# Mathematics II

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You can find the last version of these course materials at

https://github.com/mbujosab/MatematicasII/tree/main/Eng







## 1 Highlights of **Lesson 4**

### Highlights of Lesson 4

- Elementary transformations
- Indentifying singular matrices by elimination
- Matrix multiplication of Elementary matrices





## 2 Elementary transformations of a matrix

Type *I*: $\mathbf{A}_{\underset{[(\lambda)\boldsymbol{i}+\boldsymbol{j}]}{\boldsymbol{\tau}}}$ (with $i \neq j$)

add $\lambda$ times $i$-th column ($\lambda\mathbf{A}_{|i}$) to $j$-th column ($\mathbf{A}_{|j}$)

$$\begin{bmatrix} 1 & -3 & 0 \\ 1 & -6 & 3 \end{bmatrix}_{\underset{[(-2)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}} = \begin{bmatrix} 1 & -3 & -2 \\ 1 & -6 & 1 \end{bmatrix}$$

Type *II*: $\mathbf{A}_{\underset{[(\alpha)\boldsymbol{i}]}{\boldsymbol{\tau}}}$ (with $\alpha \neq 0$)

multiply by $\alpha$ the $i$-th column

$$\begin{bmatrix} 1 & -3 & 0 \\ 1 & -6 & 3 \end{bmatrix}_{\underset{[(10)\mathbf{2}]}{\boldsymbol{\tau}}} = \begin{bmatrix} 1 & -30 & 0 \\ 1 & -60 & 3 \end{bmatrix}$$

## 3 Elimination and pre-echelon form of a matrix

- *Pivot* is the first non-zero component of each column.
- *Elimination*: modifies a matrix until all components at the right-hand side of each pivot are zeros

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 8 & 4 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\tau}} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix} \xrightarrow{\underset{[(-2)\mathbf{2}+\mathbf{3}]}{\tau}} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & -2 & 5 \end{bmatrix} = \mathbf{L}$$





## 4 Elimination

### Elimination algorithm on $\mathbf{A}$

modifies $\mathbf{A}$ using a sequence of *elementary transformations*

### Goal

to get a *(pre)echelon* form

- *pre-echelon*: all components on the right side of each pivot are zero.
- *echelon*: if any column before a non-null column $\mathbf{A}_{|j}$ is non-null column and its pivot is above the pivot of $\mathbf{A}_{|j}$.

It is always possible to find a (pre)echelon form by elimination

**Rank** $(\mathrm{rg})$: the number of pivots in any of its pre-echelon forms

$\underset{n\times n}{\mathbf{A}}$ is *singular* if its pre-echelon forms have null-columns $(\mathrm{rg} < n)$





## 5 Elimination: When can't we find $n$ pivots?

$n \times n$ matrices are **singular** if less than $n$ pivots after elimination

$$\begin{bmatrix} 0 & 1 & 3 \\ 4 & 2 & 8 \\ 1 & 1 & 1 \end{bmatrix}$$

Has this matrix $n$ pivots? $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ 1 & 1 & 1 \end{bmatrix}$

and this one? $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

and this one? $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & -4 \end{bmatrix}$





## 6 Matrix multiplication: elementary matrices

$$\underbrace{\begin{bmatrix} 1 & 3 & 0 \\ 2 & 8 & 4 \\ 1 & 1 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \phantom{0} & \phantom{0} & \phantom{0} \\ & & \\ & & \end{bmatrix}}_{\mathbf{I}_{\tau}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix}}_{\mathbf{A}_{\tau}}$$

We call $\mathbf{I}_{\tau}$ "Elementary matrix":

$$\mathbf{A}(\mathbf{I}_{\tau}) = \mathbf{A}_{\tau}$$

This specific elementary matrix $\mathbf{I}_{\tau}$ is written as $\mathbf{I}_{\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\tau}}$

$$\mathbf{A}\left(\mathbf{I}_{\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\tau}}\right) = \mathbf{A}_{\underset{[(-3)\mathbf{1}+\mathbf{2}]}{\tau}}$$

