Mathematics II

Marcos Bujosa

Universidad Complutense de Madrid

16/02/2023

You can find the last version of these course materials at

https://github.com/mbujosab/MatematicasII/tree/main/Eng

1 Highlights of Lesson 1

Highlights of Lesson 1

- Vector and matrix operations
  - Addition and scalar multiplication
  - Some properties of these operations

2 Vectors in $\mathbb{R}^n$

Vector in $\mathbb{R}^n$ is an ordered list of n real numbers

Example

$\boldsymbol{v} \in \mathbb{R}^3$: first component: 5, the second: 1 and the third: 10

$$\boldsymbol{v} = \begin{cases} v_1 = 5 \\ v_2 = 1 \\ v_3 = 10 \end{cases} \quad ; \qquad \boldsymbol{v} = \begin{pmatrix} 5 \\ 1 \\ 10 \end{pmatrix} = \begin{pmatrix} 5, & 1, & 10, \end{pmatrix}.$$

Notation

- $\boldsymbol{a}$, $\boldsymbol{x}$, $\boldsymbol{0}$
- $\text{elem}_3\left(\boldsymbol{v}\right) \equiv {}_{3|}\boldsymbol{v} \equiv \boldsymbol{v}_{|3} \equiv v_3 = 10$

A parenthesis around a list of numbers denotes a vector.

3 Basic operations with vectors

Vector addition: $(\boldsymbol{a}+\boldsymbol{b})_{|i} = \boldsymbol{a}_{|i} + \boldsymbol{b}_{|i}$

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad \text{add to} \quad \boldsymbol{a}+\boldsymbol{b} = \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \end{pmatrix}.$$

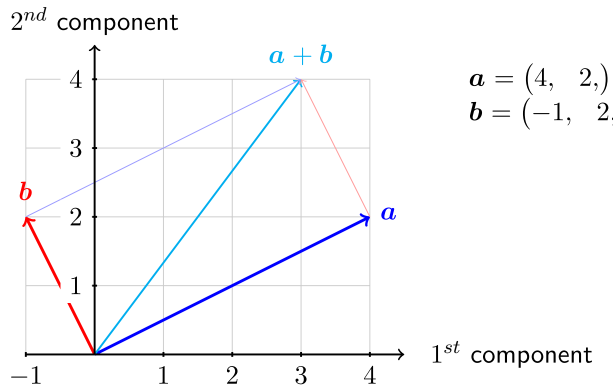
Scalar multiplication: $(\lambda \boldsymbol{a})_{|i} = \lambda(\boldsymbol{a}_{|i})$

$$2\boldsymbol{a} = \begin{pmatrix} 2a_1 \\ 2a_2 \end{pmatrix} \qquad \text{and} \qquad (-1)\boldsymbol{a} = \begin{pmatrix} -a_1 \\ -a_2 \end{pmatrix} \equiv -\boldsymbol{a}$$

(Hence, the operator " $|i$ " is linear)

$\boldsymbol{a}$ and $\boldsymbol{b}$ (with n components) are equal when:
$\boldsymbol{a}_{|i} = \boldsymbol{b}_{|i}, \quad i = 1:n.$

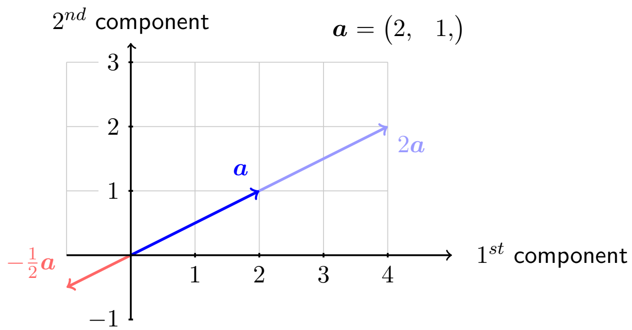
4 Vector addition



$\boldsymbol{a} = (4, \;\; 2,)$
$\boldsymbol{b} = (-1, \;\; 2,)$

5 Scalar multiplication

$\boldsymbol{a} = (2, \;\; 1,)$



What is the picture of all multiples of $\boldsymbol{a}$?
Is $\boldsymbol{0}$ a multiple of $\boldsymbol{a}$?

6 Addition and scalar multiplication

$$\boxed{(\boldsymbol{a}+\boldsymbol{b})_{|i} = \boldsymbol{a}_{|i} + \boldsymbol{b}_{|i}} \qquad \boxed{(\lambda \boldsymbol{a})_{|i} = \lambda(\boldsymbol{a}_{|i})}$$

Let us recall some properties of scalars

Scalars	Vectors
1. $a+b=b+a$	1. $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$
2. $a+(b+c)=(a+b)+c$	2. $\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})=(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}$
3. $a+0=a$	3. $\boldsymbol{a}+\boldsymbol{0}=\boldsymbol{a}$
4. $a+(-a)=0$	4. $\boldsymbol{a}+(-\boldsymbol{a})=\boldsymbol{0}$
5. $ab=ba$	5. $\lambda(\boldsymbol{a}+\boldsymbol{b})=\lambda\boldsymbol{a}+\lambda\boldsymbol{b}$
6. $a(b+c)=ab+ac$	6. $(\lambda+\eta)\boldsymbol{a}=\lambda\boldsymbol{a}+\eta\boldsymbol{a}$
7. $a(bc)=(ab)c$	7. $\lambda(\eta\boldsymbol{a})=(\lambda\eta)\boldsymbol{a}$
8. $1a=a$	8. $1\boldsymbol{a}=\boldsymbol{a}$

