of X to get PPF

$$\begin{cases} X = 4L_x \\ Y = 3L_y \\ L_x + L_y = 80 \end{cases} \rightarrow \begin{cases} X - 4L_x = 0 \\ Y - 3L_y = 0 \\ L_x + L_y = 80 \end{cases}$$

("in terms of" X means X free)

$$\left[\begin{array}{cccc|c} 1 & 0 & -4 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 1 & -80 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(3)\mathbf{2}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(4)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -80 \\ \hline 1 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(80)\mathbf{3}+\mathbf{5}]}]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 4 & -4 & 320 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{pmatrix} X \\ Y \\ L_x \\ L_y \end{pmatrix} = \begin{pmatrix} 320 \\ 0 \\ 80 \\ 0 \end{pmatrix} + a \begin{pmatrix} -4 \\ 3 \\ -1 \\ 1 \end{pmatrix} \Rightarrow a = L_y$$

$\boxed{6}$ A problem from Microeconomics

Solve Y in terms of X to get PPF

$$\begin{cases} X & = 4L_x \\ \quad Y & = 3L_y \\ \quad\quad L_x + L_y & = 80 \end{cases} \rightarrow \begin{cases} X \quad - 4L_x & = 0 \\ \quad Y \quad - 3L_y & = 0 \\ \quad L_x + \ L_y & = 80 \end{cases}$$

("in terms of" X means X free)

$$\left[\begin{array}{cccc|c} 1 & 0 & -4 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 1 & -80 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(3)\mathbf{2}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(4)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -80 \\ \hline 1 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(80)\mathbf{3}+\mathbf{5}]}]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 4 & -4 & 320 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{pmatrix} X \\ Y \\ L_x \\ L_y \end{pmatrix} = \begin{pmatrix} 320 \\ 0 \\ 80 \\ 0 \end{pmatrix} + a \begin{pmatrix} -4 \\ 3 \\ -1 \\ 1 \end{pmatrix} \Rightarrow a = L_y \Rightarrow \begin{pmatrix} X \\ Y \\ L_x \\ L_y \end{pmatrix} = \begin{pmatrix} 320 - 4L_y \\ 3L_y \\ 80 - L_y \\ L_y \end{pmatrix}$$ "in terms of" L_y

7 Free variable

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 4 & -4 & 320 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & \color{blue}{1} & \color{blue}{0} \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

7 Free variable

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 4 & -4 & 320 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & {\color{blue}1} & {\color{blue}0} \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-320)\mathbf{4}+\mathbf{5}]}]{\substack{\boldsymbol{\tau} \\ \left[\left(\frac{-1}{4}\right)\mathbf{4}\right]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 4 & {\color{blue}1} & {\color{blue}0} \\ 0 & 1 & 0 & -3/4 & 240 \\ 0 & 0 & 1 & 1/4 & 0 \\ 0 & 0 & 0 & -1/4 & 80 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

7 Free variable

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 4 & -4 & 320 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & \color{blue}{1} & \color{blue}{0} \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-320)\mathbf{4}+\mathbf{5}]}]{\substack{\boldsymbol{\tau} \\ \left[\left(\frac{-1}{4}\right)\mathbf{4}\right]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 4 & \color{blue}{1} & \color{blue}{0} \\ 0 & 1 & 0 & -3/4 & 240 \\ 0 & 0 & 1 & 1/4 & 0 \\ 0 & 0 & 0 & -1/4 & 80 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{pmatrix} X \\ Y \\ L_x \\ L_y \end{pmatrix} = \begin{pmatrix} 0 \\ 240 \\ 0 \\ 80 \end{pmatrix} + a \begin{pmatrix} 1 \\ -\frac{3}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \end{pmatrix}$$

7 Free variable

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 4 & -4 & 320 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(\text{-}320)\mathbf{4}+\mathbf{5}]}]{\boldsymbol{\tau} \atop \left[\left(\frac{-1}{4}\right)\mathbf{4}\right]} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 4 & 1 & 0 \\ 0 & 1 & 0 & -3/4 & 240 \\ 0 & 0 & 1 & 1/4 & 0 \\ 0 & 0 & 0 & -1/4 & 80 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{pmatrix} X \\ Y \\ L_x \\ L_y \end{pmatrix} = \begin{pmatrix} 0 \\ 240 \\ 0 \\ 80 \end{pmatrix} + a \begin{pmatrix} 1 \\ -\frac{3}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \end{pmatrix} \quad \Rightarrow \quad a = X$$

7 Free variable

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 4 & -4 & 320 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-320)\mathbf{4}+\mathbf{5}]}]{\substack{\boldsymbol{\tau} \\ \left[\left(\frac{-1}{4}\right)\mathbf{4}\right]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 4 & 1 & 0 \\ 0 & 1 & 0 & -3/4 & 240 \\ 0 & 0 & 1 & 1/4 & 0 \\ 0 & 0 & 0 & -1/4 & 80 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{pmatrix} X \\ Y \\ L_x \\ L_y \end{pmatrix} = \begin{pmatrix} 0 \\ 240 \\ 0 \\ 80 \end{pmatrix} + a \begin{pmatrix} 1 \\ -\frac{3}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \end{pmatrix} \Rightarrow a = X \quad \Rightarrow \begin{pmatrix} X \\ Y \\ L_x \\ L_y \end{pmatrix} = \begin{pmatrix} X \\ 240 - \frac{3}{4}X \\ \frac{1}{4}X \\ 80 - \frac{1}{4}X \end{pmatrix}$$

"in terms of" X

8 Free variables

$$\begin{cases} x + 2y - z + w = -1 \\ -x - 2y + 3z + 5w = -5 \\ -x - 2y - z - 7w = 7 \end{cases}$$

1. Solve in terms of y and w
2. Solve in terms of x and w
3. Solve in terms of x and z
4. Solve in terms of x and y

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & -1 \\ -1 & -2 & 3 & 5 & -5 \\ -1 & -2 & -1 & -7 & 7 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(1)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{4}] \\ [(1)\mathbf{1}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 6 & -6 \\ -1 & 0 & -2 & -6 & 6 \\ \hline 1 & -2 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{\boldsymbol{\tau} \\ [(-3)\mathbf{3}+\mathbf{4}] \\ [(3)\mathbf{3}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ -1 & 0 & -2 & 0 & 0 \\ \hline 1 & -2 & 1 & -4 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$
\left[\begin{array}{cc|c} -2 & -4 & 4 \\ 1 & 0 & 0 \\ 0 & -3 & 3 \\ 0 & 1 & 0 \end{array}\right]
\begin{cases}
\xrightarrow[{[(-4)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ \left[\left(\frac{-1}{2}\right)\mathbf{1}\right] \\ [(4)\mathbf{1}+\mathbf{2}]}}
\left[\begin{array}{cc|c} 1 & 0 & 0 \\ \frac{-1}{2} & -2 & 2 \\ 0 & -3 & 3 \\ 0 & 1 & 0 \end{array}\right]
\xrightarrow[{[(-1)\mathbf{2}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ \left[\left(\frac{-1}{3}\right)\mathbf{1}\right]}}
\left[\begin{array}{cc|c} 1 & 0 & 0 \\ \frac{-1}{2} & \frac{2}{3} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{-1}{3} & 1 \end{array}\right] \\
\xrightarrow[{[(-4)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ \left[\left(\frac{-1}{2}\right)\mathbf{1}\right] \\ [(4)\mathbf{1}+\mathbf{2}]}}
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\xrightarrow[{[(-2)\mathbf{2}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ \left[\left(\frac{-1}{2}\right)\mathbf{2}\right] \\ \left[\left(\frac{1}{2}\right)\mathbf{2}+\mathbf{1}\right]}}
\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{3}{4} & \frac{3}{2} & 0 \\ \frac{-1}{4} & \frac{-1}{2} & 1 \end{array}\right]
\end{cases}
$$

Questions of the Optional Lecture 1

(L-Opt-1) Question 1.

(a) What is the smallest subspace of 3 by 3 matrices which contains all symmetric matrices and all lower triangular matrices?
(b) What is the largest subspace which is contained in both of those subspaces?

(Strang, 1988, exercise 4 from section 1.2.)

(L-Opt-1) Question 2. For each of these statements, say whether the claim is true or false and give a brief justification.

(a) **True/False:** The set of 3×3 non-invertible matrices forms a subspace of the set of all 3×3 matrices.
(b) **True/False:** If the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has no solution then $\mathbf{A}$ does not have full row rank.
(c) **True/False:** There exist $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{B}$ is not invertible but $\mathbf{AB}$ is invertible.
(d) **True/False:** For any permutation matrix $\mathbf{P}$, we have that $\mathbf{P}^2 = \mathbf{I}$.

MIT Course 18.06 Quiz 1, October 4, 2004

(L-Opt-1) Question 3.

(a) Sean los vectores $\boldsymbol{u}$, $\boldsymbol{v}$ y $\boldsymbol{w}$ en $\mathbb{R}^7$ ¿Cuál es la dimensión (o cuáles son las posibles dimensiones) del espacio generado por estos tres vectores?

(b) Sea una matriz cuadrada $\mathbf{A}$. Si su espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$ está compuesto únicamente por el vector nulo $\mathbf{0}$, ¿Cuál es el espacio nulo de su traspuesta (espacio nulo por la izquierda $\mathcal{N}\left(\mathbf{A}^\intercal\right)$?

(c) Piense en el espacio vectorial de todas las matrices de orden 5 por 5, $\mathbb{R}^{5\times 5}$. Piense en el subconjunto de matrices 5 por 5 que son invertibles ¿es este subconjunto un sub-espacio vectorial? Si lo es, explique el motivo; si no lo es encuentre un contraejemplo.

(d) Indique si la siguiente aseveración es verdadera o falsa. Si es verdadera explique el motivo, si es falsa encuentre un contraejemplo: *"Si $\mathbf{B}^2 = \mathbf{0}$, entonces necesariamente $\mathbf{B} = \mathbf{0}$"*

(e) Si intercambio dos columnas de la matriz $\mathbf{A}$ ¿qué espacios fundamentales siguen siendo iguales?

(f) Si intercambio dos filas de la matriz $\mathbf{A}$ ¿qué espacios fundamentales siguen siendo iguales?

(g) ¿Por qué el vector $\boldsymbol{v} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ no puede estar en el espacio nulo de una matriz $\mathbf{A}$ y simultáneamente ser una fila de dicha matriz?

(L-Opt-1) Question 4. Empleando la definición de sub-espacio vectorial, verifique si los siguientes subconjuntos son sub-espacios vectoriales del espacio vectorial que los contiene.

(a) $\mathcal{V}$ es el espacio vectorial de todas las matrices 2×2 de números reales, con las operaciones habituales de suma y producto por un escalar; y el conjunto $\mathcal{W}$ son todas las matrices de la forma

$$\begin{bmatrix} a & b \\ 0 & b \end{bmatrix}$$

donde a y b son números reales.

(b) $\mathcal{V}$ es el espacio vectorial $C[0,1]$ de todas las funciones continuas en el intervalo $[0,1]$; y el conjunto $\mathcal{W}$ son todas las funciones $f \in C[0,1]$ tales que $f(0) = 2$.

(L-Opt-1) Question 5. Encuentre una base (de dimensión infinita) para el espacio de todos los polinomios

$$\mathcal{P} = \left\{ a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \mid \text{ para todo } n \right\}.$$

(L-Opt-1) Question 6. ¿Cuál es la dimensión de los siguientes espacios?

(a) El conjunto de matrices simétricas de orden 2×2, $\mathbf{A} = \mathbf{A}^\intercal$.

$$\mathbf{A} = \begin{bmatrix} a & b \\ b & d \end{bmatrix},$$

(b) El conjunto de matrices simétricas de orden 2×2

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

tales que $a + d = 0$.

(c) El conjunto de vectores de $\mathbb{R}^4$ de la forma
$\left\{ \Big(x,\ y,\ (x-3y),\ (2y-x),\Big)\ \mid\ x, y \in \mathbb{R} \right\}$.

Highlights of Lesson 11

- Orthogonal vectors and subspaces
- Nullspace $\perp$ row space
$$\mathcal{N}\left(\mathbf{A}\right) \perp \mathcal{C}\left(\mathbf{A}^\intercal\right)$$
- left nullspace $\perp$ column space
$$\mathcal{N}\left(\mathbf{A}^\intercal\right) \perp \mathcal{C}\left(\mathbf{A}\right)$$
- From parametric to Cartesian (or implicit) equations

2 Some definitions

- Dot product

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^{n} a_i b_i$$

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- Unit vector: $\|\boldsymbol{a}\| = 1$ $\qquad \frac{1}{\|\boldsymbol{x}\|} \cdot \boldsymbol{x}$

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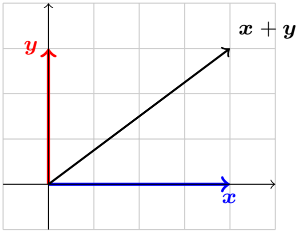
- Unit vector: $\|\boldsymbol{a}\| = 1$ $\frac{1}{\|\boldsymbol{x}\|} \cdot \boldsymbol{x}$

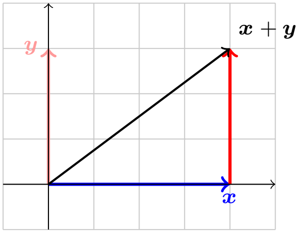
- Orthogonal (perpendicular) vectors: $\boldsymbol{x} \cdot \boldsymbol{y} = 0.$

3 Orthogonal vectors



$$\boldsymbol{x} \cdot \boldsymbol{y} = 0 \iff \boldsymbol{x} \perp \boldsymbol{y}$$

3 Orthogonal vectors



$$\boldsymbol{x}\cdot\boldsymbol{y} = 0 \iff \boldsymbol{x}\perp\boldsymbol{y}$$

Pythagoras Thm.: $\boldsymbol{x}\cdot\boldsymbol{y} = 0 \iff \|\boldsymbol{x}\|^2+\|\boldsymbol{y}\|^2 = \|\boldsymbol{x}+\boldsymbol{y}\|^2$

$$\boldsymbol{x}\cdot\boldsymbol{x}+\boldsymbol{y}\cdot\boldsymbol{y} = (\boldsymbol{x}+\boldsymbol{y})\cdot(\boldsymbol{x}+\boldsymbol{y}).$$

4 Squared length of a vector

$$\|\boldsymbol{v}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v}$$

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$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \rightarrow \quad \|\boldsymbol{x}\|^2 = \qquad ;$$

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$$\|\boldsymbol{v}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v}$$

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \rightarrow \quad \|\boldsymbol{x}\|^2 = 14;$$

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$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \rightarrow \quad \|\boldsymbol{x}\|^2 = 14; \qquad \boldsymbol{y} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \quad \rightarrow \quad \|\boldsymbol{y}\|^2 =$$

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Are these vectors orthogonal?

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Are these vectors orthogonal? $\quad \boldsymbol{x} \cdot \boldsymbol{y} = 0;$ $\quad$ yes

$$\boldsymbol{x} + \boldsymbol{y} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}; \qquad \|\boldsymbol{x} + \boldsymbol{y}\|^2 = 19;$$

$\boxed{4}$ Squared length of a vector

$$\|\boldsymbol{v}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v}$$

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \rightarrow \quad \|\boldsymbol{x}\|^2 = 14; \qquad \boldsymbol{y} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \quad \rightarrow \quad \|\boldsymbol{y}\|^2 = 5;$$

Are these vectors orthogonal? $\boldsymbol{x} \cdot \boldsymbol{y} = 0;$ yes

$$\boldsymbol{x} + \boldsymbol{y} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}; \qquad \|\boldsymbol{x} + \boldsymbol{y}\|^2 = 19;$$

(Pythagoras) (Orthogonality)

$$\boldsymbol{x} \cdot \boldsymbol{x} + \boldsymbol{y} \cdot \boldsymbol{y} = (\boldsymbol{x} + \boldsymbol{y}) \cdot (\boldsymbol{x} + \boldsymbol{y}) \quad \Longleftrightarrow \quad \boldsymbol{x} \cdot \boldsymbol{y} = 0.$$

5 Orthogonal subspaces

When subspace $\mathcal{S}$ is orthogonal to subspace $\mathcal{T}$:

$$\boxed{\text{Every vector in } \mathcal{S} \text{ is orthogonal to every vector in } \mathcal{T}}$$

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Are the plane of the *blackboard* and the floor orthogonal?

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Are the plane of the *blackboard* and the floor orthogonal? NO

6 Nullspace orthogonal to row space

- $\mathcal{N}(\mathbf{A}) \perp$ rows of $\mathbf{A}$

$$\mathbf{A}\boldsymbol{x} = \mathbf{0} \quad \Longrightarrow \quad \begin{pmatrix} ({}_{1|}\mathbf{A}) \cdot \boldsymbol{x} \\ \vdots \\ ({}_{m|}\mathbf{A}) \cdot \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

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$$\boldsymbol{x} \in \mathcal{N}(\mathbf{A}) \quad \Rightarrow \quad \boldsymbol{d}\mathbf{A}\boldsymbol{x} = \boldsymbol{d} \cdot \mathbf{0} = 0.$$

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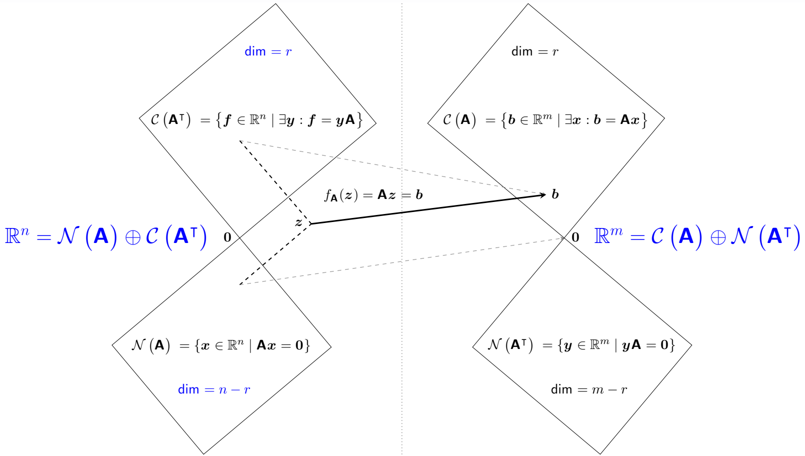
$$\boxed{\text{nullspace} \perp \text{row space} \qquad \mathcal{N}(\mathbf{A}) \perp \mathcal{C}(\mathbf{A}^\intercal)}$$

Also: $\quad \boldsymbol{x}\mathbf{A} = \mathbf{0} \quad \Rightarrow \quad \mathcal{N}(\mathbf{A}^\intercal) \perp \mathcal{C}(\mathbf{A})$

7 The big picture: direct sum of orthogonal complements



$$\mathcal{C}(\mathbf{A}^\mathsf{T}) \perp \mathcal{N}(\mathbf{A}) \qquad \mathcal{C}(\mathbf{A}) \perp \mathcal{N}(\mathbf{A}^\mathsf{T})$$

$$\boldsymbol{f} \cdot \boldsymbol{x} = \boldsymbol{y}\mathbf{A}\boldsymbol{x} = \boldsymbol{y} \cdot \mathbf{0} \qquad \boldsymbol{y} \cdot \boldsymbol{b} = \boldsymbol{y}\mathbf{A}\boldsymbol{x} = \mathbf{0} \cdot \boldsymbol{x}$$

8 Revisiting the Gaussian elimination

It's an algorithm to find a basis for the orthogonal complement

Give me some vectors (I write them as rows of **M**) and ...

$$\left[\frac{\mathbf{M}}{\mathbf{I}}\right] = \left[\begin{array}{cccc} 1 & -3 & 0 & -1 \\ 0 & -1 & 1 & 1 \\ 1 & -4 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(3)\mathbf{1}+\mathbf{2}] \\ [(1)\mathbf{1}+\mathbf{4}] \\ [(1)\mathbf{2}+\mathbf{3}] \\ [(1)\mathbf{2}+\mathbf{4}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 1 & 3 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\frac{\mathbf{L}}{\mathbf{E}}\right]$$

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Basis for the span of the given (row) vectors: $\mathcal{V}$

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Basis for the span of the given (row) vectors: $\mathcal{V}$

Basis for orthogonal complement: $\mathcal{V}^{\perp}$

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Give me some vectors (I write them as rows of **M**) and ...

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Basis for the span of the given (row) vectors: $\mathcal{V}$

Basis for orthogonal complement: $\mathcal{V}^{\perp}$

$$\mathbf{M}\mathbf{N} = \mathbf{0}$$

8 Revisiting the Gaussian elimination

It's an algorithm to find a basis for the orthogonal complement

Give me some vectors (I write them as rows of **M**) and . . .

$$\left[\frac{\mathbf{M}}{\mathbf{I}}\right] = \left[\begin{array}{cccc} 1 & -3 & 0 & -1 \\ 0 & -1 & 1 & 1 \\ 1 & -4 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{2+4}]}]{\substack{\boldsymbol{\tau} \\ [(3)\mathbf{1+2}] \\ [(1)\mathbf{1+4}] \\ [(1)\mathbf{2+3}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 1 & 3 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\frac{\mathbf{L}}{\mathbf{E}}\right] = \left[\begin{array}{cc} \mathbf{C} & \mathbf{0} \\ \hline \mathbf{D} & \mathbf{N} \end{array}\right]$$

Basis for the span of the given (row) vectors: $\mathcal{V}$

Basis for orthogonal complement: $\mathcal{V}^{\perp}$

$\mathbf{MN} = \mathbf{0}$

If you had given me $\mathbf{N}_{|1}$ and $\mathbf{N}_{|2}$, after Gaussian elimination would have obtained a basis for. . .

8 Revisiting the Gaussian elimination

It's an algorithm to find a basis for the orthogonal complement

Give me some vectors (I write them as rows of **M**) and ...

$$\left[\begin{array}{c}\mathbf{M}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{cccc} 1 & -3 & 0 & -1 \\ 0 & -1 & 1 & 1 \\ 1 & -4 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(3)\mathbf{1}+\mathbf{2}]\\ [(1)\mathbf{1}+\mathbf{4}]\\ [(1)\mathbf{2}+\mathbf{3}]\\ [(1)\mathbf{2}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 1 & 3 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c}\mathbf{L}\\ \hline \mathbf{E}\end{array}\right] = \left[\begin{array}{cc}\mathbf{C} & \mathbf{0}\\ \hline \mathbf{D} & \mathbf{N}\end{array}\right]$$

Basis for the span of the given (row) vectors: $\mathcal{V}$

Basis for orthogonal complement: $\mathcal{V}^{\perp}$ $\qquad \mathbf{M}\mathbf{N} = \mathbf{0}$

If you had given me $\mathbf{N}_{|1}$ and $\mathbf{N}_{|2}$, after Gaussian elimination would have obtained a basis for... $(\mathcal{V}^{\perp})^{\perp} = \mathcal{V}$

9 Cartesian (implicit) and parametric equations of lines and planes

Cartesian (implicit) equation $\{\boldsymbol{x} \in \mathbb{R}^n \mid \mathbf{A}\boldsymbol{x} = \boldsymbol{b}\}$:

For example

$$\left\{\boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \text{sol. set of } \begin{cases} x_1 - x_2 + x_3 = 1 \\ x_3 = 1 \end{cases}$$

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Parametric equation:

for the above set

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1 : \boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \boldsymbol{p} \right\}$$

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In this case *dimension* 1 A line (there is only one parameter a)

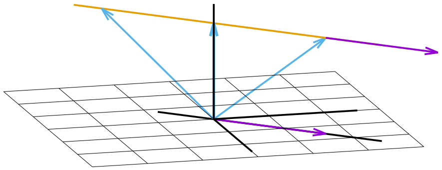
line line

or

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1 : \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \boldsymbol{p} \right\}$$

or

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1 : \boldsymbol{x} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \boldsymbol{p} \right\}$$



10 Cartesian (implicit) and parametric equations of lines and planes

Cartesian (implicit) equation $\{\boldsymbol{x} \in \mathbb{R}^n \mid \mathbf{A}\boldsymbol{x} = \boldsymbol{b}\}$:

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$$\left\{\boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \boldsymbol{x} = \left(1,\right) \right\} = \text{sol. set of } \left\{ x_1 - x_2 + x_3 = 1 \right.$$

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Parametric equation:

for the above set

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 : \boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}$$

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Parametric equation:

for the above set

$$\left\{\boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 : \boldsymbol{x} = \begin{pmatrix}0\\0\\1\end{pmatrix} + \begin{bmatrix}1 & -1\\1 & 0\\0 & 1\end{bmatrix} \boldsymbol{p} \right\}$$

In this case *dimension* 2
plane

A plane (two parameters a and b)
plane

or

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 : \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}$$

but also

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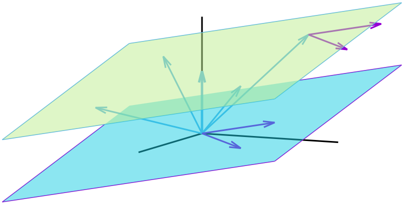

11 From parametric to Cartesian equations

$$\boxed{\mathcal{C}\left(\mathbf{A}^\intercal\right) \perp \mathcal{N}\left(\mathbf{A}\right)}$$

Consider

$$C = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^k : \boldsymbol{x} = \boldsymbol{s} + \left[\boldsymbol{n}_1;\ \ldots\ \boldsymbol{n}_k;\right]\boldsymbol{p} \right\}.$$

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If we find $\mathbf{A}$ such that $\mathbf{A}\boldsymbol{n}_i = \mathbf{0}$ then if $\boldsymbol{x} \in C$

$$\mathbf{A}\boldsymbol{x} = \mathbf{A}\boldsymbol{s} + \underbrace{\mathbf{A}\left[\boldsymbol{n}_1; \;\ldots\; \boldsymbol{n}_k;\right]}_{\mathbf{0}} \boldsymbol{p} \quad \Rightarrow \quad \mathbf{A}\boldsymbol{x} = \boldsymbol{b}, \quad \text{where } \boldsymbol{b} = \mathbf{A}\boldsymbol{s}.$$

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Therefore

$$C = \left\{\boldsymbol{x} \in \mathbb{R}^n \mid \mathbf{A}\boldsymbol{x} = \boldsymbol{b}\right\}.$$

12 From the set of solution to a linear system

Find the implicit equations of the plane P parallel to the spam of $(1, 2, 0, -2)$ and $(0, 0, 1, 3)$, that goes through $\boldsymbol{s} = (1, 3, 1, 1)$.

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$$P = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \;\middle|\; \exists a, b \in \mathbb{R} : \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix} + a \begin{pmatrix} 1 \\ 2 \\ 0 \\ -2 \end{pmatrix} + b \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \end{pmatrix} \right\}$$

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$$=\left\{\boldsymbol{x}\in\mathbb{R}^4\;\middle|\;\exists\boldsymbol{p}\in\mathbb{R}^2:\ \boldsymbol{x}=\begin{pmatrix}1\\3\\1\\1\end{pmatrix}+\begin{bmatrix}1&0\\2&0\\0&1\\-2&3\end{bmatrix}\boldsymbol{p}\right\}$$

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We need vectors perpendicular to $(1, 2, 0, -2)$ and $(0, 0, 1, 3)$

13 From the set of solution to a linear system

$$\boldsymbol{x} = (x, y, z, w,); \qquad \boldsymbol{s} = (1, 3, 1, 1,).$$

$$\left[\begin{array}{cccc} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(2)\mathbf{1}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ \hline 1 & -2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-3)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 1 & -2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

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So $\mathbf{A} = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 2 & 0 & -3 & 1 \end{bmatrix};$

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So $\mathbf{A} = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 2 & 0 & -3 & 1 \end{bmatrix}$; and then $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} -2x + y \\ 2x + w - 3z \end{pmatrix}$

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$\boldsymbol{b} = \mathbf{A}\boldsymbol{s} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Hence $\begin{cases} -2x + y & = 1 \\ 2x \quad - 3z + w & = 0 \end{cases}$

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So $\mathbf{A} = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 2 & 0 & -3 & 1 \end{bmatrix}$; and then $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} -2x + y \\ 2x + w - 3z \end{pmatrix}$ and

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Questions of the Lecture 11

(L-11) QUESTION 1. Describe the set of vectors in $\mathbb{R}^3$ orthogonal to this one $\begin{pmatrix}1\\3\\-1\end{pmatrix}$

(Hefferon, 2008, exercise 2.15 from section II.2.)

(L-11) QUESTION 2.

(a) Find a parametric representation for the line passing through the points $\boldsymbol{x}_P = \begin{pmatrix}1\\2\end{pmatrix}$ y $\boldsymbol{x}_Q = \begin{pmatrix}3\\1\end{pmatrix}$.

(b) Find a implicit representation for the same line.

(L-11) QUESTION 3.

(a) Find a parametric representation for the line passing through the points $\boldsymbol{x}_P = \begin{pmatrix}1, & -3, & 1,\end{pmatrix}$ and $\boldsymbol{x}_Q = \begin{pmatrix}-2, & 4, & 5,\end{pmatrix}$.

(b) Find a implicit representation (Cartesian equations) for the same line.

(Lang, 1986, Example 1 in Section 1.5)

(L-11) QUESTION 4. Is there any vector perpendicular to itself?
(Hefferon, 2008, exercise 2.17 from section II.2.)

(L-11) QUESTION 5.

(a) Parametric equation of a line parallel to $2x - 3y = 5$ that goes through $(1, 1)$.

(b) Find a implicit representation for the line.

(L-11) QUESTION 6. Find the length of each vector

(a) $\begin{pmatrix}1\\3\end{pmatrix}$.

(b) $\begin{pmatrix}-1\\2\end{pmatrix}$.

(c) $\begin{pmatrix}4\\1\\1\end{pmatrix}$.

(d) $\begin{pmatrix}0\\0\\0\end{pmatrix}$.

(e) $\begin{pmatrix}1\\-1\\1\\0\end{pmatrix}$.

(Hefferon, 2008, exercise 2.11 from section II.2.)

(L-11) QUESTION 7. Find a unit vector with the same direction as $\boldsymbol{v} = (2, -1, 0, 4, -2)$.

(L-11) QUESTION 8. Find k so that these two vectors are perpendicular.

$$\begin{pmatrix}k, & 1,\end{pmatrix}, \qquad \begin{pmatrix}4, & 3,\end{pmatrix}.$$

(Hefferon, 2008, exercise 2.14 from section II.2.)

(L-11) QUESTION 9. Construc a matrix with the required property or say why that is impossible:

(a) Column space contains $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$, nullspace contains $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(b) Row space contains $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$, and nullspace contains $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(c) $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ has a solution and $\mathbf{A}^\mathsf{T} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(d) Every row is orthogonal to every column (**A** is not the zero matrix)

(e) Columns add up to a column of zeros, rows add up to a row of 1's.

(Strang, 2003, exercise 3 from section 4.1.)

(L-11) QUESTION 10. If $\mathbf{AB} = \mathbf{0}$, the columns of **B** are in the ________ of **A**. The rows of **A** are in the __________ of **B**. Why can't **A** and **B** be 3 by 3 matrices of rank 2?

(Strang, 2003, exercise 4 from section 4.1.)

(L-11) QUESTION 11. Suppose that $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{w}$ and $\boldsymbol{u} \neq \mathbf{0}$. Must $\boldsymbol{v} = \boldsymbol{w}$?
(Hefferon, 2008, exercise 2.20 from section II.2.)

(L-11) QUESTION 12.

(a) If $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution and $\mathbf{A}^\intercal \boldsymbol{y} = \mathbf{0}$, then $\boldsymbol{y}$ is perpendicular to ____.
(b) If $\mathbf{A}^\intercal \boldsymbol{y} = \boldsymbol{c}$ has a solution and $\mathbf{A}\boldsymbol{x} = \mathbf{0}$, then $\boldsymbol{x}$ is perpendicular to ____.

(Strang, 2003, exercise 5 from section 4.1.)

(L-11) Question 13. Demuestre, in $\mathbb{R}^n$, that if $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular then $||\boldsymbol{u} + \boldsymbol{v}||^2 = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2$.
(Hefferon, 2008, exercise 2.33 from section II.2.)

(L-11) Question 14.

(a) Find parametric equations of the plane that goes through the point (0,1,1) and parallel to the vectors (0,1,2) and (1,1,0)
(b) Write the implicit equation of the same plane.

(L-11) Question 15.

(a) Find a parametric equation of the plane through the point $\begin{pmatrix}2, & 1, & 3,\end{pmatrix}$ with normal vector $\begin{pmatrix}3, & 1, & 1,\end{pmatrix}$.
(b) Write the implicit equation of the same plane.

(L-11) Question 16. Find a 1 by 3 matrix whose nullspace consists of all vectors in $\mathbb{R}^3$ such that $x_1 + 2x_2 + 4x_3 = 0$. Find a 3 by 3 matrix with that same nullspace.
(Strang, 2006, exercise 9 from section 2.4.)

(L-11) QUESTION 17. Consider the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}.$$

(a) (1^{pts}) Find the solution to the system.
(b) (0.5^{pts}) Explain why the solution set is a line in $\mathbb{R}^5$. Find a direction vector (a vector parallel to the line) and any point on that line.
(c) (1^{pts}) Find the set of vectors perpendicular to the solution set. Prove that set is a four dimensional subspace. Find a basis for that subspace.

(L-11) QUESTION 18. Consider $\underset{4\times 2}{\mathbf{A}}$ with exactly two special solutions for $\boldsymbol{x}\mathbf{A} = \mathbf{0}$:

$\boldsymbol{s}_1 = \begin{pmatrix}3, & 1, & 0, & 0,\end{pmatrix}$, and $\boldsymbol{s}_2 = \begin{pmatrix}6, & 0, & 2, & 1,\end{pmatrix}$.

(a) Find the reduced row echelon form $\mathbf{R}$ of $\mathbf{A}$.
(b) What is the row space of $\mathbf{A}$?
(c) What is the complete solution to $\boldsymbol{x}\mathbf{R} = \begin{pmatrix}3, & 6,\end{pmatrix}$?
(d) Find a combination of rows 2, 3, 4 that equals $\mathbf{0}$. (Not OK to use $0({}_{2|}\mathbf{A}) + 0({}_{3|}\mathbf{A}) + 0({}_{4|}\mathbf{A})$. The problem is to show that these rows are dependent.)

basado en MIT Course 18.06 Quiz 1, March 4, 2013

Highlights of Lesson 12

- Projections
- Projection matrices

2 Direct sum of subspaces

$\mathbb{R}^n$ is a *direct sum* of $\mathcal{A}$ and $\mathcal{B}$ $\quad (\mathbb{R}^n = \mathcal{A} \oplus \mathcal{B})$

if every $\boldsymbol{x} \in \mathbb{R}^n$ has a **unique** representation $\boldsymbol{x} = \boldsymbol{a} + \boldsymbol{b}$,

with $\boldsymbol{a} \in \mathcal{A}$ and $\boldsymbol{b} \in \mathcal{B}$.

$\boxed{2}$ Direct sum of subspaces

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Example

$$\boxed{\mathbb{R}^n = \mathcal{C}\left(\mathbf{A}^\intercal\right) \oplus \mathcal{N}\left(\mathbf{A}\right)}$$

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{ccc} 1 & 2 & 5 \\ 2 & 4 & 10 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc} 1 & 0 & 0 \\ 2 & 0 & 0 \\ \hline 1 & -2 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \Rightarrow \text{Basis of } \mathbb{R}^3; \left[\begin{pmatrix}1\\2\\5\end{pmatrix}; \begin{pmatrix}-2\\1\\0\end{pmatrix}; \begin{pmatrix}-5\\0\\1\end{pmatrix}\right]$$

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Example $\boxed{\mathbb{R}^n = \mathcal{C}\left(\mathbf{A}^\mathsf{T}\right) \oplus \mathcal{N}\left(\mathbf{A}\right)}$

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$$\forall \boldsymbol{x} \in \mathbb{R}^3,\ \exists c_1, c_2, c_3 \ \Big|\ \boldsymbol{x} = c_1 \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} = \boldsymbol{a} + \boldsymbol{b}$$

where $\boldsymbol{a} \in \mathcal{C}\left(\mathbf{A}^\mathsf{T}\right)$ and $\boldsymbol{b} \in \mathcal{N}\left(\mathbf{A}\right)$.

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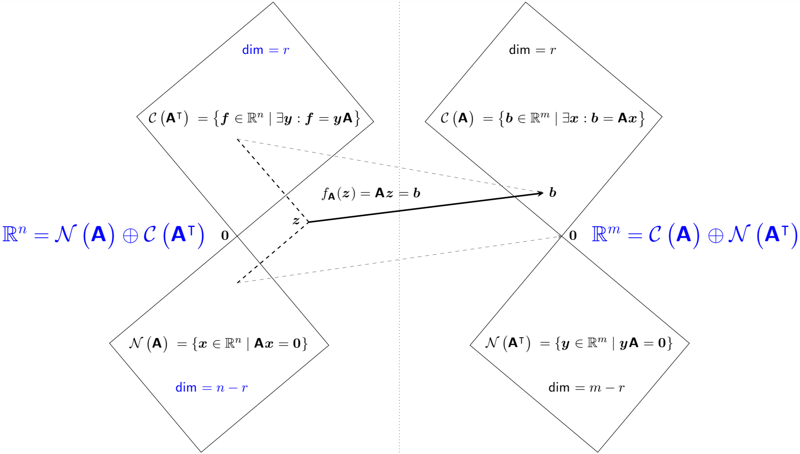
where $\boldsymbol{a} \in \mathcal{C}\left(\mathbf{A}^\intercal\right)$ and $\boldsymbol{b} \in \mathcal{N}\left(\mathbf{A}\right)$.

También $\boxed{\mathbb{R}^m = \mathcal{C}\left(\mathbf{A}\right) \oplus \mathcal{N}\left(\mathbf{A}^\intercal\right)}$

$\boxed{3}$ The big picture: direct sum of orthogonal complements



$$\mathcal{C}(\mathbf{A}^\mathsf{T}) \perp \mathcal{N}(\mathbf{A}) \qquad\qquad \mathcal{C}(\mathbf{A}) \perp \mathcal{N}(\mathbf{A}^\mathsf{T})$$

$$\boldsymbol{f} \cdot \boldsymbol{x} = \boldsymbol{y}\mathbf{A}\boldsymbol{x} = \boldsymbol{y} \cdot \mathbf{0} \qquad\qquad \boldsymbol{y} \cdot \boldsymbol{b} = \boldsymbol{y}\mathbf{A}\boldsymbol{x} = \mathbf{0} \cdot \boldsymbol{x}$$

4 Orthogonal Projection onto $\mathcal{C}(\mathbf{A})$

Consider $\underset{m\times n}{\mathbf{A}}$; since $\mathbb{R}^m = \mathcal{C}(\mathbf{A}) \oplus \mathcal{N}(\mathbf{A}^\intercal)$, for any $\boldsymbol{y} \in \mathbb{R}^m$

$$\boldsymbol{y} = \boldsymbol{p}_y + \boldsymbol{e}; \qquad (\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{p}_y)$$

where $\boxed{\boldsymbol{p}_y \in \mathcal{C}(\mathbf{A}) \text{ and } \boldsymbol{e} \perp \boldsymbol{p}_y}$, so $\boldsymbol{e} \in \mathcal{N}(\mathbf{A}^\intercal)$.

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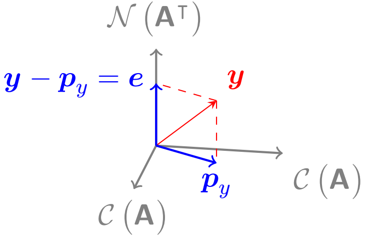

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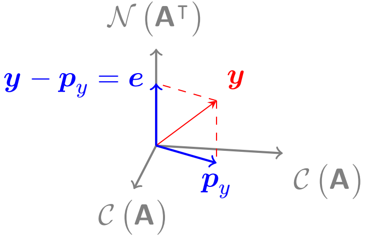

How to compute $\boldsymbol{p}_y \in \mathcal{C}(\mathbf{A})$?

5 Normal equations

Consider $\underset{m\times n}{\mathbf{A}}$. We want to find the descoposition $\boxed{\boldsymbol{y} = \boldsymbol{p}_y + \boldsymbol{e}}$

where

$$\boldsymbol{p}_y \in \mathcal{C}\left(\mathbf{A}\right) \qquad \text{and} \qquad (\boldsymbol{p}_y - \boldsymbol{y}) \in \mathcal{N}\left(\mathbf{A}^\intercal\right)$$

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Then

$$\mathbf{A}\widehat{\boldsymbol{x}} = \boldsymbol{p}_y \qquad \Leftrightarrow \qquad (\mathbf{A}\widehat{\boldsymbol{x}} - \boldsymbol{y}) \in \mathcal{N}\left(\mathbf{A}^\intercal\right)$$

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$$\mathbf{A}\widehat{\boldsymbol{x}} = \boldsymbol{p}_y \quad \Leftrightarrow \quad \mathbf{A}^\intercal(\mathbf{A}\widehat{\boldsymbol{x}} - \boldsymbol{y}) = \mathbf{0} \quad \Leftrightarrow \quad \boxed{(\mathbf{A}^\intercal\mathbf{A})\widehat{\boldsymbol{x}} = \mathbf{A}^\intercal\boldsymbol{y}}$$

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Equivalent systems!

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Equivalent systems! $\Rightarrow \mathcal{N}\left(\mathbf{A}\right) = \mathcal{N}\left(\mathbf{A}^\intercal\mathbf{A}\right)$

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Equivalent systems! $\Rightarrow\ \mathcal{N}\left(\mathbf{A}\right) = \mathcal{N}\left(\mathbf{A}^\intercal\mathbf{A}\right)\ \Rightarrow\ \mathrm{rg}(\mathbf{A}) = \mathrm{rg}(\mathbf{A}^\intercal\mathbf{A})$

5 Normal equations

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Therefore

$$\mathbf{A}\widehat{\boldsymbol{x}} = \boldsymbol{p}_y \quad \Leftrightarrow \quad \mathbf{A}^\mathsf{T}(\mathbf{A}\widehat{\boldsymbol{x}} - \boldsymbol{y}) = \mathbf{0} \quad \Leftrightarrow \quad \boxed{(\mathbf{A}^\mathsf{T}\mathbf{A})\widehat{\boldsymbol{x}} = \mathbf{A}^\mathsf{T}\boldsymbol{y}}$$

Equivalent systems! $\Rightarrow \mathcal{N}(\mathbf{A}) = \mathcal{N}(\mathbf{A}^\mathsf{T}\mathbf{A}) \Rightarrow \mathrm{rg}(\mathbf{A}) = \mathrm{rg}(\mathbf{A}^\mathsf{T}\mathbf{A})$

$\boxed{\text{unique solution } \widehat{\boldsymbol{x}} \text{ if and only if } \mathbf{A} \text{ is full column rank}}$

6 The solution to the normal equations (full column rank)

$$\boxed{\mathbf{A}^\intercal \mathbf{A}\widehat{\boldsymbol{x}} = \mathbf{A}^\intercal \boldsymbol{y}} \qquad (\mathbf{A} \text{ is full column rank})$$

The solution $\widehat{\boldsymbol{x}} = (\mathbf{A}^\intercal \mathbf{A})^{-1}\mathbf{A}^\intercal \boldsymbol{y}$

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The solution $\widehat{\boldsymbol{x}} = (\mathbf{A}^\intercal \mathbf{A})^{-1}\mathbf{A}^\intercal \boldsymbol{y}$

The projection $\boldsymbol{p} = \mathbf{A}\widehat{\boldsymbol{x}}$

6 The solution to the normal equations (full column rank)

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The solution $\qquad \widehat{\boldsymbol{x}} = (\mathbf{A}^\mathsf{T}\mathbf{A})^{-1}\mathbf{A}^\mathsf{T}\boldsymbol{y}$

The projection $\qquad \boldsymbol{p} = \mathbf{A}\widehat{\boldsymbol{x}} = \mathbf{A}(\mathbf{A}^\mathsf{T}\mathbf{A})^{-1}\mathbf{A}^\mathsf{T}\boldsymbol{y}$

6 The solution to the normal equations (full column rank)

$$\boxed{\mathbf{A}^\intercal \mathbf{A}\widehat{\boldsymbol{x}} = \mathbf{A}^\intercal \boldsymbol{y}} \qquad (\mathbf{A} \text{ is full column rank})$$

The solution $\widehat{x} = (\mathbf{A}^\intercal \mathbf{A})^{-1}\mathbf{A}^\intercal y$

The projection $p = \mathbf{A}\widehat{x} = \mathbf{A}(\mathbf{A}^\intercal \mathbf{A})^{-1}\mathbf{A}^\intercal y$

The projection matrix $\mathbf{P} = \mathbf{A}(\mathbf{A}^\intercal \mathbf{A})^{-1}\mathbf{A}^\intercal$

6 The solution to the normal equations (full column rank)

$$\boxed{\mathbf{A}^\top \mathbf{A}\widehat{\boldsymbol{x}} = \mathbf{A}^\top \boldsymbol{y}} \qquad (\mathbf{A} \text{ is full column rank})$$

The solution $\widehat{x} = (\mathbf{A}^\top \mathbf{A})^{-1}\mathbf{A}^\top y$

The projection $p = \mathbf{A}\widehat{x} = \mathbf{A}(\mathbf{A}^\top \mathbf{A})^{-1}\mathbf{A}^\top y$

The projection matrix $\mathbf{P} = \mathbf{A}(\mathbf{A}^\top \mathbf{A})^{-1}\mathbf{A}^\top$

$$\boldsymbol{p} = \mathbf{P}\boldsymbol{y}$$

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The projection matrix $\mathbf{P} = \mathbf{A}(\mathbf{A}^\intercal \mathbf{A})^{-1} \mathbf{A}^\intercal$

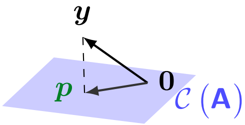
$$p = \mathbf{P} y$$

$\mathbf{P}$: Symetric and idempotent.

7 Projection matrix

$$\boxed{\mathbf{P} = \mathbf{A}\left(\mathbf{A}^\top\mathbf{A}\right)^{-1}\mathbf{A}^\top}$$

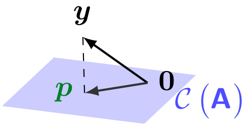
Projection $\mathbf{P}\boldsymbol{y}$ *is the point* $\boldsymbol{p}$ *of* $\mathcal{C}\left(\mathbf{A}\right)$ *closest to* $\boldsymbol{y}$



7 Projection matrix

$$\boxed{\mathbf{P} = \mathbf{A}\left(\mathbf{A}^\intercal\mathbf{A}\right)^{-1}\mathbf{A}^\intercal}$$

Projection $\mathbf{P}\boldsymbol{y}$ *is the point* $\boldsymbol{p}$ *of* $\mathcal{C}\left(\mathbf{A}\right)$ *closest to* $\boldsymbol{y}$



Extreme cases:

- If $\boldsymbol{y} \in \mathcal{C}\left(\mathbf{A}\right)$ then $\mathbf{P}\boldsymbol{y} =$

7 Projection matrix

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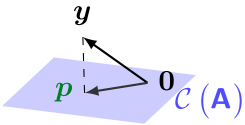

Extreme cases:

- If $\boldsymbol{y} \in \mathcal{C}\left(\mathbf{A}\right)$ then $\mathbf{P}\boldsymbol{y} = \boldsymbol{y}$

7 Projection matrix

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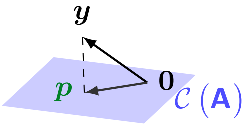
Projection $\mathbf{P}\boldsymbol{y}$ *is the point* $\boldsymbol{p}$ *of* $\mathcal{C}\left(\mathbf{A}\right)$ *closest to* $\boldsymbol{y}$



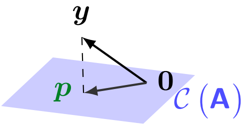
Extreme cases:

- If $\boldsymbol{y} \in \mathcal{C}\left(\mathbf{A}\right)$ then $\mathbf{P}\boldsymbol{y} = \boldsymbol{y}$
- If $\boldsymbol{y} \perp \mathcal{C}\left(\mathbf{A}\right)$ then $\mathbf{P}\boldsymbol{y} =$

7 Projection matrix

$$\boxed{\mathbf{P} = \mathbf{A}\left(\mathbf{A}^\intercal\mathbf{A}\right)^{-1}\mathbf{A}^\intercal}$$

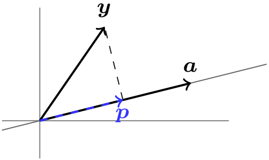
Projection $\mathbf{P}\boldsymbol{y}$ *is the point* $\boldsymbol{p}$ *of* $\mathcal{C}\left(\mathbf{A}\right)$ *closest to* $\boldsymbol{y}$



Extreme cases:

- If $\boldsymbol{y} \in \mathcal{C}\left(\mathbf{A}\right)$ then $\mathbf{P}\boldsymbol{y} = \boldsymbol{y}$
- If $\boldsymbol{y} \perp \mathcal{C}\left(\mathbf{A}\right)$ then $\mathbf{P}\boldsymbol{y} = \mathbf{0}$

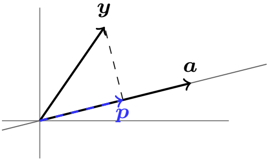
8 Projection onto a line



line$= \mathcal{C}\left(\mathbf{A}\right)$; $\quad \mathbf{A} = \left[\boldsymbol{a}\right]$; $\quad \boldsymbol{a} \neq \mathbf{0}$

I'd like to find the point $\boldsymbol{p}$ on that line closest to $\boldsymbol{y}$

8 Projection onto a line



line$= \mathcal{C}\left(\mathbf{A}\right)$; $\mathbf{A} = [\boldsymbol{a}]$; $\boldsymbol{a} \neq \mathbf{0}$

I'd like to find the point $\boldsymbol{p}$ on that line closest to $\boldsymbol{y}$

$\boldsymbol{p} \in \mathcal{C}\left(\left[\boldsymbol{a}\right]\right)$

$\boldsymbol{p}$ is some multiple of $\boldsymbol{a}$: $\boldsymbol{p} = \left[\boldsymbol{a}\right](\widehat{x},)$

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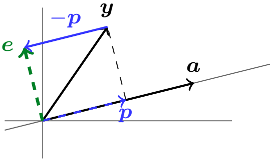

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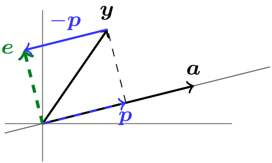

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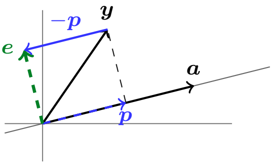

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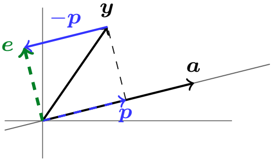

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The projection $\quad \boldsymbol{p} = [\boldsymbol{a}]\widehat{\boldsymbol{x}} = [\boldsymbol{a}]\left([\boldsymbol{a}]^\mathsf{T}[\boldsymbol{a}]\right)^{-1}[\boldsymbol{a}]^\mathsf{T}\boldsymbol{y}$

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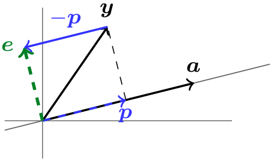

line$=\mathcal{C}(\mathbf{A})$; $\mathbf{A}=[a]$; $a\neq 0$

I'd like to find the point p on that line closest to y

$p\in\mathcal{C}([a]) \quad \perp \quad e=(y-p)\in\mathcal{N}([a]^\top)$.

p is some multiple of a: $p=[a](\widehat{x},)$

How: $[a]^\top[a]\widehat{x}=[a]^\top y$

The solution $\widehat{x}= ([a]^\top[a])^{-1}[a]^\top y$

The projection $p=[a]\widehat{x}=[a]([a]^\top[a])^{-1}[a]^\top y$

The projection matrix $\mathbf{P}=[a]([a]^\top[a])^{-1}[a]^\top$

9 Projection onto a plane

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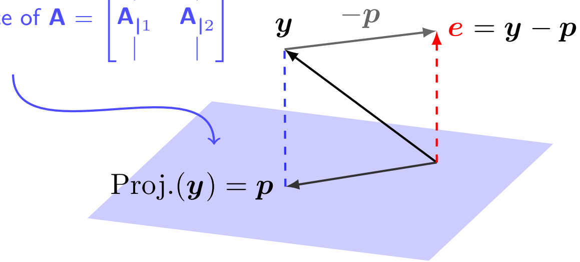
$$\mathbf{A}x = \Big(\text{Proj. of } \boldsymbol{y} \text{ onto } \mathcal{C}(\mathbf{A})\Big).$$

$\mathcal{C}(\mathbf{A})$: Column space of $\mathbf{A} = \begin{bmatrix} | & | \\ \mathbf{A}_{|1} & \mathbf{A}_{|2} \\ | & | \end{bmatrix}$



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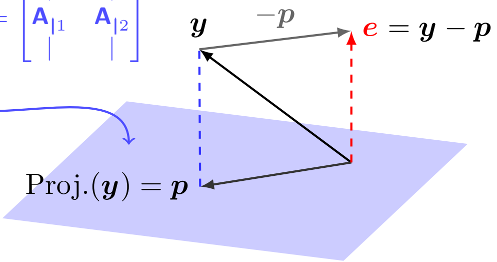
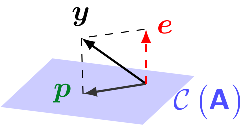


$(\boldsymbol{y} - \boldsymbol{p}) = \boldsymbol{e} \perp \mathcal{C}(\mathbf{A})$ ... that's the crucial fact.

10 Normal equations

What's the projection of $\boldsymbol{y}$ onto the column space of $\mathbf{A} = \begin{bmatrix} | & | \\ \mathbf{A}_{|1} & \mathbf{A}_{|2} \\ | & | \end{bmatrix}$?



$$\boldsymbol{p} = \left(\mathbf{A}_{|1}\right)\widehat{x_1} + \left(\mathbf{A}_{|2}\right)\widehat{x_2} \; = \; \mathbf{A}\widehat{\boldsymbol{x}}$$

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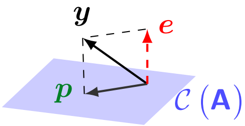

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"Find the right combination of the columns so $\boldsymbol{e} \perp \mathcal{C}\left(\mathbf{A}\right)$"

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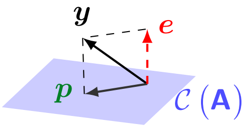

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$$\boldsymbol{e} \perp \mathcal{C}(\mathbf{A}) \quad \Rightarrow \quad \boldsymbol{e} \in$$

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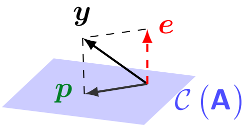

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"Find the right combination of the columns so $\boldsymbol{e} \perp \mathcal{C}(\mathbf{A})$"

$$\boldsymbol{e} \perp \mathcal{C}(\mathbf{A}) \quad \Rightarrow \quad \boldsymbol{e} \in \mathcal{N}(\mathbf{A}^\mathsf{T})$$

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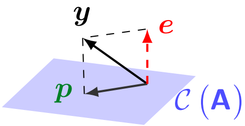

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"Find the right combination of the columns so $\boldsymbol{e} \perp \mathcal{C}(\mathbf{A})$*"*

$$\boldsymbol{e} \perp \mathcal{C}(\mathbf{A}) \quad \Rightarrow \quad \boldsymbol{e} \in \mathcal{N}(\mathbf{A}^\mathsf{T}) \quad \Rightarrow \quad \mathbf{A}^\mathsf{T}\boldsymbol{e} = \mathbf{0}$$

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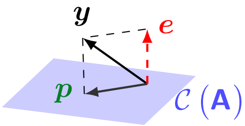

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$$\mathbf{A}^\mathsf{T}\boldsymbol{e} = \mathbf{A}^\mathsf{T}(\boldsymbol{y} - \boldsymbol{p}) \quad = \quad \mathbf{A}^\mathsf{T}(\boldsymbol{y} - \mathbf{A}\widehat{\boldsymbol{x}}) = \mathbf{0}$$

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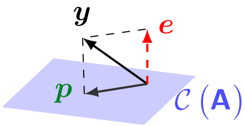
What's the projection of $\boldsymbol{y}$ onto the column space of $\mathbf{A} = \begin{bmatrix} | & | \\ \mathbf{A}_{|1} & \mathbf{A}_{|2} \\ | & | \end{bmatrix}$?



$$\boldsymbol{p} = (\mathbf{A}_{|1})\widehat{x_1} + (\mathbf{A}_{|2})\widehat{x_2} = \mathbf{A}\widehat{\boldsymbol{x}}$$

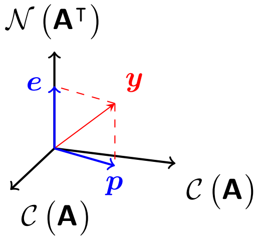
"Find the right combination of the columns so $\boldsymbol{e} \perp \mathcal{C}(\mathbf{A})$*"*

$$\boldsymbol{e} \perp \mathcal{C}(\mathbf{A}) \quad \Rightarrow \quad \boldsymbol{e} \in \mathcal{N}(\mathbf{A}^\top) \quad \Rightarrow \quad \mathbf{A}^\top \boldsymbol{e} = \mathbf{0}$$

$$\mathbf{A}^\top \boldsymbol{e} = \mathbf{A}^\top(\boldsymbol{y} - \boldsymbol{p}) \quad = \quad \mathbf{A}^\top(\boldsymbol{y} - \mathbf{A}\widehat{\boldsymbol{x}}) = \mathbf{0} \quad \Leftrightarrow \quad \boxed{(\mathbf{A}^\top\mathbf{A})\widehat{\boldsymbol{x}} = \mathbf{A}^\top\boldsymbol{y}}$$

$\boxed{11}$ Two projections

$\boldsymbol{y}$ has a component $\boldsymbol{p}$ in $\mathcal{C}(\mathbf{A})$, and another component $\boldsymbol{e}$ in $\mathcal{C}(\mathbf{A})^{\perp}$.



11 Two projections

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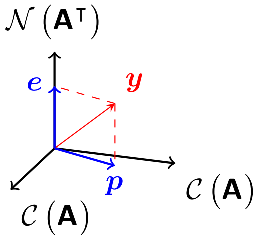


$$\boldsymbol{p} + \boldsymbol{e} = \boldsymbol{y}$$

$$\boldsymbol{p} = \mathbf{P}\boldsymbol{y} \qquad \text{projection onto } \mathcal{C}(\mathbf{A})$$

$$\boldsymbol{e} = (\mathbf{I} - \mathbf{P})\boldsymbol{y} \qquad \text{projection onto } \mathcal{C}(\mathbf{A})^{\perp}$$

Questions of the Lecture 12

(L-12) Question 1. Project the first vector orthogonally into the line spanned by the second vector. Check that $\boldsymbol{e}$ is perpendicular to $\boldsymbol{a}$. Find the projection matrix $\mathbf{P} = [\boldsymbol{a}]\left([\boldsymbol{a}]^{\mathsf{T}}[\boldsymbol{a}]\right)^{-1}[\boldsymbol{a}]^{\mathsf{T}}$ onto the line through each vector $\boldsymbol{a}$. Verify in each case that $\mathbf{P}^2 = \mathbf{P}$. Multiply $\mathbf{P}\boldsymbol{b}$ in each case to compute the projection $\boldsymbol{p}$.

(a) $\boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$; $\boldsymbol{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

(b) $\boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$; $\boldsymbol{a} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

(c) $\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$; $\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

(d) $\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$; $\boldsymbol{a} = \begin{pmatrix} 3 \\ 3 \\ 12 \end{pmatrix}$.

(Hefferon, 2008, exercise 1.6 from section VI.1.)

(L-12) Question 2. Project the vector orthogonally into the line.

(a) $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$, The line : $\left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix} \boldsymbol{p} \right\}$.

(b) $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$, the line $y = 3x$.

(L-12) QUESTION 3. Although pictures guided our development, we are not restricted to spaces that we can draw. In $\mathbb{R}^4$ project this vector into this line.

$$\begin{pmatrix}1\\2\\1\\3\end{pmatrix}; \quad \left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{bmatrix}-1\\1\\-1\\1\end{bmatrix} \boldsymbol{p} \right\}.$$

(L-12) QUESTION 4.

(a) Project the vector $\boldsymbol{b} = \left(1, \quad 1,\right)$ onto the lines through $\boldsymbol{a}_1 = \left(1, \quad 0,\right)$ and $\boldsymbol{a}_2 = \left(1, \quad 2,\right)$. Add the projections: $\boldsymbol{p}_1 + \boldsymbol{p}_2$. The projections do not add to $\boldsymbol{b}$ because $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are not orthogonal.

(b) The projection of $\boldsymbol{b}$ onto the plane of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ will equal $\boldsymbol{b}$. Find $\mathbf{P} = \mathbf{A}(\mathbf{A}^\mathsf{T}\mathbf{A})^{-1}\mathbf{A}^\mathsf{T}$ for $\mathbf{A} = \left[\boldsymbol{a}_1;\ \boldsymbol{a}_2;\right]$.

(Strang, 2003, exercise 8–9 from section 4.2.)

(L-12) QUESTION 5.

(a) If $\mathbf{P}^2 = \mathbf{P}$ show that $(\mathbf{I} - \mathbf{P})^2 = \mathbf{I} - \mathbf{P}$. When $\mathbf{P}$ projects onto the column space of $\mathbf{A}$, $(\mathbf{I} - \mathbf{P})$ projects onto the ________.

(b) If $\mathbf{P}^\mathsf{T} = \mathbf{P}$ show that $(\mathbf{I} - \mathbf{P})^\mathsf{T} = \mathbf{I} - \mathbf{P}$.

(Strang, 2003, exercise 17 from section 4.2.)

(L-12) QUESTION 6.

(a) Compute the projection matrices $\mathbf{P} = [\boldsymbol{a}]\left([\boldsymbol{a}]^\mathsf{T}[\boldsymbol{a}]\right)^{-1}[\boldsymbol{a}]^\mathsf{T}$ onto the lines through $\boldsymbol{a}_1 = (-1, \quad 2, \quad 2,)$ and $\boldsymbol{a}_2 = (2, \quad 2, \quad -1,)$. Show that $\boldsymbol{a}_1 \perp \boldsymbol{a}_2$. Multiply those projection matrices and explain why their product $\mathbf{P}_1\mathbf{P}_2$ is what it is.

(b) Project $\boldsymbol{b} = (1, \quad 0, \quad 0,)$ onto the lines through $\boldsymbol{a}_1$, and $\boldsymbol{a}_2$ and also onto $\boldsymbol{a}_3 = (2, \quad -1, \quad 2,)$. Add up the three projections $\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3$.

(c) Find the projection matrix $\mathbf{P}_3$ onto $\mathcal{L}([\boldsymbol{a}_3;]) = \mathcal{L}([(2, \quad -1, \quad 2,);])$. Verify that $\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 = \mathbf{I}$. The basis $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ is orthogonal!

(Strang, 2003, exercise 5–7 from section 4.2.)

(L-12) QUESTION 7. Project $\boldsymbol{b}$ onto the column space of $\mathbf{A}$ by solving $\mathbf{A}^\mathsf{T}\mathbf{A}\widehat{\boldsymbol{x}} = \mathbf{A}^\mathsf{T}\boldsymbol{b}$ and $\boldsymbol{p} = \mathbf{A}\widehat{\boldsymbol{x}}$. Find $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$.

(a) $\mathbf{A}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\boldsymbol{b}_1 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

(b) $\mathbf{A}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\boldsymbol{b}_2 = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix}$

(c) Compute the projection matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ onto the column spaces. Verify that $\mathbf{P}_1\boldsymbol{b}_1$ gives the first projection $\boldsymbol{p}_1$. Also verify $(\mathbf{P}_2)^2 = \mathbf{P}_2$.

(Strang, 2003, exercise 11–12 from section 4.2.)

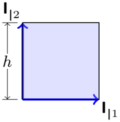
Highlights of Lesson 13

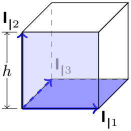
- Determinant: $\det(\mathbf{A}) \equiv |\mathbf{A}|$ $[\det: \quad \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}]$
  - Volume vs determinant
  - Properties: **1, 2, 3**

- We will deduce properties: **4 – 9**

$\boxed{2}$ Area or volume

1. $\mathrm{Vol}(\underset{n\times n}{\mathbf{I}}) = 1.$

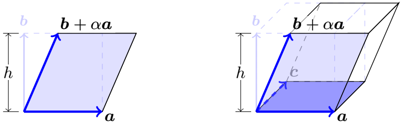




2 Area or volume

1. $\mathrm{Vol}\big(\underset{n\times n}{\mathbf{I}}\big) = 1.$

2. $\mathrm{Vol}\big(\mathbf{A}\big) \;=\; \mathrm{Vol}\Big(\mathbf{A}_{\underset{[(\alpha)\boldsymbol{k}+\boldsymbol{j}]}{\boldsymbol{\tau}}}\Big)$ for $i \neq k$.



2 Area or volume

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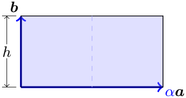
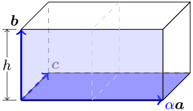
2. $\mathrm{Vol}(\mathbf{A}) \;=\; \mathrm{Vol}\Big(\mathbf{A}_{\underset{[(\alpha)\boldsymbol{k}+\boldsymbol{j}]}{\tau}}\Big)$ for $i \neq k$.

3. $|\alpha| \cdot \mathrm{Vol}(\mathbf{A}) \;=\; |\alpha| \cdot \mathrm{Vol}\big[\ldots;\mathbf{A}_{|k};\ldots\big] \;=\; \mathrm{Vol}\big[\ldots;\alpha\mathbf{A}_{|k};\ldots\big]$





3 Determinant: 3 properties that define the function

P-1 **Determinant of identity matrices:**

$$\det \underset{n\times n}{\mathbf{I}} = 1$$

P-2 **Type I elemen. transf. do not change the determinant:**

$$\det \mathbf{A} = \det\left(\mathbf{A}_{\underset{[(\alpha)\boldsymbol{k}+\boldsymbol{j}]}{\boldsymbol{\tau}}}\right)$$

P-3 **Multiplying a column by an scalar multiplies the det.**

$\alpha \cdot \det \mathbf{A} \;=\; \det\left[\ldots; \alpha\mathbf{A}_{|k}; \ldots\right]$ for any $k \in \{1:n\}$ and $\alpha \in \mathbb{R}$

$$\boxed{\text{Absolute value of } \det \mathbf{A} = \operatorname{Vol} \mathbf{A}}$$

Example

Then, we know that in $\mathbb{R}^3$:

$$\begin{vmatrix} a_1 & (b_1 + \alpha c_1) & c_1 \\ a_2 & (b_2 + \alpha c_2) & c_2 \\ a_3 & (b_3 + \alpha c_3) & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix};$$

$$\det\left[\boldsymbol{a};\ (\boldsymbol{b} + \alpha \boldsymbol{c});\ \boldsymbol{c};\right] = \det\left[\boldsymbol{a};\ \boldsymbol{b};\ \boldsymbol{c};\right];$$

and also

$$\begin{vmatrix} a_1 & \alpha\, b_1 & c_1 \\ a_2 & \alpha\, b_2 & c_2 \\ a_3 & \alpha\, b_3 & c_3 \end{vmatrix}; = \alpha \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix};$$

$$\det\left[\boldsymbol{a};\ \alpha\boldsymbol{b};\ \boldsymbol{c};\right] = \alpha \det\left[\boldsymbol{a};\ \boldsymbol{b};\ \boldsymbol{c};\right];$$

4 Determinant of a matrix with a zero column

P-4 **Det. of a matrix A with a zero column**

If $\mathbf{A}$ *has a zero column* $\mathbf{0}$, *then*

$$\boxed{\det(\mathbf{A}) = 0}$$

prove **P-4**

5 Elementary matrices

We already know

$$\det\left(\mathbf{A}_{\substack{\boldsymbol{\tau}\\[(\alpha)\boldsymbol{k}+\boldsymbol{j}]}}\right) = |\mathbf{A}|; \qquad \det\left(\mathbf{A}_{\substack{\boldsymbol{\tau}\\[(\alpha)\boldsymbol{k}]}}\right) = \alpha|\mathbf{A}|.$$

5 Elementary matrices

We already know

$$\det\left(\mathbf{A}\underset{[(\alpha)\boldsymbol{k}+\boldsymbol{j}]}{\boldsymbol{\tau}}\right) = |\mathbf{A}|; \qquad \det\left(\mathbf{A}\underset{[(\alpha)\boldsymbol{k}]}{\boldsymbol{\tau}}\right) = \alpha|\mathbf{A}|.$$

Determinant of elementary matrices

$$\det\left(\mathbf{I}\underset{[(\alpha)\boldsymbol{k}+\boldsymbol{j}]}{\boldsymbol{\tau}}\right) = 1 \qquad \text{and} \qquad \det\left(\mathbf{I}\underset{[(\alpha)\boldsymbol{j}]}{\boldsymbol{\tau}}\right) = \alpha.$$

5 Elementary matrices

We already know

$$\det\left(\mathbf{A}_{\underset{[(\alpha)\boldsymbol{k}+\boldsymbol{j}]}{\boldsymbol{\tau}}}\right) = |\mathbf{A}|; \qquad \det\left(\mathbf{A}_{\underset{[(\alpha)\boldsymbol{k}]}{\boldsymbol{\tau}}}\right) = \alpha|\mathbf{A}|.$$

Determinant of elementary matrices

$$\det\left(\mathbf{I}_{\underset{[(\alpha)\boldsymbol{k}+\boldsymbol{j}]}{\boldsymbol{\tau}}}\right) = 1 \qquad \text{and} \qquad \det\left(\mathbf{I}_{\underset{[(\alpha)\boldsymbol{j}]}{\boldsymbol{\tau}}}\right) = \alpha.$$

Hence, since $\mathbf{A}_{\boldsymbol{\tau}} = \mathbf{A}(\mathbf{I}_{\boldsymbol{\tau}})$, then

$$\left|\mathbf{A}(\mathbf{I}_{\boldsymbol{\tau}})\right| = |\mathbf{A}| \cdot |\mathbf{I}_{\boldsymbol{\tau}}| \tag{1}$$

where $\mathbf{I}_{\boldsymbol{\tau}}$ is an elementary matrix

EXERCISE 8. Prove the following propositions

(a) $\det(\mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}) = |\mathbf{A}| \cdot |\mathbf{I}_{\boldsymbol{\tau}_1}| \cdots |\mathbf{I}_{\boldsymbol{\tau}_k}|$.

