

Mathematics II

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1 Highlights of Lesson 4

Highlights of Lesson 4

- Elementary transformations
- Identifying singular matrices by elimination
- Matrix multiplication of Elementary matrices

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2 Elementary transformations of a matrix

Type I: $\mathbf{A}_{[(\lambda)\mathbf{i}+j]}$ (with $i \neq j$)

add λ times i -th column ($\lambda \mathbf{A}_{|i}$) to j -th column ($\mathbf{A}_{|j}$)

$$\begin{bmatrix} 1 & -3 & 0 \\ 1 & -6 & 3 \end{bmatrix}_{[(-2)\mathbf{1}+3]} = \begin{bmatrix} 1 & -3 & -2 \\ 1 & -6 & 1 \end{bmatrix}$$

Type II: $\mathbf{A}_{[(\alpha)\mathbf{i}]}$ (with $\alpha \neq 0$)

multiply by α the i -th column

$$\begin{bmatrix} 1 & -3 & 0 \\ 1 & -6 & 3 \end{bmatrix}_{[(10)\mathbf{2}]} = \begin{bmatrix} 1 & -30 & 0 \\ 1 & -60 & 3 \end{bmatrix}$$

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3 Elimination and pre-echelon form of a matrix

- **Pivot** is the first non-zero component of each column.
- **Elimination**: modifies a matrix until all **components at the right-hand side of each pivot are zeros**

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 8 & 4 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{[(-3)\tau_1+2]} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix} \xrightarrow{[(-2)\tau_2+3]} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & -2 & 5 \end{bmatrix} = \mathbf{L}$$

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4 Elimination

Elimination algorithm on \mathbf{A}

modifies \mathbf{A} using a sequence of *elementary transformations*

Goal

to get a (pre)echelon form

- **pre-echelon**: all components on the right side of each pivot are zero.
- **echelon**: if any column before a non-null column $\mathbf{A}_{|j}$ is non-null column and its pivot is above the pivot of $\mathbf{A}_{|j}$.

It is always possible to find a (pre)echelon form by elimination

Rank (rg): the number of pivots in any of its pre-echelon forms

\mathbf{A} is **singular** if its pre-echelon forms have null-columns (rg < n)
 $n \times n$

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5 Elimination: When can't we find n pivots?

$n \times n$ matrices are **singular** if we can't find n pivots \rightarrow ☹️

$$\begin{bmatrix} 0 & 1 & 3 \\ 4 & 2 & 8 \\ 1 & 1 & 1 \end{bmatrix}$$

Has this matrix n pivots? $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ 1 & 1 & 1 \end{bmatrix}$

and this one? $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

and this one? $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & -4 \end{bmatrix}$

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6 Matrix multiplication: elementary matrices

$$\underbrace{\begin{bmatrix} 1 & 3 & 0 \\ 2 & 8 & 4 \\ 1 & 1 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\left[\begin{array}{c} \\ \\ \end{array} \right]}_{\mathbf{I}_\tau} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix}}_{\mathbf{A}_\tau}$$

We call \mathbf{I}_τ "Elementary matrix":

$$\mathbf{A}(\mathbf{I}_\tau) = \mathbf{A}_\tau$$

This specific elementary matrix \mathbf{I}_τ is written as $\mathbf{I}_{\tau, [(-3)\tau_1+2]}$

$$\mathbf{A}(\mathbf{I}_{\tau, [(-3)\tau_1+2]}) = \mathbf{A}_{\tau, [(-3)\tau_1+2]}$$

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7 Matrix multiplication: elementary matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & -2 & 5 \end{bmatrix}$$

This specific elementary matrix \mathbf{I}_τ is written as $\mathbf{I}_{\tau_{[(-2)2+3]}}$

$$\mathbf{A} \left(\mathbf{I}_{\tau_{[(-2)2+3]}} \right) = \mathbf{A}_{\tau_{[(-2)2+3]}}$$

