METIS

Introduction to Probability

Probability is the mathematical way of quantifying uncertainty.

Put another way: probability is the study of theoretical possibilities and their likelihood of occuring.

Sample Spaces and Events

- **Sample space (*S*):** the set of all possible outcomes of our model or experiment
- **Elements:** the points in the sample space
- **Event:** a subset of the sample space

Sample Spaces and Events

- **Sample space (*S*):** the set of all possible outcomes of our model or experiment
- **Elements:** the points in the sample space
- **Event:** a subset of the sample space

Example: toss a coin twice

- **Sample space:** {HH, TT, HT, TH}
- **Elements:** {H, T}
- **Event that both tosses are the same:** {HH, TT}

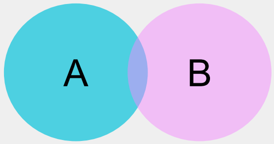
Sample Spaces and Events

- **Complement (A^C):** Everything not in set A; for any event A, $P(A^C) = 1 - P(A)$
- **Union ($A \cup B$):** add up all events in A and B
- **Intersection ($A \cap B$):** all events that fall in both A and B

Sample Spaces and Events

- **Complement (A^C):** Everything not in set A; for any event A, $P(A^C) = 1 – P(A)$
- **Union ($A \cup B$):** add up all events in A and B
- **Intersection ($A \cap B$):** all events that fall in both A and B
- **Disjoint events**: the sets don't share any common events
  - P(A or B) = P(A) + P(B)

- **Joint events**: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



Probability

Every **event** A gets a **probability P(A)**, which is a real number. Probabilities have some rules:

- $P(A) \geq 0$ *for every A*
- $P(Set) = 1$
- If $A_1, A_2, \ldots, A_i$ are disjoint, then $P(\bigcup A_1, A_2, \ldots, A_i) = \sum P(A_i)$

Two Ways of Interpreting Probabilities

- **Frequencies:** if we repeat enough trials, P(A) is the proportion of times we'll see A being true
  - E.g. If we say a fair coin has P(tossing heads) = .5, then tossing a coin lots (and lots and lots!) of times will get us 50% heads in the long term
- **Degrees of belief (Bayesian inference):** P(A) is our degree of belief that A is true (no repeated experiments necessary)
  - E.g. if we have a fair coin, we believe P(tossing heads) = .5

The difference starts to matter once we get to **inference** in a few days.

Independence

When events are independent (i.e. the occurrence of one event does not influence the occurrence of another event), the probabilities can be multiplied:

$$P(A,B) = P(A) \times P(B)$$

Independence

When events are independent (i.e. the occurrence of one event does not influence the occurrence of another event), the probabilities can be multiplied:

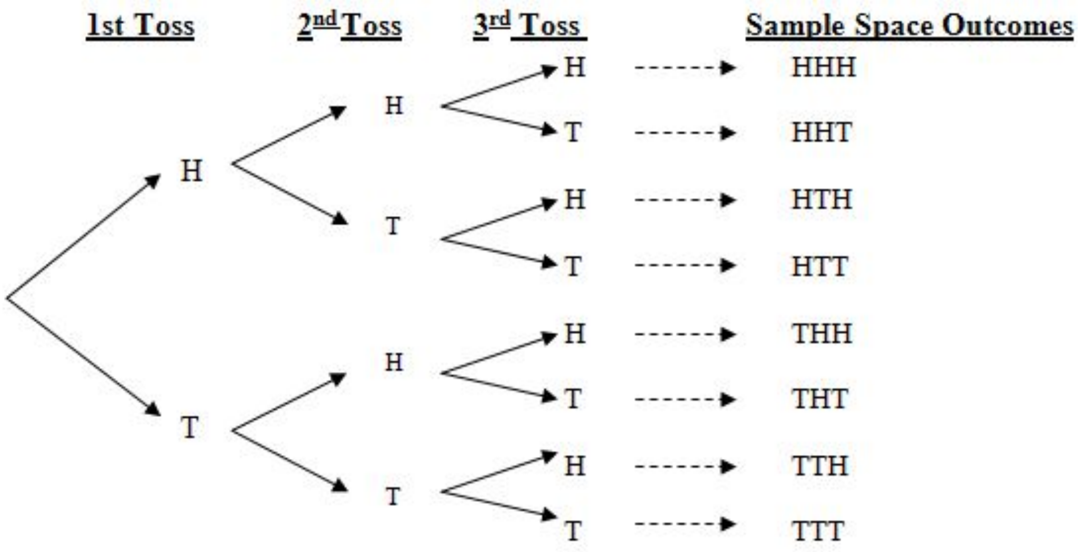
$$P(A,B) = P(A) \times P(B)$$

Independence can be:

- **Assumed:** We assume tosses of a fair coin are independent
- **Verified**: We derive then verify that $P(A,B) = P(A) \times P(B)$

Independence: Example

What's the probability of flipping three (fair) coins and getting (exactly) two heads?



$$P(H)P(H)P(T) + P(H)P(T)P(H) + P(T)P(H)P(H) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

Dependence

When two events are dependent, the probability of one event occurring influences the likelihood of the other event.

Dependence

When two events are dependent, the probability of one event occurring influences the likelihood of the other event.

- What's the probability of drawing an ace from a deck of 52 cards?
- If we don't replace the drawn card, what's the probability of drawing a second ace?

$$P(A_1) = \frac{4}{52}$$

$$P(A_2|A_1) = \frac{3}{51}$$

Unions and Intersections of Events

- What's the **intersection** of two events?
  - If they're **independent**: $P(A \text{ and } B) = P(A) \times P(B)$
  - If they're **dependent**: $P(A \text{ and } B) = P(A) \times P(B|A)$

Unions and Intersections of Events

- What's the **intersection** of two events?
  - If they're **independent**: P(A and B) = P(A) x P(B)
  - If they're **dependent**: P(A and B) = P(A) x P(B|A)

- What's the **union** of two events?
  - If they're **independent**: P(A or B) = P(A) + P(B)
  - If they're **dependent**: P(A or B) = P(A) + P(B) - P(A and B)

Unions and Intersections of Events

- What's the **intersection** of two events?
  - If they're **independent**: P(A and B) = P(A) x P(B)
  - If they're **dependent**: P(A and B) = P(A) x P(B|A)

- What's the **union** of two events?
  - If they're **independent**: P(A or B) = P(A) + P(B)
  - If they're **dependent**: P(A or B) = P(A) + P(B) - P(A and B)

Draw some Venn diagrams!

Conditional Probability

Conditional probability is the probability of event A happening, given that event B has already happened. Define:

$$P(A|B) = P(A,B) / P(B)$$

Conditional Probability

Conditional probability is the probability of event A happening, given that event B has already happened. Define:

$$P(A|B) = P(A,B) / P(B)$$

Conditional probability example

- Test for a disease (D^+,$D–$) and get test results that are positive/negative (T^+, T^-)

$$P(T^+|D^+) = P(T^+ \cap D^+) / P(D)$$

Random Variables

Random variables are rules that assign a real number value to each element.

Random Variables

Random variables are rules that assign a real number value to each element.

- Toss a coin twice and let the **random variable X** be the **number of heads**.

	Probability	X
HH	1/4	2
TT	1/4	0
TH	1/4	1
HT	1/4	1

→

X	P(X = x)
0	1/4
1	1/2
2	1/4

Random Variables: Important Quantities

Discrete random variable: there are only finitely many values attained by the variable. (These are always thought of as functions on a fixed probability space!)