7 Matrices

Matrix in $\mathbb{R}^{m\times n}$ is an ordered list of n vectors in $\mathbb{R}^m$

Example

Three vectors in $\mathbb{R}^2$: $\boldsymbol{a} = \begin{pmatrix}4\\2\end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix}-1\\2\end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix}0\\7\end{pmatrix}$

$$\mathbf{A} = [\boldsymbol{a};\ \ \boldsymbol{b};\ \ \boldsymbol{c};] = \begin{bmatrix}4 & -1 & 0\\ 2 & 2 & 7\end{bmatrix} \neq [\boldsymbol{c};\ \ \boldsymbol{b};\ \ \boldsymbol{a};]$$

Two vectors in $\mathbb{R}^3$: $\boldsymbol{x} = (4,\ \ -1,\ \ 0,)$ and $\boldsymbol{y} = (2,\ \ 2,\ \ 7,)$

$$\mathbf{B} = [\boldsymbol{x};\ \ \boldsymbol{y};]$$

Notation

- $\mathbf{A},\ \mathbf{B},\ \mathbf{0}$
- $\underset{2\times 3}{\mathbf{A}},\ \underset{3\times 2}{\mathbf{B}};$ $\qquad \underset{2\times 3}{\mathbf{A}} \neq \underset{3\times 2}{\mathbf{B}}$

A bracket around a vector list denotes a matrix.

8 More notation

$$\mathbf{A} = \begin{bmatrix}1 & 2 & 1\\ 7 & 0 & 3\end{bmatrix}$$

Picking operators

- $\mathrm{elem}_{21}(\mathbf{A}) \ = \ {}_{2|}\mathbf{A}_{|1} \ = \ a_{21}: \qquad 7$
- $\mathrm{row}_1(\mathbf{A}) \ = \ {}_{1|}\mathbf{A} \ : \qquad (1,\ \ 2,\ \ 1,) = \begin{pmatrix}1\\2\\1\end{pmatrix}$
- $\mathrm{col}_1(\mathbf{A}) \ = \ \mathbf{A}_{|1} \ : \qquad \begin{pmatrix}1\\7\end{pmatrix} = (1,\ \ 7,)$

9 Basic operations with matrices

Matrix addition: $(\mathbf{A}+\mathbf{B})_{|j} = \mathbf{A}_{|j} + \mathbf{B}_{|j}$

$$\mathbf{A} = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix} \text{ add to } \begin{bmatrix}a_{11}+b_{11} & a_{12}+b_{12}\\ a_{21}+b_{21} & a_{22}+b_{22}\end{bmatrix}$$

Scalar multiplication: $(\lambda\mathbf{A})_{|j} = \lambda(\mathbf{A}_{|j})$

$$7\mathbf{A} = \begin{bmatrix}7a_{11} & 7a_{12}\\ 7a_{21} & 7a_{22}\end{bmatrix} \text{ and } (-1)\mathbf{A} = \begin{bmatrix}-a_{11} & -a_{12}\\ -a_{21} & -a_{22}\end{bmatrix} = -\mathbf{A}.$$

(Hence, the operator " $|i$ " is linear)

$\mathbf{A}$ and $\mathbf{B}$ (with same order) are equal when: $\quad \mathbf{A}_{|j} = \mathbf{B}_{|j}$

10 Addition and scalar multiplication

$$\boxed{(\mathbf{A}+\mathbf{B})_{|j} = \mathbf{A}_{|j} + \mathbf{B}_{|j}}; \qquad \boxed{(\lambda\mathbf{A})_{|j} = \lambda(\mathbf{A}_{|j})}$$

Vectors

1. $\boldsymbol{a}+\boldsymbol{b} = \boldsymbol{b}+\boldsymbol{a}$
2. $\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c}) = (\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}$
3. $\mathbf{0}+\boldsymbol{a} = \boldsymbol{a}$
4. $\boldsymbol{a}+(-\boldsymbol{a}) = \mathbf{0}$
5. $\lambda(\boldsymbol{a}+\boldsymbol{b}) = \lambda\boldsymbol{a}+\lambda\boldsymbol{b}$
6. $(\lambda+\eta)\boldsymbol{a} = \lambda\boldsymbol{a}+\eta\boldsymbol{a}$
7. $\lambda(\eta\boldsymbol{a}) = (\lambda\eta)\boldsymbol{a}$
8. $1\boldsymbol{a} = \boldsymbol{a}$