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8 Elimination by elementary matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 8 & 4 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{[(-3)1+2]} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix} \xrightarrow{[(-2)2+3]} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & -2 & 5 \end{bmatrix} = \mathbf{L}$$

$$\mathbf{A}_{\tau_{\begin{bmatrix} [(-3)1+2] \\ [(-2)2+3] \end{bmatrix}}} = \mathbf{A}_{\begin{bmatrix} [(-3)1+2] \\ [(-2)2+3] \end{bmatrix}} = \left(\mathbf{A} \left(\mathbf{I}_{\tau_{[(-3)1+2]}} \right) \right) \left(\mathbf{I}_{\tau_{[(-2)2+3]}} \right) = \mathbf{L}$$

there is a matrix that does the whole job **at once**

$$\mathbf{A}_{\tau_{\begin{bmatrix} [(-3)1+2] \\ [(-2)2+3] \end{bmatrix}}} = \mathbf{A} \left(\left(\mathbf{I}_{\tau_{[(-3)1+2]}} \right) \left(\mathbf{I}_{\tau_{[(-2)2+3]}} \right) \right) = \mathbf{A} \mathbf{I}_{\tau_{\begin{bmatrix} [(-3)1+2] \\ [(-2)2+3] \end{bmatrix}}} = \mathbf{L}$$

$$\mathbf{A}_{\tau_1 \dots \tau_k} = \mathbf{A} \left(\mathbf{I}_{\tau_1 \dots \tau_k} \right)$$

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9 how do I get from \mathbf{L} back to \mathbf{A} ? Inverses

How do I reverse the first step? (it was subtract 3 times \mathbf{A}_{11} from \mathbf{A}_{12})

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{I}_{\tau_{[(-\lambda)i+j]}} \text{ "undo" } \mathbf{I}_{\tau_{[(\lambda)i+j]}}$$

How to undo $\mathbf{I}_{\tau_{[(\alpha)i]}}$?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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10 Interchange or swap matrices

Which matrix exchanges the columns?

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} \\ \end{bmatrix} = \begin{bmatrix} c & a \\ d & b \end{bmatrix}$$

Which matrix exchanges the rows? where do we put that matrix?

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} b & d \\ a & c \end{bmatrix}$$

Matrix multiplication is not commutative!

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11 Interchange of columns

Interchange of columns:

$\mathbf{A}_{\tau_{[i \rightleftharpoons j]}}$ → swicht columns i and j of \mathbf{A}

$$\begin{bmatrix} 1 & -3 & 0 \\ 1 & -6 & 3 \end{bmatrix}_{\tau_{[2 \rightleftharpoons 3]}} = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 3 & -6 \end{bmatrix}$$

We can switch two columns by a sequence of elementary transformations

Matrix $\mathbf{I}_{\tau_{[i \rightleftharpoons j]}}$ is call a exchange matrix

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12 Permutation matrices

Product between exchange matrices $\mathbf{I}_{\tau_{[i \rightleftharpoons j]}}$ is a permutation matrix $\mathbf{I}_{\tau_{[\mathfrak{S}]}}$.

$\mathbf{I}_{\tau_{[\mathfrak{S}]}}$ = Identity matrix \mathbf{I} with rearranged columns

Let's see the 3×3 case

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad \mathbf{I}_{\tau_{[1 \rightleftharpoons 2]}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

How many 3×3 pemutations can we find?

what happens if \mathbf{I} multiply two permutation matrices?

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Questions of the Lecture 4

(L-4) QUESTION 1.

(a) Which three matrices $\mathbf{I}_{\tau_{[(x)1+2]}}$, $\mathbf{I}_{\tau_{[(y)1+3]}}$ and $\mathbf{I}_{\tau_{[(z)2+3]}}$ put $\mathbf{A} = \begin{bmatrix} 1 & 4 & -2 \\ 1 & 6 & 2 \\ 0 & 1 & 0 \end{bmatrix}$

into an echelon form?

(b) Multiply those \mathbf{I}_{τ_i} to get one matrix \mathbf{E} that does elimination: $\mathbf{AE} = \mathbf{K}$.

Based on (Strang, 1988, exercise 24 from section 1.4.)

(L-4) QUESTION 2. Consider the matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ -1 & -3 & -2 \\ 0 & 1 & c \end{bmatrix}$$

For what value(s) of c the matrix is singular (we can't find three pivots)?

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(L-4) QUESTION 3. Consider the following 3 by 3 matrices.

