Analysis of geometric flows, with applications to optimal homogeneous geometries

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Abstract. This dissertation considers several problems related to Ricci flow, including the existence and behavior of solutions. First, we consider Ricci flow on two classes of nilpotent Lie groups that generalize the three-dimensional Heisenberg group: the higher-dimensional classical Heisenberg groups, and the groups of real unitriangular matrices. Each group is known to admit a Ricci soliton, but we construct them explicitly on each group. In the former case, this is done using Lott's blowdown method, whereby we demonstrate convergence of arbitrary diagonal metrics to the solitons. In the latter case, which is more complicated, we obtain the solitons using a suitable ansatz.

Next, we consider solsoliton involving the nilsolitons in the Heisenberg case above. This uses work of Lauret, which characterizes solsolitons as certain extensions of nilsolitons, and work of Will, which demonstrates that the space of solsolitons extensions of a given nilsoliton is parametrized by the quotient of a Grassmannian by a finite group. We determine these spaces of solsoliton extensions of Heisenberg nilsolitons, and we also explicitly describe many-parameter families of these solsolitons in dimensions greater than three.

Finally, we explore Ricci flow coupled with harmonic map flow, both as it arises naturally in certain bundle constructions related to Ricci flow and as a geometric flow in its own right. In the first case, we generalize a theorem of Knopf that demonstrates convergence and stability of certain locally $\mathbb{R}^N$-invariant Ricci flow solutions. In the second case, we prove a version of Hamilton's compactness theorem for the coupled flow, and then generalize it to the category of étale Riemannian groupoids. We also provide a detailed example of solutions to the flow on the Lie group Nil^3.

Contents

CHAPTER 1

Introduction

1. Optimal metrics and Ricci flow

The search for "optimal" Riemannian metrics on smooth manifolds has a rich history that dates back to the 1960's [2]. One type of optimal metric is an *Einstein metric*, which is a metric g_0 such that

$$\operatorname{Rc}(g_0) = cg_0, \qquad \text{for some } c \in \mathbb{R}. \tag{1.1}$$

These widely-studied metrics [4] are the most natural higher-dimensional analogs of constant curvature metrics on surfaces, and are critical points of the Einstein-Hilbert action. However, not all manifolds admit Einstein metrics.

Hamilton introduced Ricci flow in 1982 [15] as a tool to "improve" a given metric g_0 on a compact manifold $\mathcal{M}^n$, in a way analogous to heat flow for functions:

$$\begin{aligned} &\frac{\partial}{\partial t} g = -2\operatorname{Rc} + \frac{2}{n}\frac{\int_M \operatorname{scal}\, d\mu}{\int_M d\mu} g, \\ &g(0) = g_0. \end{aligned} \tag{1.2}$$

The fixed points of this flow are Einstein metrics. If $\mathcal{M}$ is not compact, the flow is

$$\frac{\partial}{\partial t} g = -2\operatorname{Rc},$$

and the fixed points are Ricci-flat metrics. The scarcity of Einstein and Ricci-flat metrics means that, even if a solution exists for all time, *a priori* one cannot expect convergence to anything reasonable. Hence, "improvement" is unclear.

The flow is invariant under diffeomorphisms and scaling, so one is led to consider solutions that evolve exactly in these ways. A metric g_0 is a *Ricci soliton* if there exists a function $\sigma(t)$ and a family $\{\eta_t\} \subset \operatorname{Diff}(\mathcal{M})$ such that

$$g(t) = \sigma(t)\, \eta_t^* g_0 \tag{1.3}$$

is a solution of Ricci flow. A metric g_0 on $\mathcal{M}$ is a Ricci soliton if and only if

$$\operatorname{Rc}(g_0) = cg_0 + \mathcal{L}_X g_0, \qquad \text{for some } c \in \mathbb{R}, X \in C^\infty(T\mathcal{M}) \text{ complete}, \tag{1.4}$$

which is a generalization of the Einstein condition (1.1).

In the time since Perelman's high-profile solution of Thurston's geometrization conjecture and Poincaré conjecture [37–39], the Ricci flow has become an increasingly important tool in geometry and topology. For example, Böhm and Wilking [5] used Ricci flow to dramatically extend Hamilton's original result on manifolds with positive Ricci curvature. Brendle and Schoen [6] used related techniques to prove their strong version of the Differentiable Sphere Theorem.

The usefulness of the Ricci flow in solving these and other problems should not obscure the complexity of this system of PDE. Many open questions remain, both in the analysis of Ricci flow itself and in its applications to other mathematical problems. For example, we have seen that there are issues of convergence of solutions, due to the lack of genuine fixed points. Suppose $\mathcal{M}$ does not admit an Einstein metric.

(1.5) What is the "proper" notion of a limiting object in this case?

(1.6) What is the asymptotic behavior of such solutions?