(b) If $\mathbf{B}$ is a full rank matrix, i.e., if $\mathbf{B} = \mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}$, then $|\mathbf{B}| = |\mathbf{I}_{\boldsymbol{\tau}_1}| \cdots |\mathbf{I}_{\boldsymbol{\tau}_k}|$, and therefore $|\mathbf{B}| \neq 0$.

(c) *If* $\mathbf{A}$ *and* $\mathbf{B}$ *have order* n *and* $\mathbf{B}$ *is full rank, then*

$$\det(\mathbf{AB}) = |\mathbf{A}| \cdot |\mathbf{B}| \tag{2}$$

6 Determinant after a sequence of elementary transformations

Example

a sequence $\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k$ of *Type I* elementary transformations does not change the determinant.

$$|\mathbf{A}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k}| = |\mathbf{A}(\mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k})| = |\mathbf{A}| \cdot |\mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k}| = |\mathbf{A}| \cdot 1 = |\mathbf{A}|$$

Example

but a sequence of *Type II* can.

$$\begin{vmatrix} 2a & 3c \\ 2b & 3d \end{vmatrix} = \_\_ \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

6 Determinant after a sequence of elementary transformations

Example

a sequence $\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k$ of *Type I* elementary transformations does not change the determinant.

$$|\mathbf{A}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k}| = |\mathbf{A}(\mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k})| = |\mathbf{A}| \cdot |\mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k}| = |\mathbf{A}| \cdot 1 = |\mathbf{A}|$$

Example

but a sequence of *Type II* can.

$$\begin{vmatrix} 2a & 3c \\ 2b & 3d \end{vmatrix} = \underline{\ 6\ } \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

7 Antisymmetric property

P-5 **[Antisymmetric property]**

Column exchange changes the sign of the determinant.

Proof.

Column exchange is a sequence of *Type I* transformation and just only one *Type II* transformation that multiplies a column by -1 □

Therefore:

$$\begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix} = (-1) \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

8 Singular matrices. Inverse of a matrix

P-6 **If A is singular then $|\mathbf{A}| = 0$.**

P-7 $\det(\mathbf{A}^{-1}) = \left(\det \mathbf{A}\right)^{-1}$.

Proof.

Consider $\underset{n\times n}{\mathbf{A}}$ and $\mathbf{E}$ such that $\mathbf{AE} = \mathbf{R}$ (where $\mathbf{E} = \mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}$).

Then: $|\mathbf{A}|\cdot|\mathbf{E}| = |\mathbf{R}|$ and two cases are possible:

$$\begin{cases} \mathbf{A} \text{ singular } (\mathbf{R}_{|n} = \mathbf{0}\ ): & |\mathbf{A}|\cdot|\mathbf{E}| = 0 \;\Rightarrow\; |\mathbf{A}| = 0 \\ \\ \mathbf{A} \text{ not singular } (\mathbf{R} = \mathbf{I}): & |\mathbf{A}|\cdot|\mathbf{E}| = 1 \;\Rightarrow\; |\mathbf{E}| = \left|\mathbf{A}^{-1}\right| = \left(|\mathbf{A}|\right)^{-1} \end{cases}.$$

□

Example

For $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$:

$$\left[\begin{array}{cc} 1 & 2 \\ 2 & 2 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] \xrightarrow[TypeI]{\overset{\tau}{[(-2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc} 1 & 0 \\ 2 & -2 \\ \hline 1 & -2 \\ 0 & 1 \end{array}\right] \xrightarrow[\color{red}{TypeII}]{\overset{\tau}{[(-1/2)\mathbf{2}]}} \left[\begin{array}{cc} 1 & 0 \\ 2 & 1 \\ \hline 1 & 1 \\ 0 & -1/2 \end{array}\right] \xrightarrow[TypeI]{\overset{\tau}{[(-2)\mathbf{2}+\mathbf{1}]}} \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline -1 & 1 \\ 1 & -1/2 \end{array}\right]$$

So

$$\left|\mathbf{A}^{-1}\right| = \left|\mathbf{I}_{\overset{\tau}{[(-2)\mathbf{1}+\mathbf{2}]}}\right| \cdot \left|\mathbf{I}_{\overset{\tau}{[(\color{red}{-1/2})\mathbf{2}]}}\right| \cdot \left|\mathbf{I}_{\overset{\tau}{[(-2)\mathbf{2}+\mathbf{1}]}}\right| = 1 \cdot \color{red}{\frac{-1}{2}} \cdot 1 = \frac{-1}{2};$$

that is $\boxed{|\mathbf{A}| = -2.}$

9 Determinant of a product

P-8 **[Determinant of a product of matrices]**

$$\boxed{\det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B}).} \tag{3}$$

$$\begin{cases} \mathbf{B} \text{ singular, then so it is } \mathbf{AB} \Rightarrow & \det(\mathbf{AB}) = 0 = \det(\mathbf{A}) \cdot \det(\mathbf{B}) \\ \\ \mathbf{B} = \mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k} \Rightarrow & \det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B}) \end{cases}$$

EXERCISE 9. **[Transposed matrices]**

(a) What is the relation between the determinant of an elementary matrix $\mathbf{I}_{\tau}$ and the determinant of its transpose ${}_{\tau}\mathbf{I}$?

(b) Consider $\mathbf{B}$, a full rank matrix, proof that $|\mathbf{B}| = |\mathbf{B}^{\intercal}|$.

P-9 Determinant of a transpose

$$|\mathbf{A}| = |\mathbf{A}^\intercal|.$$

Proof.

$$\begin{cases} \text{if } \mathbf{A} \text{ singular:} & \mathbf{A}^\intercal \text{ singular } \Rightarrow \det \mathbf{A}^\intercal = \det \mathbf{A} = 0 \\ \\ \text{if } \mathbf{A} \text{ NO singular:} & \mathbf{A} = \mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k} \Rightarrow \det \mathbf{A}^\intercal = \det \mathbf{A} \end{cases}.$$

□

Questions of the Lecture 13

(L-13) Question 1. Complete the proofs of this lecture.

(L-13) Question 2. Knowing that $|\mathbf{BC}| = |\mathbf{B}||\mathbf{C}|$; prove that for any invertible matrix $\mathbf{A}$ (so $\det \mathbf{A} \neq 0$)

$$\det(\mathbf{A}^{-1}) = \Big(\det(\mathbf{A})\Big)^{-1}.$$

(L-13) Question 3. Consider $\underset{3\times 3}{\mathbf{A}}$ and $\underset{3\times 3}{\mathbf{B}}$ such that $\det(\mathbf{A}) = 2$ and $\det(\mathbf{B}) = -2$

(a) (0.5^{pts}) Compute the determinants of $\mathbf{A}(\mathbf{B})^2$ and $(\mathbf{AB})^{-1}$
(b) (0.5^{pts}) Is it possible to compute the rank of $\mathbf{A}+\mathbf{B}$? and the rank of $\mathbf{AB}$?

(L-13) Question 4. Use the Gauss-Jordan method to compute the determinant

(a) $\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\mathbf{A}_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

(c) $\mathbf{A}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(L-13) QUESTION 5. The 3 by 3 matrix **A** reduces to the identity matrix **I** by the following three column operations (in order):

$\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}$: Subtract 4 times column 1 from column 2.

$\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}$: Subtract 3 times column 1 from column 3.

$\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}$: Subtract column 3 from column 2.

Find the determinant of **A**.

(L-13) QUESTION 6.

(a) Find the determinant of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(b) Find the determinant of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & d \end{bmatrix}$ **using Gauss-Jordan**.

Highlights of Lesson 14

- Computing $|\mathbf{A}|$ by gaussian elimination
- **P-10** — Multilinear property
- Expansion of $\det \mathbf{A}$ in Cofactors (Laplace expansion).
- Application of determinants
  - Cramer's rule for solving linear equations
  - Computing the inverse of $\mathbf{A}$

$\boxed{2}$ Extended matrix

Extended matrix of $\mathbf{B}$: $\begin{bmatrix} \mathbf{B} & \\ & 1 \end{bmatrix}$

2 Extended matrix

Extended matrix of $\mathbf{B}$: $\begin{bmatrix} \mathbf{B} & \\ & 1 \end{bmatrix}$

1. Given $\boldsymbol{\tau}$: $\begin{bmatrix} \mathbf{B}_{\boldsymbol{\tau}} & \\ & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \\ & 1 \end{bmatrix}_{\boldsymbol{\tau}}.$

$\boxed{2}$ Extended matrix

$$\text{Extended matrix of } \mathbf{B} : \quad \begin{bmatrix} \mathbf{B} & \\ & 1 \end{bmatrix}$$

1. Given $\boldsymbol{\tau}$: $\quad \begin{bmatrix} \mathbf{B}_{\boldsymbol{\tau}} & \\ & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \\ & 1 \end{bmatrix}_{\boldsymbol{\tau}}.$
2. Since $\begin{bmatrix} \mathbf{I} & \\ & 1 \end{bmatrix}_{\boldsymbol{\tau}}$ and $\mathbf{I}_{\boldsymbol{\tau}}$ same *type* Elem. Mat. $\Rightarrow$ same det.

2 Extended matrix

Extended matrix of $\mathbf{B}$: $\begin{bmatrix} \mathbf{B} & \\ & 1 \end{bmatrix}$

1. Given $\boldsymbol{\tau}$: $\begin{bmatrix} \mathbf{B}_{\boldsymbol{\tau}} & \\ & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \\ & 1 \end{bmatrix}_{\boldsymbol{\tau}}$.
2. Since $\begin{bmatrix} \mathbf{I} & \\ & 1 \end{bmatrix}_{\boldsymbol{\tau}}$ and $\mathbf{I}_{\boldsymbol{\tau}}$ same *type* Elem. Mat. $\Rightarrow$ same det.

Applying 1. k times, and then 2.

$$\left| \begin{bmatrix} \mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k} & \\ & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} \mathbf{I} & \\ & 1 \end{bmatrix}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k} \right| = \left| \begin{bmatrix} \mathbf{I} & \\ & 1 \end{bmatrix}_{\boldsymbol{\tau}_1} \cdots \begin{bmatrix} \mathbf{I} & \\ & 1 \end{bmatrix}_{\boldsymbol{\tau}_k} \right|$$
$$= \left| \mathbf{I}_{\boldsymbol{\tau}_1} \right| \cdots \left| \mathbf{I}_{\boldsymbol{\tau}_k} \right| = \left| \mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k} \right|.$$

2 Extended matrix

Extended matrix of **B** : $\begin{bmatrix} \mathbf{B} & \\ & 1 \end{bmatrix}$

1. Given $\boldsymbol{\tau}$: $\begin{bmatrix} \mathbf{B}_{\boldsymbol{\tau}} & \\ & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \\ & 1 \end{bmatrix}_{\boldsymbol{\tau}}$.

2. Since $\begin{bmatrix} \mathbf{I} & \\ & 1 \end{bmatrix}_{\boldsymbol{\tau}}$ and $\mathbf{I}_{\boldsymbol{\tau}}$ same *type* Elem. Mat. $\Rightarrow$ same det.

Applying 1. k times, and then 2.

$$\left| \begin{bmatrix} \mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k} & \\ & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} \mathbf{I} & \\ & 1 \end{bmatrix}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k} \right| = \left| \begin{bmatrix} \mathbf{I} & \\ & 1 \end{bmatrix}_{\boldsymbol{\tau}_1} \cdots \begin{bmatrix} \mathbf{I} & \\ & 1 \end{bmatrix}_{\boldsymbol{\tau}_k} \right|$$

$$= \left| \mathbf{I}_{\boldsymbol{\tau}_1} \right| \cdots \left| \mathbf{I}_{\boldsymbol{\tau}_k} \right| = \left| \mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k} \right|.$$

If **A** is the extended matrix of **B** $\begin{cases} \text{If } \mathbf{B} \text{ singular} & |\mathbf{B}| = 0 = |\mathbf{A}| \\ \text{If } \mathbf{B} \text{ invertible} & |\mathbf{B}| = |\mathbf{A}| \end{cases}$

EXERCISE 7. **[Triangular matrices]**

(a) Find the determinant of a full rank lower triangular matrix $\mathbf{L}$

(b) Find the determinant of a triangular matrix with a zero entry in the main diagonal

(c) Find the determinant of an upper triangular matrix $\mathbf{U}$

In addition $\begin{vmatrix} \mathbf{A} & \underset{m \times n}{\mathbf{0}} \\ \underset{n \times m}{\mathbf{C}} & \mathbf{B} \end{vmatrix} = \left|\mathbf{A}\right| \cdot \left|\mathbf{B}\right|.$

3 Computing by Gaussian elimination

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} : \quad \left[\begin{array}{cc|c} 1 & 5 & \\ 2 & 3 & \\ \hline & & 1 \end{array}\right] \xrightarrow[{[(-5)\mathbf{1}+\mathbf{2}]}]{\tau} \left[\begin{array}{cc|c} 1 & 0 & \\ 2 & -7 & \\ \hline & & 1 \end{array}\right] \quad \boxed{|\mathbf{A}| = -7}$$

3 Computing by Gaussian elimination

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} : \quad \left[\begin{array}{cc|c} 1 & 5 & \\ 2 & 3 & \\ \hline & & 1 \end{array}\right] \xrightarrow[{[(-5)\mathbf{1}+\mathbf{2}]}]{\tau} \left[\begin{array}{cc|c} 1 & 0 & \\ 2 & -7 & \\ \hline & & 1 \end{array}\right] \quad \boxed{|\mathbf{A}| = -7}$$

Example

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 9 & 6 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\overset{\tau}{[(\frac{1}{2})\mathbf{4}]}]{\substack{\overset{\tau}{[(2)\mathbf{3}]} \\ [(-1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc|c} 0 & 2 & 0 & 0 \\ 9 & 6 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{2} \end{array}\right] \xrightarrow[\overset{\tau}{[(-1)\mathbf{4}]}]{\overset{\tau}{[\mathbf{1}\rightleftharpoons\mathbf{2}]}} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 6 & 9 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & -\frac{1}{2} \end{array}\right]$$

$$\begin{vmatrix} 0 & 2 & 1 \\ 9 & 6 & 3 \\ 0 & 1 & 1 \end{vmatrix} = -9,$$

Matrices of order 1, $\mathbf{A} = \left[a\right]$:

$$\left[\begin{array}{c|c} a & 0 \\ \hline 0 & 1 \end{array}\right] \Rightarrow |\mathbf{A}| = a.$$

Matrices of order 1, $\mathbf{A} = \left[a\right]$:

$$\left[\begin{array}{c|c} a & 0 \\ \hline 0 & 1 \end{array}\right] \Rightarrow |\mathbf{A}| = a.$$

Matrices of order 2:

$$\left[\begin{array}{cc|c} a & b & 0 \\ c & d & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow{\overset{\tau}{[(-\frac{b}{a})\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc|c} a & 0 & 0 \\ c & d - \frac{bc}{a} & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

$$|\mathbf{A}| = ad - bc = a \det \left[d\right] - b \det \left[c\right].$$

Matrices of order 1, $\mathbf{A} = \left[a\right]$:

$$\left[\begin{array}{c|c} a & 0 \\ \hline 0 & 1 \end{array}\right] \Rightarrow |\mathbf{A}| = a.$$

Matrices of order 2:

$$\left[\begin{array}{cc|c} a & b & 0 \\ c & d & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow{[(-\frac{b}{a})\mathbf{1}+\mathbf{2}]^{\tau}} \left[\begin{array}{cc|c} a & 0 & 0 \\ c & d-\frac{bc}{a} & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

$$|\mathbf{A}| = ad - bc = a\det\left[d\right] - b\det\left[c\right].$$

Matrices of order 3:

$$\left[\begin{array}{ccc|c} a & b & c & 0 \\ d & e & f & 0 \\ g & h & i & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-\frac{c}{a})\mathbf{1}+\mathbf{3}]}]{[(-\frac{b}{a})\mathbf{1}+\mathbf{2}]^{\tau}} \left[\begin{array}{ccc|c} a & 0 & 0 & 0 \\ d & e-\frac{bd}{a} & f-\frac{cd}{a} & 0 \\ g & h-\frac{bg}{a} & i-\frac{cg}{a} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{[(\frac{-af+cd}{ae-bd})\mathbf{2}+\mathbf{3}]^{\tau}}$$

$$\left[\begin{array}{ccc|c} a & 0 & 0 & 0 \\ d & e-\frac{bd}{a} & 0 & 0 \\ g & h-\frac{bg}{a} & \frac{aei-afh-bdi+bfg+cdh-ceg}{ae-bd} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$|\mathbf{A}| = \underbrace{aei - afh - bdi + bfg + cdh - ceg}_{\text{(Rule of Sarrus)}} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

Matrices of order 4:

$$\left[\begin{array}{cccc|c} a & b & c & d & 0 \\ e & f & g & h & 0 \\ i & j & k & l & 0 \\ m & n & o & p & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\left[\left(\frac{-afl+ahj+bel-bhi-dej+dfi}{afk-agj-bek+bgi+cej-cfi}\right)\mathbf{3+4}\right]]{\substack{\boldsymbol{\tau} \\ \left[\left(-\frac{b}{a}\right)\mathbf{1+2}\right] \\ \left[\left(-\frac{c}{a}\right)\mathbf{1+3}\right] \\ \left[\left(-\frac{d}{a}\right)\mathbf{1+4}\right] \\ \left[\left(\frac{-ag+ce}{af-be}\right)\mathbf{2+3}\right] \\ \left[\left(\frac{-ah+de}{af-be}\right)\mathbf{2+4}\right]}}$$

$$\left[\begin{array}{ccccc} a & 0 & 0 & 0 \\ e & f-\frac{be}{a} & 0 & 0 \\ i & j-\frac{bi}{a} & \frac{afk-agj-bek+bgi+cej-cfi}{af-be} & 0 \\ m & n-\frac{bm}{a} & \frac{afo-agn-beo+bgm+cen-cfm}{af-be} & p+\frac{\left(-l+\frac{\left(h-\frac{de}{a}\right)\left(j-\frac{bi}{a}\right)}{f-\frac{be}{a}}+\frac{di}{a}\right)\left(-o+\frac{\left(g-\frac{ce}{a}\right)\left(n-\frac{bm}{a}\right)}{f-\frac{be}{a}}+\frac{cm}{a}\right)}{-k+\frac{\left(g-\frac{ce}{a}\right)\left(j-\frac{bi}{a}\right)}{f-\frac{be}{a}}+\frac{ci}{a}}-\frac{\left(h-\frac{de}{a}\right)\left(n-\frac{bm}{a}\right)}{f-\frac{be}{a}}- \\ \hline 0 & 0 & 0 & 0 \end{array}\right.$$

$|\mathbf{A}| =$
$afkp-aflo-agjp+agln+ahjo-ahkn-bekp+belo+bgip-bglm-bhio+bhkm+cejp-celn-cfip+cflm+chin-chjm-dejo+dekn+dfio-dfkm-dgin+dgjm$

Matrices of order 4:

$$\left[\begin{array}{cccc|c} a & b & c & d & 0 \\ e & f & g & h & 0 \\ i & j & k & l & 0 \\ m & n & o & p & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ \left[\left(-\frac{b}{a}\right)\mathbf{1}+\mathbf{2}\right] \\ \left[\left(-\frac{c}{a}\right)\mathbf{1}+\mathbf{3}\right] \\ \left[\left(-\frac{d}{a}\right)\mathbf{1}+\mathbf{4}\right] \\ \left[\left(\frac{-ag+ce}{af-be}\right)\mathbf{2}+\mathbf{3}\right] \\ \left[\left(\frac{-ah+de}{af-be}\right)\mathbf{2}+\mathbf{4}\right] \\ \left[\left(\frac{-afl+ahj+bel-bhi-dej+dfi}{afk-agj-bek+bgi+cej-cfi}\right)\mathbf{3}+\mathbf{4}\right]}}$$

$$\left[\begin{array}{cccc} a & 0 & 0 & 0 \\ e & f-\frac{be}{a} & 0 & 0 \\ i & j-\frac{bi}{a} & \frac{afk-agj-bek+bgi+cej-cfi}{af-be} & 0 \\ m & n-\frac{bm}{a} & \frac{afo-agn-beo+bgm+cen-cfm}{af-be} & p+\frac{\left(-l+\frac{\left(h-\frac{de}{a}\right)\left(j-\frac{bi}{a}\right)}{f-\frac{be}{a}}+\frac{di}{a}\right)\left(-o+\frac{\left(g-\frac{ce}{a}\right)\left(n-\frac{bm}{a}\right)}{f-\frac{be}{a}}+\frac{cm}{a}\right)}{-k+\frac{\left(g-\frac{ce}{a}\right)\left(j-\frac{bi}{a}\right)}{f-\frac{be}{a}}+\frac{ci}{a}}-\frac{\left(h-\frac{de}{a}\right)\left(n-\frac{bm}{a}\right)}{f-\frac{be}{a}}- \\ \hline 0 & 0 & 0 & 0 \end{array}\right.$$

$|\mathbf{A}| =$
$afkp-aflo-agjp+agln+ahjo-ahkn-bekp+belo+bgip-bglm-bhio+bhkm+$
$cejp-celn-cfip+cflm+chin-chjm-dejo+dekn+dfio-dfkm-dgin+dgjm$

$$= a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c\begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

P-10 Multilinear property

$$\det\left[\ldots;(\beta \boldsymbol{b}+\psi \boldsymbol{c});\ldots\right]=\beta\det\left[\ldots;\boldsymbol{b};\ldots\right]+\psi\det\left[\ldots;\boldsymbol{c};\ldots\right]$$

Example

Then, in the 2 dimensional case $\mathbb{R}^2$

$$\begin{vmatrix} a+\alpha & c \\ b+\beta & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} + \begin{vmatrix} \alpha & c \\ \beta & d \end{vmatrix};$$

P-10 Multilinear property

$$\det\left[\,\ldots;(\beta \boldsymbol{b}+\psi \boldsymbol{c});\ldots\,\right]=\beta\det\left[\,\ldots;\boldsymbol{b};\ldots\,\right]+\psi\det\left[\,\ldots;\boldsymbol{c};\ldots\,\right]$$

Example

Then, in the 2 dimensional case $\mathbb{R}^2$

$$\begin{vmatrix} a+\alpha & c \\ b+\beta & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} + \begin{vmatrix} \alpha & c \\ \beta & d \end{vmatrix};$$

therefore

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = \begin{vmatrix} a & c \\ 0 & d \end{vmatrix} + \begin{vmatrix} & c \\ & d \end{vmatrix}.$$

P-10 Multilinear property

$$\det\left[\ldots;(\beta \boldsymbol{b}+\psi \boldsymbol{c});\ldots\right]=\beta\det\left[\ldots;\boldsymbol{b};\ldots\right]+\psi\det\left[\ldots;\boldsymbol{c};\ldots\right]$$

Example

Then, in the 2 dimensional case $\mathbb{R}^2$

$$\begin{vmatrix} a+\alpha & c \\ b+\beta & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} + \begin{vmatrix} \alpha & c \\ \beta & d \end{vmatrix};$$

therefore

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = \begin{vmatrix} a & c \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & c \\ b & d \end{vmatrix}.$$

Definition minors and cofactors

We denote a submatrix of **A** *obtained by deleting row* i *and column* j *of* **A** *by*

$$ {}^{i\urcorner}\mathbf{A}^{\ulcorner j}; $$

Its determinant is called the minor of a_{ij}.

5 minors and cofactors

Definition minors and cofactors

We denote a submatrix of $\mathbf{A}$ *obtained by deleting row* i *and column* j *of* $\mathbf{A}$ *by*

$$ {}^{i\urcorner}\mathbf{A}^{\ulcorner j}; $$

Its determinant is called the minor of a_{ij}. *And*

$$ \operatorname{cof}_{ij}\left(\mathbf{A}\right) = (-1)^{i+j} \det\left({}^{i\urcorner}\mathbf{A}^{\ulcorner j}\right) $$

is called the cofactor of a_{ij}.

Example

For $\quad \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},$

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$$ {}_{1\urcorner}\mathbf{A}^{\ulcorner 2} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} $$

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$$ {}^{1\Lsh}\mathbf{A}^{\Rsh 2} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}, \qquad {}^{3\Lsh}\mathbf{A}^{\Rsh 3} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} $$

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hence

$$ \operatorname{cof}_{12}(\mathbf{A}) = (-1)^{1+2} \det\left({}^{1\Lsh}\mathbf{A}^{\Rsh 2}\right) = (-1) \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}. $$

Example

For $\quad \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, we have

$$ {}^{1\!\!\uparrow}\mathbf{A}^{\uparrow 2} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}, \qquad {}^{3\!\!\uparrow}\mathbf{A}^{\uparrow 3} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} $$

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and

$$ \operatorname{cof}_{33}(\mathbf{A}) = (-1)^{3+3} \det\left({}^{3\!\!\uparrow}\mathbf{A}^{\uparrow 3}\right) = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}. $$

Theorem [Laplace expansion]

For $\mathbf{A}$ n *by* n, $\det(\mathbf{A})$ *may be computed as the sum of the products of the elements of any column (row) of* $\mathbf{A}$ *by their cofactors:*

$$\det(\mathbf{A}) = \sum_{i=1}^{n} a_{ij} \operatorname{cof}_{ij}(\mathbf{A}), \qquad \textit{the expansion by the } j\textit{th column}$$

or

$$\det(\mathbf{A}) = \sum_{j=1}^{n} a_{ij} \operatorname{cof}_{ij}(\mathbf{A}), \qquad \textit{the expansion by the } i\textit{th row}$$

EXERCISE 8. Compute $\det \mathbf{A} = \begin{vmatrix} 2 & 0 & 3 & 2 \\ 5 & 1 & 2 & 4 \\ 3 & 0 & 1 & 2 \\ 5 & 3 & 2 & 1 \end{vmatrix}$

7 Cramer's Rule

$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b}; \qquad |\mathbf{A}| \neq 0$$

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$$x_j = \frac{\det\Big[\mathbf{A}_{|1};\ \ldots\ \overbrace{\boldsymbol{b}}^{\text{pos. } j};\ \ldots\ \mathbf{A}_{|n}\Big]}{\det(\mathbf{A})}.$$

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$$x_j = \frac{\det\left[\mathbf{A}_{|1};\ \ldots\ \overbrace{\ \boldsymbol{b}\ }^{\text{pos. } j};\ \ldots\ \mathbf{A}_{|n}\right]}{\det(\mathbf{A})}.$$

Computational issues when $\det \mathbf{A} \simeq 0$ (tiny angle between vectors)

8 The inverse of a matrix

$$[\mathbf{Adj}(\mathbf{A})] \cdot \mathbf{A} =$$

$$\begin{bmatrix} \mathrm{cof}_{11}(\mathbf{A}) & \mathrm{cof}_{21}(\mathbf{A}) & \cdots & \mathrm{cof}_{n1}(\mathbf{A}) \\ \mathrm{cof}_{12}(\mathbf{A}) & \mathrm{cof}_{22}(\mathbf{A}) & \cdots & \mathrm{cof}_{n2}(\mathbf{A}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{cof}_{1n}(\mathbf{A}) & \mathrm{cof}_{2n}(\mathbf{A}) & \cdots & \mathrm{cof}_{nn}(\mathbf{A}) \end{bmatrix} \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{\mathbf{A}}$$

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$$= \begin{bmatrix} |\mathbf{A}| & 0 & \cdots & 0 \\ 0 & |\mathbf{A}| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |\mathbf{A}| \end{bmatrix}$$

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$$= \begin{bmatrix} |\mathbf{A}| & 0 & \cdots & 0 \\ 0 & |\mathbf{A}| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |\mathbf{A}| \end{bmatrix} = |\mathbf{A}| \cdot \mathbf{I}.$$

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$$\left(\frac{1}{|\mathbf{A}|} \cdot \mathbf{Adj}(\mathbf{A}) \right) \cdot \mathbf{A} = \mathbf{I}$$

Questions of the Lecture 14

(L-14) QUESTION 1. Complete the proofs of the exercises of this lecture.

(L-14) QUESTION 2. Consider $\mathbf{A} = \left[\mathbf{A}_{|1}; \quad \mathbf{A}_{|2}; \quad \mathbf{A}_{|3};\right]$ with $\det \mathbf{A} = 2$.

(a) What are $\det(2\mathbf{A})$ and $\det \mathbf{A}^{-1}$?

(b) What is $\det \left[(3\mathbf{A}_{|1} + 2\mathbf{A}_{|2}); \quad \mathbf{A}_{|3}; \quad \mathbf{A}_{|2};\right]$

(L-14) QUESTION 3. The determinant of the 1000 by 1000 matrix $\mathbf{A}$ is 12. What is the determinant of $-\mathbf{A}^\intercal$? (Careful: No credit for the wrong sign.)
(*MIT Course 18.06 Quiz 2, Fall, 2008*)

(L-14) QUESTION 4. Consider the squared matrix $\mathbf{A}$. True or false? (to receive full credit you must explain your answer in a clear and concise way)
$|\mathbf{A}\mathbf{A}^\intercal| = |\mathbf{A}|^2$.

(L-14) QUESTION 5. We have a 3×3 matrix $\mathbf{A} = \begin{bmatrix} a & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$ with $\det \mathbf{A} = 3$.

Compute the determinant of the following matrices:

(a) (0.5 pts) $\begin{bmatrix} a-2 & 1 & 2 \\ b-4 & 3 & 4 \\ c-6 & 5 & 6 \end{bmatrix}$

(b) (0.5 pts) $\begin{bmatrix} 7a & 7 & 14 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$

(c) (1 pts) $(2\mathbf{A})^{-1}\mathbf{A}^\intercal$

(d) (0.5 pts) $\begin{bmatrix} a-2 & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$

(L-14) QUESTION 6.

(a) Escalone la matriz $\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 5 & 2 \\ 4 & 6 & 0 \end{bmatrix}$.

(b) ¿Es $\mathbf{A}$ invertible?

(c) En caso afirmativo calcule $|\mathbf{A}^{-1}|$; en caso contrario calcule $|\mathbf{A}|$

(d) La matriz $\mathbf{C}$ es igual al producto de $\mathbf{A}$ con la *traspuesta* de la matriz $\mathbf{B}$, es decir

$$\mathbf{C} = \mathbf{A}\mathbf{B}^\intercal \qquad \text{donde} \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

¿Cuánto vale el determinante de $\mathbf{C}$? ¿Es $\mathbf{C}$ invertible?

(L-14) Question 7. What is the determinant of the following matrices using Laplace expansions.

(a) $\begin{bmatrix} 1 & 2 \\ -4 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & -2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 1 & -2 \end{bmatrix}$

(L-14) Question 8. Compute the following determinant using Laplace expansions:

$$\begin{vmatrix} 0 & 0 & 0 & 3 & 0 \\ -2 & 0 & 0 & 2 & 0 \\ 8 & -1 & 0 & -7 & 2 \\ -1 & 2 & 2 & 3 & 2 \\ 2 & 2 & 3 & 6 & 4 \end{vmatrix}$$

(L-14) Question 9. Compute $\det \mathbf{A} = \begin{vmatrix} 2 & 2 & 0 & 0 \\ 5 & 5 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{vmatrix}$

(L-14) Question 10. Compute the value of $\det \mathbf{A}$ using Laplace expansion

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 3 & \cdots & 3 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & n \end{bmatrix}$$

(L-14) Question 11. Consider a n by n matrix $\mathbf{A}_n$ full of 3s in its diagonal, and twos just below the diagonal, and another 2 at the position $(1, n)$; for example, for $n = 4$:

$$\mathbf{A}_4 = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 2 & 3 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix}.$$

(a) Find, using the cofactors of the first row, the determinant of $\mathbf{A}_4$.
(b) Find the determinant of $\mathbf{A}_n$ for $n > 4$.

(L-14) QUESTION 12. Consider the following block matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}$$

Prove $|\mathbf{A}| = |\mathbf{B}||\mathbf{C}|$.

Hint

$$\begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}$$

(L-14) QUESTION 13. Solve the following linear systems using Cramer's Rule

(a) $\begin{cases} 2x + 5y = 1 \\ x + 4y = 2 \end{cases}$.

(b) $\begin{cases} 2x + y = 1 \\ x + 2y + z = 0 \\ y + 2z = 0 \end{cases}$.

(exercise 13 from section 4.4 of Strang (2006))

(L-14) QUESTION 14. Find the inverse of the following matrices using the *adjoint matrix*

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

(exercise 18 from section 4.4 of Strang (2006))

(L-14) QUESTION 15. Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 2 & 3 & 3 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & a \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}; \text{ and the vector } \boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

(a) (0.5^{pts}) For wich values of a the matrix $\mathbf{A}$ is invertible?

(b) (1^{pts}) Consider $a = 5$. Using the Cramer's rule, compute the fourth coordinate x_4 of $\boldsymbol{x}$ for linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.

(c) (1^{pts}) Compute $\mathbf{B}^{-1}$. Use the matrix $\mathbf{B}^{-1}$ to solve $\mathbf{B}\boldsymbol{x} = \boldsymbol{b}$.

1 Highlights of **Lesson 15**

Always squared matrices in this topic

Highlights of Lesson 15

- **Eigenvalues, eigenvectors** (prefix eigen is the German word for innate, distinct, self)
- $\left|\mathbf{A} - \lambda \mathbf{I}\right| = 0$ *Characteristic equation*
- $\operatorname{tr}(\mathbf{A}),\ \det \mathbf{A}$ (demo in the next lesson)

2 Eigenvalues and eigenvectors

Consider the equation

$$\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x} \qquad\qquad (\text{with} \quad \boldsymbol{x} \neq \mathbf{0})$$

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$$\mathbf{A}\boldsymbol{x} = \lambda \boldsymbol{x} \qquad \text{(with} \quad \boldsymbol{x} \neq \mathbf{0}\text{)}$$

- *Eigenvalue* is any λ such that there are solutions.

$\boxed{2}$ Eigenvalues and eigenvectors

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- Such *non-null* solutions $\boldsymbol{x}$ are called *eigenvectors*.

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When λ is 0, What are the eigenvectors?

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When λ is 0, What are the eigenvectors? $\mathcal{N}\left(\mathbf{A}\right)$

When $\mathbf{A}$ is singular 0 is an eigenvalue, and $\mathcal{N}\left(\mathbf{A}\right)$ has the corresponding eigenvectors (plus $\mathbf{0}$).

3 Example: projection matrix

- Orthogonal projection
- Which vectors are eigenvectors?
  What vectors are projected in the same starting direction?

$\boxed{3}$ Example: projection matrix

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- Which vectors are eigenvectors?
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  any vector in the plane (we can find two linearly independent)

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  any vector in the plane (we can find two linearly independent)
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  $\mathbf{P}\boldsymbol{x} = \boldsymbol{x} \quad \Rightarrow \quad \lambda = 1$

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- are there any other eigenvectors? with what eigenvalue?

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Eigenspace (Set of eigenvectors + $\mathbf{0}$):

$$\mathcal{E}_{\lambda}(\mathbf{A}) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \middle| \mathbf{A}\boldsymbol{x} = \lambda \boldsymbol{x} \right\} = \mathcal{N}\left(\mathbf{A} - \lambda \mathbf{I}\right).$$

Spectrum: set $\{\lambda_1, \ldots \lambda_k\}$ of eigenvalues $\big($roots of $P_{\mathbf{A}}(\lambda)\big)$

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$\lambda = 3$ is repeated twice, but $\dim \mathcal{E}_3(\mathbf{A}) = 1$

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$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)(3-\lambda) = 0 \begin{cases} \lambda_1 = 3 \\ \lambda_2 = 3 \end{cases}$$

- Eigenvectors
  - for λ_1: $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{x} = \mathbf{0} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x}_1$; $\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
  - for λ_2: The same **There isn't second independent eigenvector**

$\lambda = 3$ is repeated twice, but $\dim \mathcal{E}_3(\mathbf{A}) = 1$

$$\mu(3) = 2 \neq 1 = \gamma(3)$$

Summary:

1. The eigenvalues are those numbers λ that makes the matrix $(\mathbf{A} - \lambda\mathbf{I})$ singular. In other words, they are the roots of the Characteristic polynomial: $\det(\mathbf{A} - \lambda\mathbf{I})$.

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5. The product of eigenvalues of a matrix equals its determinant
6. The eigenvectors associated with λ are the non-zero vectors in $\mathcal{N}\left(\mathbf{A} - \lambda \mathbf{I}\right)$.

Questions of the Lecture 15

(L-15) QUESTION 1. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} -3 & 4 & -4 \\ -3 & 5 & -3 \\ -1 & 2 & 0 \end{bmatrix}$$

(a) The three eigenvalues of $\mathbf{A}$ are -1, 1 and 2; and two of its eigenvectors are

$$\boldsymbol{v} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}; \qquad \boldsymbol{w} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Check that both vectors are eigenvenctors of $\mathbf{A}$. What are the corresponding eigenvalues?

(b) Find a third linearly independent eigenvector.

(L-15) QUESTION 2. Find the eigenvalues and eigenvectors of

(a)

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

(b)

$$\mathbf{B} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

(Strang, 2006, exercise 12 from section 5.1.)

(L-15) QUESTION 3. If **B** has eigenvalues $1, 2, 3$, **C** has eigenvalues $4, 5, 6$, and **D** has eigenvalues $7, 8, 9$, what are the eigenvalues of the 6 by 6 matrix $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}$? where **B**, **C**, **D** are upper triangular matrices.
(Strang, 2006, exercise 13 from section 5.1.)

(L-15) QUESTION 4. Find the eigenvalues and eigenvectors of

(a)

$$\mathbf{A} = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

(b)

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

(Strang, 2006, exercise 5 from section 5.1.)

(L-15) QUESTION 5. The eigenvalues of $\mathbf{A}$ equal the eigenvalues of $\mathbf{A}^\intercal$. This is because $\det(\mathbf{A} - \lambda\mathbf{I})$ equals $\det(\mathbf{A}^\intercal - \lambda\mathbf{I})$.

(a) That is true because ____________

(b) Show by an example that, nevertheless, the eigenvectors of $\mathbf{A}$ and $\mathbf{A}^\intercal$ are not the same.

(Strang, 2006, exercise 11 from section 5.1.)

(L-15) QUESTION 6. Consider the matrix $\mathbf{B}$ and its eigenvector $\boldsymbol{x}$ associated to the eigenvalue λ, that is $\mathbf{B}\boldsymbol{x} = \lambda\boldsymbol{x}$; and also consider the matrix $\mathbf{A} = (\mathbf{B} + \alpha\mathbf{I})$. Prove that $\boldsymbol{x}$ is also an eigenvector of $\mathbf{A}$ with eigenvalue $(\lambda + \alpha)$.

(L-15) QUESTION 7.

(a) Encuentre los autovalores y los auto-vectores de la matriz $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$.

Compruebe que la traza es igual a la suma de los autovalores, y que el determinante es igual a su producto.

(b) Si consideramos una nueva matriz, generada a partir de la anterior como

$$\mathbf{B} = (\mathbf{A} - 7\mathbf{I}) = \begin{bmatrix} -6 & -1 \\ 2 & -3 \end{bmatrix}.$$

¿Cuáles son los autovalores y auto-vectores de la nueva matriz, y como están relacionados con los de $\mathbf{A}$?

(Strang, 2006, exercise 1 and 3 from section 5.1.)

(L-15) QUESTION 8. Suponga que λ es un auto-valor de $\mathbf{A}$, y que $\boldsymbol{x}$ es un auto-vector tal que $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$.

(a) Demuestre que ese mismo $\boldsymbol{x}$ es un auto-vector de $\mathbf{B} = \mathbf{A} - 7\mathbf{I}$, y encuentre el correspondiente auto-valor de $\mathbf{B}$.
(b) Suponga que $\lambda \neq 0$ (y que $\mathbf{A}$ es invertible), demuestre que $\boldsymbol{x}$ también es un auto-vector de $\mathbf{A}^{-1}$, y encuentre el correspondiente auto-valor. ¿Qué relación tiene con λ?

(Strang, 2006, exercise 7 from section 5.1.)

(L-15) QUESTION 9. Suponga que $\mathbf{A}$ es una matriz de dimensiones $n \times n$, y que $\mathbf{A}^2 = \mathbf{A}$. ¿Qué posibles valores pueden tomar los autovalores de $\mathbf{A}$?

(L-15) QUESTION 10. Suponga la matriz $\underset{3\times 3}{\mathbf{A}}$ con autovalores 1, 2 y 3. Si $\boldsymbol{v}_1$ es un auto-vector asociado al auto-valor 1, $\boldsymbol{v}_2$ al auto-valor 2 y $\boldsymbol{v}_3$ al auto-valor 3; entonces ¿cuanto es $\mathbf{A}(\boldsymbol{v}_1 + \boldsymbol{v}_2 - \boldsymbol{v}_3)$?

(L-15) QUESTION 11. Proporcione un ejemplo que muestre que los auto-valores pueden cambiar cuando un múltiplo de una columna se resta de otra. ¿Por qué los pasos de eliminación no modifican los autovalores nulos?
(Strang, 2006, exercise 6 from section 5.1.)

(L-15) QUESTION 12. El polinomio característico de una matriz $\mathbf{A}$ se puede factorizar como

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda).$$

Demuestre, partiendo de esta factorización, que el determinante de $\mathbf{A}$ es igual al producto de sus valores propios (autovalores). Para ello haga una elección inteligente del valor de λ.
(Strang, 2006, exercise 8 from section 5.1.)

(L-15) QUESTION 13. Calcule los valores característicos (autovalores o valores propios) y los vectores característicos de $\mathbf{A}$ y $\mathbf{A}^2$:

$$\mathbf{A} = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \qquad \text{y} \qquad \mathbf{A}^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}$$

$\mathbf{A}^2$ tiene los mismos ________ que $\mathbf{A}$. Cuando los autovalores de $\mathbf{A}$ son λ_1 y λ_2, los autovalores de $\mathbf{A}^2$ son ________.
(Strang, 2006, exercise 22 from section 5.1.)

(L-15) QUESTION 14. Suponga que los valores característicos de $\underset{3\times 3}{\mathbf{A}}$ son 1, 2 y 4, ¿cuál es la traza de $\mathbf{A}^2$? ¿Cuál es el determinante de $\left(\mathbf{A}^{-1}\right)^{\mathsf{T}}$?
(Strang, 2006, exercise 10 from section 5.2.)

(L-15) QUESTION 15. The equation $(\mathbf{A}^2 - 4\mathbf{I})\boldsymbol{x} = \boldsymbol{b}$ has no solution for some right-hand side $\boldsymbol{b}$. Give as much information as possible about the eigenvalues of the matrix **A** (the matrix **A** is diagonalizable). *MIT Course 18.06 Quiz 3. Spring, 2009*

(L-15) QUESTION 16. You are given the matrix

$$\mathbf{A} = \begin{bmatrix} 0.5 & 0.2 & 0.2 \\ 0.1 & 0.5 & 0.5 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$$

One of the eigenvalues is $\lambda = 1$. What are the eigenvalues of **A**? [Hint: Very little calculation required! You should be able to see another eigenvalue by inspection of the form of **A**, and the third by an easy calculation. You shouldn't need to compute $\det(\mathbf{A} - \lambda\mathbf{I})$ unless you really want to do it the hard way.] *MIT Course 18.06 Quiz 3. Spring, 2009*

Highlights of Lesson 16

- Similar matrices: $\mathbf{C} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$
- Triangular block diagonalizing a matrix

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] \xrightarrow[esp(\boldsymbol{\tau}_1^{-1}\cdots\boldsymbol{\tau}_p^{-1})]{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_p} \left[\begin{array}{c}\mathbf{C}\\ \hline \mathbf{S}\end{array}\right] \qquad \text{where} \quad \mathbf{S} = \mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_p}.$$

- Diagonalizable matrices: when $\mathbf{C}$ is diagonal.

Similarity

A and **C** are *similar* if there is an invertible **S** such that

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If **A** and **C** are similar (*see demos in the book*):

- The same determinant: $\det \mathbf{A} = \det \mathbf{C}$
- The same caracteristic polinomial: $|\mathbf{A} - \lambda\mathbf{I}| = |\mathbf{C} - \lambda\mathbf{I}|$
- The same eigenvalues (same *algebraic* and *geometric* multiplicities).
- The same trace.

2 Similar matrices

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- The same eigenvalues (same *algebraic* and *geometric* multiplicities).
- The same trace.

Mirror inverse transf.: $\left(\mathbf{I}_{(\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k)}\right)^{-1} = {}_{esp(\boldsymbol{\tau}_k^{-1}\cdots\boldsymbol{\tau}_1^{-1})}\mathbf{I}$

$$\mathbf{I} = {}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\overset{\boldsymbol{\tau}}{[(-\alpha)\boldsymbol{j}+\boldsymbol{i}]}{}_{\boldsymbol{\tau}_{(k+1)}\cdots\boldsymbol{\tau}_p}\overset{\boldsymbol{\tau}}{[(\alpha)\boldsymbol{i}+\boldsymbol{j}]}\mathbf{I} = {}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\overset{\boldsymbol{\tau}}{\left[\left(\tfrac{1}{\alpha}\right)\boldsymbol{j}\right]}{}_{\boldsymbol{\tau}_{(k+1)}\cdots\boldsymbol{\tau}_p}\overset{\boldsymbol{\tau}}{[(\alpha)\boldsymbol{j}]}\mathbf{I}$$

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3 Block diagonalizing a matrix (toothed matrix)

Consider $$\mathbf{A} = \left[\begin{array}{c||c} \mathbf{C} & \\ \hline \ast & \mathbf{L} \end{array}\right] \in \mathbb{C}^{n\times n}$$ where

C (of order m) is singular and **L** is full rank lower triangular;

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$$\mathbf{AR} = \left[\begin{array}{c|c||cccc} \underset{m\times(m-1)}{*} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & & & & \\ \hline & d_{m+1} & \beta_{m+1} & & & \\ * & d_{m+2} & * & \beta_{m+2} & & \\ & \vdots & * & * & \ddots & \\ & d_n & * & * & \cdots & \beta_n \end{array}\right]$$

$$\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k \qquad \boldsymbol{\tau}_{(k+1)} \cdots \boldsymbol{\tau}_p \left(\cdots \boldsymbol{\tau}_{[(\alpha_j)\boldsymbol{j}+\boldsymbol{m}]} \cdots \right)\mathbf{A}; \qquad \boldsymbol{j} = 1, \ldots, m$$

$\boxed{3}$ Block diagonalizing a matrix (toothed matrix)

Consider $\quad \mathbf{A} = \left[\begin{array}{c||c} \mathbf{C} & \\ \hline \ast & \mathbf{L} \end{array}\right] \in \mathbb{C}^{n\times n} \quad$ where

C (of order m) is singular and **L** is full rank lower triangular; then there exists an invertible **R** such that

$$\mathbf{R}^{-1}\mathbf{A}\mathbf{R} = \left[\begin{array}{c|c||cccc} \underset{m\times(m-1)}{\ast} & \begin{matrix}0\\ \vdots \\ 0\end{matrix} & & & & \\ \hline & d_{m+1} & \beta_{m+1} & & & \\ & d_{m+2} & \ast & \beta_{m+2} & & \\ \ast & \vdots & \ast & \ast & \ddots & \\ & d_n & \ast & \ast & \cdots & \beta_n \end{array}\right]$$

$$\underset{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}{\left(\cdots \underset{[(-\alpha_j)\boldsymbol{m}+\boldsymbol{j}]}{\boldsymbol{\tau}} \cdots\right)} \underset{\boldsymbol{\tau}_{(k+1)}\cdots\boldsymbol{\tau}_p}{\left(\cdots \underset{[(\alpha_j)\boldsymbol{j}+\boldsymbol{m}]}{\boldsymbol{\tau}} \cdots\right)} \mathbf{A}; \qquad \boldsymbol{j} = 1, \ldots, m$$

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$$_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}\Big(\cdots \boldsymbol{\tau}_{[(-\alpha_j)\boldsymbol{m}+\boldsymbol{j}]} \cdots\Big)_{\boldsymbol{\tau}_{(k+1)}\cdots\boldsymbol{\tau}_p}\Big(\cdots \boldsymbol{\tau}_{[(\alpha_j)\boldsymbol{j}+\boldsymbol{m}]} \cdots\Big)\mathbf{A}; \qquad \boldsymbol{j} = 1,\ldots,m$$

4 Block diagonalizing a matrix (toothed matrix)

Consider $\quad \mathbf{A} = \left[\begin{array}{c||c} \mathbf{C} & \\ \hline \ast & \mathbf{L} \end{array}\right] \in \mathbb{C}^{n\times n} \quad$ where

C (of order m) is singular and **L** is full rank lower triangular, then there exists $\mathbf{S} = \mathbf{RP}$ (invertible) such that

$$\mathbf{R}^{-1}\mathbf{A}\mathbf{R}\mathbf{P} = \left[\begin{array}{c|c||cccc} \underset{m\times(m-1)}{\ast} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & & & & \\ \hline & 0 & \beta_{m+1} & & & \\ & 0 & \ast & \beta_{m+2} & & \\ \ast & \vdots & \ast & \ast & \ddots & \\ & 0 & \ast & \ast & \cdots & \beta_n \end{array}\right]$$

$$\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k \qquad \boldsymbol{\tau}_{(k+1)} \cdots \boldsymbol{\tau}_p \left(\cdots \boldsymbol{\tau}_{[(\alpha_j)\boldsymbol{j}+\boldsymbol{m}]} \cdots \right) \mathbf{R}^{-1}\mathbf{A}\mathbf{R}; \qquad \boldsymbol{j} = m$$

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$$\mathbf{P}^{-1}\mathbf{R}^{-1}\mathbf{ARP} = \left[\begin{array}{c|c||cccc} & 0 & & & & \\ \underset{m\times(m-1)}{\ast} & \vdots & & & & \\ & 0 & & & & \\ \hline & 0 & \beta_{m+1} & & & \\ & 0 & \ast & \beta_{m+2} & & \\ \ast & \vdots & \ast & \ast & \ddots & \\ & 0 & \ast & \ast & \cdots & \beta_n \end{array}\right]$$

$$\left(\cdots \underset{[(-\alpha_j)\boldsymbol{m}+\boldsymbol{j}]}{\boldsymbol{\tau}} \cdots\right)_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k} \left(\cdots \underset{[(\alpha_j)\boldsymbol{j}+\boldsymbol{m}]}{\boldsymbol{\tau}} \cdots\right)_{\boldsymbol{\tau}_{(k+1)}\cdots\boldsymbol{\tau}_p} \mathbf{R}^{-1}\mathbf{AR}; \qquad \boldsymbol{j} = m\cdots$$

5 A very simple example

Example

Consider $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ with eigenvalues 0, 1 and 1.

$$\begin{bmatrix} \mathbf{A} \\ \hline \mathbf{I} \end{bmatrix} \xrightarrow[0\mathbf{I}]{(-)} \left[\begin{array}{ccc} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{1}+\mathbf{2}] \\ [(2)\mathbf{3}+\mathbf{2}] \\ [\mathbf{2}\rightleftharpoons\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

$\boxed{5}$ A very simple example

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Consider $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ with eigenvalues 0, 1 and 1.

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$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{diagonal}}$$

6 A not so simple example

Example

Consider $\mathbf{A} = \begin{bmatrix} -2 & 0 & 3 \\ 3 & -2 & -9 \\ -1 & 2 & 6 \end{bmatrix}$ with eigenvalues 1, 1 and 0.

$$\xrightarrow[1\mathbf{I}]{(-)} \left[\begin{array}{ccc} -3 & 0 & 3 \\ 3 & -3 & -9 \\ -1 & 2 & 5 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{1}+\mathbf{3}] \\ [(-2)\mathbf{2}+\mathbf{3}]}}$$

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Consider $\mathbf{A} = \begin{bmatrix} -2 & 0 & 3 \\ 3 & -2 & -9 \\ -1 & 2 & 6 \end{bmatrix}$ with eigenvalues 1, 1 and 0.

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Example

Consider $\mathbf{A} = \begin{bmatrix} -2 & 0 & 3 \\ 3 & -2 & -9 \\ -1 & 2 & 6 \end{bmatrix}$ with eigenvalues 1, 1 and 0.

$$\xrightarrow[1\mathbf{I}]{(-)} \left[\begin{array}{ccc} -3 & 0 & 3 \\ 3 & -3 & -9 \\ -1 & 2 & 5 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{1}+\mathbf{3}] \\ [(-2)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} -3 & 0 & 0 \\ 3 & -3 & 0 \\ -1 & 2 & 0 \\ \hline 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{\boldsymbol{\tau} \\ [(2)\mathbf{3}+\mathbf{2}] \\ [(-1)\mathbf{3}+\mathbf{1}]}]{} \left[\begin{array}{ccc} -2 & -2 & 0 \\ 1 & 1 & 0 \\ -1 & 2 & 0 \\ \hline 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array}\right]$$

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$\boxed{7}$ Every matrix is similar to a toothed matrix

For every **A** there exists **S** such that

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{C}$$

where **C**, toothed, has the eigenvalues on the diagonal

Example

$$\begin{bmatrix} 6 & -1 & 1 \\ -9 & 1 & -2 \\ 4 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -2 & 0 & 3 \\ 3 & -2 & -9 \\ -1 & 2 & 6 \end{bmatrix} \begin{bmatrix} 6 & -1 & 1 \\ -9 & 1 & -2 \\ 4 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}}_{\text{toothed}}$$

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Consequences

- $\sum \lambda_i = \operatorname{tr}(\mathbf{A})$ and $\prod \lambda_i = \det \mathbf{A}$

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For every **A** there exists **S** such that

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Consequences

- $\sum \lambda_i = \operatorname{tr}(\mathbf{A})$ and $\prod \lambda_i = \det \mathbf{A}$

- $\mathbf{A}\mathbf{S}_{|j} = \mathbf{S}\mathbf{C}_{|j} \qquad \Rightarrow \qquad$ for j such that $\mathbf{C}_{|j} = \lambda_i \mathbf{I}_{|j}$:

$$\mathbf{A}\left(\mathbf{S}_{|j}\right) = \lambda_i\left(\mathbf{S}_{|j}\right) \quad \Rightarrow \quad \mathbf{S}_{|j} \text{ is an eigenvector.}$$

8 Back to the simple, "toothless" example

Consider $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ with eigenvalues 0, 1 and 1.

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] \xrightarrow[0\mathbf{I}]{(-)} \left[\begin{array}{ccc} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{1}+\mathbf{2}] \\ [(2)\mathbf{3}+\mathbf{2}] \\ [\mathbf{2}\rightleftharpoons\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow[\substack{\boldsymbol{\tau} \\ [\mathbf{2}\rightleftharpoons\mathbf{3}] \\ [(-2)\mathbf{2}+\mathbf{3}] \\ [(-1)\mathbf{2}+\mathbf{1}]}]{} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow[0\mathbf{I}]{(+)} \left[\frac{\mathbf{C}}{\mathbf{S}}\right]$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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9 Diagonalizable matrices

- A matrix is diagonalizable **if and only if** *algebraic* and *geometric* multiplicities are equal for each eigenvalue

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- If there are no repeated eigenvalues, there are no "teeth" either

- When there are no repeated eigenvalues $\underset{n\times n}{\mathbf{A}}$ is diagonalizable (is sure to have n independent eigenvectors)

10 Diagonalizing a matrix

- Find the spectrum: $\{\lambda_1, \lambda_2, \ldots\}$
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2. . . . or for every λ_i
   - find the eigenspace

$$\mathcal{E}_{\lambda_i}(\mathbf{A}) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \middle| \mathbf{A}\boldsymbol{x} = \lambda_i \boldsymbol{x} \right\} = \mathcal{N}(\mathbf{A} - \lambda_i \mathbf{I}).$$

   - check $\mu(\lambda_i) = \dim \mathcal{E}_{\lambda_i}(\mathbf{A})$ (algebraic and geometric multiplicities are equal)

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix}; \quad \mathbf{S} = \left[\text{Basis for } \mathcal{E}_{\lambda_1}(\mathbf{A}) \# \cdots \# \text{Basis for } \mathcal{E}_{\lambda_k}(\mathbf{A}) \right]$$

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D} \quad \Leftrightarrow \quad \mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$$

11 Matrix powers

If $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ then $\mathbf{A}^2\boldsymbol{x} =$

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In a matrix form (if $\mathbf{A}$ is diagonalizable, $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$):

$$\mathbf{A}^2 = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}\mathbf{S}\mathbf{D}\mathbf{S}^{-1} = \mathbf{S}\mathbf{D}^2\mathbf{S}^{-1}$$

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In a matrix form (if $\mathbf{A}$ is diagonalizable, $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$):

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In general, for, $n \in \mathbb{Z}$, $n \geq 0\ldots \quad \mathbf{A}^n =$

11 Matrix powers

If $\mathbf{A}\boldsymbol{x} = \lambda \boldsymbol{x}$ then $\mathbf{A}^2\boldsymbol{x} = \mathbf{A}\mathbf{A}\boldsymbol{x} = \mathbf{A}(\lambda \boldsymbol{x}) = \lambda \mathbf{A}\boldsymbol{x} = \lambda^2 \boldsymbol{x}$

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In general, for, $n \in \mathbb{Z}$, $n \geq 0 \ldots$ $\mathbf{A}^n = \mathbf{S}\mathbf{D}^n\mathbf{S}^{-1}$
what about $\mathbf{A}$ both diagonalizable and invertible?

Questions of the Lecture 16

(L-16) QUESTION 1. Factor these two matrices into $\mathbf{SDS}^{-1}$;

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

(Strang, 2006, exercise 15 from section 5.2.)

(L-16) QUESTION 2. Which of these matrices cannot be diagonalized?

(a)

$$\mathbf{A}_1 = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$$

(b)

$$\mathbf{A}_2 = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}$$

(c)

$$\mathbf{A}_3 = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

(Strang, 2006, exercise 5 from section 5.2.)

(L-16) QUESTION 3. If $\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ find $\mathbf{A}^{100}$ by diagonalizing $\mathbf{A}$.

(Strang, 2006, exercise 7 from section 5.2.)

(L-16) QUESTION 4. If the eigenvalues of $\underset{3\times3}{\mathbf{A}}$ are 1, 1 and 2, which of the following are certain to be true? Give a reason if true or a counterexample if false:

(a) **A** is invertible.
(b) **A** is diagonalizable.
(c) **A** is not diagonalizable

(Strang, 2006, exercise 11 from section 5.2.)

(L-16) QUESTION 5. Let **A** be the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

(a) (1^{pts}) Determine if **A** is diagonalizable, and if so, diagonalize it.
(b) (0.5^{pts}) Compute $\left(\mathbf{A}^6\right)\boldsymbol{v}$, where $\boldsymbol{v} = (0, \quad 0, \quad 0, \quad 1,)$.
(c) (0.5^{pts}) Using the the eigenvalues found in part (a) justify that **A** is invertible.
(d) (0.5^{pts}) What is the relation between the eigenvalues of **A** and the eigenvalues of $\mathbf{A}^{-1}$?

(L-16) QUESTION 6. Si $\mathbf{A} = \mathbf{SDS}^{-1}$; entonces $\mathbf{A}^3 = (\qquad)(\qquad)(\qquad)$ y $\mathbf{A}^{-1} = (\qquad)(\qquad)(\qquad)$.
(Strang, 2006, exercise 16 from section 5.2.)

(L-16) QUESTION 7. Considere la matriz

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

(a) Encuentre los autovalores de **A**
(b) Encuentre los auto-vectores de **A**
(c) Diagonalice **A**: escríbalo como $\mathbf{A} = \mathbf{SDS}^{-1}$.

(L-16) QUESTION 8. ¿Falso o verdadero? Si los autovalores de **A** son 2, 2 y 3 entonces sabemos que la matriz es

(a) Invertible
(b) Diagonalizable
(c) No diagonalizable.

(L-16) QUESTION 9. Sean las matrices

$$\mathbf{A}_1 = \begin{bmatrix} 8 & \\ & 2 \end{bmatrix}; \qquad \mathbf{A}_2 = \begin{bmatrix} 9 & 4 \\ & 1 \end{bmatrix}; \qquad \mathbf{A}_3 = \begin{bmatrix} 10 & 5 \\ -5 & \end{bmatrix}$$

(a) Complete dichas matrices de modo que en los tres casos $\det \mathbf{A}_i = 25$. Así, la traza es en todos los casos igual a 10, y por tanto para las tres matrices el único auto-valor $\lambda = 5$ está repetido dos veces ($\lambda^2 = 25$ y $\lambda + \lambda = 10$ implica $\lambda = 5$).

(b) Encuentre un vector característico con $\mathbf{A}\boldsymbol{x} = 5\boldsymbol{x}$. Estas tres matrices no son diagonalizable porque no hay un segundo auto-vector linealmente independiente del primero.

(Strang, 2006, exercise 27 from section 5.2.)

(L-16) QUESTION 10. Factorice las siguientes matrices en **S D S**$^{-1}$

(a) $\mathbf{A} = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix}2 & 1\\0 & 0\end{bmatrix}$

(Strang, 2006, exercise 1 from section 5.2.)

(L-16) QUESTION 11. Encuentre la matriz **A** cuyos autovalores son 1 y 4, cuyos autovectores son $\begin{pmatrix}3\\1\end{pmatrix}$ y $\begin{pmatrix}2\\1\end{pmatrix}$ respectivamente.
(Strang, 2006, exercise 2 from section 5.2.)

(L-16) QUESTION 12. Si los elementos diagonales de una matriz triangular superior de orden 3×3 son 1, 2 y 7, ¿puede saber si la matriz es diagonalizable? ¿Quién es **D**?
(Strang, 2006, exercise 4 from section 5.2.)

(L-16) QUESTION 13.

(a) Encuentre los autovalores y auto-vectores de la matriz $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

(b) Explique por qué (o por qué no) la matriz $\mathbf{A}$ es diagonalizable.

(L-16) QUESTION 14. Sea $\mathbf{A}$ una matriz 3×3. Asuma que sus autovalores son 1 y 0, que una base de los autovectores asociados a $\lambda = 1$ son $[1, 0, 1]$ y $[0, 0, 1]$; mientras que los asociados a $\lambda = 0$ son paralelos a $[1, 1, 2]$.

(a) ¿Es $\mathbf{A}$ diagonalizable? En caso afirmativo escriba la matriz diagonal $\mathbf{D}$ y la matriz $\mathbf{S}$ tales que $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$.

(b) Encuentre $\mathbf{A}$.

(L-16) QUESTION 15. Let $\mathbf{A}$ be a 2×2 matrix such that $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ is an eigenvector for $\mathbf{A}$ with eigenvalue 2, and $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is another eigenvector for $\mathbf{A}$ with eigenvalue -2. If $\boldsymbol{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, compute $\left(\mathbf{A}^3\right)\boldsymbol{v}$.

Highlights of Lesson 17

- Symetric matrices $\mathbf{A} = \mathbf{A}^{\mathsf{T}}$
  - Eigenvalues and eigenvectors
- Introd. positive Definiteness matrices

$\boxed{2}$ Symmetric matrices $\mathbf{A} = \mathbf{A}^\mathsf{T}$

what's special about $\mathbf{A}\boldsymbol{x} = \lambda \boldsymbol{x}$ when $\underset{n\times n}{\mathbf{A}}$ is symmetric?

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$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D} \quad \longleftrightarrow \quad \mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$$

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Symmetric case:

I can choose perpendicular unit eigenvectors (ortho*normal* columns of $\mathbf{S} = \mathbf{Q}$)

$$(\text{if } \mathbf{A} = \mathbf{A}^\intercal) \qquad \mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}^{-1} =$$

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Orthogonally diagonalizable.

3 Eigenspaces are orthogonal for symmetric matrices

Eigenvectors (corresponding to different eigenvalues) of a symmetric matrix are orthogonal.

Proof.

Consider $\mathbf{A}\boldsymbol{x} = \lambda_1 \boldsymbol{x}$ and $\mathbf{A}\boldsymbol{y} = \lambda_2 \boldsymbol{y}$ (with $\lambda_1 \neq \lambda_2$). then

$$\lambda_1 \boldsymbol{x} \cdot \boldsymbol{y} \;=\; \mathbf{A}\boldsymbol{x} \cdot \boldsymbol{y} \;=\; \boldsymbol{x}(\mathbf{A}^{\mathsf{T}})\boldsymbol{y} \;=\; \boldsymbol{x}\mathbf{A}\boldsymbol{y} \;=\; \big(\boldsymbol{x} \cdot \boldsymbol{y}\big)\lambda_2.$$

Since $\lambda_1 \neq \lambda_2$ then:

$$\lambda_1(\boldsymbol{x}\cdot\boldsymbol{y}) - \lambda_2(\boldsymbol{x}\cdot\boldsymbol{y}) = 0 \quad\Longrightarrow\quad \big(\lambda_1 - \lambda_2\big)\boldsymbol{x}\cdot\boldsymbol{y} = 0 \quad\Longrightarrow\quad \boldsymbol{x}\cdot\boldsymbol{y} = 0.$$

□

4 Quadratic forms

Quadratic form:

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Positive definite quadratic form:

$$\boldsymbol{x}\mathbf{A}\boldsymbol{x} > 0 \quad \forall \boldsymbol{x} \neq \mathbf{0} \qquad \Longleftrightarrow \qquad \lambda_i > 0, \quad i = 1:n.$$

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then we also say **A** is positive definite.

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the answer must be. . .

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are all λ_i positive? are they negative?

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7 Symmetric matrices: signs of eigenvalues

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Computing eigenvalues of $\underset{5\times 5}{\mathbf{A}}$ is impossible in general! (5th degree polynomial)

Good news: The signs of the pivots of echelon form are the same as the signs of the eigenvalues λ_i
(if we do not change the sign of the determinant with *Type II* elementary transformations)

$$\text{num. of positive pivots} = \text{num. of positive eigenvalues}$$

8 Positive definite symmetric matrices

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Pivots: 5, $\frac{11}{5}$
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Pivots: 5, $\frac{11}{5}$
Hint: Product of the pivots is: $\det(\mathbf{A})$

What is the sign of each eigenvalue?

8 Positive definite symmetric matrices

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Pivots: 5, $\frac{11}{5}$
Hint: Product of the pivots is: $\det(\mathbf{A})$

What is the sign of each eigenvalue? $\lambda_1 \cdot \lambda_2 = 11, \; \lambda_1 + \lambda_2 = 8$

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Hint: Product of the pivots is: $\det(\mathbf{A})$

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positive

$$\lambda^2 - 8\lambda + 11 = 0 \rightarrow \qquad \lambda = 4 \pm \sqrt{5} > 0$$

Summary (for symmetric matrices):

1. Symmetric matrices have *real eigenvalues* and *perpendicular eigenvectors* can be choosen

2. $\mathbf{A} = \mathbf{QDQ}^\intercal$ where $\mathbf{Q}$ is orthogonal

3. $\mathbf{A}$ is symmetric if and only if it is *orthogonally* diagonalizable

4. The signs of the pivots in the echelon form are same as the signs of the eigenvalues λ_i (only if we do not change the sign of the determinant with *Type II* elementary transformations)

Questions of the Lecture 17

(L-17) QUESTION 1. Write **A**, **B** and **C** in the form $\mathbf{QDQ}^\intercal$ of the spectral theorem:

(a) $\mathbf{A} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\mathbf{C} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

(Strang, 2006, exercise 11 from section 5.5.)

