Quantitative Risk Management - VaR Analysis

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Abstract

This project focuses on the construction and risk analysis of a portfolio using the **Value at Risk (VaR)** framework, along with additional measures such as the **Capital Asset Pricing Model (CAPM)** and the **Sharpe Ratio**. The VaR methodology is employed to estimate the potential losses in the portfolio under adverse market conditions, providing a quantifiable measure of risk. Expected stock returns are calculated using CAPM, integrating factors like the risk-free rate, stock betas, and market return. A **Risk-Parity Portfolio** is constructed, assigning weights to balance risk contributions across assets. The portfolio's expected return and volatility are computed using the covariance matrix of stock returns. Finally, the Sharpe Ratio is used to assess the portfolio's risk-adjusted return. The project offers a comprehensive risk analysis, with a focus on the portfolio's potential losses and expected returns.

Disclaimer

This is in no way investment advice; it's merely a formalization of results from some independent research I conducted. This work is in no way related to any company or organization. If you decide to trade based on this research, just know you're taking advice from someone with a mere math undergrad and limited trading experience.

Import Portfolio Asset Data

The first step is to import historical stock data for a specified set of tickers between a given start and end date (in this case, **January 1, 2019, to January 1, 2024)** using the Yahoo Finance API through the python yfinance library. The portfolio is then constructed, consisting of the following 10 stocks: **AAPL, MSFT, GOOG, JNJ, XOM, TSLA, JPM, UNH, NVDA, PG**, represent a diversified selection across multiple sectors: technology, healthcare, energy, consumer goods, and financials. These stocks were selected to provide a broad view of the market, reducing unsystematic risk through diversification, which is critical for risk measures.

Five years of data (2019-2024) is sufficient for this analysis because it captures a full market cycle, including various economic conditions (pre-pandemic, pandemic, and recovery phases). This period provides a comprehensive range of data, which is necessary for accurate risk and return calculations, including volatility and correlation measures, while avoiding the consequences of too short or outdated data periods.

Data Preparation

To analyze the stock performance, we first compute the daily returns for each stock. The daily return represents the percentage change in stock price from one trading day to the next and is calculated using the following formula:

$$R_{i,t} = \frac{P_{i,t} - P_{i,t-1}}{P_{i,t-1}}$$

Where:

- $R_{i,t}$ is the return of stock i on day t,
- $P_{i,t}$ is the price of stock i on day t,
- $P_{i,t-1}$ is the price of stock i on day $t-1$.

In our Python implementation, we utilize the `pct_change()` function to efficiently compute the daily returns for each stock.

Once the daily returns are computed, we measure the risk associated with each stock by calculating its volatility. Volatility (σ) reflects the distribution of returns over time, and is expressed as the standard deviation of daily returns. To provide a more practical measure, we annualize the volatility to 252 trading days per year in order to obtain an estimate of how much each stock's price is expected to fluctuate over a year. The volatility was computed for each stock using the following formula:

$$\sigma = \text{std}(R) \times \sqrt{252}$$

Naive Risk-Parity Portfolio Construction

The **Naive Risk-Parity Portfolio** method is used to determine the weight (w_i) for the total number (N) of assets in the portfolio. The goal is to allocate more weight to less volatile stocks and less weight to more volatile stocks, thereby equalizing the contribution of each stock to the total portfolio risk [**1**]. The calculation is based on each stock's volatility (σ_i), which reflects the standard deviation of its returns. The weights are computed by first summing the inverse of the volatilities of all stocks. Each stock's weight is then determined by dividing the inverse of its volatility by the sum of all the inverses. The weight assigned to each stock is inversely proportional to its volatility, ensuring that stocks with lower volatility carry more weight in the portfolio. The formula for calculating the weight of each stock i is given by [**1**]:

$$w_i = \frac{\frac{1}{\sigma_i}}{\sum_{j=1}^{N} \frac{1}{\sigma_j}}$$

This approach ensures that stocks with lower volatility are given higher weights, thus contributing equally to the overall portfolio risk. The sum of the weights is confirmed to equal 1, maintaining the integrity of the portfolio. The weights were calculated to be:

Ticker	AAPL	MSFT	GOOG	JNJ	XOM	TSLA	JPM	UNH	NVDA	PG
Weight	0.0963	0.0976	0.1563	0.0974	0.1018	0.0599	0.1467	0.0480	0.1052	0.0906