There are also questions about the existence of generalized fixed points –soliton metrics– and how these model solutions near singularities.

(1.7) When does a manifold admit or not admit soliton metrics?

One hopes that metrics do improve under the flow, so there are questions of asymptotically stability. Suppose a manifold admits homogeneous (or locally homogeneous) metrics.

(1.8) Will arbitrary solutions converge to or otherwise "approach" such metrics?

These are some of the questions guiding the research in this dissertation.

2. Solitons on homogeneous spaces

One major area of research is the study of Ricci solitons on homogeneous spaces, where the geometry reduces to data on Lie algebras. The search for Einstein metrics here is quite active, and there are many open questions. For example, Alekseevskii has conjectured that any connected homogeneous Riemannian manifold with Einstein metric of negative scalar curvature is diffeomorphic to Euclidean space. See [4] for more information.

One contribution of Ricci flow, which is a system of ODE in the homogeneous setting, is the increased importance of Ricci soliton inner products. The only non-Einstein homogeneous Ricci solitons are non-compact, expanding, and non-gradient [40]. In fact, all known examples of homogeneous solitons are left-invariant metrics on simply connected solvable Lie groups, such that

$$\operatorname{Ric}(g) = cI + D, \tag{2.1}$$

where Ric is the Ricci $(1,1)$-tensor, $c \in \mathbb{R}$ and D is a derivation of the Lie algebra. This *algebraic soliton* condition relates the algebra and geometry of the group. On nilpotent and solvable Lie groups, such metrics are called *nilsolitons* and *solsolitons*, respectively. It is known that equations (1.4) and (2.1) are equivalent in the completely solvable case [27]. Regarding Question 1.7, however, not all such Lie groups admit Ricci solitons.

Is is not known if every homogeneous Ricci soliton is isometric to an algebraic soliton. One reason for studying algebraic solitons, however, is that they could give an answer to Alekseevskii's conjecture. Lauret has proved that an algebraic soliton on a non-solvable homogenous space would disprove the conjecture [25].

Much work on algebraic solitons has been done in both the nilpotent and solvable cases. For example, Lauret has shown that nilsolitons are unique whenever they exist, are the critical points of a certain Riemannian functional, and are the nilpotent parts of Einstein solvmanifolds [24]. The relationship in the last fact goes much deeper, as Lauret also showed that all solsolitons arise from nilsolitons in a natural way [27]. Jablonski has also obtained a number of related results [19].

One benefit of working in the homogeneous category is that concrete computations are sometimes feasible. This was a major factor in the results of Chapter 2, which addresses Questions 1.7 and 1.6. Here, I analyze behavior of Ricci flow solutions on the the classical Heisenberg groups, which includes determination of asymptotic behavior and explicit description of the soliton metrics (see Theorem 1.1). I also explicitly describe the soliton metrics on the groups of upper-triangular matrices (see Theorem 1.2). In a continuation of this work in Chapter 3, I use results of Lauret [27] and Will [43] to describe the space of solsoliton extensions of the Heisenberg groups, and I give explicit descriptions of the relevant geometric data (see Theorem 1.4).

3. Interactions with the harmonic map flow

Another main area of research involves both the Ricci and harmonic map flows. The harmonic map flow for maps between Riemannian manifolds, introduced by Eells and Sampson [11], was one of the first geometric flows and pioneered the use of parabolic PDE in solving geometric problems. Given $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ and a map ϕ_0 between them, it is

$$\begin{aligned} \frac{\partial}{\partial t}\phi &= \tau_{g,h}\phi, \\ \phi(0) &= \phi_0, \end{aligned} \tag{3.1}$$

where $\tau_{g,h}\phi$ is the tension field of ϕ. Though weakly parabolic, the Ricci flow is similar in spirit. In fact, these flows are also related beyond formal similarities. For example, the DeTurck trick modifies Ricci flow by diffeomorphisms to make it strictly parabolic, and was used to prove short-time existence and uniqueness of Ricci flow on closed manifolds in [10]. Hamilton subsequently observed that these diffeomorphisms actually solve a harmonic map flow [17].

Motivated in part by the these results and certain problems in general relativity, List [29] and Müller [34] recently studied the coupling of these two flows:

$$\begin{aligned} \frac{\partial}{\partial t}g &= -2\operatorname{Rc} + 2c\,\nabla\phi\otimes\nabla\phi \\ \frac{\partial}{\partial t}\phi &= \tau_{g,h}\phi \end{aligned} \tag{3.2}$$

where $c = c(t) \geq 0$ is a function, and $\mathcal{M}$ and $\mathcal{N}$ are usually taken to be closed. This flow has several desirable properties, and can actually be more well-behaved than either individual flow. For example, choosing c large enough can avoid any energy concentration for ϕ. There is also a notion of soliton, similar to (1.4).