Matrices

1. $\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$
2. $\mathbf{A}+(\mathbf{B}+\mathbf{C}) = (\mathbf{A}+\mathbf{B})+\mathbf{C}$
3. $\mathbf{0}+\mathbf{A} = \mathbf{A}$
4. $\mathbf{A}+(-\mathbf{A}) = \mathbf{0}$
5. $\lambda(\mathbf{A}+\mathbf{B}) = \lambda\mathbf{A}+\lambda\mathbf{B}$
6. $(\lambda+\eta)\mathbf{A} = \lambda\mathbf{A}+\eta\mathbf{A}$
7. $\lambda(\eta\mathbf{A}) = (\lambda\eta)\mathbf{A}$
8. $1\mathbf{A} = \mathbf{A}$

11 Rewriting Rules

Distributive rules

$$(\boldsymbol{a}+\boldsymbol{b})_{|i} = \boldsymbol{a}_{|i} + \boldsymbol{b}_{|i} \qquad {}_{i|}(\boldsymbol{a}+\boldsymbol{b}) = {}_{i|}\boldsymbol{a} + {}_{i|}\boldsymbol{b}$$

$$(\mathbf{A}+\mathbf{B})_{|j} = \mathbf{A}_{|j} + \mathbf{B}_{|j} \qquad {}_{i|}(\mathbf{A}+\mathbf{B}) = {}_{i|}\mathbf{A} + {}_{i|}\mathbf{B}$$

In addition, if we allow $\lambda\boldsymbol{a} = \boldsymbol{a}\lambda$ and $\lambda\mathbf{A} = \mathbf{A}\lambda$, then we get

Associative rules (moving parentheses)

$$(\lambda\boldsymbol{b})_{|i} = \lambda(\boldsymbol{b}_{|i}) \qquad {}_{i|}(\boldsymbol{b}\lambda) = ({}_{i|}\boldsymbol{b})\lambda$$

$$(\lambda\mathbf{A})_{|j} = \lambda(\mathbf{A}_{|j}) \qquad {}_{i|}(\mathbf{A}\lambda) = ({}_{i|}\mathbf{A})\lambda$$

Scalar and operator interchange

$$(\boldsymbol{b}\lambda)_{|i} = (\boldsymbol{b}_{|i})\lambda \qquad {}_{i|}(\lambda\boldsymbol{b}) = \lambda({}_{i|}\boldsymbol{b})$$

$$(\mathbf{A}\lambda)_{|j} = (\mathbf{A}_{|j})\lambda \qquad {}_{i|}(\lambda\mathbf{A}) = \lambda({}_{i|}\mathbf{A})$$

Questions of the Lecture 1

You should always complete the exercises in the theoretical sections previous to each lecture

(L-1) Question 1. Give 3 by 3 examples (not just the zero matrix) of:

(a) A diagonal matrix: ${}_{i|}\mathbf{A}_{|j} = 0$ if $i \neq j$. (b) A symmetric matrix: $\mathbf{A}_{|j} = {}_{j|}\mathbf{A}$.
(c) An upper triangular matrix: ${}_{i|}\mathbf{A}_{|j} = 0$ if $i > j$. (d) A skew-symmetric matrix: ${}_{i|}\mathbf{A}_{|j} = -{}_{j|}\mathbf{A}_{|i}$.

(Strang, 1988, exercise 7 from section 1.4.)

1 Highlights of Lesson 2

Highlights of Lesson 2

- Dot product
- linear combinations
- The column picture of the Geometry of linear equations

2 Dot product

$$\boldsymbol{x}\cdot\boldsymbol{y} = x_1y_1 + x_2y_2 + x_3y_3 + \cdots + x_ny_n = \sum_{i=1}^{n} x_iy_i.$$

Symmetric

$$\boldsymbol{x}\cdot\boldsymbol{y} = \boldsymbol{y}\cdot\boldsymbol{x}$$

Linear in the first argument

$$(a\boldsymbol{x})\cdot\boldsymbol{y} = a(\boldsymbol{x}\cdot\boldsymbol{y})$$

$$(\boldsymbol{x}+\boldsymbol{y})\cdot\boldsymbol{z} = \boldsymbol{x}\cdot\boldsymbol{z} + \boldsymbol{y}\cdot\boldsymbol{z}$$

Positive

$$\boldsymbol{x}\cdot\boldsymbol{x} \geq 0$$

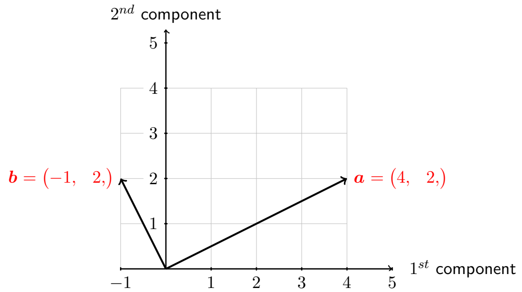
Definite

$$\boldsymbol{x}\cdot\boldsymbol{x} = 0 \Leftrightarrow \boldsymbol{x} = 0$$

3 Linear combinations

The sum of $x\boldsymbol{a}$ and $y\boldsymbol{b}$ is a linear combination of $\boldsymbol{a}$ and $\boldsymbol{b}$

