

# Mathematics II

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## 1 Highlights of Lesson 4

### Highlights of Lesson 4

- Elementary transformations
- Identifying singular matrices by elimination
- Matrix multiplication of Elementary matrices

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You can find the last version of these course materials at

<https://mbujosab.github.io/MatematicasII/>

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## 2 Elementary transformations of a matrix

Type I:  $\mathbf{A}_{\tau_{[(\lambda)i+j]}}$  (with  $i \neq j$ )

add  $\lambda$  times  $i$ -th column ( $\lambda \mathbf{A}_{|i}$ ) to  $j$ -th column ( $\mathbf{A}_{|j}$ )

$$\begin{bmatrix} 1 & -3 & 0 \\ 1 & -6 & 3 \end{bmatrix}_{\tau_{[(-2)1+3]}} = \begin{bmatrix} 1 & -3 & -2 \\ 1 & -6 & 1 \end{bmatrix}$$

Type II:  $\mathbf{A}_{\tau_{[(\alpha)i]}}$  (with  $\alpha \neq 0$ )

multiply by  $\alpha$  the  $i$ -th column

$$\begin{bmatrix} 1 & -3 & 0 \\ 1 & -6 & 3 \end{bmatrix}_{\tau_{[(10)2]}} = \begin{bmatrix} 1 & -30 & 0 \\ 1 & -60 & 3 \end{bmatrix}$$

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### 3 Elimination and pre-echelon form of a matrix

- **Pivot** is the first non-zero component of each column.
- **Elimination**: modifies a matrix until all **components at the right-hand side of each pivot are zeros**

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 8 & 4 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{[(-3)\tau_1+2]} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix} \xrightarrow{[(-2)\tau_2+3]} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & -2 & 5 \end{bmatrix} = \mathbf{L}$$

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### 4 Elimination

#### Elimination algorithm on $\mathbf{A}$

modifies  $\mathbf{A}$  using a sequence of *elementary transformations*

#### Goal

to get a (pre)echelon form

- **pre-echelon**: all components on the right side of each pivot are zero.
- **echelon**: if any column before a non-null column  $\mathbf{A}_{|j}$  is non-null column and its pivot is above the pivot of  $\mathbf{A}_{|j}$ .

It is always possible to find a (pre)echelon form by elimination

**Rank** (rg): the number of pivots in any of its pre-echelon forms

$\mathbf{A}$  is **singular** if its pre-echelon forms have null-columns (rg < n)  
 $n \times n$

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### 5 Elimination: When can't we find $n$ pivots?

$n \times n$  matrices are **singular** if less than  $n$  pivots after elimination

$$\begin{bmatrix} 0 & 1 & 3 \\ 4 & 2 & 8 \\ 1 & 1 & 1 \end{bmatrix}$$

Has this matrix  $n$  pivots?  $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ 1 & 1 & 1 \end{bmatrix}$

and this one?  $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

and this one?  $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & -4 \end{bmatrix}$

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### 6 Matrix multiplication: elementary matrices

$$\underbrace{\begin{bmatrix} 1 & 3 & 0 \\ 2 & 8 & 4 \\ 1 & 1 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\left[ \begin{array}{c} \\ \\ \end{array} \right]}_{\mathbf{I}_\tau} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix}}_{\mathbf{A}_\tau}$$

We call  $\mathbf{I}_\tau$  "Elementary matrix":

$$\mathbf{A}(\mathbf{I}_\tau) = \mathbf{A}_\tau$$

This specific elementary matrix  $\mathbf{I}_\tau$  is written as  $\mathbf{I}_{\tau_{[(-3)\tau_1+2]}}$

$$\mathbf{A}(\mathbf{I}_{\tau_{[(-3)\tau_1+2]}}) = \mathbf{A}_{\tau_{[(-3)\tau_1+2]}}$$

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### 7 Matrix multiplication: elementary matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \phantom{0} \\ \phantom{0} \\ \phantom{0} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & -2 & 5 \end{bmatrix}$$