## 7 Matrix multiplication: elementary matrices

$$\begin{bmatrix}1&0&0\\2&2&4\\1&-2&1\end{bmatrix}\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}=\begin{bmatrix}1&0&0\\2&2&0\\1&-2&5\end{bmatrix}$$

This specific elementary matrix $\mathbf{I}_{\tau}$ is written as $\mathbf{I}_{\substack{\tau\\ [(-2)\mathbf{2}+\mathbf{3}]}}$

$$\mathbf{A}\left(\mathbf{I}_{\substack{\tau\\ [(-2)\mathbf{2}+\mathbf{3}]}}\right)=\mathbf{A}_{\substack{\tau\\ [(-2)\mathbf{2}+\mathbf{3}]}}$$





## 8 Elimination by elementary matrices

$$\mathbf{A}=\begin{bmatrix}1&3&0\\2&8&4\\1&1&1\end{bmatrix}\xrightarrow{\substack{\tau\\ [(-3)\mathbf{1}+\mathbf{2}]}}\begin{bmatrix}1&0&0\\2&2&4\\1&-2&1\end{bmatrix}\xrightarrow{\substack{\tau\\ [(-2)\mathbf{2}+\mathbf{3}]}}\begin{bmatrix}1&0&0\\2&2&0\\1&-2&5\end{bmatrix}=\mathbf{L}$$

$$\mathbf{A}_{\substack{\tau\\ [(-3)\mathbf{1}+\mathbf{2}]\\ [(-2)\mathbf{2}+\mathbf{3}]}}=\mathbf{A}_{\substack{\tau\\ [(-3)\mathbf{1}+\mathbf{2}][(-2)\mathbf{2}+\mathbf{3}]}}=\left(\mathbf{A}\left(\mathbf{I}_{\substack{\tau\\ [(-3)\mathbf{1}+\mathbf{2}]}}\right)\right)\left(\mathbf{I}_{\substack{\tau\\ [(-2)\mathbf{2}+\mathbf{3}]}}\right)=\mathbf{L}$$

there is a matrix that does the whole job at once

$$\mathbf{A}_{\substack{\tau\\ [(-3)\mathbf{1}+\mathbf{2}]\\ [(-2)\mathbf{2}+\mathbf{3}]}}=\mathbf{A}\left(\left(\mathbf{I}_{\substack{\tau\\ [(-3)\mathbf{1}+\mathbf{2}]}}\right)\left(\mathbf{I}_{\substack{\tau\\ [(-2)\mathbf{2}+\mathbf{3}]}}\right)\right)=\mathbf{A}\mathbf{I}_{\substack{\tau\\ [(-3)\mathbf{1}+\mathbf{2}]\\ [(-2)\mathbf{2}+\mathbf{3}]}}=\mathbf{L}$$

$$\boxed{\mathbf{A}_{\tau_1\cdots\tau_k}=\mathbf{A}\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)}$$





## 9 how do I get from **L** back to **A**? Inverses

How do I reverse the first step? (it was subtract 3 times $\mathbf{A}_{|1}$ from $\mathbf{A}_{|2}$)

$$\begin{bmatrix}1&-3&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&3&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

$$\mathbf{I}_{\substack{\tau\\ [(-\lambda)\boldsymbol{i}+\boldsymbol{j}]}}\ \text{“undo”}\ \mathbf{I}_{\substack{\tau\\ [(\lambda)\boldsymbol{i}+\boldsymbol{j}]}}$$

How to undo $\mathbf{I}_{\substack{\tau\\ [(\alpha)\boldsymbol{i}]}}$ ?

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}\begin{bmatrix}1& & \\ &1& \\ & &1/2\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$





## 10 Interchange or swap matrices

Which matrix exchanges the columns?

$$\begin{bmatrix}a&c\\b&d\end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix}=\begin{bmatrix}c&a\\d&b\end{bmatrix}$$

Which matrix exchanges the rows? where do we put that matrix?

$$\begin{bmatrix}a&c\\b&d\end{bmatrix}\qquad=\begin{bmatrix}b&d\\a&c\end{bmatrix}$$

Matrix multiplication is not commutative!