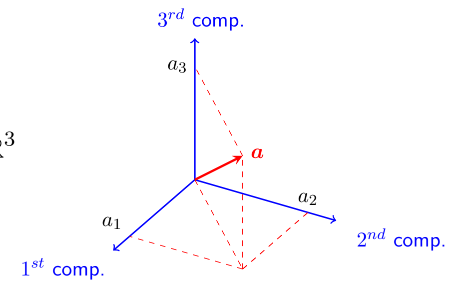
$$x\boldsymbol{a}+y\boldsymbol{b} = x\begin{pmatrix}4\\2\end{pmatrix}+y\begin{pmatrix}-1\\2\end{pmatrix} = \left[\boldsymbol{a};\ \boldsymbol{b};\right]\begin{pmatrix}x\\y\end{pmatrix} = \textbf{MATRIX}\times\boldsymbol{vector}$$



Is $\mathbf{0}$ a linear combination of $\boldsymbol{a}$ and $\boldsymbol{b}$? What is the picture of all linear combinations of $\boldsymbol{a}$ and $\boldsymbol{b}$?

4 Linear combinations in $\mathbb{R}^3$

$$\boldsymbol{a} = (a_1,\quad a_2,\quad a_3,)\ \in\ \mathbb{R}^3$$



What is the picture of all multiples of $\boldsymbol{a}$?
What is the picture of all linear combinations of two vectors in $\mathbb{R}^3$?
(linear combination)

5 Matrix times vector

$$\mathbf{A}\boldsymbol{b} = b_1\mathbf{A}_{|1} \quad + \quad b_2\mathbf{A}_{|2} \quad + \cdots + \ b_n\mathbf{A}_{|n}$$

$$= b_1\begin{pmatrix}a_{11}\\ \vdots\\ a_{i1}\\ \vdots\\ a_{m1}\end{pmatrix} + b_2\begin{pmatrix}a_{12}\\ \vdots\\ a_{i2}\\ \vdots\\ a_{m2}\end{pmatrix} + \cdots + b_n\begin{pmatrix}a_{1n}\\ \vdots\\ a_{in}\\ \vdots\\ a_{mn}\end{pmatrix} = \begin{pmatrix}({}_{1|}\mathbf{A})\cdot\boldsymbol{b}\\ \vdots\\ ({}_{i|}\mathbf{A})\cdot\boldsymbol{b}\\ \vdots\\ ({}_{n|}\mathbf{A})\cdot\boldsymbol{b}\end{pmatrix}$$

Hence,

$$\boxed{{}_{i|}(\mathbf{A}\boldsymbol{b}) = ({}_{i|}\mathbf{A})\cdot\boldsymbol{b}}$$

if we omit the period, we can simply write: ${}_{i|}\mathbf{A}\boldsymbol{b}$

6 Matrix times vector

If $\boldsymbol{b}\in\mathbb{R}^n$ then $(\underset{m\times n}{\mathbf{A}}\ \boldsymbol{b})\in\mathbb{R}^m$; where ${}_{i|}(\mathbf{A}\boldsymbol{b}) = ({}_{i|}\mathbf{A})\cdot\boldsymbol{b}$

Matrix times vector

1. $\mathbf{I}\boldsymbol{a} = \boldsymbol{a}$
2. $\mathbf{A}(\mathbf{I}_{|j}) = \mathbf{A}_{|j}$
3. $\mathbf{A}(\boldsymbol{b}+\boldsymbol{c}) = \mathbf{A}\boldsymbol{b}+\mathbf{A}\boldsymbol{c}$
4. $\mathbf{A}(\lambda\boldsymbol{b}) = \lambda(\mathbf{A}\boldsymbol{b})$
5. $\mathbf{A}(\lambda\boldsymbol{b}) = (\lambda\mathbf{A})\boldsymbol{b}$
6. $\mathbf{A}(\mathbf{B}\boldsymbol{c}) = \left[\mathbf{A}(\mathbf{B}_{|1});\ldots\ \ \mathbf{A}(\mathbf{B}_{|n});\right]\boldsymbol{c}$
7. $(\mathbf{A}+\mathbf{B})\boldsymbol{c} = \mathbf{A}\boldsymbol{c}+\mathbf{B}\boldsymbol{c}$

Prove the above propositions
(follow the rewriting rules and properties of the dot product)

7 Example of linear system: 2 equations and 2 unknowns

$$\begin{cases} 2x - \ \ y = 0 \\ -x + 2y = 3 \end{cases}$$

$$\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{pmatrix} \quad \\ \quad \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$$

$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$$

$$\underbrace{\mathbf{A}\boldsymbol{x}}_{\text{which linear combination}} \quad \underbrace{= \boldsymbol{b}}_{\text{equals this vector?}}$$

$$x\left(\mathbf{A}_{|1}\right) + y\left(\mathbf{A}_{|2}\right) \qquad = \boldsymbol{b}$$

8 Geometry of linear systems: Linear combination of columns

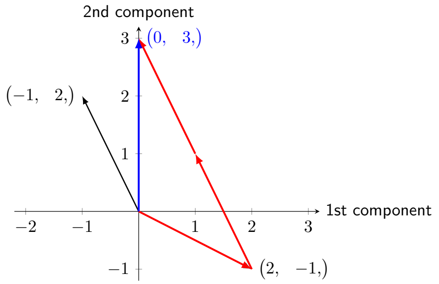
$$\begin{cases} 2x - \ \ y = 0 \\ -x + 2y = 3 \end{cases}$$

$$x\begin{pmatrix} \quad \\ \quad \end{pmatrix} + y\begin{pmatrix} \quad \\ \quad \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$$

9 2 equations and 2 unknowns: Column picture

$$x\begin{pmatrix} 2 \\ -1 \end{pmatrix} + y\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

Which linear combination of $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ gives $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$?