For these we can compute the following important quantities:

- Expected value: $E(X) := \mu_X := \sum_{\text{values } x \text{ of } X} P(X=x) \times x$
- Variance: $$\begin{aligned} &Var(X) := (\sigma_X)^2 \\ &:= \sum_{\text{values } x \text{ of } X} P(X=x) \times (x - E(X))^2 \\ &= E\left((X - E(X))^2\right) \end{aligned}$$
- Standard deviation: $\sigma_X := \sqrt{Var(X)}$

Random Variables: Important Quantities

For two discrete random variables on the same probability space, we can talk about their **covariance**:

$$\mathrm{Cov}(X, Y) = \sum P(X = x, Y = y)(x - E(X))(y - E(Y))$$

Law of Large Numbers

Often when we're working with probabilities, we're interested in what happens asymptotically (in the limit). We have some laws about this.

Law of Large Numbers:

- Informal: The average value of a large number of independent samples of a random variable X gets arbitrarily close to its expected value E(X).
- Formal: Suppose that $X_1, \ldots, X_n$ are independent random variables with the same probability densities as the random variable X, then:

$$\lim_{n\to\infty} \frac{X_1+X_2+\cdots+X_n}{n} = E(X)$$

Central Limit Theorem

Whereas the LLN was about samples from a distribution, the **Central Limit Theorem** is about the value of sample means from any distribution.

- Informal: Suppose X is a random variable with mean zero and finite variance. Then the sum of n trials of X divided by sqrt{n} approaches the normal distribution with mean zero and the same variance as X.
- Formal: Suppose X is a random variable with E(X) = 0 and variance less than infinity, and X_1, ..., X_n are independent random variables with the same probability distribution as X. Then the limit

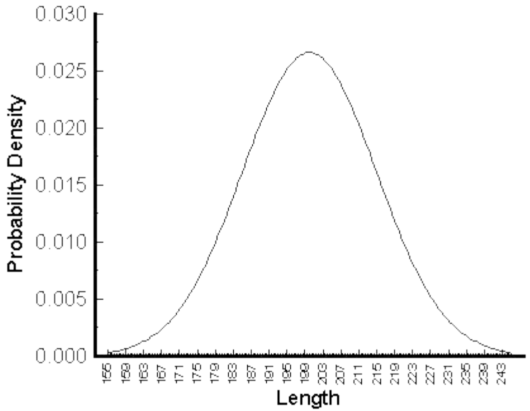
$$\lim_{n\to\infty} \frac{X_1+X_2+\cdots+X_n}{\sqrt{n}} = \mathcal{N}\left(0, \sigma^2\right)$$

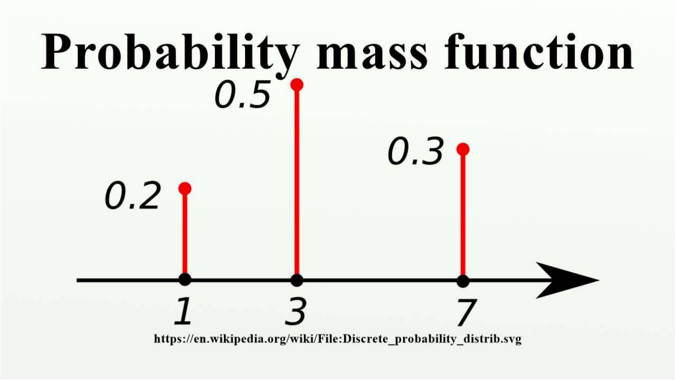
in probability.

Probability Functions of Random Variables

Probability Density Function (PDF): For continuous random variables, a function whose value at any given sample can be interpreted as a relative likelihood that the value of the random value would equal that sample

Probability Mass Function (PMF): For discrete random variables, a function that gives the probability the the random variable X is exactly equal to some value.