(L-17) QUESTION 2. Find the eigenvalues and the unit eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(Strang, 2003, exercise 3 from section 6.4.)

(L-17) QUESTION 3. Find an orthonormal **Q** that diagonalizes this symmetric matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$$

(Strang, 2003, exercise 5 from section 6.4.)

(L-17) QUESTION 4. Suppose **A** is a symmetric 3 by 3 matrix with eigenvalues 0, 1,2.

(a) What properties can be guaranteed for the corresponding unit eigenvectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$
(b) In terms of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, describe the nullspace, left nullspace, row space, and column space of **A**.
(c) Find a vector $\boldsymbol{x}$ that satisfies $\mathbf{A}\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$. Is $\boldsymbol{x}$ unique?
(d) Under what conditions on $\boldsymbol{b}$ does $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ have a solution?
(e) If $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are the columns of **S**, what are $\mathbf{S}^{-1}$ and $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$.

(Strang, 2006, exercise 13 from section 5.5.)

(L-17) QUESTION 5. Escriba un hecho destacado sobre los valores característicos de cada uno de estos tipos de matrices:

(a) Una matriz simétrica real.
(b) Una matriz diagonalizable tal que $\mathbf{A}^n \to \mathbf{0}$ cuando $n \to \infty$.
(c) Una matriz no diagonalizable
(d) Una matriz singular

(Strang, 2006, exercise 16 from section 5.5.)

(L-17) Question 6. Sean

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}; \qquad \mathbf{B} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

(a) Encuentre los valores característicos de **A** (recuerde que $i^2 = -1$).
(b) Encuentre los valores característicos de **B** (en este caso quizá le resulte más sencillo encontrar primero los autovectores, y deducir entonces los autovalores).
(c) De los siguientes tipos de matrices: ortogonales, invertibles, permutación, hermíticas, de rango 1. diagonalizables, de Markov ¿a qué tipos pertenece **A**?
(d) ¿y **B**?

(Strang, 2006, exercise 14 from section 5.5.)

(L-17) Question 7. Si $\mathbf{A}^3 = \mathbf{0}$ entonces los autovalores de **A** deben ser ________. De un ejemplo tal que $\mathbf{A} \neq \mathbf{0}$. Ahora bien, si **A** es además simétrica, demuestre que entonces $\mathbf{A}^3$ es necesariamente $\mathbf{0}$.

(L-17) QUESTION 8. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} a & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

(a) Prove that **A** is not diagonalizable when $a = 3$.

(b) Is **A** diagonalizable when $a = 2$? (explain). If it is diagonalizable, find an eigenvalue diagonal matrix **D** and an eigenvector matrix **S** such as $\mathbf{A} = \mathbf{SDS}^{-1}$.

(c) Is $\mathbf{A}^\intercal\mathbf{A}$ diagonalizable for any value a? Is it possible to find a full set of orthonormal eigenvectors of $\mathbf{A}^\intercal\mathbf{A}$?

(d) Find all posible values a such as **A** is invertible and diagonalizable.

(L-17) QUESTION 9. Sea la matriz

$$\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};$$

(a) Exprese **B** en la forma $\mathbf{B} = \mathbf{A} = \mathbf{QDQ}^\intercal$ del teorema espectral.

(b) ¿Es **B** diagonalizable? Si no lo es, diga las razones; y en caso contrario genere una matriz **S** que diagonalice a **B**.

Highlights of Lesson 18

- Positive and Negative (semi)definite matrices
- Completing the squares
- Diagonalization by congruence

$\boxed{2}$ Quadratic forms

- Positive definite: $\quad \forall \boldsymbol{x} \neq \mathbf{0} \ \Rightarrow \quad \boldsymbol{x}\mathbf{A}\boldsymbol{x} > 0.$

2 Quadratic forms

- Positive definite: $\forall \boldsymbol{x} \neq \mathbf{0} \;\Rightarrow\; \boldsymbol{x}\mathsf{A}\boldsymbol{x} > 0.$

- Positive semi-definite: $\forall \boldsymbol{x} \neq \mathbf{0} \;\Rightarrow\; \boldsymbol{x}\mathsf{A}\boldsymbol{x} \geq 0.$

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- Negative definite: $\forall \boldsymbol{x} \neq \mathbf{0} \;\Rightarrow\; \boldsymbol{x}\mathbf{A}\boldsymbol{x} < 0.$
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2 Quadratic forms

- Positive definite: $\forall \boldsymbol{x} \neq \mathbf{0} \Rightarrow \quad \boldsymbol{x}\mathbf{A}\boldsymbol{x} > 0.$
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- Negative definite: $\forall \boldsymbol{x} \neq \mathbf{0} \Rightarrow \quad \boldsymbol{x}\mathbf{A}\boldsymbol{x} < 0.$
- Negative semi-definite: $\forall \boldsymbol{x} \neq \mathbf{0} \Rightarrow \quad \boldsymbol{x}\mathbf{A}\boldsymbol{x} \leq 0.$
- Indefinite: neither positive semi-definite, nor negative semi-definite.

Example

What number do I have to put there for the matrix **A** to be singular?

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & \end{bmatrix}$$

Example

What number do I have to put there for the matrix **A** to be singular?

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$$

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- Eigenvalues:

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- Eigenvalues: $\lambda = 0, \; \begin{pmatrix} -3 \\ 1 \end{pmatrix};$

Example

What number do I have to put there for the matrix **A** to be singular?

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$$

- Eigenvalues: $\lambda = 0, \begin{pmatrix} -3 \\ 1 \end{pmatrix}; \quad \lambda = 20 \text{ (tr)} \quad \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Example

What number do I have to put there for the matrix $\mathbf{A}$ to be singular?

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- Eigenvalues: $\lambda = 0, \begin{pmatrix} -3 \\ 1 \end{pmatrix}; \quad \lambda = 20 \text{ (tr)} \quad \begin{pmatrix} 1 \\ 3 \end{pmatrix}$
- Leading principal minors:

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What number do I have to put there for the matrix $\mathbf{A}$ to be singular?

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$$

- Eigenvalues: $\lambda = 0, \begin{pmatrix} -3 \\ 1 \end{pmatrix}; \quad \lambda = 20 \text{ (tr)} \quad \begin{pmatrix} 1 \\ 3 \end{pmatrix}$
- Leading principal minors: $2,$

Example

What number do I have to put there for the matrix **A** to be singular?

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- Eigenvalues: $\lambda = 0, \begin{pmatrix} -3 \\ 1 \end{pmatrix}; \quad \lambda = 20 \ (\mathrm{tr}) \quad \begin{pmatrix} 1 \\ 3 \end{pmatrix}$
- Leading principal minors: $2, \quad 0$

Example

What number do I have to put there for the matrix **A** to be singular?

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- Leading principal minors: $2, \quad 0$
- For the following quadratic form

$$q_{\mathbf{A}}(\boldsymbol{x}) = \boldsymbol{x}\mathbf{A}\boldsymbol{x} = (x, \quad y,) \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} =$$

Example

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Example

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$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$$

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Is there a $\boldsymbol{x} \neq \mathbf{0}$ such that $\boldsymbol{x}\mathbf{A}\boldsymbol{x} = 0$?

Example

What number do I have to put there for the matrix **A** to be singular?

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$$

Positive **semi**-definite.

- Eigenvalues: $\lambda = 0, \begin{pmatrix} -3 \\ 1 \end{pmatrix}; \quad \lambda = 20 \text{ (tr)} \quad \begin{pmatrix} 1 \\ 3 \end{pmatrix}$
- Leading principal minors: $2, \quad 0$
- For the following quadratic form

$$q_{\mathbf{A}}(\boldsymbol{x}) = \boldsymbol{x}\mathbf{A}\boldsymbol{x} = (x, \quad y,) \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 12xy + 18y^2$$

Is there a $x \neq \mathbf{0}$ such that $x\mathbf{A}x = 0$? $\quad (-3, 1,) \Rightarrow \boldsymbol{x}\mathbf{A}\boldsymbol{x} = 0$

(graph)

Example

If $\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$

Example

If $\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$ then $(x, \quad y,) \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 12xy + \textbf{?}\, y^2$

Example

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- Are there numbers x and y that make $\boldsymbol{x}\mathbf{A}\boldsymbol{x}$ negative?

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  $x = 1,\ y = -1$

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- Are there numbers x and y that make $\boldsymbol{x}\mathbf{A}\boldsymbol{x}$ negative?
  $x = 1, \ y = -1$
- Does the function go through the origin?

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If $\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$ then $(x, \quad y,) \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 12xy + 7y^2$

- Are there numbers x and y that make $\boldsymbol{x}\mathbf{A}\boldsymbol{x}$ negative?
  $x = 1, \ y = -1$
- Does the function go through the origin? YES

Example

$$\text{If } \mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix} \quad \text{then} \quad (x, \quad y,) \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 12xy + 7y^2$$

- Are there numbers x and y that make $\boldsymbol{x}\mathbf{A}\boldsymbol{x}$ negative?
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- Does the function go through the origin? YES
- When $y = 0$ and $x = 1$, is it possitive? (and when $x = -1$?)

Example

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- Does the function go through the origin? YES
- When $y = 0$ and $x = 1$, is it possitive? (and when $x = -1$?)
- When $x = 0$ and $y = 1$, is it possitive? (and when $y = -1$?)
- Is it always positive?

Example

If $\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$ then $(x, \quad y,) \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 12xy + 7y^2$

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- When $x = 0$ and $y = 1$, is it possitive? (and when $y = -1$?)
- Is it always positive? **No**, in some directions it "goes" upwards, in others downwards. (graph)

Example

If $\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$ then $(x, \quad y,) \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 12xy + 7y^2$

Indefinite.

- Are there numbers x and y that make $x\mathbf{A}x$ negative?
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- Does the function go through the origin? YES
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- When $x = 0$ and $y = 1$, is it possitive? (and when $y = -1$?)
- Is it always positive? **No**, in some directions it "goes" upwards, in others downwards. (graph)

$(0, 0,)$ **saddle point**: minimum in some directions, maximum in others.

$$\lambda_1 = -2, \quad \begin{pmatrix} -6 \\ 4 \end{pmatrix}; \qquad \lambda_1 = 11, \quad \begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

Example

If $\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$ then $(x, \quad y,) \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 12xy + 20y^2$

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Does it pass the tests?

- Are the leading principal minors positive?
- Are the eigenvalues positive?

Example

If $\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$ then $(x, \quad y,) \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 12xy + 20y^2$

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- Are the leading principal minors positive?
- Are the eigenvalues positive?
  $\lambda_1 \cdot \lambda_2 = \det(\mathbf{A}) = 4$

Example

If $\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$ then $(x, \quad y,) \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 12xy + 20y^2$

Does it pass the tests?

- Are the leading principal minors positive?
- Are the eigenvalues positive?
  $\lambda_1 \cdot \lambda_2 = \det(\mathbf{A}) = 4 \quad$ y $\lambda_1 + \lambda_2 = \text{traza}(\mathbf{A}) = 22$

Example

If $\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$ then $(x, \quad y,) \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 12xy + 20y^2$

Positive definite.

Does it pass the tests?

- Are the leading principal minors positive?
- Are the eigenvalues positive?

  $\lambda_1 \cdot \lambda_2 = \det(\mathbf{A}) = 4$ y $\lambda_1 + \lambda_2 = \text{traza}(\mathbf{A}) = 22$

$$q_{\mathbf{A}}(\boldsymbol{x}) = \boldsymbol{x}\mathbf{A}\boldsymbol{x} > 0 \qquad \text{for all } \boldsymbol{x} \neq \mathbf{0}$$

$\boxed{3}$ Completing the squares

If we could express $q(\boldsymbol{x})$ as a sum of squares, we would know whether $q(\boldsymbol{x})$ is positive definite.

3 Completing the squares

If we could express $q(\boldsymbol{x})$ as a sum of squares, we would know whether $q(\boldsymbol{x})$ is positive definite.

Let's complete the square!

- $q(x, y) = 2x^2 + 12xy + 20y^2 \; = \; 2(x + \textbf{?}\, y)^2$

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- $q(x, y) = 2x^2 + 12xy + 18y^2$

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- $q(x,y) = 2x^2 + 12xy + 200y^2$ (graph)

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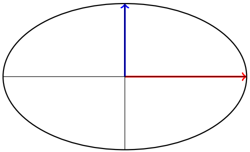
- $q(x,y) = 2x^2 + 12xy + 200y^2$ (graph)

If positive definite: $q(x,y) = a; \quad a > 0$: ellipse



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- $q(x,y) = 2x^2 + 12xy + 200y^2$ (graph)

is

$q(x,y,z,w,t) = 2t^2 - 2tx - 2tz + w^2 - 2wy + 2x^2 - 2xy + 2y^2 + z^2$

positive definite?

3 Completing the squares

If we could express $q(\boldsymbol{x})$ as a sum of squares, we would know whether $q(\boldsymbol{x})$ is positive definite.

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$q(x, y, z, w, t) = 2t^2 - 2tx - 2tz + w^2 - 2wy + 2x^2 - 2xy + 2y^2 + z^2$

positive definite? ☹☹☹☹☹!!!???

4 Congruent matrices

A and **C** are congruent if there exists an invertible **B** such that

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Diagonalization by congruence

For each **A** (symmetric) exists $\mathbf{B} = \mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k}$ (invertible) such that

$$\mathbf{D} = \mathbf{B}^\intercal \mathbf{A} \mathbf{B} \qquad \text{is diagonal} \qquad \left(\mathbf{B}^\intercal = {}_{\boldsymbol{\tau}_k \cdots \boldsymbol{\tau}_1}\mathbf{I}\right)$$

4 Congruent matrices

A and **C** are congruent if there exists an invertible **B** such that

$\boxed{\mathbf{C} = \mathbf{B}^\intercal\mathbf{A}\mathbf{B}}$

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For each **A** (symmetric) exists $\mathbf{B} = \mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}$ (invertible) such that

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Spectral Theorem: ¡Diagonalization by similarity and congruence!

$$\mathbf{D} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \mathbf{Q}^\intercal\mathbf{A}\mathbf{Q}.$$

4 Congruent matrices

$\mathbf{A}$ and $\mathbf{C}$ are congruent if there exists an invertible $\mathbf{B}$ such that

$$\boxed{\mathbf{C} = \mathbf{B}^\intercal \mathbf{A} \mathbf{B}}$$

Diagonalization by congruence

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Hence, every quadratic form can be written as a sum of squares

$$\boldsymbol{x} \mathbf{A} \boldsymbol{x} \;=\; \boldsymbol{x} \left(\mathbf{B}^{-1}\right)^\intercal \mathbf{D} \mathbf{B}^{-1} \boldsymbol{x}$$

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$$\boldsymbol{x}\mathbf{A}\boldsymbol{x} \;=\; \boldsymbol{x}\left(\mathbf{B}^{-1}\right)^\intercal\mathbf{D}\mathbf{B}^{-1}\boldsymbol{x} \;=\; \boldsymbol{y}\mathbf{D}\boldsymbol{y}; \qquad \text{where} \qquad \boldsymbol{y} = \mathbf{B}^{-1}\boldsymbol{x}.$$

$\boxed{5}$ Completing the squares

$$2x^2 + 12xy + 20y^2$$

5 Completing the squares

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$$2x^2 + 12xy + 20y^2$$

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$$2x^2 + 12xy + 20y^2$$

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therefore, we get:

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \mathbf{D} = \mathbf{E}^\intercal \mathbf{A} \mathbf{E} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

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hence $\mathbf{A} = (\mathbf{E}^{\top})^{-1}\mathbf{D}\mathbf{E}^{-1}$ so

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$$\boldsymbol{x}\mathbf{A}\boldsymbol{x} = \begin{pmatrix} x, & y, \end{pmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \left(\boldsymbol{x}\,(\mathbf{E}^{-1})^{\mathsf{T}}\right)\mathbf{D}\left(\mathbf{E}^{-1}\boldsymbol{x}\right)$$

5 Completing the squares

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$$= \Big((x + 3y), \quad y, \Big) \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{pmatrix} (x + 3y) \\ y \end{pmatrix} = 2(x + 3y)^2 + 2y^2$$

5 Completing the squares

$$2x^2 + 12xy + 20y^2 \quad > 0 \ (\forall \boldsymbol{x} \neq \mathbf{0})$$

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$\boxed{6}$ example 3 by 3

Is $\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ positive definite?

6 example 3 by 3

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- Pivots: $2, \quad 3/2, \quad 4/3$

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- Pivots: $2, \quad 3/2, \quad 4/3$
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- Leading principal minors:

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- Leading principal minors: $2, \quad 3, \quad 4$

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$$\boldsymbol{x}\mathbf{A}\boldsymbol{x} = 2x^2 + 2y^2 + 2z^2 - 2xy - 2yz > 0$$

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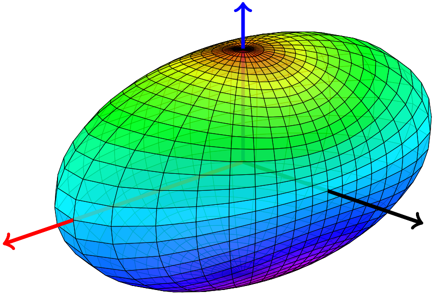
$$\boldsymbol{x}\mathbf{A}\boldsymbol{x} = 2x^2 + 2y^2 + 2z^2 - 2xy - 2yz > 0$$

$\boldsymbol{x}\mathbf{A}\boldsymbol{x} = 1$: (ellipsoid) axes are eigenvectors $\mathbf{A} = \mathbf{Q}^\intercal \boldsymbol{\lambda} \mathbf{Q}$

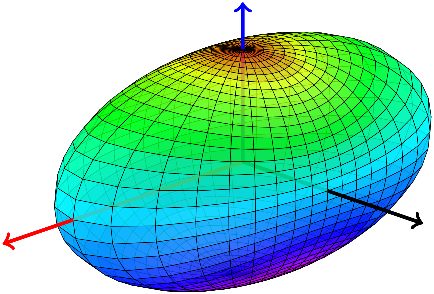
7 Positive definite matrices and ellipsoids: example 3 by 3

- The region $(\boldsymbol{x}\mathbf{A}\boldsymbol{x} = a)$ is an (ellipsoid).



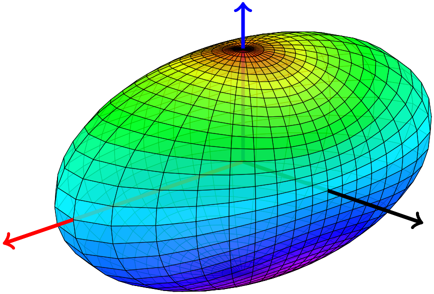
7 Positive definite matrices and ellipsoids: example 3 by 3

- The region $(\boldsymbol{x}\mathbf{A}\boldsymbol{x} = a)$ is an (ellipsoid).
- The eigenvectors of $\mathbf{Q}$ are in the direction of the three principal axes.



7 Positive definite matrices and ellipsoids: example 3 by 3

- The region $(\boldsymbol{x}\mathbf{A}\boldsymbol{x} = a)$ is an (ellipsoid).
- The eigenvectors of $\mathbf{Q}$ are in the direction of the three principal axes.
- Lengths of axes determined by the eigenvalues



$\boxed{8}$ Another example 3 by 3

Is $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ positive definite?

8 Another example 3 by 3

$$\text{Is } \mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ positive definite?}$$

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- pivots:

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- pivots: $2, \quad -1/2$

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$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow[{[(1)\overset{\tau}{\mathbf{3}}+\mathbf{1}]}]{[(1)\overset{\tau}{\mathbf{3}}+\mathbf{1}]} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow[{\left[\left(-\frac{1}{2}\right)^{\tau}\mathbf{1}+\mathbf{3}\right]}]{\left[\left(-\frac{1}{2}\right)^{\tau}\mathbf{1}+\mathbf{3}\right]} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \xrightarrow[{[\mathbf{2}\overset{\tau}{\rightleftharpoons}\mathbf{3}]}]{[\mathbf{2}\overset{\tau}{\rightleftharpoons}\mathbf{3}]} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- pivots: $2, \quad -1/2$
- eigenvalues:

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Indefinite matrix

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- pivots: $2, \quad -1/2$
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Indefinite matrix

9 "Classification" of quadratic

$$\boxed{\boldsymbol{x}\mathbf{A}\boldsymbol{x} \begin{smallmatrix}\le\\=\\>\end{smallmatrix} 0; \quad \text{for all } \boldsymbol{x} \neq \mathbf{0}}$$

Methods

Check the signs of

1. Elem. diag.: $\mathbf{D} = \mathbf{B}^\intercal \mathbf{A} \mathbf{B}$ (Diagonalization by congruence)

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(Orthogonal diagonalization $\mathbf{D} = \mathbf{Q}^\intercal\mathbf{A}\mathbf{Q}$ is a special case)

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Law of inertia

the number of positive, negative and zero entries of the diagonal of $\mathbf{D}$ is an invariant of $\mathbf{A}$, i.e. it does not depend on $\mathbf{B}$

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Methods

Check the signs of

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Questions of the Lecture 18

(L-18) Question 1. Decide for or against the positive definiteness of these matrices, and write out the corresponding quadratic form $f = \boldsymbol{x}\mathbf{A}\boldsymbol{x}$:

(a) $\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} -1 & 2 \\ 2 & -8 \end{bmatrix}$

(e) The determinant in (b) is zero; along what line is $f(x, y) = 0$?

(Strang, 2006, exercise 2 from section 6.1.)

(L-18) Question 2. What is the quadratic $f = ax^2 + 2bxy + cy^2$ for each of these matrices? Complete the square to write f as a sum of one or two squares $d_1(\quad)^2 + d_2(\quad)^2$.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$

(Strang, 2006, exercise 15 from section 6.1.)

(L-18) QUESTION 3. Which one of the following matrices has two positive eigenvalues? Test $a > 0$ and $ac > b^2$, don't compute the eigenvalues. $\boldsymbol{x}\mathbf{A}\boldsymbol{x} < 0$.

(a) $\mathbf{A} = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix}$

(c) $\mathbf{C} = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix}$

(d) $\mathbf{D} = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}$

(Strang, 2006, exercise 14 from section 6.1.)

(L-18) QUESTION 4. Show that $f(x, y) = x^2 + 4xy + 3y^2$ does not have a minimum at $(0, 0)$ even though it has positive coefficients. Write $f(x, y)$ as a difference of squares and find a point (x, y) where $f(x, y)$ is negative.
(Strang, 2006, exercise 16 from section 6.1.)

(L-18) QUESTION 5. Show from the eigenvalues that if $\mathbf{A}$ is positive definite, so is $\mathbf{A}^2$ and so is$\mathbf{A}^{-1}$.
(Strang, 2006, exercise 4 from section 6.2.)

(L-18) QUESTION 6. Consider the following quadratic forms

$$\begin{aligned} q_1(x,y,z) &= x^2+4y^2+5z^2-4xy. \\ q_2(x,y,z) &= -x^2+4y^2+z^2+2xy-2axz. \end{aligned}$$

(a) Show that $q_1(x,y,z)$ is positive semi-definite.
(b) Find, if it is possible, any value of a such as $q_2(x,y,z)$ is negative definite.

(L-18) QUESTION 7. Decide for or against the positive definiteness of

(a) $\mathbf{A} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

(c) $\mathbf{C} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}^2$

(Strang, 2006, exercise 2 from section 6.2.)

(L-18) Question 8. Consider the following quadratic form

$$q(x, y, z) = x^2 + 6xy + y^2 + az^2;$$

Decide for which values a the quadratic form is positive definite, negative definite, semidefinite, or indefinite.

(L-18) Question 9. Si $\mathbf{A} = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ es definida positiva, pruebe que $\mathbf{A}^{-1}$ es definida positiva.
(Strang, 2006, exercise 8 from section 6.1.)

(L-18) Question 10. Si una matriz simétrica de 2 por 2 satisface $a > 0$, y $ac > b^2$, demuestre que sus autovalores son reales y positivos (definida positiva). Emplee la ecuación característica y el hecho de que el producto de los autovalores es igual al determinante.
(Strang, 2006, exercise 3 from section 6.1.)

(L-18) QUESTION 11. Decida si las siguientes matrices son definidas positivas, definidas negativas, semi-definidas, o indefinidas.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 6 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 3 \end{bmatrix}$

(c) $\mathbf{C} = -\mathbf{B}$

(d) $\mathbf{D} = \mathbf{A}^{-1}$

(L-18) QUESTION 12. Una matriz definida positiva no puede tener un cero (o incluso peor; un número negativo) en su diagonal principal. Demuestre que esta matriz no cumple $\boldsymbol{x}\mathbf{A}\boldsymbol{x} > 0$, para todo $\boldsymbol{x} \neq \mathbf{0}$:

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{no es positiva cuando} \quad \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} = \begin{pmatrix} \quad & \quad & \quad \end{pmatrix}$$

(Strang, 2006, exercise 21 from section 6.2.)

(L-18) QUESTION 13. Demuestre que si **A** y **B** son definidas positivas entonces $\mathbf{A} + \mathbf{B}$ también es definida positiva. Para esta demostración los pivotes y los valores característicos no son convenientes. Es mejor emplear $\boldsymbol{x}(\mathbf{A} + \mathbf{B})\boldsymbol{x} > 0$
(Strang, 2006, exercise 5 from section 6.2.)

(L-18) QUESTION 14. Find the $\dot{\mathbf{L}}\mathbf{D}\dot{\mathbf{L}}^\intercal$ factorization for the following symmetric matrices.

(a)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

(L-18) QUESTION 15. La forma cuadrática $f(x, y) = 3(x + 2y)^2 + 4y^2$ es definida positiva. Encuentre la matriz **A**, factorícela en $\mathbf{L}\mathbf{D}\mathbf{L}^\intercal$, y relacione los elementos en **D** y **L** con 3, 2 y 4 en f.
(Strang, 2006, exercise 9 from section 6.1.)

(L-18) QUESTION 16. Consider the following matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & a & a \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) (0.5^{pts}) Compute the eigenvalues of **A**.
(b) (0.5^{pts}) Prove that when $a = 2$ the matrix **A** is not diagonalisable.
(c) (1^{pts}) For matrix **B**, find a diagonal matrix **D** and an orthonormal matrix **P** such as $\mathbf{B} = \mathbf{P}\mathbf{D}\mathbf{P}^{\intercal}$.
(d) (0.5^{pts}) Find the quadratic form $f(x, y, z)$ associated to **B**, and prove it is positive defined.

Versión de un ejercicio proporcionado por Mercedes Vazquez

(L-18) QUESTION 17. Given the matrix $\mathbf{A} = \begin{pmatrix} a & 3/5 \\ b & 4/5 \end{pmatrix}$, compute the values (if they exist) of a and b such as

(a) (0.5^{pts}) **A** is ortho-normal.
(b) (0.5^{pts}) Columns of **A** are linearly independent.
(c) (0.5^{pts}) $\lambda = 0$ is an eigenvalue of **A**.
(d) (0.5^{pts}) **A** is a symmetric definite negative matrix.

(L-18) Question 18.

(a) Consider the quadratic form $q(x, y, z) = x^2 + 2xy + ay^2 + 8z^2$ and find its corresponding symmetric matrix **Q**; determine if **Q** is positive-definite, positive-semidefinite, negative-definite, negative-semidefinite or indefinite when the parameter a is equal to one $(a = 1)$.

(b) If $a \neq 1$, determine whether the matrix is positive-definite, positive-semidefinite, negative-definite, negative-semidefinite or indefinite.

Questions of the Optional Lecture 2

(L-Opt-2) Question 1. Consider the following matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(a) (0.5^{pts}) Prove $\mathbf{A}$ is invertible if and only if $a \neq 0$.
(b) (0.5^{pts}) Is $\mathbf{A}$ positive definite when $a = 1$? Explain your answer.
(c) (1^{pts}) Compute $\mathbf{A}^{-1}$ when $a = 2$.
(d) (0.5^{pts}) How many variables can be chosen as pivot (or exogenous) variables in the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{o}$ when $a = 0$? Which ones?

(L-Opt-2) Question 2. True or false (to receive full credit you must explain your answers in a clear and concise way)

(a) If $\mathbf{A}$ is symetric, then so it is $\mathbf{A}^2$.
(b) If $\mathbf{A}^2 = \mathbf{A}$ then $(\mathbf{I} - \mathbf{A})^2 = (\mathbf{I} - \mathbf{A})$ where $\mathbf{I}$ is the identity matrix.
(c) If $\lambda = 0$ is an eigenvalue of the squared matrix $\mathbf{A}$, then the linear system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ is is always solvable and has only one solution.
(d) If $\lambda = 0$ is an eigenvalue of the squared matrix $\mathbf{A}$, then the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ could be unsolvable.
(e) If a matrix is orthogonal (perpendicular columns of norm one), then so it is the inverse of that matrix.

(f) If 1 is the only eigenvalue of a 2×2 matrix **A**, then **A** must be the identity matrix **I**.

(L-Opt-2) Question 3. complete los blancos, o responda **V**erdadero/**F**also.

(a) Cualquier sistema generador de un espacio vectorial contiene una base del espacio (V/F)

____________________.

(b) Que los vectores $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ sean linealmente independientes significa que

____

(c) El conjunto que sólo contiene el vector $\mathbf{0}$ es un conjunto linealmente independiente. (V/F)

____________________.

(d) Una matriz cuadrada de orden n por n es diagonalizable cuando:

____

(e) Si $\boldsymbol{u} = (1, 2, -1, 1)$, entonces $||\boldsymbol{u}|| =$ ________________.

(f) Si $\boldsymbol{u} = (1, 2, -1, 1)$ y $\boldsymbol{v} = (-2, 1, 0, 0)$, entonces $\boldsymbol{u} \cdot \boldsymbol{v} =$ ________________.

(L-Opt-2) Question 4. En las preguntas siguientes **A** y **B** son matrices $n \times n$. Indique si las siguientes afirmaciones son verdaderas o falsas (incluya una breve explicación, o un contra ejemplo que justifique su respuesta):

(a) Si **A** no es cero entonces $\det(\mathbf{A}) \neq 0$

(b) Si $\det(\mathbf{AB}) \neq 0$ entonces **A** es invertible.

(c) Si intercambio las dos primeras filas de **A** sus autovalores cambian.

(d) Si **A** es real y simétrica, entonces sus autovalores son reales **(aquí no es necesaria una justificación)**.

(e) Si la forma reducida de echelon de $(\mathbf{A} - 5\mathbf{I})$ es la matriz identidad, entonces 5 no es un autovalor de **A**.

(f) Sea $\boldsymbol{b}$ un vector columna de $\mathbb{R}^n$. Si el sistema $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ no tiene solución, entonces $\det(\mathbf{A}) \neq 0$

(g) Sea **C** de orden 3×5. El rango de **C** puede ser 4.

(h) Sea **C** de orden $n \times m$, y $\boldsymbol{b}$ un vector columna de $\mathbb{R}^n$. Si $\mathbf{C}\boldsymbol{x} = \boldsymbol{b}$ no tiene solución, entonces $\operatorname{rg}\left(\mathbf{C}\right) < n$.

(i) Toda matriz diagonalizable es invertible.

(j) Si **A** es invertible, entonces su forma reducida de echelon es la matriz identidad.

(L-Opt-2) Question 5. Sean

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & 5 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 0 & 4 \\ 0 & 0 & 6 \end{bmatrix}; \quad \mathbf{D} = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Los autovalores de **B** son 0 y 2. Use esta información para responder a las siguientes cuestiones. Para cada matriz debe dar una explicación. Puede haber más de una matriz que cumpla la condición:

(a) ¿Qué matrices son invertibles?
(b) ¿Qué matrices tienen un autovalor repetido?
(c) ¿Qué matrices tienen rango menor a tres?
(d) ¿Qué matrices son diagonalizables?
(e) ¿Para qué matrices diagonalizables podemos encontrar tres autovectores ortogonales entre si?

(L-Opt-2) Question 6. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

(a) Find the eigenvalues and eigenvectors of **A**.
(b) Is **A** diagonalizable?

(c) Is it possible to find a matrix $\mathbf{P}$ such as $\mathbf{A} = \mathbf{PDP}^\intercal$, where $\mathbf{D}$ is diagonal?
(d) Find $|\mathbf{A}^{-1}|$.

(L-OPT-2) QUESTION 7. Consider a 3 by 3 matrix $\mathbf{A}$ with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = -1$; and let $\boldsymbol{v}_1 = (1, 0, 1)^\intercal$ and $\boldsymbol{v}_2 = (1, 1, 1)^\intercal$ be the corresponding eigenvectors to λ_1 and λ_2.

(a) Is $\mathbf{A}$ diagonalizable?
(b) Is $\boldsymbol{v}_3 = (-1, 0, -1)^\intercal$ an eigenvector associated to the eigenvalue $\lambda_3 = -1$?
(c) Compute $\mathbf{A}(\boldsymbol{v}_1 - \boldsymbol{v}_2)$.

(L-OPT-2) QUESTION 8.

(a) (0.5^{pts}) Find an homogeneous system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ such as its solutions set is

$$\left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \exists \alpha, \beta, \gamma \in \mathbb{R} \quad \text{such that} \quad \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \right\}$$

(b) (0.5^{pts}) If the characteristic polynomial of a matrix $\mathbf{A}$ is $p(\lambda) = \lambda^5 + 3\lambda^4 - 24\lambda^3 + 28\lambda^2 - 3\lambda + 10$, find the rank of $\mathbf{A}$.

(L-OPT-2) QUESTION 9. Suponga una matriz cuadrada e *invertible* $\underset{n\times n}{\mathbf{A}}$.

(a) ¿Cuáles son sus espacios columna $\mathcal{C}\left(\mathbf{A}\right)$ y espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$? (no responda con la definición, diga qué conjunto de vectores compone cada espacio).

(b) Suponga que $\mathbf{A}$ puede ser factorizada en $\mathbf{A} = \mathbf{LU}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 7 & 3 & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{12} & u_{13} \\ 0 & 0 & u_{13} \end{bmatrix}$$

Describa el primer paso de eliminación en la reducción de $\mathbf{A}$ a $\mathbf{U}$. ¿porqué sabe que $\mathbf{U}$ es también una matriz invertible? ¿Cuanto vale el determinante de $\mathbf{A}$?

(c) Encuentre una matriz particular de dimensiones 3×3 e invertible $\mathbf{A}$ que no pueda ser factorizada en la forma $\mathbf{LU}$ (sin permutar previamente las filas). ¿Qué factorización es todavía posible en su ejemplo? (no es necesario que realice la factorización). ¿Cómo sabe que su matriz $\mathbf{A}$ es invertible?

Highlights of Lesson 19

- Mean
- Standard deviation and variance
- Ordinary Least Squares (OLS)

2 Restriction in statistics and probability

Norm of constant vector "one" is 1

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This fails using the dot product in $\mathbb{R}^m$ $(m > 1)$

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New scalar product in $\mathbb{R}^m$ for statistics

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_s = \frac{1}{m}(\boldsymbol{x} \cdot \boldsymbol{y})$$

(so: $\|\mathbf{1}\|^2 = \frac{1}{m}\Big(\mathbf{1} \cdot \mathbf{1}\Big) = 1$)

3 Mean

The mean $\mu_{\boldsymbol{y}}$ is the scalar product of $\boldsymbol{y}$ and $\mathbf{1}$

$$\mu_{\boldsymbol{y}} = \frac{1}{m}\left(\mathbf{1} \cdot \boldsymbol{y}\right), \qquad \text{so,} \quad \mu_{\boldsymbol{y}} = \frac{1}{m}\sum_{i=1}^{m} y_i$$

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The mean $\mu_{\boldsymbol{y}}$ is the *value* by which to multiply $\mathbf{1}$ to get the orthogonal projection of $\boldsymbol{y}$ onto $\mathcal{L}\big(\big[\mathbf{1};\big]\big)$

$\boldsymbol{\mu_y}$: projection of $\boldsymbol{y} \in \mathbb{R}^m$ onto the line $\mathcal{L}\big(\big[\mathbf{1};\big]\big) \subset \mathbb{R}^m$

$$\boxed{\boldsymbol{\mu_y} = \mathbf{1}\widehat{a}} \quad \text{and} \quad \boxed{(\boldsymbol{y} - \boldsymbol{\mu_y}) \perp \mathbf{1} \Rightarrow \tfrac{1}{m}(\boldsymbol{y} - \boldsymbol{\mu_y}) \cdot \mathbf{1} = 0}$$

3 Mean

The mean $\mu_{\boldsymbol{y}}$ is the scalar product of $\boldsymbol{y}$ and $\mathbf{1}$

$$\mu_{\boldsymbol{y}} = \frac{1}{m}\Big(\mathbf{1}\cdot\boldsymbol{y}\Big), \qquad \text{so,} \quad \mu_{\boldsymbol{y}} = \frac{1}{m}\sum_{i=1}^{m} y_i$$

The mean $\mu_{\boldsymbol{y}}$ is the *value* by which to multiply $\mathbf{1}$ to get the orthogonal projection of $\boldsymbol{y}$ onto $\mathcal{L}\big(\big[\mathbf{1};\big]\big)$

$\boldsymbol{\mu}_{\boldsymbol{y}}$: projection of $\boldsymbol{y} \in \mathbb{R}^m$ onto the line $\mathcal{L}\big(\big[\mathbf{1};\big]\big) \subset \mathbb{R}^m$

$$\boxed{\boldsymbol{\mu}_{\boldsymbol{y}} = \mathbf{1}\widehat{a}} \quad \text{and} \quad \boxed{(\boldsymbol{y}-\boldsymbol{\mu}_{\boldsymbol{y}}) \perp \mathbf{1} \;\Rightarrow\; \tfrac{1}{m}(\boldsymbol{y}-\boldsymbol{\mu}_{\boldsymbol{y}})\cdot\mathbf{1} = 0}$$

$$\frac{1}{m}(\boldsymbol{y}-\mathbf{1}\widehat{a})\cdot\mathbf{1} = 0 \;\Leftrightarrow\; \frac{1}{m}\Big(\boldsymbol{y}\cdot\mathbf{1}\Big) - \frac{1}{m}\Big(\mathbf{1}\cdot\mathbf{1}\Big)\widehat{a} = 0;$$

3 Mean

The mean μ_y is the scalar product of $\boldsymbol{y}$ and $\mathbf{1}$

$$\mu_{\boldsymbol{y}} = \frac{1}{m}\left(\mathbf{1}\cdot\boldsymbol{y}\right), \qquad \text{so,} \quad \mu_{\boldsymbol{y}} = \frac{1}{m}\sum_{i=1}^{m} y_i$$

The mean $\mu_{\boldsymbol{y}}$ is the *value* by which to multiply $\mathbf{1}$ to get the orthogonal projection of $\boldsymbol{y}$ onto $\mathcal{L}\left(\left[\mathbf{1};\right]\right)$

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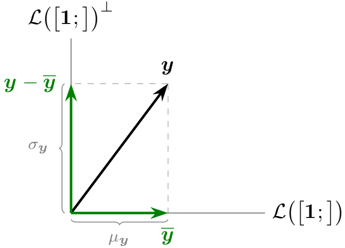
$$\boxed{\boldsymbol{\mu_y} = \mathbf{1}\widehat{a}} \quad \text{and} \quad \boxed{(\boldsymbol{y}-\boldsymbol{\mu_y}) \perp \mathbf{1} \;\Rightarrow\; \tfrac{1}{m}(\boldsymbol{y}-\boldsymbol{\mu_y})\cdot\mathbf{1} = 0}$$

$$\frac{1}{m}(\boldsymbol{y}-\mathbf{1}\widehat{a})\cdot\mathbf{1} = 0 \;\Leftrightarrow\; \frac{1}{m}\left(\boldsymbol{y}\cdot\mathbf{1}\right) - \frac{1}{m}\left(\mathbf{1}\cdot\mathbf{1}\right)\widehat{a} = 0;$$

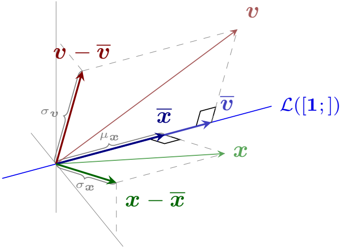
Therefore

$$\widehat{a} = \frac{1}{m}\left(\boldsymbol{y}\cdot\mathbf{1}\right) = \mu_{\boldsymbol{y}}$$

4 Mean



$\mathcal{L}([\mathbf{1};])^\perp$
$\boldsymbol{y}$
$\boldsymbol{y}-\overline{\boldsymbol{y}}$
$\sigma_{\boldsymbol{y}}$
$\mathcal{L}([\mathbf{1};])$
$\mu_{\boldsymbol{y}}$
$\overline{\boldsymbol{y}}$

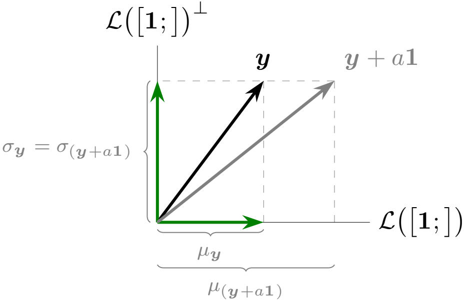


$\boldsymbol{v}$
$\boldsymbol{v}-\overline{\boldsymbol{v}}$
$\overline{\boldsymbol{v}}$
$\mathcal{L}([\mathbf{1};])$
$\overline{\boldsymbol{x}}$
$\sigma_{\boldsymbol{v}}$
$\mu_{\boldsymbol{x}}$
$\boldsymbol{x}$
$\sigma_{\boldsymbol{x}}$
$\boldsymbol{x}-\overline{\boldsymbol{x}}$

5 Standard deviation

$$\sigma_{\boldsymbol{y}} = \|\boldsymbol{y} - \boldsymbol{\mu}_{\boldsymbol{y}}\|.$$

Adding a constant vector $a\mathbf{1}$ to $\boldsymbol{y}$ does not change the standard deviation.

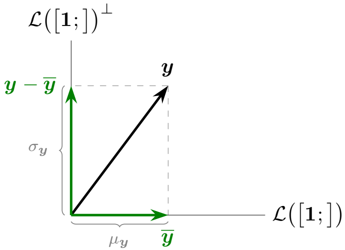


6 Variance and the Pythagorean theorem

$$\sigma_{\boldsymbol{y}}^2 = \|\boldsymbol{y} - \boldsymbol{\mu}_{\boldsymbol{y}}\|^2 = \frac{1}{m}(\boldsymbol{y} - \boldsymbol{\mu}_{\boldsymbol{y}}) \cdot (\boldsymbol{y} - \boldsymbol{\mu}_{\boldsymbol{y}}) = \frac{1}{m}\sum_i (y_i - \mu_{\boldsymbol{y}})^2.$$

6 Variance and the Pythagorean theorem

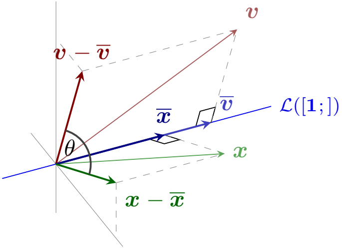
$$\sigma_{\boldsymbol{y}}^2 \;=\; \|\boldsymbol{y}-\boldsymbol{\mu_y}\|^2 \;=\; \frac{1}{m}(\boldsymbol{y}-\boldsymbol{\mu_y})\cdot(\boldsymbol{y}-\boldsymbol{\mu_y}) \;=\; \frac{1}{m}\sum_i (y_i-\mu_{\boldsymbol{y}})^2.$$

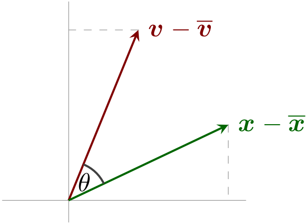


$$\sigma_{\boldsymbol{y}}^2 \;=\; \|\boldsymbol{y}-\boldsymbol{\mu_y}\|^2 \;=\; \|\boldsymbol{y}\|^2-\|\boldsymbol{\mu_y}\|^2 \;=\; \frac{1}{m}\Big(\boldsymbol{y}\cdot\boldsymbol{y}\Big)-\mu_{\boldsymbol{y}}^2, \;=\; \frac{\sum_i y_i^2}{m}-\mu_{\boldsymbol{y}}^2.$$

7 Covariance and correlation

$$\sigma_{\boldsymbol{x}\boldsymbol{y}} = \frac{1}{m}(\boldsymbol{x} - \boldsymbol{\mu}_{\boldsymbol{x}}) \cdot (\boldsymbol{y} - \boldsymbol{\mu}_{\boldsymbol{y}});$$





$$\rho_{\boldsymbol{x}\boldsymbol{y}} = \frac{\frac{1}{m}(\boldsymbol{x} - \boldsymbol{\mu}_{\boldsymbol{x}}) \cdot (\boldsymbol{y} - \boldsymbol{\mu}_{\boldsymbol{y}})}{\|(\boldsymbol{x} - \boldsymbol{\mu}_{\boldsymbol{x}})\| \cdot \|(\boldsymbol{y} - \boldsymbol{\mu}_{\boldsymbol{y}})\|} = \frac{\sigma_{\boldsymbol{x}\boldsymbol{y}}}{\sqrt{\sigma_{\boldsymbol{x}}\sigma_{\boldsymbol{y}}}} = \cos(\theta).$$

8 Ordinary Least Squares (OLS)

Let $\mathbf{X}$ suach that $\mathcal{L}\left(\left[\mathbf{1};\right]\right) \subset \mathcal{C}\left(\mathbf{X}\right)$.

$\widehat{\boldsymbol{y}}$ is the orthogonal projection of $\boldsymbol{y} \in \mathbb{R}^m$ onto $\mathcal{C}\left(\mathbf{X}\right)$

$\boxed{\widehat{\boldsymbol{y}} = \mathbf{X}\widehat{\boldsymbol{\beta}}}$ and $\boxed{(\boldsymbol{y} - \widehat{\boldsymbol{y}}) \perp \mathcal{C}\left(\mathbf{X}\right) \Rightarrow \frac{1}{m}\mathbf{X}^\mathsf{T}(\boldsymbol{y} - \widehat{\boldsymbol{y}}) = \mathbf{0}}$

8 Ordinary Least Squares (OLS)

Let $\mathbf{X}$ suach that $\mathcal{L}([\mathbf{1};]) \subset \mathcal{C}(\mathbf{X})$.

$\widehat{\boldsymbol{y}}$ is the orthogonal projection of $\boldsymbol{y} \in \mathbb{R}^m$ onto $\mathcal{C}(\mathbf{X})$

$$\boxed{\widehat{\boldsymbol{y}} = \mathbf{X}\widehat{\boldsymbol{\beta}}} \quad \text{and} \quad \boxed{(\boldsymbol{y} - \widehat{\boldsymbol{y}}) \perp \mathcal{C}(\mathbf{X}) \Rightarrow \tfrac{1}{m}\mathbf{X}^\intercal(\boldsymbol{y} - \widehat{\boldsymbol{y}}) = \mathbf{0}}$$

$$\frac{1}{m}\mathbf{X}^\intercal(\boldsymbol{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}) = \mathbf{0} \iff \frac{1}{m}\mathbf{X}^\intercal\boldsymbol{y} - \frac{1}{m}\mathbf{X}^\intercal\mathbf{X}\widehat{\boldsymbol{\beta}} = \mathbf{0}.$$

8 Ordinary Least Squares (OLS)

Let **X** suach that $\mathcal{L}([\mathbf{1};]) \subset \mathcal{C}(\mathbf{X})$.

$\widehat{\boldsymbol{y}}$ is the orthogonal projection of $\boldsymbol{y} \in \mathbb{R}^m$ onto $\mathcal{C}(\mathbf{X})$

$\boxed{\widehat{\boldsymbol{y}} = \mathbf{X}\widehat{\beta}}$ and $\boxed{(\boldsymbol{y} - \widehat{\boldsymbol{y}}) \perp \mathcal{C}(\mathbf{X}) \;\Rightarrow\; \frac{1}{m}\mathbf{X}^\intercal(\boldsymbol{y} - \widehat{\boldsymbol{y}}) = \mathbf{0}}$

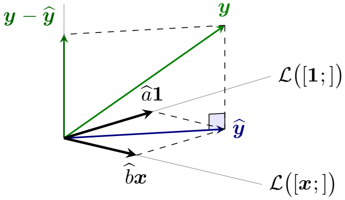
$$\frac{1}{m}\mathbf{X}^\intercal(y - \mathbf{X}\widehat{\beta}) = \mathbf{0} \iff \frac{1}{m}\mathbf{X}^\intercal y - \frac{1}{m}\mathbf{X}^\intercal\mathbf{X}\widehat{\beta} = \mathbf{0}.$$

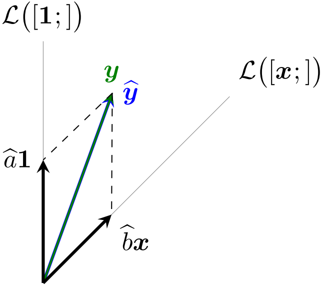
Therefore

$$\left(\frac{1}{m}\mathbf{X}^\intercal\mathbf{X}\right)\widehat{\boldsymbol{\beta}} = \frac{1}{m}\mathbf{X}^\intercal\boldsymbol{y}.$$

9 Ordinary Least Squares (OLS)

If $\mathbf{X} = \left[\mathbf{1};\, \boldsymbol{x};\right]$ has rank 2.



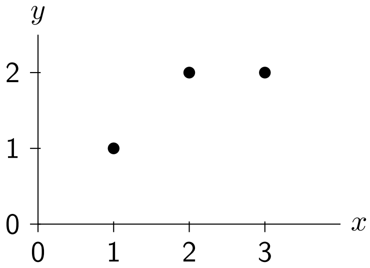


$$\left(\frac{1}{m}\mathbf{X}^\intercal\mathbf{X}\right)\begin{pmatrix}\widehat{a}\\ \widehat{b}\end{pmatrix} = \frac{1}{m}\mathbf{X}^\intercal\boldsymbol{y}.$$

10 Application: Least Squares (Fitting by a line)

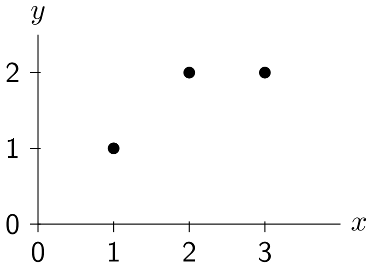
"looking for the best fitting line $\widehat{y} = \widehat{a} + \widehat{b}x$"

Points $(x, y,)$: $(1, 1,)$; $(2, 2,)$; $(3, 2,)$



10 Application: Least Squares (Fitting by a line)

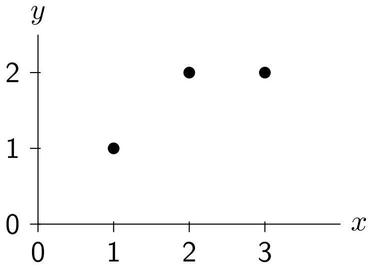
"looking for the best fitting line $\widehat{y} = \widehat{a} + \widehat{b}x$"
Points $(x, y,)$: $(1, 1,)$; $(2, 2,)$; $(3, 2,)$



$$\begin{cases} a + 1b & = 1 \\ a + 2b & = 2 \\ a + 3b & = 2 \end{cases}$$

10 Application: Least Squares (Fitting by a line)

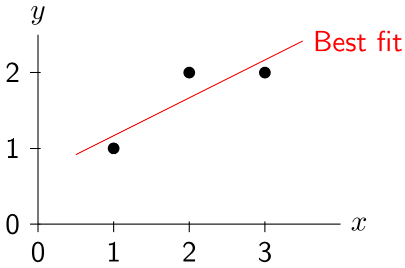
"looking for the best fitting line $\widehat{y} = \widehat{a} + \widehat{b}x$"
Points $(x, y,)$: $(1, 1,)$; $(2, 2,)$; $(3, 2,)$



$$\begin{cases} a + 1b &= 1 \\ a + 2b &= 2 \\ a + 3b &= 2 \end{cases} \quad \rightarrow \quad \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

10 Application: Least Squares (Fitting by a line)

"looking for the best fitting line $\widehat{y} = \widehat{a} + \widehat{b}x$"
Points $(x, y,)$: $(1, 1,)$; $(2, 2,)$; $(3, 2,)$



$$\begin{cases} a + 1b &= 1 \\ a + 2b &= 2 \\ a + 3b &= 2 \end{cases} \to \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$ ($\mathbf{X}\boldsymbol{\beta} = \boldsymbol{y}$ No solution)

11 Application: Least Squares (Fitting by a line)

$$\mathbf{X}\boldsymbol{\beta} = \boldsymbol{y} \quad \text{(No solution)} \;\rightarrow\; \left(\frac{1}{m}\mathbf{X}^\intercal\mathbf{X}\right)\widehat{\boldsymbol{\beta}} = \frac{1}{m}\mathbf{X}^\intercal\boldsymbol{y} \quad \rightarrow \quad \widehat{\boldsymbol{y}} = \mathbf{X}\widehat{\boldsymbol{\beta}}.$$

11 Application: Least Squares (Fitting by a line)

$$\mathbf{X}\beta = \boldsymbol{y} \quad \text{(No solution)} \;\rightarrow\; \left(\frac{1}{m}\mathbf{X}^\intercal\mathbf{X}\right)\widehat{\boldsymbol{\beta}} = \frac{1}{m}\mathbf{X}^\intercal\boldsymbol{y} \quad \rightarrow \quad \widehat{\boldsymbol{y}} = \mathbf{X}\widehat{\boldsymbol{\beta}}.$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} \widehat{a} \\ \widehat{b} \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

11 Application: Least Squares (Fitting by a line)

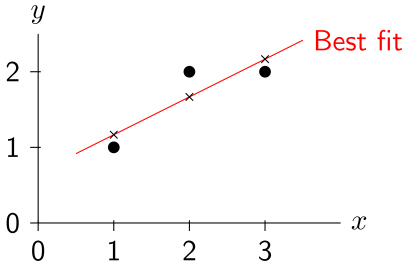
$$\mathbf{X}\boldsymbol{\beta} = \boldsymbol{y} \quad \text{(No solution)} \;\rightarrow\; \left(\frac{1}{m}\mathbf{X}^{\mathsf{T}}\mathbf{X}\right)\widehat{\boldsymbol{\beta}} = \frac{1}{m}\mathbf{X}^{\mathsf{T}}\boldsymbol{y} \quad \rightarrow \quad \widehat{\boldsymbol{y}} = \mathbf{X}\widehat{\boldsymbol{\beta}}.$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} \widehat{a} \\ \widehat{b} \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{pmatrix} \widehat{a} \\ \widehat{b} \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \end{pmatrix} \quad \Rightarrow \quad \widehat{a} = \frac{2}{3}; \quad \widehat{b} = \frac{1}{2}.$$

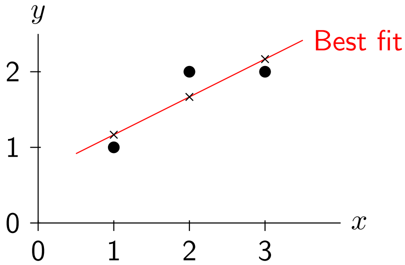
$$\boxed{\text{Best solution: } \tfrac{2}{3} + \tfrac{1}{2}x}$$

12 Application: Least Squares (Fitting by a line)



$$\widehat{\boldsymbol{y}} = \mathbf{X}\widehat{\boldsymbol{\beta}} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} \widehat{a} \\ \widehat{b} \end{pmatrix} \longrightarrow \widehat{\boldsymbol{y}} = \begin{pmatrix} 7/6 \\ 10/6 \\ 13/6 \end{pmatrix} \longrightarrow \widehat{\boldsymbol{e}} = \begin{pmatrix} -1/6 \\ 2/6 \\ -1/6 \end{pmatrix}$$

12 Application: Least Squares (Fitting by a line)



$$\widehat{\boldsymbol{y}} = \mathbf{X}\widehat{\boldsymbol{\beta}} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} \widehat{a} \\ \widehat{b} \end{pmatrix} \longrightarrow \widehat{\boldsymbol{y}} = \begin{pmatrix} 7/6 \\ 10/6 \\ 13/6 \end{pmatrix} \longrightarrow \widehat{\boldsymbol{e}} = \begin{pmatrix} -1/6 \\ 2/6 \\ -1/6 \end{pmatrix}$$

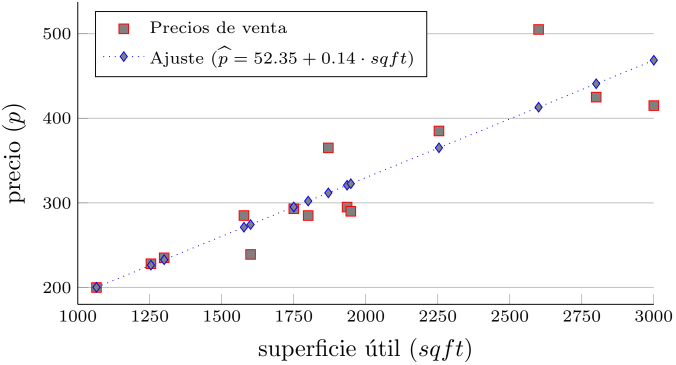
$$\boldsymbol{y} = \widehat{\boldsymbol{y}} + \widehat{\boldsymbol{e}} \quad \text{and} \quad \begin{cases} \widehat{\boldsymbol{e}} \cdot \widehat{\boldsymbol{y}} & = 0 \\ \widehat{\boldsymbol{e}}\mathbf{X} & = \mathbf{0} \end{cases}.$$

13 Application: Least Squares (Fitting by a line)

Selling price and living area of single family homes in University City community of San Diego, in 1990.

price = Sale price is in thousands of dollars

sqft = Square feet of living area (Ramanathan, 2002, pp. 78)



Questions of the Lecture 19

(L-19) QUESTION 1. With the measurements $\boldsymbol{y} = (0, 8, 8, 20,)$ at $\boldsymbol{x} = (0, 1, 3, 4,)$,

(a) Set up and solve the normal equations $\mathbf{A}^\intercal \mathbf{A}\widehat{\boldsymbol{\beta}} = \mathbf{A}^\intercal \boldsymbol{y}$.
(b) For the best straight line, find its four fits p_i and four errors e_i.
(c) What is the value of the square of the norm of the error vector $\|\boldsymbol{e}\|^2 = e_1^2 + e_2^2 + e_3^2 + e_4^2$?
(d) Draw the regression line
(e) Change the measurements to $\boldsymbol{p} = (1, 5, 13, 17,)$ write down the four equations $\mathbf{A}\boldsymbol{\beta} = \boldsymbol{p}$. Find an exact solution to $\mathbf{A}\boldsymbol{\beta} = \boldsymbol{p}$
(f) Check that $\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{p} = (-1, 3, -5, 3,)$ is perpendicular to both columns of the same matrix $\mathbf{A}$.
(g) What is the shortest distance $\|\boldsymbol{e}\|$ from $\boldsymbol{y}$ to the column space of $\mathbf{A}$?

(Strang, 2003, exercise 1–3 from section 4.3.)

(L-19) QUESTION 2.

(a) Write down three equations $y = \alpha + \beta x$ given the data: $y = 7$ at $x = -1$, $y = 7$ at $x = 1$, and $y = 21$ at $x = 2$. Find the least squares solution $\widehat{\boldsymbol{\beta}} = (\hat{\alpha}, \hat{\beta})$ and draw the closest line.
(b) Find the projection $\boldsymbol{p} = \mathbf{A}\widehat{\boldsymbol{\beta}}$. This gives the three heights of the closest line. Show that the error vector is $\boldsymbol{e} = (2, -6, 4,)$. Why is $\mathbf{P}\boldsymbol{e} = \mathbf{0}$?

(L-19) QUESTION 3. Our measurements at times $t = 1, 2, 3$ are $b = 1, 4$, and b_3. We want to fit those points by the nearest line $C + Dt$, using least squares.

(a) Which value for b_3 will put the three measurements on a straight line? *Which line is it?* Will least squares choose that line if the third measurement is $b_3 = 9$? (Yes or no).

(b) What is the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ that would be solved exactly for $\boldsymbol{x} = (C, D)$ if the three points do lie on a line? Compute the projection matrix **P** onto the column space of **A**.

(c) What is the rank of that projection matrix **P**? How is the column space of **P** related to the column space of **A**? (You can answer with or without the entries of **P** computed in (b).)

(d) Suppose $b_3 = 1$. Write down the equation for the best least squares solution $\widehat{\boldsymbol{x}}$, and show that the best straight line is horizontal.

MIT 18.06 - Quiz 2, November 2, 2005

Hefferon, J. (2008). *Linear Algebra*. Jim Hefferon, Colchester, Vermont USA. This text is Free.
URL
ftp://joshua.smcvt.edu/pub/hefferon/book/book.pdf

Lang, S. (1986). *Introduction to Linear Algebra*. Springer-Verlag, second ed.

Poole, D. (2004). *Álgebra lineal. Una introducción moderna*. Thomson Learning, Mexico D.F. ISBN 970-686-272-2.

Ramanathan, R. (2002). *Introductory Econometrics with applications*. South-Western, Mason, Ohio, fifth ed. ISBN 0-03-034186-8.

Strang, G. (1988). *Linear algebra and its applications*. Thomson Learning, Inc., third ed. ISBN 0-15-551005-3.

Strang, G. (2003). *Introduction to Linear Algebra*. Wellesley-Cambridge Press, Wellesley, Massachusetts. USA, third ed. ISBN 0-9614088-9-8.

Strang, G. (2006). *Linear algebra and its applications*. Thomson Learning, Inc., fourth ed. ISBN 0-03-010567-6.

(L-1) Question 1(a)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
A = Matrix([ [1,0,0],[0,2,0],[0,0,1] ])
display(A)
A.es_diagonal()
```

(L-1) Question 1(b)

$$\begin{bmatrix} 1 & -1 & 3 \\ -1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

```
A = Matrix([ [1,-1,3],[-1,2,0],[3,0,1] ])
display(A)
A == ~A
```

(L-1) Question 1(c)

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
A = Matrix([ [1,-1,3],[0,2,0],[0,0,1] ])
display(A)
A.es_triangularSup()
```

□

(L-1) Question 1(d)

$$\begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix}$$

```
Matrix([ [0,1,3],[-1,0,0],[-3,0,0] ])
```

□

(L-2) Question 1(a)

$$\begin{bmatrix} 4 & 1 \\ 4 & 1 \\ 6 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix} 1 + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} 3 = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ 9 \end{pmatrix}$$

```
a = Vector( [4, 4, 6] );  b = Vector( [1, 1, 1] )
A = Matrix( [ a, b ] )
x = Vector( [ 1, 3 ] )
```

```
display( A*x )
display( (A|1)*(x|1) + (A|2)*(x|2) )
display(     a *(x|1) +     b *(x|2) )
```

□

(L-2) Question 1(b)

$$\begin{pmatrix}1\\4\\7\end{pmatrix}0+\begin{pmatrix}2\\5\\8\end{pmatrix}1+\begin{pmatrix}3\\6\\9\end{pmatrix}0=\begin{pmatrix}0\\0\\0\end{pmatrix}+\begin{pmatrix}2\\5\\8\end{pmatrix}+\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}2\\5\\8\end{pmatrix}$$

□

(L-2) Question 1(c)

$$\begin{pmatrix}4\\6\\8\end{pmatrix}\frac{1}{2}+\begin{pmatrix}3\\6\\9\end{pmatrix}\frac{1}{3}=\begin{pmatrix}2\\3\\4\end{pmatrix}+\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}3\\5\\7\end{pmatrix}$$

□

(L-2) Question 2. For the right hand vector $\boldsymbol{b}=\begin{pmatrix}2\\2\\1\end{pmatrix}$ the system has no solution.

When $\boldsymbol{b} = 0\mathbf{A}_{|1} + 0\mathbf{A}_{|2} = \begin{pmatrix}0\\0\\0\end{pmatrix}$. We have lots of posibilities...

If $\boldsymbol{b} = 2\mathbf{A}_{|1} + 0\mathbf{A}_{|2} = \begin{pmatrix}2\\2\\0\end{pmatrix}$ then $x = 2$ and $y = 0$. If $\boldsymbol{b} = 0\mathbf{A}_{|1} + 1\mathbf{A}_{|2} = \begin{pmatrix}2\\-1\\1\end{pmatrix}$ then $x = 0$ and $y = 1$. If $\boldsymbol{b} = 3\mathbf{A}_{|1} + 1\mathbf{A}_{|2} = \begin{pmatrix}5\\2\\1\end{pmatrix}$ then $x = 3$ and $y = 1$.

In fact, there are infinite possibilities!... Choose values for x and y and compute the vector

$$x\begin{pmatrix}1\\1\\0\end{pmatrix} + y\begin{pmatrix}2\\-1\\1\end{pmatrix} = \boldsymbol{b};$$

with that $\boldsymbol{b}$ you will get a linear system with solution at point $(x, y,)$.

```
c  = sympy.symbols('c')  # usemos variables simbólicas
A  = Matrix( [ [3,-6,0],[0,2,-2],[1,-1,-1] ] )
x  = Vector( [2,1,1] )
y  = c*x
display(y)
display( (2*c)*(A|1) + (c*A|2) + (c*A|3) )
Sistema( [ A*x , A*y ] )
```

(L-2) Question 3.

$$\begin{bmatrix} 3 & -6 & 0 \\ 0 & 2 & -2 \\ 1 & -1 & -1 \end{bmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Any vector $\boldsymbol{x} = \begin{pmatrix} 2c \\ c \\ c \end{pmatrix}$ for any value of c is a solution of $\mathbf{A}\boldsymbol{x} = \mathbf{0}$. Therefore, there are infinite solutions.

□

(L-2) Question 4. Since $\boldsymbol{v}$ are $\boldsymbol{w}$ solutions, we know that $\mathbf{A}\boldsymbol{v} = \boldsymbol{b}$ and $\mathbf{A}\boldsymbol{w} = \boldsymbol{b}$; then

$$\begin{aligned} \mathbf{A}\left(\tfrac{1}{2}(\boldsymbol{v} + \boldsymbol{w})\right) &= \tfrac{1}{2}\mathbf{A}(\boldsymbol{v} + \boldsymbol{w}) \\ &= \tfrac{1}{2}(\mathbf{A}\boldsymbol{v} + \mathbf{A}\boldsymbol{w}) \\ &= \tfrac{1}{2}(\boldsymbol{b} + \boldsymbol{b}) && \text{because } \boldsymbol{v} \text{ and } \boldsymbol{w} \text{ are solutions} \\ &= \tfrac{1}{2}(2\boldsymbol{b}) \quad = \quad \boldsymbol{b}. \end{aligned}$$

Therefore $\frac{1}{2}(\boldsymbol{v} + \boldsymbol{w})$ is a solution of $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.
Nevertheless, if we do not divide by 2, then we have that $\boldsymbol{v} + \boldsymbol{w}$ is solution of $\mathbf{A}\boldsymbol{x} = 2\boldsymbol{b}$; a completely different system since $\boldsymbol{b} \neq 2\boldsymbol{b}$.

□

(L-2) Question 5(a) If $\mathbf{A}\boldsymbol{v} = \boldsymbol{b}$ and $\mathbf{A}\boldsymbol{w} = \boldsymbol{b}$ then, if $c + d \neq 0$

$$\begin{aligned} \mathbf{A}(c\boldsymbol{v}) =& c\boldsymbol{b} \\ \mathbf{A}(d\boldsymbol{w}) =& d\boldsymbol{b} \end{aligned}$$

adding the two equations

$$\begin{aligned} \mathbf{A}(c\boldsymbol{v}) + \mathbf{A}(d\boldsymbol{w}) =& (c+d)\boldsymbol{b} \\ \mathbf{A}(c\boldsymbol{v} + d\boldsymbol{w}) =& (c+d)\boldsymbol{b} \\ \mathbf{A}\left(\frac{c\boldsymbol{v} + d\boldsymbol{w}}{c+d}\right) =& \boldsymbol{b}; \end{aligned}$$

and therefore any combination of solutions $\frac{c\boldsymbol{v}+d\boldsymbol{w}}{c+d}$ is also a solution. For example, for $c = 3$ and $d = -2$

$$\mathbf{A}\left(\frac{3\boldsymbol{v} - 2\boldsymbol{w}}{1}\right) = \mathbf{A}3\boldsymbol{v} - \mathbf{A}2\boldsymbol{w} = 3\boldsymbol{b} - 2\boldsymbol{b} = \boldsymbol{b}$$

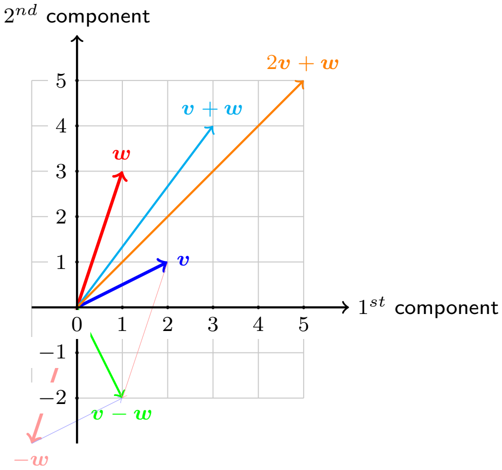
and for $c = 1$ and $d = 3$

$$\mathbf{A}\left(\frac{\boldsymbol{v} + 3\boldsymbol{w}}{4}\right) = \mathbf{A}\boldsymbol{v}\frac{1}{4} + \mathbf{A}\boldsymbol{w}\frac{3}{4} = \boldsymbol{b}$$

The EXERCISE 4 on page 110 is an example where $c = 1$ and $d = 1$.