(a) $(\mathbf{I}_{\tau_{[(-1)1+2]}})$ subtracts column 1 from column 2 and then $(\mathbf{I}_{\tau_{[2 \rightleftharpoons 3]}})$ exchanges columns 2 and 3. What matrix \mathbf{E} does both steps at once?

(b) $(\mathbf{I}_{\tau_{[2 \rightleftharpoons 3]}})$ exchanges columns 2 and 3 and then $(\mathbf{I}_{\tau_{[(-1)1+3]}})$ subtracts column 1 from column 3. What matrix $\mathbf{N} = (\mathbf{I}_{\tau_{[2 \rightleftharpoons 3]}})(\mathbf{I}_{\tau_{[(-1)1+3]}})$ does both steps at once?

Explain why \mathbf{M} and \mathbf{N} are the same but the \mathbf{I}_{τ} 's are different.

Based on (Strang, 1988, exercise 28 from section 1.4.)

(L-4) QUESTION 4. Elimination matrices $\mathbf{I}_{\tau_{[(\cdot)1+2]}}$ and $\mathbf{I}_{\tau_{[(\cdot)2+3]}}$ will reduce \mathbf{A} to triangular form. Find \mathbf{E} so that $\mathbf{AE} = \mathbf{L}$ is lower triangular (echelon), if \mathbf{A} is

$$\begin{bmatrix} 2 & 2 & 0 \\ 1 & 4 & 9 \\ 1 & 3 & 9 \end{bmatrix}$$

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(L-4) QUESTION 5. Although we will only consider as elementary the *Type I* and *II* transformations, in most of the Linear Algebra books appears a third type: the *exchange* of columns

$$\mathbf{A} \begin{smallmatrix} \tau \\ [p \rightleftharpoons s] \end{smallmatrix} \rightarrow \text{Exchanges columns } p \text{ and } s \text{ of } \mathbf{A}.$$

Prove that a column exchange is, in fact, a sequence of *Type I* and *II* elementary transformations. Try transforming $\mathbf{I}_{2 \times 2}$ in $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ by elementary transformations of the columns.

(L-4) QUESTION 6. Write down the 3 by 3 matrices that produce these elimination steps:

- (a) $\mathbf{I} \begin{smallmatrix} \tau \\ [(-5)1+2] \end{smallmatrix}$ subtracts 5 times column 1 from column 2,
 (b) $\mathbf{I} \begin{smallmatrix} \tau \\ [(-7)2+3] \end{smallmatrix}$ subtracts 7 times column 2 from column 3,
 (c) $\mathbf{I} \begin{smallmatrix} \tau \\ [\ominus] \end{smallmatrix}$ exchanges columns 1 and 2, and then columns 2 and 3.

(Strang, 2003, exercise 1 from section 2.3.)

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(L-4) QUESTION 10. If every column of \mathbf{A} is a multiple of $(1, 1, 1)$, then $\mathbf{A}\mathbf{x}$ is always a multiple of $(1, 1, 1)$. Do a 3 by 3 example. How many pivots are produced by elimination?
 (Strang, 1988, exercise 26 from section 1.4.)

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(L-4) QUESTION 7. Consider the matrices of QUESTION 6:

(a) when multiplying by $\mathbf{I} \begin{smallmatrix} \tau \\ [(-5)1+2] \end{smallmatrix}$ and then by $\mathbf{I} \begin{smallmatrix} \tau \\ [(-7)2+3] \end{smallmatrix}$ the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ we get $\mathbf{A} \begin{smallmatrix} \tau \\ [(-5)1+2] \\ [(-7)2+3] \end{smallmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}$.

(b) But, when multiplying by $\mathbf{I} \begin{smallmatrix} \tau \\ [(-5)1+2] \end{smallmatrix}$ before and then by $\mathbf{I} \begin{smallmatrix} \tau \\ [(-7)2+3] \end{smallmatrix}$ we get

$$\mathbf{A} \begin{smallmatrix} \tau \\ [(-7)2+3] \\ [(-5)1+2] \end{smallmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}.$$

(c) When $\mathbf{I} \begin{smallmatrix} \tau \\ [(-7)2+3] \end{smallmatrix}$ comes first, the column ____ feels no effect from column ____.

