Applications of Line Bundles Over Riemann Surfaces

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1 Basic Facts about Riemann Surfaces

A **Riemann surface** M is a one-dimensional complex manifold with atlas $\{\phi_\alpha : U_\alpha \to \mathbb{C}\}$ such that each composition

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is invertible and holomorphic.

$$\begin{tikzcd}
 & U_\alpha \cap U_\beta \arrow[dl, "\phi_\alpha"'] \arrow[dr, "\phi_\beta"] & \\
\phi_\alpha(U_\alpha \cap U_\beta) \arrow[rr, "\phi_\beta \circ \phi_\alpha^{-1}"] & & \phi_\beta(U_\alpha \cap U_\beta)
\end{tikzcd}$$

A map $f : M \to N$ is **holomorphic** if it is continuous, and for all charts $\phi_\alpha : U_\alpha \to \mathbb{C}$ and $\psi_\beta : V_\beta \to \mathbb{C}$, the composition $\Phi = \psi_\beta \circ f \circ \phi_\alpha^{-1}$ is holomorphic.

$$\begin{tikzcd}
U_\alpha \subset M \arrow[r, "f"] \arrow[d, "\phi_\alpha"'] & V_\beta \subset N \arrow[d, "\psi_\beta"] \\
\phi_\alpha(U_\alpha) \subset \mathbb{C} \arrow[r, "\psi_\beta \circ f \circ \phi_\alpha^{-1}"'] & \psi_\alpha(V_\alpha) \subset \mathbb{C}
\end{tikzcd}$$

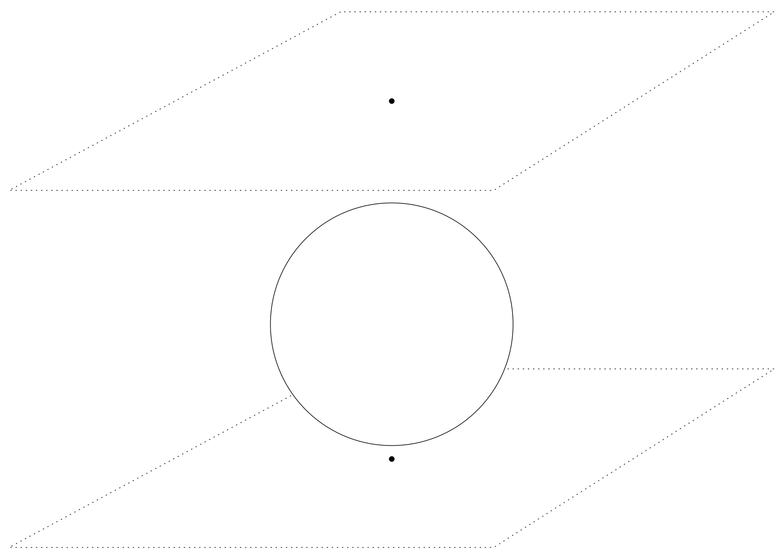

Figure 1: The Riemann Sphere

Theorem 1.1. *A two-dimensional real manifold admits a Riemann surface structure if and only if it is orientable.*

This implies that the sphere and torus (and g-holed tori) admit complex structures, while other manifolds such as the Möbius strip, Klein bottle and projective plane do not. The most important Riemann surfaces are the *compact* ones. Compactness affords us many desirable properties, such as the ability to "trap" and count zeros and poles of functions on structures related to Riemann surfaces in beneficial ways.

Theorem 1.2. *If M is connected and compact, then any holomorphic map $f : M \to \mathbb{C}$ is constant.*

This is basically a consequence of the Maximum Modulus Principle.