Variance-Covariance Matrix

The daily portfolio returns are calculated as the weighted sum of the returns of each stock for a given day. For each stock i, w_i represents the weight of the stock in the portfolio, and $R_{i,t}$ is the return of the stock on day t. To compute the portfolio return for a specific day t (ex. Day 1), you take the return of stock 1 on that day and multiply it by its corresponding weight. This process is repeated for all 10 stocks in the portfolio. Finally, the sum of these weighted returns gives the portfolio return for that day. The formula is given by:

$$R_{\text{portfolio},t} = \sum_{i=1}^{10} w_i \cdot R_{i,t}$$

Once the daily portfolio returns are calculated, we can use the values to construct the **Variance-Covariance Matrix**. The variables $R_{i,t}$ and $R_{j,t}$ represent the daily returns of stocks i and j on day t, respectively. The symbols $\bar{R_i}$ and $\bar{R_j}$ denote the average returns of stocks i and j over the entire period. Finally, T refers to the total number of time periods used in the analysis, and the formula is as follows [**2**]:

$$\text{Cov}(R_i, R_j) = \frac{1}{T-1} \sum_{t=1}^{T} (R_{i,t} - \bar{R_i})(R_{j,t} - \bar{R_j})$$

Each element in the covariance matrix NOT on the diagonal represents the covariance between two stocks, which measures how two assets returns move together. The diagonal elements of the covariance matrix represent the variance, which is the covariance of the stock with itself. Variance is a measure of the distribution of a stock's returns around its mean. A higher variance indicates that the stock's returns are more spread out, suggesting higher volatility (risk). Covariance values in the matrix are interpreted as follows [**2**]:

- **Positive Covariance**: If the covariance is positive, it means that the returns of both stocks tend to move in the same direction. When one stock's return increases, the other's return is likely to increase as well.
- **Negative Covariance**: A negative covariance indicates that the returns of the two stocks move in opposite directions. When one stock's return increases, the other's return tends to decrease.
- **Zero Covariance**: If the covariance is close to zero, the two stocks are not strongly correlated, meaning their returns move independently of each other.

Our calculated Variance-Covariance Matrix for the 10 stocks was computed to be:

	AAPL	MSFT	GOOG	JNJ	XOM	TSLA	JPM	UNH	NVDA	PG
AAPL	0.00041	0.00027	0.00010	0.00019	0.00030	0.00044	0.00012	0.00040	0.00017	0.00014
MSFT	0.00027	0.00040	0.00009	0.00018	0.00029	0.00041	0.00010	0.00033	0.00015	0.00013
GOOG	0.00010	0.00009	0.00016	0.00010	0.00010	0.00010	0.00010	0.00006	0.00012	0.00008
JNJ	0.00019	0.00018	0.00010	0.00040	0.00017	0.00025	0.00011	0.00023	0.00018	0.00026
XOM	0.00030	0.00029	0.00010	0.00017	0.00037	0.00044	0.00012	0.00036	0.00017	0.00012
TSLA	0.00044	0.00041	0.00010	0.00025	0.00044	0.00106	0.00013	0.00067	0.00021	0.00017
JPM	0.00012	0.00010	0.00010	0.00011	0.00012	0.00013	0.00018	0.00007	0.00012	0.00007
UNH	0.00040	0.00033	0.00006	0.00023	0.00036	0.00067	0.00007	0.00166	0.00016	0.00015
NVDA	0.00017	0.00015	0.00012	0.00018	0.00017	0.00021	0.00012	0.00016	0.00035	0.00015
PG	0.00014	0.00013	0.00008	0.00026	0.00012	0.00017	0.00007	0.00015	0.00015	0.00047

Using our above matrix, we can compute the portfolio variance as calculated using the formula:

$$\text{Var}(R_{\text{portfolio}}) = w^T \Sigma w$$

In this equation, w^T represents the transpose of the weight vector, Σ denotes the covariance matrix of the asset returns, and w is the vector of portfolio weights [**3**]. The formula calculates the overall portfolio variance by combining the individual variances and covariances of the assets, weighted according to their respective proportions in the portfolio. The variance of the portfolio was computed to be 0.0001822 and the volatility (the square root of the variance), was computed to be 0.01350017.

Expected Portfolio Returns

In constructing a portfolio and assessing risk, understanding how individual assets behave relative to the market is essential. This section details the process of calculating the market returns, individual stock betas, and expected portfolio returns using the **Capital Asset Pricing Model (CAPM)** [**4**]. These elements are critical in risk assessment, particularly when estimating portfolio volatility and VaR to measure potential portfolio losses over a given time frame.

