Mathematics II

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You can find the last version of these course materials at

https://github.com/mbujosab/MatematicasII/tree/main/Eng

1 Highlights of **Lesson 15**

Always squared matrices in this topic

Highlights of Lesson 15

- **Eigenvalues, eigenvectors** (prefix eigen is the German word for innate, distinct, self)
- $\left|\mathbf{A}-\lambda\mathbf{I}\right| = 0$ *Characteristic equation*
- $\operatorname{tr}(\mathbf{A})$, $\det\mathbf{A}$ (demo in the next lesson)

2 Eigenvalues and eigenvectors

Consider the equation

$$\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x} \qquad (\text{with} \quad \boldsymbol{x} \neq \mathbf{0})$$

- *Eigenvalue* is any λ such that there are solutions.
- Such *non-null* solutions $\boldsymbol{x}$ are called *eigenvectors*.

  $\boldsymbol{x} \neq \mathbf{0}$ such that $\mathbf{A}\boldsymbol{x}$ is multiple $\boldsymbol{x}$

When λ is 0, What are the eigenvectors?

.

3 Example: projection matrix

- Orthogonal projection
- Which vectors are eigenvectors?
  What vectors are projected in the same starting direction?
- What are the eigenvalues of those eigenvectors?
- are there any other eigenvectors? with what eigenvalue?
- Two eigen-spaces

4 Another example: Interchange or swap matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- A vector that does not change after interchange?
- What is the eigenvalue?
- Is there an eigenvector corresponding to $\lambda_2 = -1$?

$$\mathbf{A}\boldsymbol{x}_2 = -\boldsymbol{x}_2$$

Note: $\mathrm{tr}\left(\mathbf{A}\right) = 0 = \lambda_1 + \lambda_2; \qquad \det \mathbf{A} = -1 = \lambda_1 \cdot \lambda_2.$

5 how to find eigenvalues and eigenvectors?

How to solve

$$\mathbf{A}\boldsymbol{x} = \overbrace{\lambda}^{?}\ \overbrace{\boldsymbol{x}}^{?}\ ?$$

Here's the trick (simple idea). Bring the $\boldsymbol{x}$ s onto the same side . . .

$$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{x} =$$

idea If $\boldsymbol{x} \neq \mathbf{0}$ what kind of matrix must be $(\mathbf{A} - \lambda\mathbf{I})$?

and then its determinant must be? $|\mathbf{A} - \lambda\mathbf{I}| =$

6 how to find eigenvalues and eigenvectors?

1. Eigenvalues are λ's such that: $|\mathbf{A} - \lambda\mathbf{I}| =$
   (Characteristic polynomial $P_{\mathbf{A}}(\lambda)$)
2. How to compute $\boldsymbol{x}$ so that $\left(\mathbf{A} - \lambda\mathbf{I}\right)\boldsymbol{x} = \mathbf{0}$?

Eigenspace (Set of eigenvectors + $\mathbf{0}$):

$$\mathcal{E}_{\lambda}(\mathbf{A}) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \middle| \mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x} \right\}$$

Spectrum: set $\{\lambda_1, \ldots \lambda_k\}$ of eigenvalues $\big($roots of $P_{\mathbf{A}}(\lambda)\big)$

7 Example (we must compute the eigevalues first!)

We are looking for a null determinant (Characteristic polynomial)

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}; \qquad \det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = 0$$

Note: $\operatorname{tr}\left(\mathbf{A}\right) = 6 = \lambda_1 + \lambda_2; \qquad \det \mathbf{A} = 8 = \lambda_1 \cdot \lambda_2.$

8 Example (. . . and then the eigenspaces)

And now we compute the null space $\mathcal{N}(\mathbf{A} - \lambda\mathbf{I})$ . . . for each λ.

For $\lambda_1 = 4$

$$(\mathbf{A} - 4\mathbf{I}) = \begin{bmatrix} 3-4 & 1 \\ 1 & 3-4 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow$$

For $\lambda_2 = 2$

$$(\mathbf{A} - 2\mathbf{I}) = \begin{bmatrix} 3-2 & 1 \\ 1 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow$$

Are they the only two eigenvectors?

$$\mathbf{A}\boldsymbol{x}_i = \lambda\boldsymbol{x}_i; \qquad \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\boldsymbol{x}_i = \lambda\boldsymbol{x}_i.$$

9 Another example: 90º rotation matrix

$$\mathbf{Q} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- How much do the eigenvalues add up to?
- What is the determinant?

Difficulties

$\lambda_1 + \lambda_2 = 0$ and $\lambda_1 \cdot \lambda_2 = 1$ $\qquad (+)\cdot(-) = (+)?$

What kind of vector can be parallel to itself after a 90º rotation?

$$\det\left(\mathbf{Q} - \lambda\mathbf{I}\right) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 =$$

10 There are even worse examples

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

- Eigenvalues

$$\det\left(\mathbf{A} - \lambda\mathbf{I}\right) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)(3-\lambda) = 0 \begin{cases} \lambda_1 = 3 \\ \lambda_2 = 3 \end{cases}$$

- Eigenvectors
  - for λ_1: $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{x} = \mathbf{0} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{x}_1; \qquad \boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
  - for λ_2:

$\lambda = 3$ is repeated twice, but $\dim \mathcal{E}_3(\mathbf{A}) = 1$

$$\mu(3) = 2 \neq 1 = \gamma(3)$$

Summary:

1. The eigenvalues are those numbers λ that makes the matrix $(\mathbf{A}-\lambda\mathbf{I})$ singular. In other words, they are the roots of the Characteristic polynomial: $\det(\mathbf{A}-\lambda\mathbf{I})$.
2. Any n by n matrix has a caracteristic polynomial of degree n
3. A polynomial of degree n has n roots (perhaps some repeated roots).
4. The sum of eigenvalues of a matrix equals its trace
5. The product of eigenvalues of a matrix equals its determinant
6. The eigenvectors associated with λ are the non-zero vectors in $\mathcal{N}\left(\mathbf{A}-\lambda\mathbf{I}\right)$.

Questions of the Lecture 15

(L-15) Question 1. Consider the matrix

$$\mathbf{A}=\begin{bmatrix}-3&4&-4\\-3&5&-3\\-1&2&0\end{bmatrix}$$

(a) The three eigenvalues of $\mathbf{A}$ are -1, 1 and 2; and two of its eigenvectors are

$$\boldsymbol{v}=\begin{pmatrix}2\\1\\0\end{pmatrix};\qquad \boldsymbol{w}=\begin{pmatrix}1\\0\\-1\end{pmatrix}.$$

Check that both vectors are eigenvenctors of $\mathbf{A}$. What are the corresponding eigenvalues?

(b) Find a third linearly independent eigenvector.

(L-15) Question 2. Find the eigenvalues and eigenvectors of

(a)

$$\mathbf{A}=\begin{bmatrix}3&4\\4&-3\end{bmatrix}$$

(b)

$$\mathbf{B}=\begin{bmatrix}a&b\\b&a\end{bmatrix}$$

(Strang, 2006, exercise 12 from section 5.1.)

(L-15) Question 3. If $\mathbf{B}$ has eigenvalues $1, 2, 3$, $\mathbf{C}$ has eigenvalues $4, 5, 6$, and $\mathbf{D}$ has eigenvalues $7, 8, 9$, what are the eigenvalues of the 6 by 6 matrix $\mathbf{A}=\begin{bmatrix}\mathbf{B}&\mathbf{C}\\\mathbf{0}&\mathbf{D}\end{bmatrix}$? where $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are upper triangular matrices.
(Strang, 2006, exercise 13 from section 5.1.)

(L-15) Question 4. Find the eigenvalues and eigenvectors of

(a)

$$\mathbf{A}=\begin{bmatrix}3&4&2\\0&1&2\\0&0&0\end{bmatrix}$$

(b)

$$\mathbf{B}=\begin{bmatrix}0&0&2\\0&2&0\\2&0&0\end{bmatrix}$$

(Strang, 2006, exercise 5 from section 5.1.)

(L-15) Question 5. The eigenvalues of $\mathbf{A}$ equal the eigenvalues of $\mathbf{A}^\intercal$. This is because $\det(\mathbf{A}-\lambda\mathbf{I})$ equals $\det(\mathbf{A}^\intercal-\lambda\mathbf{I})$.