This specific elementary matrix  $\mathbf{I}_\tau$  is written as  $\mathbf{I}_{\tau_{[(-2)2+3]}}$

$$\mathbf{A} \left( \mathbf{I}_{\tau_{[(-2)2+3]}} \right) = \mathbf{A}_{\tau_{[(-2)2+3]}}$$

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### 8 Elimination by elementary matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 8 & 4 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{[(-3)1+2]} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix} \xrightarrow{[(-2)2+3]} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & -2 & 5 \end{bmatrix} = \mathbf{L}$$

$$\mathbf{A}_{\tau_{\begin{bmatrix} [(-3)1+2] \\ [(-2)2+3] \end{bmatrix}}} = \mathbf{A}_{\begin{bmatrix} [(-3)1+2] \\ [(-2)2+3] \end{bmatrix}} = \left( \mathbf{A} \left( \mathbf{I}_{\tau_{[(-3)1+2]}} \right) \right) \left( \mathbf{I}_{\tau_{[(-2)2+3]}} \right) = \mathbf{L}$$

there is a matrix that does the whole job **at once**

$$\mathbf{A}_{\tau_{\begin{bmatrix} [(-3)1+2] \\ [(-2)2+3] \end{bmatrix}}} = \mathbf{A} \left( \left( \mathbf{I}_{\tau_{[(-3)1+2]}} \right) \left( \mathbf{I}_{\tau_{[(-2)2+3]}} \right) \right) = \mathbf{A} \mathbf{I}_{\tau_{\begin{bmatrix} [(-3)1+2] \\ [(-2)2+3] \end{bmatrix}}} = \mathbf{L}$$

$$\mathbf{A}_{\tau_1 \dots \tau_k} = \mathbf{A} \left( \mathbf{I}_{\tau_1 \dots \tau_k} \right)$$

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### 9 how do I get from $\mathbf{L}$ back to $\mathbf{A}$ ? Inverses

How do I reverse the first step? (it was subtract 3 times  $\mathbf{A}_{|1}$  from  $\mathbf{A}_{|2}$ )

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{I}_{\tau_{[(-\lambda)i+j]}} \text{ "undo" } \mathbf{I}_{\tau_{[(\lambda)i+j]}}$$

How to undo  $\mathbf{I}_{\tau_{[(\alpha)i]}}$ ?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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### 10 Interchange or swap matrices

Which matrix exchanges the columns?

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} \phantom{0} \\ \phantom{0} \end{bmatrix} = \begin{bmatrix} c & a \\ d & b \end{bmatrix}$$

Which matrix exchanges the rows? where do we put that matrix?

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} b & d \\ a & c \end{bmatrix}$$

Matrix multiplication is not commutative!

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## 11 Interchange of columns

Interchange of columns:

$\mathbf{A}_{\tau_{[i \rightleftharpoons j]}}$  → swicht columns  $i$  and  $j$  of  $\mathbf{A}$

$$\begin{bmatrix} 1 & -3 & 0 \\ 1 & -6 & 3 \end{bmatrix}_{\tau_{[2 \rightleftharpoons 3]}} = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 3 & -6 \end{bmatrix}$$

We can switch two columns by a sequence of elementary transformations

Matrix  $\mathbf{I}_{\tau_{[i \rightleftharpoons j]}}$  is call a exchange matrix

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## 12 Permutation matrices

Product between exchange matrices  $\mathbf{I}_{\tau_{[i \rightleftharpoons j]}}$  is a permutation matrix  $\mathbf{I}_{\tau_{[\mathfrak{S}]}}$ .

$\mathbf{I}_{\tau_{[\mathfrak{S}]}}$  = Identity matrix  $\mathbf{I}$  with rearranged columns

Let's see the  $3 \times 3$  case

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad \mathbf{I}_{\tau_{[1 \rightleftharpoons 2]}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

How many  $3 \times 3$  pemutations can we find?

what happens if I multiply two permutation matrices?