Market Returns and Expected Market Return

We first compute the market returns using the S&P 500 index, which is commonly used as a proxy for the overall market. The daily returns of the market, R_m, are computed as the percentage change in the adjusted closing prices of the S&P 500 index from January 1, 2019, to January 1, 2024. These daily returns are then annualized to estimate the expected return of the market, $R_m^{\text{annualized}}$. The formula for annualizing the returns is:

$$R_m^{\text{annualized}} = \left(1 + \overline{R_m}\right)^{252} - 1$$

where $\overline{R_m}$ is the average daily return, and 252 represents the number of trading days in a year [**5**]. This annualized return is a key input in both the CAPM formula and the calculation of Value at Risk (VaR), as it allows us to estimate market behavior over a longer time frame.

Calculating Beta for Individual Stocks

Next, we compute the **Beta** of each stock, which measures the stock's sensitivity to market movements. The beta of a stock i is calculated as:

$$\beta_i = \frac{\text{Cov}(R_i, R_m)}{\text{Var}(R_m)}$$

where $\text{Cov}(R_i, R_m)$ is the covariance between the stock's return and the market return, and $\text{Var}(R_m)$ is the variance of the market return. Stocks with a beta greater than 1 are more volatile than the market, while those with a beta less than 1 are less volatile [**6**]. Beta plays an important role in determining how much risk each stock contributes to the portfolio and thus impacts the calculation of the portfolio's VaR. The betas were computed to be:

Ticker	AAPL	MSFT	GOOG	JNJ	XOM	TSLA	JPM	UNH	NVDA	PG
Beta	1.2166	1.1317	0.5245	1.0995	1.1831	1.7302	0.5853	1.5045	0.8617	0.8856

Using CAPM to Calculate Expected Returns

Once the beta values for each stock are known, CAPM is used to estimate the expected return of each stock. CAPM is particularly useful because it ties the expected return of a stock to its level of systematic risk, represented by beta. The CAPM formula is given by:

$$R_{\text{expected}} = R_f + \beta(R_m - R_f)$$

where R_f is the risk-free rate, R_m is the expected market return, and $R_m - R_f$ is the market risk premium. CAPM assumes that investors are compensated for taking on additional risk, and thus higher beta stocks are expected to yield higher returns [**4**]. This expected return plays a crucial role in assessing the portfolio's performance and its risk profile under VaR analysis, as the expected returns are a determinant of potential portfolio losses. The expected returns computed via CAPM for each individual asset are shown below:

Ticker	AAPL	MSFT	GOOG	JNJ	XOM	TSLA	JPM	UNH	NVDA	PG
CAPM	0.1947	0.1825	0.0953	0.1779	0.1899	0.2685	0.1041	0.2361	0.1438	0.1472

Calculating Portfolio Expected Return

After calculating the expected returns of individual stocks, the portfolio's overall expected return is computed as the weighted sum of the expected returns of each stock:

$$R_{\text{portfolio}} = \sum_{i=1}^{N} w_i \cdot R_i$$

where w_i is the weight of stock i in the portfolio, and R_i is its expected return based on CAPM. The portfolio expected return allows us to assess the performance of the entire portfolio. Together with the portfolio variance, it helps determine the overall risk and return trade-off for the portfolio. This is especially important in VaR analysis, where understanding the expected return relative to risk helps in estimating potential losses. This method produced a portfolio expected return of:

Expected Portfolio Return	15.93%

Parametric VaR Calculation

The **Parametric Value at Risk (VaR)** calculation relies on the assumption that portfolio returns follow a normal distribution. VaR estimates the maximum potential loss within a certain confidence interval over a specified time horizon. The formula used for VaR is as follows:

$$\text{VaR}_{\text{Parametric}} = \mu_p - (z_\alpha \cdot \sigma_p)$$

In this equation, μ_p represents the portfolio's expected return, which is the mean return calculated based on historical data or forecasted using models such as the Capital Asset Pricing Model (CAPM). The term z_α is the z-score corresponding to the desired confidence level; for example, a 95% confidence level corresponds to a z_α value of -1.645. This value indicates how many standard deviations away from the mean we need to go to capture 95% of the data points in a normally distributed dataset. Lastly, σ_p is the portfolio's standard deviation. A higher standard deviation implies greater variability and thus higher risk [**7**].

In the case of a time horizon set to 1, this refers to one day, assuming daily return data. By adjusting the time horizon, the model can be extended to predict potential losses over longer periods, such as weeks or months. It is important to note, however, that all other calculations must align with the chosen time horizon (using daily returns for a daily time horizon, as is the case in our analysis).