(a) That is true because ____________

(b) Show by an example that, nevertheless, the eigenvectors of $\mathbf{A}$ and $\mathbf{A}^\intercal$ are not the same.

(Strang, 2006, exercise 11 from section 5.1.)

(L-15) Question 6. Consider the matrix $\mathbf{B}$ and its eigenvector $\boldsymbol{x}$ associated to the eigenvalue λ, that is $\mathbf{B}\boldsymbol{x}=\lambda\boldsymbol{x}$; and also consider the matrix $\mathbf{A}=(\mathbf{B}+\alpha\mathbf{I})$. Prove that $\boldsymbol{x}$ is also an eigenvector of $\mathbf{A}$ with eigenvalue $(\lambda+\alpha)$.

(L-15) Question 7.

(a) Encuentre los autovalores y los auto-vectores de la matriz $\mathbf{A}=\begin{bmatrix}1&-1\\2&4\end{bmatrix}$.

Compruebe que la traza es igual a la suma de los autovalores, y que el determinante es igual a su producto.

(b) Si consideramos una nueva matriz, generada a partir de la anterior como

$$\mathbf{B}=(\mathbf{A}-7\mathbf{I})=\begin{bmatrix}-6&-1\\2&-3\end{bmatrix}.$$

¿Cuáles son los autovalores y auto-vectores de la nueva matriz, y como están relacionados con los de $\mathbf{A}$?

(Strang, 2006, exercise 1 and 3 from section 5.1.)

(L-15) QUESTION 8. Suponga que λ es un auto-valor de $\mathbf{A}$, y que $\boldsymbol{x}$ es un auto-vector tal que $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$.

(a) Demuestre que ese mismo $\boldsymbol{x}$ es un auto-vector de $\mathbf{B} = \mathbf{A} - 7\mathbf{I}$, y encuentre el correspondiente auto-valor de $\mathbf{B}$.

(b) Suponga que $\lambda \neq 0$ (y que $\mathbf{A}$ es invertible), demuestre que $\boldsymbol{x}$ también es un auto-vector de $\mathbf{A}^{-1}$, y encuentre el correspondiente auto-valor. ¿Qué relación tiene con λ?

(Strang, 2006, exercise 7 from section 5.1.)

(L-15) QUESTION 9. Suponga que $\mathbf{A}$ es una matriz de dimensiones $n \times n$, y que $\mathbf{A}^2 = \mathbf{A}$. ¿Qué posibles valores pueden tomar los autovalores de $\mathbf{A}$?

(L-15) QUESTION 10. Suponga la matriz $\underset{3\times 3}{\mathbf{A}}$ con autovalores 1, 2 y 3. Si $\boldsymbol{v}_1$ es un auto-vector asociado al auto-valor 1, $\boldsymbol{v}_2$ al auto-valor 2 y $\boldsymbol{v}_3$ al auto-valor 3; entonces ¿cuanto es $\mathbf{A}(\boldsymbol{v}_1 + \boldsymbol{v}_2 - \boldsymbol{v}_3)$?

(L-15) QUESTION 11. Proporcione un ejemplo que muestre que los auto-valores pueden cambiar cuando un múltiplo de una columna se resta de otra. ¿Por qué los pasos de eliminación no modifican los autovalores nulos?
(Strang, 2006, exercise 6 from section 5.1.)

(L-15) QUESTION 12. El polinomio característico de una matriz $\mathbf{A}$ se puede factorizar como

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda).$$

Demuestre, partiendo de esta factorización, que el determinante de $\mathbf{A}$ es igual al producto de sus valores propios (autovalores). Para ello haga una elección inteligente del valor de λ.
(Strang, 2006, exercise 8 from section 5.1.)

(L-15) QUESTION 13. Calcule los valores característicos (autovalores o valores propios) y los vectores característicos de $\mathbf{A}$ y $\mathbf{A}^2$:

$$\mathbf{A} = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \qquad \text{y} \qquad \mathbf{A}^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}$$

$\mathbf{A}^2$ tiene los mismos ________ que $\mathbf{A}$. Cuando los autovalores de $\mathbf{A}$ son λ_1 y λ_2, los autovalores de $\mathbf{A}^2$ son ________.
(Strang, 2006, exercise 22 from section 5.1.)

(L-15) QUESTION 14. Suponga que los valores característicos de $\underset{3\times 3}{\mathbf{A}}$ son 1, 2 y 4, ¿cuál es la traza de $\mathbf{A}^2$? ¿Cuál es el determinante de $\left(\mathbf{A}^{-1}\right)^{\intercal}$?
(Strang, 2006, exercise 10 from section 5.2.)

(L-15) QUESTION 15. The equation $(\mathbf{A}^2 - 4\mathbf{I})\boldsymbol{x} = \boldsymbol{b}$ has no solution for some right-hand side $\boldsymbol{b}$. Give as much information as possible about the eigenvalues of the matrix $\mathbf{A}$ (the matrix $\mathbf{A}$ is diagonalizable).

(L-15) QUESTION 16. You are given the matrix

$$\mathbf{A} = \begin{bmatrix} 0.5 & 0.2 & 0.2 \\ 0.1 & 0.5 & 0.5 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$$

One of the eigenvalues is $\lambda = 1$. What are the eigenvalues of $\mathbf{A}$? [Hint: Very little calculation required! You should be able to see another eigenvalue by inspection of the form of $\mathbf{A}$, and the third by an easy calculation. You shouldn't need to compute $\det(\mathbf{A} - \lambda\mathbf{I})$ unless you really want to do it the hard way.]

1 Highlights of **Lesson 16**

Highlights of Lesson 16

- Similar matrices: $\mathbf{C} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$
- Triangular block diagonalizing a matrix

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] \xrightarrow[esp(\tau_p^{-1}\cdots\tau_1^{-1})]{\tau_1\cdots\tau_p} \left[\frac{\mathbf{C}}{\mathbf{S}}\right] \qquad \text{where} \quad \mathbf{S} = \mathbf{I}_{\tau_1\cdots\tau_p}.$$

- Diagonalizable matrices: when $\mathbf{C}$ is diagonal.

2 Similar matrices

Similarity

A and **C** are *similar* if there is an invertible **S** such that

$$\mathbf{C} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$$

If **A** and **C** are similar (*see demos in the book*):

- The same determinant: $\det \mathbf{A} = \det \mathbf{C}$
- The same caracteristic polinomial: $|\mathbf{A} - \lambda\mathbf{I}| = |\mathbf{C} - \lambda\mathbf{I}|$
- The same eigenvalues (same *algebraic* and *geometric* multiplicities).
- The same trace.

Mirror inverse transf.: $\left(\mathbf{I}_{(\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k)}\right)^{-1} = {}_{esp(\boldsymbol{\tau}_k^{-1}\cdots\boldsymbol{\tau}_1^{-1})}\mathbf{I}$

$$\mathbf{I} = {}_{\underset{[(-\alpha)\boldsymbol{j}+\boldsymbol{i}]}{\boldsymbol{\tau}}}\mathbf{I}_{\underset{[(\alpha)\boldsymbol{i}+\boldsymbol{j}]}{\boldsymbol{\tau}}} = {}_{\underset{\left[\left(\frac{1}{\alpha}\right)\boldsymbol{j}\right]}{\boldsymbol{\tau}}}\mathbf{I}_{\underset{[(\alpha)\boldsymbol{j}]}{\boldsymbol{\tau}}} \quad\Rightarrow\quad \mathbf{A} \text{ similar to } {}_{esp(\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k)^{-1}}\mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}$$

3 Block diagonalizing a matrix (toothed matrix)

Consider $\mathbf{A} = \left[\begin{array}{c||c} \mathbf{C} & \\ \hline * & \mathbf{L} \end{array}\right] \in \mathbb{C}^{n\times n}$ where

C (of order m) is singular and **L** is full rank lower triangular; then there exists an invertible **R** such that

$$\mathbf{R}^{-1}\mathbf{A}\mathbf{R} = \left[\begin{array}{c|c||ccccc} \underset{m\times(m-1)}{*} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & & & & \\ \hline & d_{m+1} & \beta_{m+1} & & & \\ & d_{m+2} & * & \beta_{m+2} & & \\ * & \vdots & * & * & \ddots & \\ & d_n & * & * & \cdots & \beta_n \end{array}\right]$$

$$\left(\cdots {}_{\underset{[(-\alpha_j)\boldsymbol{m}+\boldsymbol{j}]}{\boldsymbol{\tau}}} \cdots\right)\mathbf{A}\left(\cdots {}_{\underset{[(\alpha_j)\boldsymbol{j}+\boldsymbol{m}]}{\boldsymbol{\tau}}} \cdots\right); \qquad \boldsymbol{j} = 1,\dots,m-1.$$

