Everything You Need to Know About Linear Algebra in Twenty Minutes

Benjamin G. Levine

This Lecture

- Definition of vector and matrix
- Definition of vector-vector, matrix-vector and matrix-matrix multiplication
- Eigenvalue problems
- Definition of commutator
- Definitions of terms related to square matrices

Vectors

Vector

$$\mathbf{v} = \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_M \end{bmatrix}$$

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$$\mathbf{v}^{\dagger} = \begin{bmatrix} v_1^* & v_2^* & \dots & v_M^* \end{bmatrix}$$

Vectors

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Transpose

$$\mathbf{v}^T = \begin{bmatrix} v_1 & v_2 & \dots & v_M \end{bmatrix}$$

Vector-Vector Multiplication

Inner Product (AKA scalar product; dot product)

$$\mathbf{v}^{\dagger}\mathbf{w} = \begin{bmatrix} v_1^* & v_2^* & \dots & v_M^* \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_M \end{bmatrix} = \sum_{i=1}^M v_i^* w_i$$

Vector-Vector Multiplication

Outer Product

$$\mathbf{w}\mathbf{v}^{\dagger} = \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_N \end{bmatrix} \begin{bmatrix} v_1^* & v_2^* & \dots & v_M^* \end{bmatrix} = \begin{bmatrix} w_1v_1^* & w_1v_2^* & \dots & w_1v_M^* \\ w_2v_1^* & w_2v_2^* & \dots & w_2v_M^* \\ \dots & \dots & \dots & \dots \\ w_Nv_1^* & w_Nv_2^* & \dots & w_Nv_M^* \end{bmatrix}$$

Properties of Sets of Vectors

Orthogonal vectors

$$\mathbf{u}^{\dagger}\mathbf{v} = 0$$

- Orthonormal set of vectors
 - are orthogonal to one another
 - have unit length (are normalized)

$$\|\mathbf{v}\|_2 \equiv \sqrt{\mathbf{v}^{\dagger}\mathbf{v}} = 1$$

Matrices

Matrix (N x M)

$$\mathbf{A} = egin{bmatrix} A_{11} & A_{12} & \dots & A_{1M} \\ A_{21} & A_{22} & \dots & A_{2M} \\ \dots & \dots & \dots & \dots \\ A_{N1} & A_{N2} & \dots & A_{NM} \end{bmatrix}$$

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$$\mathbf{AB} = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1M} \\ A_{21} & A_{22} & \dots & A_{2M} \\ \dots & \dots & \dots & \dots \\ A_{N1} & A_{N2} & \dots & A_{NM} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1L} \\ B_{21} & B_{22} & \dots & B_{2L} \\ \dots & \dots & \dots & \dots \\ B_{M1} & B_{M2} & \dots & B_{ML} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1L} \\ C_{21} & C_{22} & \dots & C_{2L} \\ \dots & \dots & \dots & \dots \\ C_{N1} & C_{N2} & \dots & C_{NL} \end{bmatrix}$$

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$$\mathbf{N} \times \mathbf{M} \qquad \mathbf{M} \times \mathbf{L} \qquad \mathbf{N} \times \mathbf{L}$$

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$$C_{ij} = \sum_{k=1}^{M} A_{ik} B_{kj}$$

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- Properties of matrix-matrix multiplication
 - Commutative property does not necessarily hold

$$AB \neq BA$$

Associative property holds

$$(\mathbf{A}\mathbf{B})\mathbf{C} = \mathbf{A}(\mathbf{B}\mathbf{C})$$

Distributive property holds

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{A}\mathbf{B} + \mathbf{A}\mathbf{C}$$

 For scalar multiplication, the associative and commutative properties hold

$$c\mathbf{A} = \mathbf{A}c$$

$$c(\mathbf{A}\mathbf{B}) = (c\mathbf{A})\mathbf{B} = \mathbf{A}(c\mathbf{B})$$
 $c \text{ is a scalar}$

$$\mathbf{Ac} = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1M} \\ A_{21} & A_{22} & \dots & A_{2M} \\ \dots & \dots & \dots & \dots \\ A_{N1} & A_{N2} & \dots & A_{NM} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \dots \\ c_M \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \dots \\ d_N \end{bmatrix}$$