Dependence and Conditional Probabilities

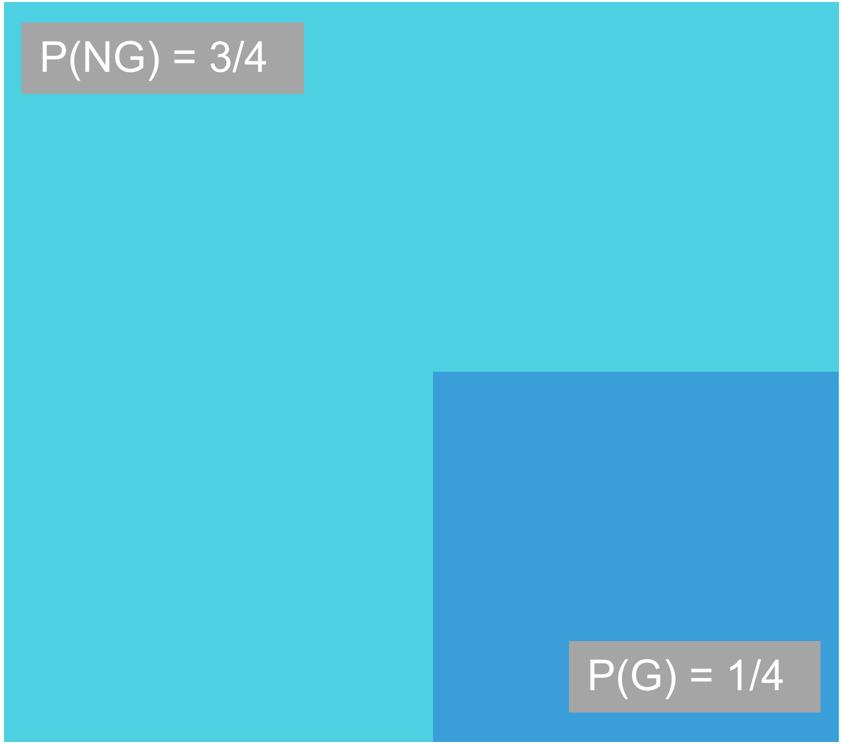
I am allergic to dogs. They make me sneeze. Sometimes dogs greet me. What is the probability that I sneeze?

P(Dog greets me) = P(G) = 1/4

P(Dog does not greet me) = P(NG) = 3/4

Visualize

P(G) = 1/4
P(NG) = 3/4



Dependence and Conditional Probabilities

Sometimes we know the **conditional probabilities** that depend on whether dogs say hello:

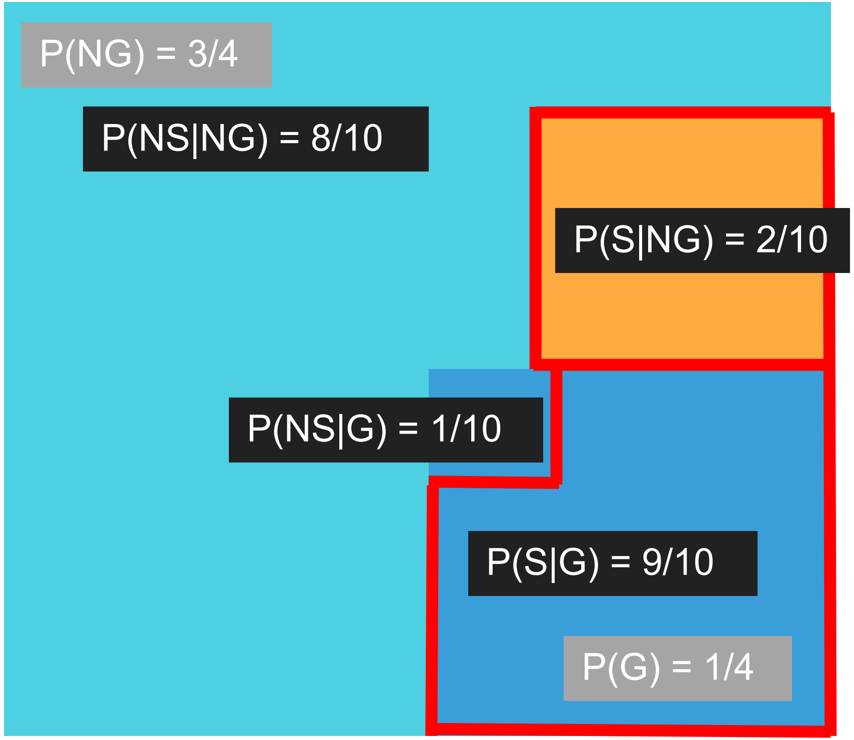
- P(Sneeze | Dog greets me) = P(S|G) = 9/10
- P(No Sneeze | Dog greets me) = P(NS|G) = 1/10
- P(Sneeze | Dog doesn't greet me) = P(S|NG) = 2/10
- P(No Sneeze | Dog doesn't greet me) = P(NS|NG) = 8/10