Example 1.3 (The Riemann Sphere). This is our most important example of a compact Riemann surface. We can put a complex structure on $S^2 \subset \mathbb{R}^3$ with the atlas $\{U_0, U_1\}$, where

$$\begin{array}{cc} U_0 = \mathbb{C}, & U_1 = (\mathbb{C} \setminus \{0\}) \cup \{\infty\} \\ \phi_0 : U_0 \to \mathbb{C}, & \phi_1 : U_1 \to \mathbb{C} \\ z \mapsto z, & z \mapsto \frac{1}{z} \end{array}$$

We can also consider the **Projective Line** $\mathbb{C}P^1 = (\mathbb{C}^2 \setminus \{0\})/\sim$, where $Z \sim \lambda Z$ for all $\lambda \in \mathbb{C}\setminus\{0\}$. This is the set of all one-dimensional subspaces of $\mathbb{C}^2$. We have homogeneous coordinates on $\mathbb{C}P^1$ given by $Z = (Z_0, Z_1)$, where Z_0 and Z_1 are not both zero, and Z is identified with $\lambda Z = (\lambda Z_0, \lambda Z_1)$. We can define a complex structure on $\mathbb{C}P^1$ with the following atlas:

$$\begin{array}{cc} V_0 = \{(Z_0, Z_1) \colon Z_0 \neq 0\} & V_1 = \{(Z_0, Z_1) \colon Z_1 \neq 0\} \\ \psi_0 : V_0 \to \mathbb{C}, & \psi_1 : V_1 \to \mathbb{C} \\ (Z_0, Z_1) \mapsto \frac{Z_1}{Z_0} = z, & (Z_0, Z_1) \mapsto \frac{Z_0}{Z_1} \end{array}$$

Note that $\psi_1 \circ \psi_0^{-1}(z) = 1/z$ for $z \in V_0 \cap V_1$.

Thus, $S^2 \cong \mathbb{C}P^1$ as Riemann Surfaces.

Remark. With this example in mind, we can consider a meromorphic function $f : M \to \mathbb{C}$ to be a holomorphic map $f : M \to \mathbb{C}P^1$, where all poles are mapped to ∞.

2 Holomorphic Line Bundles

A **rank k vector bundle** over M is a complex manifold E with holomorphic projection $\pi : E \to M$ such that

(i) $\pi^{-1}(m) \cong \mathbb{C}^k$ as vector spaces, for all $m \in M$;

(ii) Each $z \in M$ has a neighborhood U and homeomorphism ϕ_U such that the following commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi_U"] \arrow[dr, "\pi"'] & & U \times \mathbb{C}^k \arrow[dl, "\mathrm{proj}_1"] \\ & U & \end{tikzcd}$$

(iii) For neighborhoods U and V, $\phi_V \circ \phi_U^{-1}$ is of the form $(m, w) \mapsto (m, A(m)w)$ for some holomorphic $A : U \cap V \to GL(k, \mathbb{C})$.

$$\begin{tikzcd} (U \cap V) \times \mathbb{C}^k \arrow[dr, "\mathrm{proj}_1"'] & \pi^{-1}(U \cap V) \arrow[l, "\phi_V"'] \arrow[r, "\phi_U"] \arrow[d, "\pi"] & (U \cap V) \times \mathbb{C}^k \arrow[dl, "\mathrm{proj}_1"] \\ & U \cap V & \end{tikzcd}$$

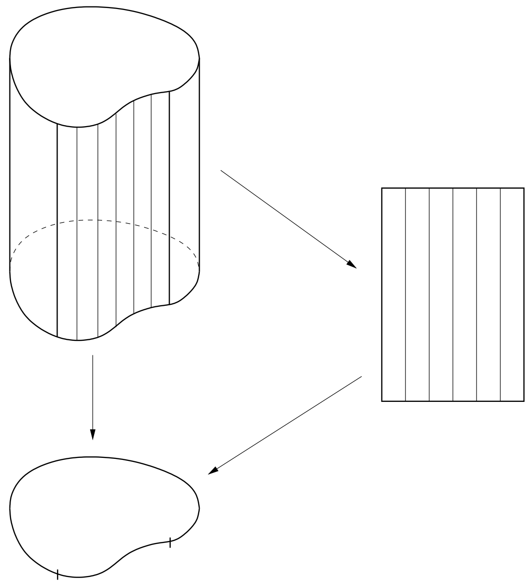

Figure 2: A Fiber Bundle

We write $A = g_{VU}$, and call this a **transition function** of E, and we call ϕ_U a **local trivialization**.

The transition functions of a vector bundle also satisfy the following **co-cycle conditions:** for $U, V, W \subset M$ with nonempty intersections,

(i) $g_{UV} \cdot g_{VU} = 1$;

(ii) $g_{UV} \cdot g_{VW} \cdot g_{WU} = 1$.

