

Mathematics II

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You can find the last version of these course materials at

<https://github.com/mbujosab/MatematicasII/tree/main/Eng>

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L-13

L-14

1 Highlights of Lesson 13

Highlights of Lesson 13

- Determinant: $\det(\mathbf{A}) \equiv |\mathbf{A}|$ $[\det : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}]$
 - Volume vs determinant
 - Properties: [1](#), [2](#), [3](#)
- We will deduce properties: **4 – 9**

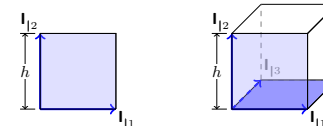
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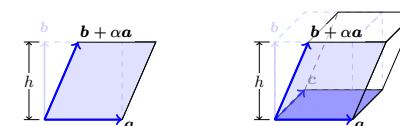
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2 Area or volume

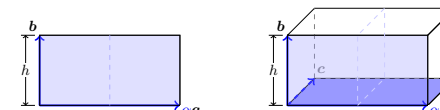
$$1. \text{Vol}(\mathbf{I}_{n \times n}) = 1.$$



$$2. \text{Vol}(\mathbf{A}) = \text{Vol}(\mathbf{A}_{[(\alpha)\tau_{k+j}]}) \text{ for } j \neq k.$$



$$3. |\alpha| \cdot \text{Vol}(\mathbf{A}) = |\alpha| \cdot \text{Vol}[\dots; \mathbf{A}_{|k}; \dots] = \text{Vol}[\dots; \alpha \mathbf{A}_{|k}; \dots]$$



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3 Determinant: 3 properties that define the function

P-1 Determinant of identity matrices:

$$\det \mathbf{I}_{n \times n} = 1$$

P-2 Type I elemen. transf. do not change the determinant:

$$\det \mathbf{A} = \det \left(\mathbf{A}_{[(\alpha)\tau, k+j]} \right)$$

P-3 Multiplying a column by an scalar multiplies the det.

$$\alpha \cdot \det \mathbf{A} = \det [\dots; \alpha \mathbf{A}_{|k}; \dots] \text{ for any } k \in \{1 : n\} \text{ and } \alpha \in \mathbb{R}$$

Absolute value of $\det \mathbf{A} = \text{Vol } \mathbf{A}$

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Example

Then, we know that in \mathbb{R}^3 :

$$\begin{vmatrix} a_1 & (b_1 + \alpha c_1) & c_1 \\ a_2 & (b_2 + \alpha c_2) & c_2 \\ a_3 & (b_3 + \alpha c_3) & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix};$$

$$\det [\mathbf{a}; (\mathbf{b} + \alpha \mathbf{c}); \mathbf{c}] = \det [\mathbf{a}; \mathbf{b}; \mathbf{c}];$$

and also

$$\begin{vmatrix} a_1 & \alpha b_1 & c_1 \\ a_2 & \alpha b_2 & c_2 \\ a_3 & \alpha b_3 & c_3 \end{vmatrix} = \alpha \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix};$$

$$\det [\mathbf{a}; \alpha \mathbf{b}; \mathbf{c}] = \alpha \det [\mathbf{a}; \mathbf{b}; \mathbf{c}];$$

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4 Determinant of a matrix with a zero column

P-4 Det. of a matrix \mathbf{A} with a zero column

If \mathbf{A} has a zero column $\mathbf{0}$, then

$\det(\mathbf{A}) = 0$

prove P-4

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5 Elementary matrices

We already know

$$\det \left(\mathbf{A}_{[(\alpha)\tau, k+j]} \right) = |\mathbf{A}|; \quad \det \left(\mathbf{A}_{[(\alpha)\tau, k]} \right) = \alpha |\mathbf{A}|.$$

Determinant of elementary matrices

$$\det \left(\mathbf{I}_{[(\alpha)\tau, k+j]} \right) = 1 \quad \text{and} \quad \det \left(\mathbf{I}_{[(\alpha)\tau, j]} \right) = \alpha.$$

Hence, since $\mathbf{A}_{\tau} = \mathbf{A}(\mathbf{I}_{\tau})$, then

$$|\mathbf{A}(\mathbf{I}_{\tau})| = |\mathbf{A}| \cdot |\mathbf{I}_{\tau}| \quad (1)$$

where \mathbf{I}_{τ} is an elementary matrix

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EXERCISE 1. Prove the following propositions

(a) $\det(\mathbf{A}_{\tau_1 \dots \tau_k}) = |\mathbf{A}| \cdot |\mathbf{I}_{\tau_1}| \cdots |\mathbf{I}_{\tau_k}|$.