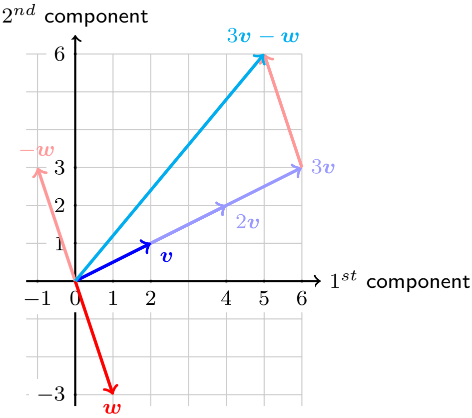
□

(L-2) Question 6. $\boldsymbol{v} + \boldsymbol{w} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$; $2\boldsymbol{v} + \boldsymbol{w} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$; $\boldsymbol{v} - \boldsymbol{w} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.



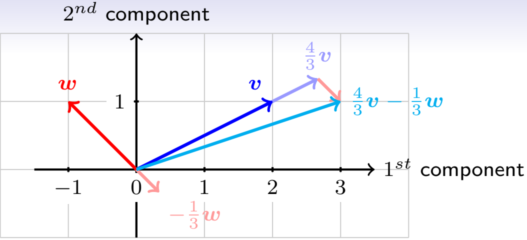
□

(L-2) Question 7.



□

(L-2) Question 8. $x = 1 + \frac{1}{3}; \quad y = -\frac{1}{3}$



$\square$

(L-3) Question 1. Since: $(\mathbf{EF})_{|1} = \mathbf{E}(\mathbf{F}_{|1}) = (\mathbf{E}_{|1})f_{11} + (\mathbf{E}_{|2})f_{21} + (\mathbf{E}_{|3})f_{31} =$

$$\begin{pmatrix}1\\a\\b\end{pmatrix}1 + \begin{pmatrix}0\\1\\0\end{pmatrix}0 + \begin{pmatrix}0\\0\\1\end{pmatrix}0 = \begin{pmatrix}1\\a\\b\end{pmatrix};$$

$$(\mathbf{EF})_{|2} = \mathbf{E}(\mathbf{F}_{|2}) = \begin{pmatrix}1\\a\\b\end{pmatrix}0 + \begin{pmatrix}0\\1\\0\end{pmatrix}1 + \begin{pmatrix}0\\0\\1\end{pmatrix}c = \begin{pmatrix}0\\1\\c\end{pmatrix};$$

$$(\mathbf{EF})_{|3} = \mathbf{E}(\mathbf{F}_{|3}) = \begin{pmatrix}1\\a\\b\end{pmatrix}0 + \begin{pmatrix}0\\1\\0\end{pmatrix}0 + \begin{pmatrix}0\\0\\1\end{pmatrix}1 = \begin{pmatrix}0\\0\\1\end{pmatrix};$$

then $\mathbf{EF} = \begin{bmatrix}1 & 0 & 0\\ a & 1 & 0\\ b & c & 1\end{bmatrix};$

Similarly: $\mathbf{FE} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ ac+b & c & 1 \end{bmatrix}, \qquad \mathbf{E}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 2b & 0 & 1 \end{bmatrix}.$

```
a, b, c = sympy.symbols('a b c')
E = Matrix([[1,0,0],[a,1,0],[b,0,1]])
F = Matrix([[1,0,0],[0,1,0],[0,c,1]])
display( E * F )
display( F * E )
display( E * E )
```

□

(L-3) Question 2(a) True. If $\mathbf{B}_{|1} = \mathbf{B}_{|3}$ then $(\mathbf{AB})_{|1} = \mathbf{A}(\mathbf{B}_{|1}) = \mathbf{A}(\mathbf{B}_{|3}) = (\mathbf{AB})_{|3}$.

□

(L-3) Question 2(b) False.

$$\mathbf{AB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -10 & -10 & -10 \end{bmatrix}..$$

□

(L-3) Question 2(c) True. If ${}_{1|}\mathbf{A} = {}_{3|}\mathbf{A}$ then ${}_{1|}(\mathbf{AB}) = ({}_{1|}\mathbf{A})\mathbf{B} = ({}_{3|}\mathbf{A})\mathbf{B} = {}_{3|}(\mathbf{AB})$.

□

(L-3) Question 2(d) False. $(\mathbf{AB})^2 = \mathbf{ABAB}$ and $\mathbf{A}^2\mathbf{B}^2 = \mathbf{AABB}$. For example if

$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -10 \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$; then

$$(\mathbf{AB})^2 = \begin{bmatrix} -9 & -9 & -9 \\ 0 & 0 & 0 \\ 90 & 90 & 90 \end{bmatrix} \neq \mathbf{A}^2\mathbf{B}^2 = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 200 & 0 & 200 \end{bmatrix}.$$

□

(L-3) Question 3(a) $\boldsymbol{a} \cdot \boldsymbol{a} = 54.$

In the topic about orthogonality we will see that $\boldsymbol{a} \cdot \boldsymbol{a}$ is the squared of the length of the vector. Therefore the length of $\boldsymbol{a}$ is $\sqrt{54} = 3\sqrt{6}$

□

(L-3) Question 3(b) $\boldsymbol{a} \cdot \boldsymbol{b} = 0.$

In the topic about orthogonality we will see that when $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ we say that $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal vectors.

□

(L-3) Question 3(c) $\left[\boldsymbol{a}\right]\left[\boldsymbol{b}\right]^\intercal = \begin{bmatrix} 3 & 5 & 1 \\ -6 & -10 & -2 \\ 21 & 35 & 7 \end{bmatrix}$

```
a = Vector([1, -2, 7])
b = Vector([3,  5, 1])
display(a*a)
display(a*b)
```

```
display(Matrix([a])* ~Matrix([b]))
```

□

(L-3) Question 4.

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \quad \text{y} \quad \mathbf{B} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \qquad \mathbf{AB} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}; \qquad \mathbf{BA} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}.$$

```
A = Matrix( [[ i+j        for i in range(1,3)] for j in range(1,3) ] )
B = Matrix( [[(-1)**(i+j) for i in range(1,3)] for j in range(1,3) ] )
Sistema( [A, B, A*B, B*A])
```

□

(L-3) Question 5. For example:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}; \quad \mathbf{AB} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}.$$

It is lower triangular since the first row of **A** is ${}_{1|}\mathbf{A} = (a_{11}, 0, 0)$ and the first row of **B** is ${}_{1|}\mathbf{B} = (b_{11}, 0, 0)$ and , and therefore first row of **AB** is

$$\left({}_{1|}\mathbf{A}\right)\mathbf{B} = \begin{pmatrix} a_{11}, & 0, & 0 \end{pmatrix}\mathbf{B} = a_{11}\begin{pmatrix} 1, & 0, & 0 \end{pmatrix}\mathbf{B} = a_{11}\left({}_{1|}\mathbf{B}\right) = \begin{pmatrix} a_{11}b_{11}, & 0, & 0 \end{pmatrix};$$

and the third column of **A** is $\begin{pmatrix} 0 \\ 0 \\ a_{33} \end{pmatrix}$ and the third column of **B** is $\begin{pmatrix} 0 \\ 0 \\ b_{33} \end{pmatrix}$; and therefore the third column of **AB** is

$$\mathbf{A}\left(\mathbf{B}_{|3}\right) = \mathbf{A}\begin{pmatrix} 0 \\ 0 \\ b_{33} \end{pmatrix} = \mathbf{A}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} b_{33} = (\mathbf{A}_{|3})b_{33} = \begin{pmatrix} 0 \\ 0 \\ a_{33}b_{33} \end{pmatrix}.$$

```
a11, a21, a31, a22, a32, a33 = sympy.symbols('a11 a21 a31 a22 a32 a33')
b11, b21, b31, b22, b32, b33 = sympy.symbols('b11 b21 b31 b22 b32 b33')
A = Matrix([ [a11,0,0], [a21,a22,0], [a31,a32,a33] ])
B = Matrix([ [b11,0,0], [b21,b22,0], [b31,b32,b33] ])
A*B
```

□

(L-3) Question 6(a) $\begin{bmatrix} 1 & 4 & -1 \\ 2 & 0 & 3 \end{bmatrix}$

□

(L-3) Question 6(b) $\begin{bmatrix} 3 & 8 \\ 4 & 3 \end{bmatrix}$

□

(L-3) Question 6(c) $\begin{bmatrix} 14 & -1 \\ 8 & -1 \\ -1 & -1 \end{bmatrix}$

□

(L-3) Question 6(d) $\begin{bmatrix} -1 & 4 \\ 10 & -5 \end{bmatrix}$

□

(L-3) Question 6(e) $\begin{bmatrix} 1 & 2 & -2 \\ 2 & 3 & 0 \\ 3 & 7 & -10 \end{bmatrix}$

□

(L-3) Question 6(f) $\begin{bmatrix} -2 & 27 & 15 \\ -2 & 15 & 9 \\ -2 & -3 & 0 \end{bmatrix}$

□

(L-3) Question 6(g) $\begin{bmatrix} 10 & -4 \\ 5 & -1 \end{bmatrix}$

□

(L-3) Question 6(h) $\begin{bmatrix} 2 & 4 \\ 1 & 7 \end{bmatrix}$

□

(L-3) Question 6(i) $\begin{bmatrix} 10 & -4 \\ 25 & -9 \\ 25 & -11 \end{bmatrix}$

□

(L-4) Question 1(a)

$$\begin{bmatrix} 1 & 4 & -2 \\ 1 & 6 & 2 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[{[(2)\mathbf{1}+\mathbf{3}]}]{\overset{\boldsymbol{\tau}}{[(-4)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{\overset{\boldsymbol{\tau}}{[(-2)\mathbf{2}+\mathbf{3}]}} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & -2 \end{bmatrix} = \mathbf{L}$$

so

$$\mathbf{I}_{\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}} = \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \mathbf{I}_{\underset{[(2)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \mathbf{I}_{\underset{[(-2)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

```
A = Matrix([ [1, 4,-2], [1, 6, 2], [0, 1, 0] ])
L = Elim(A,1)
Tr = L.TrC
display(Tr)
Sistema([ Sistema([t, I(3)&t]) for t in Tr])
```

□

(L-4) Question 1(b)

$$\mathbf{E} = \mathbf{I}_{\underset{\substack{[(-4)\mathbf{1}+\mathbf{2}] \\ [(2)\mathbf{1}+\mathbf{3}] \\ [(-2)\mathbf{2}+\mathbf{3}]}}{\boldsymbol{\tau}}} = \left(\mathbf{I}_{\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}\right)\left(\mathbf{I}_{\underset{[(2)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}}\right)\left(\mathbf{I}_{\underset{[(-2)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}}\right) = \begin{bmatrix} 1 & -4 & 10 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix};$$

$$\mathbf{AE} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & -2 \end{bmatrix} = \mathbf{K}.$$

```
ME = Sistema( [ I(3) & t for t in Tr])
E  = (ME|1)*(ME|2)*(ME|3)
display(E)
display(A*E)
A & Tr
```

□

(L-4) Question 2. Substracting 2 times first column from second and 4 times from third, and then adding 2 times the second to the third

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ -1 & -3 & -2 \\ 0 & 1 & c \end{bmatrix} \xrightarrow[{[(-4)\mathbf{1}+\mathbf{3}]}]{\overset{\tau}{[(-2)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 2 \\ 0 & 1 & c \end{bmatrix} \xrightarrow{\overset{\tau}{[(2)\mathbf{2}+\mathbf{3}]}} \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 1 & c+2 \end{bmatrix}$$

We will find only two pivots if the last column of **L** becomes a zero column, and this happens when $c = -2$.

```
c = sympy.symbols('c')
Elim( Matrix([[1,2,4],[-1,-3,-2],[0,1,c]]) )
```

□

(L-4) Question 3(a) $\mathbf{E} = \left(\mathbf{I}_{\overset{\tau}{[(-1)\mathbf{1}+\mathbf{2}]}}\right)\left(\mathbf{I}_{\overset{\tau}{[\mathbf{2}\rightleftharpoons\mathbf{3}]}}\right) =$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \mathbf{I}_{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{2}] \\ [\mathbf{2}\rightleftharpoons\mathbf{3}]}}$$

□

(L-4) Question 3(b) $\mathbf{N} = \left(\mathbf{I}_{\substack{\boldsymbol{\tau} \\ [\mathbf{2}\rightleftharpoons\mathbf{3}]}}\right)\left(\mathbf{I}_{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{3}]}}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} =$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \mathbf{I}_{\substack{\boldsymbol{\tau} \\ [\mathbf{2}\rightleftharpoons\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{3}]}}$$

the **E** and **N** are the same because the transformations are equivalent: in the first case we subtract column 1 from column 2, and then we put the result in the third column; in the second case we first exchange column 2 and 3, and then we operate on the third column.

□

(L-4) Question 4.

$$\begin{bmatrix} 2 & 2 & 0 \\ 1 & 4 & 9 \\ 1 & 3 & 9 \end{bmatrix} \xrightarrow[]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 9 \\ 1 & 2 & 9 \end{bmatrix} \xrightarrow[]{\substack{\boldsymbol{\tau} \\ [(-3)\mathbf{2}+\mathbf{3}]}} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

Therefore

$$\mathbf{I}_{\substack{\tau\\{[(-1)\mathbf{1}+\mathbf{2}]}\\{[(-3)\mathbf{2}+\mathbf{3}]}}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E}.$$

```
A      = Matrix([[2,2,0],[1,4,9],[1,3,9]])
TrfCol = Elim(A).TrC
display(TrfCol)
I(3) & TrfCol
```

□

(L-4) Question 5. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\substack{\tau\\{[(1)\mathbf{1}+\mathbf{2}]}}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{\substack{\tau\\{[(-1)\mathbf{2}+\mathbf{1}]}}} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \xrightarrow{\substack{\tau\\{[(1)\mathbf{1}+\mathbf{2}]}}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \xrightarrow{\substack{\tau\\{[(-1)\mathbf{1}]}}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

```
pasos = [ T((1,1,2)), T((-1,2,1)), T((1,1,2)),  T((-1,1))]
Math(rprElim(I(2),[[],pasos]))
# I(2) & T((1,1,2)) & T((-1,2,1)) & T((1,1,2)) & T((-1,1))
```

□

(L-4) Question 6(a)

$$\mathbf{I}_{\overset{\boldsymbol{\tau}}{[(-5)\mathbf{1}+\mathbf{2}]}} = \begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
display( T((-5,1,2)) )
I(3) & T((-5,1,2))
```

□

(L-4) Question 6(b)

$$\mathbf{I}_{\overset{\boldsymbol{\tau}}{[(-7)\mathbf{2}+\mathbf{3}]}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix}$$

```
display( T((-7,2,3)) )
I(3) & T((-7,2,3))
```

□

(L-4) Question 6(c)

$$\mathbf{I}_{\overset{\boldsymbol{\tau}}{[\mathbf{1}\rightleftharpoons\mathbf{2}]}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \qquad \mathbf{I}_{\overset{\boldsymbol{\tau}}{[\mathbf{2}\rightleftharpoons\mathbf{3}]}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The matrix that does all the exchanges at once is

$$\mathbf{I}_{\underset{[\mathbf{1}\rightleftharpoons\mathbf{2}][\mathbf{2}\rightleftharpoons\mathbf{3}]}{\tau}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}}$$

El producto de matrices intercambio es siempre una matriz cuyas columnas son como las de la matriz identidad, pero en general reordenadas en una disposición distinta; a dichas matrices la llamamos *matrices permutación* y las denotamos con: $\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}}$.

```
display( T([ {1,2}, {2,3} ]) )
I(3) & T([ {1,2}, {2,3} ])
```

□

(L-4) Question 7(a)

$$\mathbf{A}_{\underset{[(-5)\mathbf{1}+\mathbf{2}][(-7)\mathbf{2}+\mathbf{3}]}{\tau}} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -5 & 35 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 35 \end{bmatrix}.$$

```
TrfC = T([ (-5,1,2), (-7,2,3) ])
display( I(3) & TrfC )
Matrix([ [1,0,0]  ]) & TrfC
```

□

(L-4) Question 7(b)

$$\mathbf{A}_{[(-7)\overset{\tau}{\mathbf{1}}+\mathbf{2}][(-5)\overset{\tau}{\mathbf{2}}+\mathbf{3}]} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & -7\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & -5 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & -5 & 0\\0 & 1 & -7\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -5 & 0\end{bmatrix}.$$

```
TrfC = T([ (-7,2,3), (-5,1,2) ])
display( I(3) & TrfC )
Matrix([ [1,0,0]  ]) & TrfC
```

□

(L-4) Question 7(c) Then column 3 feels no effect from column 1.

□

(L-4) Question 8. On the one hand

$(1, \quad 0,)\,\mathbf{M} = (0, \quad 1,) \qquad \Rightarrow {}_{1|}\mathbf{M} = (0, \quad 1,)$; on the other

$(0, \quad 1,)\,\mathbf{M} = (1, \quad 0,) \qquad \Rightarrow {}_{2|}\mathbf{M} = (1, \quad 0,)$. Hence, the matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

□

(L-4) Question 9. The product $\mathbf{A}\left(\mathbf{I}_{\overset{\boldsymbol{\tau}}{[i \rightleftharpoons j]}}\right)$ exchange the columns of $\mathbf{A}$, but $\left({}_{\overset{\boldsymbol{\tau}}{[i \rightleftharpoons j]}}\mathbf{I}\right)\mathbf{A}$ exchange the rows. For example:

$$\left({}_{\overset{\boldsymbol{\tau}}{[\mathbf{1} \rightleftharpoons \mathbf{2}]}}\mathbf{I}\right)\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}.$$

$$\mathbf{A}\left(\mathbf{I}_{\overset{\boldsymbol{\tau}}{[\mathbf{1} \rightleftharpoons \mathbf{2}]}}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}.$$

Note that $\left(\mathbf{I}_{\overset{\boldsymbol{\tau}}{[i \rightleftharpoons j]}}\right)$ is symmetric, so that $\left(\mathbf{I}_{\overset{\boldsymbol{\tau}}{[i \rightleftharpoons j]}}\right) = \left({}_{\overset{\boldsymbol{\tau}}{[i \rightleftharpoons j]}}\mathbf{I}\right)$.

```
a, b, c, d  = sympy.symbols('a b c d')
A = Matrix([ [a,b], [c,d] ])
display( (T({1,2}) & I(2)) * A )
display( A * (I(2) & T({1,2})) )
(I(2) & T({1,2})) == (T({1,2}) & I(2))
```

□

(L-4) Question 10. For example, the first column is $(a, \quad a, \quad a,)$, the second is $(b, \quad b, \quad b,)$ and the third is $(c, \quad c, \quad c,)$; and where $a \neq 0$, then

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} a \\ a \\ a \end{pmatrix} + x_2 \begin{pmatrix} b \\ b \\ b \end{pmatrix} + x_3 \begin{pmatrix} c \\ c \\ c \end{pmatrix}$$
$$= ax_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + bx_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + cx_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = (ax_1 + bx_2 + cx_3) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

By Gaussian elimination we get

$$\begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix} \xrightarrow{\left[\left(-\frac{b}{a}\right)\mathbf{1}+\mathbf{2}\right]^{\boldsymbol{\tau}}} \begin{bmatrix} a & 0 & c \\ a & 0 & c \\ a & 0 & c \end{bmatrix} \xrightarrow{\left[\left(-\frac{c}{a}\right)\mathbf{1}+\mathbf{2}\right]^{\boldsymbol{\tau}}} \begin{bmatrix} a & 0 & 0 \\ a & 0 & 0 \\ a & 0 & 0 \end{bmatrix};$$

only one pivot.

□

(L-5) Question 1(a)

$$\left[\frac{\mathbf{A}_1}{\mathbf{I}}\right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{[(-1)\overset{\tau}{\mathbf{2}}+\mathbf{3}]} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{[(-1)\overset{\tau}{\mathbf{2}}+\mathbf{1}]} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right].$$

Therefore $\mathbf{A}_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

```
A1 = Matrix([[1,0,0],[1,1,1],[0,0,1]])
InvMat(A1, 1)
```

□

(L-5) Question 1(b)

$$\left[\begin{array}{c}\mathbf{A}_2 \\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{ccc} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{2}+\mathbf{1}]}]{\boldsymbol{\tau} \atop [(2)\mathbf{2}]} \left[\begin{array}{ccc} 2 & 0 & 0 \\ -1 & 3 & -1 \\ 0 & -2 & 2 \\ \hline 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau} \atop [(3)\mathbf{3}]} \left[\begin{array}{ccc} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 0 & -2 & 4 \\ \hline 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{array}\right] \xrightarrow[{[(1)\mathbf{3}+\mathbf{2}]}]{\boldsymbol{\tau} \atop [(2)\mathbf{2}]}$$

$$\left[\begin{array}{ccc} 2 & 0 & 0 \\ -1 & 6 & 0 \\ 0 & 0 & 4 \\ \hline 1 & 3 & 1 \\ 0 & 6 & 2 \\ 0 & 3 & 3 \end{array}\right] \xrightarrow[{[(1)\mathbf{2}+\mathbf{1}]}]{\boldsymbol{\tau} \atop [(6)\mathbf{1}]} \left[\begin{array}{ccc} 12 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \\ \hline 9 & 3 & 1 \\ 6 & 6 & 2 \\ 3 & 3 & 3 \end{array}\right] \xrightarrow[{\left[\left(\frac{2}{3}\right)\mathbf{2}\right]}]{\boldsymbol{\tau} \atop \left[\left(\frac{1}{3}\right)\mathbf{1}\right]} \left[\begin{array}{ccc} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \\ \hline 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{array}\right] \xrightarrow[{\left[\left(\frac{1}{4}\right)\mathbf{3}\right]}]{\boldsymbol{\tau} \atop {\left[\left(\frac{1}{4}\right)\mathbf{1}\right] \atop \left[\left(\frac{1}{4}\right)\mathbf{2}\right]}} \frac{1}{4}\left[\begin{array}{ccc} & 4\mathbf{I} & \\ \hline 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{array}\right]$$

Therefore $\mathbf{A}_2^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ □

(L-5) Question 1(c)

$$\left[\begin{array}{c}\mathbf{A}_3\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{ccc}0&0&1\\0&1&1\\1&1&1\\ \hline 1&0&0\\0&1&0\\0&0&1\end{array}\right] \xrightarrow[{[1 \rightleftharpoons 3]}]{\tau} \left[\begin{array}{ccc}1&0&0\\1&1&0\\1&1&1\\ \hline 0&0&1\\0&1&0\\1&0&0\end{array}\right] \xrightarrow[{[(-1)\mathbf{3}+\mathbf{2}]}]{\substack{\tau\\ {[(-1)\mathbf{2}+\mathbf{1}]}}} \left[\begin{array}{ccc}1&0&0\\0&1&0\\0&0&1\\ \hline 0&-1&1\\-1&1&0\\1&0&0\end{array}\right].$$

Therefore $\mathbf{A}_3^{-1} = \begin{bmatrix}0&-1&1\\-1&1&0\\1&0&0\end{bmatrix}$

□

(L-5) Question 2(a)

$$\begin{aligned}\mathbf{AB} &= \mathbf{AC}\\ \mathbf{A}^{-1}\mathbf{AB} &= \mathbf{A}^{-1}\mathbf{AC}\\ \mathbf{IB} &= \mathbf{IC}\\ \mathbf{B} &= \mathbf{C}.\end{aligned}$$

□

(L-5) Question 2(b)

$$\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}0&0\\5&5\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix} = \begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}0&0\\2&2\end{bmatrix}$$

(L-5) Question 3.

$$\left[\begin{array}{cc} a & b \\ c & d \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] \xrightarrow{\overset{\tau}{[(\frac{-b}{a})\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc} a & 0 \\ c & d-\frac{bc}{a} \\ \hline 1 & -b/a \\ 0 & 1 \end{array}\right] = \left[\begin{array}{cc} a & 0 \\ c & \frac{ad-bc}{a} \\ \hline 1 & -b/a \\ 0 & 1 \end{array}\right] \xrightarrow{\overset{\tau}{[(\frac{a}{ad-bc})\mathbf{2}]}} \left[\begin{array}{cc} a & 0 \\ c & 1 \\ \hline 1 & \frac{-b}{ad-bc} \\ 0 & \frac{a}{ad-bc} \end{array}\right]$$

$$\xrightarrow{\overset{\tau}{[(-c)\mathbf{2}+\mathbf{1}]}} \left[\begin{array}{cc} a & 0 \\ 0 & 1 \\ \hline 1+\frac{bc}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-ac}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] = \left[\begin{array}{cc} a & 0 \\ 0 & 1 \\ \hline \frac{ad}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-ac}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \xrightarrow{\overset{\tau}{[(\frac{1}{a})\mathbf{2}]}} \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

so

$$\mathbf{M}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The matrix is invertible only when $ad \neq bc$.

```
a, b, c, d = sympy.symbols('a b c d')
A = Matrix([ [a,b], [c,d] ])
InvMat(A, 1)
```

(L-5) Question 4.

$$\left[\begin{array}{rrr} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 6 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [\mathbf{1} \rightleftharpoons \mathbf{2}]}} \left[\begin{array}{rrr} 1 & 3 & 2 \\ 0 & -1 & 1 \\ 1 & 0 & 6 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-3)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 4 \\ \hline 0 & 1 & 0 \\ 1 & -3 & -2 \\ 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}] \\ [(-1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \\ \hline 0 & -1 & 1 \\ 1 & 3 & -5 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-3)\mathbf{3}+\mathbf{2}] \\ [(-1)\mathbf{3}+\mathbf{1}]}} \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline -1 & -4 & 1 \\ 6 & 18 & -5 \\ -1 & -3 & 1 \end{array}\right] = \left[\begin{array}{c} \mathbf{I} \\ \hline \mathbf{A}^{-1} \end{array}\right]$$

$$\left[\frac{\mathbf{B}}{\mathbf{I}}\right] = \left[\begin{array}{ccc} 1 & 2 & 1 \\ -1 & 4 & -2 \\ 1 & 3 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ -1 & 6 & -1 \\ 1 & 1 & 0 \\ \hline 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{2}]}]{\substack{\boldsymbol{\tau} \\ [\mathbf{2}\rightleftharpoons\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ -1 & 1 & 6 \\ 1 & 0 & 1 \\ \hline 1 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[{[(-6)\mathbf{2}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{2}+\mathbf{1}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ \hline 2 & 1 & -8 \\ 0 & 0 & 1 \\ -1 & -1 & 6 \end{array}\right] \xrightarrow[{[(-1)\mathbf{3}+\mathbf{1}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 10 & 1 & -8 \\ -1 & 0 & 1 \\ -7 & -1 & 6 \end{array}\right] = \left[\frac{\mathbf{I}}{\mathbf{B}^{-1}}\right]$$

□

(L-5) Question 5(a)

$$\begin{bmatrix} a_1 & b_1 & a_1 + b_1 \\ a_2 & b_2 & a_2 + b_2 \\ a_3 & b_3 & a_3 + b_3 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

□

(L-5) Question 5(b) When we sustract the first and second columns from the third we get a zero column (no pivot in that column)

$$\begin{bmatrix} a_1 & b_1 & a_1+b_1 \\ a_2 & b_2 & a_2+b_2 \\ a_3 & b_3 & a_3+b_3 \end{bmatrix} \xrightarrow[\;]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{2}+\mathbf{3}]}} \begin{bmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{bmatrix}$$

□

(L-5) Question 6(a)

$$\left[\frac{\mathbf{A}_1}{\mathbf{I}}\right] = \left[\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [\mathbf{1}\rightleftharpoons\mathbf{4}] \\ [\mathbf{2}\rightleftharpoons\mathbf{3}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(1/2)\mathbf{2}] \\ [(1/3)\mathbf{3}] \\ [(1/4)\mathbf{4}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1/4 \\ 0 & 0 & 1/3 & 0 \\ 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$$

□

(L-5) Question 6(b)

$$\left[\begin{array}{c}\mathbf{A}_2 \\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{rrrr} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\left[(1/2)\mathbf{2}+\mathbf{1}\right]]{\substack{\boldsymbol{\tau} \\ \left[(3/4)\mathbf{4}+\mathbf{3}\right] \\ \left[(2/3)\mathbf{3}+\mathbf{2}\right]}} \left[\begin{array}{rrrr} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/3 & 2/3 & 1 & 0 \\ 1/4 & 1/2 & 3/4 & 1 \end{array}\right] = \left[\begin{array}{c}\mathbf{I} \\ \hline \mathbf{A}_2^{-1}\end{array}\right]$$

$\square$

(L-5) Question 6(c) We just need to repeat, for each block, the steps of Exercise 3:

$$\left[\begin{array}{cccc} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-b/a)\mathbf{1}+\mathbf{2}] \\ \left[\left(\frac{a}{ad-bc}\right)\mathbf{2}\right] \\ [(-c)\mathbf{2}+\mathbf{1}] \\ \left[\left(\frac{1}{a}\right)\mathbf{2}\right]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \\ \hline \frac{d}{ad-bc} & \frac{-b}{ad-bc} & 0 & 0 \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{\boldsymbol{\tau} \\ [(-b/a)\mathbf{3}+\mathbf{4}] \\ \left[\left(\frac{a}{ad-bc}\right)\mathbf{4}\right] \\ [(-c)\mathbf{4}+\mathbf{3}] \\ \left[\left(\frac{1}{a}\right)\mathbf{4}\right]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline \frac{d}{ad-bc} & \frac{-b}{ad-bc} & 0 & 0 \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} & 0 & 0 \\ 0 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 0 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] = \left[\begin{array}{c} \mathbf{I} \\ \hline \mathbf{A}_3^{-1} \end{array}\right]$$

□

(L-5) Question 7.

$$\left[\begin{array}{ccc} a & b & b\\ a & a & b\\ a & a & a\\ \hline 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ \left[\left(\frac{-b}{a}\right)\mathbf{1}+\mathbf{2}\right]\\ \left[\left(\frac{-b}{a}\right)\mathbf{1}+\mathbf{3}\right]}} \left[\begin{array}{ccc} a & 0 & 0\\ a & a-b & 0\\ a & a-b & a-b\\ \hline 1 & \frac{-b}{a} & \frac{-b}{a}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ \left[\left(\frac{1}{a}\right)\mathbf{1}\right]\\ \left[\left(\frac{1}{a-b}\right)\mathbf{2}\right]\\ \left[\left(\frac{1}{a-b}\right)\mathbf{3}\right]}} \left[\begin{array}{ccc} 1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\\ \hline \frac{1}{a} & \frac{-b/a}{a-b} & \frac{-b/a}{a-b}\\ 0 & \frac{1}{a-b} & 0\\ 0 & 0 & \frac{1}{a-b}\end{array}\right]$$

$$\xrightarrow{\substack{\boldsymbol{\tau}\\ \left[(-1)\mathbf{2}+\mathbf{1}\right]\\ \left[(-1)\mathbf{3}+\mathbf{2}\right]}} \left[\begin{array}{ccc} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ \hline \frac{1}{a-b} & 0 & \frac{-b/a}{a-b}\\ \frac{-1}{a-b} & \frac{1}{a-b} & 0\\ 0 & \frac{-1}{a-b} & \frac{1}{a-b}\end{array}\right]$$

Hence $\quad \mathbf{A}^{-1} = \frac{1}{a-b}\begin{bmatrix} 1 & 0 & -b/a\\ -1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}$

There is no inverse when $a = b$, since the denominator $(a - b)$ is zero (in that case all rows in **A** are the same; and therefore there is only one pivot).
There is not inverse, either, when $a = 0$; since the third element on the first row is not defined (in that case **A** has a zero column and a zero row).

$\square$

(L-5) Question 8. Since $\mathbf{E}$ is an elementary matrix

$$\mathbf{E}^2 = \begin{bmatrix} 1 & 0 \\ 12 & 1 \end{bmatrix}; \qquad \mathbf{E}^8 = \begin{bmatrix} 1 & 0 \\ 48 & 1 \end{bmatrix}; \qquad \mathbf{E}^{-1} = \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix}; \qquad \mathbf{E}^n = \begin{bmatrix} 1 & 0 \\ 6n & 1 \end{bmatrix}$$

$\square$

(L-5) Question 9. $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. One is the transpose matrix of the other.

$\square$

(L-5) Question 10(a) Since $\mathbf{A}\left(\mathbf{I}_{\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}}\right) = \mathbf{I}$, then $\left(\mathbf{I}_{\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}}\right) = \mathbf{A}^{-1}$:

$$\mathbf{I}_{\underset{\substack{[(-4)\mathbf{1}+\mathbf{2}]\\ [(-3)\mathbf{1}+\mathbf{3}]\\ [(-1)\mathbf{3}+\mathbf{2}]}}{\boldsymbol{\tau}}} = \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \mathbf{A}^{-1}$$

```
Tr = T((-4,1,2)) & T((-3,1,3)) & T((-1,3,2))
display(Tr)
Ainv = I(3) & Tr
Ainv
```

□

(L-5) Question 10(b) Since $\left(\mathbf{I}_{\substack{\boldsymbol{\tau}\\[(-4)\mathbf{1}+\mathbf{2}]}\substack{\boldsymbol{\tau}\\[(-3)\mathbf{1}+\mathbf{3}]}\substack{\boldsymbol{\tau}\\[(-1)\mathbf{3}+\mathbf{2}]}}\right)^{-1} = \mathbf{A}$ then
$\mathbf{A} = \left(\mathbf{I}_{\substack{\boldsymbol{\tau}\\[(1)\mathbf{3}+\mathbf{2}]}\substack{\boldsymbol{\tau}\\[(3)\mathbf{1}+\mathbf{3}]}\substack{\boldsymbol{\tau}\\[(4)\mathbf{1}+\mathbf{2}]}}\right)$.

$$\mathbf{I}_{\substack{\boldsymbol{\tau}\\[(1)\mathbf{3}+\mathbf{2}]\\[(3)\mathbf{1}+\mathbf{3}]\\[(4)\mathbf{1}+\mathbf{2}]}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \mathbf{A}$$

$$\text{Check:} \quad \mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
display(Tr**-1)
A    = I(3) & Tr**-1
Sistema([A, A*Ainv] )
```

□

(L-5) Question 11(a) Remember that ${}_{\boldsymbol{\tau}}\mathbf{I}$ is the transpose of $\mathbf{I}_{\boldsymbol{\tau}}$.

Since $\left({}_{\underset{[(-1)\mathbf{3}+\mathbf{2}][(-3)\mathbf{1}+\mathbf{3}][(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}\mathbf{I}\right)\mathbf{A} = \mathbf{I}$, then $\left({}_{\underset{[(-1)\mathbf{3}+\mathbf{2}][(-3)\mathbf{1}+\mathbf{3}][(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}\mathbf{I}\right) = \mathbf{A}^{-1}$:

$$
{}_{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{3}+\mathbf{2}]\\ [(-3)\mathbf{1}+\mathbf{3}]\\ [(-4)\mathbf{1}+\mathbf{2}]}}\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ -3 & 0 & 1 \end{bmatrix} = \mathbf{A}^{-1}
$$

```
Tr   = T((-4,1,2)) & T((-3,1,3)) & T((-1,3,2))
display(Tr)
display( Sistema([t & I(3) for t in ~Tr]) )
Ainv = ~Tr & I(3)
Ainv
```

□

(L-5) Question 11(b) Since $\left({}_{\underset{[(-1)\mathbf{3}+\mathbf{2}][(-3)\mathbf{1}+\mathbf{3}][(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}\mathbf{I}\right)^{-1} = \mathbf{A}$ then

$$\mathbf{A} = \left({}_{[(4)\overset{\tau}{\mathbf{1}}+\mathbf{2}][(3)\overset{\tau}{\mathbf{1}}+\mathbf{3}][(1)\overset{\tau}{\mathbf{3}}+\mathbf{2}]}\mathbf{I} \right).$$

$$\underset{\substack{[(4)\overset{\tau}{\mathbf{1}}+\mathbf{2}]\\ [(3)\mathbf{1}+\mathbf{3}]\\ [(1)\mathbf{3}+\mathbf{2}]}}{}\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix} = \mathbf{A}$$

$$\text{Check:} \quad \mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
display(~Tr**-1)
display( Sistema([t & I(3) for t in ~Tr**-1]) )
A    = ~Tr**-1 & I(3)
Sistema([A, Ainv*A] )
```

□

(L-5) Question 12(a) The first one is an elementary matrix, its inverse is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$; the second one is a permutation matrix, its inverse is its transpose.

□

(L-5) Question 12(b)

$$\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\0&0&1&0\\a&b&c&d\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[\left[\left(\frac{1}{d}\right)\mathbf{4}\right]]{\boldsymbol{\tau}}\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\0&0&1&0\\a&b&c&1\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&\frac{1}{d}\end{array}\right]\xrightarrow[\substack{[(-c)\mathbf{4}+\mathbf{3}]\\ [(-b)\mathbf{4}+\mathbf{2}]\\ [(-a)\mathbf{4}+\mathbf{1}]}]{\boldsymbol{\tau}}\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\-\frac{a}{d}&-\frac{b}{d}&-\frac{c}{d}&\frac{1}{d}\end{array}\right]$$

```
a, b, c, d = sympy.symbols('a b c d')
C = Matrix([ [1,0,0,0], [0,1,0,0], [0,0,1,0], [a,b,c,d] ])
C.apila(I(4),1) & T((fracc(1,d),4)) & T([(-c,4,3),(-b,4,2),(-a,4,1)])
```

□

(L-5) Question 13(a) True, since

$$\mathbf{AB} = \mathbf{I} \quad\Rightarrow\quad \mathbf{B} = \mathbf{A}^{-1}.$$

and

$$\mathbf{CA} = \mathbf{I} \quad\Rightarrow\quad \mathbf{C} = \mathbf{A}^{-1}.$$

Therefore **B** and **C** are the same matrix $\mathbf{A}^{-1}$.

□

(L-5) Question 13(b) False:

$$(\mathbf{AB})^2 = (\mathbf{AB})(\mathbf{AB}) = \mathbf{ABAB}$$

is in general different from

$$\mathbf{A}^2\mathbf{B}^2 = \mathbf{AABB}.$$

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}. \qquad \mathbf{ABAB} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}; \quad \mathbf{AABB} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

□

(L-5) Question 14(a) Lets check if the rank is 4 (using column reduction):

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{cccc} 0 & 1 & 0 & 2 \\ 1 & a & 0 & 2a \\ a & 0 & 1 & 0 \\ 1 & 0 & a & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\overset{\tau}{[(-2)\mathbf{2}+\mathbf{4}]}} \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 1 & a & 0 & 0 \\ a & 0 & 1 & 0 \\ 1 & 0 & a & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\overset{\tau}{[\mathbf{1} \rightleftharpoons \mathbf{2}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 1 & a & 1 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\frac{\mathbf{L}}{\mathbf{E}}\right]$$

This matrix has rank 4 for any value a; therefore it is invertible.

□

(L-5) Question 14(b)

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] \to \left[\begin{array}{c}\mathbf{L}\\ \hline \mathbf{E}\end{array}\right] = \left[\begin{array}{cccc} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ \hline 0 & 1 & 0 & 0\\ 1 & 0 & 0 & -2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\underset{[(-1)\mathbf{4}+\mathbf{2}]}{\boldsymbol{\tau}}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \hline 0 & 1 & 0 & 0\\ 1 & 2 & 0 & -2\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 1 \end{array}\right] = \left[\begin{array}{c}\mathbf{I}\\ \hline \mathbf{A}^{-1}\end{array}\right]$$

```
a    = sympy.symbols('a')
A    = Matrix([ [0,1,0,2],[1,a,0,2*a],[a,0,1,0],[1,0,a,1] ])
Ainv = InvMat(A, 1)
Ainv.subs([(a,0)])
```

□

(L-5) Question 15. $\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 & 0 & 1\\ 0 & 1 & 0 & -\frac{1}{2}\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}$

```
A = Matrix([ [1,1,0,1], [0,1,0,-1], [0,0,1,0], [0,0,0,-2] ])
InvMat(A, 1)
```

□

(L-5) Question 16.

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{ccc}1&0&1\\1&1&0\\0&1&1\\ \hline 1&0&0\\0&1&0\\0&0&1\end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc}1&0&0\\1&1&-1\\0&1&1\\ \hline 1&0&-1\\0&1&0\\0&0&1\end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{2}+\mathbf{1}]\\ [(1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc}1&0&0\\0&1&0\\-1&1&2\\ \hline 1&0&-1\\-1&1&1\\0&0&1\end{array}\right]$$

$$\xrightarrow{\substack{\boldsymbol{\tau}\\ \left[\left(\frac{1}{2}\right)\mathbf{3}\right]\\ [(-1)\mathbf{3}+\mathbf{2}]\\ [(1)\mathbf{3}+\mathbf{1}]}} \left[\begin{array}{ccc}1&0&0\\0&1&0\\0&0&1\\ \hline 1/2&1/2&-1/2\\-1/2&1/2&1/2\\1/2&-1/2&1/2\end{array}\right] = \left[\begin{array}{c}\mathbf{I}\\ \hline \mathbf{A}^{-1}\end{array}\right]$$

$$\left[\frac{\mathbf{B}}{\mathbf{I}}\right] = \left[\begin{array}{rrr} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(2)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{rrr} 1 & 0 & 0 \\ 1 & -3 & 3 \\ -2 & 3 & -3 \\ \hline 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{rrr} 1 & 0 & 0 \\ 1 & -3 & 0 \\ -2 & 3 & 0 \\ \hline 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right] = \left[\frac{\mathbf{L}}{\mathbf{E}}\right]$$

This matrix is singular (no inverse).

□

(L-5) Question 17. Since any power of a permutation matrix is a permutation matrix, and there is a finite number of permutation matrices; the sequence $(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}})$, $(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}})^2$, $(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}})^3$, $\ldots$, $(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}})^r, \ldots$ must repeat itself. Therefore, for some m and n, $(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}})^m = (\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}})^n$. And, since permutation matrices are invertible, there exist $(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}})^{-n}$, such as $(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}})^{-n}(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}})^n = \mathbf{I}$. Then:

$$\begin{aligned}
(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}})^m &= (\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}})^n \\
(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}})^{-n}(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}})^m &= (\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}})^{-n}(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}})^n \\
(\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}})^{m-n} &= \mathbf{I}.
\end{aligned}$$

In words, the $m-n$ power of $\mathbf{I}_{\substack{\mathcal{T}\\ [\mathfrak{S}]}}$ is the identity matrix.

□

(L-6) Question 1(a) All points in the Quadrant I of $\mathbb{R}^2$, that is, $\{(x, y) : x \geq 0, y \geq 0\}$. It is close under addition, but not under multiplication by an escalar $c\boldsymbol{v}$ when $c < 0$. What about the other quadrants?

□

(L-6) Question 1(b) Take the union of two lines that goes through the origin, for example $\{(x, y) : x = y\}$ and $\{(x, y) : x = -y\}$. This set of points is close under multiplication by a scalar, since any multiple of any point is always in one line. But it isn't close under addition. What about the first and third quadrants together?

□

(L-6) Question 2(a) It is not a subspace. The addition $(1, 1) + (2, 4) = (3, 5)$ doesn't belong to the set; there fore, this set is not close under addition

□

(L-6) Question 2(b) It is a subspace

□

(L-6) Question 2(c) It is a subspace

□

(L-6) Question 2(d) It is not a subspace. It is not closed under multiplication. Consider for example $\frac{1}{2} \cdot (1, 1)$.

□

(L-6) Question 2(e) It is not a subspace. It is not closed under addition. Consider for example $(1, 0) + (0, 1)$.

□

(L-6) Question 2(f) It is a subspace

□

(L-6) Question 3. $\mathbb{R}^2$ contains vectors with *two* components —they don't belong to $\mathbb{R}^3$.

□

(L-6) Question 4. Vectors $(1, 0, -2,)$ and $(0, 1, -4,)$ belong to P, but their addition, $(1, 1, -6,)$, doesn't belong to P, since $1 - 1 - (-6) = 6 \neq 3$.

□

(L-6) Question 5. Assume (wrongly) that the set of solutions of $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, with $\boldsymbol{b} \neq \mathbf{0}$, is a subspace. Then, for any couple of solutions $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, its addition should be also a solution, and then $\mathbf{A}(\boldsymbol{x}_1 + \boldsymbol{x}_2) = \boldsymbol{b}$, but also $\mathbf{A}(\boldsymbol{x}_1 + \boldsymbol{x}_2) = \mathbf{A}\boldsymbol{x_1} + \mathbf{A}\boldsymbol{x_2} = \boldsymbol{b} + \boldsymbol{b}$; and therefore $\boldsymbol{b} = \boldsymbol{b} + \boldsymbol{b}$; something that contradicts our assumption $\boldsymbol{b} \neq \mathbf{0}$.
Therefore the solution set $\{\boldsymbol{x} \mid \mathbf{A}\boldsymbol{x} = \boldsymbol{b}\}$ does not form a subspace.

□

(L-6) Question 6(a) All the multiples of $\mathbf{A}$

$$c\mathbf{A}$$

form a subspace, and $\mathbf{B}$ does not belong to it.

Also the set of matrices:

$$\left\{ \mathbf{M} \in \mathbb{R}^{2\times 2} \middle| \exists a, b, c \in \mathbb{R}, \mathbf{M} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \right\}$$

$\square$

(L-6) Question 6(b) All the multiples of **B**

$$c\mathbf{B}$$

form a subspace, and **A** does not belong to it.
Also the set of matrices:

$$\left\{ \mathbf{M} \in \mathbb{R}^{2\times 2} \middle| \exists f, g, h \in \mathbb{R}, \mathbf{M} = \begin{bmatrix} 0 & f \\ g & h \end{bmatrix} \right\}$$

$\square$

(L-6) Question 6(c) No. All subspaces are closed under linear combinations, therefore, if it contains **A** and **B**, then it must contain $\frac{1}{2}\mathbf{A} - \frac{1}{3}\mathbf{B} = \mathbf{I}$.

$\square$

(L-6) Question 7(a) It is a subspace, since any linear combination of matrices in $\mathcal{S}$ is another symmetric matrix:
$${}_{j|}\Big((a\mathbf{A} + b\mathbf{B})^{\mathsf{T}}\Big) = (a\mathbf{A} + b\mathbf{B})_{|j} = a(\mathbf{A}_{|j}) + b\mathbf{B}_{|j} = a({}_{j|}\mathbf{A}) + b\,{}_{j|}\mathbf{B} = {}_{j|}(a\mathbf{A} + b\mathbf{B}).$$

□

(L-6) Question 7(b) It is NOT a subspace. It isn't close under addition (it is easy to find two non-symmetric matrices whose addition is a symmetric one). For example:

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}; \qquad \mathbf{A} + \mathbf{B} = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

In fact, for any $\mathbf{A}$ always $(\mathbf{A} + \mathbf{A}^{\intercal})$ is a symmetric matrix.

□

(L-6) Question 7(c) It is a subspace, since any linear combination

$$c \begin{bmatrix} 0 & -a_{12} & \dots & -a_{1m} \\ a_{12} & 0 & \dots & -a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \dots & 0 \end{bmatrix} + d \begin{bmatrix} 0 & -b_{12} & \dots & -b_{1m} \\ b_{12} & 0 & \dots & -b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1m} & b_{2m} & \dots & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & -ca_{12} - db_{12} & \dots & -ca_{1m} - db_{1m} \\ ca_{12} + db_{12} & 0 & \dots & -ca_{2m} - db_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{1m} + db_{1m} & ca_{2m} + db_{2m} & \dots & 0 \end{bmatrix}$$

is another skew-symmetric matrix.

Alternatively; since

$$\mathbf{A}^\mathsf{T} = -\mathbf{A} \quad \Leftrightarrow \quad \mathbf{A}_{|j} = \left(-\mathbf{A}^\mathsf{T}\right)_{|j} = {}_{j|}\left(-\mathbf{A}\right)$$

then ${}_{j|}\left((a\mathbf{A}+b\mathbf{B})^\mathsf{T}\right) = (a\mathbf{A}+b\mathbf{B})_{|j} = (a\mathbf{A})_{|j} + (b\mathbf{B})_{|j} = {}_{j|}(-a\mathbf{A}) + {}_{j|}(-b\mathbf{B}) = {}_{j|}\left(-(a\mathbf{A}+b\mathbf{B})\right)$.

□

(L-6) Question 8(a) A line. A plane.

□

(L-6) Question 8(b) A point. A line

□

(L-6) Question 8(c) We should check that $\boldsymbol{x}+\boldsymbol{y} \in \mathcal{S}\cap\mathcal{T}$ and $c\boldsymbol{x} \in \mathcal{S}\cap\mathcal{T}$
Since $\boldsymbol{x}$ and $\boldsymbol{y}$ belong to $\mathcal{S}\cap\mathcal{T}$, both belong to $\mathcal{S}$, and both belong to $\mathcal{T}$. Therefore,

$$\begin{aligned} \boldsymbol{x}+\boldsymbol{y} \in \mathcal{S} &\quad \text{since } \mathcal{S} \text{ is a subspace and } \boldsymbol{x},\boldsymbol{y} \in \mathcal{S} \\ \boldsymbol{x}+\boldsymbol{y} \in \mathcal{T} &\quad \text{since } \mathcal{T} \text{ is a subspace and } \boldsymbol{x},\boldsymbol{y} \in \mathcal{T} \end{aligned}$$

therefore the addition belongs to the intersection of both subspaces. Besides

$$\begin{aligned} c\boldsymbol{x} \in \mathcal{S} &\quad \text{since } \mathcal{S} \text{ is a subspace and } \boldsymbol{x} \in \mathcal{S} \\ c\boldsymbol{x} \in \mathcal{T} &\quad \text{since } \mathcal{T} \text{ is a subspace and } \boldsymbol{x} \in \mathcal{T} \end{aligned}$$

therefore $c\boldsymbol{x}$ belongs to the intersection of both subspaces.

□

(L-6) Question 9(a) It is a subspace, since any linear combination of two of those vectors also has a zero in the first component

□

(L-6) Question 9(b) No, it isn't a subspace. If we add two of them we get a vector with first component $b_1 = 2$. Also, the zero vector $\mathbf{0}$ doesn't belong top the set.

□

(L-6) Question 9(c) No, it isn't a subspace. If we add $(1, 0, 1,)$ (from the first plane) and $(1, 1, 0,)$ (from the second) we get a vector outside both planes.

□

(L-6) Question 9(d) It is a subspace, since any linear combination of zero vectors $a\mathbf{0} + b\mathbf{0} = \mathbf{0}$. is inside the set $\{\mathbf{0}\}$ (the solitary vector $\mathbf{0}$)

□

(L-6) Question 9(e) Yes, it is by construction.

□

(L-6) Question 9(f) It is a subspace, since for any linear combination

$$\boldsymbol{v} = a \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + c \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} ab_1 + cd_1 \\ ab_2 + cd_2 \\ ab_3 + cd_3 \end{pmatrix}$$

then

$$(ab_3+cd_3)-(ab_2+cd_2)+3(ab_1+cd_1) = a(b_3-b_2+3b_1)+c(d_3-d_2+3d_1) = a\cdot 0+c\cdot 0 = 0$$

which proves that the new vector also satisfy the condition $v_3 - v_2 + 3v_1 = 0$.

□

(L-6) Question 10(a)

1. $\boldsymbol{x} + \boldsymbol{y} = \Big(x_1,\ x_2,\Big) + \Big(y_1,\ y_2,\Big) = \Big((x_1 + y_1 + 1),\ (x_2 + y_2 + 1),\Big) = \Big((y_1 + x_1 + 1),\ (y_2 + x_2 + 1),\Big) = \Big(y_1,\ y_2,\Big) + \Big(x_1,\ x_2,\Big) = \boldsymbol{y} + \boldsymbol{x}$
2. $$\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = \Big(x_1,\ x_2,\Big) + \Big((y_1 + z_1 + 1),\ (y_2 + z_2 + 1),\Big) = \Big((x_1 + y_1 + z_1 + 2),\ ($$
$$= \Big((x_1 + y_1 + 1),\ (x_2 + y_2 + 1),\Big) + \Big(z$$
3. The rule is unbroken, but the new "zero vector" $\mathbf{0}$ in this case is:

$\mathbf{0} = (-1,\ -1,)$ instead of $(0,\ 0,)$; since

$$\boldsymbol{x} + \mathbf{0} = (x_1,\ x_2) + (-1,\ -1) = \Big((x_1 - 1 + 1),\ (x_2 - 1 + 1),\Big) = (x_1, x_2) = \boldsymbol{x}.$$

4. This rule is unbroken, although if $\boldsymbol{x} = (x_1, x_2,)$ then $-\boldsymbol{x}$ should be $-\boldsymbol{x} = \Big((-x_1 - 2),\ (x_2 - 2),\Big)$ since

$$\boldsymbol{x} + (-\boldsymbol{x}) = (x_1,\ x_2,) + \Big((-x_1 - 2),\ (x_2 - 2),\Big) = (-1,\ -1,) = \mathbf{0}.$$

   note that here the third rule implies $\mathbf{0} = (-1,\ -1,)$.
5. $1\boldsymbol{x} = 1(x_1,\ x_2,) = (1x_1,\ 1x_2,) = \boldsymbol{x}$
6. $ab\boldsymbol{x} = ab(x_1,\ x_2,) = (abx_1,\ abx_2,) = a(bx_1,\ bx_2,) = a(b\boldsymbol{x})$.
7. The rule $a(\boldsymbol{x} + \boldsymbol{y}) = a\boldsymbol{x} + a\boldsymbol{y}$ **is broken**. . . for example

$$2\Big((1,\ 1,)+(1,\ 1,)\Big) = 2(3,\ 3,) = (6,\ 6,) \neq 2(1,\ 1,)+2(1,\ 1,) = (2,\ 2,)+(2,\ 2,) = (5,$$

8. The rule $(a + b)\boldsymbol{x} = a\boldsymbol{x} + b\boldsymbol{x}$ **is also broken**. . . for example

$$(2+2)(1,\ 1,) = 4(1,\ 1,) = (4,\ 4,) \neq 2(1,\ 1,)+2(1,\ 1,) = (2,\ 2,)+(2,\ 2,) = (5,\ 5,).$$

$\square$

(L-6) Question 10(b)

1. $\boldsymbol{x} + \boldsymbol{y} = xy = yx = \boldsymbol{y} + \boldsymbol{x}$.
2. $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = x(yz) = (xy)z = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$.
3. If $\mathbf{0} = 1$; then $\boldsymbol{x} + \mathbf{0} = x1 = x = \boldsymbol{x}$;.
4. If $-\boldsymbol{x} = 1/x$; then $\boldsymbol{x} + (-\boldsymbol{x}) = x/x = 1 = \mathbf{0}$.
5. $1\boldsymbol{x} = x^1 = x = \boldsymbol{x}$.
6. $(ab)\boldsymbol{x} = x^{(ab)} = (x^b)^a = (b\boldsymbol{x})^a = a(b\boldsymbol{x})$.
7. $a(\boldsymbol{x} + \boldsymbol{y}) = (xy)^a = x^a \cdot y^a = a\boldsymbol{x} + a\boldsymbol{y}$.
8. $(a + b)\boldsymbol{x} = (x)^{a+b} = x^a \cdot x^b = a\boldsymbol{x} + b\boldsymbol{x}$.

□

(L-6) Question 10(c)

1. $\boldsymbol{x} + \boldsymbol{y} = \Big((x_1 + y_2),\ (x_2 + y_1),\Big) \neq \Big((y_1 + x_2),\ (y_2 + x_1),\Big) = \boldsymbol{y} + \boldsymbol{x}$. It is not satisfied. For example
$(-1,\ 1,) + (2,\ 3,) = \Big((-1+3),\ (1+2),\Big) = (2,\ 3,) \neq (3,\ 2,) = \Big((2+1),\ (3-1),\Big) = (2,\ 3,) + (-1,\ 1,)$
2. $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$. It is not satisfied. For example
$(0,\ 0,) + \Big((-1,\ 1,) + (2,\ 3,)\Big) = (0,\ 0,) + (2,\ 3,) = (3,\ 2,) \neq (4,\ 1,) = (1,\ -1,) + (2,\ 3,) = \Big((0,\ 0,) + (-1,\ 1,)\Big) + (2,\ 3,)$
3. $\boldsymbol{x} + \mathbf{0} = (x_1,\ x_2,) + (0,\ 0,) = \Big((x_1 + 0),\ (x_2 + 0),\Big) = \boldsymbol{x}$;
4. If $-\boldsymbol{x} = (-x_2,\ -x_1,)$; then
$\boldsymbol{x} + (-\boldsymbol{x}) = (x_1,\ x_2,) + (-x_2,\ -x_1,) = (0,\ 0,) = \mathbf{0}$.

5. $1\boldsymbol{x} = 1(x_1,\ x_2,) = \boldsymbol{x}.$
6. $(ab)\boldsymbol{x} = (abx_1,\ abx_2,) = a(bx_1,\ bx_2,) = a(b\boldsymbol{x}).$
7. $a(\boldsymbol{x}+\boldsymbol{y}) = a\Big((x_1+y_2),\ (x_2+y_1),\Big) = (ax_1,\ ax_2,) + (ay_1,\ ay_2,) = a\boldsymbol{x} + a\boldsymbol{y}.$
8. It is not satisfied: $(a+b)\boldsymbol{x} = \Big((a+b)x_1,\ (a+b)x_2,\Big) = \Big((ax_1+bx_1),\ (ax_2+bx_2),\Big) \neq \Big((ax_1+bx_2),\ (ax_2+bx_1),\Big) = (ax_1,\ ax_2,) + (bx_1,\ bx_2,) = a\boldsymbol{x} + b\boldsymbol{x}.$

$\square$

(L-7) Question 1(a)

$$\left[\begin{array}{cccc} 1 & 2 & 0 & 5 \\ 2 & 3 & 1 & 4 \\ -1 & -1 & -1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(-5)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 2 & -1 & 1 & -6 \\ -1 & 1 & -1 & 6 \\ \hline 1 & -2 & 0 & -5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(1)\mathbf{2}+\mathbf{3}] \\ [(-6)\mathbf{2}+\mathbf{4}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \hline 1 & -2 & -2 & 7 \\ 0 & 1 & 1 & -6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Two pivots: rank 2. The null space,

$$\mathcal{N}(\mathbf{A}) = \left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2,\ \boldsymbol{v} = \begin{bmatrix} -2 & 7 \\ 1 & -6 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\},$$

is a plane in a 4 dimensional space.

```
A = Matrix([[1,2,0,5],[2,3,1,4],[-1,-1,-1,1]])
Homogenea(A,1)
Math( SubEspacio(A).EcParametricas() )
```

□

(L-7) Question 1(b)

$$\left[\begin{array}{ccc} 1 & 2 & 1 \\ -1 & 3 & 4 \\ 2 & -1 & -3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ -1 & 5 & 5 \\ 2 & -5 & -5 \\ \hline 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ -1 & 5 & 0 \\ 2 & -5 & 0 \\ \hline 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c} \mathbf{K} \\ \hline \mathbf{E} \end{array}\right]$$

Two pivots: rank 2. It is a line in a 3 dimensional space.

$$\text{The null space,} \quad \mathcal{N}\left(\mathbf{F}\right) = \left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \boldsymbol{p} \right\}.$$

```
F = Matrix([[1,2,1],[-1,3,4],[2,-1,-3]])
Homogenea(F,1)
```

```
Math( SubEspacio(F).EcParametricas() )
```

□

(L-7) Question 1(c)

$$\left[\begin{array}{ccc} 1 & 2 & 1 \\ 0 & 3 & 1 \\ -2 & -1 & 4 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 3 & 1 \\ -2 & 3 & 6 \\ \hline 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{2}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ [(3)\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 3 & 0 \\ -2 & 3 & 15 \\ \hline 1 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{array}\right] = \left[\begin{array}{c} \mathbf{K} \\ \hline \mathbf{E} \end{array}\right]$$

Three pivots: rank 3.

The null space, $\mathcal{N}(\mathbf{A}) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$ is a point in a 3 dimensional space.

```
G = Matrix([[1,2,1],[0,3,1],[-2,-1,4]])
Homogenea(G,1)
```

□

(L-7) Question 1(d)

$$\begin{bmatrix}\mathbf{H}\\\hline\mathbf{I}\end{bmatrix}=\left[\begin{array}{cc}1&3\\2&1\\-1&-3\\\hline1&0\\0&1\end{array}\right]\xrightarrow[\,]{\substack{\boldsymbol{\tau}\\ [(-3)\mathbf{1}+\mathbf{2}]}}\left[\begin{array}{cc}1&0\\2&-5\\-1&0\\\hline1&-3\\0&1\end{array}\right]=\begin{bmatrix}\mathbf{K}\\\hline\mathbf{E}\end{bmatrix}$$

Two pivots: rank 2. There aren't zero columns.

The null space, $\mathcal{N}\left(\mathbf{H}\right)=\left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}$ is a point in a 2 dimensional space.

```
H = Matrix([[1,3],[2,1],[-1,-3]])
Homogenea(H,1)
```

□

(L-7) Question 2(a)

$$\left[\begin{array}{ccc}1&2&-2\\2&1&0\\\hline1&0&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow[\,]{\substack{\boldsymbol{\tau}\\ [(-2)\mathbf{1}+\mathbf{2}]\\ [(2)\mathbf{1}+\mathbf{3}]}}\left[\begin{array}{ccc}1&0&0\\2&-3&4\\\hline1&-2&2\\0&1&0\\0&0&1\end{array}\right]\xrightarrow[\,]{\substack{\boldsymbol{\tau}\\ [(3)\mathbf{3}]\\ [(4)\mathbf{2}+\mathbf{3}]}}\left[\begin{array}{ccc}1&0&0\\2&-3&0\\\hline1&-2&-2\\0&1&4\\0&0&3\end{array}\right]$$

Therefore $\mathcal{N}\left(\mathbf{A}\right) = \left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix} \boldsymbol{p} \right\}$ is a line in a 3 dimensional spac

□

(L-7) Question 2(b) All are pivot columns, then the sole solution to $\mathbf{F}\boldsymbol{x} = \mathbf{0}$ is the zero vector, therefore $\mathcal{N}\left(\mathbf{F}\right) = \{\mathbf{0}\}$. A point in $\mathbb{R}^2$.

□

(L-7) Question 2(c)

$$\left[\begin{array}{ccc} 1 & 2 & -4 \\ -1 & 1 & 3 \\ 1 & 5 & -5 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(4)\mathbf{1}+\mathbf{3}]}]{\overset{\boldsymbol{\tau}}{[(-2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ -1 & 3 & -1 \\ 1 & 3 & -1 \\ \hline 1 & -2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{2}+\mathbf{3}]}]{\overset{\boldsymbol{\tau}}{[(3)\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 1 & 3 & 0 \\ \hline 1 & -2 & 10 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{array}\right]$$

Therefore $\mathcal{N}\left(\mathbf{G}\right) = \left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{bmatrix} 10 \\ 1 \\ 3 \end{bmatrix} \boldsymbol{p} \right\}$ is a line in a $\mathbb{R}^3$.

□

(L-7) Question 3.

$$\left[\begin{array}{cccc} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-2)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 1 & -2 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 1 & -2 & 2 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Then, rank is 2. x_1 and x_2 are pivot variables.
The special solutions are

$$\boldsymbol{x}_a = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}; \qquad \boldsymbol{x}_b = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Finally, the complete solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ is

$$\left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2,\ \boldsymbol{v} = \begin{bmatrix} 2 & -1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}$$

□

(L-7) Question 4(a)

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\,]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\frac{\mathbf{L}}{\mathbf{E}}\right]$$

Rank 1. Pivot variable: x_1. The complete solution is

$$\mathcal{N}(\mathbf{A}) = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \,\middle|\, \text{exists } \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \text{ such that } \boldsymbol{x} = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -a-b-c \\ a \\ b \\ c \end{pmatrix}\right.$$

□

(L-7) Question 4(b)

$$\left[\begin{array}{cccc} -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(1)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{cccc} -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ \hline 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}+\mathbf{4}]}} \left[\begin{array}{cccc} -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c} \mathsf{K} \\ \hline \mathsf{E} \end{array}\right]$$

Rank 2. Pivot variables: x_1 and x_2. The complete solution is

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \text{exists } \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \text{ such that } \boldsymbol{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ -b \\ a \\ b \end{pmatrix} \right\}$$

$\square$

(L-7) Question 4(c)

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{cccc} -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{1}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cccc} -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c}\mathbf{K}\\ \hline \mathbf{E}\end{array}\right]$$

Rank 1. Pivot variable: x_1. The complete solution is

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \text{exists } \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \text{ such that } \boldsymbol{x} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a - b + c \\ a \\ b \\ c \end{pmatrix} \right\}$$

□

(L-7) Question 5(a) Since $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^3$, **A** has 3 columns.

□

(L-7) Question 5(b) Any number greater (or equal) than one.

□

(L-7) Question 5(c) Three columns and two special solutions (2 free columns) means rank 1 (only one pivot column).

□

(L-7) Question 6(a) Since **R** only has two pivots, we know the third column is a linear combination of the other two (it is a free column).

□

(L-7) Question 6(b) By column Gaussian elimination

$$\begin{bmatrix} 1 & 2 \\ 2 & a \\ 1 & 1 \\ b & 8 \end{bmatrix} \xrightarrow[]{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix} 1 & 0 \\ 2 & a-4 \\ 1 & -1 \\ b & 8-2b \end{bmatrix} \xrightarrow[a=4]{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{2}+\mathbf{1}]}} \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & -1 \\ 8-b & 8-2b \end{bmatrix} \xrightarrow[]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}]}} \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \\ 3 & 2 \end{bmatrix}$$

And since $8-b=3$, then $b=5$.

□

(L-7) Question 6(c) Since there is only one free column, we know that

$\mathbf{A}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \mathbf{0}$ then

$$\dim(\mathcal{N}(\mathbf{A})) = 1; \qquad \mathcal{N}(\mathbf{A}) = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \,\middle|\, \text{exists } c \in \mathbb{R} \text{such that } \boldsymbol{x} = c\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

```
a=Vector([1,-2,1])
SubEspacio(Sistema([a]))
```

$\square$

(L-7) Question 7(a)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{4}]}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{2}+\mathbf{3}] \\ [(-3)\mathbf{2}+\mathbf{4}]}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

```
A = Matrix([[1,1,1,1],[1,1,1,1],[0,1,2,3],[0,1,2,3]])
R = ElimGJ(A,1)
```

$\square$

(L-7) Question 7(b) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}]}} \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 1 & -2 & 0 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{2}+\mathbf{1}]}}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & -2 & 0 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-\frac{1}{2})\mathbf{2}]}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

```
B = Matrix([[1,2,1],[2,2,2],[1,0,1]])
R = ElimGJ(B,1)
```

□

(L-7) Question 7(c) $\begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(-2)\mathbf{1}+\mathbf{4}] \\ [(-1)\mathbf{1}+\mathbf{5}]}}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & -3 & 0 & -3 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}+\mathbf{4}]}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(3)\mathbf{1}] \\ [(2)\mathbf{2}+\mathbf{1}]}}$$

$$\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(\frac{1}{3})\mathbf{1}] \\ [(-\frac{1}{3})\mathbf{2}]}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \end{bmatrix}$$

□

(L-7) Question 7(d) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(-3)\mathbf{1}+\mathbf{3}]}} \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & -2 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{2}+\mathbf{3}]}}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(2)\mathbf{2}+\mathbf{1}]}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}]}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(L-7) Question 8(a) The identity matrix.

(L-7) Question 8(b)

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{2}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}+\mathbf{1}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right] = \left[\frac{\mathbf{I}}{\mathbf{E}}\right].$$ Therefore $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

```
A = Matrix([[1,0,0],[1,1,1],[0,0,1]])
InvMat(A,1)
```

(L-7) Question 9(a) The identity matrix **I**.

(L-7) Question 9(b)

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{ccc}0&0&1\\0&1&1\\1&1&1\\ \hline 1&0&0\\0&1&0\\0&0&1\end{array}\right] \xrightarrow[\boldsymbol{[1 \rightleftharpoons 3]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc}1&0&0\\1&1&0\\1&1&1\\ \hline 0&0&1\\0&1&0\\1&0&0\end{array}\right] \xrightarrow[{[(-1)\mathbf{2}+\mathbf{1}] \atop [(-1)\mathbf{3}+\mathbf{2}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc}1&0&0\\0&1&0\\0&0&1\\ \hline 0&-1&1\\-1&1&0\\1&0&0\end{array}\right] = \left[\begin{array}{c}\mathbf{I}\\ \hline \mathbf{A}^{-1}\end{array}\right]$$

□

(L-8) Question 1(a) True

□

(L-8) Question 1(b) False

□

(L-8) Question 1(c) True

□

(L-8) Question 1(d) False

□

(L-8) Question 2.

$$\left[\begin{array}{cccc|c} 1&3&1&2&-1\\ 2&6&4&8&-3\\ 0&0&2&4&-1\\ \hline 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ \hline 0&0&0&0&1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ [(-3)\mathbf{1}+\mathbf{2}]\\ [(-1)\mathbf{1}+\mathbf{3}]\\ [(-2)\mathbf{1}+\mathbf{4}]\\ [(1)\mathbf{1}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1&0&0&0&0\\ 2&0&2&4&-1\\ 0&0&2&4&-1\\ \hline 1&-3&-1&-2&1\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ \hline 0&0&0&0&1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ [(-2)\mathbf{3}+\mathbf{4}]\\ [(2)\mathbf{5}]\\ [(1)\mathbf{3}+\mathbf{5}]}}$$

$$\left[\begin{array}{cccc|c} 1&0&0&0&0\\ 2&0&2&0&0\\ 0&0&2&0&0\\ \hline 1&-3&-1&0&1\\ 0&1&0&0&0\\ 0&0&1&-2&1\\ 0&0&0&1&0\\ \hline 0&0&0&0&2 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ \left[\left(\frac{1}{2}\right)\mathbf{5}\right]}} \left[\begin{array}{cccc|c} 1&0&0&0&0\\ 2&0&2&0&0\\ 0&0&2&0&0\\ \hline 1&-3&-1&0&\frac{1}{2}\\ 0&1&0&0&0\\ 0&0&1&-2&\frac{1}{2}\\ 0&0&0&1&0\\ \hline 0&0&0&0&1 \end{array}\right]$$

$$\left\{ \boldsymbol{x}\in\mathbb{R}^4 \;\middle|\; \exists\boldsymbol{p}\in\mathbb{R}^2 \text{ tal que } \boldsymbol{x} = \begin{pmatrix}\frac{1}{2}\\0\\\frac{1}{2}\\0\end{pmatrix} + \begin{bmatrix}-3&0\\1&0\\0&-2\\0&1\end{bmatrix}\boldsymbol{p} \right\}$$

```
A = Matrix([[1,3,1,2],[2,6,4,8],[0,0,2,4]])
b = Vector([1,3,1])
SEL(A,b,1)
```

□

(L-8) Question 3.

