Computational Macro | Problem Set 2

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The Neoclassical Growth Model

1. Competitive equilibrium

An Arrow-Debreu equilibrium for this economy consists of a sequence of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ and allocations for the household $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ and the firm $\left\{y_t, k_t^d, l_t^d\right\}_{t=0}^{\infty}$ such that:

· Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, allocations $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ solve the household's problem:

$$\begin{cases} \underset{\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}}{max} & \sum_{t=0}^{\infty} \beta^t \left(\dfrac{c_t^{1-\sigma}}{1-\sigma} - \chi \dfrac{{l_t^s}^{1+\eta}}{1+\eta} \right) \\ \text{s.t.} & \sum_{t=0}^{\infty} p_t(c_t + i_t) \leq \sum_{t=0}^{\infty} p_t(w_t l_t^s + r_t k_t^s) + \Pi \\ & i_t = x_{t+1} - (1-\delta)x_t, \text{ for all } t \geq 0 \\ & c_t, x_{t+1} \geq 0, \quad 0 \leq l_t^s \leq 1, \quad 0 \leq k_t^s \leq x_t, \text{ for all } t \geq 0 \\ & x_0 = k_0 \text{ given} \end{cases}$$

· Given $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, allocations $\left\{y_t, k_t^d, l_t^d\right\}_{t=0}^{\infty}$ solve the firm's problem

$$\begin{cases} \Pi = \underset{\{y_t, k_t^d, l_t^d\}_{t=0}^{\infty}}{max} & \sum_{t=0}^{\infty} p_t(y_t - w_t l_t^d + r_t k_t^d) \\ \text{s.t.} & y_t = z {k_t^d}^{\alpha} {l_t^d}^{1-\alpha}, \text{ for all } t \geq 0 \\ & y_t, k_t^d, l_t^d \geq 0 \text{ for all } t \geq 0 \end{cases}$$

· Markets clear

$$
\begin{aligned}
&y_t = c_t + i_t,\ \text{for all } t \geq 0 && \text{(goods)}\\
&l_t^s = l_t^d,\ \text{for all } t \geq 0 && \text{(labor)}\\
&k_t^s = k_t^d,\ \text{for all } t \geq 0 && \text{(capital)}.
\end{aligned}
$$

2. Steady state

I solve the SPP:

$$
\begin{cases}
w(k_0) = \underset{\{k_{t+1}, l_t\}_{t=0}^{\infty}}{max} & \sum_{t=0}^{\infty} \beta^t \left(\dfrac{c_t^{1-\sigma}}{1-\sigma} - \chi \dfrac{l_t^{1+\eta}}{1+\eta} \right)\\
\text{s.t.} & z k_t^{\alpha} l_t^{1-\alpha} = c_t + k_{t+1} - (1-\delta) k_t,\ \text{for all } t \geq 0\\
& c_t, k_t \geq 0, \quad 0 \leq l_t \leq 1,\ \text{for all } t \geq 0\\
& k_t \leq k_0 \text{ given, for } t = 0.
\end{cases}
$$

$$
\begin{cases}
w(k_0) = \underset{\{k_{t+1}, l_t\}_{t=0}^{\infty}}{max} & \sum_{t=0}^{\infty} \beta^t \left(\dfrac{\left(z k_t^{\alpha} l_t^{1-\alpha} - k_{t+1} + (1-\delta) k_t\right)^{1-\sigma}}{1-\sigma} - \chi \dfrac{l_t^{1+\eta}}{1+\eta} \right)\\
\text{s.t.} & 0 \leq k_{t+1} \leq z k_t^{\alpha} l_t^{1-\alpha} + (1-\delta) k_t,\ \text{for all } t \geq 0\\
& 0 \leq l_t \leq 1,\ \text{for all } t \geq 0\\
& k_t \leq k_0 \text{ given, for } t = 0.
\end{cases}
$$