This property will become very important in the LU factorization!

(Strang, 2003, exercise 2 from section 2.3.)

(L-4) QUESTION 8. What matrix \mathbf{M} sends $\mathbf{v} = (1, 0)$ to $(0, 1)$, es decir $\mathbf{vM} = (0, 1)$; and also sends $\mathbf{w} = (0, 1)$ to $(1, 0)$, es decir $\mathbf{wM} = (1, 0)$?

(L-4) QUESTION 9. Consider a permutation (interchange) matrix $\mathbf{I} \begin{smallmatrix} \tau \\ [i \rightleftharpoons j] \end{smallmatrix}$, if we compute the product $\mathbf{A}(\mathbf{I} \begin{smallmatrix} \tau \\ [i \rightleftharpoons j] \end{smallmatrix})$, we get a new matrix like \mathbf{A} , but with exchanged columns. What happen if we compute the product $(\mathbf{I} \begin{smallmatrix} \tau \\ [i \rightleftharpoons j] \end{smallmatrix})\mathbf{A}$? Check your answer with a 2 by 2 example.

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1 Highlights of Lesson 5

Highlights of Lesson 5

- Inverse of \mathbf{A}
- Gauss-Jordan elimination / finding \mathbf{A}^{-1}
- Inverse of \mathbf{AB} , \mathbf{A}^T

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2 Inverse of a matrix (square matrices)

A **squared** of order n has inverse (is *invertible*) if exists **B** such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}.$$

Then

$$\mathbf{B} = \mathbf{A}^{-1} \quad \text{and} \quad \mathbf{A} = \mathbf{B}^{-1}.$$

Not all matrices have inverse

Squared matrices with no inverse are called *singular* matrices

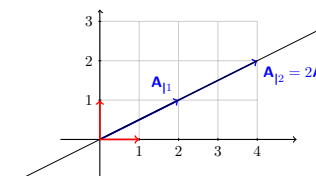
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3 Singular case (no inverse)

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

Is it possible to find a matrix **B** such that $\mathbf{AB} = \mathbf{I}$?

... columns of **I** should be linear combinations of columns of **A**... but both columns lie on the same line.



So

A is singular

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4 Singular case (no inverse)

Can we find $x \neq 0$ such that $\mathbf{Ax} = 0$?

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If $\mathbf{Ax} = 0$ and $x \neq 0 \Rightarrow$ there is no \mathbf{A}^{-1}

The existence of \mathbf{A}^{-1} leads to a **contradiction**

If $\mathbf{Ax} = 0$ and $x \neq 0 \Rightarrow \mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}0 \Rightarrow x = 0.$

When \mathbf{A}^{-1} does exist

the **only** solution to $\mathbf{Ax} = 0$ is $x = 0.$

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5 Calculating the inverse matrix

$$\mathbf{A}(\mathbf{A}^{-1}) = \mathbf{I}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So... we are solving m systems (of m equations each)

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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6 Gauss-Jordan: solving two linear systems at once

Gauss-Jordan elimination (obtaining a reduced echelon form **R**)

apply elementary transformations until a echelon matrix with only zeros to the left of each pivot (and all pivots equal to 1) is achieved

Let's solve the linear systems

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

applying Gauss-Jordan elimination on **A** stacked with **I**

$$\left[\begin{array}{c} \mathbf{A} \\ \mathbf{I} \end{array} \right] = \begin{bmatrix} 1 & 3 \\ 2 & 7 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \quad \rightarrow \quad =$$

If **R** = **I**, we have found **A**⁻¹

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8 Inverse of a product

When **A** and **B**, of order *n*, are invertible, (**AB**) is invertible.

what matrix gives me the inverse of **AB**? let's try with (**B**⁻¹**A**⁻¹):

$$\mathbf{AB}(\mathbf{B}^{-1}\mathbf{A}^{-1}) =$$

$$(\mathbf{B}^{-1}\mathbf{A}^{-1})\mathbf{AB} =$$

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7 Gauss-Jordan: Why does it work?