Meanwhile, in addressing Question 1.8 (more specifically, a conjecture of Hamilton), Lott showed that the pull-back Ricci flow solution on the universal cover of a three-manifold (with certain geometric assumptions) approaches a homogeneous expanding soliton metric [31]. For this, he considered a class of twisted principal $\mathbb{R}^N$-bundles. Metrics on such bundles $\mathcal{M} \to \mathcal{B}$ can be represented as $\mathbf{g} = (g, A, G)$, where g is a metric on the base, A is an $\mathbb{R}^N$-valued 1-form corresponding to a connection on $\mathcal{M}$, and G is an inner product on the fibers. Ricci flow on these *locally* $\mathbb{R}^N$*-invariant* metrics decomposes into a Ricci flow-type equation for g, a

Yang Mills flow-type equation for A, and a heat-type equation for G:

$$
\begin{aligned}
&\frac{\partial}{\partial t} g_{\alpha\beta} = -2R_{\alpha\beta} + \frac{1}{2} G^{ik} G^{j\ell} \nabla_\alpha G_{ij} \nabla_\beta G_{k\ell} + g^{\gamma\delta} G_{ij} (dA)^i_{\alpha\gamma} (dA)^j_{\beta\delta}, \\
&\frac{\partial}{\partial t} A^i_\alpha = -(\delta dA)^i_\alpha + G^{ij} \nabla^\beta G_{jk} (dA)^k_{\beta\alpha} \\
&\frac{\partial}{\partial t} G_{ij} = \Delta G_{ij} - G^{k\ell} \nabla_\alpha G_{ik} \nabla^\alpha G_{\ell j} - \frac{1}{2} g^{\alpha\gamma} g^{\beta\delta} G_{ik} G_{j\ell} (dA)^k_{\alpha\beta} (dA)^\ell_{\gamma\delta}.
\end{aligned}
\tag{3.3}
$$

In Chapter 4, I connect the flows (3.2) and (3.3) in a new way (that is, this is a phenomenon regarding locally $\mathbb{R}^N$-invariant Ricci flow, and is independent of any harmonic map flow connection from a DeTurck trick) with the observation that the heat-type equation for G is closely related to the harmonic map flow (see Proposition 2.5). Indeed, in the special case of a flat $\mathbb{R}^N$-vector bundle, which Lott studied in the context of expanding solitons on homogeneous spaces [30], Ricci flow is precisely the coupled flow (3.2). One application of the observation is a generalization of a theorem of Knopf [21] (which itself was an essential ingredient in the main result in [31]), demonstrating convergence and stability of certain locally $\mathbb{R}^N$-invariant Ricci flow solutions on these bundles (see Theorem 2.11).

I also consider other topics regarding the coupled flow (3.2), such as convergence of sequences of solutions. I prove a version of Hamilton's compactness theorem [16], and following Lott [30], I generalized this compactness theorem to the setting of Riemannian groupoids (see Theorem 1.2). Groupoids encompass objects like manifolds, quotients of manifolds under group actions, and orbifolds. Regarding Question 1.5, they provide a unified way to discuss convergence, for example, of Riemannian manifolds under Ricci flow when the limit is a different manifold. Finally, I consider a detailed example of a solution to the coupled flow (3.2), and see how it compares to known solutions of Ricci flow (see Propositions 4.11 and 4.16).

CHAPTER 2

Explicit Ricci solitons on nilpotent Lie groups

1. Introduction

The three-dimensional Heisenberg group, also known as Nil^3, is one of Thurston's model geometries. It is important in understanding the structure of manifolds and Ricci solitons in dimension three, and in understanding Ricci flow as a dynamical system for three-dimensional metric Lie algebras [14]. As such, it has been studied extensively: for example, by Isenberg and Jackson [18], Knopf and McLeod [22], Lott [30], Baird and Danielo [1], and Glickenstein [13]. While the behavior of Ricci flow on this Lie group is well-understood, the understanding of solitons on general nilpotent Lie groups (e.g., in higher dimensions) is nascent.

The group Nil^3 is an example of a *nilmanifold*, which is a nilpotent Lie group N together with a left-invariant metric g. We say g is a *nilsoliton* if $g(t) = c(t)\,\eta_t^* g$ is a solution of Ricci flow, for a function $c(t)$ and a one-parameter family of diffeomorphisms $\{\eta_t\}$ of N. The nilsoliton is *expanding* if $c(t) = t$. In [24], Lauret showed that a metric g on a nilmanifold is a nilsoliton if and only if $\mathrm{Ric}_g = cI + D$, where Ric_g is the Ricci endomorphism of g, $c \in \mathbb{R}$, and D is derivation of the Lie algebra $\mathfrak{n}$. Not all nilmanifolds admit nilsolitons, but Lauret also showed (among many other things) that when they exist, they are unique up to isometry and scaling. More recently, he has shown in [26] that solutions of Ricci flow on nilmanifolds exist for all time, and are Type-III: $\|\mathrm{Rm}(g(t))\| \leq C/t$ for $t > 0$. The methods used involve a flow (equivalent to Ricci flow) on the space of nilpotent Lie brackets, where the algebraic structure is more prominent.

