Computational Macro | Problem Set 1

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The Neoclassical Growth Model

Notation

Assumptions

A1. Assume $\beta \in [0,1]$ and u is continuously differentiable, strictly increasing, strictly concave, bounded, and satisfies Inada conditions,

$$lim_{c \to 0} u'(c) = \infty, \quad lim_{c \to \infty} u'(c) = 0.$$

A2. Assume $\delta \in [0,1]$ and F is continuously differentiable, homogeneous of degree 1 (constant returns to scale), strictly increasing, strictly concave, and satisfies

$$F(k,0) = F(0,l) = 0, \text{ for any } k, l > 0,$$

as well as Inada conditions

$$lim_{k \to 0} F'_k(k,l) = lim_{l \to 0} F'_l(k,l) = \infty, \quad lim_{k \to \infty} F'_k(k,l) = lim_{l \to \infty} F'_l(k,l) = 0.$$

1. Competitive equilibrium

An Arrow-Debreu equilibrium for this economy consists of a sequence of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ and allocations for the household $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ and the firm $\left\{y_t, k_t^d, l_t^d\right\}_{t=0}^{\infty}$ such that:

· Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, allocations $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ solve the household's problem:

$$\begin{cases} \max\limits_{\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t.} & \sum_{t=0}^{\infty} p_t(c_t + i_t) \leq \sum_{t=0}^{\infty} p_t(w_t l_t^s + r_t k_t^s) + \Pi \\ & i_t = x_{t+1}, \text{ for all } t \geq 0 \\ & c_t, x_{t+1} \geq 0, \quad 0 \leq l_t^s \leq 1, \quad 0 \leq k_t^s \leq x_t, \text{ for all } t \geq 0 \\ & x_0 = k_0 \text{ given} \end{cases}$$

· Given $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, allocations $\left\{y_t, k_t^d, l_t^d\right\}_{t=0}^{\infty}$ solve the firm's problem

$$\begin{cases} \Pi = \max\limits_{\{y_t, k_t^d, l_t^d\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} p_t(y_t - w_t l_t^d + r_t k_t^d) \\ \text{s.t.} & y_t = F(k_t^d, l_t^d), \text{ for all } t \geq 0 \\ & y_t, k_t^d, l_t^d \geq 0 \text{ for all } t \geq 0 \end{cases}$$

· Markets clear

$$\begin{aligned} & y_t = c_t + i_t, \text{ for all } t \geq 0 && \text{(goods)} \\ & l_t^s = l_t^d, \text{ for all } t \geq 0 && \text{(labor)} \\ & k_t^s = k_t^d, \text{ for all } t \geq 0 && \text{(capital).} \end{aligned}$$

2. Social planner's problem

The sequential social planner's problem is

$$\begin{cases} w(k_0) = \max\limits_{\{c_t, k_t, l_t\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t.} & F(k_t, l_t) = c_t + k_{t+1}, \text{ for all } t \geq 0 \\ & c_t, k_t \geq 0, \quad 0 \leq l_t \leq 1, \text{ for all } t \geq 0 \\ & k_t \leq k_0 \text{ given, for } t = 0. \end{cases}$$

3. The set of equilibrium allocations is the set of optimal allocations

General equilibrium

By market clearing conditions

$$l_t^s = l_t^d = l_t, \text{ for all } t \geq 0$$
$$k_t^s = k_t^d = k_t, \text{ for all } t \geq 0$$

As F satisfies Inada conditions for k and l, then $k_t, l_t > 0$ for all $t \geq 0$. Consequently, $y_t > 0$ for all $t \geq 0$, due to F being strictly increasing.

As a result, the problem of the firm can be rewritten as

$$\Pi = \max_{\{k_t, l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} p_t \left(F(k_t, l_t) - w_t l_t + r_t k_t\right).$$

Deriving FOCs:

$$F'_{k_t}(k_t, l_t) = r_t, \quad F'_{l_t}(k_t, l_t) = w_t.$$

Differentiating the production function,

$$\begin{aligned} F(k_t, l_t) &= F'_{l_t}(k_t, l_t) l_t + F'_{k_t}(k_t, l_t) k_t \\ &= w_t l_t + r_t k_t. \end{aligned}$$

Then $F(k_t, l_t) - w_t l_t + r_t k_t = 0$ and $\Pi = 0$.

