SOLVSOLITONS ASSOCIATED TO HEISENBERG ALGEBRAS

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Abstract. Lauret recently characterized solvsoliton inner products as certain extensions of nilsolitons. Subsequent work of Will demonstrated that the space of solvsolitons extensions of a given nilsoliton is parametrized by the quotient of a Grassmannian by a finite group. In this paper, we consider the case of a nilsoliton on a Heisenberg algebra of arbitrary odd dimension, and determine the corresponding space of solvsoliton extensions.

1. Introduction

The search for "optimal" Riemannian metrics has motivated a great amount of research in geometry. Frequent candidates for such metrics are Einstein metrics, but in recent years Ricci soliton metrics have become increasingly important. These are generalized fixed points of the Ricci flow, and are one generalization of Einstein metrics. More specifically, a metric g on a manifold $\mathcal{M}$ is a *Ricci soliton* if

$$\mathrm{Rc}(g) = cg + \mathcal{L}_X g \tag{1.1}$$

for some $c \in \mathbb{R}$ and $X \in C^\infty(T\mathcal{M})$.

While there are essentially no non-Einstein compact homogeneous Ricci solitons [18], the non-compact case is rich with examples. In fact, all known examples of non-compact homogeneous Ricci solitons are left-invariant metrics on simply connected solvable Lie groups, such that

$$\mathrm{Ric}(g) = c\,\mathrm{id} + D \tag{1.2}$$

where Ric is the Ricci $(1,1)$-tensor, $c \in \mathbb{R}$, and D is a derivation of the Lie algebra. This *algebraic soliton* condition relates the algebra and geometry of the group. On nilpotent and solvable Lie groups, such metrics are called *nilsolitons* and *solvsolitons*, respectively. It is known that equations an algebraic soliton is a Ricci soliton (see, e.g., [14]), and conversely a Ricci soliton on a solvable Lie group is isometric to a solvsoliton (on a possibly different Lie group) [7].

Lauret has proved several foundational results on algebraic solitons. Nilsolitons are unique (up to isometry and scaling) whenever they exist, are the critical points of a certain Riemannian functional, and are the nilpotent parts of Einstein solvmanifolds [13]. The relationship in the last fact goes much deeper, as all solvsolitons arise from a nilsoliton in a natural way [14]. We will revisit these results in the next subsection.

Aside from structural and uniqueness results, a common goal in the study of solitons is to find new examples—especially in families—and to describe them explicitly. For example, Payne has constructed continuous families of nilsolitons [17],

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and Kadioglu and Payne have made progress towards describing all algebras admitting nilsolitons with simple derivations in dimensions seven and eight [8]. Building on earlier work of Lauret and Will [15], Lafuente has constructed families of solvsolitons associated with certain graphs [12]. Will has also used the results of Lauret to classify all solvsoliton in dimensions less than seven [21]. Nilsolitons of dimension seven have been classified by Culma [3]. This author considered nilsolitons on the Heisenberg algebras [22], and in the current paper we use the results of Lauret to describe solvsoliton extensions of those nilsolitons.

Theorem 1.3. *Consider the Heisenberg algebra $\mathfrak{h}_{2n+1}\mathbb{R}$ with its unique nilsoliton. Up to isometry and scaling, the space of $(2n+1+k)$-dimensional solvsolitons constructed from $\mathfrak{h}_{2n+1}\mathbb{R}$ is parametrized by*

$$\operatorname{Gr}_k \mathbb{R}^{n+1} \Big/ S_n \ltimes (\mathbb{Z}/2)^n,$$

where $1 \leq k \leq n+1$.

The construction of solvsolitons in this theorem is described in Section 2 (Theorem 2.2), where we give a more detailed description on the structural results that we use to prove the theorem. In Section 3, we describe the spaces in question—Heisenberg algebras. We follow with descriptions of the corresponding Automorphisms groups in Section 4 and derivation algebras in Section 5. Finally, the proof of the theorem is completed in Section 6.