```
A = Matrix([ [1,2,-1,-2,1], [1,2,0,0,3], [2,4,1,2,9] ])
b = Vector( [0,-1,-4] )
SEL(A,b,1)
```

$$\left\{ \boldsymbol{x} \in \mathbb{R}^5 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 \text{ tal que } \boldsymbol{x} = \begin{pmatrix} 2\\0\\1\\0\\-1 \end{pmatrix} + \begin{bmatrix} 0 & -2\\ 0 & 1\\ -2 & 0\\ 1 & 0\\ 0 & 0 \end{bmatrix} \boldsymbol{p} \right\}$$

□

(L-8) Question 4. Para expresar la solución completa del sistema necesitamos una solución particular a la que sumar cualquier combinación de los vectores del espacio nulo (soluciones del sistema homogéneo).

Para obtener una solución particular podríamos aplicar la eliminación gaussiana pero en este caso una solución inmediata es asignar el valor uno a x_3 y x_4; y cero a las demás; es decir, sumar las columnas 3 y 4 de la matriz de coeficientes. Por tanto una solución particular inmediata es

$$\boldsymbol{x}_p = \begin{pmatrix} 0\\0\\1\\1\\0 \end{pmatrix}$$

Para calcular el espacio nulo aplicaremos el método de eliminación a la matriz de coeficientes del sistema de ecuaciones:

$$\left[\begin{array}{ccccc} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(\text{-}1)\mathbf{1}+\mathbf{5}]}]{\substack{\boldsymbol{\tau} \\ [(\text{-}1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & \text{-}1 \\ 0 & 0 & 1 & 1 & 0 \\ \hline 1 & 0 & \text{-}1 & 0 & \text{-}1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(\text{-}1)\mathbf{3}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(\text{-}1)\mathbf{2}+\mathbf{4}]}} \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & \text{-}1 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & \text{-}1 & 1 & \text{-}1 \\ 0 & 1 & 0 & \text{-}1 & 0 \\ 0 & 0 & 1 & \text{-}1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Así pues, la solución completa al sistema es el conjunto de vectores

$$\left\{ \boldsymbol{x} \in \mathbb{R}^5 \;\middle|\; \exists c \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$\square$

(L-8) Question 5(a)

$$\left[\begin{array}{c|c} \mathbf{A} & -\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 1 & 0 & 3 & -b_1 \\ 0 & 2 & 0 & 6 & -b_2 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-3)\mathbf{2}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & -b_1 \\ 0 & 2 & 0 & 0 & -b_2 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[{[(b_1)\mathbf{2}+\mathbf{5}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 2b_1 - b_2 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 & b_1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

The column reduced echelon form is

$$\mathbf{R} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}.$$

□

(L-8) Question 5(b) the variables x_1, x_3 and x_4 are free.

□

(L-8) Question 5(c)

$$\boldsymbol{x}_a = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{x}_b = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{x}_c = \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

□

(L-8) Question 5(d) The linear system is consistent when $b_2 = 2b_1$. In that case by gaussian elimination we get

$$\left[\begin{array}{c|c} \mathbf{R} & \mathbf{0} \\ \hline \mathbf{E} & \boldsymbol{x}_p \\ \hline \mathbf{0} & 1 \end{array}\right].$$

□

(L-8) Question 5(e) In that case ($b_2 = 2b_1$) a particular solution is

$$\boldsymbol{x}_p = \begin{pmatrix} 0 \\ b_1 \\ 0 \\ 0 \end{pmatrix}$$

And then, the complete solution to the system is

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists \begin{pmatrix} a \\ c \\ d \end{pmatrix} \text{ such that } \boldsymbol{x} = \begin{pmatrix} 0 \\ b_1 \\ 0 \\ 0 \end{pmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} a \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b_1 - 3d \\ c \\ d \end{pmatrix} \right\}$$

```
b1,b2 = sympy.symbols('b1 b2')
A   = Matrix([[0,1,0,3],[0,2,0,6]])
b   = Vector([b1,b2])
SEL(A, b, 1)
```

□

(L-8) Question 6.

$$\left[\begin{array}{c|c} \mathsf{A} & \text{-}\boldsymbol{b} \\ \hline \mathsf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{cc|c} 0 & 0 & \text{-}b_1 \\ 1 & 2 & \text{-}b_2 \\ 0 & 0 & \text{-}b_3 \\ 3 & 6 & \text{-}b_4 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-2)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \left[\begin{array}{cc|c} 0 & 0 & \text{-}b_1 \\ 1 & 0 & \text{-}b_2 \\ 0 & 0 & \text{-}b_3 \\ 3 & 0 & \text{-}b_4 \\ \hline 1 & -2 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(b_2)\mathbf{1}+\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{cc|c} 0 & 0 & \text{-}b_1 \\ 1 & 0 & 0 \\ 0 & 0 & \text{-}b_3 \\ 3 & 0 & 3b_2\text{-}b_4 \\ \hline 1 & -2 & b_2 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

Therefore, the reduced row echelon form is

$$\mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 3 & 0 \\ 0 & 0 \end{bmatrix};$$

and x_2 is a free variable. The special solution is $\boldsymbol{x}_a = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. The system is consistent when $b_1 = 0$, $b_3 = 0$ and $b_4 = 3b_2$. In that case by gaussian elimination we get

$$\left[\begin{array}{c|c} \mathbf{R} & \mathbf{0} \\ \hline \mathbf{E} & \boldsymbol{x}_p \\ \hline \mathbf{0} & 1 \end{array}\right],$$

and then a particular solution is $\boldsymbol{x}_p = \begin{pmatrix} b_2 \\ 0 \end{pmatrix}$. Then, the complete solution to the system is

$$\left\{ \boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \exists a \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} b_2 \\ 0 \end{pmatrix} + a \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$$

□

(L-8) Question 7.

$$\left[\begin{array}{c|c} \mathbf{A} & -\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & -b_1 \\ 0 & 1 & -b_2 \\ 2 & 3 & -b_3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(b_2)\mathbf{2}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ [(b_1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 2b_1 + 3b_2 - b_3 \\ \hline 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ \hline 0 & 0 & 1 \end{array}\right]$$

then, the condition is $b_3 - 2b_1 - 3b_2 = 0$; or

$$\mathcal{C}\left(\mathbf{A}\right) = \left\{\boldsymbol{b} \in \mathbb{R}^3 \;\middle|\; b_3 - 2b_1 - 3b_2 = 0\right\} = \left\{\boldsymbol{b} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} -2 & -3 & 1 \end{bmatrix} \boldsymbol{b} = \mathbf{0}\right\}.$$

The rank of $\mathbf{A}$ is 2. An attainable right-hand side $\boldsymbol{b}$ is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}.$$

The null space only has the zero vector $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (no free columns).

□

(L-8) Question 8. We must solve

$$c\begin{pmatrix}10\\30\\12\end{pmatrix}+t\begin{pmatrix}20\\75\\36\end{pmatrix}+p\begin{pmatrix}40\\135\\64\end{pmatrix}=\begin{bmatrix}10&20&40\\30&75&135\\12&36&64\end{bmatrix}\begin{pmatrix}c\\t\\p\end{pmatrix}=\begin{pmatrix}760\\2595\\1224\end{pmatrix}\quad\Longrightarrow\quad \boldsymbol{x}=\begin{pmatrix}4\\6\\15\end{pmatrix};$$

Then they paint 4 cars, 6 trains and 15 planes each week.

```
A = Matrix([ [10, 20, 40], [30, 75, 135], [12, 36, 64] ])
b = Vector( [760, 2595, 1224] )
SEL(A,b,1)
```

(L-8) Question 9(a)

$$\left[\begin{array}{c|c}\mathbf{A} & -\boldsymbol{b}\\ \hline \mathbf{I} & \mathbf{0}\\ \hline \mathbf{0} & 1\end{array}\right]=\left[\begin{array}{ccc|c}1&0&4&-6\\2&1&10&-14\\3&1&c&-20\\ \hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\ \hline 0&0&0&1\end{array}\right]\xrightarrow[{[(6)\mathbf{1}+\mathbf{4}]}]{\substack{\boldsymbol{\tau}\\ [(-4)\mathbf{1}+\mathbf{3}]}}\left[\begin{array}{ccc|c}1&0&0&0\\2&1&2&-2\\3&1&c-12&-2\\ \hline 1&0&-4&6\\0&1&0&0\\0&0&1&0\\ \hline 0&0&0&1\end{array}\right]$$

$$\xrightarrow[{[(2)\mathbf{2}+\mathbf{4}]}]{\substack{\boldsymbol{\tau}\\ [(-4)\mathbf{2}+\mathbf{1}]\\ [(-2)\mathbf{2}+\mathbf{3}]}}\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\1&1&c-14&0\\ \hline 1&0&-4&6\\-2&1&-2&2\\0&0&1&0\\ \hline 0&0&0&1\end{array}\right]=\left[\begin{array}{c|c}\mathbf{L} & \mathbf{0}\\ \hline \mathbf{E} & \boldsymbol{x}_p\\ \hline \mathbf{0} & 1\end{array}\right]$$

Vamos a operar con la matriz ampliada $\left[\mathbf{A}|-\boldsymbol{b}\right]$, para que los cálculos nos sirvan para el apartado siguiente. Por tanto, **A** tendrá menos de 3 pivotes (y por tanto no será invertible) si $c=14$.

□

(L-8) Question 9(b) Cuando $c=14$, las dos primeras variables x_1 y x_2 son pivote, y la tercera es libre. Puesto que solo hay una columna libre, sólo necesitamos una solución del sistema homogéneo para obtener una base del espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$. Así pues

$$\text{Sol.} = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists a \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix} + a \begin{pmatrix} -4 \\ -2 \\ 1 \end{pmatrix} \right\}$$

$\square$

(L-8) Question 9(c) Este sistema (con $c = 14$) tiene un espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$ de dimensión uno (infinitas soluciones). Así pues, sus filas representan tres planos en $\mathbb{R}^3$ que se cortan en una sola recta. Visto por columnas, y puesto que sólo dos de ellas son pivote —y la tercera es libre— el sistema tiene infinitas soluciones, ya que hay infinitas combinaciones de las tres columnas que general el vector del lado derecho $\boldsymbol{b}$.

$\square$

(L-8) Question 10. Let's use the first two vectors as columns one and two of **A**

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & a_{13} \\ 1 & 3 & a_{23} \\ 5 & 1 & a_{33} \end{bmatrix}$$

We know

$$\begin{bmatrix} 1 & 0 & a_{13} \\ 1 & 3 & a_{23} \\ 5 & 1 & a_{33} \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \mathbf{0}; \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

therefore

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$$

□

(L-8) Question 11(a) It has a solution for any $\boldsymbol{b}$ (three pivots in $\mathbb{R}^3$).

□

(L-8) Question 11(b) It has solution only if $b_3 = 0$.

□

(L-8) Question 12.

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & -b_1 \\ 2 & 4 & 0 & 7 & -b_2 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-3)\mathbf{1}+\mathbf{4}]}]{\overset{\tau}{[(-2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -b_1 \\ 2 & 0 & 0 & 1 & -b_2 \\ \hline 1 & -2 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\overset{\tau}{[(-2)\mathbf{4}+\mathbf{1}]}}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -b_1 \\ 0 & 0 & 0 & 1 & -b_2 \\ \hline 7 & -2 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(b_2)\mathbf{2}+\mathbf{5}]}]{\overset{\tau}{[(b_1)\mathbf{1}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 7 & -2 & 0 & -3 & -3b_2+7b_1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & b_2-2b_1 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Since the rank$(\mathbf{A})$ is 2, the system is solvable for any $\boldsymbol{b}$.
The general solution is the set of vectors

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists\, a, b \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} -3b_2+7b_1 \\ 0 \\ 0 \\ b_2-2b_1 \end{pmatrix} + c \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$\square$

(L-8) Question 13(a) **B** es de orden 3 por 4; por tanto los vectores de $\mathcal{N}\left(\mathbf{B}\right)$ pertenecen a $\mathbb{R}^4$.
Por otra parte, puesto que la segunda matriz del producto (llamemosla **E**) es de rango completo, sabemos que es producto de matrices elementales y por tanto invertible.
Si llamamos a la matriz del producto del enunciado **L**, tenemos que

$$\mathbf{B} = \mathbf{L}\mathbf{E} \Rightarrow \mathbf{B}\mathbf{E}^{-1} = \mathbf{L}$$

Por tanto, las columnas nulas de **L** son combinaciones lineales de las columnas de **B**, y las columnas de $\mathbf{E}^{-1}$ nos indican qué combinaciones son. Como **L** tiene dos pivotes, el rango de **B** es dos.
Para encontrar las soluciones del espacio nulo basta con invertir **E** y tomar sus dos últimas columnas (las que transforman las columnas de **B** en columnas de ceros. Así pues,

$$\frac{\begin{bmatrix}\mathbf{E}\end{bmatrix}}{\begin{bmatrix}\mathbf{I}\end{bmatrix}} = \left[\begin{array}{cccc}1 & 1 & -0 & 1\\ 0 & 1 & 1 & -1\\ 0 & 0 & 1 & -0\\ 0 & 0 & 0 & 1\\ \hline 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right] \rightarrow \left[\begin{array}{cccc}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \hline 1 & -1 & 1 & -2\\ 0 & 1 & -1 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right] = \frac{\begin{bmatrix}\mathbf{I}\end{bmatrix}}{\begin{bmatrix}\mathbf{E}^{-1}\end{bmatrix}}$$

El espacio nulo es

$$\mathcal{N}\left(\mathbf{B}\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists\, a, b \in \mathbb{R} \text{ such that } \boldsymbol{x} = a \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

□

(L-8) Question 13(b) A la vista de la primera columna de las matrices que intervienen en el producto, sabemos que la primera columna de **B** es precisamente $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Por tanto, una solución particular es

$$\boldsymbol{x}_p = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Y dado que conocemos el espacio nulo de **B**, la solución completa es

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists\, a, b \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$\square$

(L-8) Question 14(a)

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & -b_1 \\ 2 & 8 & 4 & -b_2 \\ -1 & -4 & -2 & -b_3 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-4)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{3}]}]{\tau} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -b_1 \\ 2 & 0 & 0 & -b_2 \\ -1 & 0 & 0 & -b_3 \\ \hline 1 & -4 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[{[(b_1)\mathbf{1}+\mathbf{4}]}]{\tau} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2b_1 - b_2 \\ -1 & 0 & -0 & -b_1 - b_3 \\ \hline 1 & -4 & -2 & b_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Solution only if $b_2 = 2b_1$ and $b_3 = -b_1$. Since the rank is one, the column space is a line. $\mathbb{R}^3$

$\square$

(L-8) Question 14(b)

$$\left[\begin{array}{cc|c} 1 & 4 & -b_1 \\ 2 & 9 & -b_2 \\ -1 & -4 & -b_3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{[(-4)\overset{\tau}{}\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{2}+\mathbf{1}]}} \left[\begin{array}{cc|c} 1 & 0 & -b_1 \\ 0 & 1 & -b_2 \\ -1 & 0 & -b_3 \\ \hline 9 & -4 & 0 \\ -2 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow{[(b_1)\overset{\tau}{}\mathbf{1}+\mathbf{3}]} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & -b_1 - b_3 \\ \hline 9 & -4 & 9b_1 - 4b_2 \\ -2 & 1 & b_2 \\ \hline 0 & 0 & 1 \end{array}\right]$$

Solution only if $b_3 = -b_1$. Since the rank is two, the column space is a plane.

□

(L-8) Question 15. Assuming that $\boldsymbol{b}$ equals

$$\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_m \end{pmatrix} \text{ then the matrix } \mathbf{A} \text{ will have the form } \quad \mathbf{A} = \begin{bmatrix} b_1 & 0 & 0 \\ b_2 & 0 & 0 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ b_n & 0 & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

□

(L-8) Question 16(a) The linear system is

$$\begin{bmatrix}1 & 6 & 1\\0 & 2 & 3\\1 & 5 & 0\end{bmatrix}\begin{pmatrix}r\\b\\g\end{pmatrix}=\begin{pmatrix}44\\24\\33\end{pmatrix},\quad \text{cuya solución es :}\qquad g=2;\ b=9;\ r=-12.$$

```
A = Matrix([ [1,6,1], [0,2,3], [1,5,0] ])
b = Vector( [44,24,33] )
SEL(A,b,1)
```

□

(L-8) Question 16(b) Old Economy Answer: The store is paying people 12 per gallon to take its red paint. Something doesn't make sense.

□

(L-8) Question 16(c) We can figure out who was undercharged by trial and error. We have to add 4 euros to the right hand side of one of the three equations above, and then solve it again. After some computation, we find that the only way we can get a nonzero solution is if we add 4 euros to the second equation. Therefore Shai was overcharged.

□

(L-8) Question 16(d) Se pueden resolver los tres sistemas del apartado anterior de una sola vez siguiendo las instrucciones. Partimos de la matriz ampliada (y aprovechamos los pasos dados en el primer apartado)

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 1 & -48 & -44 & -44 \\ 0 & 2 & 3 & -24 & -28 & -24 \\ 1 & 5 & 0 & -33 & -33 & -37 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -48 & -44 & -44 \\ 0 & 1 & 0 & -24 & -28 & -24 \\ 0 & 0 & 1 & -33 & -33 & -37 \\ \hline -15 & 5 & 16 & 0 & 0 & 0 \\ 3 & -1 & -3 & 0 & 0 & 0 \\ -2 & 1 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{\tau \\ [(48)\mathbf{1}+\mathbf{4}] \\ [(24)\mathbf{2}+\mathbf{4}] \\ [(33)\mathbf{3}+\mathbf{4}] \\ [(44)\mathbf{1}+\mathbf{5}] \\ [(28)\mathbf{2}+\mathbf{5}] \\ [(33)\mathbf{3}+\mathbf{5}] \\ [(44)\mathbf{1}+\mathbf{6}] \\ [(24)\mathbf{2}+\mathbf{6}] \\ [(37)\mathbf{3}+\mathbf{6}]}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline -15 & 5 & 16 & -72 & 8 & 52 \\ 3 & -1 & -3 & 21 & 5 & -3 \\ -2 & 1 & 2 & -6 & 6 & 10 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

como se puede ver, sólo al alterar el importe de la factura de Belén (de 24 a 28 euros), se encuentran tres precios positivos

$$r = 8; \quad a = 5; \qquad v = 6$$

Moraleja: con el método de eliminación Gauss-Jordan se pueden resolver varios sistemas de ecuaciones a la vez; siempre y cuando compartan la misma matriz de coeficientes $\mathbf{A}$.

□

(L-8) Question 17(a) Si $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ tiene solución, ello significa que existen $x_1, x_2, \dots, x_n$ tales que

$$x_1\mathbf{A}_{|1} + x_2\mathbf{A}_{|2} + \cdots + x_n\mathbf{A}_{|n} = \boldsymbol{b}$$

pero entonces, $\boldsymbol{b}$ es una combinación lineal de las columnas de $\mathbf{A}$, y por lo tanto $\boldsymbol{b} \in \mathcal{C}\left(\mathbf{A}\right)$.

□

(L-8) Question 17(b) Puesto que $\mathbf{A}\boldsymbol{x}_0 = \boldsymbol{b}$ y que $\mathbf{A}\boldsymbol{z} = \boldsymbol{0}$ sumando ambas ecuaciones tenemos

$$\begin{aligned} &\mathbf{A}\boldsymbol{x}_0 + \mathbf{A}\boldsymbol{z} = \boldsymbol{b} + \boldsymbol{0} \\ &\mathbf{A}(\boldsymbol{x}_0 + \boldsymbol{z}) = \boldsymbol{b} && \text{sacando } \mathbf{A} \text{ como factor común.} \end{aligned}$$

Por tanto el vector $\boldsymbol{x}_0 + \boldsymbol{z}$ también es solución del sistema.

□

(L-8) Question 17(c) Dado el resultado del apartado anterior, basta con demostrar que si hay dependencia lineal entre las columnas de **A**, el sistema homogéneo tiene soluciones distintas de la trivial ($\boldsymbol{x} = \mathbf{0}$).
Si hay dependencia lineal entre las columnas significa que al menos una de ellas se puede expresar como combinación lineal de las demás. Supongamos sin pérdida de generalidad que es la primera; entonces

$$\mathbf{A}_{|1} = z_2\mathbf{A}_{|2} + z_3\mathbf{A}_{|3} + \cdots + z_m\mathbf{A}_{|m}$$

pasando la expresión de la derecha al lado izquierdo de la igualdad tenemos:

$$\mathbf{A}_{|1} - z_2\mathbf{A}_{|2} - z_3\mathbf{A}_{|3} - \cdots - z_m\mathbf{A}_{|m} = \mathbf{0}$$

$$\begin{bmatrix}\mathbf{A}_{|1} & \mathbf{A}_{|2} & \cdots & \mathbf{A}_{|m}\end{bmatrix}\begin{pmatrix}1\\ -z_2\\ \vdots\\ -z_m\end{pmatrix} = \mathbf{0};$$

por tanto el vector $\begin{pmatrix}1 & -z_2 & \cdots & -z_m\end{pmatrix}$ y cualquier múltiplo de este son solución al sistema de ecuaciones homogéneo (pertenecen a $\mathcal{N}\left(\mathbf{A}\right)$). Este resultado unido al anterior demuestran que el sistema tiene más de una solución (de hecho tiene infinitas, ya que hay infinitos vectores en el espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$).

□

(L-8) Question 18. Note that the first column plus three times the third gives $\boldsymbol{b}$. Let's see how Gaussian elimination finds the solution:

$$\left[\begin{array}{ccc|c} 3 & 1 & 1 & -6 \\ 1 & -1 & -1 & 2 \\ 0 & 4 & 1 & -3 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(6)\mathbf{3}+\mathbf{4}]}]{\substack{\tau \\ [(-1)\mathbf{3}+\mathbf{2}] \\ [(-3)\mathbf{3}+\mathbf{1}]}} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 4 & 0 & -1 & -4 \\ -3 & 3 & 1 & 3 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -1 & 1 & 6 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(1)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 4 & 0 & -1 & 0 \\ -3 & 3 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -3 & -1 & 1 & 3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$\boldsymbol{x}_p = \begin{pmatrix} 1 & 0 & 3 \end{pmatrix}.$$

□

(L-8) Question 19(a)

$$\begin{cases} x = 2y \\ x + y = 39 \end{cases} \qquad \text{therefore} \qquad \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 39 \end{pmatrix}$$

Using Gaussian elimination

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & 1 & -39 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow{[(2)\overset{\tau}{\mathbf{1}}+\mathbf{2}]} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 3 & -39 \\ \hline 1 & 2 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow{[(13)\overset{\tau}{\mathbf{2}}+\mathbf{3}]} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 3 & 0 \\ \hline 1 & 2 & 26 \\ 0 & 1 & 13 \\ \hline 0 & 0 & 1 \end{array}\right]$$

Hence $x = 26$ and $y = 13$.

$\square$

(L-8) Question 19(b) Since both points lie on the line, we can substitute x and y:

$$\begin{cases} 5 = 2m + c \\ 7 = 3m + c \end{cases} \qquad \text{therefore} \qquad \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{pmatrix} m \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

By elimination

$$\left[\begin{array}{cc|c} 2 & 1 & -5 \\ 3 & 1 & -7 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(5)\mathbf{2}+\mathbf{3}]}]{[(-2)\overset{\tau}{\mathbf{2}}+\mathbf{1}]} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 1 & -2 \\ \hline 1 & 0 & 0 \\ -2 & 1 & 5 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow{[(2)\overset{\tau}{\mathbf{1}}+\mathbf{3}]} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 1 & 0 \\ \hline 1 & 0 & 2 \\ -2 & 1 & 1 \\ \hline 0 & 0 & 1 \end{array}\right]$$

Hence, $c = 1$ and $m = 2$.

□

(L-8) Question 20. The linear system is

$$\begin{aligned} a + b + c &= 4 \\ a + 2b + 4c &= 8 \\ a + 3b + 9c &= 14 \end{aligned}$$

by column reduction we get:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -4 \\ 1 & 2 & 4 & -8 \\ 1 & 3 & 9 & -14 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(4)\mathbf{1}+\mathbf{4}]}]{\overset{\tau}{[(-1)\mathbf{1}+\mathbf{2}]} \; [(-1)\mathbf{1}+\mathbf{3}]} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 3 & -4 \\ 1 & 2 & 8 & -10 \\ \hline 1 & -1 & -1 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\overset{\tau}{[(-3)\mathbf{2}+\mathbf{3}]} \; [(4)\mathbf{2}+\mathbf{4}]} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & -2 \\ \hline 1 & -1 & 2 & 0 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\overset{\tau}{[(1)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ \hline 1 & -1 & 2 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right];$$

so $a = 2$, $b = 1$ and $c = 1$. Therefore, the parabola is $y = 2 + x + x^2$.

□

(L-8) Question 21. Assume $c = 0$, so $\boldsymbol{b} = \begin{pmatrix}2, & 1, & (0+c),\end{pmatrix}$.

$$\left[\begin{array}{c|c} \mathbf{A} & \text{-}\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 1 & 2 & 3 & -1 \\ 0 & 1 & 2 & 0-c \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(2)\mathbf{1}+\mathbf{4}]}]{\tau} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 0-c \\ \hline 1 & -1 & -1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{[(-2)\mathbf{2}+\mathbf{3}] \\ [(-1)\mathbf{2}+\mathbf{4}]}]{\tau} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1-c \\ \hline 1 & -1 & 1 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{L} & -\boldsymbol{c} \\ \hline \mathbf{E} & \boldsymbol{x} \\ \hline \mathbf{0} & 1 \end{array}\right]$$

is not possible to eliminate the third component of $\boldsymbol{c}$ since there are only two pivots. We should replace the zero by a -1 in $\boldsymbol{b}$.

Then a soluton is $\boldsymbol{x}_p = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$; but we have many more, for example:

$$\boldsymbol{x} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

□

(L-8) Question 22. Applying gaussian elimination on the coefficient matrix we get

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ b & 8 \end{bmatrix} \xrightarrow{[(-2)\overset{\tau}{)}\mathbf{1}+\mathbf{2}]} \begin{bmatrix} 2 & 0 \\ b & 8-2b \end{bmatrix} = \mathbf{L}$$

The system is singular if $b = 4$, because $4x + 8y$ is 2 times $2x + 4y$.

$$\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 16 \\ g \end{pmatrix}$$

that is

$$x \begin{pmatrix} 2 \\ 4 \end{pmatrix} + y \begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 16 \\ g \end{pmatrix}$$

The system is asking us to find a linear combination of columns of **A** that equals $\begin{pmatrix} 16 \\ g \end{pmatrix}$.

But in this case, the second column is twice the first (this columns are align); hence, the set of all their linear combinations are a line (all of them lie on the same line), so the system has solution only if the right hand side vector lie on that line (only if it is a multimple of the columns of **A**. That happens when $g = 32$:

$$\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 16 \\ 32 \end{pmatrix}$$

Two possible solutions are

$$\begin{cases} x = 8 \\ y = 0 \end{cases} \qquad \text{or} \qquad \begin{cases} x = 0 \\ y = 4. \end{cases}$$

In fact there are infinite solutions.

□

(L-8) Question 23. By back substitution

$$\begin{aligned} w =& b_3 \\ v =& b_2 - w = b_2 - b_3 \\ u =& b_1 + v - w = b_1 + (b_2 - b_3) - b_3 = b_1 + b_2 - 2b_3 \end{aligned}$$

Then b_3 times the third column, plus $b_2 - b_3$ times the second one, plus $b_1 + b_2 - 2b_3$ times the first one equals $\boldsymbol{b}$.

Let's check the solution:

$$\begin{bmatrix}1 & 1 & 1\\ & 1 & 1\\ & & 1\end{bmatrix}\begin{pmatrix}b_1+b_2-2b_3\\ b_2-b_3\\ b_3\end{pmatrix} = (b_1+b_2-2b_3)\begin{pmatrix}1\\0\\0\end{pmatrix} + (b_2-b_3)\begin{pmatrix}1\\1\\0\end{pmatrix} + b_3\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}(b_1+b_2-2b_3)-(b_2-b_3)+b3\\ (b_2-b_3)+b_3\\ b_3\end{pmatrix} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \boldsymbol{b}$$

$\square$

(L-8) Question 24(a) Since there is only one solution in $\mathbb{R}^2$, both columns of **A** must be linearly independent (rank 2). We also know the second column is equal to the right hand side vector. Then, any matrix as

$$\mathbf{A} = \begin{bmatrix}a & 1\\ b & 2\\ c & 3\end{bmatrix}$$

where the vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ are linearly independent is OK; for example

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 3 \end{bmatrix}$$

□

(L-8) Question 24(b) There is not such a matrix **B**: the right hand side vector of $\mathbb{R}^2$ tells us that we have two equations; but the vector $\boldsymbol{x}$ of $\mathbb{R}^3$ says "there are three unknowns"... in that case there is no solution at all, or there are an infinite number of solutions... but the exercise claims that there is only one!

□

(L-8) Question 25. On the one hand, the right hand side vector of $\mathbb{R}^2$ indicates there are only two equations (**A** has two rows); the vector of unknowns $\boldsymbol{x}$ is also of order two: two unknows (**A** has two columns); and the particular solution tells us that the first column is $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

On the other hand, the solution to the homogeneus system is any multiple of the second column; therefore, the second column must be the zero vector. Then

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}.$$

□

(L-8) Question 26. free
is not
infinite

□

(L-8) Question 27(a) The system could have no solution; for example

$$\begin{bmatrix}1 & 1 & 0 & 0\\ 2 & 2 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix} = \begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$$

Nótese que el rango de la matriz de coeficientes **A** es uno, pero el rango de la matriz ampliada $[\mathbf{A}|\boldsymbol{b}]$ es dos.

□

(L-8) Question 27(b) Since there are more equations than unknowns, if the system has a solution, that solution is not unique.

□

(L-8) Question 27(c) The rigth hand side vector $\boldsymbol{b}$ must be a lienar combination of the columns of **A**; in other words, the matrices **A** and $[\mathbf{A}|\boldsymbol{b}]$ must have the same rank.

□

(L-8) Question 27(d) Since $\boldsymbol{b}$ belongs to $\mathbb{R}^3$, the rank of **A** must be 3.

□

(L-8) Question 28(a) The rank is 4 (there are 4 pivots in the reduced echelon form of **A**).

$$\left\{ \boldsymbol{x} \in \mathbb{R}^7 \;\middle|\; \exists\, a, b, c \in \mathbb{R} \text{ such that } \boldsymbol{x} = a \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

(parametric equation)

□

(L-8) Question 28(b) if $a = x_2,\ b = x_4$ and $c = x_6$:

$$\left\{ \boldsymbol{x} \in \mathbb{R}^7 \;\middle|\; \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_2 - 2x_4 + x_6 \\ x_2 \\ -x_4 - x_6 \\ x_4 \\ 0 \\ x_6 \\ 0 \end{pmatrix} \right\}$$

so, it is the subset of vectors in $\mathbb{R}^7$ that satifies

$$\begin{cases} x_1 = x_2 - 2x_4 + x_6 \\ x_3 = -x_4 - x_6 \\ x_5 = 0 \\ x_7 = 0 \end{cases}$$

, hence

$$\left\{ \boldsymbol{x} \in \mathbb{R}^7 \;\middle|\; \begin{bmatrix} 1 & -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \right\}$$

(cartesian equation)

□

(L-8) Question 28(c) No, since $\mathcal{C}\left(\mathbf{A}\right) = \mathbb{R}^4$ then $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has solution for any vector $\boldsymbol{b}$ in $\mathbb{R}^4$.

□

(L-8) Question 28(d)

$$\boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ -3 \\ 0 \end{pmatrix}$$

□

(L-8) Question 28(e)

$$\left\{ \boldsymbol{x} \in \mathbb{R}^7 \;\middle|\; \exists\, a, b, c \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix}1\\1\\1\\1\\1\\1\\1\end{pmatrix} + a\begin{pmatrix}1\\1\\0\\0\\0\\0\\0\end{pmatrix} + b\begin{pmatrix}-2\\0\\-1\\1\\0\\0\\0\end{pmatrix} + c\begin{pmatrix}1\\0\\-1\\0\\0\\1\\0\end{pmatrix} \right\}$$

□

(L-8) Question 29(a) El espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$ contiene infinitos vectores, es decir, su dimensión es mayor o igual a uno. Lo sabemos ya que el sistema puede tener infinitas soluciones (una particular mas cualquiera de las del espacio nulo).

□

(L-8) Question 29(b) El espacio $\mathcal{C}\left(\mathbf{A}\right)$ no puede ser todo $\mathbb{R}^m$, pues en ese caso el sistema *siempre* tendría solución, contrariamente a lo que dice el enunciado.

□

(L-8) Question 29(c) Puesto que el sistema puede no tener solución, no todas las filas son pivote (es posible encontrar ecuaciones (0=1)), es decir, que el rango r es menor que el número de filas m.

$$r < m$$

Por otra parte, cuando hay solución, hay infinitas; es decir, el espacio nulo contiene infinitos vectores, por tanto hay columnas libres, es decir, no todas las columnas son pivote.

$$r < n.$$

□

(L-8) Question 29(d) No es posible. Si $\boldsymbol{x}_p$ es solución para el lado derecho $\boldsymbol{b}$, también lo es $\boldsymbol{x}_p + \boldsymbol{x}_n$ para todo vector $\boldsymbol{x}_n$ del espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$.

□

(L-8) Question 30(a)

$$\left[\begin{array}{cccc} 2 & 3 & 1 & -1 \\ 6 & 9 & 3 & -2 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(2)\mathbf{2}] \\ [(-3)\mathbf{1}+\mathbf{2}] \\ [(2)\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(2)\mathbf{4}] \\ [(1)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{cccc} 2 & 0 & 0 & 0 \\ 6 & 0 & 0 & 2 \\ \hline 1 & -3 & -1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

El espacio nulo es el conjunto de vectores que son combinación lineal de las soluciones especiales; es decir

$$\left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2,\ \boldsymbol{v} = \begin{bmatrix} -3 & -1 \\ 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \boldsymbol{p} \right\}$$

□

(L-8) Question 30(b) Una solución particular inmediata es el vector

$$\boldsymbol{x}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

(la última columna multiplicada por -1). Así pues, la solución general es cualquier vector $\boldsymbol{x}$ que se pueda expresar como

$$\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{z},$$

donde $\boldsymbol{z}$ es un vector del espacio nulo descrito en el apartado anterior. De forma más explícita

$$\text{Conjunto de vectores: } \left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 \text{ tal que } \boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} + \begin{bmatrix} -1 & -3 \\ 0 & 2 \\ 2 & 0 \\ 0 & 0 \end{bmatrix} \boldsymbol{p} \right\}.$$

□

(L-8) Question 30(c) Cuando una matriz $\underset{m\times n}{\mathbf{A}}$ tiene rango igual a m, la dimensión del espacio columna es m, es decir, $\mathcal{C}\left(\mathbf{A}\right) = \mathbb{R}^m$; y puesto que $\boldsymbol{b} \in \mathbb{R}^m$ el sistema siempre tiene solución, sea cual sea el vector $\boldsymbol{b} \in \mathbb{R}^m$.
El número de soluciones especiales (la dimensión del espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$) es igual al número de columnas libres, es decir, igual a $n - m$ (nótese que sabemos que $n > m$ ya que el rango de matriz es m, si n fuera menor, el rango no podría ser m).

□

(L-8) Question 31. By elimination

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 2 & 2 & -1 & -1 \\ 0 & 1 & c & -2 \end{array}\right] \xrightarrow[{[(1)\mathbf{1}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 0 & -5 & 1 \\ 0 & 1 & c & -2 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(5)\mathbf{4}] \\ [(1)\mathbf{3}+\mathbf{4}]}}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 0 & -5 & 0 \\ 0 & 1 & c & c-10 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-c)\mathbf{2}+\mathbf{3}] \\ [(10-c)\mathbf{2}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 0 & -5 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

The system has always only one solution, for any value c.

```
c = sympy.symbols('c')
A = Matrix([[1,1,2],[2,2,-1],[0,1,c]])
b = Vector([-1,-1,-2])
Elim(A.concatena(Matrix([b]),1),1)
```

□

(L-8) Question 32(a)

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 2 & -3 & m & -3 \\ -1 & 2 & 3 & -2m \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-2)\mathbf{1}+\mathbf{3}]}]{\overset{\tau}{[(1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 2 & -1 & m-4 & -3 \\ -1 & 1 & 5 & -2m \\ \hline 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[{[(-m+4)\mathbf{2}+\mathbf{3}]}]{\overset{\tau}{[(-1)\mathbf{2}]}\ [(-2)\mathbf{2}+\mathbf{1}]} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 1 & -1 & m+1 & -2m \\ \hline 3 & -1 & m-6 & 0 \\ 2 & -1 & m-4 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(3)\mathbf{2}+\mathbf{4}]}]{\overset{\tau}{[(1)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & m+1 & -2(m+1) \\ \hline 3 & -1 & m-6 & 0 \\ 2 & -1 & m-4 & -1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\overset{\tau}{[(2)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & m+1 & 0 \\ \hline 3 & -1 & m-6 & 2(m-6) \\ 2 & -1 & m-4 & 2(m-4)-1 \\ 0 & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$\square$

(L-8) Question 32(b)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 3 & -1 & -7 & 0 \\ 2 & -1 & -5 & -1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Una solución particular es $\boldsymbol{x}_p = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$, y las soluciones al sistema homogéneo son los múltiplos del vector $\boldsymbol{x}_n = \begin{pmatrix} -7 \\ -5 \\ 1 \end{pmatrix}$. Por tanto, la solución completa al sistema son todos los vectores que se pueden escribir como $\boldsymbol{x} = \boldsymbol{x}_p + a\boldsymbol{x}_n$ para cualquier número real a.

□

(L-8) Question 32(c) El conjunto de puntos que son solución al sistema del apartado anterior es una recta en $\mathbb{R}^3$.
No es posible que el conjunto de soluciones sea un plano en ningún caso; para que fuera posible sería necesario que la matriz de coeficientes del sistema fuera de rango 1. Pero en este caso el rango es 2 para $m = -1$ o rango 3 cuando $m \neq -1$. En este último caso (rango 3), el conjunto de soluciones es un punto en $\mathbb{R}^3$.

□

(L-8) Question 32(d) En este caso tenemos

$$\left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 1 & -1 & 2 & 0\\ \hline 3 & -1 & -5 & -10\\ 2 & -1 & -3 & -7\\ 0 & 0 & 1 & 2\\ \hline 0 & 0 & 0 & 1\end{array}\right]$$

Por tanto la solución en este caso es

$$\boldsymbol{x} = \begin{pmatrix}-10\\ -7\\ 2\end{pmatrix}.$$

$\square$

(L-8) Question 33(a)

$$\left[\begin{array}{ccc}1 & 1 & 1\\ 1 & 0 & 2\\ \hline 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{3}]}]{\overset{\tau}{[(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc}1 & 0 & 0\\ 1 & -1 & 1\\ \hline 1 & -1 & -1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right] \xrightarrow{\overset{\tau}{[(1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc}1 & 0 & 0\\ 1 & -1 & 0\\ \hline 1 & -1 & -2\\ 0 & 1 & 1\\ 0 & 0 & 1\end{array}\right]$$

False. Since the solutions are multiples of

$$\boldsymbol{x}_0 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix};$$

the set of solutions is the whole line $c\boldsymbol{x}_0$ for all c.

□

(L-8) Question 33(b) True

□

(L-8) Question 33(c) False

□

(L-8) Question 33(d) True. It is a line through the origin.

□

(L-8) Question 33(e) True.

□

(L-8) Question 33(f) False.

□

(L-8) Question 34(f)

$$
\left[\begin{array}{cc|c}
1 & 0 & -b_1 \\
4 & 1 & -b_2 \\
2 & -1 & -b_3 \\
\hline
1 & 0 & 0 \\
0 & 1 & 0 \\
\hline
0 & 0 & 1
\end{array}\right]
\xrightarrow[{[(b_2)\mathbf{2}+\mathbf{3}]}]{\overset{\tau}{[(-4)\mathbf{2}+\mathbf{1}]}}
\left[\begin{array}{cc|c}
1 & 0 & -b_1 \\
0 & 1 & 0 \\
6 & -1 & -b_2-b_3 \\
\hline
1 & 0 & 0 \\
-4 & 1 & b_2 \\
\hline
0 & 0 & 1
\end{array}\right]
$$

$$
\xrightarrow{\overset{\tau}{[(b_1)\mathbf{1}+\mathbf{3}]}}
\left[\begin{array}{cc|c}
1 & 0 & 0 \\
0 & 1 & 0 \\
6 & -1 & 6b_1-b_2-b_3 \\
\hline
1 & 0 & b_1 \\
-4 & 1 & b_2-4b_1 \\
\hline
0 & 0 & 1
\end{array}\right]
$$

Any vector in $\mathbb{R}^3$ such that $6b_1 - b_2 - b_3 = 0$.

$\square$

(L-8) Question 35(a)

```
A = Matrix( [ [0,3,3,0], [4,3,0,0], [0,0,2,2] ] )
b = Vector( [39, 44, 22] )
SEL(A,b,1)
```

$$\left[\begin{array}{cccc|c} 0 & 3 & 3 & 0 & -39 \\ 4 & 3 & 0 & 0 & -44 \\ 0 & 0 & 2 & 2 & -22 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 0 & 12 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 \\ \hline 1 & -3 & 3 & -3 & 19/2 \\ 0 & 4 & -4 & 4 & 2 \\ 0 & 0 & 4 & -4 & 11 \\ 0 & 0 & 0 & 4 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Puesto que el rango es tres (igual al número de filas) siempre hay solución al sistema; pero como hay columnas libres, el sistema tiene infinitas soluciones. (Nota: no es necesario llegar a la forma escalonada reducida, con llegar a la forma pre-escalonada **K**, es suficiente para ver que hay tres pivotes y una columna libre. Con los pasos anteriores hemos obtenido una solución particular, pero tampoco esto es necesario para responder a la pregunta; bastaba trabajar con la matriz de coeficientes **A** y mirar su rango.). $\square$

(L-8) Question 35(b) En este caso el sistema se reduce a

$$\begin{bmatrix} 3 & 3 & 0 \\ 3 & 4 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{pmatrix} a \\ i \\ d \end{pmatrix} = \begin{pmatrix} 39 \\ 44 \\ 22 \end{pmatrix}$$

donde en este caso i hace referencia al precio tanto de la entrada infantil como a la de tercera edad (que son iguales).

$$\left[\begin{array}{ccc|c} 3 & 3 & 0 & -39 \\ 3 & 4 & 0 & -44 \\ 0 & 2 & 2 & -22 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(13)\mathbf{1}+\mathbf{4}]}]{\substack{\tau \\ [(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 3 & 1 & 0 & -5 \\ 0 & 2 & 2 & -22 \\ \hline 1 & -1 & 0 & 13 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(6)\mathbf{3}+\mathbf{4}]}]{\substack{\tau \\ [(5)\mathbf{2}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ \hline 1 & -1 & 0 & 8 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 6 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Por tanto los precios son 8 para adultos, 5 para infantiles (y tercera edad) y 6 la tarifa reducida.

□

(L-8) Question 36(a)

$$
\left[\begin{array}{cccc|c}
1 & 2 & -1 & 1 & 1 \\
-1 & -2 & 3 & 5 & 5 \\
-1 & -2 & -1 & -7 & -7 \\
\hline
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 1
\end{array}\right]
\xrightarrow[{[(-1)\mathbf{1}+\mathbf{5}]}]{\substack{\tau \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(1)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{4}]}}
\left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 2 & 6 & 6 \\
-1 & 0 & -2 & -6 & -6 \\
\hline
1 & -2 & 1 & -1 & -1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

$$
\xrightarrow{\substack{\tau \\ [(-3)\mathbf{3}+\mathbf{4}] \\ [(-3)\mathbf{3}+\mathbf{5}]}}
\left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 2 & 0 & 0 \\
-1 & 0 & -2 & 0 & 0 \\
\hline
1 & -2 & 1 & -4 & -4 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -3 & -3 \\
0 & 0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

Tanto la matriz de coeficientes, cómo la matriz ampliada tienen rango 2.

$\square$

(L-8) Question 36(b) Una solución particular es

$$\boldsymbol{x}_p = \begin{pmatrix} -4 \\ 0 \\ -3 \\ 0 \end{pmatrix},$$

Solución al sistema homogéneo es el conjunto de combinaciones lineales de los vectores

$$\boldsymbol{x}_a = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{y} \quad \boldsymbol{x}_b = \begin{pmatrix} -4 \\ 0 \\ -3 \\ 1 \end{pmatrix},$$

Así pues, la solución al sistema propuesto es

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists\, a, b \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} -4 \\ 0 \\ -3 \\ 0 \end{pmatrix} + a \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} -4 \\ 0 \\ -3 \\ 1 \end{pmatrix} \right\}$$

□

(L-8) Question 36(c) Es un **plano** paralelo al generado por las combinaciones lineales de $\boldsymbol{x}_a$ y $\boldsymbol{x}_b$ (que es la solución del sistema homogéneo) pero que pasa por el punto $\boldsymbol{x}_p = (-4, 0, -3, 0,)$ (que es uno de los infinitos vectores que resuelven el sistema completo).

□

(L-8) Question 37(a) We need a rank 3 matrix; by Gaussian elimination we get:

$$\left[\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ a & 1 & 1 & 2 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{4}]}]{\substack{\tau \\ [(-1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 \\ a & 1 & 1-a & 2-a \\ \hline 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[{[(-1)\mathbf{2}+\mathbf{4}]}]{\substack{\tau \\ [(-1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ a & 1 & -a & -a \\ \hline 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-1)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ a & 1 & -a & 0 \\ \hline 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

therefore, if $a \neq 0$ the rank of $\mathbf{A}$ is 3, and the dimension of $\mathcal{N}\left(\mathbf{A}\right)$ is one.

□

(L-8) Question 37(b) When $a = 0$; in that case $\dim \mathcal{N}\left(\mathbf{A}\right) = 2$.

□

(L-8) Question 38.