FOCs

$$
\begin{aligned}
(k_{t+1}): \quad & \left(z k_t^{\alpha} l_t^{1-\alpha} - k_{t+1} + (1-\delta) k_t\right)^{-\sigma}\\
& = \beta \left(z k_{t+1}^{\alpha} l_{t+1}^{1-\alpha} - k_{t+2} + (1-\delta) k_{t+1}\right)^{-\sigma} \left(\alpha z \left(\tfrac{l_{t+1}}{k_{t+1}}\right)^{1-\alpha} + 1 - \delta \right) && \text{for all } t \geq 0\\
(l_t): \quad & \left(z k_t^{\alpha} l_t^{1-\alpha} - k_{t+1} + (1-\delta) k_t\right)^{-\sigma} (1-\alpha) z \left(\tfrac{k_t}{l_t}\right)^{\alpha} = \chi l_t^{\eta} && \text{for all } t \geq 0\\
& z k_t^{\alpha} l_t^{1-\alpha} = c_t + k_{t+1} - (1-\delta) k_t && \text{for all } t \geq 0.
\end{aligned}
$$

In steady state, the FOCs beome

$$
\begin{aligned}
&1 = \beta \left(\alpha z \left(\tfrac{l}{k}\right)^{1-\alpha} + 1 - \delta \right)\\
&\left(z k^{\alpha} l^{1-\alpha} - \delta k\right)^{-\sigma} (1-\alpha) z \left(\tfrac{k}{l}\right)^{\alpha} = \chi l^{\eta}\\
&z k^{\alpha} l^{1-\alpha} = c - \delta k.
\end{aligned}
$$

Then the steady-state allocations are (c, l, k, y), where

$$c = \left(\frac{(1-\alpha)}{\chi} \frac{\left\{ z^{1+\eta-\alpha(\eta+\sigma)} \left[\frac{\alpha\beta}{1-\beta(1-\delta)} \right]^{\eta+\alpha} \right\}^{\frac{1}{1-\alpha}}}{\left\{ \left[\frac{\alpha\beta}{1-\beta(1-\delta)} \right]^{\alpha} + \delta \right\}^{\eta}} \right)^{\frac{1}{\eta+\delta}},$$

$$l = \left(\frac{(1-\alpha)}{\chi} \frac{\left\{ z^{1-\sigma} \left[\frac{\alpha\beta}{1-\beta(1-\delta)} \right]^{\alpha-\sigma} \right\}^{\frac{1}{1-\alpha}}}{\left\{ \left[\frac{\alpha\beta}{1-\beta(1-\delta)} \right]^{\alpha} + \delta \right\}^{\sigma}} \right)^{\frac{1}{\eta+\delta}},$$

$$k = \left(\frac{(1-\alpha)}{\chi} \frac{\left\{ z^{1+\eta} \left[\frac{\alpha\beta}{1-\beta(1-\delta)} \right]^{\alpha+\eta} \right\}^{\frac{1}{1-\alpha}}}{\left\{ \left[\frac{\alpha\beta}{1-\beta(1-\delta)} \right]^{\alpha} + \delta \right\}^{\sigma}} \right)^{\frac{1}{\eta+\delta}},$$

$$y = \left(\frac{(1-\alpha)}{\chi} \frac{\left\{ z^{1+\eta} \left[\frac{\alpha\beta}{1-\beta(1-\delta)} \right]^{\alpha(1+\eta)-(1-\alpha)\sigma} \right\}^{\frac{1}{1-\alpha}}}{\left\{ \left[\frac{\alpha\beta}{1-\beta(1-\delta)} \right]^{\alpha} + \delta \right\}^{\sigma}} \right)^{\frac{1}{\eta+\delta}}$$

To get steady-state prices (w, r), I substitute the values above in the FOCs of the firm's problem in the general equilibrium:

$$F'_k(k, l) = r, \quad F'_l(k, l) = w,$$

that is,

$$\alpha z \left(\frac{l}{k} \right)^{1-\alpha} = r, \quad (1-\alpha) z \left(\frac{k}{l} \right)^{\alpha} = w.$$