Despite this, there are still very few explicit examples of Ricci solitons on nilpotent Lie groups (to say nothing of Ricci flow solutions in general), and that is the motivation for this paper. As we will see, Lauret has answered the question, "Do they exist?" We focus on the questions, "What are they?" and "How do they behave?" Namely, we demonstrate explicit nilsoliton metrics on two classes of nilpotent Lie groups that generalize Nil^3 to higher dimensions. For one class, we also show that arbitrary diagonal metrics will converge, modulo rescaling, to such solitons.

Theorem 1.1. *Let $H_N\mathbb{R}$ be the classical Heisenberg group of dimension $N = 2n+1$, with coordinates (x^i) and coframe $\{\theta^i\}$ to be described later. Let $g(t)$ be Ricci flow solution on $H_N\mathbb{R}$, starting at a diagonal left-invariant metric g_0.*

(a) *The solution $g(t)$ has the following asymptotic behavior:*

$$g_{ii} \sim \gamma_i t^{1/n+2}$$
$$g_{NN} \sim \gamma_N t^{-n/n+2}$$

where $i = 1, \ldots, 2n$, and the γs are constants depending only on g_0 and n.

(b) *The solution $g(t)$ converges, after pullback by diffeomorphisms, to the solution $g_\infty(t)$ corresponding to the metric*

$$g_\infty = \theta^1 \otimes \theta^1 + \cdots + \theta^{2n} \otimes \theta^{2n} + \frac{1}{n+2}\,\theta^N \otimes \theta^N.$$

This is a nilsoliton with respect to the diffeomorphisms

$$\eta_t(x^1, \ldots, x^{2n}, x^N) = (t^{-\frac{1}{2}\frac{n+1}{n+2}}x^1, \ldots, t^{-\frac{1}{2}\frac{n+1}{n+2}}x^{2n}, t^{-\frac{n+1}{n+2}}x^N).$$

Theorem 1.2. *Let $\mathrm{UT}_n\,\mathbb{R}$ be the Lie group of real $n \times n$ unitriangular matrices, with coordinates (x^{ij}) and coframe $\{\theta^{ij}\}$ to be described later. Then the family of diagonal metrics $g(t) = g_{ij,ij}(t)\,\theta^{ij} \otimes \theta^{ij}$, where*

$$g_{ij,ij}(t) = \frac{1}{n^{j-i-1}}t^{1-2(j-i)/n},$$

is a Ricci flow solution on $\mathrm{UT}_n\,\mathbb{R}$. The metric $g(1)$ is a nilsoliton with respect to the diffeomorphisms η_t, where

$$(\eta_t(x))^{ij} = t^{-(j-i)/n}x^{ij}.$$

We make a few explanatory remarks regarding these results. The solitons on these spaces were shown to exist by Lauret in [24]. The solitons in Theorem 1.1 are studied by Payne [36] in the context of a related but distinct evolution equation, the "projectivized bracket flow," introduced in that paper. The existence of the metric $g(1)$ in Theorem 1.2 may also be deduced from results in [42]. The (mostly analytic) approach taken in the present paper provides the following additional features, which facilitate the study of these solitons as models of infinite-time (nonhomogeneous) Ricci flow solutions undergoing collapse:

(1) We demonstrate the existence of an explicit stably Ricci-diagonal basis[1] for both families of metrics.
(2) We construct explicit families of diffeomorphisms that exhibit both families of solitons — *a priori* solutions of a static elliptic system — as time-dependent solutions of the Ricci flow parabolic system.
(3) We deduce the asymptotic behaviors of solutions $g(t)$ in Theorem 1.1, which do not readily follow from the corresponding results for projectivized bracket flow.

The structure of this paper is as follows. In Section 2 we recall Lott's blowdown method for finding solitons, and review the three-dimensional example case. Section 3 focuses on Heisenberg groups. In it, we examine more closely the structure of the classical Heisenberg groups, and compute their Ricci tensors. Using these computations, we write down the Ricci flow equations and describe the asymptotic behavior of solutions. Then we find the solitons with the blowdown method. We conclude with analysis of collapse of compact quotients of Heisenberg groups, interpreted as Riemannian groupoids.

Section 4 focuses on the (significantly more complicated) spaces of unitriangular matrices. We review essential properties of these spaces, and compute their Ricci tensors with the aid of a computer algebra system. Finally, we analyze the Ricci flow, and construct the Ricci solitons.