```
A = Matrix([[1,3,2,4,-3], [2,6,0,-1,-2], [0,0,6,2,-1], [1,3,-1,4,2]])
b = Vector([-7,0,12,-6])
SEL(A,b,1)
```

Conjunto de vectores: $\left\{ \boldsymbol{x} \in \mathbb{R}^5 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1 \text{ tal que } \boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 3 \\ -2 \\ 2 \end{pmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \boldsymbol{p} \right\}$

□

(L-8) Question 39(a)

$$\left[\begin{array}{c|c} \mathbf{A} & -\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 3 & 2 & 2 & -2 \\ 2 & 7 & 6 & 8 & -7 \\ 3 & 9 & 6 & 7 & -7 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(2)\mathbf{1}+\mathbf{5}]}]{\substack{\tau \\ [(-3)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{3}] \\ [(-2)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 4 & -3 \\ 3 & 0 & 0 & 1 & -1 \\ \hline 1 & -3 & -2 & -2 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{\tau \\ [(-2)\mathbf{2}+\mathbf{3}] \\ [(-4)\mathbf{2}+\mathbf{4}] \\ [(3)\mathbf{2}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & -1 \\ \hline 1 & -3 & 4 & 10 & -7 \\ 0 & 1 & -2 & -4 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(1)\mathbf{4}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ \hline 1 & -3 & 4 & 10 & 3 \\ 0 & 1 & -2 & -4 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} \mathbf{L} & \mathbf{0} \\ \hline \mathbf{E} & \boldsymbol{x}_p \\ \hline \mathbf{0} & 1 \end{array}\right]$$

La solución es el subconjunto de vectores de $\mathbb{R}^4$

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists\, a \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 1 \end{pmatrix} + a \begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Dicho conjunto de soluciones es una recta, pero puesto que no pasa por el origen de coordenadas (no contiene a $\mathbf{0}$), no es espacio vectorial. Es decir, es una recta en la dirección del vector del espacio nulo $\boldsymbol{x}_a$, que pasa por el punto $\boldsymbol{x}_p$, pero no por el origen.

$\square$

(L-8) Question 39(b) Puesto que hay tres columnas pivote (tres columnas linealmente independientes), el espacio columna es todo el espacio $\mathbb{R}^3$.
Si en lugar de un 7 hubiera un 6, en el segundo paso de eliminación generaríamos una fila de ceros, y por tanto habría sólo dos pivotes y $\mathcal{C}\left(\mathbf{A}\right)$ sería tan sólo un plano

dentro de $\mathbb{R}^3$ que pasa por el origen.

$$\left[\begin{array}{c|c}\mathbf{M} & -\boldsymbol{b}\\ \hline \mathbf{I} & \mathbf{0}\\ \hline \mathbf{0} & 1\end{array}\right]=\left[\begin{array}{cccc|c}1&3&2&2&-2\\2&7&6&8&-7\\3&9&6&6&-7\\ \hline 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\ \hline 0&0&0&0&1\end{array}\right]\xrightarrow{\substack{\tau\\ [(-3)\mathbf{1}+\mathbf{2}]\\ [(-2)\mathbf{1}+\mathbf{3}]\\ [(-2)\mathbf{1}+\mathbf{4}]\\ [(2)\mathbf{1}+\mathbf{5}]}}\left[\begin{array}{cccc|c}1&0&0&0&0\\2&1&2&4&-3\\3&0&0&0&-1\\ \hline 1&-3&-2&-2&2\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\ \hline 0&0&0&0&1\end{array}\right]$$

$$\xrightarrow{\substack{\tau\\ [(-2)\mathbf{2}+\mathbf{3}]\\ [(-4)\mathbf{2}+\mathbf{4}]\\ [(3)\mathbf{2}+\mathbf{5}]}}\left[\begin{array}{cccc|c}1&0&0&0&0\\2&1&0&0&0\\3&0&0&0&-1\\ \hline 1&-3&4&10&-7\\0&1&-2&-4&3\\0&0&1&0&0\\0&0&0&1&0\\ \hline 0&0&0&0&1\end{array}\right]$$

□

(L-8) Question 39(c) La primera parte es sencilla, basta elegir como lado derecho $\boldsymbol{b}$ cualquiera de las columnas de $\mathbf{M}$. La segunda parte también es fácil empleando lo que sabemos. Si mantenemos el vector derecho $\boldsymbol{b}$ del enunciado, el sistema no tiene solución.

$\square$

(L-9) Question 1.

$$\left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \hline 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{2}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ \hline 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[{[(-1)\mathbf{3}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Since there are only three pivots, the four columns are linearly dependent (in fact if we add the first and third columns and then we substract the second one we get the fourth).

Extending the matrix with the new column $\begin{pmatrix}0, & 0, & 0, & 1,\end{pmatrix}$, we have

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{3}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{2}] \\ [(1)\mathbf{2}+\mathbf{4}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ \hline 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

It is not possible transforming the right hand side vector in a column of zeros by column elimination. Therefore, the system has no solution. There isn't any linear combination of the columns that reaches the right hand side vector in $\mathbb{R}^4$. Therefore, those four columns don't span $\mathbb{R}^4$.

□

(L-9) Question 2(a) *might not* span (if there is not a subset of 4 independent vectors).

□

(L-9) Question 2(b) Those vectors *are not* linearly independent (there are more than four!).

□

(L-9) Question 2(c) The system *might not have* a solution (if $\boldsymbol{b}$ isn't a linear combination of the columns of **A**).

□

(L-9) Question 2(d) $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ *does not have* a sole solution. There are more unknows (6) than equations(4), therefore there is no solution to the system at all, or there are infinite solutions (but never only one!).

□

(L-9) Question 3(a) Matrix **B** should have order 3×2. We also know there is only one solution to the system (the complete solution consists of only one vector), and the null space consists of only the zero vector. Therefore the rank is two (both columns are independent). One example of such a matrix is

$$\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 4 & 0 \end{bmatrix}$$

but also any other matrix such as

$$\mathbf{B} = \begin{bmatrix} 1 & a \\ 2 & b \\ 4 & c \end{bmatrix}$$

if both columns are independent.

□

(L-9) Question 3(b) There is not such a matrix:
On the one hand the order of **C** should be 2×3. (two equations and three unknowns), and then, at least one column is free. Therefore there will be infinte solutions.
But on the other hand the exercise claims that the complete solution is only one vector. . . that is impossible.

□

(L-9) Question 4. A system of vectors is independent when $x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_k\boldsymbol{v}_k = \mathbf{0}$ if and only if $x_1 = x_2 = \cdots = x_k = 0$ (no free columns).
In the first case:

$$\begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \mathbf{0}$$

has the only solution $x_1 = x_2 = x_3 = 0$. Therefore, these three vectors are independent.
But in the second case

$$\begin{bmatrix}1 & 1 & 1 & 2\\0 & 1 & 1 & 3\\0 & 0 & 1 & 4\end{bmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \mathbf{0}$$

is a system with infinite solutions; for example $\boldsymbol{x} = \begin{pmatrix}1\\1\\-4\\1\end{pmatrix}$. So

$\boldsymbol{v}_1 + \boldsymbol{v}_2 - 4\boldsymbol{v}_3 + \boldsymbol{x}_4 = \mathbf{0}$, and then, the four vectors are dependent.

□

(L-9) Question 5. If $\mathbf{A}^\intercal = 2\mathbf{A}$, then also $\mathbf{A} = 2\mathbf{A}^\intercal = 2(2\mathbf{A}) = 4\mathbf{A}$ so $\mathbf{A} = \mathbf{0}$; and, of course, the rows of $\mathbf{A}$ are then linearly dependent.

□

(L-9) Question 6(a) **No, they don't span.** only two vectors can't span the three dimensional space $\mathbb{R}^3$.

□

(L-9) Question 6(b) **Yes.** Since we can find three pivots, we known that these three vectors are linearly independent. We can always find a solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ for any vector $\boldsymbol{b}$ in $\mathbb{R}^3$, $(\mathcal{C}\left(\mathbf{A}\right) = \mathbb{R}^3)$.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{R}$$

□

(L-9) Question 6(c) **No, they don't span.** We can only find two pivots. We can't find a solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ for some vectors $\boldsymbol{b}$ in $\mathbb{R}^3$, $(\mathcal{C}\left(\mathbf{A}\right) \neq \mathbb{R}^3)$.

□

(L-9) Question 6(d) **Yes.** Since we can find three pivots, $(\mathcal{C}\left(\mathbf{A}\right) = \mathbb{R}^3)$.

□

(L-9) Question 7(a) **Dependent.** Solving the linear system

$$a\begin{pmatrix}-1\\2\\3\end{pmatrix}+b\begin{pmatrix}2\\1\\-1\end{pmatrix}=\begin{pmatrix}4\\7\\3\end{pmatrix}$$

$$\left[\begin{array}{cc|c}-1&2&-4\\2&1&-7\\3&-1&-3\\\hline 1&0&0\\0&1&0\\\hline 0&0&1\end{array}\right]\xrightarrow[{[(2)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}}\left[\begin{array}{cc|c}-1&0&-4\\2&5&-7\\3&5&-3\\\hline 1&2&0\\0&1&0\\\hline 0&0&1\end{array}\right]\xrightarrow[{[(-4)\mathbf{1}+\mathbf{3}]}]{\boldsymbol{\tau}}\left[\begin{array}{cc|c}-1&0&0\\2&5&-15\\3&5&-15\\\hline 1&2&-4\\0&1&0\\\hline 0&0&1\end{array}\right]$$

$$\xrightarrow[{[(3)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}}\left[\begin{array}{cc|c}-1&0&0\\2&5&0\\3&5&0\\\hline 1&2&2\\0&1&3\\\hline 0&0&1\end{array}\right]$$

we find that $2(-1,2,3,)+3(2,1,-1,)=(4,7,3,)$. Three vectors and only two pivots!

□

(L-9) Question 7(b) **Independent.** Two vectors, two pivots.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \mathbf{R}$$

□

(L-9) Question 7(c) **Dependent.** Three vectors, two pivots

$$\left[\begin{array}{cc|c} 1 & 2 & -8 \\ 2 & 3 & 2 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-2)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \left[\begin{array}{cc|c} 1 & 0 & -8 \\ 2 & -1 & 2 \\ \hline 1 & -2 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(8)\mathbf{1}+\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 2 & -1 & 18 \\ \hline 1 & -2 & 8 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[{[(18)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 2 & -1 & 0 \\ \hline 1 & -2 & -28 \\ 0 & 1 & 18 \\ \hline 0 & 0 & 1 \end{array}\right]$$

then $18(2, 3,) - 28(1, 2,) = (8, -2,)$.

□

(L-9) Question 7(d) **Dependent.** The fourth vector (polynomial) is the sum of the other three. Using the coeficients of the polynomials as vectors in $\mathbb{R}^4$ we can solve the system:

$$
\left[\begin{array}{ccc|c}
0 & 1 & 1 & -1 \\
1 & -1 & 0 & 0 \\
2 & 0 & 0 & -1 \\
1 & 0 & 1 & -1 \\ \hline
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\ \hline
0 & 0 & 0 & 1
\end{array}\right]
\xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}+\mathbf{3}] \\ [(1)\mathbf{2}+\mathbf{4}] \\ [(1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(1)\mathbf{1}+\mathbf{4}]}}
\left[\begin{array}{ccc|c}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
2 & 2 & -2 & 1 \\
1 & 1 & 0 & 0 \\ \hline
1 & 1 & -1 & 1 \\
0 & 1 & -1 & 1 \\
0 & 0 & 1 & 0 \\ \hline
0 & 0 & 0 & 1
\end{array}\right]
\xrightarrow{\substack{\boldsymbol{\tau} \\ [(2)\mathbf{4}] \\ [(1)\mathbf{3}+\mathbf{4}]}}
$$

$$
\left[\begin{array}{ccc|c}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
2 & 2 & -2 & 0 \\
1 & 1 & 0 & 0 \\ \hline
1 & 1 & -1 & 1 \\
0 & 1 & -1 & 1 \\
0 & 0 & 1 & 1 \\ \hline
0 & 0 & 0 & 2
\end{array}\right]
\xrightarrow{\substack{\boldsymbol{\tau} \\ [(\frac{1}{2})\mathbf{4}]}}
\left[\begin{array}{ccc|c}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
2 & 2 & -2 & 0 \\
1 & 1 & 0 & 0 \\ \hline
1 & 1 & -1 & 1/2 \\
0 & 1 & -1 & 1/2 \\
0 & 0 & 1 & 1/2 \\ \hline
0 & 0 & 0 & 1
\end{array}\right]
$$

finding the same result: the last polynomial is half the sum of the other three.

□

(L-9) Question 8. That means the null space has dimension 0 (there aren't any free columns). All the n columns are pivot columns (rank n). All columns are linearly independent.

□

(L-9) Question 9. When the columns are linear independent, it is no possible to get a zero column by Gauss-Jordan column elimination. Hence the colums are linear independent if and only all columns are pivot columns in any (pre)echelon form of the matrix. But each pivot apears in a different row, and since there are only 4 rows, there are, at most, 4 pivots. Thus, at least two columns are free columns.

□

(L-9) Question 10(a) Since the nullspace is spanned by the given three vectors, we may simply take **B** to consist of the three vectors as columns, i.e.,

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ -1 & 1 & 3 \\ 3 & 4 & 1 \end{bmatrix}$$

B need not be square.

□

(L-9) Question 10(b) For example, we may simply add a zero column to **B**:

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 2 & 1 & -1 \\ 0 & -1 & 1 & 3 \\ 0 & 3 & 4 & 1 \end{bmatrix}$$

Or, we could interchange two columns. Or we could multiply one of the columns by -1. For example:

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -3 \\ 3 & 4 & -1 \end{bmatrix}$$

Or we could replace one of the columns by a linear combination of that column with the other two columns (any invertible column operation). Or we could replace **B** by $-$**B** or 2**B**. There are many possible solutions. In any case, the solution shouldn't require any significant calculation!

□

(L-9) Question 10(c) Since any solution $\boldsymbol{x}$ to the equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is of the form $\boldsymbol{x}_p + \boldsymbol{n}$ for some vector $\boldsymbol{n}$ in the nullspace, the vector $\boldsymbol{x} - \boldsymbol{x}_p$ must lie in the nullspace $\mathcal{N}(\mathbf{A})$. Thus, we want to look at:

$$\boldsymbol{x}_Z - \boldsymbol{x}_p = \begin{pmatrix} 0 \\ -1 \\ 0 \\ -4 \end{pmatrix}, \quad \boldsymbol{x} - \boldsymbol{x}_p \begin{pmatrix} 0 \\ -1 \\ 0 \\ -3 \end{pmatrix}.$$

To determine whether a vector $\boldsymbol{y}$ lies in the nullspace $\mathcal{N}(\mathbf{A})$, we can just check whether it is in the column space of $\mathbf{B}$, i.e. check whether $\mathbf{B}\boldsymbol{z} = \boldsymbol{y}$ has a solution. As we learned in class, we can check this just by doing elimination:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 2 & 1 & -1 & -1 \\ -1 & 1 & 3 & 0 \\ 3 & 4 & 1 & -a \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[]{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{2}+\mathbf{3}] \\ [(1)\mathbf{2}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 3 & 4 & 0 & -a-4 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 3 & 4 & 0 & -a-4 \\ \hline 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

We can get a solution if and only if $a = -4$. So Zarkon is correct.

□

(L-9) Question 11(a)

$$
\left[\begin{array}{ccccc|c}
1 & 2 & -1 & 0 & 0 & -a \\
1 & 2 & 0 & 2 & 2 & -b \\
1 & 2 & -1 & 0 & 0 & -c \\
2 & 4 & 0 & 4 & 4 & -d \\
\hline
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(1)\mathbf{1}+\mathbf{3}] \\ [(a)\mathbf{1}+\mathbf{6}]}}
\left[\begin{array}{ccccc|c}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 & 2 & a-b \\
1 & 0 & 0 & 0 & 0 & a-c \\
2 & 0 & 2 & 4 & 4 & 2a-d \\
\hline
1 & -2 & 1 & 0 & 0 & a \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

$$
\xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{3}+\mathbf{4}] \\ [(-2)\mathbf{3}+\mathbf{5}] \\ [(b-a)\mathbf{3}+\mathbf{6}]}}
\left[\begin{array}{ccccc|c}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & a-c \\
2 & 0 & 2 & 0 & 0 & 2b-d \\
\hline
1 & -2 & 1 & -2 & -2 & b \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & -2 & b-a \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

Therefore

$$\left[\begin{pmatrix}1\\1\\1\\2\end{pmatrix}; \quad \begin{pmatrix}-1\\0\\-1\\0\end{pmatrix};\right] \quad \text{and also} \quad \left[\begin{pmatrix}1\\1\\1\\2\end{pmatrix}; \quad \begin{pmatrix}0\\1\\0\\2\end{pmatrix};\right].$$

□

(L-9) Question 11(b)

$$\left[\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}; \quad \begin{pmatrix}-2\\0\\-2\\1\\0\end{pmatrix}; \quad \begin{pmatrix}-2\\0\\-2\\0\\1\end{pmatrix};\right]$$

□

(L-9) Question 11(c) $c - a = 0$ and $d - 2b = 0$.

□

(L-9) Question 11(d)

$$\left\{ \boldsymbol{x}\in\mathbb{R}^5 \;\middle|\; \exists\, a,b,c\in\mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix} + a\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix} + b\begin{pmatrix}-2\\0\\-2\\1\\0\end{pmatrix} + c\begin{pmatrix}-2\\0\\-2\\0\\1\end{pmatrix} \right\}.$$

□

(L-9) Question 12. Si a una matriz **A** se le "añade" una nueva columna extra $\boldsymbol{b}$, entonces el espacio columna se vuelve más grande, a no ser que el vector $\boldsymbol{b}$ ya esté en $\mathcal{C}\left(\mathbf{A}\right)$.

Caso en que se hace más grande

$$\mathbf{A} = \begin{bmatrix}1&0\\0&0\end{bmatrix};\qquad \boldsymbol{b} = \begin{pmatrix}1\\1\end{pmatrix};\qquad [\mathbf{A}\;\boldsymbol{b}] = \begin{bmatrix}1&0&1\\0&0&1\end{bmatrix}.$$

En este caso $\mathcal{C}\left([\mathbf{A}\;\boldsymbol{b}]\right)$ es más grande que $\mathcal{C}\left(\mathbf{A}\right)$; y el sistema $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$ no tiene solución por NO pertenecer $\boldsymbol{b}$ al espacio columna de **A** (es decir, porque $\boldsymbol{b}\notin\mathcal{C}\left(\mathbf{A}\right)$). Nótese que en este caso ninguna combinación lineal de las columnas de **A** puede ser igual a $\boldsymbol{b}$. Nótese que el rango de **A** es 1, pero el de la matriz ampliada $[\mathbf{A}\mid -\boldsymbol{b}]$ es 2, así que el método de Gauss visto en clase fallaría.

Caso en el que es igual de grande

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \qquad [\mathbf{A}\ \boldsymbol{b}] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

En este caso $\mathcal{C}\left([\mathbf{A}\ \boldsymbol{b}]\right) = \mathcal{C}\left(\mathbf{A}\right)$; y el sistema $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ tiene solución por pertenecer $\boldsymbol{b}$ al espacio columna de $\mathbf{A}$ (es decir, porque $\boldsymbol{b} \in \mathcal{C}\left(\mathbf{A}\right)$).

□

(L-9) Question 13. $\mathcal{C}\left(\mathbf{A}\right) = \mathbb{R}^9$.

□

(L-9) Question 14. Puesto que todo vector $\boldsymbol{a}$ en $\mathcal{V}$, se puede expresar como $x_1\boldsymbol{v}_1 + \cdots + x_n\boldsymbol{v}_n$; es decir como

$$\boldsymbol{a} = \begin{bmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{V}\boldsymbol{x}, \qquad \text{donde} \quad \mathbf{V} = \begin{bmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{bmatrix},$$

el espacio vectorial $\mathcal{V}$ resulta ser el espacio columna de la matriz $\mathbf{V}$, es decir $\mathcal{C}\left(\mathbf{V}\right)$. Por otra parte, sabemos que sumar combinaciones lineales de columnas a otras columnas no altera el espacio columna, $\mathcal{C}\left(\mathbf{V}\right)$; y puesto que $\boldsymbol{v}_n$ es combinación lineal del resto de vectores, tenemos

$$\boldsymbol{v}_n = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_{n-1}\boldsymbol{v}_{n-1}.$$

Sabiendo esto, podemos reducir la matriz **V** a una nueva matriz con la última columna compuesta por ceros (sin alterar el espacio columna) del siguiente modo:

$$\left[\frac{\mathbf{V}}{\mathbf{I}}\right] = \left[\frac{\boldsymbol{v}_1 \quad \cdots \quad \boldsymbol{v}_n}{\mathbf{I}}\right] \rightarrow \left[\frac{\begin{matrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ h_1 & h_2 & \dots & h_{n-1} & 0 \end{matrix}}{\mathbf{E}}\right] = \left[\frac{\mathbf{W}}{\mathbf{E}}\right]$$

donde la última columna de **E** es $\left(-a_1, \quad -a_2, \quad \dots \quad -a_{n-1}, \quad 1,\right)$. Pero $\mathcal{V} = \mathcal{C}\left(\mathbf{V}\right) = \mathcal{C}\left(\mathbf{W}\right)$; donde el espacio columna de **W** está generado por las combinaciones lineales de las columnas no nulas de la matriz, por tanto está generado por los $n-1$ primeros vectores columna.
Si los vectores $\boldsymbol{v}_j$ pertenecen a $\mathbb{R}^m$ con $m < n$, el razonamiento es el mismo pero **W** tiene menos filas y si $m < n-1$ la matriz **W** tendrá más columnas de ceros al final.
Si los vectores $\boldsymbol{v}_j$ pertenecen a $\mathbb{R}^m$ con $m > n$, el razonamiento tampoco cambia, **W** tiene más filas, pero la última columna seguirá siendo nula.

□

(L-9) Question 15(a)

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 1 & 1 & 2 & -2 \\ 2 & 2 & 2 & -4 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(2)\mathbf{1}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \\ \hline 1 & -1 & -2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(2)\mathbf{2}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \\ \hline 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

The free variable is y. The general solution is

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists c \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists c \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

$\square$

(L-9) Question 15(b) A basis is $\left[\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}; \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}; \right]$ and also $\left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \right]$.

$\square$

(L-9) Question 16(a) **No.** El espacio P_3 tiene dimensión 4, y tan sólo hay tres vectores en este conjunto.

□

(L-9) Question 16(b) **No.** De nuevo no hay vectores suficientes para generar un espacio de dimensión 4.

□

(L-9) Question 16(c) **Si.** Estos cuatro vectores son linealmente independientes (cuatro pivotes).

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[{[(-1)\mathbf{2}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}+\mathbf{3}]}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{3}+\mathbf{4}]}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\substack{\boldsymbol{\tau} \\ [\mathbf{1}\rightleftharpoons\mathbf{2}] \\ [\mathbf{2}\rightleftharpoons\mathbf{3}] \\ [\mathbf{3}\rightleftharpoons\mathbf{4}]}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{R}$$

□

(L-9) Question 16(d) **No.** El cuarto vector es la combinación de los tres primeros, por lo que no hay suficientes vectores linealmente independientes para generar un espacio de dimensión 4 (sólo tres pivotes)

$$\begin{bmatrix}1&1&0&0\\0&1&-1&0\\0&0&2&4\\1&2&1&4\end{bmatrix}\xrightarrow[{[(-1)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}}\begin{bmatrix}1&0&0&0\\0&1&-1&0\\0&0&2&4\\1&1&1&4\end{bmatrix}\xrightarrow[{[(1)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&2&4\\1&1&2&4\end{bmatrix}$$
$$\xrightarrow[{[(-2)\mathbf{3}+\mathbf{4}]}]{\boldsymbol{\tau}}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&2&0\\1&1&2&0\end{bmatrix}.$$

□

(L-9) Question 17(a) Si $\boldsymbol{u}_1$ y $\boldsymbol{u}_2$ son linealmente dependientes, entonces existe un número a tal que

$$\boldsymbol{u}_1 = a\boldsymbol{u}_2;$$

es decir, que componente a componente $u_{1i} = a \cdot u_{2i}$. Puesto que las terceras componentes son iguales, dicho número debería ser $a = 1$, pero entonces la primera componente de $\boldsymbol{u}_1$ también debería ser una vez la primera componente de $\boldsymbol{u}_2$. Puesto que no es así, el primer vector no es un múltiplo del segundo. Así pues no existe tal número a y, por tanto, estos vectores son linealmente *independientes*.
Otra forma de verlo es comprobar que el rango de la matriz $\mathbf{M} = \begin{bmatrix}\boldsymbol{u}_1 & \boldsymbol{u}_2\end{bmatrix}$ es dos.

□

(L-9) Question 17(b) La respuesta es sí. Si escribiéramos los vectores en forma de columna, esta pregunta sería equivalente a preguntar si $\boldsymbol{v}$ (en forma de columna) pertenece al espacio columna de la matriz

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix};$$

es decir, a preguntar si el sistema

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

tiene solución. Puesto que 3 veces la primera columna menos la segunda da el resultado deseado, $\boldsymbol{v} = (2, 1, 2)$ pertenece al espacio generado por $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$.
Otra forma de verlo es comprobar que la matriz

$$\mathbf{N} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{v} \end{bmatrix}$$

también tiene rango 2, es decir, que al añadir el vector $\boldsymbol{v}$, el rango de la nueva matriz sigue siendo 2 (como el rango de la matriz $\mathbf{M}$ del primer apartado).

$\square$

(L-9) Question 17(c) Debemos encontrar un tercer vector linealmente independiente de $\boldsymbol{u}_1$ y $\boldsymbol{u}_2$. Si escribimos de nuevo los vectores en columna, buscamos un vector $\boldsymbol{b}$

que no pertenezca al espacio columna de $\mathbf{A}$. Por tanto queremos un $\boldsymbol{b}$ tal que $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ no tenga solución.

$$\left[\begin{array}{c|c} \mathbf{A} & \text{-}\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & -b_1 \\ 0 & -1 & -b_2 \\ 1 & 0 & -b_3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-b_2)\mathbf{2}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ [(b_1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & b_1 - b_3 \\ \hline 1 & 0 & b1 \\ 0 & 1 & -b2 \\ \hline 0 & 0 & 1 \end{array}\right]$$

Por tanto, si $b_1 \neq b_3$ el sistema no tiene solución, es decir, $\boldsymbol{b}$ no es combinación lineal de los otros dos, y por tanto tenemos tres vectores de $\mathbb{R}^3$ linealmente independientes, es decir, una base de $\mathbb{R}^3$.

□

(L-9) Question 18(a) Si. Dos vectores son dependientes si uno es un múltiplo del otro; pero en este caso no es así. Por tanto son linealmente independientes.

□

(L-9) Question 18(b) No, los vectores $\boldsymbol{v}_1,\ \boldsymbol{v}_3,\ \boldsymbol{v}_3$ y $\boldsymbol{v}_4$ no son linealmente independientes. Por ejemplo $\boldsymbol{v}_1 = \boldsymbol{v}_4$.

□

(L-9) Question 18(c) No, los vectores no son base de dicho sub-espacio.
Los vectores son linealmente independientes, pero no generan el plano descrito en el enunciado ya que $\boldsymbol{v}_3$ no está en dicho plano (no satisface la ecuación $x_1 + 2x_2 + 3x_3 + 6x_4 = 0$).

$\square$

(L-9) Question 18(d) Lo resolveremos por eliminación

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -q \\ 4 & 2 & 12 & -3 \\ 6 & 2 & 10 & -1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(1)\mathbf{1}+\mathbf{3}] \\ [(q)\mathbf{1}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 4 & 2 & 16 & 4q-3 \\ 6 & 2 & 16 & 6q-1 \\ \hline 1 & 0 & 1 & q \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{[(-8)\mathbf{2}+\mathbf{3}] \\ [(2)\mathbf{4}] \\ [(-4q+3)\mathbf{2}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ 6 & 2 & 0 & 4q+4 \\ \hline 1 & 0 & 1 & 2q \\ 0 & 1 & -8 & -4q+3 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 2 \end{array}\right]$$

Los cuatro vectores generan todo $\mathbb{R}^3$ si $q \neq -1$ (tres pivotes). Pero si $q = -1$, los cuatro vectores solo generan un subespacio de dimensión $r = 2$ (dos pivotes), es decir, generan un plano en $\mathbb{R}^3$ que pasa por el origen.

$\square$

(L-9) Question 19(a)

$$\boldsymbol{u} = \begin{pmatrix}1\\0\\0\end{pmatrix},\ \boldsymbol{v} = \begin{pmatrix}2\\0\\0\end{pmatrix},\ \boldsymbol{w} = \begin{pmatrix}3\\0\\0\end{pmatrix},\ \boldsymbol{z} = \begin{pmatrix}0\\1\\0\end{pmatrix}; \qquad \mathbf{A} = \begin{bmatrix}-1 & -1\\0 & 0\\0 & 0\end{bmatrix}.$$

□

(L-9) Question 19(b) Son el número de pivotes y el número de columnas libres respectivamente:

$$\dim(\mathcal{C}(\mathbf{A})) = 1; \qquad \dim(\mathcal{N}(\mathbf{A})) = 1.$$

□

(L-10) Question 1(a)

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{cccc}0&1&4&0\\0&2&8&0\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right] \xrightarrow[{[(-4)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc}0&1&0&0\\0&2&0&0\\\hline 1&0&0&0\\0&1&-4&0\\0&0&1&0\\0&0&0&1\end{array}\right] = \left[\frac{\mathbf{K}}{\mathbf{E}}\right]$$

From its pre-echelon form **K** we known that only the second one is a pivot column,

therefore

$$\mathcal{C}\left(\mathbf{A}\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \exists\, c \in \mathbb{R} \text{ such that } \boldsymbol{x} = c\begin{pmatrix}1\\2\end{pmatrix} \right\};$$

$$\boxed{\dim \mathcal{C}\left(\mathbf{A}\right) = 1. \quad \text{Basis for } \mathcal{C}\left(\mathbf{A}\right)\text{: } \left[\begin{pmatrix}1\\2\end{pmatrix};\right].}$$

From the columns of **E** below the null columns of **K** we see that $\boxed{\dim \mathcal{N}\left(\mathbf{A}\right) = 3}$ and

$$\mathcal{N}\left(\mathbf{A}\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists\, \boldsymbol{a} \in \mathbb{R}^3 \text{ such that } \boldsymbol{x} = \begin{bmatrix}1 & 0 & 0\\0 & -4 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\boldsymbol{a} \right\}.$$

$$\boxed{\text{Basis for } \mathcal{N}\left(\mathbf{A}\right)\text{: } \left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix};\begin{pmatrix}0\\-4\\1\\0\end{pmatrix};\begin{pmatrix}0\\0\\0\\1\end{pmatrix};\right].}$$

Since there is only one pivot row $\boxed{\dim \mathcal{C}\left(\mathbf{A}^\intercal\right) = 1}$ and

$$\mathcal{C}\left(\mathbf{A}^\intercal\right) = \left\{\boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists\, c \in \mathbb{R} \text{ such that } \boldsymbol{x} = c\left(0, \quad 1, \quad 4, \quad 0,\right)\right\}.$$

$$\boxed{\text{Basis for } \mathcal{C}\left(\mathbf{A}^\intercal\right)\text{: } \left[(0,\ 1,\ 4,\ 0,);\right].}$$

Since

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \mathbf{K} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{N}\left(\mathbf{A}^\intercal\right) = \left\{\boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \exists\, c \in \mathbb{R} \text{ such that } \boldsymbol{x} = c\left(-2, \quad 1,\right)\right\};$$

$$\boxed{\dim \mathcal{N}\left(\mathbf{A}^\intercal\right) = 1. \quad \text{Basis for } \mathcal{N}\left(\mathbf{A}^\intercal\right)\text{: } \left[(-2,\ 1,);\right].}$$

□

(L-10) Question 1(b) $\dim \mathcal{C}\left(\mathbf{A}\right) + \dim \mathcal{N}\left(\mathbf{A}^\intercal\right) = 1 + 1 = 2 = m.$
$\dim \mathcal{C}\left(\mathbf{A}^\intercal\right) + \dim \mathcal{N}\left(\mathbf{A}\right) = 1 + 3 = 4 = n$

□

(L-10) Question 1(c)

$$\left[\begin{array}{cccc} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-4)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Since only the second is a pivot column, then

$$\mathcal{C}\left(\mathbf{U}\right) = \mathcal{C}\left(\mathbf{A}\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \exists\, c \in \mathbb{R} \text{ such that } \boldsymbol{x} = c \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\};$$

$$\boxed{\dim \mathcal{C}\left(\mathbf{A}\right) = 1. \quad \text{Basis for } \mathcal{C}\left(\mathbf{A}\right)\text{: } \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}; \right].}$$

$\mathcal{N}(\mathbf{U}) = \mathcal{N}(\mathbf{A})$:

$$\mathcal{N}(\mathbf{U}) = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists\, \boldsymbol{a} \in \mathbb{R}^3 \text{ such that } \boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{a} \right\}.$$

$$\boxed{\text{Basis for } \mathcal{N}(\mathbf{A}) \text{: } \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix};\ \begin{pmatrix} 0 \\ -4 \\ 1 \\ 0 \end{pmatrix};\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix};\ \right].}$$

$\mathcal{C}(\mathbf{U}^\intercal) = \mathcal{C}(\mathbf{A}^\intercal)$:

$$\mathcal{C}(\mathbf{U}^\intercal) = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists\, c \in \mathbb{R} \text{ such that } \boldsymbol{x} = c\,(0, \quad 1, \quad 4, \quad 0,) \right\}.$$

$$\boxed{\text{Basis for } \mathcal{C}(\mathbf{A}^\intercal) \text{: } \Big[(0,\ 1,\ 4,\ 0,); \Big].}$$

Since the only free row is the second one: "a zero vector"

$$\mathcal{N}\left(\mathbf{U}^{\intercal}\right) = \left\{\boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \exists\, c \in \mathbb{R} \text{ such that } \boldsymbol{x} = c\,(0, \quad 1,)\right\};$$

$$\boxed{\dim \mathcal{N}\left(\mathbf{A}^{\intercal}\right) = 1. \quad \text{Basis for } \mathcal{N}\left(\mathbf{A}^{\intercal}\right)\text{: } \left[(0,\ 1,);\right].}$$

□

(L-10) Question 1(d) $\dim \mathcal{C}\left(\mathbf{U}\right) + \dim \mathcal{N}\left(\mathbf{U}^{\intercal}\right) = 1 + 1 = 2 = m.$
$\dim \mathcal{C}\left(\mathbf{U}^{\intercal}\right) + \dim \mathcal{N}\left(\mathbf{U}\right) = 1 + 3 = 4 = n$

□

(L-10) Question 2. The column space $\mathcal{C}\left(\mathbf{A}\right)$ is the set of linear combinations of the two last columns

$$\left\{\boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{a} \in \mathbb{R}^3 \text{ such that } \boldsymbol{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{a} \right\}$$

that is, all vectors in $\mathbb{R}^3$ with a zero as a third component (a subspace of dimension 2). The null space $\mathcal{N}\left(\mathbf{A}\right)$ is the set of all multiples

$$\left\{\boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists c \in \mathbb{R} \text{ such that } \boldsymbol{x} = c \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

that is, all vectors in $\mathbb{R}^3$ with zero as second and third components (a subspace of dimension 1).
The row space $\mathcal{C}\left(\mathbf{A}^\intercal\right)$ is the set of all the vectors $\boldsymbol{x}$ such that

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{a} \in \mathbb{R}^2 \text{ such that } \boldsymbol{x} = \boldsymbol{a} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

or the set of all vectors in $\mathbb{R}^3$ with zero as a first component (a subspace of dimension 1).
$\mathcal{N}\left(\mathbf{A}^\intercal\right)$ is the set of all the multiples

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists c \in \mathbb{R} \text{ such that } \boldsymbol{x} = c \left(0, \quad 0, \quad 1,\right) \right\}$$

that is, all vectors in $\mathbb{R}^3$ with zero as first and second components (a subspace of dimension 1).

□

(L-10) Question 3(a) Row and column space dimension $= 5$; nullspace dimension $= 4$; left nullspace dimension $= 2$; sum $= 16 = m + n$.

□

(L-10) Question 3(b) $\mathcal{C}\left(\mathbf{A}\right) = \mathbb{R}^3$; $\quad \mathcal{N}\left(\mathbf{A}^\intercal\right) = \{\mathbf{0}\}$.

□

(L-10) Question 4. **No.** Consider any two invetible matrices n by n; both have the same four subspaces.

□

(L-10) Question 5. Since **U** has two pivots, the rank is two for both matrices. The row space for both matrices is the same (column operations do not change the column space), but $\mathcal{C}\left(\mathbf{A}^\intercal\right)$ is different from $\mathcal{C}\left(\mathbf{L}^\intercal\right)$. Note that all the vectors in $\mathcal{C}\left(\mathbf{L}^\intercal\right)$ have the third and fourth components equal to zero.

$$\mathcal{C}(\mathbf{A}) = \mathcal{C}(\mathbf{L}) = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \,\middle|\, \text{exists } a, b \in \mathbb{R} \text{ such that } \boldsymbol{x} = a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$\boxed{\text{Basis for } \mathcal{C}(\mathbf{A}) \text{ and } \mathcal{C}(\mathbf{L}):\ \left[\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix};\ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \right].}$$

$$\mathcal{C}\left(\mathbf{A}^\intercal\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \,\middle|\, \text{exists } a, b \in \mathbb{R} \text{ such that } \boldsymbol{x} = a \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$\boxed{\text{Basis for } \mathcal{C}\left(\mathbf{A}^\intercal\right): \left[\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} ; \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} ; \right].}$$

$$\mathcal{C}\left(\mathbf{L}^\intercal\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \,\middle|\, \text{exists } a, b \in \mathbb{R} \text{ such that } \boldsymbol{x} = a \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\boxed{\text{Basis for } \mathcal{C}\left(\mathbf{L}^\intercal\right): \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} ; \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} ; \right].}$$

Since there are two free columns:

$$\mathcal{N}\left(\mathbf{A}\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \,\middle|\, \text{exists } a, b \in \mathbb{R} \text{ such that } \boldsymbol{x} = a \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\boxed{\text{Basis for } \mathcal{N}\left(\mathbf{A}\right): \left[\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}; \right].}$$

$$\mathcal{N}\left(\mathbf{L}\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \,\middle|\, \text{exists } a, b \in \mathbb{R} \text{ such that } \boldsymbol{x} = a \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\boxed{\text{Basis for } \mathcal{N}\left(\mathbf{A}^{\mathsf{T}}\right): \left[\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}; \right].}$$

There is only one free row and therefore the dimension of the left null space is one for both matrices; and the left null space is for both matrices.

$$\mathcal{N}\left(\mathbf{A}^{\intercal}\right) = \mathcal{N}\left(\mathbf{L}^{\intercal}\right) = \left\{\boldsymbol{x} \in \mathbb{R}^3 \,\middle|\, \text{exists } a \in \mathbb{R} \text{ such that } \boldsymbol{x} = a\left(1, \quad 0, \quad -1,\right)\right\}$$

$$\boxed{\text{Basis for } \mathcal{N}\left(\mathbf{A}^{\intercal}\right) \text{ and } \mathcal{N}\left(\mathbf{L}^{\intercal}\right)\text{:} \quad \left[\begin{pmatrix}1\\0\\-1\end{pmatrix};\right].}$$

□

(L-10) Question 6(a) Dimension 3:

$$\mathcal{V} = \left\{\boldsymbol{x} \in \mathbb{R}^4 \,\middle|\, x_1 + x_2 + x_3 + x_4 = 0\right\}$$

therefore

$$\mathcal{V} = \mathcal{N}\left(\mathbf{A}\right); \quad \text{where} \quad \mathbf{A} = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}.$$

Hence

$$\left[\begin{array}{cccc}1 & 1 & 1 & 1\\ \hline 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right] \xrightarrow[\;]{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{1}+\mathbf{2}]\\ [(-1)\mathbf{1}+\mathbf{3}]\\ [(-1)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{cccc}1 & 0 & 0 & 0\\ \hline 1 & -1 & -1 & -1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right]$$

□

(L-10) Question 6(b) Since the only vector en $\mathcal{N}(\mathbf{I})$ is the zero vector $\mathbf{0}$, the dimension is 0.

□

(L-10) Question 6(c) Dimension 16.

□

(L-10) Question 7. A basis of the row space is

$$\left[\begin{pmatrix}3\\0\\3\end{pmatrix};\begin{pmatrix}1\\1\\2\end{pmatrix};\right].$$

A basis of the column space is

$$\left[\begin{pmatrix}1\\4\\2\end{pmatrix};\begin{pmatrix}2\\5\\7\end{pmatrix};\right].$$

Because the rank is two.

□

(L-10) Question 8(a) No. This is not a vector space because $\mathbf{0}$ is not in this subspace.

□

(L-10) Question 8(b) Yes. (This is actually just the left nullspace of the matrix whose columns are $\boldsymbol{y}$ and $\boldsymbol{z}$.)

$$\boldsymbol{x}\begin{bmatrix}\boldsymbol{z} & \boldsymbol{y}\end{bmatrix} = \mathbf{0}.$$

□

(L-10) Question 8(c) No. For example, the zero matrix **0** is not in this subset.

□

(L-10) Question 8(d) Yes. If the nullspaces of $\mathbf{A}_1$ and $\mathbf{A}_2$ contain $\begin{pmatrix}1\\2\\3\end{pmatrix}$ then any linear combination of these matrices does too:

$$(a\mathbf{A}_1 + b\mathbf{A}_1)\begin{pmatrix}1\\2\\3\end{pmatrix} = a\mathbf{A}_1\begin{pmatrix}1\\2\\3\end{pmatrix} + b\mathbf{A}_2\begin{pmatrix}1\\2\\3\end{pmatrix} = a\mathbf{0} + b\mathbf{0} = \mathbf{0}.$$

□

(L-10) Question 9. Si empleamos el método de eliminación gaussiana, encontramos que sólo hay dos pivotes, que corresponden con las dos primeras columnas y las dos primeras filas. Así que sabemos que tanto el espacio columna y el espacio fila tienen dimensión dos. Por tanto, tomemos las dos primeras columnas y las dos primeras filas, puesto que son linealmente independientes.

$$\mathcal{C}\left(\mathbf{A}\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists\, c, d \in \mathbb{R} \text{ such that } \boldsymbol{x} = c\begin{pmatrix}1\\2\\1\end{pmatrix} + d\begin{pmatrix}3\\-1\\4\end{pmatrix} \right\}$$

$$\mathcal{C}\left(\mathbf{A}^\intercal\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists\, c, d \in \mathbb{R} \text{ such that } \boldsymbol{x} = c\begin{pmatrix}1 & 3 & 5 & -2\end{pmatrix} + d\begin{pmatrix}2 & -1 & 3 & -4\end{pmatrix} \right\}.$$

□

(L-10) Question 10(a) $\mathcal{N}\left(\mathbf{A}\right) = \mathbf{0}$.

□

(L-10) Question 10(b) $\dim \mathcal{N}\left(\mathbf{A}^\intercal\right) = 1$.

□

(L-10) Question 10(c) $\boldsymbol{x}_p = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$.

□

(L-10) Question 10(d) $\boldsymbol{x} = \boldsymbol{x}_p + \mathbf{0} = \boldsymbol{x}_p = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$.

□

(L-10) Question 10(e)

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & d \end{bmatrix},$$

where the last row of **R** is not possible to knowm without more information.

□

(L-10) Question 11(a) Falso. Por ejemplo para la matriz $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$; $\mathcal{C}\left(\mathbf{A}\right)$ es el sub-espacio de vectores de $\mathbb{R}^3$ con la última componente igual a cero; mientras que $\mathcal{C}\left(\mathbf{A}^\intercal\right)$ es el sub-espacio de vectores de $\mathbb{R}^3$ con la primera componente nula.

□

(L-10) Question 11(b) Verdadero.

□

(L-10) Question 11(c) Falso. Suponga dos matrices invertibles, por ejemplo

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{y} \quad \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

$\mathcal{C}\left(\mathbf{A}\right) = \mathcal{C}\left(\mathbf{I}\right) = \mathcal{C}\left(\mathbf{A}^\intercal\right) = \mathcal{C}\left(\mathbf{I}^\intercal\right) = \mathbb{R}^2$ y
$\mathcal{N}\left(\mathbf{A}\right) = \mathcal{N}\left(\mathbf{I}\right) = \mathcal{N}\left(\mathbf{A}^\intercal\right) = \mathcal{N}\left(\mathbf{I}^\intercal\right) = \{\mathbf{0}\}.$

□

(L-10) Question 11(d) Verdadero. Sea cual sea el vector $\boldsymbol{b}$, las n columnas (vectores de $\mathbb{R}^n$) son linealmente independientes, por tanto son una base de $\mathbb{R}^n$, puesto que el lado derecho $\boldsymbol{b}$ pertenece a $\mathbb{R}^n$, siempre existe una única combinación lineal de las columnas igual a $\boldsymbol{b}$ (por ser estas una base de $\mathbb{R}^n$). Dicha combinación es *"la"* solución $\boldsymbol{x}$ al sistema de ecuaciones.
Otra forma de verlo es la siguiente: si las n columnas son linealmente independientes, **A** es de rango completo y por lo tanto invertible; entonces

$$\begin{aligned} \mathbf{A}\boldsymbol{x} =& \boldsymbol{b} \\ \mathbf{A}^{-1}\mathbf{A}\boldsymbol{x} =& \mathbf{A}^{-1}\boldsymbol{b} \\ \boldsymbol{x} =& \mathbf{A}^{-1}\boldsymbol{b} \end{aligned}$$

Por tanto sabemos que para cualquier $\boldsymbol{b}$, el vector $\mathbf{A}^{-1}\ \boldsymbol{b}$ es la solución.

□

(L-10) Question 12(a) Puesto que cualquier combinación del espacio columna $\mathbf{A}\boldsymbol{x}$ es un múltiplo de $\begin{pmatrix}-2, & 1,\end{pmatrix}$, dicho espacio es un subespacio de $\mathbb{R}^2$ de dimensión 1 (una recta en $\mathbb{R}^2$); por tanto el rango de **A** es $r = 1$ (una sola columna pivote, y consecuentemente sólo una fila pivote).
Además, sabemos que las dimensiones de **A** son $m = 2$ y $n = 4$ es decir: $\underset{2\times 4}{\mathbf{A}}$.

□

(L-10) Question 12(b) Número de columnas libres, es decir, el número de columnas menos el número de columnas pivote: $\dim \mathcal{N}\left(\mathbf{A}\right) = 4 - r = 4 - 1 = 3.$

□

(L-10) Question 12(c) $\dim \mathcal{C}\left(\mathbf{A}^\intercal\right) = r = 1.$

□

(L-10) Question 12(d) El número de filas libres; por tanto $m - r = 2 - 1 = 1.$

□

(L-10) Question 12(e) Sabemos que

$$\mathbf{A}\boldsymbol{v} = \mathbf{A}\begin{pmatrix}1\\-2\\3\\1\end{pmatrix} = \begin{pmatrix}-6\\3\end{pmatrix} \qquad \text{y que} \qquad \mathbf{A}\boldsymbol{w} = \mathbf{A}\begin{pmatrix}3\\-1\\1\\2\end{pmatrix} = \begin{pmatrix}-18\\9\end{pmatrix}.$$

Por tanto,

$$-3\mathbf{A}\boldsymbol{v} + \mathbf{A}\boldsymbol{w} = \begin{pmatrix}18\\-9\end{pmatrix} + \begin{pmatrix}-18\\9\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

Es decir

$$-3\mathbf{A}\boldsymbol{v} + \mathbf{A}\boldsymbol{w} = \mathbf{A}(-3\boldsymbol{v}) + \mathbf{A}\boldsymbol{w} = \mathbf{A}(-3\boldsymbol{v} + \boldsymbol{w}) = \mathbf{0}.$$

Así pues, una solución al sistema homogéneo es:

$$\boldsymbol{x} = -3\boldsymbol{v} + \boldsymbol{w} = \begin{pmatrix} -3 \\ 6 \\ -9 \\ -3 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ -8 \\ -1 \end{pmatrix}.$$

También es solución cualquier múltiplo de dicho vector $\boldsymbol{x}$.

□

(L-10) Question 13(a)
$\operatorname{rg}(\mathbf{A}) =$ number of pivots $= 3$.
$\dim \mathcal{C}(\mathbf{A}) = \dim \mathcal{C}(\mathbf{A}^\intercal) = \operatorname{rg}(\mathbf{A}) = 3$.
$\dim \mathcal{N}(\mathbf{A}) =$ number of free columns $= 5 - 3 = 2$.

□

(L-10) Question 13(b) The rows 1, 2 and 4 of **A** (the three pivot rows of **A**).

□

(L-10) Question 13(c) The pivot columns of **R** (also the pivot columns of **A**).

□

(L-10) Question 13(d) The columns 3 and 4 of **E**.

□

(L-10) Question 13(e) $3\mathbf{A}_{|1} - 2\mathbf{A}_{|2} = \mathbf{A}_{|3}$. Note that the third column of **E** is telling us a linear combination. We can find more, but that is the only possibility if we only use the first and second columns.

□

(L-10) Question 14. Since both solutions belong to $\mathbb{R}^4$, we know **A** has 4 colunms. Since the whole nullspace consists of all linear combination of only two vectors, only two of the columns are free, and the other two are pivots (rank 2). Hence, the dimension of row space is two, and it is orthogonal to the given vectors.
Lets compute one possible answer using gaussian elimination.

$$\left[\begin{array}{cccc} 2 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[\mathbf{1} \rightleftharpoons \mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc} 1 & 2 & 2 & 0 \\ 0 & 1 & 3 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{3}] \\ [(-3)\mathbf{2}+\mathbf{3}] \\ [(-1)\mathbf{2}+\mathbf{4}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & \mathbf{1} & \mathbf{0} \\ 0 & 1 & \mathbf{-3} & \mathbf{-1} \\ 1 & -2 & \mathbf{4} & \mathbf{2} \\ 0 & 0 & \mathbf{0} & \mathbf{1} \end{array}\right]$$

Now we've got a basis for the row space. Thus, a possible answer is

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 4 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix}.$$

It's easy to check...

$$\begin{bmatrix}1 & -3 & 4 & 0\\0 & -1 & 2 & 1\end{bmatrix}\begin{bmatrix}2 & 3\\2 & 1\\1 & 0\\0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

But we can find more matrices as far as we multiply $\mathbf{A}$ on the left, and preserve the rank (preserve the number of free variables). Thus, any matrix obtained by row elementary operations (row operations without changing the rank) on $\mathbf{A}$ is a valid one

$$\underset{2\times 2}{\mathbf{E}}\ \underset{2\times 4}{\mathbf{A}}\begin{bmatrix}2 & 3\\2 & 1\\1 & 0\\0 & 1\end{bmatrix}=\mathbf{E0}=\mathbf{0};$$

where $\operatorname{rg}\left(\mathbf{EA}\right)=2.$

□

(L-10) Question 15. By column gaussian elimination from right to left we get

$$\left[\begin{array}{cccc} 4 & 3 & 2 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-4)\mathbf{4}+\mathbf{1}]}]{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{4}+\mathbf{3}] \\ [(-3)\mathbf{4}+\mathbf{2}]}} \left[\begin{array}{cccc} 0 & 0 & 0 & 1 \\ \hline \mathbf{1} & \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 \\ \mathbf{-4} & \mathbf{-3} & \mathbf{-2} & 1 \end{array}\right]$$

Then:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

But any matrix obtained by row elementary operations (row operations without changing the rank) on **A** is a valid one.

□

(L-10) Question 16(a) Suponga que el producto de **A** y **B** es la matriz nula: $\mathbf{AB} = \mathbf{0}$. Entonces el espacio *nulo* de la matriz **A** contiene el espacio *columna* de la matriz **B**. También el espacio *nulo por la izquierda* de la matriz **B** contiene el espacio *fila* de la matriz **A**.

□

(L-10) Question 16(b) La dimensión del espacio nulo de **A** es $n - r = 7 - r$. La dimensión del espacio columna de **B** es s. Puesto que el primero contiene al segundo, $7 - r \geq s$, es decir $r + s \leq 7$.

(L-10) Question 17(a) 4. There are 4 pivots in the reduced echelon form of **A**.

(L-10) Question 17(b)
$\dim \mathcal{C}\left(\mathbf{A}\right) = \dim \mathcal{C}\left(\mathbf{A}^\intercal\right) = \mathrm{rg}\left(A\right) = 4$
$\dim \mathcal{N}\left(\mathbf{A}\right) = n - \mathrm{rg}\left(A\right) = 8 - 4 = 4$
$\dim \mathcal{N}\left(\mathbf{A}^\intercal\right) = m - \mathrm{rg}\left(A\right) = 4 - 4 = 0$

(L-10) Question 17(c) $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ will have infinitely many solutions for any $\boldsymbol{b}$. There is no row of 0's in the reduced column echelon form to cause there to be no solutions for the "wrong" $\boldsymbol{b}$. There are infinitely many solutions since the nullspace, being 4-dimensional, has infinitely many elements.

(L-10) Question 17(d) Yes. The reduced column echelon form of **A** has linearly independent rows.

(L-10) Question 17(e) Columns 2, 4, 5 and 7 of **E**.

(L-10) Question 17(f) We saw that $\dim(\mathcal{N}\left(\mathbf{A}^\intercal\right)) = 0$. Hence, $\mathcal{N}\left(\mathbf{A}^\intercal\right)$ contains only the zero vector; and the isn't any basis for this space.

(L-10) Question 17(g) It is imposible, it is a singular matrix.

□

(L-10) Question 17(h) $\left[\mathbf{E}_{|1}; \mathbf{E}_{|3}; \mathbf{E}_{|6}; \mathbf{E}_{|8};\right]$.

□

(L-10) Question 18(a) Since the three columns are pivot colums (no free columns), then $\mathcal{N}\left(\mathbf{R}\right) = \{\mathbf{0}\}$.

$$\mathcal{N}\left(\mathbf{R}\right) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

□

(L-10) Question 18(b) **B** is its column reduced echelon form. Therefore, the rank is 3.

□

(L-10) Question 18(c)

$$\begin{bmatrix} \mathbf{R} & \mathbf{R} \\ \mathbf{R} & \mathbf{0} \end{bmatrix} \xrightarrow[\text{from the 5 last ones}]{\text{substracting the five first columns.}} \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{R} & -\mathbf{R} \end{bmatrix} \xrightarrow[\text{to the first ones}]{\text{adding the last ones.}}$$

$$\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & -\mathbf{R} \end{bmatrix} \xrightarrow[\text{columns by -1}]{\text{multiplying the last.}} \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}$$

□

(L-10) Question 18(d) Twice the rank of **R**, that is, rank 6.

□

(L-10) Question 18(e) Must be equal to the number of zero rows of $\underset{10\times 6}{\mathbf{C}}$ (the zero columns of **C**; since the six columns of **C** are pivot columns (rank 6); only 6 rows are pivot rows, the remaining are free; therefore, the dimension is $10 - 6 = 4$.

□

(L-10) Question 19(a) Since the right hand side vector $\boldsymbol{b}$ belongs to $\mathbb{R}^3$, then **A** has three rows. In addition, $\boldsymbol{x}$ also belongs to $\mathbb{R}^3$, thus **A** has also three columns. Besides, there are two special solutions; therefore $\operatorname{rg}\left(\mathbf{A}\right) = 3 - 2 = 1$. It follows that there is only one pivot row, hence $\dim \mathcal{C}\left(\mathbf{A}^{\mathsf{T}}\right) = 1$.

□

(L-10) Question 19(b) From the particular solution, it follows that twice the first column equals the right hand side vector $\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$, hence, the first column of **A** is $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. Because the rank is one, the other columns are multiples of the first one. From the first special solution we know that the second column must be the opposite of the first one, or $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$. Finally, from the second special solution it follows that the last column is the zero vector. Consequently,

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}.$$

□

(L-10) Question 19(c) For any vector $\boldsymbol{b}$ in the column space of $\mathbf{A}$; in other words, the system is solvable for any for any multiple of the first column.

□

(L-10) Question 20(a) **False.** Example: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

□

(L-10) Question 20(b) **False.** Example: $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

$$\mathcal{C}\left(\mathbf{A}\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \exists\, c \in \mathbb{R} \text{ such that } \boldsymbol{x} = c \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}\left(\mathbf{A}^{\mathsf{T}}\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \exists\, c \in \mathbb{R} \text{ such that } \boldsymbol{x} = c \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

□

(L-10) Question 20(c) **True.** The dimension of both spaces is the number of pivots. It is called the rank of the matrix.

□

(L-10) Question 20(d) **False**. The columns can be linearly dependent.

□

(L-10) Question 21(a) False.

□

(L-10) Question 21(b) True.

□

(L-10) Question 21(c) Then, the former statements are false in general.

□

(L-Opt-1) Question 1(a) The subspace $\mathbb{R}^{3\times 3}$ of all matrices 3 by 3, since any matrix can be express as a sum of a symmetric and a triangular matrices.

□

(L-Opt-1) Question 1(b) The intesection of both: the set of all diagonal matrices

□

(L-Opt-1) Question 2(a) **False.** Consider for example

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

The matrix on the right hand side is invertible, but the two on the left hand side are not.

□

(L-Opt-1) Question 2(b) **True.** For $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ to have no solution we must have a row of 0's in the reduced row echelon form. Hence, the number of pivots will be less than the number of rows, and so the matrix $\mathbf{A}$ does not have full rank.

□

(L-Opt-1) Question 2(c) **False.** Suppose $\mathbf{AB}$ is invertible, and consider $\mathbf{C} = (\mathbf{AB})^{-1}$ $\mathbf{A}$. Then $\mathbf{CB} = (\mathbf{AB})^{-1}\mathbf{AB} = \mathbf{I}$, so $\mathbf{C}$ is an inverse for $\mathbf{B}$.
Otra demostración alternativa: existe una matriz de rango completo $\mathbf{E}$ tal que $\mathbf{BE} = \mathbf{L}$, donde $\mathbf{L}$ es su forma escalonada. Como $\mathbf{B}$ es singular, $\mathbf{L}$ tiene una columna de ceros, entonces $(\mathbf{AB})\mathbf{E} = \mathbf{AL} = \mathbf{M}$ tiene necesariamente una columna de ceros como $\mathbf{L}$ y por tanto $\mathbf{AB}$ es singular.

□

(L-Opt-1) Question 2(d) **False.** Consider the permutation matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Then

$$\mathbf{P}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \neq \mathbf{I}.$$

(L-Opt-1) Question 3(a) Es un sub-espacio de $\mathbb{R}^7$ de dimensión:
cero si los tres son vectores nulos;
uno si $\boldsymbol{v}$ y $\boldsymbol{w}$ son múltiplos de $\boldsymbol{u}$;
dos si como máximo se pueden elegir dos vectores linealmente independientes;
o tres si los tres vectores son linealmente independientes.

(L-Opt-1) Question 3(b) También es únicamente por el vector nulo $\mathbf{0}$.

(L-Opt-1) Question 3(c) No. Por ejemplo las matrices 5 por 5 identidad (**I**) y la matriz opuesta a la identidad ($-$**I**) son de rango completo, y por tanto son invertibles, pero su suma es la matriz nula, que no es invertible. Por tanto, dicho subconjunto no es cerrado para la suma, es decir, no es un subespacio vectorial.

$$\mathbf{I} + (-\mathbf{I}) = \mathbf{0} \quad \text{que no es invertible.}$$

(L-Opt-1) Question 3(d) Falso. Ejemplos

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \quad \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}; \quad \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

(L-Opt-1) Question 3(e) El espacio columna $\mathcal{C}\left(\mathbf{A}\right)$ y el espacio nulo por la izquierda $\mathcal{N}\left(\mathbf{A}^{\intercal}\right)$.

□

(L-Opt-1) Question 3(f) El espacio fila $\mathcal{C}\left(\mathbf{A}^{\intercal}\right)$ y el espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$.

□

(L-Opt-1) Question 3(g) Si está en el espacio nulo, implica que sumar a la primera columna dos veces la segunda y tres veces la tercera nos da un vector de ceros.

$$\mathbf{A}\begin{pmatrix}1\\2\\3\end{pmatrix}=\mathbf{0}$$

Que es algo incompatible si el vector $\begin{pmatrix}1, & 2, & 3\end{pmatrix}$ es una fila,

$$\begin{bmatrix}1 & 2 & 3\\ - & - & -\\ - & - & -\end{bmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix}\neq\mathbf{0}!$$

□

(L-Opt-1) Question 4(a) Es espacio vectorial ya que

$$\begin{bmatrix}a & b\\0 & b\end{bmatrix}+\begin{bmatrix}c & d\\0 & d\end{bmatrix}=\begin{bmatrix}a+c & b+d\\0 & b+d\end{bmatrix} \text{ pertecene al conjunto; y}$$

$c\begin{bmatrix} a & b \\ 0 & b \end{bmatrix} = \begin{bmatrix} ca & cd \\ 0 & cd \end{bmatrix}$ también pertecene al conjunto.

□

(L-Opt-1) Question 4(b) No es espacio vectorial. Sean $g(\cdot)$, $h(\cdot)$ dos funciones de dicho conjunto, entonces la suma evaluada en cero es $g(0) + h(0) = 4$, y por tanto no pertenece al conjunto.

□

(L-Opt-1) Question 5.
$\{$Todos los monomios (vectores) de la forma: $a_n x^n$ para $n = 0, 1, 2...\}$

□

(L-Opt-1) Question 6(a) **Tres.** Puesto que las esquinas superior-derecha e inferior-izquierda deben ser iguales, sólo tres variables pueden variar.

□

(L-Opt-1) Question 6(b) **Tres.** Puesto que podemos re-escribir las matrices como

$$\mathbf{A} = \left[\begin{array}{cc} a & b \\ c & -a \end{array}\right],$$

está claro que el espacio es de dimensión tres como en el caso anterior.

□

(L-Opt-1) Question 6(c) **Dos.** Sólo x e y pueden variar, de hecho, dicho conjunto lo podemos expresar como:

$$\left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2,\ \boldsymbol{v} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -3 \\ -1 & 2 \end{bmatrix} \boldsymbol{p} \right\}$$

□

(L-11) Question 1. Since $\mathcal{C}\left(\mathbf{A}^{\intercal}\right) \perp \mathcal{N}\left(\mathbf{A}\right)$, we only need to find the orthogonal complement of the spam of $\begin{bmatrix} 1 & 3 & -1 \end{bmatrix}$.

$$\begin{bmatrix} \mathbf{A} \\ \hline \mathbf{I} \end{bmatrix} = \left[\begin{array}{ccc} 1 & 3 & -1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ {[(-3)\mathbf{1}+\mathbf{2}]}}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ \hline 1 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} \mathbf{L} \\ \hline \mathbf{E} \end{bmatrix}$$

Therefore, the set of vectors in $\mathbb{R}^3$ orthogonal to $\begin{pmatrix} 1, & 3, & -1 \end{pmatrix}$ is

$$\left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 \text{ tal que } \boldsymbol{v} = \begin{bmatrix} -3 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}.$$

With NAcAL there are several ways to obtain such a subspace. There are two ways to invoke `Subspace`: if the argument is a system (`Sistema`) of `Vectors` of $\mathbb{R}^n$, it returns the subspace spanned by that system.

```
a = Vector([-3,1,0])
b = Vector([1,0,1])
SubEspacio(Sistema([a,b]))
```

If the argument is a `Matrix`, it returns it's null space.

```
v = Vector([1,3,-1])
A = ~Matrix([v])      # trasponemos para obtener la matriz fila
SubEspacio(A)
```

But since we are asked for the orthogonal complement of the subspace generated by the vector, we can simply write (since in this context means the orthogonal complement):

```
~SubEspacio(Sistema([v]))
```

The representation by means of parametric or Cartesian equations is not unique, in fact, we obtain different parametric equations for the systems $[\boldsymbol{a}; \boldsymbol{b};]$ (seen above) and $[\boldsymbol{b}; \boldsymbol{a};]$.

```
SubEspacio(Sistema([b,a]))
```

It is therefore useful to be able to check whether two subspaces are the same

```
~SubEspacio(Sistema([v])) == SubEspacio(Sistema([b,a]))
```

(L-11) Question 2(a) We first have to find a vector parallel to the line. We let

$$\boldsymbol{v} = \boldsymbol{x}_P - \boldsymbol{x}_Q = \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}-2\\1\end{pmatrix}.$$

A parametric representation of the line is therefore

$$\left\{ \boldsymbol{v} \in \mathbb{R}^2 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{pmatrix}1\\2\end{pmatrix} + \begin{bmatrix} -2\\ 1 \end{bmatrix} \boldsymbol{p} \right\}.$$

With NAcAL, points, lines, planes, etc. (i.e. planar regions in R[n]) are created with `EAfin`. The required arguments to `EAfin` are a `Subspace` and a `Vector`. If instead of a `Subspace` a `System` of `Vectors` of R[n] or a `Matrix` is given, NAcAL shall use those arguments to generate the necessary subspace (the subspace generated by the system in in the first case, or the null space of the matrix in the second).
Thus, in this case we obtain the equations of the required line with:

```
p = Vector([1,2])
q = Vector([3,1])
S = SubEspacio(Sistema([p-q]))
R = EAfin(S,p)
Math( R.EcParametricas() )  # Por ahora solo quiero visualizar las Ec. Paramétr
```

(L-11) Question 2(b) We need to multiply $\boldsymbol{x} = \boldsymbol{x}_P + a\boldsymbol{v}$ by a vector perpendicular to $\boldsymbol{v}$. We will do it by elimination:

$$\left[\begin{array}{cc} -2 & 1 \\ \hline x & y \\ \hline 1 & 2 \end{array}\right] \xrightarrow[{[(1)\mathbf{1}+\mathbf{2}]}]{\substack{\boldsymbol{\tau}\\ {[(2)\mathbf{2}]}}} \left[\begin{array}{cc} -2 & 0 \\ \hline x & x+2y \\ \hline 1 & 5 \end{array}\right] \Rightarrow \text{ the solution set of } \left\{ x + 2y = 5 \right. ;$$

y therefore the line is:

$$\left\{ \boldsymbol{v} \in \mathbb{R}^2 \;\middle|\; \begin{bmatrix} 1 & 2 \end{bmatrix} \boldsymbol{v} = (5,) \right\}.$$

Let's reproduce the pencil and paper calculation with NAcAL.

```
x,y = sympy.symbols('x y')
N = ~Matrix([p-q])
M = N.apila(~Matrix([Vector([x,y])]),1).apila(~Matrix([p]),1)
Math( rprElim(M, Elim(N).pasos) )
```

Therefore the straight line is the set of vectors that solve the following system of linear equations:

```
A = Matrix([[1,2]])
b = Vector([5])
SEL(A,b).eafin
```

(note that NAcAL stores as an attribute (of type `EAfin`) the set of solutions to a system of equations)

□

(L-11) Question 3(a) We first have to find a vector in the direction of the line. We let

$$\boldsymbol{v} = \boldsymbol{x}_P - \boldsymbol{x}_Q = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \\ -4 \end{pmatrix}.$$

A parametric representation of the line is therefore

$$\left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} + \begin{bmatrix} 3 \\ -7 \\ -4 \end{bmatrix} \boldsymbol{p} \right\}.$$

□

(L-11) Question 3(b)

$$\left[\begin{array}{ccc} 3 & -7 & -4 \\ \hline x & y & z \\ \hline 1 & -3 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(3)\mathbf{2}] \\ [(7)\mathbf{1}+\mathbf{2}] \\ [(3)\mathbf{3}] \\ [(4)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc} 3 & 0 & 0 \\ \hline x & 7x+3y & 4x+3z \\ \hline 1 & -2 & 7 \end{array}\right] \Rightarrow \begin{cases} 7x + 3y & = -2 \\ 4x \quad + 3z & = 7 \end{cases};$$

Por tanto las ecuaciones cartesianas de la recta son:

$$\left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} 7 & 3 & 0 \\ 4 & 0 & 3 \end{bmatrix} \boldsymbol{v} = \begin{pmatrix} -2 \\ 7 \end{pmatrix} \right\}.$$

This system has two equations. If we take them separately, they correspond to two planes in R[3].

```
p1=SEL(Matrix([[7,3,0]]),Vector([-2])).eafin
p1
```

and

```
p1=SEL(Matrix([[7,3,0]]),Vector([-2])).eafin
p1
```

(we know that they are two planes, because the parametric equations have two parameters, and the coefficient matrices of the Cartesian equations have two free columns) The line of the exercise corresponds to the intersection of both planes, that is, to the points that belong to both planes:

```
p1 & p2
```

□

(L-11) Question 4. $\boldsymbol{a} \cdot \boldsymbol{a} = \sum_{i=1}^{n} a_i^2 = 0 \iff \boldsymbol{a} = \boldsymbol{0}$. Therefore, the answer is yes: the zero vector $\boldsymbol{0}$.

□

(L-11) Question 5(a) Since it is parallel to the line $2x - 3y = 5$, we need to find a vector $\boldsymbol{v}$ in the nullspace of the coeficient matrix of the system, for example:

$$\left[\begin{array}{cc} 2 & -3 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] \xrightarrow[{[(2)\mathbf{2}]}]{\substack{\boldsymbol{\tau} \\ [(6)\mathbf{1}]}} \left[\begin{array}{cc} 6 & -6 \\ \hline 3 & 0 \\ 0 & 2 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{2}+\mathbf{1}]}} \left[\begin{array}{cc} 6 & 0 \\ \hline 3 & {\color{blue}3} \\ 0 & {\color{blue}2} \end{array}\right]$$

therefore $$\left\{ \boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \exists a \in \mathbb{R} \text{ tal que } \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + a \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}$$

□

(L-11) Question 5(b) We only need to substitute (x, y) by $(1, 1)$ to obtain the right hand side "vector" $\boldsymbol{b}$.

$$2x - 3y = 2 \cdot 1 - 3 \cdot 1 = -1 \quad \Rightarrow \quad 2x - 3y = -1.$$

Hence
$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; \begin{bmatrix} 2 & -3 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \end{pmatrix} \right\}$$

□

(L-11) Question 6(a) $\sqrt{3^2 + 1^2} = \sqrt{10}$

□

(L-11) Question 6(b) $\sqrt{5}$

□

(L-11) Question 6(c) $\sqrt{18}$

□

(L-11) Question 6(d) 0

□

(L-11) Question 6(e) $\sqrt{3}$

□

(L-11) Question 7. $\|\boldsymbol{v}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v} = 4 + 1 + 0 + 16 + 4 = 25$ so we take $\boldsymbol{u} = \frac{1}{\|\boldsymbol{v}\|} \cdot \boldsymbol{v} = (\frac{2}{5}, \frac{-1}{5}, 0, \frac{4}{5}, \frac{-2}{5})$.

□

(L-11) Question 8. Its dot product must be zero, therefore $(k)(4) + (1)(3) = 0$ therefore $k = -3/4$.

□

(L-11) Question 9(a) $\begin{bmatrix} 1 & 2 & a \\ 2 & -3 & b \\ -3 & 5 & c \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow$

$$\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix};$$

So, $\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix}$.

□

(L-11) Question 9(b) Impossible, $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$ not orthogonal to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

□

(L-11) Question 9(c) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ in $\mathcal{C}\left(\mathbf{A}\right)$, and $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ in $\mathcal{N}\left(\mathbf{A}^{\mathsf{T}}\right)$. It is impossible: these vectors are not perpendicular.

□

(L-11) Question 9(d) This asks for $\mathbf{A} \cdot \mathbf{A} = \mathbf{0}$. Take, for example $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, or $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ for example.

□

(L-11) Question 9(e) $\left(1, \quad 1, \quad 1,\right)$ will be in the nullspace, $\mathbf{A} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{0}$; and row space, $\left(1, \quad 1, \quad 1,\right) \mathbf{A} = \left(1, \quad 1, \quad 1,\right)$, . . . no such matrix.

□

(L-11) Question 10. If $\mathbf{AB} = \mathbf{0}$, the columns of $\mathbf{B}$ are in the *nullspace* of $\mathbf{A}$. The rows of $\mathbf{A}$ are in the *left nullspace* of $\mathbf{B}$.
If rank $= 2$, all four subspaces would have dimension 2 which is impossible for 3 by 3 matrix.

$\square$

(L-11) Question 11. No. These give an example.

$$\boldsymbol{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \boldsymbol{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \boldsymbol{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$\boldsymbol{u} \cdot \boldsymbol{v} = 1 = \boldsymbol{u} \cdot \boldsymbol{w}$, but $\boldsymbol{v} \neq \boldsymbol{w}$.

$\square$

(L-11) Question 12(a) On the one hand $\mathbf{A}\boldsymbol{x} = \boldsymbol{b} \Rightarrow \boldsymbol{b} \in \mathcal{C}\left(\mathbf{A}\right)$ on the other hand $\mathbf{A}^\intercal \boldsymbol{y} = \mathbf{0} \Rightarrow \boldsymbol{y} \perp \in \mathcal{C}\left(\mathbf{A}\right)$.
If $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution and $\mathbf{A}^\intercal \boldsymbol{y} = \mathbf{0}$, then $\boldsymbol{y}$ is perpendicular to $\boldsymbol{b}$.

$$\boldsymbol{y} \cdot \boldsymbol{b} = \boldsymbol{y}\mathbf{A}\boldsymbol{b} = \mathbf{0} \cdot \boldsymbol{b} = 0.$$

$\square$

(L-11) Question 12(b) If $\mathbf{A}^\intercal \boldsymbol{y} = \boldsymbol{c}$ then $\boldsymbol{y}\mathbf{A} = \boldsymbol{c}$, also $\mathbf{A}\boldsymbol{x} = \mathbf{0}$; then $\boldsymbol{x}$ is perpendicular to $\boldsymbol{c}$.
$\boldsymbol{c}$ is in the row space, and therefore it is orthogonal to $\boldsymbol{x}$, that is a vector in the nullspace. In other words:

$$\boldsymbol{c} \cdot \boldsymbol{x} = \boldsymbol{y}\mathbf{A}\boldsymbol{x} = \boldsymbol{y} \cdot \mathbf{0} = 0.$$

$\square$

(L-11) Question 13. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular then

$$\begin{aligned}\|(\boldsymbol{u}+\boldsymbol{v})\|^2 = (\boldsymbol{u}+\boldsymbol{v})\cdot(\boldsymbol{u}+\boldsymbol{v}) = \boldsymbol{u}\cdot\boldsymbol{u} + 2\big(\boldsymbol{u}\cdot\boldsymbol{v}\big) + \boldsymbol{v}\cdot\boldsymbol{u}\\ = \boldsymbol{u}\cdot\boldsymbol{u} + \boldsymbol{v}\cdot\boldsymbol{v} = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2\end{aligned}$$

(the third equality holds because $\boldsymbol{u}\cdot\boldsymbol{v} = 0$).

□

(L-11) Question 14(a) $\left\{\boldsymbol{v}\in\mathbb{R}^3 \,\middle|\, \exists \boldsymbol{p}\in\mathbb{R}^2,\ \boldsymbol{v} = \begin{pmatrix}0\\1\\1\end{pmatrix} + \begin{bmatrix}0 & 1\\ 1 & 1\\ 2 & 0\end{bmatrix}\boldsymbol{p}\right\}.$

□

(L-11) Question 14(b)

$$\left[\begin{array}{ccc} 0 & 1 & 2\\ 1 & 1 & 0\\ \hline x & y & z\\ \hline 0 & 1 & 1\end{array}\right] \xrightarrow[{[(-2)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc} 0 & 1 & 0\\ 1 & 1 & -2\\ \hline x & y & -2y+z\\ \hline 0 & 1 & -1\end{array}\right] \xrightarrow[{[(2)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc} 0 & 1 & 0\\ 1 & 0 & 0\\ \hline x & -x+y & 2x-2y+z\\ \hline 0 & 1 & -1\end{array}\right]$$

Therefore: $\left\{\boldsymbol{v}\in\mathbb{R}^3 \,\middle|\, \begin{bmatrix} 2 & -2 & 1\end{bmatrix}\boldsymbol{v} = \big(-1,\big)\right\}.$

```
p = Vector([0,1,1])
v = Vector([0,1,2])
w = Vector([1,1,0])
```

```
S = SubEspacio(Sistema([v,w]))
EAfin(S,p)
```

□

(L-11) Question 15(a) Since the plane is in the 3 dimensinal space $\mathbb{R}^3$, in this case we need to find two vectors orthohonal to $\begin{pmatrix}2, & 1, & 3,\end{pmatrix}$. For example, $\begin{pmatrix}-1, & 3, & 1,\end{pmatrix}$ and $\begin{pmatrix}0, & -1, & 1,\end{pmatrix}$. therefore,

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 \text{ tal que } \boldsymbol{x} = \begin{pmatrix}2\\1\\3\end{pmatrix} + \begin{bmatrix}1 & 0\\-3 & -1\\0 & 1\end{bmatrix}\boldsymbol{p} \right\}.$$

□

(L-11) Question 15(b) In this case we already know a vector orthogonal to the parametric part, hence:

$$\begin{bmatrix}3 & 1 & 1\end{bmatrix}\boldsymbol{x} = \begin{bmatrix}3 & 1 & 1\end{bmatrix}\boldsymbol{s}; \quad \Rightarrow \quad \begin{bmatrix}3 & 1 & 1\end{bmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{bmatrix}3 & 1 & 1\end{bmatrix}\begin{pmatrix}2\\1\\3\end{pmatrix} = (10,);$$

$$\Longrightarrow \quad \left\{ \boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \begin{bmatrix}3 & 1 & 1\end{bmatrix}\boldsymbol{x} = (10,) \right\}.$$

```
p = Vector([2,1,3])
```

```
v = Vector([3,1,1])
S =  SubEspacio(~Matrix([v])) # esta es una alternativa
#S = ~SubEspacio(Sistema([v])) # esta es otra alternativa
EAfin(S,p)
```

□

(L-11) Question 16. We can take as row of **A**, a linear combination of a basis of the left null space of $\begin{bmatrix}1\\2\\4\end{bmatrix}$. Hence,

$$\left[\begin{array}{ccc}1&2&4\\\hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow[{[(-4)\mathbf{1}+\mathbf{3}]}]{\overset{\boldsymbol{\tau}}{[(-2)\mathbf{1}+\mathbf{2}]}}\left[\begin{array}{ccc}1&0&0\\\hline 1&-2&-4\\0&1&0\\0&0&1\end{array}\right]\qquad\Rightarrow\qquad\begin{bmatrix}-2&1&0\\-4&0&1\end{bmatrix}$$

and then

$$\underset{1\times 3}{\mathbf{A}}=\begin{bmatrix}1&1\end{bmatrix}\begin{bmatrix}-2&1&0\\-4&0&1\end{bmatrix}=\begin{bmatrix}-6&1&1\end{bmatrix}$$

but also

$$\mathbf{C}=\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix}\begin{bmatrix}-2&1&0\\-4&0&1\end{bmatrix}=\begin{bmatrix}-2&1&0\\-4&0&1\\6&-1&-1\end{bmatrix}$$

has the same nullspace. . .

□

(L-11) Question 17(a) La solución completa es:

$$\boldsymbol{b} = \left\{ \boldsymbol{v} \in \mathbb{R}^5 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ -2 \\ 4 \end{pmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \boldsymbol{p} \right\}$$

```
A = Matrix([ [1,2,0,1,1], [0,0,2,3,1], [0,0,1,4,2], [0,0,0,1,1] ])
b = Vector( [1,0,1,2] )
SEL (A, b, 1)
```

□

(L-11) Question 17(b) Puesto que la matriz de coeficientes tiene cinco columnas, el sistema tiene cinco incognitas, así pues, los vectores que pertenecen al conjunto de soluciones tienen cinco componentes (un número por columna). Así pues, el conjunto de soluciones es un subconjunto de $\mathbb{R}^5$; Y en este caso, dicho conjunto es una recta, ya que la dimensión de $\mathcal{N}\left(\mathbf{A}\right)$ es uno. Así pues, un vector director es cualquier múltiplo (excepto el vector nulo $\mathbf{0}$) de la solución especial que hemos encontrado: $\boldsymbol{n} = \begin{pmatrix}-2, & 1, & 0, & 0, & 0,\end{pmatrix}$. Y uno de los puntos por donde pasa la recta es la solución particular que obtuvimos al resolver el sistema: $\boldsymbol{s} = \begin{pmatrix}-1, & 0, & 1, & -2, & 4,\end{pmatrix}$.

□

(L-11) Question 17(c)

$$\left[\begin{array}{c}[\boldsymbol{n}]^{\mathsf{T}} \\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{ccccc} -2 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{1}+\mathbf{2}]}]{\substack{\boldsymbol{\tau} \\ [(2)\mathbf{2}]}} \left[\begin{array}{ccccc} -2 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c}\mathbf{L} \\ \hline \mathbf{E}\end{array}\right]$$

Las cuatro últimas columnas de la matriz **E** son vectores perpendiculares a $\boldsymbol{n}$; y es evidente que son cuatro, y que son linealmente independientes, así que son una base del subespacio perpendicular a $\boldsymbol{n}$.

□

(L-11) Question 18(a) Any column of **A** is orthogonal to the two special solutions given in the problem. That is,

$$\left[\begin{array}{cccc} 3 & 1 & 0 & 0 \\ 6 & 0 & 2 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-3)\mathbf{2}+\mathbf{1}] \\ [(-6)\mathbf{4}+\mathbf{1}] \\ [(-2)\mathbf{4}+\mathbf{3}]}} \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -6 & 0 & -2 & 1 \end{array}\right] \quad \text{so} \quad \mathbf{R} = \begin{bmatrix} 1 & 0 \\ -3 & 0 \\ 0 & 1 \\ -6 & -2 \end{bmatrix}.$$

□

(L-11) Question 18(b) **R** has two pivots, and therefore **A** has two pivots and $r(\mathbf{A}) = 2$. Two independent rows in $\mathbb{R}^2$ span $\mathbb{R}^2$, so $\mathcal{C}\left(\mathbf{A}^\intercal\right) = \mathbb{R}^2$.

□

(L-11) Question 18(c) Since rows 1 and 3 are pivot rows, then $\boldsymbol{x}_p = \left(3, \quad 0, \quad 6, \quad 0,\right)$ is a particular solution, so the complete solution is

$$\left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2,\ \boldsymbol{v} = \left(3, \quad 0, \quad 6, \quad 0,\right) + \boldsymbol{p} \begin{bmatrix} 3 & 1 & 0 & 0 \\ 6 & 0 & 2 & 1 \end{bmatrix} \right\}$$

since

$$\left(3, \quad 0, \quad 6, \quad 0,\right) \begin{bmatrix} 1 & 0 \\ -3 & 0 \\ 0 & 1 \\ -6 & -2 \end{bmatrix} = \left(3, \quad 6,\right)$$

and

$$\boldsymbol{p} \begin{bmatrix} 3 & 1 & 0 & 0 \\ 6 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 0 \\ 0 & 1 \\ -6 & -2 \end{bmatrix} = \boldsymbol{p} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \left(0, \quad 0,\right).$$

□

(L-11) Question 18(d) It is easy to see that

$$-2\begin{pmatrix}-3\\0\end{pmatrix}+2\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}-6\\-2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

If you dont see that, we can always use gaussian elimination

$$\left[\begin{array}{ccc}-3&0&-6\\0&1&-2\\\hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow{\overset{\tau}{[(-2)\mathbf{1}+\mathbf{3}]}}\left[\begin{array}{ccc}-3&0&0\\0&1&-2\\\hline 1&0&-2\\0&1&0\\0&0&1\end{array}\right]\xrightarrow{\overset{\tau}{[(2)\mathbf{2}+\mathbf{3}]}}\left[\begin{array}{ccc}-3&0&0\\0&1&0\\\hline 1&0&-2\\0&1&2\\0&0&1\end{array}\right].$$

□

(L-12) Question 1(a)

$$\boldsymbol{p}=[\boldsymbol{a}]\left([\boldsymbol{a}]^{\intercal}[\boldsymbol{a}]\right)^{-1}[\boldsymbol{a}]^{\intercal}\boldsymbol{b}=\begin{bmatrix}3\\2\end{bmatrix}\left(\begin{bmatrix}3&2\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}\right)^{-1}\begin{bmatrix}3&2\end{bmatrix}\begin{pmatrix}2\\1\end{pmatrix}=$$

$$\frac{1}{13}\begin{bmatrix}3\\2\end{bmatrix}\begin{bmatrix}3&2\end{bmatrix}\begin{pmatrix}2\\1\end{pmatrix}=\frac{1}{13}\begin{pmatrix}24\\16\end{pmatrix}$$

$$\boldsymbol{e}=\boldsymbol{b}-\boldsymbol{p}=\begin{pmatrix}2\\1\end{pmatrix}-\frac{1}{13}\begin{pmatrix}24\\16\end{pmatrix}=\frac{1}{13}\begin{pmatrix}2\\-3\end{pmatrix}$$

$$\boldsymbol{a}\cdot\boldsymbol{e} = \begin{pmatrix}3, & 2,\end{pmatrix}\begin{pmatrix}2\\-3\end{pmatrix}\frac{1}{13} = 0\frac{1}{13} = 0.$$

$$\mathsf{P} = \frac{1}{13}\begin{bmatrix}9 & 6\\6 & 4\end{bmatrix}; \quad \mathsf{P}^2 = \frac{1}{13}\cdot\frac{1}{13}\begin{bmatrix}117 & 78\\78 & 52\end{bmatrix} = \frac{1}{13}\begin{bmatrix}9 & 6\\6 & 4\end{bmatrix} = \mathsf{P};$$

$$\mathsf{P}\boldsymbol{b} = \frac{1}{13}\begin{bmatrix}9 & 6\\6 & 4\end{bmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \frac{1}{13}\begin{pmatrix}24\\16\end{pmatrix}.$$

```
a      = Vector([3,2]);  b = Vector([2,1]);   A = Matrix([a])
P      = A*InvMat((~A)*A)*(~A)
p      = P*b;   e = b-p
Sistema([p,e,P])
```

□

(L-12) Question 1(b)

$$\boldsymbol{p} = [\boldsymbol{a}]\left([\boldsymbol{a}]^\intercal[\boldsymbol{a}]\right)^{-1}[\boldsymbol{a}]^\intercal\boldsymbol{b} = \begin{bmatrix}3\\0\end{bmatrix}\left(\begin{bmatrix}3 & 0\end{bmatrix}\begin{bmatrix}3\\0\end{bmatrix}\right)^{-1}\begin{bmatrix}3 & 0\end{bmatrix}\begin{pmatrix}2\\1\end{pmatrix} =$$

$$\frac{1}{9}\begin{bmatrix}3\\0\end{bmatrix}\begin{bmatrix}3 & 0\end{bmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \frac{1}{9}\begin{pmatrix}18\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}.$$

$$\boldsymbol{e} = \boldsymbol{b}-\boldsymbol{p} = \begin{pmatrix}2\\1\end{pmatrix} - \frac{1}{9}\begin{pmatrix}18\\0\end{pmatrix} = \frac{1}{9}\begin{pmatrix}0\\9\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\boldsymbol{a} \cdot \boldsymbol{e} = \begin{pmatrix} 3, & 0, \end{pmatrix} \begin{pmatrix} 0 \\ 9 \end{pmatrix} \frac{1}{9} = 0 \frac{1}{9} = 0.$$

$$\mathbf{P} = \frac{1}{9} \begin{bmatrix} 9 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \quad \mathbf{P}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{P};$$

$$\mathbf{P}\boldsymbol{b} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

□

(L-12) Question 1(c)

$$\boldsymbol{p} = [\boldsymbol{a}] \left([\boldsymbol{a}]^\mathsf{T} [\boldsymbol{a}] \right)^{-1} [\boldsymbol{a}]^\mathsf{T} \boldsymbol{b} =$$

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} =$$

$$\frac{1}{6} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

$$\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 7 \\ 8 \\ 23 \end{pmatrix}$$

$$\boldsymbol{a} \cdot \boldsymbol{e} = (1, \quad 2, \quad -1,) \begin{pmatrix} 7 \\ 8 \\ 23 \end{pmatrix} \frac{1}{6} = 0\frac{1}{6} = 0.$$

$$\mathbf{P} = \frac{1}{6}\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix}; \quad \mathbf{P}^2 = \frac{1}{6}\cdot\frac{1}{6}\begin{bmatrix} 6 & 12 & -6 \\ 12 & 24 & -12 \\ -6 & -12 & 6 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix}.$$

$$\mathbf{P}\boldsymbol{b} = \frac{1}{6}\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}.$$

□

(L-12) Question 1(d)

$$\boldsymbol{p} = [\boldsymbol{a}]\left([\boldsymbol{a}]^\intercal[\boldsymbol{a}]\right)^{-1}[\boldsymbol{a}]^\intercal\boldsymbol{b} =$$

$$\begin{bmatrix} 3 \\ 3 \\ 12 \end{bmatrix} \left(\begin{bmatrix} 3 & 3 & 12 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 12 \end{bmatrix} \right)^{-1} \begin{bmatrix} 3 & 3 & 12 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} =$$

$$\frac{1}{162} \begin{bmatrix} 3 \\ 3 \\ 12 \end{bmatrix} \begin{bmatrix} 3 & 3 & 12 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

$$\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boldsymbol{a} \cdot \boldsymbol{e} = \boldsymbol{a} \cdot \boldsymbol{0} = 0.$$

$$\mathsf{P} = \frac{1}{18} \begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & 4 \\ 4 & 4 & 16 \end{bmatrix}; \quad \mathsf{P}^2 = \frac{1}{18}\frac{1}{18} \begin{bmatrix} 18 & 18 & 72 \\ 18 & 18 & 72 \\ 72 & 72 & 288 \end{bmatrix} = \mathsf{P};$$

$$\mathsf{P}\boldsymbol{b} = \frac{1}{18} \begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & 4 \\ 4 & 4 & 16 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}.$$

(L-12) Question 2(a)

$$\boldsymbol{p} = [\boldsymbol{a}]\left([\boldsymbol{a}]^\intercal[\boldsymbol{a}]\right)^{-1}[\boldsymbol{a}]^\intercal\boldsymbol{b} =$$

$$\begin{bmatrix}-3\\1\\-3\end{bmatrix}\left(\begin{bmatrix}-3 & 1 & -3\end{bmatrix}\begin{bmatrix}-3\\1\\-3\end{bmatrix}\right)^{-1}\begin{bmatrix}-3 & 1 & -3\end{bmatrix}\begin{pmatrix}2\\-1\\4\end{pmatrix} =$$

$$\frac{1}{19}\begin{bmatrix}-3\\1\\-3\end{bmatrix}\begin{bmatrix}-3 & 1 & -3\end{bmatrix}\begin{pmatrix}2\\-1\\4\end{pmatrix} = \frac{1}{19}\begin{bmatrix}9 & -3 & 9\\-3 & 1 & -3\\9 & -3 & 9\end{bmatrix}\begin{pmatrix}2\\-1\\4\end{pmatrix} = \begin{pmatrix}3\\-1\\3\end{pmatrix}$$

```
b       = Vector([2,-1,4]); a = Vector([-3,1,-3]); A = Matrix([a])
P       = A*InvMat((~A)*A)*(~A)      # Matriz proyección
p1      = P*b                        # Alternativa 1
x       = SEL( (~A)*A, (~A)*b ).solP # Solución Ecuaciones Normales
p2      = A*x                        # Alternativa 2
Sistema([p1,p2])
```

(L-12) Question 2(b) The line is the set of solutions to $3x - y = 0$:

$$\left[\begin{array}{c}\mathsf{A}\\\hline\mathsf{I}\end{array}\right] = \left[\begin{array}{cc} 3 & -1 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] \xrightarrow[]{\substack{\boldsymbol{\tau} \\ [(3)\mathbf{2}] \\ [(1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc} 3 & 0 \\ \hline 1 & 1 \\ 0 & 3 \end{array}\right] = \left[\begin{array}{c}\mathsf{L}\\\hline\mathsf{E}\end{array}\right];$$

so we should project into the line

$$\text{The line : } \left\{ \boldsymbol{v} \in \mathbb{R}^2 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \boldsymbol{p} \right\}$$

$$\boldsymbol{p} = [\boldsymbol{a}]\left([\boldsymbol{a}]^{\mathsf{T}}[\boldsymbol{a}]\right)^{-1}[\boldsymbol{a}]^{\mathsf{T}}\boldsymbol{b} =$$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \left(\begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} =$$

$$\frac{1}{10} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \frac{1}{10} \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -2 \\ -6 \end{pmatrix};$$

```
b       = Vector([-1,-1])
B       = Matrix([[3,-1]])
```

```
a       = Homogenea(B).sgen|1
# a       = Homogenea(B).enulo.sgen|1 # alternativa equivalente
# a       = EAfin(B, V0(2)).S.sgen|1  # alternativa equivalente

A       = Matrix([a])
P       = A*InvMat((~A)*A)*(~A)        # Matriz proyección
p1      = P*b                          # Alternativa 1
x       = SEL( (~A)*A, (~A)*b ).solP # Solución Ecuaciones Normales
p2      = A*x                          # Alternativa 2
Sistema([p1,p2])
```

□

(L-12) Question 3.

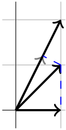
$$\boldsymbol{p} = [\boldsymbol{a}]\left([\boldsymbol{a}]^\intercal[\boldsymbol{a}]\right)^{-1}[\boldsymbol{a}]^\intercal\boldsymbol{b} =$$

$$\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}\left(\begin{bmatrix}-1&1&-1&1\end{bmatrix}\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}\right)^{-1}\begin{bmatrix}-1&1&-1&1\end{bmatrix}\begin{pmatrix}1\\2\\1\\3\end{pmatrix} =$$

$$\frac{1}{4}\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}\begin{bmatrix}-1&1&-1&1\end{bmatrix}\begin{pmatrix}1\\2\\1\\3\end{pmatrix} = \frac{1}{4}\begin{bmatrix}1&-1&1&-1\\-1&1&-1&1\\1&-1&1&-1\\-1&1&-1&1\end{bmatrix}\begin{pmatrix}1\\2\\1\\3\end{pmatrix} = \frac{1}{4}\begin{pmatrix}-3\\3\\-3\\3\end{pmatrix};$$

□

(L-12) Question 4(a) $\boldsymbol{p}_1 = \left(1, \quad 0,\right)$ and $\boldsymbol{p}_2 = \left(\frac{3}{5}, \quad \frac{6}{5},\right)$. Then $\boldsymbol{p}_1 + \boldsymbol{p}_2 \neq \boldsymbol{b}$.