2. Structure and spaces of solvsolitons

As mentioned above, the results of Lauret regarding the structure and uniqueness of solvsolitons show that solvsolitons are related to nilsolitons in a precise way, which we now describe. If $\mathfrak{n}$ is a Lie algebra and $\mathfrak{a} \subset \operatorname{Der}(\mathfrak{n})$ is a subalgebra, first recall that the *semi-direct product* $\mathfrak{s} = \mathfrak{n} \ltimes \mathfrak{a}$ is the direct sum of vector spaces $\mathfrak{n} \oplus \mathfrak{a}$ together with the Lie bracket $[\cdot,\cdot]$ given by

$$[X,Y] = [X,Y]_{\mathfrak{n}}, \quad [A,B] = [A,B]_{\mathfrak{a}}, \quad [X,A] = A(X), \tag{2.1}$$

for all $X, Y \in \mathfrak{n}$, $A, B \in \mathfrak{a}$. Also, if $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ has an inner product $\langle \cdot,\cdot \rangle$, recall that the *mean curvature vector* of $\mathfrak{s}$ is the unique $H \in \mathfrak{a}$ such that $\langle H, A \rangle = \operatorname{tr} \operatorname{ad} H$ for all $A \in \mathfrak{a}$.

Theorem 2.2 (The structure and uniqueness of solvsolitons [14])**.**

(a) *Let $(\mathfrak{n}, \langle\cdot,\cdot\rangle_{\mathfrak{n}})$ be a nilsoliton with Ricci operator $\operatorname{Ric}_{\mathfrak{n}} = cI + D_{\mathfrak{n}}$, $c < 0$, and $D_{\mathfrak{n}} \in \operatorname{Der}(\mathfrak{n})$. Consider an abelian Lie algebra $\mathfrak{a}$ of $\langle\cdot,\cdot\rangle_{\mathfrak{n}}$-symmetric derivations of $\mathfrak{n}$. Then the solvmanifold S with Lie algebra $\mathfrak{s} = \mathfrak{n} \ltimes \mathfrak{a}$ and inner product*

$$\langle\cdot,\cdot\rangle_{\mathfrak{s}} = \langle\cdot,\cdot\rangle_{\mathfrak{n}} + \langle\cdot,\cdot\rangle_{\mathfrak{a}}, \quad \textit{where } \langle A, B\rangle_{\mathfrak{a}} = -\tfrac{1}{c}\operatorname{tr}(AB),$$

is a solvsoliton with $\operatorname{Ric} = cI + D$. Here, $D \in \operatorname{Der}(\mathfrak{s})$ is defined by $D|_{\mathfrak{a}} = 0$, $D|_{\mathfrak{n}} = D_{\mathfrak{n}} - \operatorname{ad} H|_{\mathfrak{n}}$, and H is the mean curvature vector of S. Furthermore, S is Einstein if and only if $D_{\mathfrak{n}} \in \mathfrak{a}$.

(b) *All solvsolitons are of the form described in (a), with $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$.*

(c) *Let S and S' be two solvsolitons which are isomorphic as Lie groups. Then S is isometric to S' up to scaling.*

(d) *Let* $(\mathfrak{n}, \langle\cdot,\cdot\rangle_{\mathfrak{n}})$ *be a nilsoliton and consider two solvsolitons* S *and* S'*, constructed as in (a) from abelian Lie algebras* $\mathfrak{a}$ *and* $\mathfrak{a}'$ *of* $\langle\cdot,\cdot\rangle_{\mathfrak{n}}$*-symmetric derivations. Then* S *is isometric to* S' *if and only if there exists* $h \in \mathrm{Aut}(\mathfrak{n}) \cap \mathrm{O}(\mathfrak{n}, \langle\cdot,\cdot\rangle_{\mathfrak{n}})$ *such that* $\mathfrak{a}' = h\mathfrak{a}h^{-1}$*.*

Given a nilsoliton, a natural question to ask is, "How can one describe the space of associated solvsolitons?" Let us consider another example, also found in [21].

Example 2.3. The Lie groups $\mathrm{Nil}^3 = H_3\mathbb{R}$ has been studied extensively in the context of Ricci flow and Ricci solitons [1, 5, 6, 11, 16]. It admits a nilsoliton, and after applying an isometry and scaling (as in Theorem 2.2 (c)), we can assume that it is given by the standard inner product. We would like to examine the solvsolitons that can be constructed from this nilsoliton. By part (d) in the Theorem 2.2, these are parametrized by abelian subalgebras of symmetric derivations of the Lie algebra $\mathfrak{h}_3\mathbb{R}$, up to conjugacy by orthogonal automorphisms. We can describe these spaces directly.