[1]A *stably Ricci-diagonal basis* is a basis of the Lie algebra (equivalently, a left-invariant frame) such that the Ricci tensors of any family of diagonal metrics are all diagonal. Such bases do not always exist.

The appendix contains the derivation of some helpful formulas for curvature of Lie groups.

2. The blowdown method

In this section, we recall a method for finding solitons that Lott used extensively in [30]. We also review the Heisenberg soliton in three dimensions. As mentioned above, this example appears in several other places, but we include it here for completeness, to establish notation, and to motivate the procedures (adapted from the above references) that we will use in the general case.

Let M be a manifold with local coordinates $(x^1, \dots, x^n)$, local frame $\{F_1, \dots, F_n\}$, and dual coframe $\{\theta^1, \dots, \theta^n\}$. Suppose that $(M, \hat{g}(t))$ is a type III Ricci flow solution such that the metric $\hat{g}(t)$ stays diagonal, and that its asymptotic behavior is given by some other metric $g(t)$. We write

$$g(t) = g_i(t)\, \theta^i \otimes \theta^i,$$

where[2] $\hat{g}_i(t) \sim g_i(t)$ for all $i = 1, \dots, n$. Consider the *blowdown* of this solution,

$$g_s(t) = \frac{1}{s} g(st),$$

which itself is another Ricci flow solution. The behavior of $g_s(t)$ as $s \to \infty$ tells us about the behavior of the original solution $g(t)$ whenever t is large.

Note that it does not matter in which order we take a blowdown or find asymptotics. Namely,

$$\hat{g}_i(t) \sim g_i(t) \iff \frac{1}{s}\hat{g}_i(st) \sim \frac{1}{s} g_i(st).$$

The goal is to find a family of diffeomorphisms $\{\phi_s \colon M \to M\}_{s>0}$, such that $\phi_s^* g_s(t)$ is a Ricci flow solution for each s, and such that

$$g_\infty(t) = \lim_{s\to\infty} \phi_s^* g_s(t)$$

exists. By Proposition 2.5 in [30], this limit (whenever it exists) is a soliton metric on M.

Note that for the above limit to exist, it is necessary that $\phi_s^* g_i(st)/s$ is finite and positive for each fixed s and t. In explicit calculations, it is extrememly helpful to choose the family $\{\phi_s\}$ such that

$$\phi_s^* \theta^i = \alpha^i(s)\, \theta^i$$

for all i and for some functions $\alpha^i(s)$. This is usually straight-forward when the solution is diagonal.

Example 2.1. Consider the Lie group

$$\mathrm{Nil}^3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\} \subset \mathrm{SL}_3\,\mathbb{R}.$$

We obtain global coordinates (x, y, z) from the obvious diffeomorphism with $\mathbb{R}^3$. Then the group multiplication is

$$(x, y, z) \cdot (z', y', z') = (x + x', y + y', z + z' + xy').$$

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[2]We use the symbol $\sim$ to mean $a(t) \sim b(t)$ if and only if $\lim_{t\to\infty} \dfrac{a(t)}{b(t)} = 1$.

There is a frame of left-invariant vector fields,

$$F_1 = \frac{\partial}{\partial x}, \quad F_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad F_3 = \frac{\partial}{\partial z},$$

and the only nontrivial Lie bracket relation is

$$[F_1, F_2] = F_3.$$

The dual coframe is

$$\theta^1 = dx, \quad \theta^2 = dy, \quad \theta^3 = dz - xdy.$$

A family of left-invariant metrics on Nil^3 is given by

$$\hat{g}(t) = A(t)\,\theta^1 \otimes \theta^1 + B(t)\,\theta^2 \otimes \theta^2 + C(t)\,\theta^3 \otimes \theta^3,$$

and the Ricci flow is the following system of ordinary differential equations:

$$\frac{d}{dt}A = \frac{C}{B}, \quad \frac{d}{dt}B = \frac{C}{A}, \quad \frac{d}{dt}C = -\frac{C^2}{AB}.$$

It is well-known that the flow will preserve the diagonality of an initial metric, and the solution (with asymptotics) is

$$\begin{aligned} A(t) &= A_0K^{-1/3}(t+K)^{1/3} \sim A_0K^{-1/3}t^{1/3}, \\ B(t) &= B_0K^{-1/3}(t+K)^{1/3} \sim B_0K^{-1/3}t^{1/3}, \\ C(t) &= C_0K^{1/3}(t+K)^{-1/3} \sim C_0K^{1/3}t^{-1/3}, \end{aligned}$$

for the constant

$$K = \frac{A_0B_0}{3C_0}.$$

This solution exists for all time, but as $t \to \infty$, we see that $A, B \to \infty$, and $C \to 0$. This is known as the "pancake" solution, as two directions are becoming more and more spread out, while the third is collapsing.