Note that if we replace $\mathbb{C}^k$ by a general topological space F (and ignore condition (iii)), then we have a fiber bundle. Typical examples of bundles are the Möbius Strip over the unit circle, and Tangent Bundle of S^2.

Of more interest to us is the special case where $k = 1$. In this case, we call $E = L$ a **line bundle**. Note that in this case, $g_{UV} : U \cap V \to \mathbb{C}^*$. We can think of a line bundle as a collection of copies of $\mathbb{C}$, parametrized by the elements of M, glued together in a "nice" way.

A **holomorphic section** of a line bundle L over M is a holomorphic map $s : M \to L$ such that $\pi \circ s = Id_M$. Since $\pi^{-1}(m) \cong \mathbb{C}$ for all $m \in M$, we can add sections pointwise and multiply them by scalars:

$$(s+t)(m) = s(m) + t(m), \qquad (\lambda s)(m) = \lambda s(m).$$

Let $H^0(M, L)$ denote the vector space of holomorphic sections of L.

Like transition functions, sections can be represented locally, and pieced together to give global functions. For neighborhoods $U, V \subset M$, we have $s_U = g_{UV} s_V$ on $U \cap V$.

Example 2.1. Any Riemann surface M has **trivial** bundle $M \times \mathbb{C}$.

Example 2.2. Any Riemann surface M has **canonical** bundle K, which is the cotangent bundle, or the bundle of holomorphic 1-forms. We also define the **genus** of M as $\dim H^0(M, K)$.

A section of K over $\mathbb{C}P^1$ is given by $f_0(z)dz$ on U_0 and $f_1(z^{-1})d(z^{-1})$ on U_1, and these agree on $U_0 \cap U_1$. By the chain rule, we have $d(z^{-1}) = -z^{-2}dz$. The above sections agree, so

$$f_0(z)dz = -z^{-2} f_1(z^{-1})dz.$$

Expand power series to see that $f_0 = f_1 = 0$, which implies there are no nonzero global sections of K. Note that $g_{01} = z^{-2}$.

Example 2.3 (Operations on Bundles). Given an open cover $\{U_\alpha\}$ of M and maps $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to U_\alpha \cap U_\beta\}$ satisfying the cocycle conditions, we have a unique line bundle with transition functions $\{g_{\alpha\beta}\}$. Thus, we can form many new line bundles from existing ones. Let $L, \tilde{L}$ be line bundles.

- The dual bundle of L is L^{-1}, and the transition functions are $g_{\alpha\beta}^{-1}(L)$.
- The determinant bundle of a vector bundle E is denoted $\det E$, and the transition functions are $\det g_{\alpha\beta}$.
- The tensor product is denoted $L\tilde{L}$, and the transition functions are $g_{\alpha\beta}(L\tilde{L}) = g_{\alpha\beta}(L) g_{\alpha\beta}(\tilde{L})$. The bundle $L^{-1}L$ is trivial, which explains the notation L^{-1}.

Insinuated by this multiplicative notation is the interesting and powerful fact that the collection of classes of line bundles over M forms an abelian group (with operation $\otimes$), denoted $H^1(M, \mathcal{O}^*)$.

Example 2.4 (Bundle Associated with a Point). Let $p \in M$ have neighborhood U and coordinate chart z such that $z(p) = 0$. Let $V = M \setminus \{p\}$. Then z is holomorphic and nonvanishing on $U \cap V$, so we can use it as a transition function to define a bundle L_p on M. We glue together $U \times \mathbb{C}$ and $V \times \mathbb{C}$ over $U \cap V$ with the function $\varphi(m, w) = (m, z(m)w)$.

The bundle L_p has a canonical section s_p. Take the functions z on U and 1 on V, so $z = z \cdot 1 = g_{UV} \cdot 1$. Then s_p has a simple zero only at p.

Note that the line bundles L_p play the same role as divisors associated with points.

This example is very important, as it not only gives us a multitude of line bundles over a given Riemann surface, but it also provides the building blocks for other bundles, as we will see that any line bundle can be decomposed into a product of finitely many such line bundles. In higher dimensions, line bundles are not as plentiful, which makes (compact) Riemann surfaces all the more useful.