Hence,

$$r = \left\{ (\alpha z)^{\delta-\sigma} \left[\frac{1-\beta(1-\delta)}{\beta} \right]^{\sigma+\eta} \right\}^{\frac{1}{\eta+\delta}},$$

$$w = (1-\alpha) \left\{ z^{\eta+\delta-\alpha(\delta-\sigma)} \left[\frac{\alpha\beta}{1-\beta(1-\delta)} \right]^{\alpha(\sigma+\eta)} \right\}^{\frac{1}{(1-\alpha)(\eta+\delta)}}.$$

3. Social planner's dynamic programming

From the sequential SPP, I derive the dynamic programming problem:

$$\begin{cases} w(k_0) & = \max_{\substack{\left\{0\le k_{t+1}\le zk_t^\alpha l_t^{1-\alpha}+(1-\delta)k_t,\right.\\ \left.0\le l_t\le 1,\text{ given }k_0\right\}_{t=0}^\infty}} \sum_{k=0}^\infty \beta^t \left(\frac{\left(zk_t^\alpha l_t^{1-\alpha}-k_{t+1}+(1-\delta)k_t\right)^{1-\sigma}}{1-\sigma}-\chi\frac{l_t^{1+\eta}}{1+\eta}\right) \\ & = \max_{\substack{0\le k_1\le zk_0^\alpha l_0^{1-\alpha}+(1-\delta)k_0,\\ 0\le l_0\le 1,\text{ given }k_0}} \{\frac{\left(zk_0^\alpha l_0^{1-\alpha}-k_1+(1-\delta)k_0\right)^{1-\sigma}}{1-\sigma}-\chi\frac{l_0^{1+\eta}}{1+\eta} \\ & +\beta \max_{\substack{\left\{0\le k_{t+1}\le zk_t^\alpha l_t^{1-\alpha}+(1-\delta)k_t,\right.\\ \left.0\le l_t\le 1,\text{ given }k_1\right\}_{t=0}^\infty}} \sum_{k=0}^\infty \beta^t \left(\frac{\left(zk_t^\alpha l_t^{1-\alpha}-k_{t+1}+(1-\delta)k_t\right)^{1-\sigma}}{1-\sigma}-\chi\frac{l_t^{1+\eta}}{1+\eta}\right)\} \\ & = \max_{\substack{0\le k_1\le zk_0^\alpha l_0^{1-\alpha}+(1-\delta)k_0,\\ 0\le l_0\le 1,\text{ given }k_0}} \left\{\frac{\left(zk_0^\alpha l_0^{1-\alpha}-k_1+(1-\delta)k_0\right)^{1-\sigma}}{1-\sigma}-\chi\frac{l_0^{1+\eta}}{1+\eta}+\beta w(k_1)\right\} \end{cases}$$

$$v(k) = \max_{\substack{0\le k'\le zk^\alpha l^{1-\alpha}+(1-\delta)k,\\ 0\le l\le 1,\text{ given }k}} \left\{\frac{\left(zk^\alpha l^{1-\alpha}-k+(1-\delta)k\right)^{1-\sigma}}{1-\sigma}-\chi\frac{l^{1+\eta}}{1+\eta}+\beta v(k')\right\}$$

Consider $\alpha=\frac{1}{3}, z=1, \sigma=2, \eta=1$. Then,

$$v(k) = \max_{\substack{0\le k'\le k^{\frac{1}{3}}l^{\frac{2}{3}}+(1-\delta)k,\\ 0\le l\le 1,\text{ given }k}} \left\{-\left(k^{\frac{1}{3}}l^{\frac{2}{3}}-k+(1-\delta)k\right)^{-1}-\chi\frac{l^2}{2}+\beta v(k')\right\}$$

4.