Calling the asymptotic solution $g(t)$, we see that the blowdown is

$$\begin{aligned} g_s(t) = A_0K^{-1/3}&s^{-2/3}t^{1/3}\,\theta^1 \otimes \theta^1 \\ &+ B_0K^{-1/3}s^{-2/3}t^{1/3}\,\theta^2 \otimes \theta^2 \\ &\qquad + C_0K^{1/3}s^{-4/3}t^{-1/3}\,\theta^3 \otimes \theta^3. \end{aligned}$$

We now want to find the appropriate diffeomorphisms ϕ_s. Suppose that they are of the form

$$\phi_s(x, y, z) = \big(\alpha(s)x, \beta(s)y, \gamma(s)z\big).$$

It is simple, then, to see that the functions

$$\begin{aligned} \alpha(s) &= (A_0K^{-1/3})^{-1/2}s^{1/3} \\ \beta(s) &= (B_0K^{-1/3})^{-1/2}s^{1/3} \\ \gamma(s) = \alpha(s)\beta(s) &= (A_0B_0K^{-2/3})^{-1/2}s^{2/3} \end{aligned}$$

work as desired. Thus,

$$\phi_s^* g_s(t) = t^{1/3}\Big(\theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2\Big) + \frac{1}{3}t^{-1/3}\,\theta^3 \otimes \theta^3 = g_\infty(t),$$

and there is no need to take a limit. A quick check shows that this is still a solution to Ricci flow, and that it satisfies

$$g_\infty(t) = t\eta_t^* g_\infty(1)$$

for the diffeomorphisms

$$\eta_t(x, y, z) = (t^{-1/3}x, t^{-1/3}y, t^{-2/3}z).$$

The metric $g_\infty(1)$ is the unique nilsoliton in dimension three, as seen in [30], [1], and [13].

Remark 2.2. Regarding the uniqueness of these diffeomorphisms in general, it is expected that if we have two families of diffeomorphisms, $\{\phi_s\}$ and $\{\psi\}_s$, that satisfy the above properties, then

$$\lim_{s\to\infty} \psi_s^{-1} \circ \phi_s$$

exists and is a diffeomorphism, even though ϕ_s and ψ_s may not converge to diffeomorphisms individually.

3. Nilsolitons on Heisenberg groups

3.1. The classical Heisenberg groups. We now recall the construction and properties of the higher-dimensional, classical Heisenberg groups. In terms of the framework outlined in [3], these are simply connected Lie groups corresponding to *generalized Heisenberg algebras* of the form $\mathfrak{n} = V \oplus Z$, where Z is one-dimensional.

To begin, let W be an n-dimensional real vector space, and set $V = W \oplus W$. Write

$$v = \begin{pmatrix} x \\ y \end{pmatrix} \in V, \quad \text{where } x, y \in W.$$

There is a natural involution $\iota\colon V \to V$ given by

$$\iota \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}.$$

Suppose V has a bilinear, skew-symmetric form ω, i.e., a symplectic form, and a trivial Lie bracket. Let $N = 2n + 1$ and consider a 1-dimensional central extension of V:

$$0 \longrightarrow Z \longrightarrow \mathfrak{h}_N\mathbb{R} \xrightarrow{\pi} V \longrightarrow 0.$$

This extension is a sum $\mathfrak{h}_N\mathbb{R} = V \oplus Z$, and we write an element as

$$\xi = \begin{pmatrix} v \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \text{where } v \in V, x, y \in W, z \in z.$$

The projection map is $\pi \begin{pmatrix} v \\ z \end{pmatrix} = v$. One can also think of this as the usual quotient map, with $\pi(\xi) = \xi + Z$. The Lie bracket on $\mathfrak{h}_N\mathbb{R}$ defined by this extension is expressed in terms of the symplectic form on V. Namely,

$$[\xi, \eta] = \omega\big(\pi(\xi), \pi(\eta)\big),$$

or

$$\left[\begin{pmatrix} v \\ x \end{pmatrix}, \begin{pmatrix} v' \\ x' \end{pmatrix} \right] = \begin{pmatrix} 0 \\ \omega(v, v') \end{pmatrix}.$$

This implies $\mathfrak{h}_N\mathbb{R}$ is a two-step nilpotent Lie algebra, since $[V, V] = Z$. Call $\mathfrak{h}_N\mathbb{R}$ the *Heisenberg algebra* of dimension N. The simply connected nilpotent Lie group corresponding to $\mathfrak{h}_N\mathbb{R}$ is $H_N\mathbb{R}$, the *Heisenberg group* of dimension N.