Example 2.5. Consider $\mathbb{C}P^1$. For $n \in \mathbb{Z}$, the transition function $g_{01} = z^n$ defines a line bundle $\mathcal{O}(n)$. Let s be a section represented locally by s_0 and s_1, so they are related by $s_0(z) = z^n s_1(1/z)$. Examining the Taylor series expansions, we see that all sections of this bundle are polynomials of degree less than or equal to n. Therefore $\dim H^0(\mathbb{C}P^1, \mathcal{O}(n)) = n + 1$.

Example 2.6. In Example 2.2, we saw that $g_{01} = z^{-2}$, so $K \cong \mathcal{O}(-2)$, and of course $H^0(\mathbb{C}p^1, K) = 0$.

Example 2.7. If L is a bundle over M, $U \subset M$, $p \in M$, then $\mathcal{O}_p(L)$ represents the sections of L over $U \cap \{p\}$. The space of global sections is $\pi^{-1}(p) \cong \mathbb{C}$.

Example 2.8 (Measures on Manifolds). If M is a smooth n-dimensional real manifold, let $x = (x_1, \ldots, x_n)$ be local coordinates in some neighboordhood $U \subset M$. Consider the Lebesque measure $dx = dx_1 \cdots dx_n$ on U. We can change coordinates to $y = (y_1, \ldots, y_n)$ via

$$dy = \left| \det \left(\frac{\partial y}{\partial x} \right) \right| dx,$$

where $(\partial y / \partial x)$ is the matrix of partial derivatives, and $dy/dx = |\det(\partial y/\partial x)|$ is the Radon-Nikodym derivative. A **smooth measure** on M is a Borel measure μ which has the form

$$d\mu = \phi^x dx,$$

where x is the local coordinate chart and ϕ^x is a smooth, nonnegative function. We can change coordinates via

$$\phi^x = \left|\det\left(\frac{\partial y}{\partial x}\right)\right| \phi^y.$$

The functions $|\det(\partial y/\partial x)|$ are the transition functions for a line bundle over M. A section of this line bundle is called a **density** on M, and is represented locally by the functions ϕ^x. Thus, there is a correspondence between smooth measures and nonnegative densities.

3 Sheaves and Cohomology

We introduce sheaves associated with surfaces or bundles, usually as collections of functions or sections on opens sets. We also define sheaf cohomology, which allows us to translate certain analytic problems into linear algebraic problems, and a result we prove is the Riemann-Roch Theorem. Historically, one previous approach was to consider harmonic forms and divisors on M. While line bundles and divisors are completely analogous, our approach fits into a much more general framework, which, after decades of development, makes our work appear smoother.

A **sheaf** $\mathcal{S}$ on M associates to each open $U \subset M$ an abelian group $\mathcal{S}(U)$, and to $U \subset V$ a restriction map $r_{VU} : \mathcal{S}(V) \to \mathcal{S}(U)$ such that

(i) If $U \subset V \subset W$, then $r_{WU} = r_{VU} \circ r_{WV}$;

(ii) If $\sigma \in \mathcal{S}(U), \tau \in \mathcal{S}(V)$, and $r_{UU\cap V}(\sigma) = r_{VU\cap V}(\tau)$, then there exists $\rho \in \mathcal{S}(U \cup V)$ such that $r_{U\cup VU}(\rho) = \sigma$ and $r_{U\cup VV}(\rho) = \tau$;

(iii) If $\sigma \in \mathcal{S}(U \cap V)$ is such that $r_{U\cup VU}(\sigma) = 0 = r_{U\cup VV}(\sigma)$, then $\sigma = 0$.

Remark. A common approach, which is actually simpler, is to consider **presheaves** over X. A presheaf only requires that condition (i) hold, which is really the most important of the three.

In categorical terms, a sheaf is a contravariant functor $\mathcal{T}op(X) \to \mathcal{A}b$, where $\mathcal{T}op(X)$ is the category whose objects are open sets in X, and whose morphisms are inclusion maps.