After choosing a basis, it is easy to see that the group of orthogonal automorphisms is

$$\mathrm{Aut}(\mathfrak{h}_3) \cap \mathrm{O}(3) = \left\{ \begin{pmatrix} T & 0 \\ 0 & \det T \end{pmatrix} \;\middle|\; T \in \mathrm{O}(2) \right\} \cong \mathrm{O}(2).$$

It is also easy to compute that the space of derivations of the Lie algebra is

$$\mathrm{Der}(\mathfrak{h}_3\mathbb{R}) = \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & a+d \end{pmatrix} \;\middle|\; a, b, c, d, e, f \in \mathbb{R} \right\},$$

and so the space of symmetric derivations is

$$\mathrm{Der}(\mathfrak{h}_3\mathbb{R}) \cap \mathrm{sym}(3) = \left\{ \begin{pmatrix} a & c & \\ c & b & \\ & & a+b \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}.$$

It is a fact that any two abelian subspaces of $\mathrm{Der}(\mathfrak{h}_3\mathbb{R}) \cap \mathrm{sym}(3)$ can be conjugated into a maximal abelian algebra using an orthogonal derivation, so we restrict our attention to subspaces of

$$\mathfrak{a} = \left\{ \begin{pmatrix} a & & \\ & b & \\ & & a+b \end{pmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}.$$

The last fact to keep in mind is that this maximal abelian subalgebra may have symmetries, that is, permutations of elements of this subalgebra. The only nontrivial permutation is exchanging a and b, so these form a group isomorphic to $\mathbb{Z}/2$. Also, as $\mathfrak{a}$ is a two-dimensional vector space, we have

$$\mathrm{Gr}_2(\mathfrak{a}) \cong \mathbb{P}(\mathfrak{a}) \cong \mathbb{P}^1,$$

and the space of solvsoliton extensions of Nil^3 is therefore parametrized by

$$\mathbb{P}^1 \big/ \mathbb{Z}/2.$$

This can be identified with the set

$$\{(a, b) \in S^1 \mid |a| \leq b\},$$

where the correspondence is given by

$$(a,b) \longleftrightarrow \mathbb{R}\begin{pmatrix} a & & \\ & b & \\ & & a+b \end{pmatrix}.$$

Write $E(a,b)$ for an element of $\mathfrak{a}$. Then the Lie algebra structure is

$$\begin{aligned} [E_1,E_2] &= E_3, \qquad & [E(a,b),E_1] &= aE_1, \\ [E(a,b),E_2] &= bE_2, \qquad & [E(a,b),E_3] &= (a+b)E_3, \end{aligned}$$

and the solvsoliton metric is, for $i=1,2$,

$$\langle E_i, E_j\rangle = \delta_{ij}, \qquad \langle E(a,b), E(a,b)\rangle = 4(a^2+b^2+ab)$$

according to Theorem 2.2.

In a similar manner, Will has described the spaces of such solvsolitons, up to scaling and isometry, on groups of low dimension. We quickly recall the general phenomenon discussed in [21, Section 3]. Letting μ represent a Lie bracket structure, suppose $(\mathbb{R}^n, \mu)$ is a nilpotent Lie algebra with inner product $\langle\cdot,\cdot\rangle$. We write $\mathrm{O}(n), \mathfrak{so}(n), \mathrm{sym}(n)$ to refer to the orthogonal, skew-symmetric, and symmetric linear maps on $\mathbb{R}^n$, with respect to $\langle\cdot,\cdot\rangle$. Let

$$\begin{aligned} G_\mu &= \{g \in \mathrm{Aut}(\mu) \mid g^t \in \mathrm{Aut}(\mu)\} = K_\mu \exp(\mathfrak{p}_\mu), \\ \mathfrak{g}_\mu &= \{A \in \mathrm{Der}(\mu) \mid A^t \in \mathrm{Der}(\mu)\} = \mathfrak{k}_\mu \oplus \mathfrak{p}_\mu. \end{aligned}$$

The group G_μ is real reductive, and the above decompositions are the Cartan decompositions, where

$$\begin{aligned} K_\mu &= G_\mu \cap \mathrm{O}(n) = \mathrm{Aut}(\mu) \cap \mathrm{O}(n), \\ \mathfrak{k}_\mu &= \mathfrak{g}_\mu \cap \mathfrak{so}(n) = \mathrm{Der}(\mu) \cap \mathfrak{so}(n), \\ \mathfrak{p}_\mu &= \mathfrak{g}_\mu \cap \mathrm{sym}(n) = \mathrm{Der}(\mu) \cap \mathrm{sym}(n). \end{aligned}$$

Assume that $\langle\cdot,\cdot\rangle$ is a nilsoliton. By Theorem 2.2 (d), solvsoliton extensions of this nilsoliton are parametrized by K_μ-conjugacy classes of abelian subspaces of $\mathfrak{p}_\mu$. Let $\mathfrak{a}_\mu \subseteq \mathfrak{p}_\mu$ be a maximal abelian subalgebra. Define the *rank* of the nilpotent Lie algebra to be $\mathrm{rank}(\mu) = \dim(\mathfrak{a}_\mu)$ and let $0 < k \leq \mathrm{rank}(\mu)$.