4 Block diagonalizing a matrix (toothed matrix)

Consider $\mathbf{A} = \left[\begin{array}{c||c} \mathbf{C} & \\ \hline * & \mathbf{L} \end{array}\right] \in \mathbb{C}^{n\times n}$ where

C (of order m) is singular and **L** is full rank lower triangular, then there exists $\mathbf{S} = \mathbf{R}\mathbf{P}$ (invertible) such that

$$\mathbf{P}^{-1}\mathbf{R}^{-1}\mathbf{A}\mathbf{R}\mathbf{P} = \left[\begin{array}{c|c||ccccc} \underset{m\times(m-1)}{*} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & & & & \\ \hline & 0 & \beta_{m+1} & & & \\ & 0 & * & \beta_{m+2} & & \\ * & \vdots & * & * & \ddots & \\ & 0 & * & * & \cdots & \beta_n \end{array}\right]$$

$$\left(\cdots {}_{\underset{[(-\alpha_j)\boldsymbol{m}+\boldsymbol{j}]}{\boldsymbol{\tau}}} \cdots\right)\mathbf{R}^{-1}\mathbf{A}\mathbf{R}\left(\cdots {}_{\underset{[(\alpha_j)\boldsymbol{j}+\boldsymbol{m}]}{\boldsymbol{\tau}}} \cdots\right); \qquad \boldsymbol{j} = m+1,\dots,n.$$

5 A very simple example

Example

Consider $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ with eigenvalues 0, 1 and 1.

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] \xrightarrow[0\mathbf{I}]{(-)} \left[\begin{array}{ccc} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ [(1)\mathbf{1}+\mathbf{2}]\\ [(2)\mathbf{3}+\mathbf{2}]\\ [\mathbf{2}\rightleftharpoons\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow[\substack{\boldsymbol{\tau}\\ [\mathbf{2}\rightleftharpoons\mathbf{3}]\\ [(-2)\mathbf{2}+\mathbf{3}]\\ [(-1)\mathbf{2}+\mathbf{1}]}]{} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow[0\mathbf{I}]{(+)} \left[\begin{array}{c}\mathbf{C}\\ \hline \mathbf{S}\end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{diagonal}}$$

6 A not so simple example

Example

Consider $\mathbf{A} = \begin{bmatrix} -2 & 0 & 3 \\ 3 & -2 & -9 \\ -1 & 2 & 6 \end{bmatrix}$ with eigenvalues 1, 1 and 0.

$$\xrightarrow[1\mathbf{I}]{(-)} \left[\begin{array}{ccc} -3 & 0 & 3 \\ 3 & -3 & -9 \\ -1 & 2 & 5 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(1)\mathbf{1}+\mathbf{3}] \\ [(-2)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} -3 & 0 & 0 \\ 3 & -3 & 0 \\ -1 & 2 & 0 \\ \hline 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{\tau \\ [(2)\mathbf{3}+\mathbf{2}] \\ [(-1)\mathbf{3}+\mathbf{1}]}]{} \left[\begin{array}{ccc} -2 & -2 & 0 \\ 1 & 1 & 0 \\ -1 & 2 & 0 \\ \hline 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[1\mathbf{I}]{(+)}$$

$$\xrightarrow[1\mathbf{I}]{(-)} \left[\begin{array}{ccc} -2 & -2 & 0 \\ 1 & 1 & 0 \\ \hline -1 & 2 & 0 \\ \hline 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc} -2 & 0 & 0 \\ 1 & 0 & 0 \\ \hline -1 & 3 & 0 \\ \hline 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{\tau \\ [(1)\mathbf{2}+\mathbf{1}]}]{} \left[\begin{array}{ccc} -1 & 0 & 0 \\ 1 & 0 & 0 \\ \hline -1 & 3 & 0 \\ \hline 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[1\mathbf{I}]{(+)}$$

$$\xrightarrow[0\mathbf{I}]{(-)} \left[\begin{array}{ccc} 0 & 0 & 0 \\ \hline 1 & 1 & 0 \\ -1 & 3 & 1 \\ \hline 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-1)\mathbf{2}+\mathbf{1}] \\ [(4)\mathbf{3}+\mathbf{1}]}} \left[\begin{array}{ccc} 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 3 & 1 \\ \hline 6 & -1 & 1 \\ -9 & 1 & -2 \\ 4 & 0 & 1 \end{array}\right] \xrightarrow[\substack{\tau \\ [(-4)\mathbf{1}+\mathbf{3}] \\ [(1)\mathbf{1}+\mathbf{2}]}]{} \left[\begin{array}{ccc} 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 3 & 1 \\ \hline 6 & -1 & 1 \\ -9 & 1 & -2 \\ 4 & 0 & 1 \end{array}\right] \xrightarrow[0\mathbf{I}]{(+)}$$

7 Every matrix is similar to a toothed matrix

For every **A** there exists **S** such that

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{C} \qquad \Rightarrow \qquad \mathbf{A}\mathbf{S} = \mathbf{S}\mathbf{C}$$

where **C**, toothed, has the eigenvalues on the diagonal

Example

$$\begin{bmatrix} 6 & -1 & 1 \\ -9 & 1 & -2 \\ 4 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -2 & 0 & 3 \\ 3 & -2 & -9 \\ -1 & 2 & 6 \end{bmatrix} \begin{bmatrix} 6 & -1 & 1 \\ -9 & 1 & -2 \\ 4 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}}_{\text{toothed}}$$

Consequences

- $\sum \lambda_i = \operatorname{tr}(\mathbf{A})$ and $\prod \lambda_i = \det \mathbf{A}$
- $\mathbf{A}\mathbf{S}_{|j} = \mathbf{S}\mathbf{C}_{|j} \quad \Rightarrow \quad$ for j such that $\mathbf{C}_{|j} = \lambda_i \mathbf{I}_{|j}$:

$$\mathbf{A}(\mathbf{S}_{|j}) = \lambda_i(\mathbf{S}_{|j}) \quad \Rightarrow \quad \mathbf{S}_{|j} \text{ is an eigenvector.}$$

8 Back to the simple, "toothless" example

Consider $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ with eigenvalues 0, 1 and 1.

$$\left[\begin{array}{c} \mathbf{A} \\ \hline \mathbf{I} \end{array}\right] \xrightarrow[0\mathbf{I}]{(-)} \left[\begin{array}{ccc} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(1)\mathbf{1}+\mathbf{2}] \\ [(2)\mathbf{3}+\mathbf{2}] \\ [\mathbf{2}\rightleftharpoons\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow[\substack{\tau \\ [\mathbf{2}\rightleftharpoons\mathbf{3}] \\ [(-2)\mathbf{2}+\mathbf{3}] \\ [(-1)\mathbf{2}+\mathbf{1}]}]{} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow[0\mathbf{I}]{(+)} \left[\begin{array}{c} \mathbf{C} \\ \hline \mathbf{S} \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}(\mathbf{S}_{|j}) = \lambda_i(\mathbf{S}_{|j}) \quad \Rightarrow \quad \mathbf{S}_{|j} \text{ is an eigenvector.}$$

9 Diagonalizable matrices

- A matrix is diagonalizable **if and only if** *algebraic* and *geometric* multiplicities are equal for each eigenvalue
- If there are no repeated eigenvalues, there are no "teeth" either
- When there are no repeated eigenvalues $\underset{n\times n}{\mathbf{A}}$ is diagonalizable (is sure to have n independent eigenvectors)

10 Diagonalizing a matrix

- Find the spectrum: $\{\lambda_1, \lambda_2, \ldots\}$
- Find the *algebraic multiplicity* of each eigenvalue: $\mu(\lambda_i)$

then choose one of these alternatives:

1. teething the matrix (implemented in NAcAL)
2. . . . or for every λ_i
   - find the eigenspace

$$\mathcal{E}_{\lambda_i}(\mathbf{A}) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \middle| \mathbf{A}\boldsymbol{x} = \lambda_i \boldsymbol{x} \right\} = \mathcal{N}(\mathbf{A} - \lambda_i \mathbf{I}).$$