$$c = \left(\frac{2}{3}\frac{1}{\chi}\frac{\left\{\frac{\beta}{3[1-\beta(1-\delta)]}\right\}^2}{\left\{\frac{\beta}{3[1-\beta(1-\delta)]}\right\}^{\frac{1}{3}}+\delta}\right)^{\frac{1}{1+\delta}},$$

$$l = \left(\frac{2}{3}\frac{1}{\chi}\frac{\left\{\frac{3[1-\beta(1-\delta)]}{\beta}\right\}^{\frac{5}{2}}}{\left(\left\{\frac{\beta}{3[1-\beta(1-\delta)]}\right\}^{\frac{1}{3}}+\delta\right)^2}\right)^{\frac{1}{1+\delta}},$$

$$k = \left(\frac{2}{3}\frac{1}{\chi}\left[\frac{\frac{\beta}{3[1-\beta(1-\delta)\}}}{\left\{\frac{\beta}{3[1-\beta(1-\delta)]}\right\}^{\frac{1}{3}}+\delta}\right]^2\right)^{\frac{1}{1+\delta}},$$

$$y = \left(\frac{2}{3}\frac{1}{\chi}\frac{\frac{3[1-\beta(1-\delta)]}{\beta}}{\left(\left\{\frac{\beta}{3[1-\beta(1-\delta)]}\right\}^{\frac{1}{3}}+\delta\right)^2}\right)^{\frac{1}{1+\delta}},$$

$$r = \left\{\left(\frac{1}{3}\right)^{\delta-2}\left[\frac{1-\beta(1-\delta)}{\beta}\right]^3\right\}^{\frac{1}{1+\delta}},$$

$$w = \frac{2}{3}\left\{\frac{\beta}{3\left[1-\beta(1-\delta)\right]}\right\}^{\frac{3}{(2+\delta)}}.$$

Assume further, $\beta = 0.9$, $\delta = 0.1$. Then,

$$c = \left(\frac{1}{\chi} \frac{\frac{60}{361}}{\left(\frac{30}{19}\right)^{\frac{1}{3}} + \frac{1}{10}} \right)^{\frac{10}{11}},$$

General equilibrium

By market clearing conditions

$$l_t^s = l_t^d = l_t, \text{ for all } t \geq 0$$
$$k_t^s = k_t^d = k_t, \text{ for all } t \geq 0$$

As F satisfies Inada conditions for k and l, then $k_t, l_t > 0$ for all $t \geq 0$. Consequently, $y_t > 0$ for all $t \geq 0$, due to F being strictly increasing.

As a result, the problem of the firm can be rewritten as

$$\Pi = \max_{\{k_t, l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} p_t \left(F(k_t, l_t) - w_t l_t + r_t k_t \right).$$

Deriving FOCs:

$$F'_{k_t}(k_t, l_t) = r_t, \quad F'_{l_t}(k_t, l_t) = w_t.$$

Differentiating the production function,

$$\begin{aligned} F(k_t, l_t) &= F'_{l_t}(k_t, l_t) l_t + F'_{k_t}(k_t, l_t) k_t \\ &= w_t l_t + r_t k_t. \end{aligned}$$

Then $F(k_t, l_t) - w_t l_t - r_t k_t = 0$, so $\Pi = 0$.

Since u is satisfies Inada conditions with respect to c, then $c_t > 0$ for all $t \geq 0$. u depends on capital only through consumption, so households supply all their capital,

$$k_t = x_t > 0, \text{ for all } t \geq 0.$$

Therefore, investment can be rewritten as

$$i_t = k_{t+1} - (1-\delta)k_t, \text{ for all } t \geq 0.$$

Moreover, u is strictly increasing. Hence, the budget constraint holds with equality (recall that profits are zero):

$$\sum_{t=0}^{\infty} p_t \left(c_t + k_{t+1} - (1-\delta)k_t\right) = \sum_{t=0}^{\infty} p_t(w_t + r_t k_t).$$

The problem of the household can then be written as

$$\begin{cases} \underset{\{c_t, k_t\}_{t=0}^{\infty}}{max} & \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta}\right) \\ \text{s.t.} & \sum_{t=0}^{\infty} p_t \left(c_t + k_{t+1} - (1-\delta)k_t\right) = \sum_{t=0}^{\infty} p_t(w_t l_t + r_t k_t) \\ & 0 \leq l_t \leq 1, \text{ for all } t \geq 0 \\ & k_0 \text{ given.} \end{cases}$$

Consider the Lagrange function

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta}\right) + \lambda \left[\sum_{t=0}^{\infty} p_t \left(w_t l_t + r_t k_t - c_t - k_{t+1} + (1-\delta)k_t\right)\right]$$