Proposition 2.4 ([21])**.** *The set of $(n+k)$-dimensional solvsoliton extensions of $(\mathbb{R}^n, \mu, \langle\cdot,\cdot\rangle)$, up to isometry and scaling, is parametrized by*

$$\mathrm{Gr}_k(\mathfrak{a}_\mu)/W_\mu,$$

where W_μ is the Weyl group of G_μ.

Here

$$\begin{aligned} N_{K_\mu}(\mathfrak{a}_\mu) &= \{g \in K_\mu \mid \mathrm{Ad}(g)\mathfrak{a}_\mu \subseteq \mathfrak{a}_\mu\}, \\ Z_{K_\mu}(\mathfrak{a}_\mu) &= \{g \in K_\mu \mid \mathrm{Ad}(g)A = A \text{ for all } A \in \mathfrak{a}_\mu\}, \\ W_\mu &= N_{K_\mu}(\mathfrak{a}_\mu)/Z_{K_\mu}(\mathfrak{a}_\mu). \end{aligned}$$

The main goal of this paper is to use the above results of Lauret and Will to describe all solvsolitons that are constructed from nilsolitons on the classical Heisenberg algebras. The main tasks are to compute or estimate the rank r and to determine the Weyl group W_μ.

Remark 2.5. Solvsolitons arising from Heisenberg algebras have been previously explored in low dimensions. The graphs in [12] generate certain nilpotent Lie algebras, and the three-dimensional Heisenberg algebra is such an algebra. Also, the classification of solvsolitons in low dimensions [21] includes the study of the three- and five-dimensional Heisenberg algebras.

3. Heisenberg algebras

Let $\mathfrak{w}$ be an n-dimensional real vector space, and set $\mathfrak{v} = \mathfrak{w} \oplus \mathfrak{w}$. Write

$$v = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathfrak{v}, \quad \text{where } x, y \in \mathfrak{w}.$$

There is a natural involution $\iota : \mathfrak{v} \to \mathfrak{v}$ given by

$$\iota \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}.$$

Suppose $\mathfrak{v}$ has a bilinear, skew-symmetric form ω, i.e., a symplectic form, and a trivial Lie bracket. Let $N = 2n + 1$ and consider a 1-dimensional central extension of $\mathfrak{v}$:

$$0 \longrightarrow \mathfrak{z} \longrightarrow \mathfrak{h}_N\mathbb{R} \xrightarrow{\pi} \mathfrak{v} \longrightarrow 0.$$

This extension can be considered as a sum $\mathfrak{h}_N\mathbb{R} = \mathfrak{v} \oplus \mathfrak{z}$, and we write an element as

$$\xi = \begin{pmatrix} v \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \text{where } v \in \mathfrak{v},\ x, y \in \mathfrak{w},\ z \in \mathfrak{z}.$$

The projection map is $\pi \begin{pmatrix} v \\ z \end{pmatrix} = v$. One can also think of this as the usual quotient map, with $\pi(\xi) = \xi + \mathfrak{z}$. The Lie bracket on $\mathfrak{h}_N\mathbb{R}$ defined by this extension is expressed in terms of the symplectic form on $\mathfrak{v}$. Namely,

$$[\xi, \eta] = \omega\big(\pi(\xi), \pi(\eta)\big),$$

or

$$\left[\begin{pmatrix} v \\ x \end{pmatrix}, \begin{pmatrix} v' \\ x' \end{pmatrix} \right] = \begin{pmatrix} 0 \\ \omega(v, v') \end{pmatrix}.$$

This implies $\mathfrak{h}_N\mathbb{R}$ is a two-step nilpotent Lie algebra, since $[\mathfrak{v}, \mathfrak{v}] = \mathfrak{z}$. Call $\mathfrak{h}_N\mathbb{R}$ the *Heisenberg algebra* of dimension N. The simply connected nilpotent Lie group corresponding to $\mathfrak{h}_N\mathbb{R}$ is $H_N\mathbb{R}$, the *Heisenberg group* of dimension N.