Recall that F is homogeneous of degree 1, then

$$F(k, l) = l^{-1} F(k/l, 1),$$

and F'_k is homogeneous of degree 0:

$$F'_k(k, l) = l^0 F'_k(k/l, 1) = F'_k(k/l, 1).$$

Define

$$f(k_t) := F(k_t, 1).$$

Then,

$$F'_{k_t}(k_t, 1) = f'(k_t).$$

Hence, by (k_t),

$$r_t = f'(k_t).$$

Since u is satisfies Inada conditions with respect to c, then $c_t > 0$ for all $t \geq 0$. The only argument in u is c. u depends on capital only through consumption and leisure is not an argument in u, so households supply all their capital and devote no time to leisure,

$$l_t = 1, \quad k_t = x_t > 0, \text{ for all } t \geq 0.$$

Therefore, investment can be rewritten as

$$i_t = k_{t+1}, \text{ for all } t \geq 0.$$

Moreover, u is strictly increasing. Hence, the budget constraint holds with equality (recall that profits are zero):

$$\sum_{t=0}^{\infty} p_t \left(c_t + k_{t+1}\right) = \sum_{t=0}^{\infty} p_t(w_t + r_t k_t).$$

The problem of the household can then be written as

$$\begin{cases} \underset{\{c_t, k_t\}_{t=0}^{\infty}}{max} & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t.} & \sum_{t=0}^{\infty} p_t \left(c_t + k_{t+1}\right) = \sum_{t=0}^{\infty} p_t(w_t + r_t k_t) \\ & k_0 \text{ given.} \end{cases}$$

Consider the Lagrange function

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda \left[\sum_{t=0}^{\infty} p_t \left(w_t + r_t k_t - c_t + k_{t+1}\right)\right]$$

Taking FOCs:

$$
\begin{aligned}
c_t: &\quad \beta^t u'(c_t) = \lambda p_t && \text{for all } t \geq 0 \\
c_{t+1}: &\quad \beta^{t+1} u'(c_{t+1}) = \lambda p_{t+1} && \text{for all } t \geq 0 \\
k_{t+1}: &\quad p_t = p_{t+1} r_t && \text{for all } t \geq 0.
\end{aligned}
$$

From (c_t) and (c_{t+1}),

$$\beta u'(c_{t+1}) = \frac{p_{t+1}}{p_t} u'(c_t), \quad \text{for all } t \geq 0.$$

Combining with (k_{t+1}), I derive the Euler equation:

$$u'(c_t) = \beta u'(c_{t+1}) r_t, \quad \text{for all } t \geq 0.$$

By market clearing in the market of goods,

$$F(k_t, 1) = c_t + k_{t+1}, \quad \text{for all } t \geq 0.$$

As a result,

$$c_t = F(k_t, 1) - k_{t+1}, \quad \text{for all } t \geq 0.$$

Equivalently,

$$c_t = f(k_t) - k_{t+1}, \quad \text{for all } t \geq 0.$$

Substituting in the Euler equation,

$$u'\left(f(k_t) - k_{t+1}\right) = \beta u'\left(f(k_{t+1}) - k_{t+2}\right) r_t, \quad \text{for all } t \geq 0.$$

By (k_t) in the problem of the firm, in equilibrium the Euler equation can be written as

$$u'\left(f(k_t) - k_{t+1}\right) = \beta u'\left(f(k_{t+1}) - k_{t+2}\right) f'(k_t), \quad \text{for all } t \geq 0.$$

In equilibrium, capital stock $k_0, \{k_{t+1}\}_{t=0}^{\infty}$ solve the Euler equations. All other equilibrium allocations satisfy the equations above. In particular, in equilibrium, $l_t = 1$ and $c_t = f(k_t) - k_{t+1},$ for all $t \geq 0$.