(b) If \mathbf{B} is a full rank matrix, i.e., if $\mathbf{B} = \mathbf{I}_{\tau_1 \dots \tau_k}$, then $|\mathbf{B}| = |\mathbf{I}_{\tau_1}| \cdots |\mathbf{I}_{\tau_k}|$, and therefore $|\mathbf{B}| \neq 0$.

(c) If \mathbf{A} and \mathbf{B} have order n and \mathbf{B} is full rank, then

$$\det(\mathbf{AB}) = |\mathbf{A}| \cdot |\mathbf{B}| \quad (2)$$

6 Determinant after a sequence of elementary transformations

Example

a sequence $\tau_1 \cdots \tau_k$ of *Type I* elementary transformations does not change the determinant.

$$|\mathbf{A}_{\tau_1 \dots \tau_k}| = |\mathbf{A}(\mathbf{I}_{\tau_1 \dots \tau_k})| = |\mathbf{A}| \cdot |\mathbf{I}_{\tau_1 \dots \tau_k}| = |\mathbf{A}| \cdot 1 = |\mathbf{A}|$$

Example

but a sequence of *Type II* can.

$$\begin{vmatrix} 2a & 3c \\ 2b & 3d \end{vmatrix} = ? \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

7 Antisymmetric property

P-5 [Antisymmetric property]

Column exchange changes the sign of the determinant.

Proof.

Column exchange is a sequence of *Type I* transformation and just only one *Type II* transformation that multiplies a column by -1 \square

Therefore:

$$\begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix} = (-1) \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

8 Singular matrices. Inverse of a matrix

P-6 If \mathbf{A} is singular then $|\mathbf{A}| = 0$.

P-7 $\det(\mathbf{A}^{-1}) = (\det \mathbf{A})^{-1}$.

Proof.

Let $\mathbf{A}_{\tau_1 \dots \tau_k} = \mathbf{R}_{n \times n}$ be a reduced equelon form (and $\mathbf{E} = \mathbf{I}_{\tau_1 \dots \tau_k}$).

Since $\mathbf{AE} = \mathbf{R}$, then: $|\mathbf{A}| \cdot |\mathbf{E}| = |\mathbf{R}|$; with only two cases:

$$\begin{cases} \mathbf{A} \text{ singular } (\mathbf{R}_{|n} = \mathbf{0}) : & |\mathbf{A}| \cdot |\mathbf{E}| = 0 \Rightarrow |\mathbf{A}| = 0 \\ \mathbf{A} \text{ not singular } (\mathbf{R} = \mathbf{I}) : & |\mathbf{A}| \cdot |\mathbf{E}| = 1 \Rightarrow |\mathbf{E}| = |\mathbf{A}^{-1}| = (|\mathbf{A}|)^{-1} \end{cases}$$

\square

Example

For $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$:

$$\begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[\text{Type I}]{[(-2)\tau_1+2]} \begin{bmatrix} 1 & 0 \\ 2 & -2 \\ 1 & -2 \\ 0 & 1 \end{bmatrix} \xrightarrow[\text{Type II}]{[(-1/2)\tau_2]} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 1 \\ 0 & -1/2 \end{bmatrix} \xrightarrow[\text{Type I}]{[(-2)\tau_2+1]} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 1 & -1/2 \end{bmatrix}$$

So

$$|\mathbf{A}^{-1}| = \left| \mathbf{I}_{[(-2)\tau_1+2]} \right| \cdot \left| \mathbf{I}_{[(-1/2)\tau_2]} \right| \cdot \left| \mathbf{I}_{[(-2)\tau_2+1]} \right| = 1 \cdot \frac{-1}{2} \cdot 1 = \frac{-1}{2};$$

that is

$$|\mathbf{A}| = -2.$$

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EXERCISE 2. [Transposed matrices]

(a) What is the relation between the determinant of an elementary matrix \mathbf{I}_τ and the determinant of its transpose ${}_\tau\mathbf{I}$?