Taking FOCs:

$$\begin{aligned} (c_t): \quad & \beta^t c_t^{-\sigma} = \lambda p_t && \text{for all } t \geq 0 \\ (c_{t+1}): \quad & \beta^{t+1} c_{t+1}^{-\sigma} = \lambda p_{t+1} && \text{for all } t \geq 0 \\ (k_{t+1}): \quad & p_t = p_{t+1}\left(r_{t+1} + 1 - \delta\right) && \text{for all } t \geq 0 \\ (l_t): \quad & \beta^t \chi l_t^{\eta} = \lambda p_t w_t && \text{for all } t \geq 0. \end{aligned}$$

From (c_t) and (c_{t+1}),

$$c_{t+1}^{\sigma} = \frac{p_{t+1}}{p_t} \beta c_t^{\sigma}, \quad \text{for all } t \geq 0.$$

Combining with (k_{t+1}), I derive the Euler equation:

$$c_{t+1} = \left[\beta\left(1 + r_{r+1} - \delta\right)\right]^{\frac{1}{\sigma}} c_t, \quad \text{for all } t \geq 0.$$

From (l_t),

$$l_t = \left(\frac{w_t}{\chi c_t^{\sigma}}\right)^{\frac{1}{\eta}}.$$

Plug in (l_t) from the firm's problem,

By market clearing in the market of goods,

$$c_t = F(k_t, l_t) - k_{t+1} + (1-\delta)k_t, \quad \text{for all } t \geq 0.$$

Substituting in the Euler equation,

$$\left[F(k_{t+1}, l_{t+1}) - k_{t+2} + (1-\delta)k_{t+1}\right] = \left[\beta\left(1 + r_{r+1} - \delta\right)\right]^{\frac{1}{\sigma}}\left[F(k_t, l_t) - k_{t+1} + (1-\delta)k_t\right], \quad \text{for all } t \geq 0.$$

In steady state, $k_t = k$, $l_t = l$, for all $t \geq 0$.Then,

$$\left[\beta\left(1 + r - \delta\right)\right]^{\frac{1}{\sigma}} = 1,$$

so

$$r = \frac{1}{\beta} - (1-\delta).$$

By (k_t) in the problem of the firm, in equilibrium the Euler equation can be written as

$$u'\left(f(k_t) - k_{t+1}\right) = \beta u'\left(f(k_{t+1}) - k_{t+2}\right) f'(k_t), \quad \text{for all } t \geq 0.$$

In equilibrium, capital stock $k_0, \{k_{t+1}\}_{t=0}^{\infty}$ solve the Euler equations. All other equilibrium allocations satisfy the equations above. In particular, in equilibrium, $l_t = 1$ and $c_t = f(k_t) - k_{t+1},$ for all $t \geq 0$.

Social planner

In the problem defined in **2**, since u and F satisfy Inada conditions with respect to c and k, l, respectively, it follows that

$$c_t > 0 \text{ for all } t \geq 0, \text{ and}$$

$$k_t, l_t > 0, \text{ for all } t \geq 0, \quad k_t = k_0 \text{ for } t = 0.$$

Besides, as u is strictly increasing in c, and c_t satisfies

$$c_t = F(k_t, l_t) - k_{t+1}, \text{ for all } t \geq 0,$$

F being strictly increasing in l implies c_t and consequently u are as well. Hence,

$$l_t = 1, \text{ for all } t \geq 0.$$

Then, considering that

$$f(k_t) = F(k_t, 1), \text{ for all } t \geq 0,$$

it follows that

$$f(k_t) = c_t + k_{t+1} \text{ for all } t \geq 0.$$

In turn, $c_t, k_t > 0$ for all $t \geq 0$ and f strictly increasing entail

$$0 \leq k_{t+1} \leq f(k_t).$$

Therefore, I may rewrite the sequential social planner's problem as

$$\begin{cases} w(k_0) = \underset{\{k_{t+1}\}_{t=0}^{\infty}}{max} & \sum_{t=0}^{\infty} \beta^t u\left(f(k_t) - k_{t+1}\right) \\ \text{s.t.} & 0 \leq k_{t+1} \leq f(k_t)., \text{ for all } t \geq 0 \\ & k_t \leq k_0 \text{ given, for } t = 0. \end{cases}$$