Visualize

P(G) = 1/4
P(NG) = 3/4

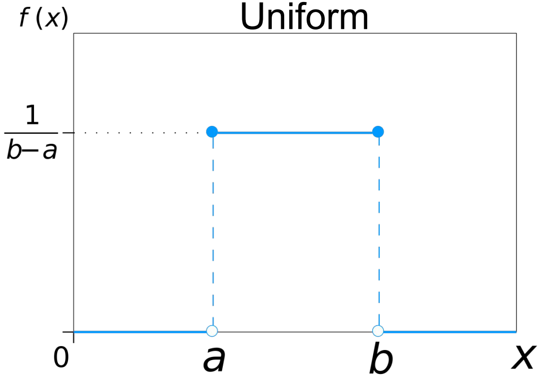
P(S|G) = 9/10
P(NS|G) = 1/10
P(S|NG) = 2/10
P(NS|NG) = 8/10

$$P(S) = P(S|G)\,P(G) + P(S|NG)\,P(NG)$$
$$= \frac{9}{10}\frac{1}{4} + \frac{2}{10}\frac{3}{4} = 0.375$$

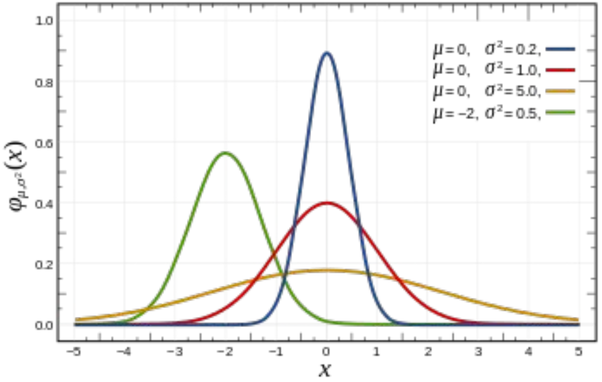


Commonly Used Distributions

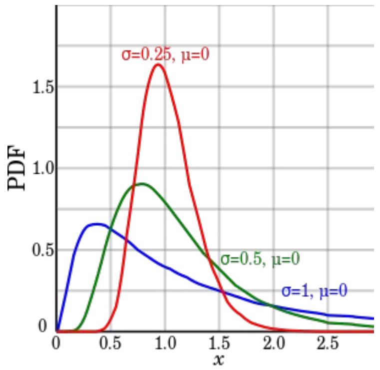
Uniform



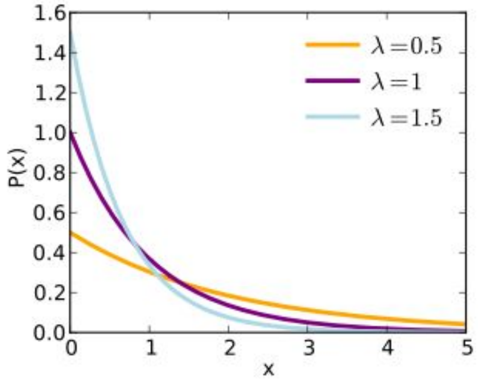
Gaussian/Normal



Log - Normal



Exponential



Poisson