The standard basis of $\mathbb{R}^N$ gives a vector space isomorphism $\mathfrak{h}_N\mathbb{R} \cong \mathbb{R}^N$, and we take this basis to be ordered as follows:

$$\mathcal{E}_N = \{\underbrace{E_1, \dots, E_n}_{E_i}, \underbrace{E_{n+1}, \dots, E_{2n}}_{E_{n+i}}, E_N\}. \tag{3.1}$$

With respect to this basis, we may describe the additional structures on $\mathfrak{v}$ and $\mathfrak{h}_N\mathbb{R}$ as the standard ones. First,

$$\iota = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \quad \omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad \langle \cdot, \cdot \rangle = I_N.$$

Additionally, the Lie bracket on $\mathfrak{h}_N\mathbb{R}$ has the following relations:

$$[E_i, E_j] = [E_i, E_N] = [E_{n+i}, E_{n+j}] = [E_{n+i}, E_N] = 0, \qquad [E_i, E_{n+j}] = \delta_{ij} E_N,$$

where $1 \le i, j \le n$. Thus, the only non-vanishing stucture constants are of the form

$$c^N_{i,n+i} = 1. \tag{3.2}$$

Let $\mathfrak{w}$ have an inner product (written $\cdot$) that induces an inner product on $\mathfrak{v} = \mathfrak{w} \oplus \mathfrak{w}$. Furthermore, suppose that $\mathfrak{h}_N\mathbb{R}$ has an inner product $\langle \cdot, \cdot \rangle$ whose restriction to $\mathfrak{v}$ is the one just described, and for which $\mathfrak{v} \perp \mathfrak{z}$.

4. Automorphisms

The condition for a map $\tau \in \mathrm{End}(\mathfrak{h}_N\mathbb{R})$ to be an automorphism is

$$\tau[\xi, \eta] = [\tau\xi, \tau\eta] \tag{4.1}$$

for all $\xi, \eta \in \mathfrak{h}_N\mathbb{R}$. *A priori*, we can write $\tau \in \mathrm{Aut}(\mathfrak{h}_N\mathbb{R})$ as

$$\tau = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \text{where} \quad \begin{matrix} \alpha : \mathfrak{v} \longrightarrow \mathfrak{v}, & \beta : \mathfrak{z} \longrightarrow \mathfrak{v}, \\ \gamma : \mathfrak{v} \longrightarrow \mathfrak{z}, & \delta : \mathfrak{z} \longrightarrow \mathfrak{z}. \end{matrix}$$

Since τ satisfies (4.1), we know that τ must preserve $\mathfrak{z}$. Therefore $\beta = 0$, and

$$\tau\xi = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} v \\ x \end{pmatrix} = \begin{pmatrix} \alpha(v) \\ \gamma(v) + \delta(x) \end{pmatrix}.$$

We must also have $\alpha \in \mathrm{GL}(\mathfrak{v})$ and $\delta \in \mathrm{GL}(\mathfrak{z})$, since $\det(\tau) = \det(\alpha)\det(\delta) \neq 0$. We can say much more about elements of $\mathrm{Aut}(\mathfrak{h}_N\mathbb{R})$, however.

Theorem 4.2. *Each element of* $\mathrm{Aut}(\mathfrak{h}_N\mathbb{R})$ *can be written uniquely as*

$$\tau = \tau_1\tau_2\tau_3\tau_4,$$

where the τ_i *are of the following form:*

$$\tau_1 \in \{g_S \mid S \in \mathrm{Sp}(\mathfrak{v}, \omega)\}, \qquad \textit{where } g_S = \begin{pmatrix} S & 0 \\ 0 & \mathrm{id}_{\mathfrak{z}} \end{pmatrix}$$

$$\tau_2 \in \{g_\gamma \mid \gamma \in \mathrm{Hom}(\mathfrak{v}, \mathfrak{z})\}, \qquad \textit{where } g_\gamma = \begin{pmatrix} \mathrm{id}_{\mathfrak{v}} & 0 \\ \gamma & \mathrm{id}_{\mathfrak{z}} \end{pmatrix},$$

$$\tau_3 \in \{g_r \mid r > 0\}, \qquad \textit{where } g_r = \begin{pmatrix} r\,\mathrm{id}_{\mathfrak{v}} & 0 \\ 0 & r^2\,\mathrm{id}_{\mathfrak{z}} \end{pmatrix},$$

$$\tau_4 \in \{\mathrm{id}, g_\iota\}, \qquad \textit{where } g_\iota = \begin{pmatrix} \iota & 0 \\ 0 & -\mathrm{id}_{\mathfrak{z}} \end{pmatrix}.$$