   - check $\mu(\lambda_i) = \dim \mathcal{E}_{\lambda_i}(\mathbf{A})$ (algebraic and geometric multiplicities are equal)

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix}; \quad \mathbf{S} = \left[\text{Basis for } \mathcal{E}_{\lambda_1}(\mathbf{A}) \# \cdots \# \text{Basis for } \mathcal{E}_{\lambda_k}(\mathbf{A}) \right]$$

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D} \quad \Leftrightarrow \quad \mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$$

11 Matrix powers

If $\mathbf{A}\boldsymbol{x} = \lambda \boldsymbol{x}$ then $\mathbf{A}^2\boldsymbol{x} = \mathbf{A}\mathbf{A}\boldsymbol{x} = \mathbf{A}(\lambda \boldsymbol{x}) = \lambda \mathbf{A}\boldsymbol{x} =$

- What can I say about the eigenvectors?
- What is the relationship between the eigenvalues of $\mathbf{A}$ and those of $\mathbf{A}^2$

In a matrix form (if $\mathbf{A}$ is diagonalizable, $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$):

$$\mathbf{A}^2 = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}\mathbf{S}\mathbf{D}\mathbf{S}^{-1} = \mathbf{S}\mathbf{D}^2\mathbf{S}^{-1}$$

In general, for, $n \in \mathbb{Z}$, $n \geq 0 \ldots$ $\quad \mathbf{A}^n =$
what about $\mathbf{A}$ both diagonalizable and invertible?

Questions of the Lecture 16

(L-16) QUESTION 1. Factor these two matrices into $\mathbf{S}\mathbf{D}\mathbf{S}^{-1}$;

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

(Strang, 2006, exercise 15 from section 5.2.)

(L-16) QUESTION 2. Which of these matrices cannot be diagonalized?

(a)

$$\mathbf{A}_1 = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$$

(b)

$$\mathbf{A}_2 = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}$$

(c)

$$\mathbf{A}_3 = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

(Strang, 2006, exercise 5 from section 5.2.)

(L-16) QUESTION 3. If $\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ find $\mathbf{A}^{100}$ by diagonalizing $\mathbf{A}$.
(Strang, 2006, exercise 7 from section 5.2.)

(L-16) QUESTION 4. If the eigenvalues of $\underset{3\times 3}{\mathbf{A}}$ are 1, 1 and 2, which of the following are certain to be true? Give a reason if true or a counterexample if false:

(a) $\mathbf{A}$ is invertible.
(b) $\mathbf{A}$ is diagonalizable.
(c) $\mathbf{A}$ is not diagonalizable

(Strang, 2006, exercise 11 from section 5.2.)

(L-16) QUESTION 5. Let $\mathbf{A}$ be the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

(a) (1^{pts}) Determine if $\mathbf{A}$ is diagonalizable, and if so, diagonalize it.
(b) (0.5^{pts}) Compute $\left(\mathbf{A}^6\right)\boldsymbol{v}$, where $\boldsymbol{v} = (0, \quad 0, \quad 0, \quad 1,)$.
(c) (0.5^{pts}) Using the the eigenvalues found in part (a) justify that $\mathbf{A}$ is invertible.
(d) (0.5^{pts}) What is the relation between the eigenvalues of $\mathbf{A}$ and the eigenvalues of $\mathbf{A}^{-1}$?

(L-16) QUESTION 6. Si $\mathbf{A} = \mathbf{SDS}^{-1}$; entonces $\mathbf{A}^3 = (\quad)(\quad)(\quad)$ y $\mathbf{A}^{-1} = (\quad)(\quad)(\quad)$.
(Strang, 2006, exercise 16 from section 5.2.)

(L-16) QUESTION 7. Considere la matriz

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

(a) Encuentre los autovalores de **A**
(b) Encuentre los auto-vectores de **A**
(c) Diagonalice **A**: escríbalo como $\mathbf{A} = \mathbf{SDS}^{-1}$.

(L-16) QUESTION 8. ¿Falso o verdadero? Si los autovalores de **A** son 2, 2 y 3 entonces sabemos que la matriz es

(a) Invertible
(b) Diagonalizable
(c) No diagonalizable.

(L-16) QUESTION 9. Sean las matrices

$$\mathbf{A}_1 = \begin{bmatrix} 8 & \\ & 2 \end{bmatrix}; \qquad \mathbf{A}_2 = \begin{bmatrix} 9 & 4 \\ & 1 \end{bmatrix}; \qquad \mathbf{A}_3 = \begin{bmatrix} 10 & 5 \\ -5 & \end{bmatrix}$$

(a) Complete dichas matrices de modo que en los tres casos $\det \mathbf{A}_i = 25$. Así, la traza es en todos los casos igual a 10, y por tanto para las tres matrices el único auto-valor $\lambda = 5$ está repetido dos veces ($\lambda^2 = 25$ y $\lambda + \lambda = 10$ implica $\lambda = 5$).
(b) Encuentre un vector característico con $\mathbf{A}\boldsymbol{x} = 5\boldsymbol{x}$. Estas tres matrices no son diagonalizable porque no hay un segundo auto-vector linealmente independiente del primero.

(Strang, 2006, exercise 27 from section 5.2.)

(L-16) QUESTION 10. Factorice las siguientes matrices en $\mathbf{S}\ \mathbf{D}\ \mathbf{S}^{-1}$

(a) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
(b) $\mathbf{B} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$

(Strang, 2006, exercise 1 from section 5.2.)

(L-16) QUESTION 11. Encuentre la matriz **A** cuyos autovalores son 1 y 4, cuyos autovectores son $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ y $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ respectivamente.
(Strang, 2006, exercise 2 from section 5.2.)

(L-16) QUESTION 12. Si los elementos diagonales de una matriz triangular superior de orden 3×3 son 1, 2 y 7, ¿puede saber si la matriz es diagonalizable? ¿Quién es **D**?
(Strang, 2006, exercise 4 from section 5.2.)

(L-16) QUESTION 13.

(a) Encuentre los autovalores y auto-vectores de la matriz $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.
(b) Explique por qué (o por qué no) la matriz **A** es diagonalizable.

(L-16) QUESTION 14. Sea **A** una matriz 3×3. Asuma que sus autovalores son 1 y 0, que una base de los autovectores asociados a $\lambda = 1$ son $[1, 0, 1]$ y $[0, 0, 1]$; mientras que los asociados a $\lambda = 0$ son paralelos a $[1, 1, 2]$.

(a) ¿Es **A** diagonalizable? En caso afirmativo escriba la matriz diagonal **D** y la matriz **S** tales que $\mathbf{A} = \mathbf{SDS}^{-1}$.
(b) Encuentre **A**.

(L-16) QUESTION 15. Let **A** be a 2×2 matrix such that $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ is an eigenvector for **A** with eigenvalue 2, and $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is another eigenvector for **A** with eigenvalue -2. If $\boldsymbol{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, compute $\left(\mathbf{A}^3\right)\boldsymbol{v}$.

1 Highlights of **Lesson 17**

Highlights of Lesson 17

- Symetric matrices $\mathbf{A} = \mathbf{A}^\intercal$
  - Eigenvalues and eigenvectors
- Introd. positive Definiteness matrices

2 Symmetric matrices $\mathbf{A} = \mathbf{A}^\top$

what's special about $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ when $\underset{n\times n}{\mathbf{A}}$ is symmetric?

1. A symmetric matrix has only REAL EIGENVALUES
2. n EIGENVECTORS *can be choosen* ORTHOGONAL

(always diagonalizable)

The usual diagonalizable case:

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D} \quad \longleftrightarrow \quad \mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$$

Symmetric case:

I can choose perpendicular unit eigenvectors (ortho*normal* columns of $\mathbf{S} = \mathbf{Q}$)

$$(\text{if } \mathbf{A} = \mathbf{A}^\top) \qquad \mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}^{-1} = \mathbf{Q}\mathbf{D}\mathbf{Q}^\top \qquad \text{Spectral thm.}$$

Orthogonally diagonalizable.

3 Eigenspaces are orthogonal for symmetric matrices

Eigenvectors (corresponding to different eigenvalues) of a symmetric matrix are orthogonal.

Proof.