(b) Consider \mathbf{B} , a full rank matrix, proof that $|\mathbf{B}| = |\mathbf{B}^\top|$.

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9 Determinant of a product**P-8 [Determinant of a product of matrices]**

$$\det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B}). \quad (3)$$

$$\begin{cases} \mathbf{B} \text{ singular, then so it is } \mathbf{AB} \Rightarrow \det(\mathbf{AB}) = 0 = \det(\mathbf{A}) \cdot \det(\mathbf{B}) \\ \mathbf{B} = \mathbf{I}_{\tau_1 \dots \tau_k} \Rightarrow \det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B}) \end{cases}$$

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10 Determinant of a transpose**P-9 Determinant of a transpose**

$$|\mathbf{A}| = |\mathbf{A}^\top|.$$

Proof.

$$\begin{cases} \text{if } \mathbf{A} \text{ singular:} & \mathbf{A}^\top \text{ singular} \Rightarrow \det \mathbf{A}^\top = \det \mathbf{A} = 0 \\ \text{if } \mathbf{A} \text{ NO singular:} & \mathbf{A} = \mathbf{I}_{\tau_1 \dots \tau_k} \Rightarrow \det \mathbf{A}^\top = \det \mathbf{A} \end{cases}.$$

□

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Questions of the Lecture 13

(L-13) QUESTION 1. Complete the proofs of this lecture.

(L-13) QUESTION 2. Knowing that $|\mathbf{BC}| = |\mathbf{B}||\mathbf{C}|$; prove that for any invertible matrix \mathbf{A} (so $\det \mathbf{A} \neq 0$)

$$\det(\mathbf{A}^{-1}) = (\det(\mathbf{A}))^{-1}.$$

(L-13) QUESTION 3. Consider \mathbf{A} and \mathbf{B} such that $\det(\mathbf{A}) = 2$ and $\det(\mathbf{B}) = -2$

3×3 3×3

(a) (0.5pts) Compute the determinants of $\mathbf{A}(\mathbf{B})^2$ and $(\mathbf{AB})^{-1}$

(b) (0.5pts) Is it possible to compute the rank of $\mathbf{A} + \mathbf{B}$? and the rank of \mathbf{AB} ?

(L-13) QUESTION 4. Use the Gauss-Jordan method to compute the determinant

(a) $\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\mathbf{A}_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

(c) $\mathbf{A}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

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(L-13) QUESTION 5. The 3 by 3 matrix \mathbf{A} reduces to the identity matrix \mathbf{I} by the following three column operations (in order):

$[(\tau_{-4})1+2]$: Subtract 4 times column 1 from column 2.

$[(\tau_{-3})1+3]$: Subtract 3 times column 1 from column 3.

$[(-1)3+2]$: Subtract column 3 from column 2.

Find the determinant of \mathbf{A} .

(L-13) QUESTION 6.

(a) Find the determinant of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(b) Find the determinant of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & d \end{bmatrix}$ using Gauss-Jordan.

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1 Highlights of Lesson 14

Highlights of Lesson 14

- Computing $|\mathbf{A}|$ by gaussian elimination
- P-10 — Multilinear property
- Expansion of $\det \mathbf{A}$ in Cofactors (Laplace expansion).
- Application of determinants
 - Cramer's rule for solving linear equations
 - Computing the inverse of \mathbf{A}

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2 Extended matrix

Extended matrix of \mathbf{B} : $\begin{bmatrix} \mathbf{B} \\ 1 \end{bmatrix}$

1. Given τ : $\begin{bmatrix} \mathbf{B}_\tau \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{B} \\ 1 \end{bmatrix}_\tau$.