The proof is a combination of [4, Theorem 1.22] and the fact that $\mathrm{Aut}(H_N\mathbb{R}) = \mathrm{Aut}(\mathfrak{h}_N\mathbb{R})$. Next we consider the compact subgroup of orthogonal automorphisms.

Corollary 4.3. *Each* $\tau \in \mathrm{Aut}(\mathfrak{h}_N\mathbb{R}) \cap \mathrm{O}(N)$ *can be written uniquely as*

$$\tau = g_U g,$$

where $U \in \mathrm{U}(n)$ *and* $g \in \{\mathrm{id}, g_\iota\}$.

Proof. We take automorphisms of each type and see what happens when they are assumed to be orthogonal. First, suppose that for some $\tau \in \mathrm{Aut}(\mathfrak{h}_N\mathbb{R})$, we have

$$\langle \tau\xi, \tau\eta \rangle = \langle \xi, \eta \rangle \tag{4.4}$$

for all $\xi, \eta \in \mathfrak{h}_N\mathbb{R}$.

For $\tau = g_S$ with $S \in \mathrm{Sp}(\mathfrak{v}, \omega)$, (4.4) implies that

$$\langle Sv, Sv' \rangle_{\mathfrak{v}} = \langle v, v' \rangle_{\mathfrak{v}},$$

which means $S \in \mathrm{O}(\mathfrak{v}) \cap \mathrm{Sp}(\mathfrak{v}, \omega) = \mathrm{U}(n)$.

For $\tau = g_\gamma$ with $\gamma \in \mathrm{Hom}(\mathfrak{v}, \mathfrak{z})$, (4.4) implies that

$$\langle \gamma(v), \gamma(v')\rangle_{\mathfrak{z}} + \langle \gamma(v), z'\rangle_{\mathfrak{z}} + \langle z, \gamma(v')\rangle_{\mathfrak{z}} = 0.$$

Taking $z = 0$ gives $\gamma = 0$. Hence, the only orthogonal automorphism of the form g_γ is $g_0 = \mathrm{id}_{\mathfrak{h}_N\mathbb{R}}$.

For $\tau = g_r$ with $r > 0$, (4.4) implies that

$$(r^2 - 1)\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \right\rangle_{\mathfrak{v}} + (r^4 - 1)\langle z, z'\rangle_{\mathfrak{z}} = 0.$$

If we take $x = x' \neq 0$, $y = y' = 0$, $z = z' \neq 0$, then we have a polynomial equation in r:

$$|z|^2 r^4 + |x|^2 r^2 - |x|^2 - |z|^2 = 0.$$

For (4.4) to hold, the polynomial equation must hold for all possible $|x|^2, |z|^2 \in \mathbb{R}^+$, which is clearly impossible. Hence, the only orthogonal automorphisms of the form g_r is is $g_1 = \mathrm{id}_{\mathfrak{h}_N\mathbb{R}}$.

Finally, the identity map is clearly orthogonal, and it is simple to check that g_ι is as well. $\square$

Since $g_\iota^2 = \mathrm{id}$, we have $\{\mathrm{id}, g_\iota\} \cong \mathbb{Z}/2$. Identifying $\mathrm{U}(n)$ with the subgroup

$$\{g_U \mid U \in \mathrm{U}(n)\} \subset \mathrm{Aut}(\mathfrak{h}_N\mathbb{R}) \cap \mathrm{O}(N),$$

we can write

$$\mathrm{Aut}(\mathfrak{h}_N\mathbb{R}) \cap \mathrm{O}(N) \cong \mathrm{U}(n) \ltimes \mathbb{Z}/2.$$