## $\boxed{11}$ Interchange of columns

Interchange of columns:

$\mathbf{A}_{\underset{[i\rightleftharpoons j]}{\boldsymbol{\tau}}} \to$ swicht columns $i$ and $j$ of $\mathbf{A}$

$$\begin{bmatrix} 1 & -3 & 0 \\ 1 & -6 & 3 \end{bmatrix}_{\underset{[\mathbf{2}\rightleftharpoons\mathbf{3}]}{\boldsymbol{\tau}}} = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 3 & -6 \end{bmatrix}$$

We can switch two columns by a sequence of elementary transformations

Matrix $\mathbf{I}_{\underset{[i\rightleftharpoons j]}{\boldsymbol{\tau}}}$ is call a exchange matrix





## $\boxed{12}$ Permutation matrices

Product between exchange matrices $\mathbf{I}_{\underset{[\text{-}\rightleftharpoons\text{-}]}{\boldsymbol{\tau}}}$ is a permutation matrix $\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}$.

$$\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}} = \text{ Identity matrix } \mathbf{I} \text{ with rearranged columns}$$

Let's see the $3 \times 3$ case

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad \mathbf{I}_{\underset{[\mathbf{1}\rightleftharpoons\mathbf{2}]}{\boldsymbol{\tau}}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

How many $3 \times 3$ pemutations can we find?

what happens if I multiply two permutation matrices?





## Questions of the Lecture 4

(L-4) QUESTION 1.

(a) Which three matrices $\mathbf{I}_{\underset{[(x)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}$, $\mathbf{I}_{\underset{[(y)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}}$ and $\mathbf{I}_{\underset{[(z)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}}$ put $\mathbf{A} = \begin{bmatrix} 1 & 4 & -2 \\ 1 & 6 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ into an echelon form?

(b) Multiply those $\mathbf{I}_{\boldsymbol{\tau}_i}$ to get one matrix $\mathbf{E}$ that does elimination: $\mathbf{AE} = \mathbf{K}$.

Based on (Strang, 1988, exercise 24 from section 1.4.)

(L-4) QUESTION 2. Consider the matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ -1 & -3 & -2 \\ 0 & 1 & c \end{bmatrix}$$

For what value(s) of $c$ the matrix is singular (we can't find three pivots)?





(L-4) QUESTION 3. Consider the following 3 by 3 matrices.

(a) $(\mathbf{I}_{\underset{[(-1)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}})$ subtracts column 1 from column 2 and then $(\mathbf{I}_{\underset{[\mathbf{2}\rightleftharpoons\mathbf{3}]}{\boldsymbol{\tau}}})$ exchanges columns 2 and 3. What matrix $\mathbf{E}$ does both steps at once?

(b) $(\mathbf{I}_{\underset{[\mathbf{2}\rightleftharpoons\mathbf{3}]}{\boldsymbol{\tau}}})$ exchanges columns 2 and 3 and then $\mathbf{I}_{\underset{[(-1)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}}$ subtracts column 1 from column 3. What matrix $\mathbf{N} = (\mathbf{I}_{\underset{[\mathbf{2}\rightleftharpoons\mathbf{3}]}{\boldsymbol{\tau}}})(\mathbf{I}_{\underset{[(-1)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}})$ does both steps at once? Explain why $\mathbf{M}$ and $\mathbf{N}$ are the same but the $\mathbf{I}_{\boldsymbol{\tau}}$'s are different.

Based on (Strang, 1988, exercise 28 from section 1.4.)

