THE COMPLETE GUIDE TO Option Pricing Formulas

SECOND EDITION

ESPEN GAARDER HAUG

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1.1 BLACK-SCHOLES-MERTON

Provided here are the various versions of the Black-Scholes-Merton formula presented in the literature. All formulas in this section are originally derived based on the underlying asset S follows a geometric Brownian motion

$$dS = \mu S dt + \sigma S dz,$$

where μ is the expected instantaneous rate of return on the underlying asset, σ is the instantaneous volatility of the rate of return, and dz is a Wiener process.

1.1.1 The Black-Scholes Option Pricing Formula

The formula derived by Black and Scholes (1973) can be used to value a European option on a stock that does not pay dividends before the option's expiration date.[1] Letting c and p denote the price of European call and put options, respectively, the formula states that

$$c = SN(d_1) - Xe^{-rT}N(d_2) \tag{1.1}$$

$$p = Xe^{-rT}N(-d_2) - SN(-d_1), \tag{1.2}$$

where

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S/X) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

S = Stock price.
X = Strike price of option.
r = Risk-free interest rate.
T = Time to expiration in years.
σ = Volatility of the relative price change of the underlying stock price.
$N(x)$ = The cumulative normal distribution function, described in Chapter 13.

[1] The Black-Scholes formula can also be used to price American call options on a nondividend-paying stock, since it will never be optimal to exercise the option before expiration.

Example
Consider a European call option with three months to expiry. The stock price is 60, the strike price is 65, the risk-free interest rate is 8% per year, and the volatility is 30% per annum. Thus, $S = 60$, $X = 65$, $T = 0.25$, $r = 0.08$, $\sigma = 0.3$,

$$d_1 = \frac{\ln(60/65) + (0.08 + 0.3^2/2)0.25}{0.3\sqrt{0.25}} = -0.3253$$

$$d_2 = d_1 - 0.3\sqrt{0.25} = -0.4753$$

The value of the cumulative normal distribution $N(\cdot)$ can be found using the approximation function in Chapter 13:

$$N(d_1) = N(-0.3253) = 0.3725 \qquad N(d_2) = N(-0.4753) = 0.3173$$

$$c = 60N(d_1) - 65e^{-0.08\times0.25}N(d_2) = 2.1334$$

Computer algorithm
The *BlackScholes*$(\cdot)$ function returns the call price if the *CallPutFlag* is set equal to "c" or the put price when set equal to "p." In the computer code $v = \sigma$.

```
Function BlackScholes(CallPutFlag As String, S As Double, X _
                As Double, T As Double, r As Double, v As Double) As Double

    Dim d1 As Double, d2 As Double

    d1 = (Log(S / X) + (r + v^2 / 2) * T) / (v * Sqr(T))
    d2 = d1 - v * Sqr(T)
    If CallPutFlag = "c" Then
        BlackScholes = S * CND(d1) - X * Exp(-r * T) * CND(d2)
    ElseIf CallPutFlag = "p" Then
        BlackScholes = X * Exp(-r * T) * CND(-d2) - S * CND(-d1)
    End If

End Function
```

where *CND*$(\cdot)$ is the cumulative normal distribution function described in Chapter 13. Example: *BlackScholes*("c", 60, 65, 0.25, 0.08, 0.3) will return a call price of 2.1334 as in the numerical example above.

Black-Scholes PDE
An alternative way to find the Black-Scholes option value is to solve the Black-Scholes partial differential equation (PDE). This can be done numerically using several different methods and is covered in Chapter 7. The PDE is given by

$$\left[\frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2 + r\frac{\partial c}{\partial S}S\right]dt = rc$$