2. Since $\begin{bmatrix} \mathbf{I} \\ 1 \end{bmatrix}_\tau$ and \mathbf{I}_τ same type Elem. Mat. \Rightarrow same det.

Applying 1. k times, and then 2.

$$\begin{aligned} \left| \begin{bmatrix} \mathbf{I}_{\tau_1 \dots \tau_k} \\ 1 \end{bmatrix} \right| &= \left| \begin{bmatrix} \mathbf{I} \\ 1 \end{bmatrix}_{\tau_1 \dots \tau_k} \right| = \left| \begin{bmatrix} \mathbf{I} \\ 1 \end{bmatrix}_{\tau_1} \dots \begin{bmatrix} \mathbf{I} \\ 1 \end{bmatrix}_{\tau_k} \right| \\ &= |\mathbf{I}_{\tau_1}| \dots |\mathbf{I}_{\tau_k}| = |\mathbf{I}_{\tau_1 \dots \tau_k}|. \end{aligned}$$

If \mathbf{A} is the extended matrix of \mathbf{B} $\begin{cases} \text{If } \mathbf{B} \text{ singular} & |\mathbf{B}| = 0 = |\mathbf{A}| \\ \text{If } \mathbf{B} \text{ invertible} & |\mathbf{B}| = |\mathbf{A}| \end{cases}$

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EXERCISE 7. [Triangular matrices]

- (a) Find the determinant of a full rank lower triangular matrix **L**
 (b) Find the determinant of a triangular matrix with a zero entry in the main diagonal
 (c) Find the determinant of an upper triangular matrix **U**

$$\text{In addition } \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{B} \end{bmatrix}_{n \times m}^{m \times n} = |\mathbf{A}| \cdot |\mathbf{B}|.$$

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3 Computing by Gaussian elimination

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} : \left[\begin{array}{cc|c} 1 & 5 & 0 \\ 2 & 3 & 1 \end{array} \right] \xrightarrow{[(-5)1+2]} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 2 & -7 & 1 \end{array} \right] \quad |\mathbf{A}| = -7$$

Example

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 9 & 6 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\begin{matrix} [(2)3] \\ [(-1)2+3] \\ [(\frac{1}{2})4] \end{matrix}} \left[\begin{array}{ccc|c} 0 & 2 & 0 & 0 \\ 9 & 6 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{array} \right] \xrightarrow{\begin{matrix} [1 \rightleftharpoons 2] \\ [(-1)4] \end{matrix}} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 6 & 9 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{array} \right]$$

$$\begin{vmatrix} 0 & 2 & 1 \\ 9 & 6 & 3 \\ 0 & 1 & 1 \end{vmatrix} = -9,$$

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Matrices of order 1, $\mathbf{A} = [a]$:

$$\left[\begin{array}{c|c} a & 0 \\ 0 & 1 \end{array} \right] \Rightarrow |\mathbf{A}| = a.$$

Matrices of order 2:

$$\left[\begin{array}{cc|c} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{array} \right] \xrightarrow{[(-\frac{b}{a})1+2]} \left[\begin{array}{cc|c} a & 0 & 0 \\ c & d - \frac{bc}{a} & 0 \\ 0 & 0 & 1 \end{array} \right]$$

$$|\mathbf{A}| = ad - bc = a \det[d] - b \det[c].$$

Matrices of order 3:

$$\left[\begin{array}{ccc|c} a & b & c & 0 \\ d & e & f & 0 \\ g & h & i & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\begin{matrix} [(-\frac{b}{a})1+2] \\ [(-\frac{c}{a})1+3] \end{matrix}} \left[\begin{array}{ccc|c} a & 0 & 0 & 0 \\ d & e - \frac{bd}{a} & f - \frac{cd}{a} & 0 \\ g & h - \frac{bg}{a} & i - \frac{cg}{a} & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{[(\frac{-a f + c d}{a e - b d})2+3]} \left[\begin{array}{ccc|c} a & 0 & 0 & 0 \\ d & e - \frac{bd}{a} & f - \frac{cd}{a} & 0 \\ g & h - \frac{bg}{a} & i - \frac{cg}{a} & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} a & 0 & 0 & 0 \\ d & e - \frac{bd}{a} & f - \frac{cd}{a} & 0 \\ g & h - \frac{bg}{a} & i - \frac{cg}{a} & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\begin{matrix} aei - afh - bdi + bfg + cdh - ceg \\ ae - bd \end{matrix}} \left[\begin{array}{ccc|c} a & 0 & 0 & 0 \\ d & e - \frac{bd}{a} & f - \frac{cd}{a} & 0 \\ g & h - \frac{bg}{a} & i - \frac{cg}{a} & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$|\mathbf{A}| = \underbrace{aei - afh - bdi + bfg + cdh - ceg}_{\text{(Rule of Sarrus)}} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

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Matrices of order 4:

$$\left[\begin{array}{cccc|c} a & b & c & d & 0 \\ e & f & g & h & 0 \\ i & j & k & l & 0 \\ m & n & o & p & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\begin{matrix} [(-\frac{b}{a})1+2] \\ [(-\frac{c}{a})1+3] \\ [(-\frac{d}{a})1+4] \\ [(\frac{-ag+ce}{af-be})2+3] \\ [(\frac{-ah+de}{af-be})2+4] \end{matrix}} \left[\begin{array}{cccc|c} a & 0 & 0 & 0 & 0 \\ e & f - \frac{be}{a} & g - \frac{ce}{a} & h - \frac{de}{a} & 0 \\ i & j - \frac{bi}{a} & k - \frac{ci}{a} & l - \frac{di}{a} & 0 \\ m & n - \frac{bm}{a} & o - \frac{cm}{a} & p - \frac{dm}{a} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{[(\frac{-afl+ahj+bel-bhi-dej+dfi}{afk-agj-bek+bgi+cej-cfi})3+4]} \left[\begin{array}{cccc|c} a & 0 & 0 & 0 & 0 \\ e & f - \frac{be}{a} & g - \frac{ce}{a} & h - \frac{de}{a} & 0 \\ i & j - \frac{bi}{a} & k - \frac{ci}{a} & l - \frac{di}{a} & 0 \\ m & n - \frac{bm}{a} & o - \frac{cm}{a} & p - \frac{dm}{a} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$|\mathbf{A}| =$$

$$afkp - aflo - agjp + agln + ahjo - ahkn - bekp + belo + bgip - bgln - bhio + bhkm + cejp - celn - cfip + cflm + chin - chjm - dejo + dekn + dfio - dfkm - dgin + dgjm$$

$$= a \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b \begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

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4 Multilinear property

P-10 Multilinear property

$$\det [\dots; (\beta \mathbf{b} + \psi \mathbf{c}); \dots] = \beta \det [\dots; \mathbf{b}; \dots] + \psi \det [\dots; \mathbf{c}; \dots]$$

Example

Then, in the 2 dimensional case \mathbb{R}^2

$$\begin{vmatrix} a + \alpha & c \\ b + \beta & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} + \begin{vmatrix} \alpha & c \\ \beta & d \end{vmatrix};$$

therefore

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = \begin{vmatrix} a & c \\ 0 & d \end{vmatrix} + \begin{vmatrix} \alpha & c \\ \beta & d \end{vmatrix}.$$

Example

For $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, we have

$${}^1\mathbf{A}^2 = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}, \quad {}^3\mathbf{A}^3 = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

hence

$$\text{cof}_{12}(\mathbf{A}) = (-1)^{1+2} \det({}^1\mathbf{A}^2) = (-1) \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}.$$

and

$$\text{cof}_{33}(\mathbf{A}) = (-1)^{3+3} \det({}^3\mathbf{A}^3) = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}.$$

5 minors and cofactors

Definition minors and cofactors

We denote a submatrix of \mathbf{A} obtained by deleting row i and column j of \mathbf{A} by

$${}^i\mathbf{A}^j;$$

Its determinant is called the minor of a_{ij} . And

$$\text{cof}_{ij}(\mathbf{A}) = (-1)^{i+j} \det({}^i\mathbf{A}^j)$$

is called the cofactor of a_{ij} .