(L-4) QUESTION 4. Elimination matrices $\mathbf{I}_{\underset{[(?)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}$ and $\mathbf{I}_{\underset{[(?)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}}$ will reduce $\mathbf{A}$ to triangular form. Find $\mathbf{E}$ so that $\mathbf{AE} = \mathbf{L}$ is lower triangular (echelon), if $\mathbf{A}$ is

$$\begin{bmatrix} 2 & 2 & 0 \\ 1 & 4 & 9 \\ 1 & 3 & 9 \end{bmatrix}$$

(L-4) QUESTION 5. Although we will only consider as elementary the *Type I* and *II* transformations, in most of the Linear Algebra books appears a third type: the *exchange* of columns

$$\mathbf{A}_{\underset{[\boldsymbol{p}\rightleftharpoons\boldsymbol{s}]}{\boldsymbol{\tau}}} \to \text{Exchanges columns } p \text{ and } s \text{ of } \mathbf{A}.$$

Prove that a column exchange is, in fact, a sequence of *Type I* and *II* elementary transformations. Try transforming $\underset{2\times 2}{\mathbf{I}}$ in $\begin{bmatrix}0&1\\1&0\end{bmatrix}$ by elementary transformations of the columns.

(L-4) QUESTION 6. Write down the 3 by 3 matrices that produce these elimination steps:

(a) $\mathbf{I}_{\underset{[(-5)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}$ substracts 5 times column 1 from column 2,

(b) $\mathbf{I}_{\underset{[(-7)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}}$ substracts 7 times column 2 from column 3,

(c) $\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}$ exchanges columns 1 and 2, and then columns 2 and 3.

(Strang, 2003, exercise 1 from section 2.3.)





(L-4) QUESTION 7. Consider the matrices of QUESTION 6:

(a) when multiplying by $\mathbf{I}_{\underset{[(-5)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}$ and then by $\mathbf{I}_{\underset{[(-7)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}}$ the matrix $\mathbf{A} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}$ we get $\mathbf{A}_{\underset{\substack{[(-5)\mathbf{1}+\mathbf{2}]\\ [(-7)\mathbf{2}+\mathbf{3}]}}{\boldsymbol{\tau}}} = [\quad;\quad;\quad].$

(b) But, when multiplying by $\mathbf{I}_{\underset{[(-5)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}}$ before and then by $\mathbf{I}_{\underset{[(-7)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}}$ we get $\mathbf{A}_{\underset{\substack{[(-7)\mathbf{2}+\mathbf{3}]\\ [(-5)\mathbf{1}+\mathbf{2}]}}{\boldsymbol{\tau}}} = [\quad;\quad;\quad].$

(c) When $\underset{[(-7)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}$ comes first, the column ____ feels no effect from column ____.

This property will become very important in the LU factorization!

(Strang, 2003, exercise 2 from section 2.3.)

(L-4) QUESTION 8. What matrix $\mathbf{M}$ sends $\boldsymbol{v} = (1, \quad 0,)$ to $(0, \quad 1,)$, es decir $\boldsymbol{v}\mathbf{M} = (0, \quad 1,)$ ; and also sends $\boldsymbol{w} = (0, \quad 1,)$ to $(1, \quad 0,)$, es decir $\boldsymbol{w}\mathbf{M} = (1, \quad 0,)$?

(L-4) QUESTION 9. Consider a permutation (interchange) matrix $\mathbf{I}_{\underset{[\boldsymbol{i}\rightleftharpoons\boldsymbol{j}]}{\boldsymbol{\tau}}}$, if we compute the product $\mathbf{A}(\mathbf{I}_{\underset{[\boldsymbol{i}\rightleftharpoons\boldsymbol{j}]}{\boldsymbol{\tau}}})$, we get a new matrix like $\mathbf{A}$, but with exchanged columns. What happen if we compute the product $(\mathbf{I}_{\underset{[\boldsymbol{i}\rightleftharpoons\boldsymbol{j}]}{\boldsymbol{\tau}}})\mathbf{A}$? Check your answer with a 2 by 2 example.





(L-4) QUESTION 10. If every column of $\mathbf{A}$ is a multiple of $(1, \quad 1, \quad 1,)$ , then $\mathbf{A}\boldsymbol{x}$ is always a multiple of $(1, \quad 1, \quad 1,)$ . Do a 3 by 3 example. How many pivots are produced by elimination?
(Strang, 1988, exercise 26 from section 1.4.)