Deriving the FOCs:

$$(k_{t+1}): \qquad -\beta^t u'\left(f(k_t) - k_{t+1}\right) + \beta^{t+1} u'\left(f(k_{t+1}) - k_{t+2}\right) f'(k_{t+1}) = 0 \text{ for all } t \geq 0,$$

I may get the Euler equation,

$$u'\left(f(k_t)-k_{t+1}\right)=\beta u'\left(f(k_{t+1})-k_{t+2}\right)f'(k_{t+1}) \text{ for all } t\geq 0.$$

Note that this is the same equation as obtained in the general equilibrium.

Suppose the transversality condition,

$$lim_{t\rightarrow\infty}\beta^t u'\left(f(k_t)-k_{t+1}\right)f'(k_t)k_t=0,$$

holds, where $\beta^t u'\left(f(k_t)-k_{t+1}\right)f'(k_t)$ is the value in terms of discounted utility of one more unit of capital stock. In other words, this condition means that the shadow value of capital stock converges to zero.

Let u satisfy **A1**, $\beta\in(0,1)$, f,F satisfy **A2**, $\delta\in(0,1)$. Then an allocation $\{k_{t+1}\}_{t=0}^{\infty}$ that satisfies both the Euler equation and the transversality condition solves the sequential social planner problem for a given initial capital stock k_0.

Under these assumptions, $k_0,\{k_{t+1}\}_{t=0}^{\infty}$ are optimal. Notice that this allocation is the same as in general equilibrium. Also optimal labor and optimal consumption are the same as in equilibrium, l_t and $c_t=f(k_t)-k_{t+1}$ for all $t\geq 0$, respectively.

4. Social planner's dynamic programming problem

I derive the recursive problem of the social planner:

$$\begin{aligned}
w(k_0) &= \max_{\substack{\{0\leq k_{t+1}\leq f(k_t)\}_{t=0}^{\infty},\\ k_0 \text{ given}}} \left\{\textstyle\sum_{t=0}^{\infty}\beta^t u\left(f(k_t)-k_{t+1}\right)\right\}\\
&= \max_{\substack{\{0\leq k_{t+1}\leq f(k_t)\}_{t=0}^{\infty},\\ k_0 \text{ given}}} \left\{u\left(f(k_0)-k_1\right)+\max_{\substack{\{0\leq k_{t+1}\leq f(k_t)\}_{t=1}^{\infty},\\ k_1 \text{ given}}}\textstyle\sum_{t=1}^{\infty}\beta^t u\left(f(k_t)-k_{t+1}\right)\right\}\\
&= \max_{\substack{\{0\leq k_{t+1}\leq f(k_t)\}_{t=0}^{\infty},\\ k_0 \text{ given}}} \left\{u\left(f(k_0)-k_1\right)+\max_{\substack{\{0\leq k_{t+1}\leq f(k_t)\}_{t=1}^{\infty},\\ k_1 \text{ given}}}\textstyle\sum_{t=0}^{\infty}\beta^{t+1} u\left(f(k_t)-k_{t+1}\right)\right\}\\
&= \max_{\substack{\{0\leq k_{t+1}\leq f(k_t)\}_{t=0}^{\infty},\\ k_0 \text{ given}}} \left\{u\left(f(k_0)-k_1\right)+\beta\max_{\substack{\{0\leq k_{t+1}\leq f(k_t)\}_{t=1}^{\infty},\\ k_1 \text{ given}}}\textstyle\sum_{t=0}^{\infty}\beta^{t} u\left(f(k_t)-k_{t+1}\right)\right\}\\
&= \max_{\substack{\{0\leq k_{t+1}\leq f(k_t)\}_{t=0}^{\infty},\\ k_0 \text{ given}}} \left\{u\left(f(k_0)-k_1\right)+\beta w(k_1)\right\}.
\end{aligned}$$