Example 3.1. We are mainly concerned with sheaves of functions or sections. If $U \subset M$ and L is a line bundle, then some sheaves are

- holomorphic functions $\mathcal{O}(U)$ (or continuous, smooth, etc.);
- locally constant functions with values in $\mathbb{Z}$ or $\mathbb{C}$;
- nonzero holomorphic functions $\mathcal{O}^*(U)$;
- holomorphic sections $\mathcal{O}(L)(U)$;

We will often suppress reference to the open set or line bundle, and simply write $\mathcal{O}, \mathcal{O}^*, \mathbb{Z}$, etc.

To define the sheaf cohomology over M, let M have open cover $\{U_\alpha\}$. Let

$$C^0 = \bigoplus_{\alpha} S(U_\alpha),$$

$$C^1 = \bigoplus_{\alpha<\beta} S(U_\alpha \cap U_\beta),$$

$$C^2 = \bigoplus_{\alpha<\beta<\gamma} S(U_\alpha \cap U_\beta \cap U_\gamma).$$

We define the homomorphism $\delta_0 : C^0 \to C^1$ by

$$(\delta_0 f)_{\alpha\beta} = f_\beta\Big|_{U_\alpha \cap U_\beta} - f_\alpha\Big|_{U_\alpha \cap U_\beta},$$

and the homomorphism $\delta_1 : C^1 \to C^2$ by

$$(\delta_1 f)_{\alpha\beta\gamma} = f_{\beta\gamma}\Big|_{U_\alpha \cap U_\beta \cap U_\gamma} - f_{\alpha\gamma}\Big|_{U_\alpha \cap U_\beta \cap U_\gamma} + f_{\alpha\beta}\Big|_{U_\alpha \cap U_\beta \cap U_\gamma}.$$

We then define the **p-th cohomology group of $\mathcal{S}$** as

$$H^p(M, \mathcal{S}) = \frac{\ker \delta^p}{\operatorname{im} \delta^{p-1}}.$$

Remark. This definition depends on the cover of M, but this is resolved by choosing a "good" cover, where cohomology of all intersections vanishes for $p > 0$. Cohomology is "good" cover-independent.

Example 3.2. The space of holomorphic sections of a line bundle is in fact isomorphic to the cohomology group $H^0(M, \mathcal{O}(L))$. The group of line bundle classes over M is in fact isomorphic to the cohomology group $H^1(M, \mathcal{O}^*)$. These facts follow from the cocycle conditions above.

We could have defined the cohomology groups for all $p \geq 0$, but it turns out that only $p = 0, 1, 2$ are necessary on a Riemann surface:

Theorem 3.3. *If $\mathcal{S} = \mathcal{O}(L)$ is the sheaf of holomorphic sections of L over $U \subset M$, then $H^p(M, \mathcal{S}) = 0$ for $p > 1$. If $\mathcal{S} = \mathbb{C}$ or $\mathbb{Z}$, then $H^p(M, \mathcal{S}) = 0$ for $p > 2$.*

Theorem 3.4 (Serre Duality). *If L is a line bundle and M is compact, then*

$$H^1(M, L) \cong H^0(M, KL^{-1})^*.$$

Recall from algebra that given a sequence of homomorphisms between vector spaces (or groups, rings, modules, etc.)

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0,$$

we say this is a **short exact sequence** if $\operatorname{im} f = \ker g$. A few easy consequences of exactness are

- if the above sequence is short exact, then $fg = 0$;
- $0 \to A \xrightarrow{f} B$ exact means f is injective;
- $B \xrightarrow{g} C \to 0$ exact means g is surjective;
- $0 \to A \xrightarrow{f} B \to 0$ exact means f is an isomorphism;
- $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ exact means $C \cong B/A$.

We can also define **long exact** sequences $\cdots \to A^i \xrightarrow{f_i} A^{i+i} \xrightarrow{f_{i+1}} \cdots$, where we require exactness at each term: $\operatorname{im} f_i = \ker f_{i+1}$.

Lemma 3.5. *Given a long exact sequence of vector spaces*

$$0 \longrightarrow A^0 \xrightarrow{d_0} A^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{k-1}} A^k \longrightarrow 0,$$

the alternating sum of dimensions is 0, that is, $\displaystyle\sum_{i=0}^{k} (-1)^i \dim A^i = 0$.

A basic theorem from general homological algebra holds for sheaf cohomology. The proof is standard diagram chasing.