$$\left[\begin{array}{c} \mathbf{A} \\ \mathbf{I} \end{array} \right] = \begin{bmatrix} 1 & 3 \\ 2 & 7 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{[(-3)\mathbf{I}_1 + \mathbf{I}_2]} \quad \xrightarrow{[(-2)\mathbf{I}_2 + \mathbf{I}_1]}$$

that is, since $\mathbf{A}_{\tau_1 \dots \tau_k} = \mathbf{A}(\mathbf{I}_{\tau_1 \dots \tau_k})$:

$$\left[\begin{array}{c} \mathbf{A} \\ \mathbf{I} \end{array} \right]_{\tau_1 \dots \tau_k} = \left[\begin{array}{c} \mathbf{A}_{\tau_1 \dots \tau_k} \\ \mathbf{I}_{\tau_1 \dots \tau_k} \end{array} \right] = \left[\begin{array}{c} \mathbf{A}(\mathbf{I}_{\tau_1 \dots \tau_k}) \\ \mathbf{I}_{\tau_1 \dots \tau_k} \end{array} \right] = \left[\begin{array}{c} \mathbf{I} \\ \mathbf{I}_{\tau_1 \dots \tau_k} \end{array} \right],$$

who is $\mathbf{I}_{\tau_1 \dots \tau_k}$?

therefore **A**⁻¹ =

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9 Inverse of a transpose matrix

$$\mathbf{AA}^{-1} = \mathbf{I}$$

let me transpose both sides

$$((\mathbf{A}^{-1})^T) \mathbf{A}^T = \mathbf{I}$$

then

the inverse of **A**^T is

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10 Interchanges and permutations

Are interchange matrices $\mathbf{I}_{\tau_{[i \rightleftharpoons j]}}$, invertible?

It is easy to check that

$$\left(\mathbf{I}_{\tau_{[i \rightleftharpoons j]}}\right)^T \left(\mathbf{I}_{\tau_{[i \rightleftharpoons j]}}\right) = \mathbf{I} \quad \Rightarrow$$

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Questions of the Lecture 5

(L-5) QUESTION 1. Use the Gauss-Jordan method to invert

(a) $\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) $\mathbf{A}_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

(c) $\mathbf{A}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

(Strang, 1988, exercise 6 from section 1.6.)

(L-5) QUESTION 2.

(a) If \mathbf{A} is invertible and $\mathbf{AB} = \mathbf{AC}$, prove quickly that $\mathbf{B} = \mathbf{C}$.

(b) If $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, find an example with $\mathbf{AB} = \mathbf{AC}$, but $\mathbf{B} \neq \mathbf{C}$.

(Strang, 1988, exercise 4 from section 1.6.)

(L-5) QUESTION 3. Use the Gauss-Jordan method to invert the generic matrix 2×2

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The matrix is invertible (not singular) only when ...

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11 Caracterización de invertible matrices

Given \mathbf{A} of order n , the following statements are equivalent

1. No zero columns in $\mathbf{A}_{\tau_1 \dots \tau_p} = \mathbf{K}$ (pre-echelon matrix).
2. \mathbf{A} has inverse.
3. \mathbf{A} is product of elementary matrices.

$$\mathbf{A}_{\tau_1 \dots \tau_k} = \mathbf{A}(\mathbf{I}_{\tau_1 \dots \tau_k}) = \mathbf{I} \quad \Rightarrow \quad \mathbf{A} = (\mathbf{I}_{\tau_1 \dots \tau_k})^{-1}$$

where

$$(\mathbf{I}_{\tau_1 \dots \tau_k})^{-1} = ((\mathbf{I}_{\tau_1}) \cdots (\mathbf{I}_{\tau_k}))^{-1} = (\mathbf{I}_{\tau_k^{-1}}) \cdots (\mathbf{I}_{\tau_1^{-1}}) = \mathbf{I}_{\tau_k^{-1} \dots \tau_1^{-1}}$$

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(L-5) QUESTION 4. Use the Gauss-Jordan method to invert the following matrices.

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 6 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 4 & -2 \\ 1 & 3 & 1 \end{bmatrix}$$

(L-5) QUESTION 5. If the 3 by 3 matrix \mathbf{A} has $\mathbf{A}_{|1} + \mathbf{A}_{|2} = \mathbf{A}_{|3}$, show that \mathbf{A} is not invertible, by two different methods:

(a) Find a nonzero solution \mathbf{x} to $\mathbf{Ax} = \mathbf{0}$.