Let v denote de Bellman equation,

$$v(k) = \max_{\{0 \le k' \le f(k)\}_{t=0}^{\infty}, k \text{ given}} \{u(f(k) - k') + \beta v(k')\}.$$

5. Solution to the dynamic programming problem

Assuming $u(c) = log(c)$ and $f(k, l) = zk^{\alpha} l^{1-\alpha}$, $(\delta = 0)$ the Bellman equation is

$$v(k) = \max_{\{0 \le k' \le f(k)\}_{t=0}^{\infty}, k \text{ given}} \{log(zk^{\alpha} - k') + \beta v(k')\}.$$

I use the method of undetermined coefficients to solve this problem. Suppose the value function satisfies

$$v(k) = A + Bln(k).$$

Substituting in the RHS of the Bellman equation,

$$\max_{\{0 \le k' \le f(k)\}_{t=0}^{\infty}, k \text{ given}} \{log(zk^{\alpha} - k') + \beta(A + Blog(k'))\}.$$

By FOC,

$$\frac{1}{zk^{\alpha} - k'} = \frac{\beta B}{k'},$$

$$k' = \frac{\beta B z}{1 + \beta B} k^{\alpha}.$$

Now, I plug k' into the Bellman equation,

$$\begin{aligned}
v(k) &= log\left(zk^{\alpha} - \frac{\beta B z}{1+\beta B} k^{\alpha}\right) + \beta\left[A + Blog\left(\frac{\beta B z}{1+\beta B} k^{\alpha}\right)\right] \\
&= log\left(\frac{z}{1+\beta B} k^{\alpha}\right) + \beta A + \beta Blog\left(\frac{\beta B z}{1+\beta B} k^{\alpha}\right) \\
&= log\left(\frac{z}{1+\beta B}\right) + \alpha log(k) + \beta A + \beta Blog\left(\frac{\beta B z}{1+\beta B}\right) + \alpha\beta Blog(k) \\
&= log(z) - log(1+\beta B) + \beta A + \beta Blog(\beta B z) - \beta Blog(1+\beta B) + \alpha(1+\beta B)log(k).
\end{aligned}$$

Denote

$$A = log(z) - log(1+\beta B) + \beta A + \beta B log(\beta B z) - \beta B log(1+\beta B),$$

and

$$B = \alpha(1+\beta B).$$

Solving for B,

$$B = \frac{\alpha}{1-\alpha\beta}.$$

Plugging in B in A:

$$A = \frac{1}{1-\beta}\left[log\left(\frac{z}{1-\alpha\beta}\right) + \frac{\alpha\beta}{1-\alpha\beta}log(z\alpha\beta)\right].$$

Then plugging B in k', the solution to the social planner's dynamic programming problem is

$$k' = \frac{\alpha\beta z}{1-\alpha\beta}k^{\alpha}.$$

6. Steady state

Let $k_t = k$, for all $t \geq 0$ be the steady-state capital stock. Then k satisfies

$$k = \left(\frac{\alpha\beta z}{1-\alpha\beta}\right)^{1-\alpha}$$

Plug k in the Euler equation:

$$u'\left(f(k) - k\right) = \beta u'\left(f(k) - k\right) f'(k).$$

Then

$$f'(k) = 1 + \rho,$$

where ρ satisfies $\beta = \dfrac{1}{1+\rho}$.

By FOC in the firm's problem (k_t):

$$r = 1 + \rho.$$

Recall that

$$F(k, l) = wl + rk,$$

so plugging in steady-state labor $l = 1$,

$$f(k) = w + rk,$$

where

$$f(k) = zk^{\alpha}.$$

Therefore, plugging in k, steady-state output is

$$f(k) = z\left(\frac{\alpha\beta z}{1-\alpha\beta}\right)^{\alpha(1-\alpha)},$$

and steady-state wages are

$$w = \left(\frac{\alpha\beta z}{1-\alpha\beta}\right)^{1-\alpha}\left[z\left(\frac{\alpha\beta z}{1-\alpha\beta}\right)^{\alpha} - (1+\rho)\right].$$