Theorem 3.6. *If* $0 \to \mathcal{S} \to \mathcal{T} \to \mathcal{U} \to 0$ *is a short exact sequence of sheaves, then there is a long exact sequence of cohomology groups*

$$\begin{aligned} 0 \longrightarrow H^0(M,\mathcal{S}) \longrightarrow H^0(M,\mathcal{T}) \longrightarrow H^0(M,\mathcal{U}) \xrightarrow{\delta_0} \\ \longrightarrow H^1(M,\mathcal{S}) \longrightarrow H^1(M,\mathcal{T}) \longrightarrow H^1(M,\mathcal{U}) \xrightarrow{\delta_1} \\ \longrightarrow H^2(M,\mathcal{S}) \longrightarrow H^2(M,\mathcal{T}) \longrightarrow H^2(M,\mathcal{U}) \xrightarrow{\delta_2} \cdots \end{aligned}$$

We also have a powerful result limiting the size of the cohomology spaces.

Theorem 3.7. *If* M *is compact, then* $H^p(M, L)$ *is finite dimensional for all* p*.*

4 The Riemann-Roch Theorem

Let M be a compact Riemann surface of genus g, and consider the sheaves of integer-valued functions, holomorphic functions, and nonzero holomorphic functions as in Example 3.1. We have a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp(2\pi i f)} \mathcal{O}^* \longrightarrow 1.$$

Theorem 3.6 gives us a long exact sequence of cohomology spaces. We use abbreviated notation:

$$\begin{array}{cccccccc} 0 & \longrightarrow & H^0(\mathbb{Z}) & \longrightarrow & H^0(\mathcal{O}) & \longrightarrow & H^0(\mathcal{O}^*) & \longrightarrow \\ & \longrightarrow & H^1(\mathbb{Z}) & \longrightarrow & H^1(\mathcal{O}) & \longrightarrow & H^1(\mathcal{O}^*) & \longrightarrow \\ & \longrightarrow & H^2(\mathbb{Z}) & \longrightarrow & H^2(\mathcal{O}) & \longrightarrow & H^2(\mathcal{O}^*) & \longrightarrow \quad \cdots. \end{array}$$

Recall that by Theorem 1.2, the only holomorphic functions on a compact surface are constants, which simplifies the first row. By Theorem 3.3, only the first term in the bottom row is nonzero, and if fact it is $\mathbb{Z}$. As for the second row, we have $H^1(\mathbb{Z}) \cong \mathbb{Z}^{2g}$ and $H^1(\mathcal{O}) \cong \mathbb{C}^g$. This gives

$$\begin{array}{cccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^* & \xrightarrow{g} \\ & \xrightarrow{g} & \mathbb{Z}^{2g} & \xrightarrow{h} & \mathbb{C}^g & \longrightarrow & H^1(\mathcal{O}^*) & \xrightarrow{\delta} \\ & \xrightarrow{\delta} & \mathbb{Z} & \longrightarrow & 0. & & & \end{array}$$

Now, the exponential map $\mathbb{C} \xrightarrow{\exp} \mathbb{C}^*$ is surjective, so by exactness, $\operatorname{im}\exp = \mathbb{C}^* = \ker g$. Exactness also implies that $0 = \operatorname{im} g = \ker h$, so the map $\mathbb{Z}^{2g} \xrightarrow{h} \mathbb{C}^g$ is injective. We can rewrite the second row:

$$0 \longrightarrow \frac{\mathbb{C}^g}{\mathbb{Z}^{2g}} \longrightarrow H^1(M, \mathcal{O}^*) \xrightarrow{\delta} \mathbb{Z} \to 0. \tag{1}$$

From Example 3.2, we know that $H^1(M, \mathcal{O}^*)$ is the group of classes of line bundles on M. Thus, each line bundles has an integer invariant given by the map δ.

We define the **degree** of L as $\deg L = \delta([L]) \in \mathbb{Z}$. If E is a vector bundle, then we define $\deg E = \deg\det(E)$. Some facts about $\deg L$:

(i) Since δ is a homomorphism, $\deg(L\tilde{L}) = \deg L + \deg \tilde{L}$;

(ii) $\deg L_p = 1$;

(iii) If $\deg L < 0$, then $H^0(M, L) = 0$, i.e., L has no nontrivial sections.