1 Highlights of **Lesson 5**

## Highlights of Lesson 5

- Inverse of $\mathbf{A}$
- Gauss-Jordan elimination / finding $\mathbf{A}^{-1}$
- Inverse of $\mathbf{AB}$, $\mathbf{A}^{\intercal}$

## 2 Inverse of a matrix (square matrices)

**A** squared of order $n$ has inverse (is *invertible*) if exists **B** such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}.$$

Then

$$\mathbf{B} = \mathbf{A}^{-1} \qquad \text{and} \qquad \mathbf{A} = \mathbf{B}^{-1}.$$

Not all matrices have inverse

*Squared matrices with no inverse* are called *singular* matrices

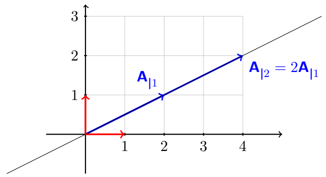




## 3 Singular case (no inverse)

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

Is it possible to find a matrix **B** such that **AB** =**I**?
. . . columns of **I** should be linear combinations of columns of **A**. . . but both columns lie on the same line.




So

$$\boxed{\mathbf{A} \text{ is singular}}$$





## 4 Singular case (no inverse)

Can we find $\boldsymbol{x} \neq \mathbf{0}$ such that $\mathbf{A}\boldsymbol{x} = \mathbf{0}$?

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} \; \\ \; \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If $\mathbf{A}\boldsymbol{x} = 0$ and $\boldsymbol{x} \neq 0 \quad \Rightarrow \quad$ there is no $\mathbf{A}^{-1}$

$$\boxed{\text{The existence of } \mathbf{A}^{-1} \text{ leads to a contradiction}}$$

If $\mathbf{A}\boldsymbol{x} = 0$ and $\boldsymbol{x} \neq \mathbf{0} \quad \Rightarrow \quad \mathbf{A}^{-1}\mathbf{A}\boldsymbol{x} = \mathbf{A}^{-1}\mathbf{0} \quad \Rightarrow \quad \boldsymbol{x} = \mathbf{0}.$

When $\mathbf{A}^{-1}$ does exist

the only solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$.





## 5 Calculating the inverse matrix

$$\mathbf{A}(\mathbf{A}^{-1}) = \mathbf{I}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So. . . we are solving $m$ systems (of $m$ equations each)

$$\begin{bmatrix} 1 & 3 \\ 2 & \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 2 & \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

## 6 Gauss-Jordan: solving two linear systems at once

### Gauss-Jordan elimination (obtaining a reduced echelon form $\mathbf{R}$)

apply elementary transformations until a echelon matrix with only zeros to the left of each pivot (and all pivots equal to 1) is achieved

Let's solve the linear systems

$$\begin{bmatrix}1 & 3\\ 2 & 7\end{bmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}1\\ 0\end{pmatrix} \qquad \begin{bmatrix}1 & 3\\ 2 & 7\end{bmatrix}\begin{pmatrix}c\\ d\end{pmatrix}=\begin{pmatrix}0\\ 1\end{pmatrix}$$

applying Gauss-Jordan elimination on $\mathbf{A}$ stacked with $\mathbf{I}$

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right]=\left[\begin{array}{cc}1 & 3\\ 2 & 7\\ \hline 1 & 0\\ 0 & 1\end{array}\right] \rightarrow \qquad \rightarrow \qquad =$$

If $\mathbf{R}=\mathbf{I}$, we have found $\mathbf{A}^{-1}$





## 7 Gauss-Jordan: Why does it work?

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right]=\left[\begin{array}{cc}1 & 3\\ 2 & 7\\ \hline 1 & 0\\ 0 & 1\end{array}\right] \xrightarrow{\tau_{[(-3)\mathbf{1}+\mathbf{2}]}} \qquad \xrightarrow{\tau_{[(-2)\mathbf{2}+\mathbf{1}]}}$$

that is, since $\mathbf{A}_{\tau_1\cdots\tau_k}=\mathbf{A}\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)$:

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right]_{\tau_1\cdots\tau_k}=\left[\frac{\mathbf{A}_{\tau_1\cdots\tau_k}}{\mathbf{I}_{\tau_1\cdots\tau_k}}\right]=\left[\frac{\mathbf{A}\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)}{\mathbf{I}_{\tau_1\cdots\tau_k}}\right]=\left[\frac{\mathbf{I}}{\mathbf{I}_{\tau_1\cdots\tau_k}}\right],$$

who is $\mathbf{I}_{\tau_1\cdots\tau_k}$?

therefore $\mathbf{A}^{-1}=$





## 8 Inverse of a product

When $\mathbf{A}$ and $\mathbf{B}$, of order $n$, are invertible, $(\mathbf{AB})$ is invertible.

what matrix gives me the inverse of $\mathbf{AB}$? lets try with $(\mathbf{B}^{-1}\mathbf{A}^{-1})$ :

$$\mathbf{AB}\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right)=$$

$$\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right)\mathbf{AB}=$$





## 9 Inverse of a transpose matrix

$$\mathbf{A}\mathbf{A}^{-1}=\mathbf{I}$$

let me transpose both sides

$$\left((\mathbf{A}^{-1})^{\mathsf{T}}\right)\mathbf{A}^{\mathsf{T}}=\mathbf{I}$$

then

the inverse of $\mathbf{A}^{\mathsf{T}}$ is

## $\boxed{10}$ Interchanges and permutations

Are interchange matrices $\mathbf{I}_{\underset{[i \rightleftharpoons j]}{\tau}}$, invertible?

It is easy to check that

$$\left(\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}}\right)^{\mathsf{T}}\left(\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}}\right) = \mathbf{I} \qquad \Longrightarrow$$





## $\boxed{11}$ Caracterización of invertible matrices

Given $\mathbf{A}$ of order $n$, the following statements are equivalent

1. No zero columns in $\mathbf{A}_{\tau_1\cdots\tau_p} = \mathbf{K}$ (pre-echelon matrix).
2. $\mathbf{A}$ has inverse.
3. $\mathbf{A}$ is product of elementary matrices.

$$\mathbf{A}_{\tau_1\cdots\tau_k} = \mathbf{A}\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right) = \mathbf{I} \qquad \Rightarrow \qquad \mathbf{A} = \left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)^{-1}$$

where

$$\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)^{-1} = \left((\mathbf{I}_{\tau_1})\cdots(\mathbf{I}_{\tau_k})\right)^{-1} = (\mathbf{I}_{\tau_k^{-1}})\cdots(\mathbf{I}_{\tau_1^{-1}}) = \mathbf{I}_{\tau_k^{-1}\cdots\tau_1^{-1}}$$





## Questions of the Lecture 5

(L-5) Question 1. Use the Gauss-Jordan method to invert

(a) $\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) $\mathbf{A}_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

(c) $\mathbf{A}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

(Strang, 1988, exercise 6 from section 1.6.)

(L-5) Question 2.

(a) If $\mathbf{A}$ is invertible and $\mathbf{AB} = \mathbf{AC}$, prove quickly that $\mathbf{B} = \mathbf{C}$.
(b) If $\mathbf{A} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, find an example with $\mathbf{AB} = \mathbf{AC}$, but $\mathbf{B} \neq \mathbf{C}$.

(Strang, 1988, exercise 4 from section 1.6.)

(L-5) Question 3. Use the Gauss-Jordan method to invert the generic matrix $2 \times 2$

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The matrix is invertible (not singular) only when ...





(L-5) Question 4. Use the Gauss-Jordan method to invert the following matrices.

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 6 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 4 & -2 \\ 1 & 3 & 1 \end{bmatrix}$$

(L-5) Question 5. If the 3 by 3 matrix $\mathbf{A}$ has $\mathbf{A}_{|1} + \mathbf{A}_{|2} = \mathbf{A}_{|3}$, show that $\mathbf{A}$ is not invertible, by two different methods:

(a) Find a nonzero solution $\boldsymbol{x}$ to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.
(b) Elimination keeps $column\,1 + column\,2 = column\,3$. Explain why there is no third pivot.