Consider $\mathbf{A}\boldsymbol{x} = \lambda_1\boldsymbol{x}$ and $\mathbf{A}\boldsymbol{y} = \lambda_2\boldsymbol{y}$ (with $\lambda_1 \neq \lambda_2$). then

$$\lambda_1\boldsymbol{x}\cdot\boldsymbol{y} = \mathbf{A}\boldsymbol{x}\cdot\boldsymbol{y} = \boldsymbol{x}(\mathbf{A}^\top)\boldsymbol{y} = \boldsymbol{x}\mathbf{A}\boldsymbol{y} = (\boldsymbol{x}\cdot\boldsymbol{y})\lambda_2.$$

Since $\lambda_1 \neq \lambda_2$ then:

$$\lambda_1(\boldsymbol{x}\cdot\boldsymbol{y}) - \lambda_2(\boldsymbol{x}\cdot\boldsymbol{y}) = 0 \implies (\lambda_1-\lambda_2)\boldsymbol{x}\cdot\boldsymbol{y} = 0 \implies \boldsymbol{x}\cdot\boldsymbol{y} = 0.$$

□

4 Quadratic forms

Quadratic form:

$$\boldsymbol{x}\mathbf{A}\boldsymbol{x}; \quad \text{with} \quad \mathbf{A}^\top = \mathbf{A}$$

Since $\mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}^\top$ (with $\mathbf{Q}^\top\mathbf{Q} = \mathbf{Q}\mathbf{Q}^\top = \mathbf{I}$), then

$$\boldsymbol{x}\mathbf{A}\boldsymbol{x} = \boldsymbol{x}\mathbf{Q}\mathbf{D}\mathbf{Q}^\top\boldsymbol{x} = (\mathbf{Q}^\top\boldsymbol{x})\mathbf{D}(\mathbf{Q}^\top\boldsymbol{x}) \text{ (weighted sum of squares)}$$

Positive definite quadratic form:

$$\boldsymbol{x}\mathbf{A}\boldsymbol{x} > 0 \quad \forall\boldsymbol{x} \neq \mathbf{0} \qquad \Longleftrightarrow \qquad \lambda_i > 0, \quad i = 1:n.$$

then we also say $\mathbf{A}$ is positive definite.

5 Positive definite matrices

Meaning:

$$\boldsymbol{x}\mathbf{A}\boldsymbol{x} > 0 \qquad (\text{except for } \boldsymbol{x} = \mathbf{0})$$

Some properties

Consider a positive definite symmetric $\mathbf{A}$: What about $\mathbf{A}^{-1}$?

$\mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}^{-1} = \mathbf{Q}\mathbf{D}\mathbf{Q}^\top$

Consider two positive definite symmetric matrices $\mathbf{A}$, $\mathbf{B}$: What about $\mathbf{A} + \mathbf{B}$?

the answer must be. . .

6 The matrix product $\mathbf{A}^\intercal\mathbf{A}$

Consider the rectangular matrix $\underset{m\times n}{\mathbf{A}}$. Is $\mathbf{A}^\intercal\mathbf{A}$ positive definite?

$$\boldsymbol{x}(\mathbf{A}^\intercal\mathbf{A})\boldsymbol{x} =$$

It can only be 0 when $\mathbf{A}\boldsymbol{x}$ is $\mathbf{0}$

How can we guarantee that $\mathbf{A}\boldsymbol{x} \neq \mathbf{0}$ when $\boldsymbol{x} \neq \mathbf{0}$?

7 Symmetric matrices: signs of eigenvalues

are all λ_i positive? are they negative?

Computing eigenvalues of $\underset{5\times 5}{\mathbf{A}}$ is impossible in general! (5th degree polynomial)

Good news: The signs of the pivots of echelon form are the same as the signs of the eigenvalues λ_i
(if we do not change the sign of the determinant with *Type II* elementary transformations)

$$\text{num. of positive pivots} = \text{num. of positive eigenvalues}$$

8 Positive definite symmetric matrices

- All eigenvalues are:
- All pivots are:

$$\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$$

Pivots:

What is the sign of each eigenvalue?

$$\lambda^2 - 8\lambda + 11 = 0 \rightarrow \qquad \lambda = 4 \pm \sqrt{5} > 0$$

Summary (for symmetric matrices):

1. Symmetric matrices have *real eigenvalues* and *perpendicular eigenvectors* can be choosen
2. $\mathbf{A} = \mathbf{QDQ}^\intercal$ where $\mathbf{Q}$ is orthogonal
3. $\mathbf{A}$ is symmetric if and only if it is *orthogonally* diagonalizable
4. The signs of the pivots in the echelon form are same as the signs of the eigenvalues λ_i (only if we do not change the sign of the determinant with *Type II* elementary transformations)

Questions of the Lecture 17

(L-17) QUESTION 1. Write $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ in the form $\mathbf{QDQ}^\intercal$ of the spectral theorem:

(a) $\mathbf{A} = \begin{bmatrix}1/2 & 1/2\\1/2 & 1/2\end{bmatrix}$
(b) $\mathbf{B} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$
(c) $\mathbf{C} = \begin{bmatrix}3 & 4\\4 & -3\end{bmatrix}$

(Strang, 2006, exercise 11 from section 5.5.)

(L-17) QUESTION 2. Find the eigenvalues and the unit eigenvectors of

$$\mathbf{A} = \begin{bmatrix}1 & 1 & 1\\1 & 0 & 0\\1 & 0 & 0\end{bmatrix}$$

(Strang, 2003, exercise 3 from section 6.4.)

(L-17) QUESTION 3. Find an orthonormal $\mathbf{Q}$ that diagonalizes this symmetric matrix:

$$\mathbf{A} = \begin{bmatrix}1 & 0 & 2\\0 & -1 & -2\\2 & -2 & 0\end{bmatrix}$$

(Strang, 2003, exercise 5 from section 6.4.)

(L-17) QUESTION 4. Suppose $\mathbf{A}$ is a symmetric 3 by 3 matrix with eigenvalues 0, 1,2.

(a) What properties can be guaranteed for the corresponding unit eigenvectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$
(b) In terms of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, describe the nullspace, left nullspace, row space, and column space of $\mathbf{A}$.
(c) Find a vector $\boldsymbol{x}$ that satisfies $\mathbf{A}\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$. Is $\boldsymbol{x}$ unique?
(d) Under what conditions on $\boldsymbol{b}$ does $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ have a solution?
(e) If $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are the columns of $\mathbf{S}$, what are $\mathbf{S}^{-1}$ and $\mathbf{S}^{-1}\mathbf{AS}$.

(Strang, 2006, exercise 13 from section 5.5.)

(L-17) QUESTION 5. Escriba un hecho destacado sobre los valores característicos de cada uno de estos tipos de matrices:

(a) Una matriz simétrica real.
(b) Una matriz diagonalizable tal que $\mathbf{A}^n \to \mathbf{0}$ cuando $n \to \infty$.
(c) Una matriz no diagonalizable
(d) Una matriz singular

(Strang, 2006, exercise 16 from section 5.5.)

(L-17) QUESTION 6. Sean

$$\mathbf{A} = \begin{bmatrix}0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\1 & 0 & 0 & 0\end{bmatrix}; \qquad \mathbf{B} = \frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\end{bmatrix}$$

(a) Encuentre los valores característicos de $\mathbf{A}$ (recuerde que $i^2 = -1$).
(b) Encuentre los valores característicos de $\mathbf{B}$ (en este caso quizá le resulte más sencillo encontrar primero los autovectores, y deducir entonces los autovalores).
(c) De los siguientes tipos de matrices: ortogonales, invertibles, permutación, hermíticas, de rango 1. diagonalizables, de Markov ¿a qué tipos pertenece $\mathbf{A}$?
(d) ¿y $\mathbf{B}$?

(Strang, 2006, exercise 14 from section 5.5.)

(L-17) QUESTION 7. Si $\mathbf{A}^3 = \mathbf{0}$ entonces los autovalores de $\mathbf{A}$ deben ser ________. De un ejemplo tal que $\mathbf{A} \neq \mathbf{0}$. Ahora bien, si $\mathbf{A}$ es además simétrica, demuestre que entonces $\mathbf{A}^3$ es necesariamente $\mathbf{0}$.