This definition is analogous to the definition of degree for divisors.

Also, the first term of (1) above, $\mathbb{C}^g/\mathbb{Z}^{2g}$, is a complex torus of dimension g, called the **Jacobian** J^g of M. This means that the line bundles of fixed degree are a torus. As a group, the space $H^1(M, \mathcal{O}^*)$ is called the **Picard Group**.

Example 4.1. The genus of $\mathbb{C}P^1$ is 0, so in (1) above, $\mathbb{C}^g/\mathbb{Z}^{2g} = 0$, so

$$0 \longrightarrow H^1(\mathcal{O}^*) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0$$

is an isomorphism. This means the line bundles $\mathcal{O}(n)$ we saw in Example 2.5 are in fact the only line bundles over $\mathbb{C}P^1$.

Theorem 4.2 (Riemann-Roch). *If E is a vector bundle on a compact Riemann surface M of genus g, then*

$$\dim H^0(E) - \dim H^1(E) = \deg E + (1-g)\operatorname{rank} E. \tag{2}$$

Proof. The proof is by induction on the rank of E. We will prove the base case, where $E = L$ is a line bundle. The general case involves a similar argument.

First, suppose L is trivial. Then let $\mathcal{O} = \mathcal{O}(L)$ be the holomorphic sections of L. Considering the left side of (2), we have

$$\dim H^0(\mathcal{O}) - \dim H^1(\mathcal{O}) = \dim \mathbb{C} - \dim \mathbb{C}^g = 1 - g.$$

On the other hand, since L is trivial, we have

$$\deg \mathcal{O} + (1-g)\operatorname{rank} \mathcal{O} = 0 + (1-g) \cdot 1 = 1 - g,$$

so (2) holds.

Now we show that if (2) holds for some L, then it holds for LL_p and LL_p^{-1}. Recall Example 2.7 and consider the short exact sequence

$$0 \longrightarrow \mathcal{O}(L) \xrightarrow{s_p} \mathcal{O}(LL_p) \longrightarrow \mathcal{O}_p(LL_p) \longrightarrow 0,$$

and the corresponding long exact sequence induced by cohomology,

$$0 \longrightarrow H^0(L) \longrightarrow H^0(LL_p) \longrightarrow \mathbb{C} \longrightarrow H^1(L) \longrightarrow H^1(LL_p) \longrightarrow 0.$$

Using Lemma 3.5, we obtain

$$\begin{aligned} \dim H^0(LL_p) - \dim H^1(LL_p) &= \dim H^0(L) - \dim H^1(L) + 1 \\ &= \deg L + (1-g)\operatorname{rank} L + 1 \qquad \text{(by assumption)} \\ &= \deg LL_p + (1-g). \end{aligned}$$

In the last line we used the fact that

$$\deg LL_p = \deg(L \otimes L_p) = \deg L + \deg L_p = \deg L + 1.$$

Next, we claim that every line bundle is isomorphic to some product $L_{p_1} \cdots L_{p_m} L_{q_1}^{-1} \cdots L_{q_n}^{-1}$.

Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}(L) \xrightarrow{s_p^n} \mathcal{O}(LL_p^n) \longrightarrow \mathcal{S} \longrightarrow 0,$$

where $\mathcal{S}$ is the quotient sheaf described locally as the quotient space of $f(z) \mapsto z^n f(z)$ with coordinate $z(p) = 0$, with s_p as in Example 2.4 above.

If we write $f(z) = \sum a_k z^k$, then $z^n f(z) = \sum a_k z^{k+n} = a_0 z^n + \cdots$. Thus, the elements in quotient will have power series with $a_k = 0$ for $k \geq n$, so the cohomology is n-dimensional. We form the long exact sequence of cohomology

$$0 \longrightarrow H^0(L) \longrightarrow H^0(LL_p^n) \longrightarrow \mathbb{C}^n \longrightarrow H^1(L) \longrightarrow H^1(LL_p^n) \longrightarrow 0,$$

and again we apply Lemma 3.5:

$$\begin{aligned} \dim H^0(LL_p^n) &= n + \dim H^1(LL_p^n) + \dim H^0(L) - \dim H^1(L) \\ &\geq n + \dim H^0(L) - \dim H^1(L). \end{aligned}$$

If we choose sufficiently large n, the right side becomes positive, which implies the existence of sections of LL_p^n.