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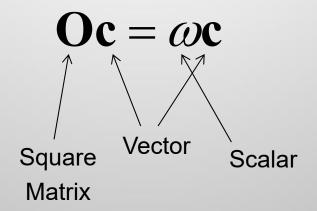
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$$\mathbf{Oc} = \omega \mathbf{c}$$

The Schrodinger equation is an eigenvalue problem

$$\mathbf{H}\mathbf{\Psi} = E\mathbf{\Psi}$$

Commutator

$$[A,B] \equiv AB - BA$$

if
$$[\mathbf{A}, \mathbf{B}] = 0$$
 then

$$AB = BA$$

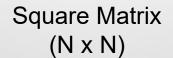
A and B have the same eigenvectors

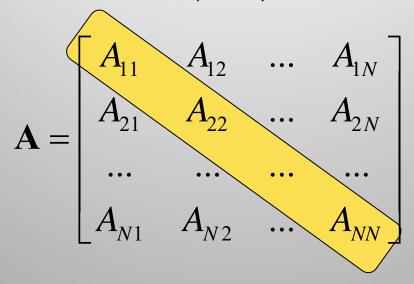
Square Matrices

Square Matrix (N x N)

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1N} \\ A_{21} & A_{22} & \dots & A_{2N} \\ \dots & \dots & \dots & \dots \\ A_{N1} & A_{N2} & \dots & A_{NN} \end{bmatrix}$$

Square Matrices





Trace

$$\operatorname{tr} \mathbf{A} \equiv \sum_{i=1}^{N} A_{ii}$$

$$|\mathbf{A}| = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{12}A_{21}$$

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$$\left| \mathbf{A} \right| = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{12}A_{21}$$

Most Important Property:

The interchange of any two rows or columns of A changes the sign of |A|.

- Diagonal Matrix $A_{ij} = 0$ for $i \neq j$
 - $-A_{ii}$ are the eigenvalues of A

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$$-1A = A1 = A$$

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• Inverse Matrix $A^{-1}A = AA^{-1} = 1$

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- Real eigenvalues, orthogonal eigenvectors

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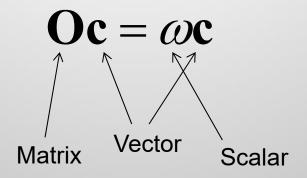
$$\mathbf{A} = \mathbf{A}^{\dagger} \text{ or } A_{ij} = A_{ji}^{*}$$

- Real eigenvalues, orthogonal eigenvectors

Unitary Matrix

$$\mathbf{U}^{\text{-1}} = \mathbf{U}^{\dagger}$$

- $-\mathbf{U}^{\dagger}\mathbf{U}=\mathbf{1}$
- Unitary transformation $A = U^{\dagger}BU$
 - A and B have the same eigenvalues, same trace, same determinant
- Preserves length of vector $\|\mathbf{v}\|_2 = \|\mathbf{U}\mathbf{v}\|_2$



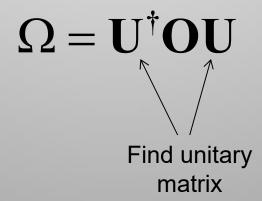
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 Solving an eigenvalue problem is diagonalizing a matrix

$$\Omega = \mathbf{U}^{\dagger} \mathbf{O} \mathbf{U}$$

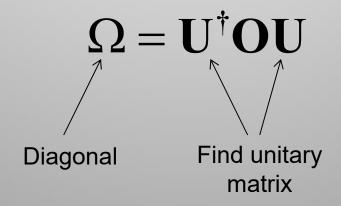
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Columns of unitary matrix give us eigenvectors