6 Expansion by cofactors

Theorem [Laplace expansion]

For \mathbf{A} n by n , $\det(\mathbf{A})$ may be computed as the sum of the products of the elements of any column (row) of \mathbf{A} by their cofactors:

$$\det(\mathbf{A}) = \sum_{i=1}^n a_{ij} \text{cof}_{ij}(\mathbf{A}), \quad \text{the expansion by the } j\text{th column}$$

or

$$\det(\mathbf{A}) = \sum_{j=1}^n a_{ij} \text{cof}_{ij}(\mathbf{A}), \quad \text{the expansion by the } i\text{th row}$$

EXERCISE 8. Compute $\det \mathbf{A} = \begin{vmatrix} 2 & 0 & 3 & 2 \\ 5 & 1 & 2 & 4 \\ 3 & 0 & 1 & 2 \\ 5 & 3 & 2 & 1 \end{vmatrix}$

7 Cramer's Rule

$$\mathbf{A}\mathbf{x} = \mathbf{b}; \quad |\mathbf{A}| \neq 0 \quad \text{then}$$

$$\mathbf{b} = (\mathbf{A}_{|1})x_1 + \cdots + (\mathbf{A}_{|j})x_j + \cdots + (\mathbf{A}_{|n})x_n.$$

$$\det[\mathbf{A}_{|1}; \dots \overset{\text{pos. } j}{\mathbf{b}}; \dots \mathbf{A}_{|n}] = x_j \cdot \det(\mathbf{A}).$$

$$x_j = \frac{\det[\mathbf{A}_{|1}; \dots \overset{\text{pos. } j}{\mathbf{b}}; \dots \mathbf{A}_{|n}]}{\det(\mathbf{A})}.$$

Computational issues when $\det \mathbf{A} \simeq 0$ (tiny angle between vectors)

8 The inverse of a matrix

$$[\text{Adj}(\mathbf{A})] \cdot \mathbf{A} = \begin{bmatrix} \text{cof}_{11}(\mathbf{A}) & \text{cof}_{21}(\mathbf{A}) & \cdots & \text{cof}_{n1}(\mathbf{A}) \\ \text{cof}_{12}(\mathbf{A}) & \text{cof}_{22}(\mathbf{A}) & \cdots & \text{cof}_{n2}(\mathbf{A}) \\ \vdots & \vdots & \ddots & \vdots \\ \text{cof}_{1n}(\mathbf{A}) & \text{cof}_{2n}(\mathbf{A}) & \cdots & \text{cof}_{nn}(\mathbf{A}) \end{bmatrix} \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{\mathbf{A}}$$

Questions of the Lecture 14

(L-14) QUESTION 1. Complete the proofs of the exercises of this lecture.

(L-14) QUESTION 2. Consider $\mathbf{A} = [\mathbf{A}_{|1}; \mathbf{A}_{|2}; \mathbf{A}_{|3}]$ with $\det \mathbf{A} = 2$.

(a) What are $\det(2\mathbf{A})$ and $\det \mathbf{A}^{-1}$?

(b) What is $\det[(3\mathbf{A}_{|1} + 2\mathbf{A}_{|2}); \mathbf{A}_{|3}; \mathbf{A}_{|2}]$?

(L-14) QUESTION 3. The determinant of the 1000 by 1000 matrix \mathbf{A} is 12. What is the determinant of $-\mathbf{A}^T$? (Careful: No credit for the wrong sign.)
(MIT Course 18.06 Quiz 2, Fall, 2008)

(L-14) QUESTION 4. Consider the squared matrix \mathbf{A} . True or false? (to receive full credit you must explain your answer in a clear and concise way)
 $|\mathbf{A}\mathbf{A}^T| = |\mathbf{A}|^2$.

(L-14) QUESTION 5. We have a 3×3 matrix $\mathbf{A} = \begin{bmatrix} a & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$ with $\det \mathbf{A} = 3$.