Parametric VaR Results

For our chosen portfolio, the Parametric VaR at a 95% confidence level is calculated to be -0.0228, this means that, under normal conditions, there is a 95% probability that the portfolio will not lose more than 2.28% of its value in a single day. This approach provides holders of this portfolio the ability to asses the probability of maximum losses and make more informed risk management decisions.

Parametric VaR Distribution Plot

The final stage involves plotting the normal distribution curve of parametric VaR. In the graph, the normal distribution curve represents the theoretical distribution of the portfolio returns assuming they follow a normal (Gaussian) distribution. The curve is generated using the mean (expected return, μ) and standard deviation (volatility, σ) of the portfolio's daily returns. The x-axis represents a range of possible portfolio returns (centered around the expected return μ), and the y-axis represents the probability density (the likelihood of observing a particular return given the normal distribution) [**8**]. The normal distribution curve is generated and shown below:

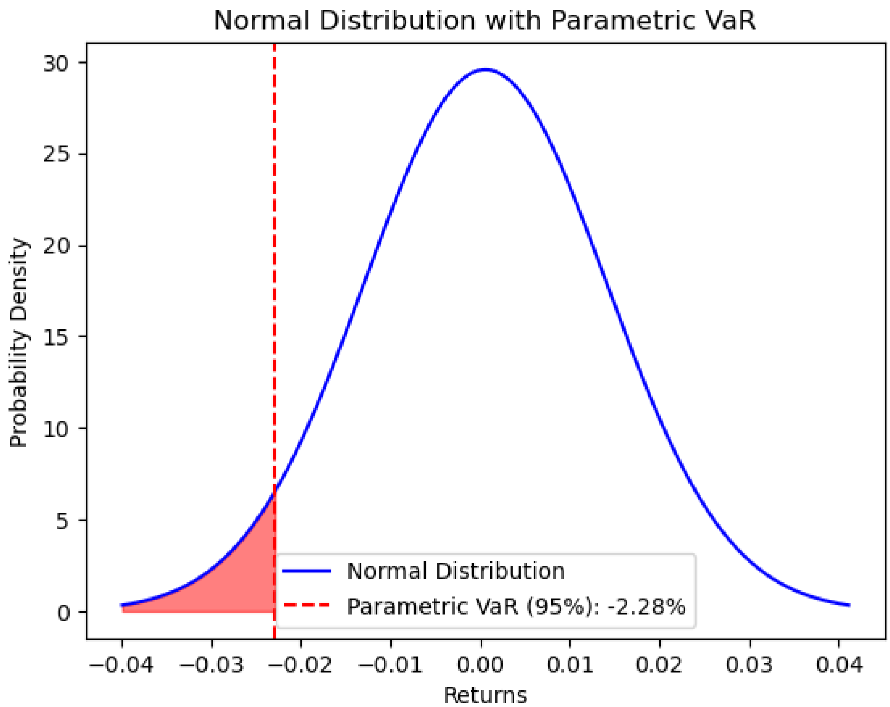

Figure 1: Graph of Parametric VaR Distribution

Variance-Covariance (VCV) VaR Calculation

The **Variance-Covariance (VCV) Value at Risk (VaR)** calculation, like the Parametric VaR, assumes that portfolio returns follow a normal distribution. However, it specifically leverages the portfolio's variance-covariance matrix to account for the relationships between individual assets in the portfolio. The calculated VCV VaR represents the maximum expected loss under normal market conditions, over a specified time horizon. In this model, with a time horizon of 1 day, the VaR calculation estimates the portfolio's potential loss for a single trading day, based on the portfolio's current structure and historical performance of its assets. The formula used for VCV VaR is as follows:

$$\text{VaR}_{\text{VCV}} = z_\alpha \cdot \sigma_p \cdot \sqrt{\text{time horizon}}$$

In this equation, z_α represents the z-score corresponding to the desired confidence level. For instance, a 95% confidence level corresponds to a z_α value of -1.645. This value indicates how many standard deviations away from the mean we need to go to capture 95% of the data points in a normally distributed dataset. The term σ_p is the portfolio's volatility, calculated using the covariance matrix of the portfolio's assets [**8**]. This takes into account how each asset's returns correlate with one another, offering a more comprehensive view of risk than a simple parametric calculation.

Parametric VaR Results

For our chosen portfolio, the VCV VaR at a 95% confidence level is calculated to be -0.0222. This means that, under normal market conditions, there is a 95% probability that the portfolio will not lose more than 2.22% of its value in a single day. This model, by incorporating the covariances between individual asset returns, provides a more refined estimate of risk and potential loss, compared to the Parametric VaR model.