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## Questions of the Lecture 4

(L-4) QUESTION 1.

(a) Which three matrices  $\mathbf{I}_{\tau_{[(x)1+2]}}$ ,  $\mathbf{I}_{\tau_{[(y)1+3]}}$  and  $\mathbf{I}_{\tau_{[(z)2+3]}}$  put  $\mathbf{A} = \begin{bmatrix} 1 & 4 & -2 \\ 1 & 6 & 2 \\ 0 & 1 & 0 \end{bmatrix}$  into an echelon form?

(b) Multiply those  $\mathbf{I}_{\tau_i}$  to get one matrix  $\mathbf{E}$  that does elimination:  $\mathbf{AE} = \mathbf{K}$ .

Based on (Strang, 1988, exercise 24 from section 1.4.)

(L-4) QUESTION 2. Consider the matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ -1 & -3 & -2 \\ 0 & 1 & c \end{bmatrix}$$

For what value(s) of  $c$  the matrix is singular (we can't find three pivots)?

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(L-4) QUESTION 3. Consider the following 3 by 3 matrices.

(a)  $(\mathbf{I}_{\tau_{[(-1)1+2]}})$  subtracts column 1 from column 2 and then  $(\mathbf{I}_{\tau_{[2 \rightleftharpoons 3]}})$  exchanges columns 2 and 3. What matrix  $\mathbf{E}$  does both steps at once?

(b)  $(\mathbf{I}_{\tau_{[2 \rightleftharpoons 3]}})$  exchanges columns 2 and 3 and then  $(\mathbf{I}_{\tau_{[(-1)1+3]}})$  subtracts column 1 from column 3. What matrix  $\mathbf{N} = (\mathbf{I}_{\tau_{[2 \rightleftharpoons 3]}})(\mathbf{I}_{\tau_{[(-1)1+3]}})$  does both steps at once?

Explain why  $\mathbf{M}$  and  $\mathbf{N}$  are the same but the  $\mathbf{I}_{\tau}$ 's are different.

Based on (Strang, 1988, exercise 28 from section 1.4.)

(L-4) QUESTION 4. Elimination matrices  $\mathbf{I}_{\tau_{[(?)1+2]}}$  and  $\mathbf{I}_{\tau_{[(?)2+3]}}$  will reduce  $\mathbf{A}$  to triangular form. Find  $\mathbf{E}$  so that  $\mathbf{AE} = \mathbf{L}$  is lower triangular (echelon), if  $\mathbf{A}$  is

$$\begin{bmatrix} 2 & 2 & 0 \\ 1 & 4 & 9 \\ 1 & 3 & 9 \end{bmatrix}$$

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(L-4) QUESTION 5. Although we will only consider as elementary the *Type I* and *II* transformations, in most of the Linear Algebra books appears a third type: the *exchange* of columns

$$\mathbf{A} \begin{smallmatrix} \tau \\ [p \rightleftharpoons s] \end{smallmatrix} \rightarrow \text{Exchanges columns } p \text{ and } s \text{ of } \mathbf{A}.$$

Prove that a column exchange is, in fact, a sequence of *Type I* and *II* elementary transformations. Try transforming  $\mathbf{I}_{2 \times 2}$  in  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  by elementary transformations of the columns.

(L-4) QUESTION 6. Write down the 3 by 3 matrices that produce these elimination steps:

- (a)  $\mathbf{I} \begin{smallmatrix} \tau \\ [(-5)1+2] \end{smallmatrix}$  subtracts 5 times column 1 from column 2,  
 (b)  $\mathbf{I} \begin{smallmatrix} \tau \\ [(-7)2+3] \end{smallmatrix}$  subtracts 7 times column 2 from column 3,  
 (c)  $\mathbf{I} \begin{smallmatrix} \tau \\ [\leftrightarrow] \end{smallmatrix}$  exchanges columns 1 and 2, and then columns 2 and 3.