Compute the determinant of the following matrices:

(a) (0.5 pts) $\begin{bmatrix} a-2 & 1 & 2 \\ b-4 & 3 & 4 \\ c-6 & 5 & 6 \end{bmatrix}$

- (b) (0.5 pts) $\begin{bmatrix} 7a & 7 & 14 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$
- (c) (1 pts) $(2\mathbf{A})^{-1}\mathbf{A}^T$
- (d) (0.5 pts) $\begin{bmatrix} a-2 & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$

(L-14) QUESTION 6.

- (a) Escalone la matriz $\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 5 & 2 \\ 4 & 6 & 0 \end{bmatrix}$.
- (b) ¿Es \mathbf{A} invertible?
- (c) En caso afirmativo calcule $|\mathbf{A}^{-1}|$; en caso contrario calcule $|\mathbf{A}|$
- (d) La matriz \mathbf{C} es igual al producto de \mathbf{A} con la *traspuesta* de la matriz \mathbf{B} , es decir

$$\mathbf{C} = \mathbf{A}\mathbf{B}^T \quad \text{donde} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

¿Cuánto vale el determinante de \mathbf{C} ? ¿Es \mathbf{C} invertible?

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(L-14) QUESTION 7. What is the determinant of the following matrices using Laplace expansions.

- (a) $\begin{bmatrix} 1 & 2 \\ -4 & 3 \end{bmatrix}$
- (b) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & -2 \end{bmatrix}$
- (c) $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 1 & -2 \end{bmatrix}$

(L-14) QUESTION 8. Compute the following determinant using Laplace expansions:

$$\begin{vmatrix} 0 & 0 & 0 & 3 & 0 \\ -2 & 0 & 0 & 2 & 0 \\ 8 & -1 & 0 & -7 & 2 \\ -1 & 2 & 2 & 3 & 2 \\ 2 & 2 & 3 & 6 & 4 \end{vmatrix}$$

(L-14) QUESTION 9. Compute $\det \mathbf{A} = \begin{vmatrix} 2 & 2 & 0 & 0 \\ 5 & 5 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{vmatrix}$

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(L-14) QUESTION 10. Compute the value of $\det \mathbf{A}$ using Laplace expansion

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 3 & \cdots & 3 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & n \end{bmatrix}$$

(L-14) QUESTION 11. Consider a n by n matrix \mathbf{A}_n full of 3s in its diagonal, and twos just below the diagonal, and another 2 at the position $(1, n)$; for example, for $n = 4$:

$$\mathbf{A}_4 = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 2 & 3 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix}.$$

- (a) Find, using the cofactors of the first row, the determinant of \mathbf{A}_4 .
- (b) Find the determinant of \mathbf{A}_n for $n > 4$.

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(L-14) QUESTION 12. Consider the following block matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}$$

Prove $|\mathbf{A}| = |\mathbf{B}||\mathbf{C}|$.

Hint

$$\begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}$$

(L-14) QUESTION 13. Solve the following linear systems using Cramer's Rule

- (a) $\begin{cases} 2x + 5y = 1 \\ x + 4y = 2 \end{cases}$
- (b) $\begin{cases} 2x + y = 1 \\ x + 2y + z = 0 \\ y + 2z = 0 \end{cases}$

(exercise 13 from section 4.4 of ?)

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(L-14) QUESTION 14. Find the inverse of the following matrices using the *adjoint matrix*

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

(exercise 18 from section 4.4 of ?)

(L-14) QUESTION 15. Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 2 & 3 & 3 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & a \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}; \quad \text{and the vector } \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

(a) (0.5pts) For which values of a the matrix \mathbf{A} is invertible?

(b) (1pts) Consider $a = 5$. Using the Cramer's rule, compute the fourth coordinate x_4 of \mathbf{x} for linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$.

(c) (1pts) Compute \mathbf{B}^{-1} . Use the matrix \mathbf{B}^{-1} to solve $\mathbf{B}\mathbf{x} = \mathbf{b}$.