(b) Elimination keeps *column 1 + column 2 = column 3*. Explain why there is no third pivot.

(Strang, 1988, exercise 26 from section 1.6.)

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(L-5) QUESTION 6. Find the inverses of

$$(a) \mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}.$$

$$(b) \mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}.$$

$$(c) \mathbf{A}_3 = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}.$$

(Strang, 1988, exercise 10 from section 1.6.)

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(L-5) QUESTION 7. Find the inverse of

$$\mathbf{A} = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}$$

What values of a and b make the matrix singular?
(Strang, 1988, exercise 42 from section 1.6.)

(L-5) QUESTION 8. Find \mathbf{E}^2 , \mathbf{E}^8 and \mathbf{E}^{-1} if $\mathbf{E} = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$
(Strang, 1988, exercise 6 from section 1.5.)

(L-5) QUESTION 9. Consider the following permutation matrix:

$$\mathbf{I}_{\tau_{[\mathfrak{S}]}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Find $\mathbf{I}_{\tau_{[\mathfrak{S}]}}^{-1}$. Can you say something else about the relationship between $\mathbf{I}_{\tau_{[\mathfrak{S}]}}$ and $\mathbf{I}_{\tau_{[\mathfrak{S}]}^{-1}}$?

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(L-5) QUESTION 10. The 3 by 3 matrix \mathbf{A} reduces to the identity matrix \mathbf{I} by the following three column operations (in order):

$\tau_{[(-4)1+2]}$: Subtract 4 times column 1 from column 2.

$\tau_{[(-3)1+3]}$: Subtract 3 times column 1 from column 3.

$\tau_{[(-1)3+2]}$: Subtract column 3 from column 2.

(a) Write \mathbf{A}^{-1} in terms of elementary matrices \mathbf{I}_{τ} . Then compute \mathbf{A}^{-1} .

(b) What is the original matrix \mathbf{A} ?

(Based on MIT Course 18.06 Quiz 1, October 4, 2006)

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(L-5) QUESTION 11. The 3 by 3 matrix \mathbf{A} reduces to the identity matrix \mathbf{I} by the following three row operations (in order):

$\tau_{[(-4)1+2]}$: Subtract 4 times row 1 from row 2.

$\tau_{[(-3)1+3]}$: Subtract 3 times row 1 from row 3.

$\tau_{[(-1)3+2]}$: Subtract row 3 from row 2.

(a) Write \mathbf{A}^{-1} in terms of the \mathbf{E} 's. Then compute \mathbf{A}^{-1} .

(b) What is the original matrix \mathbf{A} ?

(MIT Course 18.06 Quiz 1, October 4, 2006)

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(L-5) QUESTION 12.

(a) Find the inverse of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(b) Find the inverse of the following matrix **using the Gauss-Jordan method**

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & d \end{bmatrix}$$

(Poole, 2004, exercise 36, 38 and 59 from section 3.3.)

(L-5) QUESTION 13. Consider the squared matrices **A**, **B**, and **C**. True or false?

(a) If $\mathbf{AB} = \mathbf{I}$ and $\mathbf{CA} = \mathbf{I}$ then $\mathbf{B} = \mathbf{C}$.

(b) $(\mathbf{AB})^2 = \mathbf{A}^2\mathbf{B}^2$.

(L-5) QUESTION 14. Consider the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & a & 0 & 2a \\ a & 0 & 1 & 0 \\ 1 & 0 & a & 1 \end{bmatrix}$

(a) Prove that **A** is invertible for any value of a .

(b) Compute \mathbf{A}^{-1} when $a = 0$.

(L-5) QUESTION 15. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$. Find \mathbf{A}^{-1} .

(L-5) QUESTION 16. Find (if it is possible) the inverse of the following inverses

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}.$$

(L-5) QUESTION 17. There is a finite number $(n!)$ of $n \times n$ permutation matrices. In addition, any power of a permutation matrix is a another permutation matrix. Use these facts to prove that $(\mathbf{I}_{\mathcal{T}})_{[\mathcal{S}]}^r = \mathbf{I}$ for some integer numbers r .

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