```
b  = Vector([1,1])
a1 = Vector([1,0])
a2 = Vector([1,2])

A1 = Matrix([a1])
p1 = A1 * SEL((~A1)*A1,(~A1*b)).solP

A2 = Matrix([a2])
p2 = A2 * SEL((~A2)*A2,(~A2*b)).solP
Sistema([p1,p2])
```

□

(L-12) Question 4(b) Since $\mathbf{A}$ is invertible, the projection matrix $\mathbf{P} = \mathbf{A}(\mathbf{A}^{\intercal}\mathbf{A})^{-1}\mathbf{A}^{\intercal} = \mathbf{I}$ projects onto all of $\mathbb{R}^2$. Therefore $\boldsymbol{p} = \mathbf{P}\boldsymbol{b}_1 = \boldsymbol{b}_1$.

```
A3 = Matrix([a1,a2])
P  = A3*InvMat((~A3)*A3)*(~A3)
```

```
p3 = P*b
Sistema([p1,p2,p3,P])
```

□

(L-12) Question 5(a) $\mathbf{P}^2 = \mathbf{P}$ and therefore
$(\mathbf{I}-\mathbf{P})^2 = (\mathbf{I}-\mathbf{P})(\mathbf{I}-\mathbf{P}) = \mathbf{I} - \mathbf{PI} - \mathbf{IP} + \mathbf{P}^2 = \mathbf{I} - \mathbf{P}$.
When $\mathbf{P}$ projects onto the column space of $\mathbf{A}$, $(\mathbf{I}-\mathbf{P})$ projects onto the *left nullspace* of $\mathbf{A}$.

□

(L-12) Question 5(b) $\mathbf{P}^\intercal = \mathbf{P}$ and therefore $(\mathbf{I}-\mathbf{P})^\intercal = (\mathbf{I}^\intercal - \mathbf{P}^\intercal) = \mathbf{I} - \mathbf{P}$.

□

(L-12) Question 6(a)

$$\mathbf{P}_1 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}; \qquad \mathbf{P}_1 = \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}.$$

$\mathbf{P}_1\mathbf{P}_2 =$ zero matrix because $\boldsymbol{a}_1 \perp \boldsymbol{a}_2$.

□

(L-12) Question 6(b) $\boldsymbol{p}_1 = \frac{1}{9}\begin{pmatrix}1, & -2, & -2,\end{pmatrix}$, $\boldsymbol{p}_2 = \frac{1}{9}\begin{pmatrix}4, & 4, & -2,\end{pmatrix}$, $\boldsymbol{p}_3 = \frac{1}{9}\begin{pmatrix}4, & -2, & 4,\end{pmatrix}$. Then $\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3 = \begin{pmatrix}1, & 0, & 0,\end{pmatrix} = \boldsymbol{b}$. Note that $\boldsymbol{a}_3 \perp \boldsymbol{a}_1$ and $\boldsymbol{a}_3 \perp \boldsymbol{a}_2$.

□

(L-12) Question 6(c)

$$\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 =$$
$$\frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix} = \mathbf{I}.$$

□

(L-12) Question 7(a) $\boldsymbol{p}_1 = \mathbf{A}_1 \left(\mathbf{A}_1^\intercal \mathbf{A}_1\right)^{-1} (\mathbf{A}_1^\intercal) \boldsymbol{b}_1 = \left(2, \quad 3, \quad 0,\right)$ and $\boldsymbol{e}_1 = \left(0, \quad 0, \quad 4,\right)$.

□

(L-12) Question 7(b) $\boldsymbol{p}_2 = \mathbf{A}_2 \left(\mathbf{A}_2^\intercal \mathbf{A}_2\right)^{-1} (\mathbf{A}_2^\intercal) \boldsymbol{b}_2 = \left(4, \quad 4, \quad 6,\right)$ and $\boldsymbol{e}_2 = \left(0, \quad 0, \quad 0,\right)$.

□

(L-12) Question 7(c)

$\mathbf{P}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ projection on xy plane. $\mathbf{P}_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \left(\mathbf{P}_2\right)^2.$

□

Exercise 8(a) Since $\mathbf{A}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k} = \mathbf{A}(\mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k}) = \mathbf{A}(\mathbf{I}_{\boldsymbol{\tau}_1}) \cdots (\mathbf{I}_{\boldsymbol{\tau}_k})$, applying repeatedly (1) we get

$$\begin{aligned}
\det(\mathbf{A}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k}) &= \left|\mathbf{A}(\mathbf{I}_{\boldsymbol{\tau}_1}) \cdots (\mathbf{I}_{\boldsymbol{\tau}_k})\right| \\
&= \left|\mathbf{A}(\mathbf{I}_{\boldsymbol{\tau}_1}) \cdots (\mathbf{I}_{\boldsymbol{\tau}_{(k-1)}})\right| \cdot |\mathbf{I}_{\boldsymbol{\tau}_k}| \\
&= \left|\mathbf{A}(\mathbf{I}_{\boldsymbol{\tau}_1}) \cdots (\mathbf{I}_{\boldsymbol{\tau}_{(k-2)}})\right| \cdot |\mathbf{I}_{\boldsymbol{\tau}_{(k-1)}}| \cdot |\mathbf{I}_{\boldsymbol{\tau}_k}| \\
&\;\vdots \\
&= |\mathbf{A}| \cdot |\mathbf{I}_{\boldsymbol{\tau}_1}| \cdots |\mathbf{I}_{\boldsymbol{\tau}_k}|.
\end{aligned}$$

□

Exercise 8(b)

$$|\mathbf{B}| = \det(\mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k}) = |\mathbf{I}_{\boldsymbol{\tau}_1}| \cdots |\mathbf{I}_{\boldsymbol{\tau}_k}|.$$

and, since the determinants of elementary matrices are not zero $|\mathbf{B}| \neq 0$.

□

Exercise 8(c) If $\mathbf{B}$ is full rank then it is the product of k elementary matrices $\mathbf{B} = \mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k}$. Hence

$$\det(\mathbf{AB}) = \det(\mathbf{A}\mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k}) = \det(\mathbf{A}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k}) = |\mathbf{A}| \cdot \left(|\mathbf{I}_{\boldsymbol{\tau}_1}| \cdots |\mathbf{I}_{\boldsymbol{\tau}_k}|\right) = |\mathbf{A}| \cdot |\mathbf{B}|.$$

$\square$

Exercise 9(a) Since they are elementary matrices of the same type, $\det(\mathbf{I}_{\boldsymbol{\tau}}) = \det({}_{\boldsymbol{\tau}}\mathbf{I})$.

$\square$

Exercise 9(b) Since $\mathbf{B} = \mathbf{I}_{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_k} = (\mathbf{I}_{\boldsymbol{\tau}_1}) \cdots (\mathbf{I}_{\boldsymbol{\tau}_k})$, the determinant is

$$|\mathbf{B}| = \det\left(\mathbf{I}_{\boldsymbol{\tau}_1}\right) \cdots \det\left(\mathbf{I}_{\boldsymbol{\tau}_k}\right) = \prod_{i=1}^{k} \det\left(\mathbf{I}_{\boldsymbol{\tau}_i}\right).$$

But we also know that $\mathbf{B}^{\mathsf{T}} = {}_{\boldsymbol{\tau}_k \cdots \boldsymbol{\tau}_1}\mathbf{I} = \left({}_{\boldsymbol{\tau}_k}\mathbf{I}\right) \cdots \left({}_{\boldsymbol{\tau}_1}\mathbf{I}\right)$, and then its determinant is

$$|\mathbf{B}^{\mathsf{T}}| \;=\; \prod_{i=1}^{k} \det\left({}_{\boldsymbol{\tau}_i}\mathbf{I}\right) \;=\; \prod_{i=1}^{k} \det\left(\mathbf{I}_{\boldsymbol{\tau}_i}\right) \;=\; |\mathbf{B}|.$$

$\square$

(L-13) Question 2. Since $\mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$, we know that

$$1 = |\mathbf{I}| = |\mathbf{A}||\mathbf{A}^{-1}|;$$

and therefore $\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$.

□

(L-13) Question 3(a) $\left|\mathbf{A}(\mathbf{B})^2\right| = 2\cdot(-2)^2 = 8.$
$\left|(\mathbf{AB})^{-1}\right| = \left(|\mathbf{AB}|\right)^{-1} = \frac{1}{-4}.$

□

(L-13) Question 3(b) There is no enough information to compute the determinant of $\mathbf{A}+\mathbf{B}$. "Not enough information" means we can find two pairs of examples $(\mathbf{A}_1,\mathbf{B}_1)$ and $(\mathbf{A}_1,\mathbf{B}_1)$ that satisfies the hypothesis: $\det\mathbf{A}_1 = 2 = \det\mathbf{B}_1$ and $\det\mathbf{A}_2 = 2 = \det\mathbf{B}_2$; but $\operatorname{rg}\left(\mathbf{A}_1+\mathbf{B}_1\right) \neq \operatorname{rg}\left(\mathbf{A}_2+\mathbf{B}_2\right)$.
On the other hand, since $|\mathbf{AB}| = -4 \neq 0$, we know $\underset{3\times 3}{\mathbf{AB}}$ is a full rank matrix; therefore its rank is 3.

□

(L-13) Question 4(a)

$$\left[\mathbf{A}_1\right] = \begin{bmatrix}1&0&0\\1&1&1\\0&0&1\end{bmatrix} \xrightarrow[{[(-1)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix} \xrightarrow[{[(-1)\mathbf{2}+\mathbf{1}]}]{\boldsymbol{\tau}} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = [\mathbf{I}] \quad\Rightarrow\quad \det\mathbf{A}_1 = -$$

```
A =  Matrix([ [1,0,0], [1,1,1], [0,0,1] ])
Determinante(A)      # esta es una opción
A.determinante()     # esta es otra opción
```

□

(L-13) Question 4(b)

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow[\ ]{\substack{\boldsymbol{\tau} \\ [(2)\mathbf{2}] \\ [(1)\mathbf{1+2}]}} \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & -1 \\ 0 & -2 & 2 \end{bmatrix} \xrightarrow[\ ]{\substack{\boldsymbol{\tau} \\ [(3)\mathbf{3}] \\ [(1)\mathbf{2+3}]}} \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 0 & -2 & 4 \end{bmatrix} \xrightarrow[\ ]{\substack{\boldsymbol{\tau} \\ [(3)\mathbf{1}] \\ [(1)\mathbf{2+1}]}} \begin{bmatrix} 6 \\ 0 \\ -2 \end{bmatrix}$$

$$\xrightarrow[\ ]{\substack{\boldsymbol{\tau} \\ [(2)\mathbf{1}] \\ [(1)\mathbf{3+1}] \\ [(2)\mathbf{2}] \\ [(1)\mathbf{3+2}]}} \begin{bmatrix} 12 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow[\ ]{\substack{\boldsymbol{\tau} \\ [(\frac{1}{12})\mathbf{1}] \\ [(\frac{1}{6})\mathbf{2}] \\ [(\frac{1}{4})\mathbf{3}]}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore $\det \mathbf{A}_2 = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot 12 \cdot 6 \cdot 4 = 4$

$\square$

(L-13) Question 4(c)

$$\mathbf{A}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[\ ]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2+3}]}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow[\ ]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1+2}]}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow[\ ]{\substack{\boldsymbol{\tau} \\ [\mathbf{1 \rightleftharpoons 3}]}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [\mathbf{I}$$

$\square$

(L-13) Question 5. Since all applied trasformations are *Type I*, then $\mathbf{AE} = \mathbf{I} \Rightarrow \det(\mathbf{A}) \cdot 1 = 1.$

□

(L-13) Question 6(a) The first one is an elementary matrix, its determinant is 1. The second one is a permutation matrix that exchanges two vectors, its determinant is -1.

□

(L-13) Question 6(b)

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\a&b&c&d\end{bmatrix}\xrightarrow[{[(\frac{1}{d})\mathbf{4}]}]{\boldsymbol{\tau}}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\a&b&c&1\end{bmatrix}\xrightarrow[{\substack{[(-c)\mathbf{4}+\mathbf{3}]\\ [(-b)\mathbf{4}+\mathbf{2}]\\ [(-a)\mathbf{4}+\mathbf{1}]}}]{\boldsymbol{\tau}}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$ Hence, the determinant is d.

□

Exercise 7(a) Como la matriz de orden n es de rango completo, los n elementos de la diagonal principal son pivotes (i.e., distintos de cero).

$$\mathbf{L}=\begin{bmatrix}*_1&&&\\ \vdots&*_2&&\\ &&\ddots&\\ \vdots&\vdots&&\\ \vdots&\vdots&\vdots&*_n\end{bmatrix}$$ donde $*_j$ son números distintos de cero.

Dividiendo cada columna $j$ésima por su pivote $*_j$ para normalizar los pivotes (y compensando dichas transformaciones multiplicado la última fila por cada pivote); y aplicando, en una segunda fase, la eliminación de izquierda a derecha con transformaciones de Tipo I para anular todo lo que queda a la izquierda de los pivotes (ahora basta multiplicar la última fila por 1), llegamos a:

$$\left[\begin{array}{cccc|c} *_1 & & & & \\ \vdots & *_2 & & & \\ \vdots & \vdots & \ddots & & \\ \vdots & \vdots & \vdots & *_n & \\ \hline & & & & 1 \end{array}\right] \xrightarrow[\substack{\boldsymbol{\tau} \\ [(*_1)(\mathbf{n+1})] \\ \vdots \\ [(*_n)(\mathbf{n+1})]}]{\substack{\boldsymbol{\tau} \\ \left[\left(\frac{1}{*_1}\right)\mathbf{1}\right] \\ \vdots \\ \left[\left(\frac{1}{*_n}\right)\mathbf{n}\right]}} \left[\begin{array}{cccc|c} 1 & & & & \\ \vdots & 1 & & & \\ \vdots & \vdots & \ddots & & \\ \vdots & \vdots & \vdots & 1 & \\ \hline & & & & *_1 \cdot *_2 \cdots *_n \end{array}\right] \xrightarrow[\substack{\boldsymbol{\tau} \\ [(1)(\mathbf{n+1})]}]{\substack{\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_q \\ \text{(de Tipo I)}}} \left[\right.$$

por tanto, si la matriz es triangular inferior es de rango completo, su determinante es igual al producto de sus pivotes; es decir, al producto de los elementos de la diagonal.

$$\det(\mathbf{L}) = \text{ producto de los elementos de la diagonal}$$

□

Exercise 7(b) Una matriz de orden n y triangular solo puede ser de rango completo si los n elementos de la diagonal son distintos de cero. Por tanto, si la matriz tringular es singular, necesariamente tiene algún cero en su diagonal principal. Como su determinante es cero, por ser singular, su determinatne es igual al producto de los elementos de la diagonal (donde uno de ellos es cero).

□

Exercise 7(c)

$$\det(\mathbf{U}) = \det\left(\mathbf{U}^\intercal\right) = \text{ producto de los elementos de la diagonal}$$

por ser $\mathbf{U}^\intercal$ triangular inferior.

□

Exercise 8. Expanding by the second column we get

$$\det \mathbf{A} = -0\begin{vmatrix}5&2&4\\3&1&2\\5&2&1\end{vmatrix} + \begin{vmatrix}2&3&2\\3&1&2\\5&2&1\end{vmatrix} - 0\begin{vmatrix}2&3&2\\5&2&4\\5&2&1\end{vmatrix} + 3\begin{vmatrix}2&3&2\\5&2&4\\3&1&2\end{vmatrix}$$
$$= 0 + 17 - 0 + 3\times 4 = 29$$

□

(L-14) Question 2(a) $\det\left(2 \underset{3\times 3}{\mathbf{A}}\right) = 2^3\cdot\det\mathbf{A} = 16.$ $\quad \det\mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}} = 1/2.$

□

(L-14) Question 2(b)

$$\det\left[(3\mathbf{A}_{|1} + 2\mathbf{A}_{|2});\quad \mathbf{A}_{|3};\quad \mathbf{A}_{|2};\right] = \det\left[3\mathbf{A}_{|1};\quad \mathbf{A}_{|3};\quad \mathbf{A}_{|2};\right] =$$
$$= 3\det\left[\mathbf{A}_{|1};\quad \mathbf{A}_{|3};\quad \mathbf{A}_{|2};\right] = -3\det\mathbf{A} = -6.$$

□

(L-14) Question 3.

$$\det(-\mathbf{A}^\intercal) = \det(-\mathbf{A}) = (-1)^n \mathbf{A} = \det(\mathbf{A})$$

since n is an even number.

□

(L-14) Question 4. True, since

$$|\mathbf{A}\mathbf{A}^\intercal| = |\mathbf{A}||\mathbf{A}^\intercal| = |\mathbf{A}||\mathbf{A}| = |\mathbf{A}|^2.$$

□

(L-14) Question 5(a)

$$\begin{vmatrix} a-2 & 1 & 2 \\ b-4 & 3 & 4 \\ c-6 & 5 & 6 \end{vmatrix} = \begin{vmatrix} a & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{vmatrix} = 3$$

□

(L-14) Question 5(b)

$$\begin{vmatrix} 7a & 7 & 14 \\ b & 3 & 4 \\ c & 5 & 6 \end{vmatrix} = 7 \begin{vmatrix} a & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{vmatrix} = 7 \times 3 = 21$$

□

(L-14) Question 5(c)

$$\left|(2\mathbf{A})^{-1}\mathbf{A}^{\mathsf{T}}\right| = \frac{1}{\det 2\mathbf{A}} \det \mathbf{A} = \frac{1}{2^3 \det \mathbf{A}} \det \mathbf{A} = \frac{1}{8}.$$

□

(L-14) Question 5(d)

$$\begin{vmatrix} a-2 & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{vmatrix} = \begin{vmatrix} a & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{vmatrix} - \begin{vmatrix} 2 & 1 & 2 \\ 0 & 3 & 4 \\ 0 & 5 & 6 \end{vmatrix} = 3 + 4 = 7$$

□

(L-14) Question 6(a)

$$\left[\begin{array}{c} \mathbf{A} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{ccc} 2 & 4 & 4 \\ 3 & 5 & 6 \\ 1 & 2 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-2)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc} 2 & 0 & 0 \\ 3 & -1 & 0 \\ 1 & 0 & -2 \\ \hline 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c} \mathbf{L} \\ \hline \mathbf{E} \end{array}\right]$$

$\square$

(L-14) Question 6(b) Puesto que **L** tiene tres pivotes, **A** es invertible.

$\square$

(L-14) Question 6(c)

$$|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|} = \frac{1}{\text{producto de los pivotes de } \mathbf{L}} = \frac{1}{2(-1)(-2)} = \frac{1}{4}.$$

$\square$

(L-14) Question 6(d)

$$|\mathbf{C}| = |\mathbf{A}\mathbf{B}^{\intercal}| = |\mathbf{A}||\mathbf{B}^{\intercal}| = 4 \cdot 0 = 0;$$

ya que **B** tiene dos filas iguales. Por tanto **C** no es invertible.

$\square$

(L-14) Question 7(a)

$$\begin{vmatrix} 1 & 2 \\ -4 & 3 \end{vmatrix} = 1 \cdot 3 - (-4) \cdot 2 = 11$$

$\square$

(L-14) Question 7(b) Expanding by the first row

$$\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & -2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} - (-1) \begin{vmatrix} 0 & 2 \\ 2 & -2 \end{vmatrix} + 0 = -8$$

□

(L-14) Question 7(c) Expanding by the second column we get a minor equal to the determinant in the previous exercise:

$$\begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 1 & -2 \end{vmatrix} = -0 + 1 \cdot \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & -2 \end{vmatrix} - 0 + 0 = -8$$

□

(L-14) Question 8.

$$\begin{vmatrix} 0 & 0 & 0 & 3 & 0 \\ -2 & 0 & 0 & 2 & 0 \\ 8 & -1 & 0 & -7 & 2 \\ -1 & 2 & 2 & 3 & 2 \\ 2 & 2 & 3 & 6 & 4 \end{vmatrix} = -3 \begin{vmatrix} -2 & 0 & 0 & 0 \\ 8 & -1 & 0 & 2 \\ -1 & 2 & 2 & 2 \\ 2 & 2 & 3 & 4 \end{vmatrix} =$$

$$= (-3) \cdot (-2) \begin{vmatrix} -1 & 0 & 2 \\ 2 & 2 & 2 \\ 2 & 3 & 4 \end{vmatrix} = (-3) \cdot (-2) \cdot (2) = 12$$

□

(L-14) Question 9.

$$\det \mathbf{A} = \begin{vmatrix} 2 & 2 & 0 & 0 \\ 5 & 5 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 2 & 0 \\ 5 & 5 & 0 \\ 3 & 0 & 1 \end{vmatrix} = 1 \cdot 1 \cdot \begin{vmatrix} 2 & 2 \\ 5 & 5 \end{vmatrix} = 1 \cdot 1 \cdot 0 = 0.$$

Note that the first column is a linear combination of the others.

□

(L-14) Question 10. Expanding by the first column

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 3 & \cdots & 3 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & n \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 2 & \cdots & 2 \\ 0 & 3 & \cdots & 3 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & n \end{vmatrix} - 0 + 0 - \cdots 0$$

expanding by the first column again

$$= 1 \cdot 2 \cdot \begin{vmatrix} 3 & \cdots & 3 \\ \vdots & \ddots & \\ 0 & \cdots & n \end{vmatrix} - 0 + 0 - \cdots 0$$

and again. . . and again. . .

$$1 \cdot 2 \cdot 3 \cdots (n-2) \cdot \begin{vmatrix} (n-1) & (n-1) \\ 0 & n \end{vmatrix} = n!.$$

□

(L-14) Question 11(a)

$$|\mathbf{A}_4| = \begin{vmatrix} 3 & 0 & 0 & 2 \\ 2 & 3 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 3 \end{vmatrix} = 3 \begin{vmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 2 & 3 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{vmatrix} = 3 \cdot 27 - 2 \cdot 8 = 65$$

□

(L-14) Question 11(b) In general $|\mathbf{A}_n| = 3^n + (-1)^{n-1} 2^n.$

□

(L-14) Question 12. Expanding by the last column the first matrix, and expanding by the first column the second matrix, and repeating the expansions with the minors (as in the previous exercise), we get

$$\begin{vmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{vmatrix} = |\mathbf{B}|$$

$$\begin{vmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{vmatrix} = |\mathbf{C}|.$$

Therefore

$$\det \mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} = \begin{vmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{vmatrix} \begin{vmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{vmatrix} = \det \mathbf{B} \det \mathbf{C}.$$

□

(L-14) Question 13(a) On the one hand,

$$\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}; \qquad \boldsymbol{b} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \end{bmatrix},$$

on the other hand,

$$\det(\mathbf{A}) = 3; \qquad \begin{vmatrix} \mathbf{1} & 5 \\ \mathbf{2} & 4 \end{vmatrix} = -6; \qquad \begin{vmatrix} 2 & \mathbf{1} \\ 1 & \mathbf{2} \end{vmatrix} = 3.$$

Therefore

$$x = \frac{\begin{vmatrix} \mathbf{1} & 5 \\ \mathbf{2} & 4 \end{vmatrix}}{\det(\mathbf{A})} = \frac{-6}{3} = -2; \qquad y = \frac{\begin{vmatrix} 2 & \mathbf{1} \\ 1 & \mathbf{2} \end{vmatrix}}{\det(\mathbf{A})} = \frac{3}{3} = 1.$$

$\square$

(L-14) Question 13(b) On the one hand,

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}; \qquad \boldsymbol{b} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

on the other hand,

$$\det(\mathbf{A}) = 4; \qquad \begin{vmatrix} \mathbf{1} & 1 & 0 \\ \mathbf{0} & 2 & 1 \\ \mathbf{0} & 1 & 2 \end{vmatrix} = 3; \qquad \begin{vmatrix} 2 & \mathbf{1} & 0 \\ 1 & \mathbf{0} & 1 \\ 0 & \mathbf{0} & 2 \end{vmatrix} = -2; \qquad \begin{vmatrix} 2 & 1 & \mathbf{1} \\ 1 & 2 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \end{vmatrix} = 1.$$

Therefore, $x = \frac{3}{4}; \qquad y = \frac{-2}{4} = \frac{-1}{2}; \qquad z = \frac{1}{4}.$

□

(L-14) Question 14(a) $\quad \mathbf{Adj}(\mathbf{A}) = \begin{bmatrix} 3 & -2 & 0 \\ -0 & 1 & -0 \\ 0 & -4 & 3 \end{bmatrix}; \qquad \det(\mathbf{A}) = 3;$

$$\mathbf{A}^{-1} = \frac{\mathbf{Adj}(\mathbf{A})}{|\mathbf{A}|} = \frac{1}{3} \begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 3 \end{bmatrix}.$$

□

(L-14) Question 14(b) $\quad \mathbf{Adj}(\mathbf{B}) = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}; \qquad \det(\mathbf{B}) = 4$

$$\mathbf{B}^{-1} = \frac{\mathbf{Adj}(\mathbf{B})}{|\mathbf{B}|} = \frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Note the inverse of a symmetric matrix is also symmetric.

□

(L-14) Question 15(a)

$$\begin{bmatrix}1&4&2&3\\2&3&3&7\\0&1&0&3\\0&2&0&a\end{bmatrix}\to\begin{bmatrix}1&4&2&3\\0&-5&-1&1\\0&1&0&3\\0&2&0&a\end{bmatrix}\to\begin{bmatrix}1&4&2&3\\0&1&0&3\\0&-5&-1&1\\0&2&0&a\end{bmatrix}$$
$$\to\begin{bmatrix}1&4&2&3\\0&1&0&3\\0&0&-1&16\\0&0&0&a-6\end{bmatrix}$$

In order to have a full rank matrix, the parameter a must be different from 6.

□

(L-14) Question 15(b) On the one hand

$$\det \mathbf{A} = \begin{vmatrix}1&4&2&3\\2&3&3&7\\0&1&0&3\\0&2&0&5\end{vmatrix} = 1\begin{vmatrix}3&3&7\\1&0&3\\2&0&5\end{vmatrix} - 2\begin{vmatrix}4&2&3\\1&0&3\\2&0&5\end{vmatrix} =$$
$$= -3\begin{vmatrix}1&3\\2&5\end{vmatrix} - 2\cdot(-2)\begin{vmatrix}1&3\\2&5\end{vmatrix} = \begin{vmatrix}1&3\\2&5\end{vmatrix} = -1.$$

On the other hand

$$\begin{vmatrix} 1 & 4 & 2 & 1 \\ 2 & 3 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{vmatrix} = -1 \cdot \begin{vmatrix} 2 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{vmatrix} = -2 \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} = 0$$

Consequently, $x_4 = \frac{0}{-1} = 0.$

□

(L-14) Question 15(c)

$$\left[\begin{array}{cccc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{4}]}]{\overset{\tau}{[(-1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ \hline 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[\mathbf{3}\rightleftharpoons\mathbf{4}]}]{\overset{\tau}{[\mathbf{1}\rightleftharpoons\mathbf{2}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ \hline 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[{[(-1)\mathbf{4}]}]{\overset{\tau}{[(1)\mathbf{4}+\mathbf{1}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Hence $\quad \mathbf{B}^{-1} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ and thus, multiplying by $\mathbf{B}^{-1}$ we get

$\mathbf{B}\boldsymbol{x} = \boldsymbol{b} \quad \Rightarrow \quad \mathbf{B}^{-1}\mathbf{B}\boldsymbol{x} = \mathbf{B}^{-1}\boldsymbol{b} \quad \Rightarrow \quad \boldsymbol{x} = \mathbf{B}^{-1}\boldsymbol{b}$:

$$\boldsymbol{x} = \mathbf{B}^{-1}\boldsymbol{b} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

□

(L-15) Question 1(a)

$$\begin{bmatrix} -3 & 4 & -4 \\ -3 & 5 & -3 \\ -1 & 2 & 0 \end{bmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

Therefore, the corresponding eigenvalue to $\boldsymbol{v}$ is -1.

$$\begin{bmatrix} -3 & 4 & -4 \\ -3 & 5 & -3 \\ -1 & 2 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Therefore, the correspondant eigenvalue to $\boldsymbol{w}$ is 1.

□

(L-15) Question 1(b) We need to find a vector in the null space of $(\mathbf{A} - 2\mathbf{I})$, in other words, we need to find a solution to

$$\begin{bmatrix} -5 & 4 & -4 \\ -3 & 3 & -3 \\ -1 & 2 & -2 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We can find a solution by gaussian elimination, but as the third column equals minus the second one, it is easy to realize that one solution is:

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$\square$

(L-15) Question 2(a) The eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

are the solutions to the characteristic equation

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = (3-\lambda)(-3-\lambda) - 16 = 0; \quad \Rightarrow \quad \lambda_1 = 5;\ \lambda_2 = -5.$$

We can also use

$$\begin{cases} \lambda_1 \cdot \lambda_2 = & \det \mathbf{A} = -25 \\ \lambda_1 + \lambda_2 = & \operatorname{tr} \mathbf{A} = 0 \end{cases}$$

with the same result.
For $\lambda_1 = 5$, the null space of

$$(\mathbf{A} - \lambda_1 \mathbf{I}) = \begin{bmatrix} 3-5 & 4 \\ 4 & -3-5 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix}$$

consists of all multiples of $\boldsymbol{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Hence, the eigenvectors are the non-null multiples of $\boldsymbol{x}_1$
For $\lambda_2 = -5$, the null space of

$$(\mathbf{A} - \lambda_2 \mathbf{I}) = \begin{bmatrix} 3+5 & 4 \\ 4 & -3+5 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}$$

consists of all multiples of $\boldsymbol{x}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Hence, the eigenvectors are the non-null multiples of $\boldsymbol{x}_2$

□

(L-15) Question 2(b) The eigenvalues of

$$\mathbf{B} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

are the solutions to the characteristic equation

$$\det(\mathbf{B} - \lambda\mathbf{I}) = \begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} = (a-\lambda)^2 - b^2 = \lambda^2 - 2a\lambda + (a^2 - b^2) = 0.$$

Therefore

$$\lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2 - b^2)}}{2} = a \pm b.$$

For $\lambda_1 = a + b$, the null space of

$$(\mathbf{B} - \lambda_1\mathbf{I}) = \begin{bmatrix} -b & b \\ b & -b \end{bmatrix}$$

consists of all multiples of $\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Hence, the eigenvectors are the non-null multiples of $\boldsymbol{x}_1$.

For $\lambda_2 = a - b$, the null space of

$$(\mathbf{B} - \lambda_2 \mathbf{I}) = \begin{bmatrix} b & b \\ b & b \end{bmatrix}$$

consists of all multiples of $\boldsymbol{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Hence, the eigenvectors are the non-null multiples of $\boldsymbol{x}_2$.

□

(L-15) Question 3. The numbers on the diagonal of **A**: 1, 2, 3, 7, 8, and 9.

□

(L-15) Question 4(a) Since the matrix is triangular, the numbers on the diagonal are the eigenvectors of this matrix For $\lambda_1 = 3$ we need to compute a basis for the nullspace of $(\mathbf{A} - 3\mathbf{I})$.

$$\left[\begin{array}{c}\mathbf{A}-3\mathbf{I}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{ccc} 0 & 4 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & -3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(\frac{-1}{2})\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc} 0 & 4 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & -3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{array}\right].$$

Eigenvector: the non-null multiples of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

For $\lambda_2 = 1$ we need to compute a basis for the nullspace of $(\mathbf{A} - \mathbf{I})$.

$$\left[\begin{array}{c}\mathbf{A}-\mathbf{I}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{ccc} 2 & 4 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{\substack{[(-2)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}]}}]{\boldsymbol{\tau}} \left[\begin{array}{ccc} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \\ \hline 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

Eigenvector: the non-null multiples of $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$.

For $\lambda_3 = 0$ we need to compute a basis for the nullspace of $\big(\mathbf{A} - 0\mathbf{I}\big)$.

$$\left[\begin{array}{c}\mathbf{A}-0\mathbf{I}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{ccc} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-2)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ [(3)\mathbf{2}] \\ [(-4)\mathbf{1}+\mathbf{2}] \\ [(3)\mathbf{3}]}} \left[\begin{array}{ccc} 3 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \\ \hline 1 & -4 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \\ \hline 1 & -4 & 6 \\ 0 & 3 & -6 \\ 0 & 0 & 3 \end{array}\right]$$

The eigenvectors are non-null multiples of $\begin{pmatrix} 6 \\ -6 \\ 3 \end{pmatrix}$.

In addition $\lambda_1 + \lambda_2 + \lambda_3 = 3 + 1 + 0 = 4 = \operatorname{tr}\big(\mathbf{A}\big)$ and $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 3 \cdot 1 \cdot 0 = 0 = |\mathbf{A}|$.

```
A =  Matrix([[3,4,2],[0,1,2],[0,0,0]])
espectro = {3,1,0}
for l in espectro:
    display(l)
    L   = Elim(  (A-l*I(3)).apila(I(3),1) ,1)
    cL0 = tuple([c+1 for c,i in enumerate( (1,2,3)|L ) if i.es_nulo()])
    display((4,5,6)|L|cL0)
```

□

(L-15) Question 4(b) The characteristic equation is

$$\begin{vmatrix} -\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = (\lambda^2)(2-\lambda) - 4(2-\lambda) = 0$$

It is clear that $\lambda_1 = 2$ is one eigenvalue.
Dividing the characteristic equation by $(2-\lambda)$ we get $\lambda^2 = 4$; therefore $\lambda_2 = 2$ and $\lambda_3 = -2$
For $\lambda = 2$ we need to compute a basis for the nullspace of

$$\left(\mathbf{A} - 2\mathbf{I}\right) = \begin{bmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{bmatrix}$$

We get two independent eigenvectors for $\lambda = 2$ (a basis for the null space of $\left(\mathbf{A} - 2\mathbf{I}\right)$):

$$\text{Eigenvectors: all non-null linear combinations of} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} ; \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda_3 = -2$ we need to compute a basis for the nullspace of

$$(\mathbf{A} + 2\mathbf{I}) = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

The corresponding eigenvectors are all non-null multiples of $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

In addition $\lambda_1 + \lambda_2 + \lambda_3 = 2 + 2 - 2 = 2 = \operatorname{tr}\left(\mathbf{A}\right)$ and
$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 2 \cdot 2 \cdot (-2) = -8 = |\mathbf{A}|$.

```
B = Matrix([[0,0,2],[0,2,0],[2,0,0]])
l = sympy.symbols('\lambda')
d = Determinante( (B-l*I(3)) ,1)
p = sympy.poly(d.valor)
e = sympy.real_roots(p)
display(Sistema([d, e]))
for i in set(e):
    caso = "\lambda = %d\n" % i
    display(Math(caso))
    L   = Elim(  (B-i*I(3)).apila(I(3),1),1 )
    cL0 = tuple([c+1 for c,v in enumerate((1,2,3)|L) if v.es_nulo()])
    display( (4,5,6)|L|cL0 )
```

(L-15) Question 5(a) when a matrix is transposed, the determinant doesn't change $(\det \mathbf{B} = \det \mathbf{B}^\mathsf{T})$; therefore

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det(\mathbf{A} - \lambda\mathbf{I})^\mathsf{T} = \det(\mathbf{A}^\mathsf{T} - \lambda\mathbf{I})$$

(L-15) Question 5(b) The eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

are the solutions to the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) = 0;$$

in this case, the eigenvalues are equal to the numbers on the diagonal (since this matrix is triangular) .
For $\lambda_1 = 1$, the null space of

$$(\mathbf{A} - \lambda_1\mathbf{I}) = \begin{bmatrix} 1-1 & 2 \\ 0 & 3-1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$$

consists of all multiples of the eigenvector $\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

For $\lambda_2 = 3$, the null space of

$$(\mathbf{A} - \lambda_2 \mathbf{I}) = \begin{bmatrix} 1-3 & 2 \\ 0 & 3-3 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}$$

consists of all multiples of the eigenvector $\boldsymbol{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Nevertheless, for the transposed matrix

$$\mathbf{A}^\mathsf{T} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix},$$

with the same eigevalues (same diagonal), we get
For $\lambda_1 = 1$, the null space of

$$(\mathbf{A}^\mathsf{T} - \lambda \mathbf{I}) = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}$$

consists of all multiples of the eigenvector $\boldsymbol{x}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = 3$, the null space of

$$(\mathbf{A}^\intercal - \lambda\mathbf{I}) = \begin{bmatrix} -2 & 0 \\ 2 & 0 \end{bmatrix}$$

consists of all multiples of the eigenvector $\boldsymbol{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Therefore, the eigenvectors of $\mathbf{A}$ and $\mathbf{A}^\intercal$ are not the same.

□

(L-15) Question 6.

$$\begin{aligned} \mathbf{A}\boldsymbol{x} =& \big(\mathbf{B} + \alpha\mathbf{I}\big)\boldsymbol{x} \\ =& \mathbf{B}\boldsymbol{x} + \alpha\mathbf{I}\boldsymbol{x} \\ =& \lambda\boldsymbol{x} + \alpha\boldsymbol{x} \\ =& (\lambda + \alpha)\boldsymbol{x}; \end{aligned}$$

therefore, $\boldsymbol{x}$ is an eigenvector of $\mathbf{A}$ associated to the eigenvalue $(\lambda + \alpha)$.

□

(L-15) Question 7(a) Primero calculemos los autovalores:

$$\begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = 0; \quad \Rightarrow \quad \begin{cases} \lambda_1 &= 2 \\ \lambda_2 &= 3 \end{cases}$$

Y ahora los auto-vectores.
Para $\lambda_1 = 2$ buscamos una base del espacio nulo de la matriz

$$\begin{bmatrix} 1-2 & -1 \\ 2 & 4-2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$$

Puesto que las dos columnas son iguales, los autovectores son los múltiplos no nulos de $\boldsymbol{x}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.
Para $\lambda_2 = 3$ buscamos una base del espacio nulo de la matriz

$$\begin{bmatrix} 1-3 & -1 \\ 2 & 4-3 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}$$

Puesto que la primera columna es el doble de la segunda, los correspondientes autovectores son los múltiplos no nulos de $\boldsymbol{x}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$
Por último, el determinante de **A** es 6 que es igual a 2×3 y la traza es 5 que es igual a $2 + 3$.

□

(L-15) Question 7(b) Calculemos los autovalores:

$$\begin{vmatrix} -6-\lambda & -1 \\ 2 & -3-\lambda \end{vmatrix} = (-6-\lambda)(-3-\lambda)+2 = \lambda^2+9\lambda+20 = 0;$$

$$\Rightarrow \quad \begin{cases} \lambda_1 & = -5 \\ \lambda_2 & = -4 \end{cases}.$$

Y ahora los auto-vectores.
Para $\lambda_1 = -5$ buscamos una base del espacio nulo de la matriz

$$\begin{bmatrix} -6-(-5) & -1 \\ 2 & -3-(-5) \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}.$$

Al igual que en el apartado anterior, los correspondientes autovectores son los múltiplos no nulos de $\boldsymbol{x}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.
Para $\lambda_2 = -4$ buscamos una base del espacio nulo de la matriz

$$\begin{bmatrix} -6-(-4) & -1 \\ 2 & -3-(-4) \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}$$

De manera idéntica al apartado (a), los correspondientes autovectores son los múltiplos no nulos de $\boldsymbol{x}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Así pues, tras restar 7 veces la matriz identidad ($\mathbf{B} = \mathbf{A} - 7\mathbf{I}$), los autovalores son los de la matriz original menos 7; es decir, $\lambda_1 = 2 - 7 = -5$ y $\lambda_2 = 3 - 7 = -4$; y los auto-vectores son idénticos a los de la matriz original $\mathbf{A}$.

□

(L-15) Question 8(a) Sea un vector $\boldsymbol{x}$ tal que verifica $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$; entonces

$$\mathbf{B}\boldsymbol{x} = (\mathbf{A} - 7\mathbf{I})\boldsymbol{x} = \mathbf{A}\boldsymbol{x} - 7\boldsymbol{x} = (\lambda - 7)\boldsymbol{x}$$

por tanto $\boldsymbol{x}$ también es auto-vector de $\mathbf{B}$ con un auto-valor asociado igual a $(\lambda - 7)$.

□

(L-15) Question 8(b)

$$\begin{aligned} \mathbf{A}\boldsymbol{x} =& \lambda\boldsymbol{x} \\ \boldsymbol{x} =& \lambda(\mathbf{A}^{-1})\boldsymbol{x} \\ \frac{1}{\lambda}\boldsymbol{x} =& (\mathbf{A}^{-1})\boldsymbol{x} \end{aligned}$$

La última igualdad $(\mathbf{A}^{-1})\boldsymbol{x} = (1/\lambda)\boldsymbol{x}$ implica que $\boldsymbol{x}$ es también auto-vector de $\mathbf{A}^{-1}$ con un auto-vector asociado igual a $1/\lambda$ para el caso de $\mathbf{A}^{-1}$.

□

(L-15) Question 9. Si λ es un auto-valor de $\mathbf{A}$, entonces $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$. Entonces

$$\mathbf{A}^2\boldsymbol{x} = \mathbf{A}\mathbf{A}\boldsymbol{x} = \mathbf{A}\lambda\boldsymbol{x} = \lambda\mathbf{A}\boldsymbol{x} = \lambda^2\boldsymbol{x}.$$

Pero, puesto que $\mathbf{A}^2 = \mathbf{A}$ entonces

$$\mathbf{A}^2\boldsymbol{x} = \mathbf{A}\boldsymbol{x} = \lambda^2\boldsymbol{x} = \lambda\boldsymbol{x},$$

por tanto

$$\lambda^2 = \lambda.$$

Los dos únicos valores posibles son, o bien 0, o bien 1.

□

(L-15) Question 10.

$$\mathbf{A}(\boldsymbol{v}_1 + \boldsymbol{v}_2 - \boldsymbol{v}_3) = \mathbf{A}\boldsymbol{v}_1 + \mathbf{A}\boldsymbol{v}_2 - \mathbf{A}\boldsymbol{v}_3 = \boldsymbol{v}_1 + 2\boldsymbol{v}_2 - 3\boldsymbol{v}_3.$$

□

(L-15) Question 11. La ecuación característica de la matriz

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

es $(1-\lambda)^2 - 1 = 0$. Por tanto la matriz tiene autovalores $\lambda = 0$ y $\lambda = 2$.

Sin embargo, la ecuación característica de su forma de escalonada es

$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

es $(1-\lambda)(-\lambda) = 0$. Por tanto los nuevos autovalores son $\lambda = 0$ y $\lambda = 1$.
Los autovalores nulos no cambian. Hay tantos cómo el número de columnas menos el rango de la matriz (es decir, tantos como numero de columnas libres); y ni el número de columnas ni el rango de la matriz cambia al aplicar transformaciones elementales. Por tanto el número de autovalores nulos se mantiene tras aplicar el método de eliminación.

$\square$

(L-15) Question 12. Basta con igualar λ a cero.

$\square$

(L-15) Question 13. Para $\mathbf{A}$, la suma de λ_1 y λ_2 es -1 (la traza) y el producto es -6 (el determinante), por tanto $\lambda_1 = -3$ y $\lambda_2 = 2$
Para $\lambda_1 = -3$; una base del espacio nulo de $(\mathbf{A} + 3\mathbf{I}) = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$ es $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$. Sus múltiplos no nulos son los correspondientes autovectores.
Y para $\lambda_2 = 2$; una base del espacio nulo de $(\mathbf{A} - 2\mathbf{I}) = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}$ es $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Sus múltiplos no nulos son los correspondientes autovectores.
Por otra parte, para $\mathbf{A}^2$, la suma de λ_1 y λ_2 es 13 (la traza) y el producto es 36 (el determinante), por tanto $\lambda_1 = 9$ y $\lambda_2 = 4$

Para $\lambda_1 = 9$; una base del espacio nulo de $(\mathbf{A} - 9\mathbf{I}) = \begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix}$ es $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$. Sus múltiplos no nulos son los correspondientes autovectores.
Y para $\lambda_2 = 4$; una base del espacio nulo de $(\mathbf{A} - 4\mathbf{I}) = \begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix}$ es $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Sus múltiplos no nulos son los correspondientes autovectores.
$\mathbf{A}^2$ tiene los mismos **autovectores** que $\mathbf{A}$. Si los autovalores de $\mathbf{A}$ son λ_1 y λ_2, los autovalores de $\mathbf{A}^2$ son **el cuadrado de los anteriores** (λ_1^2 y λ_2^2).

□

(L-15) Question 14. Los autovalores de $\mathbf{A}^2$ son el cuadrado de los autovalores de $\mathbf{A}$, por lo que

$$\operatorname{tr}\left(\mathbf{A}^2\right) = 1^2 + 2^2 + 4^2 = 21$$

Razonando de la misma manera

$$\left|(\mathbf{A}^{-1})^{\mathsf{T}}\right| = \det \mathbf{A}^{-1} = (1^{-1})(2^{-1})(4^{-1}) = 1 \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}.$$

□

(L-15) Question 15. The condition says that $(\mathbf{A}^2 - 4\mathbf{I})$ is singular. But we know that, if $\lambda_1, \dots \lambda_n$ are eigenvalues of $\mathbf{A}$, then the eigenvalues of $(\mathbf{A}^2 - 4\mathbf{I})$ are $\lambda_1^2 - 4, \lambda_1^2 - 4, \dots, \lambda_n^2 - 4$. The condition $(\mathbf{A}^2 - 4\mathbf{I})$ being singular says that one of $\lambda_i^2 - 4$ is zero, and hence $\lambda_i = 2$ or $\lambda_i = -2$. That is to say $\mathbf{A}$ has an eigenvalue 2 or -2.

□

(L-15) Question 16. First, the last two columns of **A** are the same. Hence **A** is singular and it must have an eigenvalue $\lambda_1 = 0$. Also, we observe that **A** is a Markov matrix. This means that $\lambda_2 = 1$ is an eigenvalue of **A**. Finally, we know the trace of **A** is the sum of its three eigenvalues. So, $\operatorname{tr}\left(\mathbf{A}\right) = 0.5 + 0.5 + 0.3 = 1.3$ and the last eigenvalue is $\lambda_3 = 1.3 - 1 - 0 = 0.3$.

□

(L-16) Question 1(a) From Question 5 on page 981 (L-19) we know the eigenvalues and eigenvectors of **A**; therefore we know

$$\mathbf{S} = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}; \qquad \mathbf{D} = \begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}.$$

We only need to compute $\mathbf{S}^{-1}$:

$$\left[\begin{array}{c}\mathbf{S}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{cc}1 & 1\\ 0 & 1\\ \hline 1 & 0\\ 0 & 1\end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \left[\begin{array}{cc}1 & 0\\ 0 & 1\\ \hline 1 & -1\\ 0 & 1\end{array}\right] = \left[\begin{array}{c}\mathbf{I}\\ \hline \mathbf{S}^{-1}\end{array}\right]$$

Therefore

$$\mathbf{A} = \begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix} = \mathbf{SDS}^{-1}.$$

O aplicando el algoritmo de diagonalización por bloques (que en este caso sabemos que arrojará una matriz diagonal, puesto que los autovalores no se repiten)

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] \xrightarrow[3\mathbf{I}]{(-)} \left[\begin{array}{cc}-2&2\\0&0\\ \hline 1&0\\0&1\end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ [(1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc}-2&0\\0&0\\ \hline 1&1\\0&1\end{array}\right] \xrightarrow[\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{2}+\mathbf{1}]}]{} \left[\begin{array}{cc}-2&0\\0&0\\ \hline 1&1\\0&1\end{array}\right] \xrightarrow[3\mathbf{I}]{(+)}$$

$$\left[\begin{array}{cc}1&0\\0&3\\ \hline 1&1\\0&1\end{array}\right] \xrightarrow[1\mathbf{I}]{(-)} \left[\begin{array}{cc}0&0\\0&2\\ \hline 1&1\\0&1\end{array}\right] \xrightarrow[1\mathbf{I}]{(+)} \left[\begin{array}{cc}1&0\\0&3\\ \hline 1&1\\0&1\end{array}\right]$$

```
A = Matrix([[1,2],[0,3]])
lambdas = [1,3]
D = DiagonalizaS(A,lambdas,1)
display(D)
display(D.S)
```

□

(L-16) Question 1(b) First we are going to solve the characteristic equation

$\det(\mathbf{B}-\lambda\mathbf{I})=0$ in order to find the eigenvalues

$$\begin{vmatrix}1-\lambda & 1\\ 2 & 2-\lambda\end{vmatrix}=(1-\lambda)(2-\lambda)-2=\lambda^2-3\lambda=0;\quad\Rightarrow\quad\begin{cases}\lambda_1 &=0\\ \lambda_2 &=3\end{cases}$$

(Why should we know one eigenvalue is zero before solving the characteristic equation?)

- for $\lambda_1=0$, the null space of $(\mathbf{B}-0\mathbf{I})=\mathbf{B}=\begin{bmatrix}1&1\\2&2\end{bmatrix}$ consists of all multiples of the eigenvector
$$\boldsymbol{x}_1=\begin{pmatrix}-1\\1\end{pmatrix}$$
- for $\lambda_2=3$, the null space of $(\mathbf{B}-\lambda\mathbf{I})=\begin{bmatrix}-2&1\\2&-1\end{bmatrix}$ consists of all the multiples of
$$\boldsymbol{x}_2=\begin{pmatrix}1\\2\end{pmatrix}$$

Therefore:
$$\mathbf{S}=\begin{bmatrix}1&-1\\2&1\end{bmatrix};\qquad\mathbf{D}=\begin{bmatrix}3&0\\0&0\end{bmatrix}.$$

O aplicando el algoritmo de diagonalización por bloques (que en este caso sabemos que arrojará una matriz diagonal, puesto que los autovalores no se repiten)

$$\xrightarrow[0\mathbf{I}]{(-)}\left[\begin{array}{cc}1&1\\2&2\\\hline 1&0\\0&1\end{array}\right]\xrightarrow{\substack{\boldsymbol{\tau}\\{[(-1)\mathbf{1}+\mathbf{2}]}}}\left[\begin{array}{cc}1&0\\2&0\\\hline 1&-1\\0&1\end{array}\right]\xrightarrow[\substack{\boldsymbol{\tau}\\{[(1)\mathbf{2}+\mathbf{1}]}}]{}\left[\begin{array}{cc}3&0\\2&0\\\hline 1&-1\\0&1\end{array}\right]\xrightarrow[0\mathbf{I}]{(+)}\left[\begin{array}{cc}3&0\\2&0\\\hline 1&-1\\0&1\end{array}\right]$$

$$\xrightarrow[3\mathbf{I}]{(-)}\left[\begin{array}{cc}0&0\\\hline 2&-3\\\hline 1&-1\\0&1\end{array}\right]\xrightarrow{\substack{\boldsymbol{\tau}\\{[(3)\mathbf{1}]}\\{[(2)\mathbf{2}+\mathbf{1}]}}}\left[\begin{array}{cc}0&0\\\hline 0&-3\\\hline 1&-1\\2&1\end{array}\right]\xrightarrow[\substack{\boldsymbol{\tau}\\{[(1/3)\mathbf{1}]}\\{[(-2)\mathbf{1}+\mathbf{2}]}}]{}\left[\begin{array}{cc}0&0\\\hline 0&-3\\\hline 1&-1\\2&1\end{array}\right]\xrightarrow[3\mathbf{I}]{(+)}\left[\begin{array}{cc}3&0\\0&0\\\hline 1&-1\\2&1\end{array}\right]=\left[\begin{array}{c}\mathbf{D}\\\hline\mathbf{S}\end{array}\right]$$

Finally we compute $\mathbf{S}^{-1}$:

$$\left[\begin{array}{cc}1&-1\\2&1\\\hline 1&0\\0&1\end{array}\right]\xrightarrow{\substack{\boldsymbol{\tau}\\{[(1)\mathbf{1}+\mathbf{2}]}}}\left[\begin{array}{cc}1&0\\2&3\\\hline 1&1\\0&1\end{array}\right]\xrightarrow{\substack{\boldsymbol{\tau}\\{[(3)\mathbf{1}]}\\{[(-2)\mathbf{2}+\mathbf{1}]}}}\left[\begin{array}{cc}3&0\\0&3\\\hline 1&1\\-2&1\end{array}\right]\xrightarrow{\substack{\boldsymbol{\tau}\\{[(\frac{1}{3})\mathbf{1}]}\\{[(\frac{1}{3})\mathbf{2}]}}}\left[\begin{array}{cc}1&0\\0&1\\\hline \frac{1}{3}&\frac{1}{3}\\\frac{-2}{3}&\frac{1}{3}\end{array}\right]$$

The factorization is

$$\mathbf{B}=\begin{bmatrix}1&1\\2&2\end{bmatrix}=\begin{bmatrix}1&-1\\2&1\end{bmatrix}\begin{bmatrix}3&0\\0&0\end{bmatrix}\begin{bmatrix}1/3&1/3\\-2/3&1/3\end{bmatrix}=\mathbf{SDS}^{-1}.$$

```
B = Matrix([[1,1],[2,2]])
l = sympy.symbols('\lambda')
d = Determinante( (B-l*I(B.n)) ,1)
p = sympy.poly(d.valor)
e = sympy.real_roots(p)
display(Sistema([d, e]))
S = Sistema([])
D = tuple()
for i in set(e):
    caso = "\lambda = %d\n" % i
    display(Math(caso))
    L   = Elim( (B-i*I(B.n)).apila(I(B.n),1), 1)
    cL0 = tuple([c+1 for c,v in enumerate(slice(1,B.n)|L) if v.es_nulo()])
    S = S.concatena(slice(B.n+1,None)|L|cL0).sis()
    D = D+(i,)*e.count(i)