(Strang, 2003, exercise 1 from section 2.3.)

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(L-4) QUESTION 10. If every column of  $\mathbf{A}$  is a multiple of  $(1, 1, 1)$ , then  $\mathbf{A}\mathbf{x}$  is always a multiple of  $(1, 1, 1)$ . Do a 3 by 3 example. How many pivots are produced by elimination?  
 (Strang, 1988, exercise 26 from section 1.4.)

(L-4) QUESTION 7. Consider the matrices of QUESTION 6:

(a) when multiplying by  $\mathbf{I} \begin{smallmatrix} \tau \\ [(-5)1+2] \end{smallmatrix}$  and then by  $\mathbf{I} \begin{smallmatrix} \tau \\ [(-7)2+3] \end{smallmatrix}$  the matrix  $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$  we get  $\mathbf{A} \begin{smallmatrix} \tau \\ [(-5)1+2] \\ [(-7)2+3] \end{smallmatrix} = \begin{bmatrix} \phantom{0} \\ \phantom{0} \\ \phantom{0} \end{bmatrix}$ .

(b) But, when multiplying by  $\mathbf{I} \begin{smallmatrix} \tau \\ [(-5)1+2] \end{smallmatrix}$  before and then by  $\mathbf{I} \begin{smallmatrix} \tau \\ [(-7)2+3] \end{smallmatrix}$  we get

$$\mathbf{A} \begin{smallmatrix} \tau \\ [(-7)2+3] \\ [(-5)1+2] \end{smallmatrix} = \begin{bmatrix} \phantom{0} \\ \phantom{0} \\ \phantom{0} \end{bmatrix}.$$

(c) When  $\mathbf{I} \begin{smallmatrix} \tau \\ [(-7)2+3] \end{smallmatrix}$  comes first, the column \_\_\_\_ feels no effect from column \_\_\_\_.

This property will become very important in the LU factorization!

(Strang, 2003, exercise 2 from section 2.3.)

(L-4) QUESTION 8. What matrix  $\mathbf{M}$  sends  $\mathbf{v} = (1, 0)$  to  $(0, 1)$ , es decir  $\mathbf{vM} = (0, 1)$ ; and also sends  $\mathbf{w} = (0, 1)$  to  $(1, 0)$ , es decir  $\mathbf{wM} = (1, 0)$ ?

(L-4) QUESTION 9. Consider a permutation (interchange) matrix  $\mathbf{I} \begin{smallmatrix} \tau \\ [i \rightleftharpoons j] \end{smallmatrix}$ , if we compute the product  $\mathbf{A}(\mathbf{I} \begin{smallmatrix} \tau \\ [i \rightleftharpoons j] \end{smallmatrix})$ , we get a new matrix like  $\mathbf{A}$ , but with exchanged columns. What happen if we compute the product  $(\mathbf{I} \begin{smallmatrix} \tau \\ [i \rightleftharpoons j] \end{smallmatrix})\mathbf{A}$ ? Check your answer with a 2 by 2 example.

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## 1 Highlights of Lesson 5

### Highlights of Lesson 5

- Inverse of  $\mathbf{A}$
- Gauss-Jordan elimination / finding  $\mathbf{A}^{-1}$
- Inverse of  $\mathbf{AB}$ ,  $\mathbf{A}^T$

## 2 Inverse of a matrix (square matrices)

A **squared** of order  $n$  has inverse (is *invertible*) if exists **B** such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}.$$

Then

$$\mathbf{B} = \mathbf{A}^{-1} \quad \text{and} \quad \mathbf{A} = \mathbf{B}^{-1}.$$

Not all matrices have inverse

*Squared matrices with no inverse* are called *singular* matrices

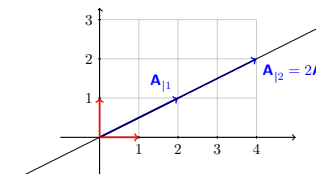
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## 3 Singular case (no inverse)

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

Is it possible to find a matrix **B** such that  $\mathbf{AB} = \mathbf{I}$ ?