5. Derivations

The derivation algebra of $\mathfrak{h}_N\mathbb{R}$ is the Lie algebra of $\mathrm{Aut}(\mathfrak{h}_N\mathbb{R})$. We will need an explicit description of this space, so we think of a derivation $\phi \in \mathrm{Der}(\mathfrak{h}_N\mathbb{R})$ as an $N \times N$ matrix relative to the basis $\mathcal{E}_N$ that satisfies

$$\phi[E_I, E_J] = [\phi E_I, E_J] + [E_I, \phi E_J]. \tag{5.1}$$

If we let $\{E^I\}$ denote the dual basis, then we can write $\phi = \phi^I_J\, E_I \otimes E^J$. The division of $\mathcal{E}_N$ into three parts as in (3.1) gives ϕ the following structure:

$$\phi = \left(\begin{array}{c:c:c} \phi^i_j & \phi^i_{j+n} & \phi^i_N \\ \hdashline \phi^{i+n}_j & \phi^{i+n}_{j+n} & \phi^{i+n}_N \\ \hdashline \phi^N_j & \phi^N_{j+n} & \phi^N_N \end{array}\right).$$

Now (5.1) applied to each pair of basis elements gives us constraints on the matrix components,

$$\begin{aligned} \phi^{i+n}_j &= \phi^{j+n}_i, \\ \phi^j_{i+n} &= \phi^i_{j+n}, \\ \phi^{i+n}_{j+n} &= -\phi^j_i \\ \phi^N_N &= \phi^j_i + \phi^{i+n}_{j+n}, \\ 0 &= \phi^i_N = \phi^{i+j}_N. \end{aligned}$$

From this, we see that

$$\operatorname{Der}(\mathfrak{h}_N\mathbb{R}) \cong \left\{ \left(\begin{array}{c|c|c} A & B & \vec{0} \\ \hline C & cI_n - A^T & \vec{0} \\ \hline \vec{a}^T & \vec{b}^T & c \end{array} \right) \;\middle|\; \begin{array}{c} \vec{a}, \vec{b} \in \mathbb{R}^n, c \in \mathbb{R}, \\ A \in M_n\mathbb{R}, \\ B, C \in \operatorname{sym}(n) \end{array} \right\}.$$

The space has dimension $2n^2 + 3n + 1$.

We consider two subspaces of this algebra. First, the subalgebra of symmetric derivations is

$$\operatorname{Der}(\mathfrak{h}_N\mathbb{R}) \cap \operatorname{sym}(N) \cong \left\{ \left(\begin{array}{c|c|c} A & B & \\ \hline B & cI_n - A & \\ \hline & & c \end{array} \right) \;\middle|\; \begin{array}{c} c \in \mathbb{R}, \\ A, B \in \operatorname{sym}(n) \end{array} \right\},$$

and it has dimension n^2+n+1. Second, the abelian algebra of diagonal derivations is

$$\begin{aligned} \mathfrak{d}(\mathfrak{h}_N\mathbb{R}) &= \left\{ \left(\begin{array}{c|c|c} A & & \\ \hline & cI_n - A & \\ \hline & & c \end{array} \right) \;\middle|\; \begin{array}{c} c \in \mathbb{R}, \\ A \text{ diagonal} \end{array} \right\} \\ &= \{\mathbf{d}(a_1, \ldots, a_n, c - a_1, \ldots, c - a_n, c) \mid a_i, c \in \mathbb{R}\} \\ &\subset \operatorname{Der}(\mathfrak{h}_N\mathbb{R}) \cap \operatorname{sym}(N), \end{aligned}$$

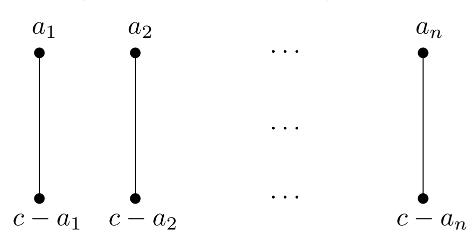
where $\mathbf{d}$ refers to the diagonal matrix with given diagonal entries. This is is a maximal abelian subalgebra of $\operatorname{Der}(\mathfrak{h}_N\mathbb{R})$, and therefore $\operatorname{rank}(\mathfrak{h}_N\mathbb{R}) = n + 1$.

