Everything You Need to Know About Linear Algebra in Twenty Minutes

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This Lecture

- Definition of vector and matrix
- Definition of vector-vector, matrix-vector and matrix-matrix multiplication
- Eigenvalue problems
- Definition of commutator
- Definitions of terms related to square matrices

Vectors

Vector

$$\mathbf{v} = \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_M \end{bmatrix}$$

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Transpose

$$\mathbf{v}^T = \begin{bmatrix} v_1 & v_2 & \dots & v_M \end{bmatrix}$$

Vector-Vector Multiplication

Inner Product
(AKA scalar
product; dot
product)

$$\mathbf{v}^{\dagger}\mathbf{w} = \begin{bmatrix} v_1^* & v_2^* & \dots & v_M^* \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_M \end{bmatrix} = \sum_{i=1}^{M} v_i^* w_i$$

Vector-Vector Multiplication

Outer Product

$$\mathbf{w}\mathbf{v}^{\dagger} = \begin{bmatrix} w_1 \\ w_2 \\ \ldots \\ w_N \end{bmatrix} \begin{bmatrix} v_1^* & v_2^* & \ldots & v_M^* \end{bmatrix} = \begin{bmatrix} w_1 v_1^* & w_1 v_2^* & \ldots & w_1 v_M^* \\ w_2 v_1^* & w_2 v_2^* & \ldots & w_2 v_M^* \\ \ldots & \ldots & \ldots & \ldots \\ w_N v_1^* & w_N v_2^* & \ldots & w_N v_M^* \end{bmatrix}$$

Properties of Sets of Vectors

- Orthogonal vectors $\mathbf{u}^{\dagger}\mathbf{v} = 0$
- Orthonormal set of vectors
  - are orthogonal to one another
  - have unit length (are normalized)

$$\|\mathbf{v}\|_2 \equiv \sqrt{\mathbf{v}^{\dagger}\mathbf{v}} = 1$$

Matrices

Matrix (N x M)

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1M} \\ A_{21} & A_{22} & \dots & A_{2M} \\ \dots & \dots & \dots & \dots \\ A_{N1} & A_{N2} & \dots & A_{NM} \end{bmatrix}$$

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Matrix-Matrix Multiplication

$$\mathbf{AB} = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1M} \\ A_{21} & A_{22} & \ldots & A_{2M} \\ \ldots & \ldots & \ldots & \ldots \\ A_{N1} & A_{N2} & \ldots & A_{NM} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & \ldots & B_{1L} \\ B_{21} & B_{22} & \ldots & B_{2L} \\ \ldots & \ldots & \ldots & \ldots \\ B_{M1} & B_{M2} & \ldots & B_{ML} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & \ldots & C_{1L} \\ C_{21} & C_{22} & \ldots & C_{2L} \\ \ldots & \ldots & \ldots & \ldots \\ C_{N1} & C_{N2} & \ldots & C_{NL} \end{bmatrix}$$

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N x M M x L N x L

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$$C_{ij} = \sum_{k=1}^{M} A_{ik} B_{kj}$$

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$$C_{ij} = \sum_{k=1}^{M} A_{ik} B_{kj}$$

Matrix-Matrix Multiplication

- Properties of matrix-matrix multiplication
  - Commutative property does not necessarily hold

$$\mathbf{AB} \neq \mathbf{BA}$$

  - Associative property holds

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$

Matrix-Matrix Multiplication

- Distributive property holds

$$\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC}$$

- For scalar multiplication, the associative and commutative properties hold

$$c\mathbf{A}=\mathbf{A}c$$

$$c\left(\mathbf{AB}\right)=\left(c\mathbf{A}\right)\mathbf{B}=\mathbf{A}\left(c\mathbf{B}\right)$$

c is a scalar

Matrix-Vector Multiplication

$$\mathbf{Ac} = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1M} \\ A_{21} & A_{22} & \ldots & A_{2M} \\ \ldots & \ldots & \ldots & \ldots \\ A_{N1} & A_{N2} & \ldots & A_{NM} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \ldots \\ c_M \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \ldots \\ d_N \end{bmatrix}$$

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$$d_i = \sum_{j=1}^{M} A_{ij} c_j$$

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N x M   M   N

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Eigenvalue Problems

$$\mathbf{Oc} = \omega\mathbf{c}$$

Square Matrix

Vector

Scalar

Eigenvalue Problems

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- The Schrodinger equation is an eigenvalue problem

$$\mathbf{H\Psi} = E\mathbf{\Psi}$$

Commutator

$$[\mathbf{A},\mathbf{B}] \equiv \mathbf{AB} - \mathbf{BA}$$

$$\text{if } [\mathbf{A},\mathbf{B}] = 0 \text{ then}$$

$$\mathbf{AB} = \mathbf{BA}$$

$$\mathbf{A} \text{ and } \mathbf{B} \text{ have the same eigenvectors}$$

Square Matrices

Square Matrix
(N x N)

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{bmatrix}$$

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Trace

$$\mathrm{tr}\mathbf{A} \equiv \sum_{i=1}^{N} A_{ii}$$

Determinant

$$|\mathbf{A}| = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{12}A_{21}$$

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Most Important Property:

The interchange of any two rows or columns of $\mathbf{A}$ changes the sign of $\left|\mathbf{A}\right|$.

Types of Square Matrices

- Diagonal Matrix $A_{ij} = 0 \text{ for } i \neq j$
  - A_{ii} are the eigenvalues of $\mathbf{A}$

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- Inverse Matrix $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{1}$

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- Unitary Matrix $\mathbf{U}^{-1} = \mathbf{U}^{\dagger}$
  - $\mathbf{U}^{\dagger}\mathbf{U} = \mathbf{1}$
  - Unitary transformation $\mathbf{A} = \mathbf{U}^{\dagger}\mathbf{B}\mathbf{U}$
    - $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues, same trace, same determinant
  - Preserves length of vector $\|\mathbf{v}\|_2 = \|\mathbf{U}\mathbf{v}\|_2$

Eigenvalue Problems

$$\mathbf{Oc} = \omega\mathbf{c}$$

Matrix

Vector

Scalar

Eigenvalue Problems

$$\mathbf{Oc} = \omega\mathbf{c}$$

- Solving an eigenvalue problem is diagonalizing a matrix

$$\Omega = \mathbf{U}^{\dagger}\mathbf{OU}$$

Eigenvalue Problems

$$\mathbf{Oc} = \omega\mathbf{c}$$

- Solving an eigenvalue problem is diagonalizing a matrix

$$\Omega = \mathbf{U}^{\dagger}\mathbf{O}\mathbf{U}$$

Find unitary matrix

Eigenvalue Problems

$$\mathbf{Oc} = \omega\mathbf{c}$$

- Solving an eigenvalue problem is diagonalizing a matrix

$$\underbrace{\Omega}_{\text{Diagonal}} = \underbrace{\mathbf{U}^{\dagger}\mathbf{O}\mathbf{U}}_{\text{Find unitary matrix}}$$

Eigenvalue Problems

$$\mathbf{Oc} = \omega\mathbf{c}$$

- Solving an eigenvalue problem is diagonalizing a matrix

$$\Omega = \mathbf{U}^{\dagger}\mathbf{O}\mathbf{U}$$

Diagonal elements give us the eigenvalues

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Diagonal elements give us the eigenvalues

Columns of unitary matrix give us eigenvectors