What is the set of all possible combinations?

10 Example: 3 equations and 3 unknowns

$$\begin{cases} 2x - \ \ y \qquad\ \ = 0 \\ -x + 2y - \ \ z = -1 \\ \qquad - 3y + 4z = 4 \end{cases}$$

$$\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}\begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$$

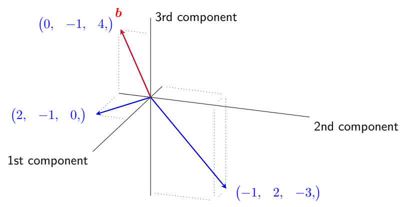
$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$$

$$\underbrace{\mathbf{A}\boldsymbol{x}}_{\text{which linear combination}} \quad \underbrace{= \boldsymbol{b}}_{\text{equals this vector?}}$$

11 3 equations and 3 unknowns: Column picture

$$x\begin{pmatrix}2\\-1\\0\end{pmatrix}+y\begin{pmatrix}-1\\2\\-3\end{pmatrix}+z\begin{pmatrix}0\\-1\\4\end{pmatrix}=\begin{pmatrix}0\\-1\\4\end{pmatrix}$$

Which linear combination of the columns gives $\boldsymbol{b}$?



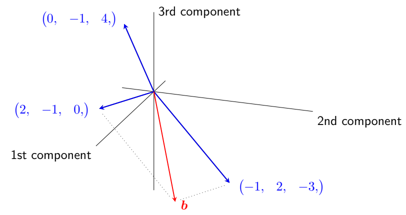
$$\left\{x= \quad ; \quad y= \quad ; \quad z= \quad \right\}$$

What happens with a different $\boldsymbol{b}$?... let's see

12 3 equations and 3 unknowns: Column picture

$$x\begin{pmatrix}2\\-1\\0\end{pmatrix}+y\begin{pmatrix}-1\\2\\-3\end{pmatrix}+z\begin{pmatrix}0\\-1\\4\end{pmatrix}=\begin{pmatrix}1\\1\\-3\end{pmatrix}$$

Which linear combination of the columns gives this new $\boldsymbol{b}$?



$$\left\{x= \quad ; \quad y= \quad ; \quad z= \quad \right\}$$

13 What does Ax=b mean?

$\mathbf{A}\boldsymbol{x}$ is a *linear combination* of columns of $\mathbf{A}$:

Example

$$\begin{bmatrix}2&5\\1&3\end{bmatrix}\begin{pmatrix}1\\2\end{pmatrix}=1\begin{pmatrix}2\\1\end{pmatrix}+2\begin{pmatrix}5\\3\end{pmatrix}=\begin{pmatrix}12\\7\end{pmatrix}$$

"$\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$" is asking for a particular linear combination:

$$\begin{bmatrix}2&5\\1&3\end{bmatrix}\begin{pmatrix}\quad\\\quad\end{pmatrix}=\begin{pmatrix}12\\7\end{pmatrix}$$

To solve linear systems we will first learn to transform coeficient matrices by elimination (next lectures)

Questions of the Lecture 2

You must complete the exercises from the corresponding sections of the book

(L-2) QUESTION 1. Working a column at a time, compute the following products

(a)

$$\begin{bmatrix}4&1\\4&1\\6&1\end{bmatrix}\begin{pmatrix}1\\3\end{pmatrix}$$

(b)

$$\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}$$

(c)

$$\begin{bmatrix}4&3\\6&6\\8&9\end{bmatrix}\begin{pmatrix}1/2\\1/3\end{pmatrix}$$

(Strang, 1988, exercise 2 from section 1.4.)

(L-2) QUESTION 2. Can the three equations be solved simultaneusly?

$$\begin{aligned} x+2y&=2\\ x-y&=2\\ y&=1. \end{aligned}$$

What happens if all right hand sides are zero? Is there any non-zero choice of right hand sides which allows the three equations to have a solution? How many non-zero choices have we?
(Strang, 1988, exercise 4 from section 1.2.)

(L-2) QUESTION 3. Compute the product $\mathbf{A}\boldsymbol{x}$ with
$\mathbf{A}=\begin{bmatrix}3&-6&0\\0&2&-2\\1&-1&-1\end{bmatrix}$ and $\boldsymbol{x}=\begin{pmatrix}2\\1\\1\end{pmatrix}$. For this matrix $\mathbf{A}$, find a solution vector $\boldsymbol{x}$ to the system $\mathbf{A}\boldsymbol{x}=\mathbf{0}$, with zeros on the right side of all three equations. Can you find more than one solution?
(Strang, 1988, exercise 5 from section 1.4.)