6. The Weyl group

In this section we compute the Weyl group of $G_\mu \subset \operatorname{Aut}(\mathfrak{h}_N\mathbb{R})$, which we can think of as a subgroup of $\operatorname{Aut}(\mathfrak{h}_N\mathbb{R}) \cap \mathrm{O}(N)$. This will prove Theorem 1.3. One way to characterize the Weyl group is by its action as permutations, in this case on the elements of $\mathfrak{d}(\mathfrak{h}_N\mathbb{R})$. In other words, it consists of those orthogonal automorphisms that permute the entries of the diagonal derivations while perserving their overall structure. Consider one such derivation,

$$\mathbf{d}(a_1, \ldots, a_n, c - a_1, \ldots, c - a_n, c) \in \mathfrak{d}(\mathfrak{h}_N\mathbb{R}).$$

To understand *all* possible structure-preserving permutations, we consider the equivalent problem of determining the automorphism group of the following graph $\mathcal{G}$.



We see that $\operatorname{Aut}(\mathcal{G})$ is generated by maps that exchange a_i and $c - a_i$ (i.e., reverse the orientation of an edge), and those that exchange a_i with a_j and $c - a_i$ with $c - a_j$ (i.e., tranpose two connected components). Thus $\operatorname{Aut}(\mathcal{G}) \cong S_n \ltimes (\mathbb{Z}/2)^n$, where S_n acts on $(\mathbb{Z}/2)^n$ by permuting indices. This automorphism group is of course a subgroup of S_{2n}, which can be thought of as permuting the $2n$ vertices of $\mathcal{G}$ (or the first $2n$ diagonal entries of elements of $\mathfrak{d}(\mathfrak{h}_N\mathbb{R})$). Writing transpositions as $\sigma_{i,j}$, we can describe the generators of $\mathrm{Aut}(\mathcal{G})$ described above as

$$\sigma_{i,n+i}, \quad \sigma_{i,j} \cdot \sigma_{n+i,n+j},$$

for $1 \leq i, j, \leq n$.

If we can show that the these generators can all be realized as elements of $\mathrm{U}(n)$, thought of as a subgroup of $\mathrm{Aut}(\mathfrak{h}_N\mathbb{R}) \cap \mathrm{O}(N)$, then this will prove that the Weyl group is indeed isomorphic to $S_n \ltimes (\mathbb{Z}/2)^n$. All of the generators can be realized as $2n \times 2n$ permutation matrices, but we must check that such matrices are actually unitary. First, recall that

$$\mathrm{U}(n) \cong \left\{ \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \;\middle|\; X, Y \in M_n\mathbb{R}, \begin{matrix} X^TX + Y^TY = I \\ X^TY = Y^TX \end{matrix} \right\} \subset M_{2n}\mathbb{R}.$$

Then consider the generator $\sigma_{i,n+i}$ and its corresponding permutation matrix

$$P_{i,n+i} = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}.$$

Here, X is the identity matrix, except with $(i, i) = 0$, and Y is the zero matrix, except with $(i, i) = 1$. It is easy to see that

$$X^TX + Y^TY = I, \quad X^TY = Y^TX,$$

so $P_{i,n+i}$ is unitary.

Next, consider the $n \times n$ permutation matrix $P_{i,j}$ corresponding to $\sigma_{i,j}$. This is the identity matrix, except with $(i, i) = (j, j) = 0$ and $(i, j) = (j, i) = 1$. Then the permutation matrix corresponding to the generator $\sigma_{i,j} \cdot \sigma_{n+i,n+j}$ is

$$U_{i,j} = \begin{pmatrix} P_{i,j} & 0 \\ 0 & P_{i,j} \end{pmatrix}.$$

This is unitary since $P_{i,j}$ is orthogonal.

Hence, Theorem 1.3 follows.

Remark 6.1. Recalling that $\dim \mathrm{Gr}_k \mathbb{R}^{n+1} = k(n+1-k)$, and ignoring the $k = n+1$ case, since it results in a single (Einstein) solvsoliton (see [21, Section 3.2]) we have the following.

For fixed $n \geq 1$ and $N = 2n + 1$, there is a space, depending on

$$\sum_{k=1}^{n} \dim \mathrm{Gr}_k \mathbb{R}^{n+1} = \frac{1}{6}n(n+1)(n+8)$$

parameters, of solvsoliton extensions of $\mathfrak{h}_N\mathbb{R}$ with dimensions $N + k$, for $1 \leq k \leq n$.

Remark 6.2. The Heisenberg algebras described here fall into a larger class of spaces called *generalized Heisenberg algebras* [2, 9, 10]. These all admit nilsolitons (by [13, Example 3.3(i)]), so it would be interesting to determine the corresponding spaces of associated solvsolitons. There is information on the automorphisms and derivations of these algebras, but the picture is significantly more complicated; see, e.g., [19, 20].

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