(Strang, 1988, exercise 26 from section 1.6.)

(L-5) QUESTION 6. Find the inverses of

(a) $\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}$.

(b) $\mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}$.

(c) $\mathbf{A}_3 = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$.

(Strang, 1988, exercise 10 from section 1.6.)





(L-5) QUESTION 7. Find the inverse of

$$\mathbf{A} = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}$$

What values of $a$ and $b$ make the matrix singular?
(Strang, 1988, exercise 42 from section 1.6.)

(L-5) QUESTION 8. Find $\mathbf{E}^2$, $\mathbf{E}^8$ and $\mathbf{E}^{-1}$ if $\mathbf{E} = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$
(Strang, 1988, exercise 6 from section 1.5.)

(L-5) QUESTION 9. Consider the following permutation matrix:

$$\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Find $\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}{}^{-1}$. Can you say something else about the relationship between $\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}$ and $\mathbf{I}_{\underset{[\mathfrak{S}]}{\boldsymbol{\tau}}}{}^{-1}$?





(L-5) QUESTION 10. The 3 by 3 matrix **A** reduces to the identity matrix **I** by the following three column operations (in order):

$\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}$: Subtract 4 times column 1 from column 2.

$\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}$: Subtract 3 times column 1 from column 3.

$\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}$: Subtract column 3 from column 2.

(a) Write $\mathbf{A}^{-1}$ in terms of elementary matrices $\mathbf{I}_{\boldsymbol{\tau}}$. **Then compute $\mathbf{A}^{-1}$.**
(b) What is the original matrix **A**?

(Based on *MIT Course 18.06 Quiz 1, October 4, 2006*)





(L-5) QUESTION 11. The 3 by 3 matrix **A** reduces to the identity matrix **I** by the following three **row** operations (in order):

$\underset{[(-4)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}$: Subtract 4 times row 1 from row 2.

$\underset{[(-3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}$: Subtract 3 times row 1 from row 3.

$\underset{[(-1)\mathbf{3}+\mathbf{2}]}{\boldsymbol{\tau}}$: Subtract row 3 from row 2.

(a) Write $\mathbf{A}^{-1}$ in terms of the **E**'s. **Then compute $\mathbf{A}^{-1}$.**
(b) What is the original matrix **A** ?

(*MIT Course 18.06 Quiz 1, October 4, 2006*)

(L-5) QUESTION 12.

(a) Find the inverse of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(b) Find the inverse of the following matrix **using the Gauss-Jordan method**

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & d \end{bmatrix}$$

(Poole, 2004, exercise 36, 38 and 59 from section 3.3.)

(L-5) QUESTION 13. Consider the squared matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. True or false?

(a) If $\mathbf{AB} = \mathbf{I}$ and $\mathbf{CA} = \mathbf{I}$ then $\mathbf{B} = \mathbf{C}$.

(b) $(\mathbf{AB})^2 = \mathbf{A}^2\mathbf{B}^2$.





(L-5) QUESTION 14. Consider the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & a & 0 & 2a \\ a & 0 & 1 & 0 \\ 1 & 0 & a & 1 \end{bmatrix}$

(a) Prove that $\mathbf{A}$ is invertible for any value of $a$.

(b) Compute $\mathbf{A}^{-1}$ when $a = 0$.

(L-5) QUESTION 15. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$. Find $\mathbf{A}^{-1}$.

(L-5) QUESTION 16. Find (if it is possible) the inverse of the following inverses

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}.$$

(L-5) QUESTION 17. There is a finite number ($n!$) of $n \times n$ permutation matrices. In addition, any power of a permutation matrix is a another permutation matrix. Use these facts to prove that $\left(\mathbf{I}_{\underset{[\mathfrak{S}]}{\tau}}\right)^r = \mathbf{I}$ for some integer numbers $r$.





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