VCV VaR Distribution Plot

Plotting the normal distribution curve of the VCV VaR follows the exact method of Parametric VaR, in that the curve represents the theoretical distribution of the portfolio returns assuming they follow a normal (Gaussian) distribution [**8**]. The normal distribution curve is generated and shown below:

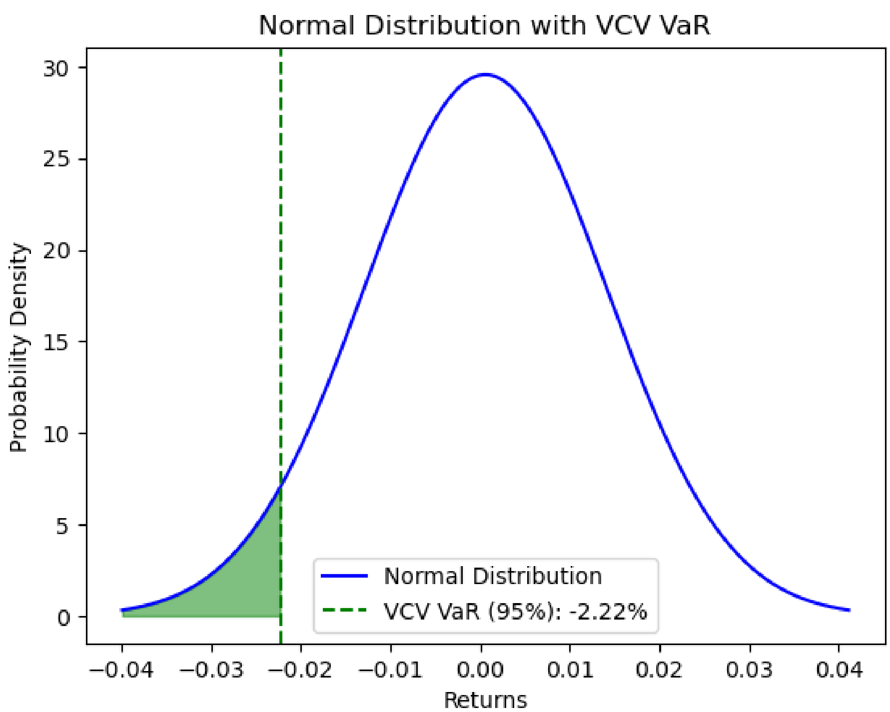

Figure 2: Graph of Variance-Covariance VaR Distribution

Conclusion

In this project, we implemented and analyzed two distinct methods for calculating Value at Risk (VaR): Parametric VaR and Variance-Covariance (VCV) VaR. Both approaches assume that portfolio returns follow a normal distribution but differ in their calculation methodology and the information they incorporate.

The Parametric VaR method uses the expected portfolio return and volatility, assuming independence among asset returns. It provides a quick, simplified estimate of risk by assuming that returns are normally distributed, but it doesn't account for the correlations between assets. In our analysis, the Parametric VaR at a 95% confidence level was calculated to be 2.28%, meaning that under normal market conditions, the portfolio is unlikely to lose more than 2.28% of its value in a single day.

On the other hand, the Variance-Covariance (VCV) VaR method uses the full covariance matrix of asset returns, providing a more comprehensive view of portfolio risk. It accounts for the relationship between individual assets by incorporating their covariances, leading to a more refined and often more accurate estimate of potential losses. In this case, the VCV VaR at a 95% confidence level was found to be 2.22%, slightly lower than the Parametric VaR, suggesting that the inclusion of correlations between assets reduced the risk.

While both methods offer valuable insights, the VCV VaR provides a more in depth picture of portfolio risk by considering the interactions between assets, which can lead to a more reliable risk estimate in diversified portfolios. However, the Parametric VaR is still a useful and fast alternative when correlations between assets are minimal or the covariance matrix is unavailable.

Model Enhancements

- **Incorporate non-normal distribution**: Parametric and VCV VaR assumes normal distribution of returns, but financial returns often exhibit fat tails. Consider using alternative distributions like t-distribution or skewed distributions.
- **Stress testing and scenario analysis**: Introduce stress testing to evaluate how the portfolio performs under extreme market conditions. Scenario analysis can help in understanding the impact of rare events.
- **Historical simulation VaR**: Instead of assuming a specific distribution, use historical data to simulate possible outcomes and calculate VaR based on actual past events.
- **Rolling window backtesting**: Implement more robust backtesting through the usage of rolling windows to test the performance of the VaR model over different time periods. By using rolling windows, you can continuously update the VaR calculation based on the most recent data, ensuring that the model adapts to changes in market conditions.

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