... columns of **I** should be linear combinations of columns of **A**... but both columns lie on the same line.



So

**A is singular**

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## 4 Singular case (no inverse)

Can we find  $x \neq 0$  such that  $\mathbf{Ax} = 0$ ?

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{pmatrix} \phantom{0} \\ \phantom{0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If  $\mathbf{Ax} = 0$  and  $x \neq 0 \Rightarrow$  there is no  $\mathbf{A}^{-1}$

The existence of  $\mathbf{A}^{-1}$  leads to a **contradiction**

If  $\mathbf{Ax} = 0$  and  $x \neq 0 \Rightarrow \mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}0 \Rightarrow x = 0.$

When  $\mathbf{A}^{-1}$  does exist

the **only** solution to  $\mathbf{Ax} = 0$  is  $x = 0.$

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## 5 Calculating the inverse matrix

$$\mathbf{A}(\mathbf{A}^{-1}) = \mathbf{I}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & \phantom{0} \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So... we are solving  $m$  systems (of  $m$  equations each)

$$\begin{bmatrix} 1 & 3 \\ 2 & \phantom{0} \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & \phantom{0} \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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## 6 Gauss-Jordan: solving two linear systems at once

### Gauss-Jordan elimination (obtaining a reduced echelon form $\mathbf{R}$ )

apply elementary transformations until a echelon matrix with only zeros to the left of each pivot (and all pivots equal to 1) is achieved

Let's solve the linear systems

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

applying Gauss-Jordan elimination on  $\mathbf{A}$  stacked with  $\mathbf{I}$

$$\left[ \begin{array}{c} \mathbf{A} \\ \mathbf{I} \end{array} \right] = \begin{bmatrix} 1 & 3 \\ 2 & 7 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \quad \rightarrow \quad =$$

If  $\mathbf{R} = \mathbf{I}$ , we have found  $\mathbf{A}^{-1}$

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## 8 Inverse of a product

When  $\mathbf{A}$  and  $\mathbf{B}$ , of order  $n$ , are invertible,  $(\mathbf{AB})$  is invertible.

what matrix gives me the inverse of  $\mathbf{AB}$ ? let's try with  $(\mathbf{B}^{-1}\mathbf{A}^{-1})$ :

$$\mathbf{AB}(\mathbf{B}^{-1}\mathbf{A}^{-1}) =$$

$$(\mathbf{B}^{-1}\mathbf{A}^{-1})\mathbf{AB} =$$

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## 7 Gauss-Jordan: Why does it work?

$$\left[ \begin{array}{c} \mathbf{A} \\ \mathbf{I} \end{array} \right] = \begin{bmatrix} 1 & 3 \\ 2 & 7 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{[(-3)\mathbf{I}_1 + \mathbf{I}_2]} \quad \xrightarrow{[(-2)\mathbf{I}_2 + \mathbf{I}_1]}$$

that is, since  $\mathbf{A}_{\tau_1 \dots \tau_k} = \mathbf{A}(\mathbf{I}_{\tau_1 \dots \tau_k})$ :

$$\left[ \begin{array}{c} \mathbf{A} \\ \mathbf{I} \end{array} \right]_{\tau_1 \dots \tau_k} = \left[ \begin{array}{c} \mathbf{A}_{\tau_1 \dots \tau_k} \\ \mathbf{I}_{\tau_1 \dots \tau_k} \end{array} \right] = \left[ \begin{array}{c} \mathbf{A}(\mathbf{I}_{\tau_1 \dots \tau_k}) \\ \mathbf{I}_{\tau_1 \dots \tau_k} \end{array} \right] = \left[ \begin{array}{c} \mathbf{I} \\ \mathbf{I}_{\tau_1 \dots \tau_k} \end{array} \right],$$

who is  $\mathbf{I}_{\tau_1 \dots \tau_k}$ ?