(L-17) QUESTION 8. Consider the matrix

$$\mathbf{A} = \begin{bmatrix}a & 1 & 1\\0 & 3 & 1\\0 & 0 & 2\end{bmatrix}.$$

(a) Prove that $\mathbf{A}$ is not diagonalizable when $a = 3$.
(b) Is $\mathbf{A}$ diagonalizable when $a = 2$? (explain). If it is diagonalizable, find an eigenvalue diagonal matrix $\mathbf{D}$ and an eigenvector matrix $\mathbf{S}$ such as $\mathbf{A} = \mathbf{SDS}^{-1}$.
(c) Is $\mathbf{A}^\intercal\mathbf{A}$ diagonalizable for any value a? Is it possible to find a full set of orthonormal eigenvectors of $\mathbf{A}^\intercal\mathbf{A}$?
(d) Find all posible values a such as $\mathbf{A}$ is invertible and diagonalizable.

(L-17) QUESTION 9. Sea la matriz

$$\mathbf{B} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix};$$

(a) Exprese $\mathbf{B}$ en la forma $\mathbf{B} = \mathbf{A} = \mathbf{QDQ}^\intercal$ del teorema espectral.
(b) ¿Es $\mathbf{B}$ diagonalizable? Si no lo es, diga las razones; y en caso contrario genere una matriz $\mathbf{S}$ que diagonalice a $\mathbf{B}$.

1 Highlights of Lesson 18

Highlights of Lesson 18

- Positive and Negative (semi)definite matrices
- Completing the squares
- Diagonalization by congruence

2 Quadratic forms

- Positive definite: $\forall \boldsymbol{x} \neq \mathbf{0} \;\Rightarrow\; \boldsymbol{x}\mathbf{A}\boldsymbol{x} > 0.$
- Positive semi-definite: $\forall \boldsymbol{x} \neq \mathbf{0} \;\Rightarrow\; \boldsymbol{x}\mathbf{A}\boldsymbol{x} \geq 0.$
- Negative definite: $\forall \boldsymbol{x} \neq \mathbf{0} \;\Rightarrow\; \boldsymbol{x}\mathbf{A}\boldsymbol{x} < 0.$
- Negative semi-definite: $\forall \boldsymbol{x} \neq \mathbf{0} \;\Rightarrow\; \boldsymbol{x}\mathbf{A}\boldsymbol{x} \leq 0.$
- Indefinite: neither positive semi-definite, nor negative semi-definite.

Example

What number do I have to put there for the matrix **A** to be singular?

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & \end{bmatrix}$$

- Eigenvalues:
- Leading principal minors:
- For the following quadratic form

$$q_{\mathbf{A}}(\boldsymbol{x}) = \boldsymbol{x}\mathbf{A}\boldsymbol{x} = (x, \quad y,) \begin{bmatrix} 2 & 6 \\ 6 & \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 12xy + \quad y^2$$

Is there a $\boldsymbol{x} \neq \mathbf{0}$ such that $\boldsymbol{x}\mathbf{A}\boldsymbol{x} = 0$?

Example

If $\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$ then $(x, \quad y,) \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 12xy + \,?\, y^2$

- Are there numbers x and y that make $\boldsymbol{x}\mathbf{A}\boldsymbol{x}$ negative?
- Does the function go through the origin?
- When $y = 0$ and $x = 1$, is it positive? (and when $x = -1$?)
- When $x = 0$ and $y = 1$, is it positive? (and when $y = -1$?)
- Is it always positive?

$(0, 0,)$ **saddle point**: minimum in some directions, maximum in others.

$$\lambda_1 = -2, \quad \begin{pmatrix} -6 \\ 4 \end{pmatrix}; \qquad \lambda_1 = 11, \quad \begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

Example

If $\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$ then $(x, \quad y,) \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 12xy + 20y^2$

Positive definite.
Does it pass the tests?

- Are the leading principal minors positive?
- Are the eigenvalues positive?

$$q_{\mathbf{A}}(\boldsymbol{x}) = \boldsymbol{x}\mathbf{A}\boldsymbol{x} > 0 \qquad \text{for all } \boldsymbol{x} \neq \mathbf{0}$$

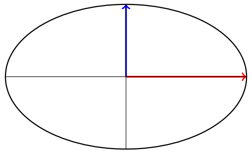
$\boxed{3}$ Completing the squares

If we could express $q(\boldsymbol{x})$ as a sum of squares, we would know whether $q(\boldsymbol{x})$ is positive definite.

Let's complete the square!

- $q(x,y) = 2x^2 + 12xy + 20y^2 \;=\; 2(x + \,\mathbf{?}\, y)^2 + \mathbf{?}$
- $q(x,y) = 2x^2 + 12xy + 7y^2$
- $q(x,y) = 2x^2 + 12xy + 18y^2$
- $q(x,y) = 2x^2 + 12xy + 200y^2$ (graph)

If positive definite: $q(x,y) = a; \quad a > 0$: ellipse



is
$q(x,y,z,w,t) = 2t^2 - 2tx - 2tz + w^2 - 2wy + 2x^2 - 2xy + 2y^2 + z^2$
positive definite? ☹☹☹☹☹!!!???

$\boxed{4}$ Congruent matrices

A and **C** are congruent if there exists an invertible **B** such that
$\boxed{\mathbf{C} = \mathbf{B}^{\mathsf{T}}\mathbf{A}\mathbf{B}}$

Diagonalization by congruence

For each **A** (symmetric) exists $\mathbf{B} = \mathbf{I}_{\tau_1 \cdots \tau_k}$ (invertible) such that

$$\mathbf{D} = \mathbf{B}^{\mathsf{T}}\mathbf{A}\mathbf{B} \qquad \text{is diagonal} \qquad (\mathbf{B}^{\mathsf{T}} = {}_{\tau_k \cdots \tau_1}\mathbf{I})$$

Spectral Theorem: ¡Diagonalization by similarity and congruence!

$$\mathbf{D} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \mathbf{Q}^{\mathsf{T}}\mathbf{A}\mathbf{Q}.$$

Hence, every quadratic form can be written as a sum of squares

$$\boldsymbol{x}\mathbf{A}\boldsymbol{x} \;=\; \boldsymbol{x}\left(\mathbf{B}^{-1}\right)^{\mathsf{T}}\mathbf{D}\mathbf{B}^{-1}\boldsymbol{x} \;=\; \boldsymbol{y}\mathbf{D}\boldsymbol{y}; \qquad \text{where} \qquad \boldsymbol{y} = \mathbf{B}^{-1}\boldsymbol{x}.$$

$\boxed{5}$ Completing the squares

$$2x^2 + 12xy + 20y^2$$

$$\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-3)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix} 2 & 0 \\ 6 & 2 \end{bmatrix} \xrightarrow[\substack{\boldsymbol{\tau} \\ [(-3)\mathbf{1}+\mathbf{2}]}]{} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix};$$

therefore, we get:

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \mathbf{D} = \mathbf{E}^{\mathsf{T}}\mathbf{A}\mathbf{E} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

hence $\mathbf{A} = (\mathbf{E}^{\mathsf{T}})^{-1}\mathbf{D}\mathbf{E}^{-1}$ so

$$\boldsymbol{x}\mathbf{A}\boldsymbol{x} = (x, \quad y,) \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \left(\boldsymbol{x}\,(\mathbf{E}^{-1})^{\mathsf{T}}\right)\mathbf{D}\left(\mathbf{E}^{-1}\boldsymbol{x}\right)$$

$$= \left((x+3y), \quad y,\right) \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{pmatrix} (x+3y) \\ y \end{pmatrix} = 2(x+3y)^2 + 2y^2$$

6 example 3 by 3

$$\text{Is } \mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \text{ positive definite?}$$

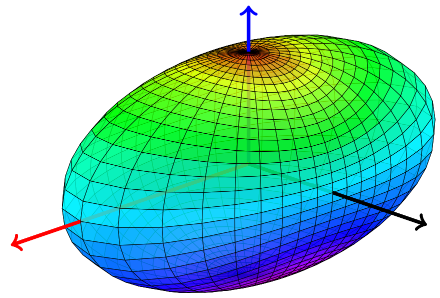
$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow[{[(\frac{1}{2})\overset{\tau}{\mathbf{1}+\mathbf{2}}]}]{[(\frac{1}{2})\overset{\tau}{\mathbf{1}+\mathbf{2}}]} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow[{[(\frac{2}{3})\overset{\tau}{\mathbf{2}+\mathbf{3}}]}]{[(\frac{2}{3})\overset{\tau}{\mathbf{2}+\mathbf{3}}]} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

$$\boldsymbol{x}\mathbf{A}\boldsymbol{x} = 2x^2 + 2y^2 + 2z^2 - 2xy - 2yz > 0$$

$\boldsymbol{x}\mathbf{A}\boldsymbol{x} = 1$: (ellipsoid) axes are eigenvectors $\mathbf{A} = \mathbf{Q}^{\intercal}\boldsymbol{\lambda}\mathbf{Q}$