(L-2) QUESTION 4. Suppose $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$ has two solutions $\boldsymbol{v}$ and $\boldsymbol{w}$ (with $\boldsymbol{b}\neq\mathbf{0}$). Then show that $\frac{1}{2}(\boldsymbol{v}+\boldsymbol{w})$ is also a solution, although $\boldsymbol{v}+\boldsymbol{w}$ is not.

Hint

Use the following properties: $\mathbf{A}(\boldsymbol{b}+\boldsymbol{c})=\mathbf{A}\boldsymbol{b}+\mathbf{A}\boldsymbol{c}$ and $\mathbf{A}(c\boldsymbol{b})=c(\mathbf{A}\boldsymbol{b})$.

(L-2) QUESTION 5. *"It is impossible for a system of linear equations to have exactly two solutions"*. Explain why (answering the next question):

(a) If $\boldsymbol{v}$ y $\boldsymbol{w}$ are two solutions, what is another one?

(Strang, 2003, exercise 19 from section 2.2.)

(L-2) QUESTION 6. Draw $\boldsymbol{v}=\begin{pmatrix}2\\1\end{pmatrix}$ and $\boldsymbol{w}=\begin{pmatrix}1\\3\end{pmatrix}$, along with $\boldsymbol{v}+\boldsymbol{w}$, $2\boldsymbol{v}+\boldsymbol{w}$, and $\boldsymbol{v}-\boldsymbol{w}$ in a plane (first component on the horizontal axis and second component on the vertical axis).

(L-2) QUESTION 7. draw the column picture of the following system with solution $x=3$ and $y=-1$.

$$\begin{cases}2x+y=5\\x-3y=6\end{cases}$$

(L-2) QUESTION 8. draw the column picture of the following system.

$$\begin{cases}2x-y=3\\x+y=1\end{cases};\qquad\left(\text{the solution is :}\qquad x=1+\frac{1}{3},\quad y=-\frac{1}{3}\right).$$

No deje de hacer los ejercicios del libro.

1 Highlights of **Lesson 3**

Highlights of Lesson 3

- Matrix multiplication: $(\mathbf{AB})_{|j}=\mathbf{A}(\mathbf{B}_{|j})$
  - Properties
- Transpose of a matrix
- $\mathbf{A}\boldsymbol{x}$ and $\boldsymbol{x}\mathbf{A}$ (linear combinations)
- Other ways to compute the product
- Transpose of $\mathbf{AB}$

2 Matrix multiplication (by **columns**)

Column j of ($\underset{m\times p}{\mathbf{A}}$ times $\underset{p\times n}{\mathbf{B}}$) is:

$$\boxed{(\mathbf{AB})_{|j} = \mathbf{A}(\mathbf{B}_{|j})} \quad \longrightarrow \quad \mathbf{AB}_{|j}$$

Each column of **AB** is a linear combination of the p columns of **A**

Example

$$\begin{bmatrix} 2 & 1 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{A}(\mathbf{B}_{|1}); & \mathbf{A}(\mathbf{B}_{|2}); \end{bmatrix}$$

$$= \left[\begin{bmatrix} 2 & 1 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \begin{bmatrix} 2 & 1 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{pmatrix} 6 \\ 0 \end{pmatrix}; \right] = \begin{bmatrix} 3 & 12 \\ 11 & 18 \\ 13 & 24 \end{bmatrix}$$

3 Matrix multiplication properties

MATRIX $\times$ MATRIX $=$ MATRIX

1. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ remember $\mathbf{A}(\mathbf{B}\boldsymbol{c}) = \left[\mathbf{A}(\mathbf{B}_{|1}); \dots \; \mathbf{A}(\mathbf{B}_{|n});\right] \boldsymbol{c}$
2. $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$.
3. $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC}$.
4. $\mathbf{A}(\lambda\mathbf{B}) = (\lambda\mathbf{A})\mathbf{B} = \lambda(\mathbf{AB})$.
5. $\mathbf{IA} = \mathbf{A}$.
6. $\mathbf{AI} = \mathbf{A}$.

4 Transposing a matrix

Transpose

$$(\text{column } i \text{ of } \mathbf{A}^\intercal) = (\text{row } i \text{ of } \mathbf{A}) \quad \leftrightarrow \quad (\mathbf{A}^\intercal)_{|i} = {}_{i|}\mathbf{A}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 4 & 1 \end{bmatrix}; \quad \mathbf{A}^\intercal =$$

$${}_{i|}\mathbf{A}_{|j} = {}_{j|}(\mathbf{A}^\intercal)_{|i}; \qquad (\mathbf{A}^\intercal)^\intercal = \mathbf{A}; \qquad {}_{j|}(\mathbf{A}^\intercal) = \mathbf{A}_{|j}$$

Symmetric matrices $\quad \mathbf{A}^\intercal = \mathbf{A}$

$$\begin{bmatrix} 3 & 1 & 7 \\ & 2 & 9 \\ & & 1 \end{bmatrix}$$

5 Vectors, row matrices, column matrices

$$(1, \; 3, \; -10,) = \begin{pmatrix} 1 \\ 3 \\ -10 \end{pmatrix}; \quad \text{but} \quad \begin{bmatrix} 1 & 3 & -10 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 3 \\ -10 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}; \quad {}_{2|}\mathbf{A} = (2, \; 3,) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}; \quad \mathbf{A}_{|1} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = (1, \; 2, \; 4,)$$