The standard basis of $\mathbb{R}^N$ gives a vector space isomorphism $\mathfrak{h}_N\mathbb{R} \cong \mathbb{R}^N$, and we take this basis to be ordered as follows:

$$\mathcal{E}_N = \{\underbrace{E_1, \ldots, E_n}_{E_i}, \underbrace{E_{n+1}, \ldots, E_{2n}}_{E_{n+i}}, E_N\}. \tag{3.1}$$

With respect to this basis, we may describe the additional structures on V and $\mathfrak{h}_N\mathbb{R}$ as the standard ones. First,

$$\iota = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \quad \omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad \langle \cdot, \cdot \rangle = I_N.$$

Additionally, the Lie bracket on $\mathfrak{h}_N\mathbb{R}$ has the following relations:

$$[E_i, E_j] = [E_i, E_N] = [E_{n+i}, E_{n+j}] = [E_{n+i}, E_N] = 0, \qquad [E_i, E_{n+j}] = \delta_{ij} E_N,$$

where $1 \leq i, j \leq n$. Thus, the only non-vanishing stucture constants are of the form

$$c_{i,n+i}^N = 1. \tag{3.2}$$

We can represent this Lie algebra in terms of $(n+2)\times(n+2)$ matrices. Namely,

$$\mathfrak{h}_N\mathbb{R} \cong \left\{ \begin{pmatrix} 0 & \vec{X}^T & Z \\ \vec{0} & 0_n & \vec{Y} \\ 0 & \vec{0}^T & 0 \end{pmatrix} \;\middle|\; \vec{X}, \vec{Y} \in \mathbb{R}^n, Z \in \mathbb{R} \right\},$$

where 0_n is the $n \times n$ zero matrix.

If $\{e_i\}$ is the standard basis for $\mathbb{R}^n$, there is a basis for $\mathfrak{h}_N\mathbb{R}$ given by

$$E_i = \begin{pmatrix} 0 & e_i^T & 0 \\ \vec{0} & 0_n & \vec{0} \\ 0 & \vec{0}^T & 0 \end{pmatrix}, \quad E_{i+n} = \begin{pmatrix} 0 & \vec{0}^T & 0 \\ \vec{0} & 0_n & e_i \\ 0 & \vec{0}^T & 0 \end{pmatrix}, \quad E_N = \begin{pmatrix} 0 & \vec{0}^T & 1 \\ \vec{0} & 0_n & \vec{0} \\ 0 & \vec{0}^T & 0 \end{pmatrix},$$

where $1 \leq i \leq n$.

In what follows, lower case Roman indices will always range over $1, \ldots, n$ (or sometimes $1, \ldots, 2n$) and Capital Roman indices (with the exception of N, which is fixed) will range over $1, \ldots, N$.

The simply connected Lie group associated with $\mathfrak{h}_N\mathbb{R}$ can also be respresented in terms of $(n+2)\times(n+2)$ matrices:

$$H_N\mathbb{R} = \left\{ \begin{pmatrix} 1 & \vec{a}^T & c \\ \vec{0} & I_n & \vec{b} \\ 0 & \vec{0}^T & 1 \end{pmatrix} \;\middle|\; \vec{a}, \vec{b} \in \mathbb{R}^n, c \in \mathbb{R} \right\} \subset \mathrm{SL}_{n+2}\,\mathbb{R},$$

where I_n is the $n \times n$ identity matrix and $\vec{0} \in \mathbb{R}^n$ is the zero vector. Group multiplication is again matrix multiplication:

$$\begin{pmatrix} 1 & \vec{a_1}^T & c_1 \\ \vec{0} & I_n & \vec{b_1} \\ 0 & \vec{0}^T & 1 \end{pmatrix} \begin{pmatrix} 1 & \vec{a_2}^T & c_2 \\ \vec{0} & I_n & \vec{b_2} \\ 0 & \vec{0}^T & 1 \end{pmatrix} = \begin{pmatrix} 1 & \vec{a_1}^T + \vec{a_2}^T & c_1 + c_2 + \vec{a_1} \cdot \vec{b_2} \\ \vec{0} & I_n & \vec{b_1} + \vec{b_2} \\ 0 & \vec{0}^T & 1 \end{pmatrix},$$

or more briefly,

$$(\vec{a_1}, \vec{b_1}, c_1)(\vec{a_2}, \vec{b_2}, c_2) = (\vec{a_1} + \vec{a_2}, \vec{b_1} + \vec{b_2}, c_1 + c_2 + \vec{a_1} \cdot \vec{b_2}),$$

where $\cdot$ refers to the standard Euclidean inner product.

We have a diffeomorphism $H_N\mathbb{R} \cong \mathbb{R}^N$, which gives us coordinates:

$$\begin{pmatrix} 1 & x^1 & \cdots & x^n & x^N \\ 0 & 1 & \cdots & 0 & x^{1+n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 1 & x^{2n} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \longmapsto (x^1, \ldots, x^n, x^{1+n}, \ldots, x^{2n}, x^N). \tag{3.3}$$

With respect to these coordinates, we can find a left-invariant frame with the same bracket relations as those above, and then find its coframe.