therefore  $\mathbf{A}^{-1} =$

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## 9 Inverse of a transpose matrix

$$\mathbf{AA}^{-1} = \mathbf{I}$$

let me transpose both sides

$$((\mathbf{A}^{-1})^T)^T \mathbf{A}^T = \mathbf{I}$$

then

the inverse of  $\mathbf{A}^T$  is

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## 10 Interchanges and permutations

Are interchange matrices  $\mathbf{I}_{\tau_{[i \rightleftharpoons j]}}$ , invertible?

It is easy to check that

$$\left(\mathbf{I}_{\tau_{[i \rightleftharpoons j]}}\right)^T \left(\mathbf{I}_{\tau_{[i \rightleftharpoons j]}}\right) = \mathbf{I} \quad \Rightarrow$$

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## Questions of the Lecture 5

(L-5) QUESTION 1. Use the Gauss-Jordan method to invert

(a)  $\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ .

(b)  $\mathbf{A}_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ .

(c)  $\mathbf{A}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ .

(Strang, 1988, exercise 6 from section 1.6.)

(L-5) QUESTION 2.

(a) If  $\mathbf{A}$  is invertible and  $\mathbf{AB} = \mathbf{AC}$ , prove quickly that  $\mathbf{B} = \mathbf{C}$ .

(b) If  $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ , find an example with  $\mathbf{AB} = \mathbf{AC}$ , but  $\mathbf{B} \neq \mathbf{C}$ .

(Strang, 1988, exercise 4 from section 1.6.)

(L-5) QUESTION 3. Use the Gauss-Jordan method to invert the generic matrix  $2 \times 2$

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The matrix is invertible (not singular) only when ...

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## 11 Caracterización de invertible matrices

Given  $\mathbf{A}$  of order  $n$ , the following statements are equivalent

1. No zero columns in  $\mathbf{A}_{\tau_1 \dots \tau_p} = \mathbf{K}$  (pre-echelon matrix).
2.  $\mathbf{A}$  has inverse.
3.  $\mathbf{A}$  is product of elementary matrices.

$$\mathbf{A}_{\tau_1 \dots \tau_k} = \mathbf{A}(\mathbf{I}_{\tau_1 \dots \tau_k}) = \mathbf{I} \quad \Rightarrow \quad \mathbf{A} = (\mathbf{I}_{\tau_1 \dots \tau_k})^{-1}$$

where

$$(\mathbf{I}_{\tau_1 \dots \tau_k})^{-1} = ((\mathbf{I}_{\tau_1}) \cdots (\mathbf{I}_{\tau_k}))^{-1} = (\mathbf{I}_{\tau_k^{-1}}) \cdots (\mathbf{I}_{\tau_1^{-1}}) = \mathbf{I}_{\tau_k^{-1} \dots \tau_1^{-1}}$$

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(L-5) QUESTION 4. Use the Gauss-Jordan method to invert the following matrices.

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 6 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 4 & -2 \\ 1 & 3 & 1 \end{bmatrix}$$

(L-5) QUESTION 5. If the 3 by 3 matrix  $\mathbf{A}$  has  $\mathbf{A}_{|1} + \mathbf{A}_{|2} = \mathbf{A}_{|3}$ , show that  $\mathbf{A}$  is not invertible, by two different methods:

(a) Find a nonzero solution  $\mathbf{x}$  to  $\mathbf{Ax} = \mathbf{0}$ .

(b) Elimination keeps *column 1 + column 2 = column 3*. Explain why there is no third pivot.

(Strang, 1988, exercise 26 from section 1.6.)

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(L-5) QUESTION 6. Find the inverses of

$$(a) \mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}.$$

$$(b) \mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}.$$

$$(c) \mathbf{A}_3 = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}.$$

(Strang, 1988, exercise 10 from section 1.6.)