Social planner

In the problem defined in **2**, since u and F satisfy Inada conditions with respect to c and k, l, respectively, it follows that

$$c_t > 0 \text{ for all } t \geq 0, \text{ and}$$

$$k_t, l_t > 0, \text{ for all } t \geq 0, \quad k_t = k_0 \text{ for } t = 0.$$

Besides, as u is strictly increasing in c, and c_t satisfies

$$c_t = F(k_t, l_t) - k_{t+1}, \text{ for all } t \geq 0,$$

F being strictly increasing in l implies c_t and consequently u are as well. Hence,

$$l_t = 1, \text{ for all } t \geq 0.$$

Then, considering that

$$f(k_t) = F(k_t, 1), \text{ for all } t \geq 0,$$

it follows that

$$f(k_t) = c_t + k_{t+1} \text{ for all } t \geq 0.$$

In turn, $c_t, k_t > 0$ for all $t \geq 0$ and f strictly increasing entail

$$0 \leq k_{t+1} \leq f(k_t).$$

Therefore, I may rewrite the sequential social planner's problem as

$$\begin{cases} w(k_0) = \max\limits_{\{k_{t+1}\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t u\left(f(k_t) - k_{t+1}\right) \\ \text{s.t.} & 0 \leq k_{t+1} \leq f(k_t)., \text{ for all } t \geq 0 \\ & k_t \leq k_0 \text{ given, for } t = 0. \end{cases}$$

Deriving the FOCs:

$$(k_{t+1}): \qquad -\beta^t u'\left(f(k_t) - k_{t+1}\right) + \beta^{t+1} u'\left(f(k_{t+1}) - k_{t+2}\right) f'(k_{t+1}) = 0 \text{ for all } t \geq 0,$$

I may get the Euler equation,

$$u'\left(f(k_t) - k_{t+1}\right) = \beta u'\left(f(k_{t+1}) - k_{t+2}\right) f'(k_{t+1}) \text{ for all } t \geq 0.$$

Note that this is the same equation as obtained in the general equilibrium.

Suppose the transversality condition,

$$lim_{t \to \infty} \beta^t u'\left(f(k_t) - k_{t+1}\right) f'(k_t) k_t = 0,$$

holds, where $\beta^t u'\left(f(k_t) - k_{t+1}\right) f'(k_t)$ is the value in terms of discounted utility of one more unit of capital stock. In other words, this condition means that the shadow value of capital stock converges to zero.

Let u satisfy **A1**, $\beta \in (0,1)$, f, F satisfy **A2**, $\delta \in (0,1)$. Then an allocation $\{k_{t+1}\}_{t=0}^{\infty}$ that satisfies both the Euler equation and the transversality condition solves the sequential social planner problem for a given initial capital stock k_0.

Under these assumptions, $k_0, \{k_{t+1}\}_{t=0}^{\infty}$ are optimal. Notice that this allocation is the same as in general equilibrium. Also optimal labor and optimal consumption are the same as in equilibrium, l_t and $c_t = f(k_t) - k_{t+1}$ for all $t \geq 0$, respectively.

4. Social planner's dynamic programming problem

I derive the recursive problem of the social planner:

$$
\begin{aligned}
w(k_0) &= \max_{\substack{\{0\le k_{t+1}\le f(k_t)\}_{t=0}^{\infty},\\ k_0 \text{ given}}} \left\{\sum_{t=0}^{\infty}\beta^t u\left(f(k_t)-k_{t+1}\right)\right\} \\
&= \max_{\substack{\{0\le k_{t+1}\le f(k_t)\}_{t=0}^{\infty},\\ k_0 \text{ given}}} \left\{u\left(f(k_0)-k_1\right) + \max_{\substack{\{0\le k_{t+1}\le f(k_t)\}_{t=1}^{\infty},\\ k_1 \text{ given}}} \sum_{t=1}^{\infty}\beta^t u\left(f(k_t)-k_{t+1}\right)\right\} \\
&= \max_{\substack{\{0\le k_{t+1}\le f(k_t)\}_{t=0}^{\infty},\\ k_0 \text{ given}}} \left\{u\left(f(k_0)-k_1\right) + \max_{\substack{\{0\le k_{t+1}\le f(k_t)\}_{t=1}^{\infty},\\ k_1 \text{ given}}} \sum_{t=0}^{\infty}\beta^{t+1} u\left(f(k_t)-k_{t+1}\right)\right\} \\
&= \max_{\substack{\{0\le k_{t+1}\le f(k_t)\}_{t=0}^{\infty},\\ k_0 \text{ given}}} \left\{u\left(f(k_0)-k_1\right) + \beta \max_{\substack{\{0\le k_{t+1}\le f(k_t)\}_{t=1}^{\infty},\\ k_1 \text{ given}}} \sum_{t=0}^{\infty}\beta^{t} u\left(f(k_t)-k_{t+1}\right)\right\} \\
&= \max_{\substack{\{0\le k_{t+1}\le f(k_t)\}_{t=0}^{\infty},\\ k_0 \text{ given}}} \left\{u\left(f(k_0)-k_1\right) + \beta w(k_1)\right\}.
\end{aligned}
$$