7 Positive definite matrices and ellipsoids: example 3 by 3

- The region $(\boldsymbol{x}\mathbf{A}\boldsymbol{x} = a)$ is an (ellipsoid).
- The eigenvectors of $\mathbf{Q}$ are in the direction of the three principal axes.
- Lengths of axes determined by the eigenvalues



8 Another example 3 by 3

$$\text{Is } \mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ positive definite?}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow[{[(1)\overset{\tau}{\mathbf{3}+\mathbf{1}}]}]{[(1)\overset{\tau}{\mathbf{3}+\mathbf{1}}]} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow[{[(-\frac{1}{2})\overset{\tau}{\mathbf{1}+\mathbf{3}}]}]{[(-\frac{1}{2})\overset{\tau}{\mathbf{1}+\mathbf{3}}]} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \xrightarrow[{[\overset{\tau}{\mathbf{2}\rightleftharpoons\mathbf{3}}]}]{[\overset{\tau}{\mathbf{2}\rightleftharpoons\mathbf{3}}]} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Indefinite matrix

9 "Classification" of quadratic

$$\boxed{\boldsymbol{x}\mathbf{A}\boldsymbol{x} \lesseqqgtr 0; \quad \text{for all } \boldsymbol{x} \neq \mathbf{0}}$$

Methods

Check the signs of

1. Elem. diag.: $\mathbf{D} = \mathbf{B}^{\intercal}\mathbf{A}\mathbf{B}$ (Diagonalization by congruence) ☺
2. Computing eigenvalues: (Roots of a polynomial) ☹
3. Leading principal minors: (Sylvester's criterion) ☹

Law of inertia

the number of positive, negative and zero entries of the diagonal of $\mathbf{D}$ is an invariant of $\mathbf{A}$, i.e. it does not depend on $\mathbf{B}$

(Orthogonal diagonalization $\mathbf{D} = \mathbf{Q}^{\intercal}\mathbf{A}\mathbf{Q}$ is a special case)

Questions of the Lecture 18

(L-18) QUESTION 1. Decide for or against the positive definiteness of these matrices, and write out the corresponding quadratic form $f = \boldsymbol{x}\mathbf{A}\boldsymbol{x}$:

(a) $\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} -1 & 2 \\ 2 & -8 \end{bmatrix}$

(e) The determinant in (b) is zero; along what line is $f(x, y) = 0$?

(Strang, 2006, exercise 2 from section 6.1.)

(L-18) QUESTION 2. What is the quadratic $f = ax^2 + 2bxy + cy^2$ for each of these matrices? Complete the square to write f as a sum of one or two squares $d_1(\quad)^2 + d_2(\quad)^2$.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$

(Strang, 2006, exercise 15 from section 6.1.)

(L-18) QUESTION 3. Which one of the following matrices has two positive eigenvalues? Test $a > 0$ and $ac > b^2$, don't compute the eigenvalues. $\boldsymbol{x}\mathbf{A}\boldsymbol{x} < 0$.

(a) $\mathbf{A} = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix}$

(c) $\mathbf{C} = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix}$

(d) $\mathbf{D} = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}$

(Strang, 2006, exercise 14 from section 6.1.)

(L-18) QUESTION 4. Show that $f(x, y) = x^2 + 4xy + 3y^2$ does not have a minimum at $(0, 0)$ even though it has positive coefficients. Write $f(x, y)$ as a difference of squares and find a point (x, y) where $f(x, y)$ is negative.
(Strang, 2006, exercise 16 from section 6.1.)

(L-18) QUESTION 5. Show from the eigenvalues that if **A** is positive definite, so is $\mathbf{A}^2$ and so is$\mathbf{A}^{-1}$.
(Strang, 2006, exercise 4 from section 6.2.)

(L-18) QUESTION 6. Consider the following quadratic forms

$$\begin{aligned} q_1(x, y, z) &= x^2 + 4y^2 + 5z^2 - 4xy. \\ q_2(x, y, z) &= -x^2 + 4y^2 + z^2 + 2xy - 2axz. \end{aligned}$$

(a) Show that $q_1(x, y, z)$ is positive semi-definite.
(b) Find, if it is possible, any value of a such as $q_2(x, y, z)$ is negative definite.

(L-18) QUESTION 7. Decide for or against the positive definiteness of

(a) $\mathbf{A} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

(c) $\mathbf{C} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}^2$

(Strang, 2006, exercise 2 from section 6.2.)

(L-18) QUESTION 8. Consider the following quadratic form

$$q(x, y, z) = x^2 + 6xy + y^2 + az^2;$$

Decide for which values a the quadratic form is positive definite, negative definite, semidefinite, or indefinite.

(L-18) QUESTION 9. Si $\mathbf{A} = \left[\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right]$ es definida positiva, pruebe que $\mathbf{A}^{-1}$ es definida positiva.
(Strang, 2006, exercise 8 from section 6.1.)

(L-18) QUESTION 10. Si una matriz simétrica de 2 por 2 satisface $a > 0$, y $ac > b^2$, demuestre que sus autovalores son reales y positivos (definida positiva). Emplee la ecuación característica y el hecho de que el producto de los autovalores es igual al determinante.
(Strang, 2006, exercise 3 from section 6.1.)

(L-18) QUESTION 11. Decida si las siguientes matrices son definidas positivas, definidas negativas, semi-definidas, o indefinidas.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 6 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 3 \end{bmatrix}$

(c) $\mathbf{C} = -\mathbf{B}$

(d) $\mathbf{D} = \mathbf{A}^{-1}$

(L-18) QUESTION 12. Una matriz definida positiva no puede tener un cero (o incluso peor; un número negativo) en su diagonal principal. Demuestre que esta matriz no cumple $\boldsymbol{x}\mathbf{A}\boldsymbol{x} > 0$, para todo $\boldsymbol{x} \neq \mathbf{0}$:

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{no es positiva cuando} \quad \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} = \begin{pmatrix} \quad \end{pmatrix}$$

(Strang, 2006, exercise 21 from section 6.2.)

(L-18) QUESTION 13. Demuestre que si $\mathbf{A}$ y $\mathbf{B}$ son definidas positivas entonces $\mathbf{A} + \mathbf{B}$ también es definida positiva. Para esta demostración los pivotes y los valores característicos no son convenientes. Es mejor emplear $\boldsymbol{x}(\mathbf{A} + \mathbf{B})\boldsymbol{x} > 0$
(Strang, 2006, exercise 5 from section 6.2.)

(L-18) QUESTION 14. Find the $\dot{\mathbf{L}}\mathbf{D}\dot{\mathbf{L}}^{\intercal}$ factorization for the following symmetric matrices.

(a)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

(L-18) QUESTION 15. La forma cuadrática $f(x, y) = 3(x + 2y)^2 + 4y^2$ es definida positiva. Encuentre la matriz $\mathbf{A}$, factorícela en $\mathbf{L}\mathbf{D}\mathbf{L}^{\intercal}$, y relacione los elementos en $\mathbf{D}$ y $\mathbf{L}$ con 3, 2 y 4 en f.
(Strang, 2006, exercise 9 from section 6.1.)

(L-18) QUESTION 16. Consider the following matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & a & a \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) (0.5^{pts}) Compute the eigenvalues of $\mathbf{A}$.
(b) (0.5^{pts}) Prove that when $a = 2$ the matrix $\mathbf{A}$ is not diagonalisable.
(c) (1^{pts}) For matrix $\mathbf{B}$, find a diagonal matrix $\mathbf{D}$ and an orthonormal matrix $\mathbf{P}$ such as $\mathbf{B} = \mathbf{P}\mathbf{D}\mathbf{P}^{\intercal}$.
(d) (0.5^{pts}) Find the quadratic form $f(x, y, z)$ associated to $\mathbf{B}$, and prove it is positive defined.