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(L-5) QUESTION 7. Find the inverse of

$$\mathbf{A} = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}$$

What values of  $a$  and  $b$  make the matrix singular?  
(Strang, 1988, exercise 42 from section 1.6.)

(L-5) QUESTION 8. Find  $\mathbf{E}^2$ ,  $\mathbf{E}^8$  and  $\mathbf{E}^{-1}$  if  $\mathbf{E} = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$   
(Strang, 1988, exercise 6 from section 1.5.)

(L-5) QUESTION 9. Consider the following permutation matrix:

$$\mathbf{I}_{\tau_{[\mathbf{E}]}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Find  $\mathbf{I}_{\tau_{[\mathbf{E}]}}^{-1}$ . Can you say something else about the relationship between  $\mathbf{I}_{\tau_{[\mathbf{E}]}}$  and  $\mathbf{I}_{\tau_{[\mathbf{E}]}}^{-1}$ ?

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(L-5) QUESTION 10. The 3 by 3 matrix  $\mathbf{A}$  reduces to the identity matrix  $\mathbf{I}$  by the following three column operations (in order):

$\tau_{[(-4)1+2]}$  : Subtract 4 times column 1 from column 2.

$\tau_{[(-3)1+3]}$  : Subtract 3 times column 1 from column 3.

$\tau_{[(-1)3+2]}$  : Subtract column 3 from column 2.

(a) Write  $\mathbf{A}^{-1}$  in terms of elementary matrices  $\mathbf{I}_{\tau}$ . Then compute  $\mathbf{A}^{-1}$ .

(b) What is the original matrix  $\mathbf{A}$ ?

(Based on MIT Course 18.06 Quiz 1, October 4, 2006)

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(L-5) QUESTION 11. The 3 by 3 matrix  $\mathbf{A}$  reduces to the identity matrix  $\mathbf{I}$  by the following three row operations (in order):

$\tau_{[(-4)1+2]}$  : Subtract 4 times row 1 from row 2.

$\tau_{[(-3)1+3]}$  : Subtract 3 times row 1 from row 3.

$\tau_{[(-1)3+2]}$  : Subtract row 3 from row 2.

(a) Write  $\mathbf{A}^{-1}$  in terms of the  $\mathbf{E}$ 's. Then compute  $\mathbf{A}^{-1}$ .

(b) What is the original matrix  $\mathbf{A}$ ?

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(L-5) QUESTION 12.

(a) Find the inverse of  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$  and  $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(b) Find the inverse of the following matrix **using the Gauss-Jordan method**

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & d \end{bmatrix}$$

(Poole, 2004, exercise 36, 38 and 59 from section 3.3.)

(L-5) QUESTION 13. Consider the squared matrices **A**, **B**, and **C**. True or false?

(a) If  $\mathbf{AB} = \mathbf{I}$  and  $\mathbf{CA} = \mathbf{I}$  then  $\mathbf{B} = \mathbf{C}$ .

(b)  $(\mathbf{AB})^2 = \mathbf{A}^2\mathbf{B}^2$ .

(L-5) QUESTION 14. Consider the matrix  $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & a & 0 & 2a \\ a & 0 & 1 & 0 \\ 1 & 0 & a & 1 \end{bmatrix}$

(a) Prove that **A** is invertible for any value of  $a$ .

(b) Compute  $\mathbf{A}^{-1}$  when  $a = 0$ .

(L-5) QUESTION 15. Consider the matrix  $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ . Find  $\mathbf{A}^{-1}$ .

(L-5) QUESTION 16. Find (if it is possible) the inverse of the following inverses

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}.$$

(L-5) QUESTION 17. There is a finite number  $(n!)$  of  $n \times n$  permutation matrices. In addition, any power of a permutation matrix is a another permutation matrix. Use these facts to prove that  $(\mathbf{I}_{\mathcal{T}})_{[\mathcal{S}]}^r = \mathbf{I}$  for some integer numbers  $r$ .

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