Now, let $s \in \mathcal{O}(LL_p)$ be a section vanishing at the points $p_1, \ldots, p_k$ with multiplicities $m_1, \ldots, m_k$. Notice that $ss_{p_1}^{-m_1} \cdots s_{p_k}^{-m_k}$ is a nonvanishing section of LL_p, since we have "multiplied out" all the zeros. Note that a nonvanishing section trivializes a line bundle, since $0 \neq a \mapsto 1 \in \mathbb{C}$ gives an isomorphism $L \cong \mathbb{C}$. Thus, we have a trivialization, and using the group structure of $\mathcal{L} = H^1(\mathcal{O}^*)$ as in Example 2.3, we can write

$$1_{\mathcal{L}} \cong LL_p^n L_{p_1}^{-m_1} \cdots L_{p_k}^{-m_k},$$

and so, again using the group structure, we have

$$L \cong L_{p_1}^{m_1} \cdots L_{p_k}^{m_k} L_p^{-n}.$$

Thus, (2) holds for all line bundles. □

Example 4.3. Some consequences of Riemann-Roch:

- The symmetries of $\mathbb{C}P^1$ are $SL(2, \mathbb{C})/\mathbb{Z}_2 \cong PSL(2, \mathbb{C})$.
- The Riemann Sphere $\mathbb{C}P^1$ has a unique complex structure.

Example 4.4 (Integrable systems). Consider a rigid body with a fixed point in a constant gravitational field, i.e., a spinning top. Its behavior is described by the equations

$$
\begin{aligned}
I_1\dot{\Omega}_1 &= (I_2 - I_3)\Omega_2\Omega_3 \\
I_2\dot{\Omega}_2 &= (I_3 - I_1)\Omega_3\Omega_1 \\
I_3\dot{\Omega}_3 &= (I_1 - I_2)\Omega_1\Omega_2,
\end{aligned}
$$

where $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ is angular velocity and I_j are the principal moments of inertia. This is a system of nonlinear differential equations, and as such, one does not expect an explicit solution. However, this is an integrable system, and we will briefly describe how we can consider linear motion on the Jacobian of a Riemann surface to obtain explicit solutions. Consider the so-called **Lax Pair**

$$
\frac{dA}{dt} = [A, B],
$$

where

$$
\begin{aligned}
A(\boldsymbol{z}) &= A_0 + \boldsymbol{z}A_1 + \cdots + \boldsymbol{z}^n A_n, \\
B(\boldsymbol{z}) &= B_0 + \boldsymbol{z}B_1 + \cdots + \boldsymbol{z}^m B_m
\end{aligned}
$$

are $k \times k$ matrix-valued polynomials. The characteristic equation $\det(\boldsymbol{y} - A(\boldsymbol{z})) = 0$ defines an algebraic curve called the **spectral curve**, which is preserved by flow. Each point $(\boldsymbol{y}, \boldsymbol{z})$ on the spectral curve gives a 1-dimensional space

$$
L_{(\boldsymbol{y},\boldsymbol{z})} = \ker(\boldsymbol{y} - A(\boldsymbol{z}))
$$

that varies with time, and this forms a line bundle over the curve. But, recall that the space of line bundles of fixed degree is a g-torus,

$$
\frac{H^1(\mathcal{O})}{H^1(\mathbb{Z})} \cong J^g.
$$

The time evolution of the system is time evolution of line bundles, a straight line on the torus J^g. Thus, we have linearized a nonlinear problem.

A matrix $A(\boldsymbol{z})$ acts on $H^0(M, L) = \mathbb{C}^m$, and evaluating sections of L at a point gives a surjective vector bundle homomorphism $M \times \mathbb{C}^m \to L$. Looking at the duals we see that $L \subset M \times (\mathbb{C}^m)^*$ is the eigenspace of $A^t(\boldsymbol{z})$. Therefore,

- The spectrum of $A(\boldsymbol{z})$ is a Riemann surface M.
- The eigenspace of $A^t(\boldsymbol{z})$ is a line bundle on M.

5 References

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