When writting vectors between "square brackets" we get a matrix whose columns are those vectors

$$\left[{}_{3|}\mathbf{A}; \; {}_{1|}\mathbf{A};\right] = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}; \qquad \mathbf{A}^\intercal = \left[{}_{1|}\mathbf{A}; \; {}_{2|}\mathbf{A}; \; {}_{3|}\mathbf{A};\right]$$

6 Linear combination of rows and columns

Linear combination of columns

$$\begin{bmatrix} \diamondsuit & \clubsuit \\ \heartsuit & \spadesuit \\ \diamondsuit & \clubsuit \end{bmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3 \begin{pmatrix} \diamondsuit \\ \heartsuit \\ \diamondsuit \end{pmatrix} + 4 \begin{pmatrix} \clubsuit \\ \spadesuit \\ \clubsuit \end{pmatrix}$$

$$\textbf{MATRIX} \times \boldsymbol{vector} = \boldsymbol{vector}$$

Linear combination of rows

$$(1, \ 2, \ 7,) \begin{bmatrix} \diamondsuit & \clubsuit \\ \heartsuit & \spadesuit \\ \diamondsuit & \clubsuit \end{bmatrix} = 1\,(\diamondsuit, \ \clubsuit,) + 2\,(\heartsuit, \ \spadesuit,) + 7\,(\diamondsuit, \ \clubsuit,)$$

$$\boldsymbol{vector} \times \textbf{MATRIX} = \boldsymbol{vector}$$

Linear combinations $\longrightarrow$ $\boxed{\boldsymbol{a}\mathbf{B} = (\mathbf{B}^\mathsf{T})\boldsymbol{a}}$

7 Vector times matrix

Remember that $\quad {}_{i|}(\mathbf{A}\boldsymbol{b}) = ({}_{i|}\mathbf{A}) \cdot \boldsymbol{b}; \quad$ hence

$$(\boldsymbol{a}\mathbf{B})_{|j} = {}_{j|}(\boldsymbol{a}\mathbf{B}) = {}_{j|}\Big((\mathbf{B}^\mathsf{T})\boldsymbol{a}\Big) = \Big({}_{j|}(\mathbf{B}^\mathsf{T})\Big) \cdot \boldsymbol{a} = (\mathbf{B}_{|j}) \cdot \boldsymbol{a} = \boldsymbol{a} \cdot (\mathbf{B}_{|j})$$

Rewriting rules

$\boxed{{}_{i|}(\mathbf{A}\boldsymbol{b}) = ({}_{i|}\mathbf{A}) \cdot \boldsymbol{b}}$ and $\boxed{(\boldsymbol{a}\mathbf{B})_{|j} = \boldsymbol{a} \cdot (\mathbf{B}_{|j})}$

Thus, if we omit the period, we can simply write:

$${}_{i|}\mathbf{A}\boldsymbol{b} \quad \text{and} \quad \boldsymbol{a}\mathbf{B}_{|j}$$

8 Matrix multiplication: **rows times columns**

Consider $\underset{m\times p}{\mathbf{A}}$ and $\underset{p\times n}{\mathbf{B}}$, then:

$$\boxed{{}_{i|}(\mathbf{AB})_{|j} = ({}_{i|}\mathbf{A}) \cdot (\mathbf{B}_{|j})}$$

Proof.

Remember that $\quad {}_{i|}(\mathbf{A}\boldsymbol{b}) = ({}_{i|}\mathbf{A}) \cdot \boldsymbol{b}, \quad$ hence

$${}_{i|}(\mathbf{AB})_{|j} = {}_{i|}\Big((\mathbf{AB})_{|j}\Big) = {}_{i|}\Big(\mathbf{A}(\mathbf{B}_{|j})\Big) = ({}_{i|}\mathbf{A}) \cdot (\mathbf{B}_{|j})$$

□

Thus, if we omit the period, we can simply write:

$${}_{i|}\mathbf{AB}_{|j}$$

9 Matrix multiplication (**by rows**)

Consider $\underset{m\times p}{\mathbf{A}}$ and $\underset{p\times n}{\mathbf{B}}$, then:

$$\boxed{{}_{i|}(\mathbf{AB}) = ({}_{i|}\mathbf{A})\mathbf{B}}$$

Proof.

Let's see the jth components are equal:

$$\Big({}_{i|}(\mathbf{AB})\Big)_{|j} = {}_{i|}\Big((\mathbf{AB})_{|j}\Big) = ({}_{i|}\mathbf{A}) \cdot (\mathbf{B}_{|j}) = \Big(({}_{i|}\mathbf{A})\mathbf{B}\Big)_{|j}$$

so ${}_{i|}(\mathbf{AB}) = ({}_{i|}\mathbf{A})\mathbf{B}$. □

Thus, we can simply write:

$${}_{i|}\mathbf{AB}$$

10 Matrix multiplication (by rows)

Each row of **AB** is a linear combination of the p rows of **B**

$$\begin{bmatrix}2&1\\3&8\\4&9\end{bmatrix}\begin{bmatrix}1&6\\1&0\end{bmatrix}=\begin{bmatrix}3&12\\11&18\\13&24\end{bmatrix}\quad\text{where}\quad\begin{cases}\begin{pmatrix}2, & 1,\end{pmatrix}\begin{bmatrix}1&6\\1&0\end{bmatrix}=\begin{pmatrix}3, & 12,\end{pmatrix}\\[2ex]\begin{pmatrix}3, & 8,\end{pmatrix}\begin{bmatrix}1&6\\1&0\end{bmatrix}=\begin{pmatrix}11, & 18,\end{pmatrix}\\[2ex]\begin{pmatrix}4, & 9,\end{pmatrix}\begin{bmatrix}1&6\\1&0\end{bmatrix}=\begin{pmatrix}13, & 24,\end{pmatrix}\end{cases}$$

11 Transposing a product of matrices

Since

- $(\mathbf{A}^\intercal)_{|j} = {}_{j|}\mathbf{A}$
- $\boldsymbol{a}\mathbf{B} = (\mathbf{B}^\intercal)\boldsymbol{a}$

it follows that:

$$\boxed{(\mathbf{AB})^\intercal = (\mathbf{B}^\intercal)(\mathbf{A}^\intercal)}$$

Proof.

$$(\mathbf{AB})^\intercal{}_{|j} \;=\; {}_{j|}\mathbf{AB} \;=\; (\mathbf{B}^\intercal)\big({}_{j|}\mathbf{A}\big) \;=\; (\mathbf{B}^\intercal)(\mathbf{A}^\intercal)_{|j}.$$

□

Matrix times its transpose is always symmetric

Librería nacal para Python

Revise la implementación de las operaciones del álgebra matricial en la librería nacal para Python que acompaña al curso: Sección 1.3 de la documentación (o estudie directamente el código).

`https://github.com/mbujosab/nacallib`

Verá que el código es una traducción literal de las *definiciones* vistas aquí; pero que no hay ni una línea de código que describa las propiedades que hemos demostrado en estas tres lecciones. ¡No es necesario! Las definiciones implican las propiedades (como hemos comprobado teóricamente con las demostraciones de estas lecciones). Verifique con ejemplos que todas las propiedades se cumplen. Estudie los notebooks de Jupyter correspondientes a las tres primeras lecciones.

Questions of the Lecture 3

No deje de hacer los ejercicios del libro.

(L-3) Question 1. Multiply these matrices in the orders **EF**, **FE** and $\mathbf{E}^2$

$$\mathbf{E}=\begin{bmatrix}1&0&0\\a&1&0\\b&0&1\end{bmatrix};\qquad \mathbf{F}=\begin{bmatrix}1&0&0\\0&1&0\\0&c&1\end{bmatrix}.$$

(Strang, 1988, exercise 34 from section 1.4.)

(L-3) Question 2. True or false; give a specific counterexample when false.

(a) If the first and third columns of **B** are the same, so are the first and third columns of **AB**.
(b) If the first and third rows of **B** are the same, so are the first and third rows of **AB**.
(c) If the first and third rows of **A** are the same, so are the first and third rows of **AB**.
(d) $(\mathbf{AB})^2 = \mathbf{A}^2\mathbf{B}^2$.

(Strang, 1988, exercise 10 from section 1.4.)

(L-3) QUESTION 3. Consider the vectors
$\boldsymbol{a} = (1, \quad -2, \quad 7,)$ y $\boldsymbol{b} = (3, \quad 5, \quad 1,)$. Compute the following products

(a) $\boldsymbol{a} \cdot \boldsymbol{a}$ (b) $\boldsymbol{a} \cdot \boldsymbol{b}$ (c) $[\boldsymbol{a}]\,[\boldsymbol{b}]^{\intercal}$

(Strang, 1988, exercise 3 from section 1.4.)

(L-3) QUESTION 4. Write down the 2 by 2 matrices **A** and **B** that have entries $a_{ij} = i + j$ and $b_{ij} = (-1)^{i+j}$. Multiply them to find **AB** and **BA**.
(Strang, 1988, exercise 6 from section 1.4.)

(L-3) QUESTION 5. The product of two lower triangular matrices is again lower triangular (all its entries above the main diagonal are zero). Confirm this with a 3 by 3 example, and then explain how it follows from the laws of matrix multiplication.
(Strang, 1988, exercise 12 from section 1.4.)

(L-3) QUESTION 6. consider the matrices **A**, **B**, **C**, **D**, **E** and **F** .

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & 4 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

Compute (*in particular, ¡note that* $\mathbf{EF} \neq \mathbf{FE}$*!*)

(a) $\mathbf{B} + \mathbf{D}$ (b) $2\mathbf{E} - \mathbf{F}$ (c) **AC**
(d) **BC** (e) **CB** (f) **ACD**
(g) **EF** (h) **FE** (i) **CEF**

Strang, G. (1988). *Linear algebra and its applications*. Thomson Learning, Inc., third ed. ISBN 0-15-551005-3.

Strang, G. (2003). *Introduction to Linear Algebra*. Wellesley-Cambridge Press, Wellesley, Massachusetts. USA, third ed. ISBN 0-9614088-9-8.