S = Matrix(S)
D = Vector(D).diag()
display(Sistema([D,S,S**-1]))
display(Sistema([(S**-1)*B*S, S*D*(S**-1)]))
```

□

(L-16) Question 2(a)

$$\begin{cases} \lambda_1 + \lambda_2 = 0 \\ \lambda_1 \cdot \lambda_2 = 0 \end{cases} \quad \Rightarrow \quad \lambda_1 = \lambda_2 = 0.$$

This matrix can be diagonalized if the null space of $(\mathbf{A}_1 - 0\mathbf{I}) = \mathbf{A}_1$ has dimension 2. Since $\mathbf{A}_1$ will have only one free column $\dim \mathcal{N}\left(\mathbf{A}\right) = 1$; therefore this matrix cannot be diagonalized.

□

(L-16) Question 2(b) There are no repeated eigenvalues; therefore this matrix is diagonalizable.

□

(L-16) Question 2(c) There is a repeated eigenvalue $\lambda = 2$; therefore, this matriz is diagonalizable if the null space of $(\mathbf{A}_3 - 2\mathbf{I})$ has dimension 2 (two linearly independent eigenvectors), but $\dim \mathcal{N}\left(\mathbf{A_3 - 2I}\right) = 1$. This matrix can not be diagonalized.

□

(L-16) Question 3. Characteristic equation:
$(4-\lambda)(2-\lambda) - 3 = \lambda^2 - 6\lambda + 5 = 0; \quad \Rightarrow \quad \lambda_1 = 5; \quad \lambda_2 = 1.$

$$\lambda_1 = 5 \qquad \Longrightarrow \quad (\mathbf{A} - 5\mathbf{I}) = \begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{x} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\lambda_1 = 1 \qquad \Longrightarrow \quad (\mathbf{A} - \mathbf{I}) = \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Or in an alternative way:

$$\left[\begin{array}{c}\mathbf{A}\\\hline \mathbf{I}\end{array}\right] \xrightarrow[1\mathbf{I}]{(-)} \left[\begin{array}{cc}3&3\\1&1\\\hline 1&0\\0&1\end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc}3&0\\1&0\\\hline 1&-1\\0&1\end{array}\right] \xrightarrow[\substack{\boldsymbol{\tau}\\ [(1)\mathbf{2}+\mathbf{1}]}]{} \left[\begin{array}{cc}4&0\\1&0\\\hline 1&-1\\0&1\end{array}\right] \xrightarrow[1\mathbf{I}]{(+)} \left[\begin{array}{cc}5&0\\1&1\\\hline 1&-1\\0&1\end{array}\right]$$

$$\xrightarrow[5\mathbf{I}]{(-)} \left[\begin{array}{cc}0&0\\\hline 1&-4\\\hline 1&-1\\0&1\end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ [(4)\mathbf{1}]\\ [(1)\mathbf{2}+\mathbf{1}]}} \left[\begin{array}{cc}0&0\\\hline 0&-4\\\hline 3&-1\\1&1\end{array}\right] \xrightarrow[\substack{\boldsymbol{\tau}\\ [(1/4)\mathbf{1}]\\ [(-1)\mathbf{1}+\mathbf{2}]}]{} \left[\begin{array}{cc}0&0\\\hline 0&-4\\\hline 3&-1\\1&1\end{array}\right] \xrightarrow[5\mathbf{I}]{(+)} \left[\begin{array}{cc}5&0\\0&1\\\hline 3&-1\\1&1\end{array}\right]$$

Hence,

$$\begin{aligned}\mathbf{A}^{100} = \mathbf{S}\mathbf{D}^{100}\mathbf{S}^{-1} &= \begin{bmatrix}3&1\\1&-1\end{bmatrix}\begin{bmatrix}5^{100}&\\&1\end{bmatrix}\begin{bmatrix}3&1\\1&-1\end{bmatrix}^{-1}\\ &= \begin{bmatrix}5\cdot 5^{100}&1\\5^{100}&-1\end{bmatrix}\begin{bmatrix}1&1\\-1&3\end{bmatrix}\frac{1}{4} = \frac{1}{4}\begin{bmatrix}3\cdot 5^{100}-1 & 3\cdot 5^{100}+3\\ 5^{100}+1 & 5^{100}-3\end{bmatrix}\end{aligned}$$

□

(L-16) Question 4(a) True. $|\mathbf{A}| = 1\times 1\times 2 = 2 \neq 0$, therefore $\mathbf{A}$ is invertible.

□

(L-16) Question 4(b) We don't know. It has a repeated eigenvalue (might have 2 or 3 independent eigenvectors)

□

(L-16) Question 4(c) We don't know. It has a repeated eigenvalue (might have 2 or 3 independent eigenvectors)

□

(L-16) Question 5(a) Since the matrix is triangular, the elements on its main diagonal ($\lambda = 4$ and $\lambda = 2$) are the eigenvalues (both with algebraic multiplicity two):

$$\operatorname{rg}\left(\mathbf{A} - 4\mathbf{I}\right) = \operatorname{rg}\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix}\right) = 2;$$

$$\operatorname{rg}\left(\mathbf{A} - 2\mathbf{I}\right) = \operatorname{rg}\left(\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right) = 2.$$

Por tanto ya sabemos que **A** es diagonalizable.
Observando la matriz $(\mathbf{A} - 4\mathbf{I})$, es fácil ver que dos autovectores asociados a $\lambda = 4$ son $(2, \quad 0, \quad 0, \quad 1,)$ and $(0, \quad 1, \quad 0, \quad 0,)$; y observando la matriz $(\mathbf{A} - 2\mathbf{I})$, que dos autovectores asociados a $\lambda = 2$ son $(0, \quad 0, \quad 1, \quad 0,)$ and

$\begin{pmatrix}0, & 0, & 0, & 1,\end{pmatrix}$. Así pues, $\mathbf{D} = \begin{bmatrix}4 & & & \\ & 4 & & \\ & & 2 & \\ & & & 2\end{bmatrix}$; and $\mathbf{S} = \begin{bmatrix}2 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 1\end{bmatrix}$. □

(L-16) Question 5(b) Puesto que hemos visto que $\boldsymbol{v}$ es un autovector de $\mathbf{A}$ asociado al autovalor $\lambda = 2$, sabemos que $\mathbf{A}\boldsymbol{v} = 2\boldsymbol{v}$, y por tanto:

$$\begin{aligned}\left(\mathbf{A}^6\right)\boldsymbol{v} =& \mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\boldsymbol{v}\\ =& \mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\lambda\boldsymbol{v}\\ =& \mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\lambda^2\boldsymbol{v}\\ =& \mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\lambda^3\boldsymbol{v}\\ =& \mathbf{A}\cdot\mathbf{A}\cdot\lambda^4\boldsymbol{v}\\ =& \mathbf{A}\cdot\lambda^5\boldsymbol{v}\\ =& \lambda^6\boldsymbol{v} = 2^6\boldsymbol{v} = 64\boldsymbol{v} = \begin{pmatrix}0, & 0, & 0, & 64,\end{pmatrix}.\end{aligned}$$

□

(L-16) Question 5(c) Puesto que ningún autovalor es cero, la matriz es de rango completo, es decir, invertible.

□

(L-16) Question 5(d) Puesto que $\mathbf{A} = \mathbf{SDS}^{-1}$, entonces

$$\mathbf{A}^{-1} = \left(\mathbf{SDS}^{-1}\right)^{-1} = \left(\mathbf{DS}^{-1}\right)^{-1}\mathbf{S}^{-1} = \left(\mathbf{S}^{-1}\right)^{-1}\mathbf{D}^{-1}\mathbf{S}^{-1} = \mathbf{SD}^{-1}\mathbf{S}^{-1};$$

es decir, los autovectores $\mathbf{S}$ son los mismos, pero los autovalores $\mathbf{D}^{-1}$, son los inversos de los autovalores de la matriz $\mathbf{A}$.

□

(L-16) Question 6.

$$\mathbf{A}^3 = (\mathbf{S})(\mathbf{D}^3)(\mathbf{S}^{-1}); \quad \text{y} \quad \mathbf{A}^{-1} = (\mathbf{S})(\mathbf{D}^{-1})(\mathbf{S}^{-1}).$$

□

(L-16) Question 7(a) Debemos resolver la ecuación característica

$$\begin{bmatrix} 1-\lambda & 0 \\ -1 & 2-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda) = 0.$$

Así $\lambda = 1,\ 2$ (por ser una matriz triangular).

□

(L-16) Question 7(b) Para encontrar los auto-vectores correspondientes a λ, debemos encontrar el espacio nulo de $(\mathbf{A} - \lambda\mathbf{I})$. Hay dos casos:

- $\lambda = 1$. Aquí debemos resolver $\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Así, un auto-vector es $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
- $\lambda = 2$. Aquí tenemos que resolver $\begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. En este caso un auto-vector es $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$\square$

(L-16) Question 7(c) De los apartados anteriores concluimos que:

$$\mathbf{S} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}; \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Por ser **S** una matriz elemental, sabemos que su inversa es:

$$\mathbf{S}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Por tanto

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \mathbf{SDS}^{-1}.$$

Pero también podiamos haberlo hecho así

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] \xrightarrow[2\mathbf{I}]{(-)} \left[\begin{array}{cc}-1 & 0\\ -1 & 0\\ \hline 1 & 0\\ 0 & 1\end{array}\right] \xrightarrow[2\mathbf{I}]{(+)} \left[\begin{array}{cc}1 & 0\\ -1 & 2\\ \hline 1 & 0\\ 0 & 1\end{array}\right]$$

$$\xrightarrow[1\mathbf{I}]{(-)} \left[\begin{array}{cc}0 & 0\\ \hline -1 & 1\\ \hline 1 & 0\\ 0 & 1\end{array}\right] \xrightarrow{\underset{[(1)\mathbf{2}+\mathbf{1}]}{\boldsymbol{\tau}}} \left[\begin{array}{cc}0 & 0\\ \hline 0 & 1\\ \hline 1 & 0\\ 1 & 1\end{array}\right] \xrightarrow[\underset{[(-1)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}]{} \left[\begin{array}{cc}0 & 0\\ \hline 0 & 1\\ \hline 1 & 0\\ 1 & 1\end{array}\right] \xrightarrow[1\mathbf{I}]{(+)} \left[\begin{array}{cc}1 & 0\\ 0 & 2\\ \hline 1 & 0\\ 1 & 1\end{array}\right]$$

$\square$

(L-16) Question 8(a) **Verdadero**. Puesto que ninguno de los autovalores es nulo, el determinante (que es igual al producto de los autovalores) es necesariamente distinto de cero, y por tanto la matriz es invertible.

$\square$

(L-16) Question 8(b) **Falso**. Si no hay autovalores repetidos, sabemos que necesariamente la matriz es diagonalizable, pues existen suficientes auto-vectores linealmente independientes, como para generar una matriz **S** invertible. Puesto que el auto-valor 2 está repetido, no podemos saber si existen suficientes auto-vectores linealmente independientes.

$\square$

(L-16) Question 8(c) **Falso**. No lo podemos saber. Necesitamos conocer si hay tres auto-vectores linealmente independientes (lo sabríamos si no hubiera autovalores repetidos).

□

(L-16) Question 9(a)

$$\mathbf{A}_1 = \begin{bmatrix} 8 & a \\ b & 2 \end{bmatrix}, \quad a = \frac{-9}{b}; \qquad \mathbf{A}_2 = \begin{bmatrix} 9 & 4 \\ -4 & 1 \end{bmatrix}; \qquad \mathbf{A}_3 = \begin{bmatrix} 10 & 5 \\ -5 & 0 \end{bmatrix}$$

□

(L-16) Question 9(b) El espacio nulo de la matriz $(\mathbf{A}_1 - 5\mathbf{I})$; es decir, de $\begin{bmatrix} 3 & 3 \\ -3 & -3 \end{bmatrix}$; es el conjunto de vectores múltiplos de $\boldsymbol{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, y el es mismo que el espacio nulo de las matrices $(\mathbf{A}_2 - 5\mathbf{I}) = \begin{bmatrix} 4 & 4 \\ -4 & -4 \end{bmatrix}$ y $(\mathbf{A}_2 - 5\mathbf{I}) = \begin{bmatrix} 5 & 5 \\ -5 & -5 \end{bmatrix}$. Por tanto, en los tres casos no podemos encontrar dos auto-vectores linealmente independientes, y en consecuencia estas matrices no son diagonalizables.

□

(L-16) Question 10(a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}.$

□

(L-16) Question 10(b) $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}^{-1}.$

□

(L-16) Question 11.

$$\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}^{-1} = \begin{bmatrix}1 & -2\\-1 & 3\end{bmatrix}; \qquad \begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 4\end{bmatrix}\begin{bmatrix}1 & -2\\-1 & 3\end{bmatrix} = \begin{bmatrix}-5 & 18\\-3 & 10\end{bmatrix}$$

Comprobación: Traza = 5; determinante = 4.

□

(L-16) Question 12. Sabemos que es diagonalizable, puesto que no tiene autovalores repetidos. La matriz de autovalores **D** es una matriz diagonal; con los valores 1, 2 y 7 es su diagonal principal (en cualquier orden).

□

(L-16) Question 13(a) Los tres autovalores son iguales a 1 (son los elementos de la diagonal, por ser **A** triangular).
Los auto-vectores para el único auto-valor (triple) $\lambda = 1$ se calculan partiendo de la matriz $[\mathbf{A} - 1\mathbf{I}]$:

$$\mathbf{A} - 1\mathbf{I} = \begin{bmatrix}0 & 0 & 0\\-2 & 0 & 0\\1 & 0 & 0\end{bmatrix}$$

que tiene sólo dos columnas libres; por tanto debemos encontrar dos soluciones especiales (cuyas combinaciones lineales son el espacio nulo de la matriz anterior y que constituyen los auto-vectores asociados al auto-valor $\lambda = 1$):

$$\boldsymbol{x}_a = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \qquad \boldsymbol{x}_b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

□

(L-16) Question 13(b) La matriz no es diagonalizable ya que como máximo podemos encontrar 2 auto-vectores linealmente independientes (para que lo fuera necesitaríamos encontrar 3).

□

(L-16) Question 14(a) Es diagonalizable, puesto que tiene tres auto-vectores linealmente independientes. Así pues:

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}; \qquad \mathbf{D} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}.$$

□

(L-16) Question 14(b)

$$\mathbf{A} = \mathbf{SDS}^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

□

(L-16) Question 15.

$$\begin{aligned}\left(\mathbf{A}^3\right)\boldsymbol{v} &= \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & \\ & -2 \end{bmatrix}^3 \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2^3 & \\ & -2^3 \end{bmatrix} \begin{bmatrix} 1/2 & 1 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{bmatrix} 8 & 32 \\ 0 & -8 \end{bmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -24 \\ 8 \end{pmatrix}.\end{aligned}$$

□

(L-17) Question 1(a) Characteristic equation:

$$0 = |\mathbf{A} - \lambda \mathbf{I}| = \lambda^2 - \lambda,$$

then $\lambda_1 = 0$ and $\lambda_2 = 1$

- $\lambda_1 = 0$

$$(\mathbf{A} - \mathbf{0}) = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

The corresponding unit eigenvector is $\boldsymbol{x}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

- $\lambda_1 = 1$

$$(\mathbf{A} - \mathbf{I}) = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

The corresponding unit eigenvector is $\boldsymbol{x}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Therefore

$$\mathbf{A} = \mathbf{QDQ}^\intercal = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & \\ & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}}$$

□

(L-17) Question 1(b) Characteristic equation:

$$0 = |\mathbf{B} - \lambda \mathbf{I}| = \lambda^2 - 1,$$

then $\lambda = \pm 1$

- $\lambda_1 = 1$

$$(\mathbf{B} - \mathbf{I}) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

The corresponding unit eigenvector is $\boldsymbol{x}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

- $\lambda_2 = -1$

$$(\mathbf{B}+\mathbf{I}) = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

The corresponding unit eigenvector is $\boldsymbol{x}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}$

Therefore

$$\mathbf{B} = \mathbf{QDQ}^\mathsf{T} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & \\ & -1\end{bmatrix}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}\frac{1}{\sqrt{2}}$$

□

(L-17) Question 1(c) Characteristic equation:

$$0 = |\mathbf{C}-\lambda\mathbf{I}| = (3-\lambda)(-3-\lambda)-16 = \lambda^2-25,$$

therefore $\lambda = \pm 5$

- $\lambda_1 = 5$

$$(\mathbf{C}-5\mathbf{I}) = \begin{bmatrix}-2 & 4\\4 & -8\end{bmatrix}$$

The corresponding unit eigenvector is $\boldsymbol{x}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix}$

- $\lambda_1 = -5$

$$(\mathbf{C} + 5\mathbf{I}) = \begin{bmatrix} 8 & 4 \\ 4 & -2 \end{bmatrix}$$

The corresponding unit eigenvector is $\boldsymbol{x}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Therefore

$$\mathbf{C} = \mathbf{QDQ}^\intercal = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & \\ & -5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \frac{1}{\sqrt{5}}$$

$\square$

(L-17) Question 2. Characteristic equation:

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda^3 + \lambda^2 + 2\lambda = 0$$

$$\Rightarrow \begin{cases} \lambda = 0 \\ -\lambda^2 + \lambda + 2 \Rightarrow \begin{cases} \lambda = 2 \\ \lambda = -1 \end{cases} \end{cases}$$

$\lambda = 0$, 2, -1; with unit eigenvectors

$$\boldsymbol{x}_1 = \pm\frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\end{pmatrix}; \qquad \boldsymbol{x}_2 = \pm\frac{1}{\sqrt{6}}\begin{pmatrix}2\\1\\1\end{pmatrix}; \qquad \boldsymbol{x}_3 = \pm\frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\1\end{pmatrix}.$$

since

$$\left[\begin{array}{c}\mathbf{A} - {\color{blue}0}\mathbf{I}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{ccc}1&1&1\\1&0&0\\1&0&0\\ \hline 1&0&0\\0&1&0\\0&0&1\end{array}\right] \xrightarrow{\overset{\tau}{\substack{[(-1)\mathbf{1}+\mathbf{2}]\\ [(-1)\mathbf{1}+\mathbf{3}]}}} \left[\begin{array}{ccc}1&0&0\\1&-1&-1\\1&-1&-1\\ \hline 1&-1&-1\\0&1&0\\0&0&1\end{array}\right] \xrightarrow{\overset{\tau}{[(-1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc}1&0&0\\1&-1&0\\1&-1&0\\ \hline 1&-1&\mathbf{0}\\0&1&\mathbf{-1}\\0&0&\mathbf{1}\end{array}\right]$$

$$\left[\begin{array}{c}\mathbf{A} - {\color{blue}2}\mathbf{I}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{ccc}-1&1&1\\1&-2&0\\1&0&-2\\ \hline 1&0&0\\0&1&0\\0&0&1\end{array}\right] \xrightarrow{\overset{\tau}{\substack{[(1)\mathbf{1}+\mathbf{2}]\\ [(1)\mathbf{1}+\mathbf{3}]}}} \left[\begin{array}{ccc}-1&0&0\\1&-1&1\\1&1&-1\\ \hline 1&1&1\\0&1&0\\0&0&1\end{array}\right] \xrightarrow{\overset{\tau}{[(1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc}-1&0&0\\1&-1&0\\1&1&0\\ \hline 1&1&\mathbf{2}\\0&1&\mathbf{1}\\0&0&\mathbf{1}\end{array}\right]$$

$$\left[\begin{array}{c} \mathbf{A} + 1\mathbf{I} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{ccc} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(2)\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{2}] \\ [(2)\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc} 2 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ \hline 1 & -1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{array}\right] \xrightarrow{\substack{\tau \\ [(1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \\ \hline 1 & -1 & -\mathbf{2} \\ 0 & 2 & \mathbf{2} \\ 0 & 0 & \mathbf{2} \end{array}\right]$$

$\square$

(L-17) Question 3. $|\mathbf{A} - \lambda\mathbf{I}| = \lambda\left(9 - \lambda^2\right) = 0 \;\rightarrow\; \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 3 \\ \lambda_3 = -3 \end{cases}.$

$$
\lambda_1 = -3 : \left[\begin{array}{c} \mathbf{A} + 3\mathbf{I} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{ccc} 4 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & -2 & 3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{[(2)\overset{\tau}{\mathbf{3}}] \\ [(-1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc} 4 & 0 & 0 \\ 0 & 2 & -4 \\ 2 & -2 & 4 \\ \hline 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{array}\right] \xrightarrow{[(2)\overset{\tau}{\mathbf{2}}+\mathbf{3}]} \left[\begin{array}{c} 4 \\ 0 \\ 2 \\ \hline 1 \\ 0 \\ 0 \end{array}\right.
$$

$$
\lambda_1 = 0 : \left[\begin{array}{c} \mathbf{A} - 0\mathbf{I} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{ccc} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{[(-2)\overset{\tau}{\mathbf{1}}+\mathbf{3}]} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & -2 \\ 2 & -2 & -4 \\ \hline 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{[(-2)\overset{\tau}{\mathbf{2}}+\mathbf{3}]} \left[\begin{array}{c} 1 \\ 0 \\ 2 \\ \hline 1 \\ 0 \\ 0 \end{array}\right.
$$

$$
\lambda_1 = 3 : \left[\begin{array}{c} \mathbf{A} - 3\mathbf{I} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{ccc} -2 & 0 & 2 \\ 0 & -4 & -2 \\ 2 & -2 & -3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right. \xrightarrow{[(1)\overset{\tau}{\mathbf{1}}+\mathbf{3}]} \left[\begin{array}{ccc} -2 & 0 & 0 \\ 0 & -4 & -2 \\ 2 & -2 & -1 \\ \hline 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right. \xrightarrow{\substack{[(2)\overset{\tau}{\mathbf{3}}] \\ [(-1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{c} -2 \\ 0 \\ 2 \\ \hline 1 \\ 0 \end{array}\right.
$$

Hence, for example

$$\text{If} \quad \mathbf{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad \text{then} \quad \mathbf{Q} = \frac{1}{3}\begin{bmatrix} 2 & -2 & -1 \\ 2 & 1 & 2 \\ -1 & -2 & 2 \end{bmatrix}.$$

```
B = Matrix([[1,0,2],[0,-1,-2],[2,-2,0]])
l = sympy.symbols('\lambda')
d = Determinante( (B-l*I(B.n)) ,1)
p = sympy.poly(d.valor)
e = sympy.real_roots(p)
display(Sistema([d, e]))
S=Sistema([])
D = tuple()
for i in set(e):
    caso = "\lambda = %d\n" % i
    display(Math(caso))
    L = Elim( (B-i*I(B.n)).apila(I(B.n),1), 1)
    cL0 = tuple([c+1 for c,v in enumerate(slice(1,B.n)|L) if v.es_nulo()])
    S = S.concatena(slice(B.n+1,None)|L|cL0).sis()
    D = D+(i,)*e.count(i)

Q = Matrix([v.normalizado() for v in S]).GS()
D = Vector(D).diag()
display(Sistema([D,Q,~Q,Q**-1]))
```

```
display(Sistema([(~Q)*B*Q, Q*D*(~Q)]))
~Q*Q
```

O sencillamente

```
B = Matrix([[1,0,2],[0,-1,-2],[2,-2,0]])
espectro = [0,3,-3]
D = DiagonalizaO( B, espectro )
display(Sistema([D, D.Q]))
```

□

(L-17) Question 4(a) $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are orthogonal to each other.

□

(L-17) Question 4(b) The nullspace is spanned by $\boldsymbol{u}$; the left nullspace is the same as the nullspace; the row space is spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$; the column space is the same as the row space.

$$\mathcal{C}\left(\mathbf{A}\right) = \mathcal{C}\left(\mathbf{A}^\intercal\right) \perp \mathcal{N}\left(\mathbf{A}\right) = \mathcal{N}\left(\mathbf{A}^\intercal\right).$$

□

(L-17) Question 4(c) $\boldsymbol{x} = \boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$, since

$$\mathbf{A}(\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}) = \mathbf{A}\boldsymbol{v} + \frac{1}{2}\mathbf{A}\boldsymbol{w} = \boldsymbol{v} + \frac{1}{2}(2\boldsymbol{w}) = \boldsymbol{v} + \boldsymbol{w};$$

not unique, we can add any multiple of $\boldsymbol{u}$ to $\boldsymbol{x}$.

□

(L-17) Question 4(d) Since $\boldsymbol{b} \in \mathcal{C}\left(\mathbf{A}\right) = \mathcal{C}\left(\mathbf{A}^\intercal\right)$ and $\mathcal{N}\left(\mathbf{A}\right)$ es perpendicular a $\mathcal{C}\left(\mathbf{A}^\intercal\right) = \mathcal{C}\left(\mathbf{A}\right)$, we need $\boldsymbol{b} \cdot \boldsymbol{u} = 0$.

□

(L-17) Question 4(e) $\mathbf{S}^{-1} = \mathbf{S}^\intercal$;

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 2 \end{bmatrix} = \mathbf{D}.$$

□

(L-17) Question 5(a) Autovalores reales.

□

(L-17) Question 5(b) Todos los autovalores son menores que uno en valor absoluto.

□

(L-17) Question 5(c) Tiene autovalores repetidos

□

(L-17) Question 5(d) Tiene al menos un autovalor igual a cero.

□

(L-17) Question 6(a) La ecuación característica es $|\mathbf{A} - \lambda \mathbf{I}| = 0$.

$$\begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 & 0 \\ -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \end{vmatrix}$$

$$= \lambda^4 - 1 = 0$$

por tanto $\lambda = \pm 1$ y $\lambda = \pm i$; es decir cuatro autovalores distintos.

□

(L-17) Question 6(b) La matriz tiene un espacio nulo de dimensión 3. Busquemos, por tanto, tres autovectores que sean base del espacio nulo de **B** (autovalor igual a cero):

$$\boldsymbol{x}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad \boldsymbol{x}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad \boldsymbol{x}_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Además,

$$\frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Por tanto, hay un triple autovalor igual a 0, y otro autovalor igual a 1. Y puesto que hemos encontrado cuatro autovectores linealmente independientes, esta matriz es diagonalizable.

□

(L-17) Question 6(c) Ortogonal, invertible, permutación, diagonalizable y Markov

□

(L-17) Question 6(d) Hermítica, de rango uno, diagonalizable y Markov

□

(L-17) Question 7. Si $\mathbf{A}^3 = \mathbf{0}$ entonces para todos los autovalores $\lambda^3 = 0$, es decir $\lambda = 0$ como en

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Si $\mathbf{A}$ es simétrica, entonces es diagonalizable, y por el Teorema espectral, se puede factorizar como

$$\mathbf{A}^3 = \mathbf{Q}\mathbf{D}^3\mathbf{Q}^\intercal = \mathbf{Q}\mathbf{0}\mathbf{Q}^\intercal = \mathbf{0}.$$

□

(L-17) Question 8(a) Los autovalores de

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

son los elementos de la diagonal; $\lambda = 3$ (con multiplicidad 2) y $\lambda = 2$. Pero el rango de

$$\mathbf{A} - 3\lambda = \begin{bmatrix} 3-3 & 1 & 1 \\ 0 & 3-3 & 1 \\ 0 & 0 & 2-3 \end{bmatrix}$$

es 2; por tanto la matriz no es diagonalizable.

□

(L-17) Question 8(b) Los autovalores de

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

son los elementos de la diagonal; $\lambda = 3$ y $\lambda = 2$ (con multiplicidad 2). El rango de

$$\mathbf{A} - 2\lambda = \begin{bmatrix} 2-2 & 1 & 1 \\ 0 & 3-2 & 1 \\ 0 & 0 & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

es 1; por tanto la matriz es diagonalizable.

Dos autovectores independientes correspondientes al autovalor $\lambda = 2$ son $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ y

$\begin{pmatrix}0\\1\\-1\end{pmatrix}.$

Por otra parte

$$\mathbf{A} - 3\lambda = \begin{bmatrix}2-3 & 1 & 1\\0 & 3-3 & 1\\0 & 0 & 2-3\end{bmatrix} = \begin{bmatrix}-1 & 1 & 1\\0 & 0 & 1\\0 & 0 & -1\end{bmatrix}$$

Un autovector correspondiente al autovalor $\lambda = 3$ es $\begin{pmatrix}1\\1\\0\end{pmatrix}$.

Así pues

$$\mathbf{D} = \begin{bmatrix}2 & & \\ & 2 & \\ & & 3\end{bmatrix} \quad \text{and} \quad \mathbf{S} = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\0 & -1 & 0\end{bmatrix}$$

son matrices tales que $\mathbf{A} = \mathbf{SDS}^{-1}$.

$\square$

(L-17) Question 8(c) Sea como sea $\mathbf{A}$, la matriz $\mathbf{A}^\intercal\mathbf{A}$ siempre es simétrica; y por tanto es diagonalizable, y es posible encontrar una base ortonormal de autovectores de $\mathbf{A}^\intercal\mathbf{A}$.

$\square$

(L-17) Question 8(d) Basta con encontrar los valores de a que hacen la matriz de rango completo; es decir, cualquier valor de a distinto de cero $(a \neq 0)$ (para que la matriz sea invertible) y simultáneamente distinto de tres $(a \neq 3)$ (para que la matriz sea diagonalizable).

□

(L-17) Question 9(a) Ecuación característica:

$$0 = |\mathbf{B} - \lambda \mathbf{I}| = \lambda^2 - 1,$$

por tanto $\lambda = \pm 1$

- $\lambda_1 = 1$

$$\left(\mathbf{B} - \mathbf{I}\right) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

  Con autovector unitario $\boldsymbol{x}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- $\lambda_1 = -1$

$$\left(\mathbf{B} + \mathbf{I}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

  Con autovector unitario $\boldsymbol{x}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Por tanto

$$\mathbf{B} = \mathbf{QDQ}^\intercal = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}}$$

□

(L-17) Question 9(b) Puesto que los dos autovalores son distintos los dos autovectores encontrados son linealmente independientes, y la matriz es diagonalizable. Una matriz **S** que diagonaliza **B** es

$$\mathbf{S} = \begin{bmatrix}\boldsymbol{x}_1 & \boldsymbol{x}_2\end{bmatrix} = \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix};$$

ya que $\mathbf{BS} = \mathbf{SD}$, donde

$$\mathbf{D} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

Puesto que la matriz **S** es invertible, ya que ambos auto-vectores son linealmente independientes, podemos diagonalizar **B** del siguiente modo:

$$\mathbf{S}^{-1}\mathbf{BS} = \mathbf{D};$$

donde la matriz diagonal contiene los autovalores de **B**.

□

(L-18) Question 1(a) No. $\begin{bmatrix}1 & 3\\ 3 & 5\end{bmatrix} \xrightarrow[{[(-3)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \begin{bmatrix}1 & 0\\ 3 & -4\end{bmatrix} \xrightarrow[\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}]{} \begin{bmatrix}1 & 0\\ 0 & -4\end{bmatrix}.$

$f(x,y) = x^2 + 6xy + 5y^2.$

```
A = Matrix([[1,3],[3,5]])
```

```
DiagonalizaC(A,1)
```

$\square$

(L-18) Question 1(b) No. $\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \xrightarrow{\underset{[(1)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}} \begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix} \xrightarrow[\underset{[(1)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}]{} \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}.$

$f(x,y) = x^2 - 2xy + y^2 = (x-y)^2.$

$\square$

(L-18) Question 1(c) Yes. $\begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix} \xrightarrow{\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\underset{[(2)\mathbf{2}]}{\boldsymbol{\tau}}}} \begin{bmatrix}2 & 0\\ 3 & 1\end{bmatrix} \xrightarrow[\underset{[(2)\mathbf{2}]}{\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}]{} \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}.$

$f(x,y) = 2x^2 + 6xy + 5y^2.$

$\square$

(L-18) Question 1(d) No, There are negative number in the main diagonal.

$\begin{bmatrix}-1 & 2\\ 2 & -8\end{bmatrix} \xrightarrow{\underset{[(2)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}} \begin{bmatrix}-1 & 0\\ 2 & -4\end{bmatrix} \xrightarrow[\underset{[(2)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}]{} \begin{bmatrix}-1 & 0\\ 0 & -4\end{bmatrix}.$

$f(x,y) = -x^2 + 4xy - 8y^2.$

$\square$

(L-18) Question 1(e) The function $f(x,y) = (y-x)^2$ is zero along the line $y = x$. Note that (1,1) is an eigenvector corresponding to $\lambda = 0$.

$\square$

(L-18) Question 2(a) $\begin{bmatrix}1 & 2\\ 2 & 9\end{bmatrix} \xrightarrow[]{\overset{\boldsymbol{\tau}}{[(-2)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix}1 & 0\\ 2 & 5\end{bmatrix} \xrightarrow[\underset{[(-2)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}]{} \begin{bmatrix}1 & 0\\ 0 & 5\end{bmatrix}$. Hence

$$\mathbf{D} = \begin{bmatrix}1 & 0\\ 0 & 5\end{bmatrix} = \begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & 9\end{bmatrix}\begin{bmatrix}1 & -2\\ 0 & 1\end{bmatrix} = \mathbf{B}^{\mathsf{T}}\mathbf{A}\mathbf{B}.$$

Therefore, $\boldsymbol{x}\mathbf{A}\boldsymbol{x} = \boldsymbol{x}(\mathbf{B}^{-1})^{\mathsf{T}}\mathbf{D}\mathbf{B}^{-1}\boldsymbol{x} = \begin{pmatrix}x & y\end{pmatrix}\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 5\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix} =$
$\begin{pmatrix}x+2y & y\end{pmatrix}\begin{bmatrix}1 & 0\\ 0 & 5\end{bmatrix}\begin{pmatrix}x+2y\\ y\end{pmatrix}$.
So, $f(x, y) = x^2 + 4\,x\,y + 9\,y^2 = (x+2y)^2 + 5(y)^2$.

$\square$

(L-18) Question 2(b) $\begin{bmatrix}1 & 3\\ 3 & 9\end{bmatrix} \xrightarrow[]{\overset{\boldsymbol{\tau}}{[(-3)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix} \xrightarrow[\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}]{} \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$.

$f(x, y) = x^2 + 6\,x\,y + 9\,y^2 = (x+3y)^2 + 0(y)^2$.

$\square$

(L-18) Question 3(a) Not this one. The determinant is negative.

$\square$

(L-18) Question 3(b) Not this one; since $a = -1$.

$\square$

(L-18) Question 3(c) Not this one; This one is singular $(\det \mathbf{C} = 0)$.

$\square$

(L-18) Question 3(d) **D** has two positive eigenvalues since $a = 1$ and $\det \mathbf{A} = 1$.

$\square$

(L-18) Question 4. $\begin{bmatrix}1 & 2\\2 & 3\end{bmatrix} \xrightarrow[]{\underset{[(-2)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}} \begin{bmatrix}1 & 0\\2 & -1\end{bmatrix} \xrightarrow[\underset{[(-2)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}]{} \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}.$

$$f(x,y) = x^2 + 4xy + 3y^2 = (x+2y)^2 - y^2.$$

It is negative (because $-y^2$) at $(2,-1)$:

$$f(2,-1) = -1.$$

$\square$

(L-18) Question 5. If $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x} \quad \Rightarrow \quad \mathbf{A}^2\boldsymbol{x} = \mathbf{A}\mathbf{A}\boldsymbol{x} = \mathbf{A}(\lambda\boldsymbol{x}) = \lambda^2\boldsymbol{x}$.
Si $\mathbf{A}$ tiene autovalores λ_i positivos entonces los autovalores de $\mathbf{A}^2$ son λ_i^2 y los de $\mathbf{A}^{-1}$ son $1/\lambda_i$, y por tanto también positivos.

$\square$

(L-18) Question 6(a) The corresponding matrix is

$$\begin{bmatrix} 1 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow[\overset{\tau}{[(2)\mathbf{1}+\mathbf{2}]}]{\overset{\tau}{[(2)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow[\overset{\tau}{[\mathbf{2}\rightleftharpoons\mathbf{3}]}]{\overset{\tau}{[\mathbf{2}\rightleftharpoons\mathbf{3}]}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Cuya diagonal está formada por dos números positivos y un cero.

```
A = Matrix([[1,-2,0],[-2,4,0],[0,0,5]])
DiagonalizaC(A,1)
```

□

(L-18) Question 6(b) The corresponding matrix is

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & -a \\ 1 & 4 & 0 \\ -a & 0 & 1 \end{bmatrix} \xrightarrow{\substack{\tau \\ [(1)\mathbf{1}+\mathbf{2}] \\ [(-a)\mathbf{1}+\mathbf{3}]}} \begin{bmatrix} -1 & 0 & 0 \\ 1 & 5 & -a \\ -a & -a & a^2+1 \end{bmatrix} \xrightarrow[\substack{\tau \\ [(-a)\mathbf{1}+\mathbf{3}] \\ [(1)\mathbf{1}+\mathbf{2}]}]{} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & -a \\ 0 & -a & a^2+1 \end{bmatrix} \dots$$

Since there is a 5 (positive) in the main diagonal, it can't be negative definite.

□

(L-18) Question 7(a) $\mathbf{A}$ is not positive definite since $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\intercal}$ is in its nullspace.

In fact, diagonalizing by congruence we see it is positive semidefinite

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(2)\mathbf{2}] \\ [(1)\mathbf{1}+\mathbf{2}] \\ [(2)\mathbf{3}] \\ [(1)\mathbf{1}+\mathbf{3}]}} \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & -3 \\ -1 & -3 & 3 \end{bmatrix} \xrightarrow[\substack{\boldsymbol{\tau} \\ [(1)\mathbf{1}+\mathbf{3}] \\ [(2)\mathbf{3}] \\ [(1)\mathbf{1}+\mathbf{2}] \\ [(2)\mathbf{2}]}]{} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & -6 \\ 0 & -6 & 6 \end{bmatrix}$$

$$\xrightarrow{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{2}+\mathbf{3}]}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & -6 & 0 \end{bmatrix} \xrightarrow[\substack{\boldsymbol{\tau} \\ [(1)\mathbf{2}+\mathbf{3}]}]{} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\square$

(L-18) Question 7(b) **B** is positive definite

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(2)\mathbf{2}] \\ [(1)\mathbf{1}+\mathbf{2}] \\ [(2)\mathbf{3}] \\ [(1)\mathbf{1}+\mathbf{3}]}} \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \xrightarrow[\substack{\boldsymbol{\tau} \\ [(1)\mathbf{1}+\mathbf{3}] \\ [(2)\mathbf{3}] \\ [(1)\mathbf{1}+\mathbf{2}] \\ [(2)\mathbf{2}]}]{} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 6 \end{bmatrix}$$

$$\xrightarrow{\substack{\boldsymbol{\tau} \\ [(3)\mathbf{3}] \\ [(-1)\mathbf{2}+\mathbf{3}]}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 2 & 16 \end{bmatrix} \xrightarrow[\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}+\mathbf{3}] \\ [(3)\mathbf{3}]}]{} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 48 \end{bmatrix}$$

Hence, Sylvester's method works in this case!... $\det(2) = 2;$ $\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3;$ $\det(\mathbf{B}) = 4.$

□

(L-18) Question 7(c) The eigenvalues of **C** are the square eigenvalues of

$$\begin{bmatrix}0&1&2\\1&0&1\\2&1&0\end{bmatrix};$$

hence, if this matrix is full rank, then **C** is positive definite. Let's see it by diagonalization by congruence:

$$\begin{bmatrix}0&1&2\\1&0&1\\2&1&0\end{bmatrix}\xrightarrow{\substack{\boldsymbol{\tau}\\[(1)\mathbf{2}+\mathbf{1}]}}\begin{bmatrix}1&1&2\\1&0&1\\3&1&0\end{bmatrix}\xrightarrow[\substack{\boldsymbol{\tau}\\[(1)\mathbf{2}+\mathbf{1}]}]{}\begin{bmatrix}2&1&3\\1&0&1\\3&1&0\end{bmatrix}\xrightarrow{\substack{\boldsymbol{\tau}\\[(2)\mathbf{2}]\\[(-1)\mathbf{1}+\mathbf{2}]\\[(2)\mathbf{3}]\\[(-3)\mathbf{1}+\mathbf{3}]}}\begin{bmatrix}2&0&0\\1&-1&-1\\3&-1&-9\end{bmatrix}$$

$$\xrightarrow[\substack{\boldsymbol{\tau}\\[(-3)\mathbf{1}+\mathbf{3}]\\[(2)\mathbf{3}]\\[(-1)\mathbf{1}+\mathbf{2}]\\[(2)\mathbf{2}]}]{}\begin{bmatrix}2&0&0\\0&-2&-2\\0&-2&-18\end{bmatrix}\xrightarrow{\substack{\boldsymbol{\tau}\\[(-1)\mathbf{2}+\mathbf{3}]}}\begin{bmatrix}2&0&0\\0&-2&0\\0&-2&-16\end{bmatrix}\xrightarrow[\substack{\boldsymbol{\tau}\\[(-1)\mathbf{2}+\mathbf{3}]}]{}\begin{bmatrix}2&0&0\\0&-2&0\\0&0&-16\end{bmatrix}.$$

Since there are three pivots, zero is not an eigenvalue of that matrix nor of **C**. Therefore **C** is positive definite.

□

(L-18) Question 8.

$$\begin{bmatrix}1&3&0\\3&1&0\\0&0&a\end{bmatrix}\xrightarrow[]{\substack{\boldsymbol{\tau}\\[(-3)\mathbf{1}+\mathbf{2}]}}\begin{bmatrix}1&0&0\\3&-8&0\\0&0&a\end{bmatrix}\xrightarrow[\substack{\boldsymbol{\tau}\\[(-3)\mathbf{1}+\mathbf{2}]}]{}\begin{bmatrix}1&0&0\\0&-8&0\\0&0&a\end{bmatrix};$$

la forma cuadrática no es definida (sea cual sea el valor de a).

□

(L-18) Question 9. **A** es simétrica y por lo tanto diagonalizable:

$$\mathbf{A} = \mathbf{SDS}^{-1};$$

y sabemos que en este caso

$$\mathbf{A}^n = \mathbf{SD}^n\mathbf{S}^{-1};$$

Así pues, como **A** es definida positiva, sus autovalores son positivos $\lambda_1 > 0$ y $\lambda_2 > 0$ y entonces $\lambda_1^{-1} > 0$ y $\lambda_2^{-1} > 0$ que son los autovalores de $\mathbf{A}^{-1}$.

□

(L-18) Question 10. La ecuación característica es:

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix}a-\lambda & b\\ b & c-\lambda\end{vmatrix} = \lambda^2 - (a+c)\lambda + ac - b^2 = 0;$$

cuyas soluciones son

$$\lambda = \frac{(a+c) \pm \sqrt{(a+c)^2 - 4(ac-b^2)}}{2}$$

pero

$$\begin{aligned}(a+c)^2 - 4(ac-b^2) = a^2 + c^2 + 2ac - 4ac + 4b^2 &= a^2 + c^2 - 2ac + 4b^2 \\ &= (a-c)^2 + (2b)^2\end{aligned}$$

que es una suma de cuadrados, y por lo tanto, lo que hay dentro de la raíz cuadrada es mayor o igual a cero. Así pues, los **autovalores son reales** (no hay raíces cuadradas de números negativos).

Por otra parte, si $a > 0$ y $ac > b^2$ necesariamente $c \geq 0$. Así que sabemos que $(a+c) > 0$ y por tanto

$$\lambda_1 = \frac{(a+c) + \sqrt{(a-c)^2 + (2b)^2}}{2} > 0$$

Además, puesto que $ac > b^2$, sabemos que $\det \mathbf{A} = ac - b^2 > 0$; pero, puesto que $\lambda_1 \lambda_2 = \det \mathbf{A} > 0$ y $\lambda_1 > 0$, necesariamente $\lambda_2 > 0$.

□

(L-18) Question 11(a) Miremos los signos de los pivotes

$$\begin{bmatrix}1&2&3\\2&5&4\\3&4&9\end{bmatrix}\xrightarrow[\substack{[(-2)\mathbf{1}+\mathbf{2}]\\[(-3)\mathbf{1}+\mathbf{3}]}]{\boldsymbol{\tau}}\begin{bmatrix}1&0&0\\2&1&-2\\3&-2&0\end{bmatrix}\xrightarrow[\substack{\boldsymbol{\tau}\\[(-3)\mathbf{1}+\mathbf{3}]\\[(-2)\mathbf{1}+\mathbf{2}]}]{}\begin{bmatrix}1&0&0\\0&1&-2\\0&-2&0\end{bmatrix}$$

$$\xrightarrow[{[(2)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}}\begin{bmatrix}1&0&0\\0&1&0\\0&-2&-4\end{bmatrix}\xrightarrow[\substack{\boldsymbol{\tau}\\[(2)\mathbf{2}+\mathbf{3}]}]{}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-4\end{bmatrix}$$

Puesto que hay tanto pivotes positivos como negativos, la matriz **es indefinida.**

□

(L-18) Question 11(b)

$$\begin{bmatrix}1&2&0&0\\2&6&-2&0\\0&-2&5&-2\\0&0&-2&3\end{bmatrix}\xrightarrow[{[(-2)\mathbf{1}+\mathbf{2}]}]{\tau}\begin{bmatrix}1&0&0&0\\2&2&-2&0\\0&-2&5&-2\\0&0&-2&3\end{bmatrix}\xrightarrow[{[(1)\mathbf{2}+\mathbf{3}]}]{\tau}$$

$$\begin{bmatrix}1&0&0&0\\2&2&0&0\\0&-2&3&-2\\0&0&-2&3\end{bmatrix}\xrightarrow[{[(\frac{2}{3})\mathbf{3}+\mathbf{4}]}]{\tau}\begin{bmatrix}1&0&0&0\\2&2&0&0\\0&-2&3&0\\0&0&-2&\frac{5}{3}\end{bmatrix}$$

Puesto que aplicando eliminación usando *solo* transformaciones *Tipo I* hemos llegado a una matriz triangular cuyos pivotes son positivos $\rightarrow$ **Definida positiva.**

□

(L-18) Question 11(c) Since **B** is positive definite, $-$**B** is **Negative Definite**.

□

(L-18) Question 11(d) Puesto que **A** tiene dos pivotes positivos $\lambda_1 > 0$ y $\lambda_2 > 0$ y uno negativo $\lambda_3 < 0$; los pivotes de **D** —que son los inversos de los de **A**— conservan los signos. Por tanto es: **indefinida.**
(ejercicio 14 del conjunto de problemas 6.2 del libro de texto)

□

(L-18) Question 12. $\begin{pmatrix}0 & a & 0\end{pmatrix}$, with $a \neq 0$

□

(L-18) Question 13. $\boldsymbol{x}(\mathbf{A}+\mathbf{B})\boldsymbol{x} = (\boldsymbol{x}\mathbf{A}+\boldsymbol{x}\mathbf{B})\boldsymbol{x} = \boldsymbol{x}\mathbf{A}\boldsymbol{x} + \boldsymbol{x}\mathbf{B}\boldsymbol{x} > 0,$ ya que **A** y **B** son definidas positivas.

$\square$

(L-18) Question 14(a)

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{cccc} 1&1&1&1\\ 1&2&3&4\\ 1&3&6&10\\ 1&4&10&20\\ \hline 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{4}]}]{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{1}+\mathbf{2}]\\ [(-1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cccc} 1&0&0&0\\ 1&1&2&3\\ 1&2&5&9\\ 1&3&9&19\\ \hline 1&-1&-1&-1\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ [(-2)\mathbf{2}+\mathbf{3}]\\ [(-3)\mathbf{2}+\mathbf{4}]}}$$

$$\left[\begin{array}{cccc} 1&0&0&0\\ 1&1&0&0\\ 1&2&1&3\\ 1&3&3&10\\ \hline 1&-1&1&2\\ 0&1&-2&-3\\ 0&0&1&0\\ 0&0&0&1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ [(-3)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{cccc} 1&0&0&0\\ 1&1&0&0\\ 1&2&1&0\\ 1&3&3&1\\ \hline 1&-1&1&-1\\ 0&1&-2&3\\ 0&0&1&-3\\ 0&0&0&1 \end{array}\right] = \left[\frac{\mathbf{L}}{\dot{\mathbf{E}}}\right]$$

Hence, the Gauss transformations are

$$\mathbf{G}_{1\triangleright} = \begin{bmatrix}1 & -1 & -1 & -1\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}; \quad \mathbf{G}_{2\triangleright} = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & -2 & -3\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}; \quad \mathbf{G}_{3\triangleright} = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & -3\\0 & 0 & 0 & 1\end{bmatrix};$$

$\dot{\mathbf{E}} = \dot{\mathbf{G}}_{1\triangleright}\ \dot{\mathbf{G}}_{2\triangleright}\ \dot{\mathbf{G}}_{3\triangleright}$. Matrix $\dot{\mathbf{U}}$ is the inverse of $\dot{\mathbf{E}}^{-1}$, so:
$\dot{\mathbf{U}} = \dot{\mathbf{E}}^{-1} = (\dot{\mathbf{G}}_{1\triangleright}\dot{\mathbf{G}}_{2\triangleright}\dot{\mathbf{G}}_{3\triangleright})^{-1} = \dot{\mathbf{G}}^{-1}_{3\triangleright}\dot{\mathbf{G}}^{-1}_{2\triangleright}\dot{\mathbf{G}}^{-1}_{1\triangleright}$, and the $\dot{\mathbf{L}}\mathbf{D}\dot{\mathbf{U}}$ factorization is:

$$\mathbf{A} = \begin{bmatrix}1 & 1 & 1 & 1\\1 & 2 & 3 & 4\\1 & 3 & 6 & 10\\1 & 4 & 10 & 20\end{bmatrix} = \begin{bmatrix}1 & & & \\1 & 1 & & \\1 & 2 & 1 & \\1 & 3 & 3 & 1\end{bmatrix}\begin{bmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\\ & 1 & 2 & 3\\ & & 1 & 3\\ & & & 1\end{bmatrix} = \dot{\mathbf{L}}\mathbf{D}\dot{\mathbf{U}}$$

that is equal to $\mathbf{A} = \dot{\mathbf{L}}\mathbf{D}\dot{\mathbf{L}}^{\mathsf{T}}$, since $\mathbf{A}$ is symmetric.

□

(L-18) Question 14(b)

$$\left[\begin{array}{c}\mathsf{A}\\\hline \mathsf{I}\end{array}\right]=\left[\begin{array}{cccc}1&1&1&1\\1&2&2&2\\1&2&3&3\\1&2&3&4\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{1}+\mathbf{2}]\\ [(-1)\mathbf{1}+\mathbf{3}]\\ [(-1)\mathbf{1}+\mathbf{4}]}}\left[\begin{array}{cccc}1&0&0&0\\1&1&1&1\\1&1&2&2\\1&1&2&3\\\hline 1&-1&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{2}+\mathbf{3}]\\ [(-1)\mathbf{2}+\mathbf{4}]}}$$

$$\left[\begin{array}{cccc}1&0&0&0\\1&1&0&0\\1&1&1&1\\1&1&1&2\\\hline 1&-1&0&0\\0&1&-1&-1\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{3}+\mathbf{4}]}}\left[\begin{array}{cccc}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\\\hline 1&-1&0&0\\0&1&-1&0\\0&0&1&-1\\0&0&0&1\end{array}\right]=\left[\begin{array}{c}\mathsf{L}\\\hline \dot{\mathsf{E}}\end{array}\right]$$

Hence, the Gauss transformations are

$$\dot{\mathsf{G}}_{1\triangleright}=\begin{bmatrix}1&-1&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix};\quad \dot{\mathsf{G}}_{2\triangleright}=\begin{bmatrix}1&0&0&0\\0&1&-1&-1\\0&0&1&0\\0&0&0&1\end{bmatrix};\quad \dot{\mathsf{G}}_{3\triangleright}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&-1\\0&0&0&1\end{bmatrix};$$

$\dot{\mathbf{E}} = \dot{\mathbf{G}}_{1\triangleright}\ \dot{\mathbf{G}}_{2\triangleright}\ \dot{\mathbf{G}}_{3\triangleright}$. Matrix $\dot{\mathbf{U}}$ is the inverse of $\dot{\mathbf{E}}^{-1}$, so: $\dot{\mathbf{U}} = \dot{\mathbf{E}}^{-1} = \dot{\mathbf{G}}_{3\triangleright}^{-1}\dot{\mathbf{G}}_{2\triangleright}^{-1}\dot{\mathbf{G}}_{1\triangleright}^{-1}$, and the $\dot{\mathbf{L}}\mathbf{D}\dot{\mathbf{U}}$ factorization is:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} = \dot{\mathbf{L}}\mathbf{D}\dot{\mathbf{U}}$$

that is equal to $\mathbf{A} = \dot{\mathbf{L}}\mathbf{D}\dot{\mathbf{L}}^{\mathsf{T}}$, since $\mathbf{A}$ is symmetric.

□

(L-18) Question 15.

$$\begin{bmatrix} 3 & 6 \\ 6 & 16 \end{bmatrix} \xrightarrow[{[(-2)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \begin{bmatrix} 3 & 0 \\ 6 & 4 \end{bmatrix} \xrightarrow[\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}]}]{} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$

por tanto

$$\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 6 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Los coeficientes de los cuadrados son los pivotes en $\mathbf{D}$, y los coeficientes dentro de los cuadrados son las columnas de $\mathbf{L}$.

□

(L-18) Question 16(a)

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 2-\lambda & 1 & 0 & 0 \\ 1 & 2-\lambda & 0 & 0 \\ 0 & 0 & a-\lambda & 0 \\ 0 & 0 & a & a-\lambda \end{vmatrix} = (a-\lambda)^2 \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0.$$

Therefore, the eigenvalue $\lambda = a$ is repeated twice. We can get the other two eigenvalues solving

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = 0; \qquad \Rightarrow \lambda^2 - 4\lambda + 3 = 0$$

Thus, the other two eigenvalues are 1 and 3.

$\square$

(L-18) Question 16(b) When $\lambda = a = 2$, the rank of the matrix

$$\mathbf{A} - 2\mathbf{I} = \begin{bmatrix} 2-\lambda & 1 & 0 & 0 \\ 1 & 2-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 2 & 2-\lambda \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

is 3. Therefore $\dim \mathcal{N}\left(\mathbf{A}\right) = 1$ (only one free column); hence it is not possible to find two linearly independent eigenvectors for the repeated eigenvalue $\lambda = 2$. It follows that THE MATRIX IS NOT DIAGONALISABLE.

□

(L-18) Question 16(c)

$$|\mathbf{B} - \lambda \mathbf{I}| = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda) \cdot \Big((2-\lambda)^2 - 1\Big) = 0$$

Clearly one eigenvector is $\lambda = 1$. The other two are the roots of

$$\Big((2-\lambda)^2 - 1\Big) = 4 + \lambda^2 - 4\lambda - 1 = \lambda^2 - 4\lambda + 3 = 0.$$

that is, $\lambda = 3$ and $\lambda = 1$. Thus, $\boxed{\mathbf{D} = \left[\begin{smallmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right].}$

- For $\lambda = 3$

$$\mathbf{A} - 3\lambda \mathbf{I} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix};$$

therefore $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector. Since its norm is $\sqrt{2}$, then $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is a normalised eigenvector for $\lambda = 3$.

- For $\lambda = 1$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

because the last column of $(\mathbf{A} - \lambda\mathbf{I})$, the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector (with norm 1); besides, from the other two columns, it is ease tho check that $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ is another eigenvector for $\lambda = 3$. Normalising the vector we get $\frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

It is not difficult to see that those three vector are orthogonal. Therefore:

$$\boxed{\mathbf{P} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.}$$

□

(L-18) Question 16(d) The quadratic form is

$$f(x, y, z) = \boldsymbol{b}\mathbf{X}\boldsymbol{b} = \begin{pmatrix} x & y & z \end{pmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2x^2 + 2y^2 + z^2 + 2xy,$$

and we already know it is positive defined, since the three eigenvalues of the symmetric matrix **B** are positive.

□

(L-18) Question 17(a) There are two cases:

- $a = -4/5$ and $b = 3/5$
- $a = 4/5$ and $b = -3/5$.

□

(L-18) Question 17(b) Any values of a and b such as the first column is not a multiple of the second; for example, $a = 1$ and $b = 0$.

□

(L-18) Question 17(c) This is just the opposite case..., here we need a singular matrix; therefore we can use any multiple of the second column; for example: $a = 3$ and $b = 4$.

□

(L-18) Question 17(d) By symmetry, $b = 3/5$. In addition, we need $a < 0$ and

$\det \mathbf{A} > 0$; that is $a \cdot 4/5 - (3/5)^2 > 0$, or

$$a \cdot 4/5 > (3/5)^2$$

something impossible if $a < 0$. Therefore, THERE ISN'T SUCH VALUES OF a AND b.

$\square$

(L-18) Question 18(a)

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & a & 0 \\ 0 & 0 & 8 \end{bmatrix} \xrightarrow[{}]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix} 1 & 0 & 0 \\ 1 & a-1 & 0 \\ 0 & 0 & 8 \end{bmatrix} \xrightarrow[\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{2}]}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & a-1 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Si $a = 1$ la matriz **Q** es semidefinida positiva.

$\square$

(L-18) Question 18(b)

- Si $a > 1$ la matriz **Q** es definida positiva.
- Si $a < 1$ la matriz **Q** es indefinida.

$\square$

(L-Opt-2) Question 1(a)

$$\det \mathbf{A} = \begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix} = -a;$$

therefore $|\mathbf{A}|$ is no zero if and only if $a \neq 0$.

□

(L-Opt-2) Question 1(b) No, since:

$$|1| = 1; \quad \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0; \quad \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0; \quad \begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -1,$$

when all subdeterminants should be possitive. It follows that the matrix is not definite.

□

(L-Opt-2) Question 1(c)

$$
\left[\begin{array}{cccc} 1 & 1 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]
\xrightarrow{\substack{\tau \\ [(-1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{4}] \\ [(1)\mathbf{2}+\mathbf{4}]}}
\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & -2 \\ \hline 1 & -1 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]
\xrightarrow{\substack{\tau \\ [(-1)\mathbf{2}+\mathbf{1}] \\ [(1)\mathbf{4}+\mathbf{1}] \\ [(2)\mathbf{2}] \\ [(-1)\mathbf{4}+\mathbf{2}]}}
\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \\ \hline 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right]
$$

Hence,

$$
\mathbf{A}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & 0 \\ 1 & -1/2 & 0 & -1/2 \end{bmatrix}
$$

$\square$

(L-Opt-2) Question 1(d) From the steps given in the gaussian elimination when solving the first part of the exercise, it's easy to check that $\mathrm{rg}\left(\mathbf{A}\right) = 3$ when $a = 0$; and therefore, there are three pivot variables. Hence, only one variable can be chosen as a free variable.

When $a = 0$ the second and third columns are equal (and hence dependent); it follows that we can take as free variable either the second or the third one.

$\square$

(L-Opt-2) Question 2(a) *Es verdadero.* Si $\mathbf{A}$ es simétrica, estonces $\mathbf{A}^\intercal = \mathbf{A}$, por tanto

$$(\mathbf{A}^2)^\intercal = (\mathbf{A}\mathbf{A})^\intercal = \mathbf{A}^\intercal\mathbf{A}^\intercal = \left(\mathbf{A}^\intercal\right)^2 = \mathbf{A}^2,$$

es decir, que $\mathbf{A}^2$ también es simétrica.

$\square$

(L-Opt-2) Question 2(b) *Es verdadero.* Veamoslo:

$$(\mathbf{I}-\mathbf{A})^2 = (\mathbf{I}-\mathbf{A})(\mathbf{I}-\mathbf{A}) = \mathbf{I}-\mathbf{A}-\mathbf{A}+\mathbf{A}^2 = \mathbf{I}-\mathbf{A}-\mathbf{A}+\mathbf{A} = \mathbf{I}-\mathbf{A}$$

A las matrices con esta propiedad se las denomina *"matrices idempotentes"*.

$\square$

(L-Opt-2) Question 2(c) *Es falso.* El determinate de una matriz es igual al producto de sus autovaloes; si uno de ellos es cero, necesariamente la matriz es singular. En tal caso sus columnas son linealmente dependientes y es posible encontrar una solución distinta a la trivial ($\boldsymbol{x} = \mathbf{0}$) para dicho sistema homogeneo; así que hay más de una solución y el sistema es necesariamente *indeterminado.*

$\square$

(L-Opt-2) Question 2(d) *Verdadero.* Por el mismo motivo de antes, $\underset{m\times m}{\mathbf{A}}$ es singular, lo que quiere decir que el subespacio generado por las columnas de $\mathbf{A}$ (*que llamaremos espacio columna de* $\mathbf{A}$, $\mathcal{C}\left(\mathbf{A}\right)$) es de dimensión menor que m, pero eso quiere decir que existen vectores de $\mathbb{R}^m$ que no pertenencen a $\mathcal{C}\left(\mathbf{A}\right)$. Si $\boldsymbol{b}$ fuera uno de ellos, entonces no existiría una combinación lineal de las columnas de $\mathbf{A}$ igual a $\boldsymbol{b}$, es decir, que $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ será incompatible para dicho $\boldsymbol{b} \notin \mathcal{C}\left(\mathbf{A}\right)$.

□

(L-Opt-2) Question 2(e) *True.* If $\mathbf{Q}$ is orthogonal, then $\mathbf{Q}^\intercal\mathbf{Q} = \mathbf{I}$; therefore the inverse of $\mathbf{Q}$ is its transpose ($\mathbf{Q}^\intercal = \mathbf{Q}^{-1}$), and then $\mathbf{Q}\mathbf{Q}^{-1} = \mathbf{I} = \mathbf{Q}\mathbf{Q}^\intercal$; but this means that the columns of $\mathbf{Q}^{-1}$ are orthogonal (since all the elements of $\mathbf{Q}\mathbf{Q}^\intercal = \mathbf{I}$ outside the main diagonal are zero) with norm equal to one (since $\mathbf{Q}\ \mathbf{Q}^\intercal$ has only ones in the main diagonal).

□

(L-Opt-2) Question 2(f) *False.* For the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

1 is the only eigenvalue, but $\mathbf{A}$ is not the identity matrix.

□

(L-Opt-2) Question 3(a) Verdadero.

□

(L-Opt-2) Question 3(b) $a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_n\boldsymbol{v}_n = \boldsymbol{0}$ si y sólo si todos los coeficientes a_j son iguales a cero.
Es decir, si la única solución a $\left[\boldsymbol{v}_1, \ldots \boldsymbol{v}_n\right]\boldsymbol{x} = \boldsymbol{0}$ es un vector de ceros (es decir, si $\mathcal{N}\left(\left[\boldsymbol{v}_1, \ldots \boldsymbol{v}_n\right]\right)$ es $\{\boldsymbol{0}\}$.

□

(L-Opt-2) Question 3(c) Falso, ya que $a_1\mathbf{0} = \mathbf{0}$ incluso para $a_1 \neq 0$.

□

(L-Opt-2) Question 3(d) podemos encontrar n autovectores linealmente independientes.

□

(L-Opt-2) Question 3(e) $\sqrt{7}$

□

(L-Opt-2) Question 3(f) $\boldsymbol{u} \cdot \boldsymbol{v} = 0$

□

(L-Opt-2) Question 4(a) Falso. $\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$

□

(L-Opt-2) Question 4(b) Verdadero
$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}) \neq 0 \Rightarrow \det(\mathbf{A}) \neq 0 \neq \det(\mathbf{B})$

□

(L-Opt-2) Question 4(c) Verdadero. El determinante cambia de signo, y el producto de los autovalores es igual al determinante.

□

(L-Opt-2) Question 4(d) Verdadero

□

(L-Opt-2) Question 4(e) Verdadero. En tal caso $(\mathbf{A} - 5\mathbf{I})$ es de rango completo, y $\dim \mathcal{N}(\mathbf{A} - 5\mathbf{I}) = 0$, por tanto no hay autovectores asociados al autovalor 5.

(L-Opt-2) Question 4(f) Falso. $\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ no tiene solución y $|\mathbf{A}| = 0$.

(L-Opt-2) Question 4(g) Falso. Como máximo puede tener tres pivotes.

(L-Opt-2) Question 4(h) Verdadero. $\boldsymbol{b} \in \mathbb{R}^n$ pero $\boldsymbol{b} \notin \mathcal{C}\left(\mathbf{C}\right)$; por tanto no todos los vectores de $\mathbb{R}^n$ están en $\mathcal{C}\left(\mathbf{C}\right)$. Es decir, $\operatorname{rg}\left(\mathbf{C}\right) = \dim \mathcal{C}\left(\mathbf{C}\right) < n = \dim \mathbb{R}^n$.

(L-Opt-2) Question 4(i) Falso. Si algún autovalor es igual a cero, la matriz no es invertible. Por ejemplo una matriz nula y cuadrada.

(L-Opt-2) Question 4(j) Verdadero. Si es invertible tiene rango completo —n pivotes iguales a uno con ceros por encima y por debajo. . . es decir la identidad).

(L-Opt-2) Question 5(a) **A** y **D** por tener tres pivotes (para **A** se ve directamente, y para **D** tras el primer paso de eliminación).

(L-Opt-2) Question 5(b) Sólo **B** (puesto que sólo tiene dos autovalores 0 y 2, necesariamente alguno esta repetido).

(L-Opt-2) Question 5(c) **B** y **C** por ser no invertibles (ambas tienen un autovalor igual a cero)

□

(L-Opt-2) Question 5(d) **A**, **C** y **D** por tener autovalores distintos (además **D** es simétrica).
Para **B**, el autovalor $\lambda = 2$ está repetido:

$$(\mathbf{B} - 2\mathbf{I}) = \begin{bmatrix} -1 & 1 & 3 \\ 1 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

sólo una columna libre, y por tanto sólo podemos encontrar una dirección asociada al autovalor repetido $\lambda = 2$. Es decir, *la matriz* **B** *no es diagonalizable.*

□

(L-Opt-2) Question 5(e) Para la matriz simétrica **D**.

□

(L-Opt-2) Question 6(a) Since the matrix is triangular, the eigenvalues are the numbers on the main diagonal: $\lambda_1 = 1$ and $\lambda_2 = 2$.

For $\lambda = 1$

$$(\mathbf{A} - \lambda\mathbf{I}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

The following are three linearly independent eigenvectors

$$\boldsymbol{x}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad \boldsymbol{x}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad \boldsymbol{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda = 2$

$$(\mathbf{A} - 2\lambda \mathbf{I}) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$$

The following is an eigenvector

$$\boldsymbol{x}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$\square$

(L-Opt-2) Question 6(b) Yes, since there are 4 linearly independent eigenvectors $\square$

(L-Opt-2) Question 6(c) This factorization $\mathbf{A} = \mathbf{PDP}^\intercal$ implies that $\mathbf{A}$ must be symetric; but **A** is not. Therefore, it is not possible.

□

(L-Opt-2) Question 6(d)

$$|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|} = \frac{1}{\text{product of eigenvalues of } \mathbf{A}} = \frac{1}{2}.$$

□

(L-Opt-2) Question 7(a) Yes, it is. A 3 by 3 matrix with 3 different eigenvalues.

□

(L-Opt-2) Question 7(b) No, it is not. Since $\boldsymbol{v}_3 = -\boldsymbol{v}_1$, then $\boldsymbol{v}_3$ must be an eigenvector associated to λ_1.

□

(L-Opt-2) Question 7(c)

$$\mathbf{A}(\boldsymbol{v}_1 - \boldsymbol{v}_2) = \mathbf{A}\boldsymbol{v}_1 - \mathbf{A}\boldsymbol{v}_2 = \lambda_1\boldsymbol{v}_1 - \lambda_2\boldsymbol{v}_2 = 1\begin{pmatrix}1\\0\\1\end{pmatrix} - 2\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\-2\\-1\end{pmatrix}$$

□

(L-Opt-2) Question 8(a) Since the first two vectors are the same, the dimension of $\mathcal{N}\left(\mathbf{A}\right)$ is 2. The number of the columns is 4, therefore the rank of **A** is 2.

The last vector is telling us that the last column of **A** is zero vector; and the first vector means that the first column of **A** is the opposite of the third. Then, one possibility is:

$$\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\begin{pmatrix}x\\ y\\ z\\ t\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\\ 0\end{pmatrix} \qquad \Longrightarrow \qquad \begin{cases}x-z & =0\\ y & =0\end{cases}.$$

But that is not the only possible answer; for example we can add zero rows below. The coefficient matrix $\underset{m\times 4}{\mathbf{A}}$ must be rank 2, with a fourth column full of zeros, and a first column opposite to the third one.

□

(L-Opt-2) Question 8(b) Since **A** has a characteristic polynomial of degree 5, we know that **A** is a 5×5 matrix. Since 0 is not a root of $p(\cdot)$ and so is not an eigenvalue, we know **A** is invertible so $rank(A)=5$.

□

(L-Opt-2) Question 9(a) Puesto que la matriz es invertible, el rango es 3; y el espacio columna $\mathcal{C}\left(\mathbf{A}\right)$ es todo $\mathbb{R}^3$. Por ello no hay columnas libres, es decir, el espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$ sólo contiene el vector cero $\mathbf{0}$.

□

(L-Opt-2) Question 9(b) Puesto que **L** es $\mathbf{E}^{-1}$ (la inversa del producto de matrices elementales necesarias para triangularizar **A**), el 5 en la primera posición de la segunda

fila de **L** nos dice que el primer paso fue *"restar a la segunda fila cinco veces la primera"*.
Puesto que **U** tiene tres pivotes, es de rango completo; y por tanto es invertible.
El determinante es $\det(\mathbf{A}) = u_{11} \cdot u_{22} \cdot u_{33}$.

□

(L-Opt-2) Question 9(c) La matriz

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

no puede ser triangularizada sin permutar las filas previamente.
Sin embargo, permutando la primera fila con la tercera, la matriz ya es triangular. De hecho toda matriz invertible admite la siguiente factorización:

$$\mathbf{PA} = \mathbf{LU}.$$

La matriz **A** es invertible ya que una vez se han permutado las filas, aparecen tres pivotes; es decir, la matriz es de rango completo.

□

(L-19) Question 1(a) For the $\boldsymbol{y}$ and the given points our matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}$$

The normal equations are given by $\mathbf{A}^\intercal \mathbf{A} \widehat{\boldsymbol{\beta}} = \mathbf{A}^\intercal \boldsymbol{y}$, or

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{pmatrix} \widehat{\alpha} \\ \widehat{\beta} \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}$$

or

$$\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{pmatrix} \widehat{\alpha} \\ \widehat{\beta} \end{pmatrix} = \begin{pmatrix} 36 \\ 112 \end{pmatrix}$$

which has as its solution $\widehat{\alpha} = 1$, $\widehat{\beta} = 4$.

□

(L-19) Question 1(b) So the four heights with this $\widehat{\boldsymbol{\beta}}$ are given by

$$\boldsymbol{p} = \mathbf{A}\widehat{\boldsymbol{\beta}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 13 \\ 17 \end{pmatrix}.$$

With this solution by direct calculation the error vector $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ is given by

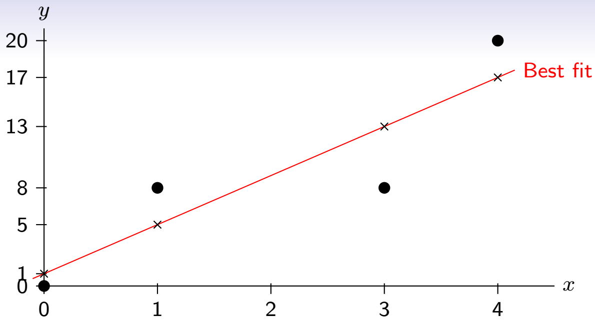
$$\boldsymbol{e} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 13 \\ 17 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -5 \\ 3 \end{pmatrix}.$$

□

(L-19) Question 1(c) It is the smallest possible value for a linear fitting $\|\boldsymbol{e}\|^2 = \boldsymbol{e} \cdot \boldsymbol{e} = (-1)^2 + 3^2 + (-5)^2 + 3^2 = 44.$

□

(L-19) Question 1(d)



□

(L-19) Question 1(e) If our mathematical model of the relationship between $\boldsymbol{y}$ and $\boldsymbol{x}$ is a line given by $\boldsymbol{y} = \alpha \cdot \mathbf{1} + \beta \cdot \boldsymbol{x}$, then if the measurements change to what is given in the text then we have

$$\begin{aligned} \alpha + 0\beta =& 1 \\ \alpha + 1\beta =& 5 \\ \alpha + 3\beta =& 13 \\ \alpha + 4\beta =& 17 \end{aligned}$$

Which has as an analytic solution given by $\alpha = 1$ and $\beta = 4$.

□

(L-19) Question 1(f)

$$\boldsymbol{e}\mathbf{A} = \begin{pmatrix}-1, & 3, & -5, & 3,\end{pmatrix} \begin{bmatrix}1 & 0\\1 & 1\\1 & 3\\1 & 4\end{bmatrix} = \begin{pmatrix}0, & 0,\end{pmatrix}.$$

□

(L-19) Question 1(g) So the shortest distance is given by $\|\boldsymbol{e}\| = \sqrt{\boldsymbol{e}\cdot\boldsymbol{e}} = \sqrt{44}$.

□

(L-19) Question 2(a) Our equations are given by

$$\begin{aligned}\alpha - \beta &= 7\\ \alpha + \beta &= 7\\ \alpha + 2\beta &= 21\end{aligned}$$

Which as a system of linear equations matrix are given by

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ 21 \end{pmatrix}$$

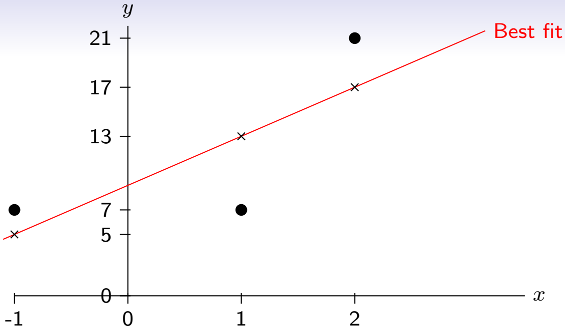
The least squares solution is given by $\mathbf{A}^\intercal \mathbf{A} \widehat{\boldsymbol{\beta}} = \mathbf{A}^\intercal \boldsymbol{y}$ which in this case simplify as follows

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} \widehat{\alpha} \\ \widehat{\beta} \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{pmatrix} 7 \\ 7 \\ 21 \end{pmatrix}$$

or

$$\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{pmatrix} \widehat{\alpha} \\ \widehat{\beta} \end{pmatrix} = \begin{pmatrix} 35 \\ 42 \end{pmatrix}$$

which gives the following $\widehat{\alpha} = 9$, $\widehat{\beta} = 4$. So the best linear fit is $\widehat{y} = 9 + 4x$.



□

(L-19) Question 2(b)

$$\boldsymbol{p} = \mathbf{A}\widehat{\boldsymbol{\beta}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} 9 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \\ 17 \end{pmatrix}.$$

that gives the values on the closest line. The error vector $\boldsymbol{e}$ is then given by

$$\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{p} = \begin{pmatrix} 7 \\ 7 \\ 21 \end{pmatrix} - \begin{pmatrix} 5 \\ 13 \\ 17 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ 4 \end{pmatrix}.$$

The matrix **P** projects onto the column space of **A**, but $\boldsymbol{e}$ is in the left null space, so it is orthogonal to $\mathcal{C}\left(\mathbf{A}\right)$.

□

(L-19) Question 3(a) The three data points lie on the same line when $b_3 = 7$. This line is $-2 + 3t$. If $b_3 = 9$, the least squares method will NOT choose this line. (A quick way to see this is from the fact that the line chosen by least squares will give the average of the given b's at the time equal to the average of the given t's; in this case, the best fit line would take the value $(1 + 3 + 9)/3 = 13/3$ at $t = (1 + 2 + 3)/3 = 2$, whereas our line gives 4 at $t = 2$.)

□

(L-19) Question 3(b) The linear system for $\boldsymbol{x} = (C, D)$ would be the following:

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ b_3 \end{pmatrix}.$$

We compute the projection matrix **P** onto the column space of **A** using the projection matrix formula:

$$\begin{aligned}\mathbf{P} =& \mathbf{A}(\mathbf{A}^\intercal\mathbf{A})^{-1}\mathbf{A}^\intercal = \frac{1}{6}\begin{bmatrix}1 & 1\\1 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}14 & -6\\-6 & 3\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\end{bmatrix}\\ =& \frac{1}{6}\begin{bmatrix}1 & 1\\1 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}8 & 4 & -4\\-3 & 0 & 3\end{bmatrix} = \frac{1}{6}\begin{bmatrix}5 & 2 & -1\\2 & 2 & 2\\-1 & 2 & 5\end{bmatrix}.\end{aligned}$$

□

(L-19) Question 3(c) The column space of $\mathbf{P}$ is the space consisting of all the vectors $\mathbf{P}\boldsymbol{b}$, i.e. all the projections of vectors in $\mathbb{R}^3$ onto the column space of $\mathbf{A}$, which is precisely the column space of $\mathbf{A}$. Thus the rank of $\mathbf{P}$ is equal to the rank of $\mathbf{A}$, which is 2.

□

(L-19) Question 3(d) The equation for the best least squares solution $\widehat{\boldsymbol{x}}$ is $\mathbf{A}^\intercal\mathbf{A}\widehat{\boldsymbol{x}} = \mathbf{A}^\intercal\boldsymbol{b}$, where $\boldsymbol{b} = \begin{pmatrix}1\\4\\1\end{pmatrix}$. Writing out this system, we get

$$\begin{bmatrix}3 & 6\\6 & 14\end{bmatrix}\widehat{\boldsymbol{x}} = \begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\end{bmatrix}\begin{pmatrix}1\\4\\1\end{pmatrix} = \begin{pmatrix}6\\12\end{pmatrix}.$$

The solution to this system is $\widehat{\boldsymbol{x}} = \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, so the best fit line is the horizontal line $b = 2$.

$\square$