Let v denote de Bellman equation,

$$v(k) = \max_{\{0\le k'\le f(k)\}_{t=0}^{\infty}, k \text{ given}} \left\{u\left(f(k)-k'\right) + \beta v(k')\right\}.$$

5. Solution to the dynamic programming problem

Assuming $u(c) = log(c)$ and $f(k,l) = zk^{\alpha}l^{1-\alpha}$, $(\delta = 0)$ the Bellman equation is

$$v(k) = \max_{\{0\le k'\le f(k)\}_{t=0}^{\infty}, k \text{ given}} \left\{log\left(zk^{\alpha}-k'\right) + \beta v(k')\right\}.$$

I use the method of undetermined coefficients to solve this problem. Suppose the value function satisfies

$$v(k) = A + Bln(k).$$

Substituting in the RHS of the Bellman equation,

$$\max_{\{0\le k'\le f(k)\}_{t=0}^{\infty}, k \text{ given}} \left\{log\left(zk^{\alpha}-k'\right) + \beta\left(A + Blog(k')\right)\right\}.$$

By FOC,

$$\frac{1}{zk^{\alpha}-k'} = \frac{\beta B}{k'},$$

$$k' = \frac{\beta B z}{1+\beta B} k^{\alpha}.$$

Now, I plug k' into the Bellman equation,

$$\begin{aligned}
v(k) &= log\left(zk^{\alpha} - \frac{\beta B z}{1+\beta B} k^{\alpha}\right) + \beta\left[A + Blog\left(\frac{\beta B z}{1+\beta B} k^{\alpha}\right)\right] \\
&= log\left(\frac{z}{1+\beta B} k^{\alpha}\right) + \beta A + \beta Blog\left(\frac{\beta B z}{1+\beta B} k^{\alpha}\right) \\
&= log\left(\frac{z}{1+\beta B}\right) + \alpha log(k) + \beta A + \beta Blog\left(\frac{\beta B z}{1+\beta B}\right) + \alpha\beta Blog(k) \\
&= log(z) - log(1+\beta B) + \beta A + \beta Blog(\beta B z) - \beta Blog(1+\beta B) + \alpha(1+\beta B)log(k).
\end{aligned}$$

Denote

$$A = log(z) - log(1+\beta B) + \beta A + \beta Blog(\beta B z) - \beta Blog(1+\beta B),$$

and

$$B = \alpha(1+\beta B).$$

Solving for B,

$$B = \frac{\alpha}{1-\alpha\beta}.$$

Plugging in B in A:

$$A = \frac{1}{1-\beta}\left[log\left(\frac{z}{1-\alpha\beta}\right) + \frac{\alpha\beta}{1-\alpha\beta} log(z\alpha\beta)\right].$$

Then plugging B in k', the solution to the social planner's dynamic programming problem is

$$k' = \frac{\alpha\beta z}{1-\alpha\beta} k^{\alpha}.$$

6. Steady state

Let $k_t = k$, for all $t \geq 0$ be the steady-state capital stock. Then k satisfies

$$k = \left(\frac{\alpha \beta z}{1 - \alpha \beta} \right)^{1-\alpha}$$

Plug k in the Euler equation:

$$u' \left(f(k) - k \right) = \beta u' \left(f(k) - k \right) f'(k).$$

Then

$$f'(k) = 1 + \rho,$$

where ρ satisfies $\beta = \dfrac{1}{1+\rho}$.

By FOC in the firm's problem (k_t):

$$r = 1 + \rho.$$

Recall that

$$F(k, l) = wl + rk,$$

so plugging in steady-state labor $l = 1$,

$$f(k) = w + rk,$$

where

$$f(k) = zk^{\alpha}.$$

Therefore, plugging in k, steady-state output is

$$f(k) = z \left(\frac{\alpha \beta z}{1 - \alpha \beta} \right)^{\alpha(1-\alpha)},$$

and steady-state wages are

$$w = \left(\frac{\alpha \beta z}{1 - \alpha \beta} \right)^{1-\alpha} \left[z \left(\frac{\alpha \beta z}{1 - \alpha \beta} \right)^{\alpha} - (1 + \rho) \right].$$