(L-18) QUESTION 17. Given the matrix $\mathbf{A} = \left(\begin{smallmatrix} a & 3/5 \\ b & 4/5 \end{smallmatrix}\right)$, compute the values (if they exist) of a and b such as

(a) (0.5^{pts}) $\mathbf{A}$ is ortho-normal.
(b) (0.5^{pts}) Columns of $\mathbf{A}$ are linearly independent.
(c) (0.5^{pts}) $\lambda = 0$ is an eigenvalue of $\mathbf{A}$.
(d) (0.5^{pts}) $\mathbf{A}$ is a symmetric definite negative matrix.

(L-18) QUESTION 18.

(a) Consider the quadratic form $q(x, y, z) = x^2 + 2xy + ay^2 + 8z^2$ and find its corresponding symmetric matrix $\mathbf{Q}$; determine if $\mathbf{Q}$ is positive-definite, positive-semidefinite, negative-definite, negative-semidefinite or indefinite when the parameter a is equal to one $(a = 1)$.
(b) If $a \neq 1$, determine whether the matrix is positive-definite, positive-semidefinite, negative-definite, negative-semidefinite or indefinite.

Questions of the Optional Lecture 2

(L-Opt-2) Question 1. Consider the following matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(a) (0.5^{pts}) Prove **A** is invertible if and only if $a \neq 0$.
(b) (0.5^{pts}) Is **A** positive definite when $a = 1$? Explain your answer.
(c) (1^{pts}) Compute $\mathbf{A}^{-1}$ when $a = 2$.
(d) (0.5^{pts}) How many variables can be chosen as pivot (or exogenous) variables in the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{o}$ when $a = 0$? Which ones?

(L-Opt-2) Question 2. True or false (to receive full credit you must explain your answers in a clear and concise way)

(a) If **A** is symetric, then so it is $\mathbf{A}^2$.
(b) If $\mathbf{A}^2 = \mathbf{A}$ then $(\mathbf{I} - \mathbf{A})^2 = (\mathbf{I} - \mathbf{A})$ where **I** is the identity matrix.
(c) If $\lambda = 0$ is an eigenvalue of the squared matrix **A**, then the linear system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ is is always solvable and has only one solution.
(d) If $\lambda = 0$ is an eigenvalue of the squared matrix **A**, then the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ could be unsolvable.
(e) If a matrix is orthogonal (perpendicular columns of norm one), then so it is the inverse of that matrix.

(f) If 1 is the only eigenvalue of a 2×2 matrix **A**, then **A** must be the identity matrix **I**.

(L-Opt-2) Question 3. complete los blancos, o responda **V**erdadero/**F**also.

(a) Cualquier sistema generador de un espacio vectorial contiene una base del espacio (V/F)

____________________.

(b) Que los vectores $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ sean linealmente independientes significa que

____

(c) El conjunto que sólo contiene el vector $\mathbf{0}$ es un conjunto linealmente independiente. (V/F)

____________________.

(d) Una matriz cuadrada de orden n por n es diagonalizable cuando:

____

(e) Si $\boldsymbol{u} = (1, 2, -1, 1)$, entonces $||\boldsymbol{u}|| =$ ________________.
(f) Si $\boldsymbol{u} = (1, 2, -1, 1)$ y $\boldsymbol{v} = (-2, 1, 0, 0)$, entonces $\boldsymbol{u} \cdot \boldsymbol{v} =$ ________________.

(L-Opt-2) Question 4. En las preguntas siguientes **A** y **B** son matrices $n \times n$. Indique si las siguientes afirmaciones son verdaderas o falsas (incluya una breve explicación, o un contra ejemplo que justifique su respuesta):

(a) Si **A** no es cero entonces $\det(\mathbf{A}) \neq 0$
(b) Si $\det(\mathbf{AB}) \neq 0$ entonces **A** es invertible.
(c) Si intercambio las dos primeras filas de **A** sus autovalores cambian.
(d) Si **A** es real y simétrica, entonces sus autovalores son reales **(aquí no es necesaria una justificación)**.
(e) Si la forma reducida de echelon de $(\mathbf{A} - 5\mathbf{I})$ es la matriz identidad, entonces 5 no es un autovalor de **A**.
(f) Sea $\boldsymbol{b}$ un vector columna de $\mathbb{R}^n$. Si el sistema $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ no tiene solución, entonces $\det(\mathbf{A}) \neq 0$
(g) Sea **C** de orden 3×5. El rango de **C** puede ser 4.
(h) Sea **C** de orden $n \times m$, y $\boldsymbol{b}$ un vector columna de $\mathbb{R}^n$. Si $\mathbf{C}\boldsymbol{x} = \boldsymbol{b}$ no tiene solución, entonces $\operatorname{rg}(\mathbf{C}) < n$.
(i) Toda matriz diagonalizable es invertible.
(j) Si **A** es invertible, entonces su forma reducida de echelon es la matriz identidad.

(L-Opt-2) Question 5. Sean

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & 5 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 0 & 4 \\ 0 & 0 & 6 \end{bmatrix}; \quad \mathbf{D} = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Los autovalores de **B** son 0 y 2. Use esta información para responder a las siguientes cuestiones. Para cada matriz debe dar una explicación. Puede haber más de una matriz que cumpla la condición:

(a) ¿Qué matrices son invertibles?
(b) ¿Qué matrices tienen un autovalor repetido?
(c) ¿Qué matrices tienen rango menor a tres?
(d) ¿Qué matrices son diagonalizables?
(e) ¿Para qué matrices diagonalizables podemos encontrar tres autovectores ortogonales entre si?

(L-Opt-2) Question 6. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

(a) Find the eigenvalues and eigenvectors of **A**.
(b) Is **A** diagonalizable?

(c) Is it possible to find a matrix $\mathbf{P}$ such as $\mathbf{A} = \mathbf{PDP}^\intercal$, where $\mathbf{D}$ is diagonal?
(d) Find $|\mathbf{A}^{-1}|$.

(L-Opt-2) Question 7. Consider a 3 by 3 matrix $\mathbf{A}$ with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = -1$; and let $\boldsymbol{v}_1 = (1, 0, 1)^\intercal$ and $\boldsymbol{v}_2 = (1, 1, 1)^\intercal$ be the corresponding eigenvectors to λ_1 and λ_2.

(a) Is $\mathbf{A}$ diagonalizable?
(b) Is $\boldsymbol{v}_3 = (-1, 0, -1)^\intercal$ an eigenvector associated to the eigenvalue $\lambda_3 = -1$?
(c) Compute $\mathbf{A}(\boldsymbol{v}_1 - \boldsymbol{v}_2)$.

(L-Opt-2) Question 8.

(a) (0.5^{pts}) Find an homogeneous system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ such as its solutions set is

$$\left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \exists \alpha, \beta, \gamma \in \mathbb{R} \quad \text{such that } \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \right\}$$

(b) (0.5^{pts}) If the characteristic polynomial of a matrix $\mathbf{A}$ is $p(\lambda) = \lambda^5 + 3\lambda^4 - 24\lambda^3 + 28\lambda^2 - 3\lambda + 10$, find the rank of $\mathbf{A}$.

(L-Opt-2) Question 9. Suponga una matriz cuadrada e *invertible* $\underset{n\times n}{\mathbf{A}}$.

(a) ¿Cuáles son sus espacios columna $\mathcal{C}(\mathbf{A})$ y espacio nulo $\mathcal{N}(\mathbf{A})$? (no responda con la definición, diga qué conjunto de vectores compone cada espacio).
(b) Suponga que $\mathbf{A}$ puede ser factorizada en $\mathbf{A} = \mathbf{LU}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 7 & 3 & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{12} & u_{13} \\ 0 & 0 & u_{13} \end{bmatrix}$$

Describa el primer paso de eliminación en la reducción de $\mathbf{A}$ a $\mathbf{U}$. ¿porqué sabe que $\mathbf{U}$ es también una matriz invertible? ¿Cuanto vale el determinante de $\mathbf{A}$?
(c) Encuentre una matriz particular de dimensiones 3×3 e invertible $\mathbf{A}$ que no pueda ser factorizada en la forma $\mathbf{LU}$ (sin permutar previamente las filas). ¿Qué factorización es todavía posible en su ejemplo? (no es necesario que realice la factorización). ¿Cómo sabe que su matriz $\mathbf{A}$ es invertible?

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