Lemma 3.4. *With respect to the coordinates from (3.3), $H_N\mathbb{R}$ has the following left-invariant frame $\{F_I\}$ and dual coframe $\{\theta^I\}$:*

$$\begin{aligned} F_i &= \partial_i, & \theta^i &= dx^i, \\ F_{i+n} &= \partial_{i+n} + x^i \partial_N, & \theta^{i+n} &= dx^{i+n} \\ F_N &= \partial_N, & \theta^N &= dx^N - \sum_{k=1}^{n} x^k dx^{k+n}. \end{aligned}$$

The frame $\{F_I\}$ satisfies the same bracket relations as the basis $\{E_I\}$.

3.2. Computing the Ricci tensor. We wish to analyze solutions to the Ricci flow

$$\frac{d}{dt} g = -2\,\mathrm{Rc},$$

starting at some initial metric g_0. By Lemma 3.4, any one-parameter family of left-invariant metrics, and therefore any Ricci flow solution, $g(t)$ on $H_N\mathbb{R}$ can be written as

$$g(t) = g_{IJ}(t)\, \theta^I \otimes \theta^J.$$

Analysis of these solutions requires a detailed understanding of the Ricci tensor. For this we will use formula (0.23) from Appendix A, which utilizes the Lie algebra structure. We break that equation apart as follows:

$$\begin{aligned} 4R_{IJ} = \quad & \underbrace{\left[2c_{KI}^{P} c_{JM}^{Q} + c_{KJ}^{P} c_{IM}^{Q} - c_{KM}^{P} c_{IJ}^{Q}\right] g^{KM} g_{PQ}}_{\langle 1 \rangle} \\ & + \underbrace{\left[c_{JM}^{P} c_{PI}^{Q} g_{QK} - c_{JM}^{P} c_{PK}^{Q} g_{QI} + c_{KI}^{P} c_{PM}^{Q} g_{QJ} - c_{KI}^{P} c_{PJ}^{Q} g_{QM}\right] g^{KM}}_{\langle 2 \rangle} \\ & + \underbrace{\left[(a_{KJ}^{P} + a_{JK}^{P})(a_{IM}^{Q} + a_{MI}^{Q}) - (a_{KM}^{P} + a_{MK}^{P})(a_{IJ}^{Q} + a_{JI}^{Q})\right] g^{KM} g_{PQ}}_{\langle 3 \rangle}. \end{aligned}$$

The computations are relatively straight-forward, though lengthy, and so we omit them. We simply remark that for each of the three pieces of $R_{IJ} = R_{IJ}\langle 1 \rangle + R_{IJ}\langle 2 \rangle + R_{IJ}\langle 3 \rangle$, one must consider six cases depending on index combinations:

$$R_{ij}, \quad R_{i,j+n}, \quad R_{iN}, \quad R_{i+n,j+n}, \quad R_{i+n,N}, \quad R_{NN}.$$

We can see this structure in the following $N \times N$ matrix:

$$R_{IJ} = \left(\begin{array}{c:c:c} R_{ij} & R_{i,j+n} & R_{iN} \\ \hdashline R_{i,j+n} & R_{i+n,j+n} & R_{i+n,N} \\ \hdashline R_{iN} & R_{i+n,N} & R_{NN} \end{array}\right).$$

If we set

$$\Sigma = \sum_{k,m=1}^{n} g^{km} g^{k+n,m+n} - \sum_{k=1}^{n} \sum_{m=n+1}^{2n} g^{km} g^{k+n,m-n},$$

then the components of the Ricci tensor are

$$\begin{aligned} R_{ij} &= -\frac{1}{2} g^{i+n,j+n} g_{NN} + \frac{1}{2} g_{iN} g_{jN} \Sigma, \\ R_{i,j+n} &= \frac{1}{2} g^{i+n,j} g_{NN} + \frac{1}{2} g_{iN} g_{j+n,N} \Sigma, \\ R_{iN} &= \frac{1}{2} g_{iN} g_{NN} \Sigma, \\ R_{i+n,j+n} &= -\frac{1}{2} g^{ij} g_{NN} + \frac{1}{2} g_{i+n,j+n} g_{\beta N} \Sigma, \\ R_{i+n,N} &= \frac{1}{2} g_{i+n,N} g_{NN} \Sigma, \\ R_{NN} &= \frac{1}{2} g_{NN}^2 \Sigma. \end{aligned}$$

3.3. The Ricci flow. Due to the complexity of the inverse of g, solving the Ricci flow system for arbitary initial data is intractable. Instead, we assume that we have diagonal initial data, and show that the flow preserves diagonality. So, if we assume that $g_{IJ} = g^{IJ} = 0$ for all $I \neq J$, then we claim that $R_{IJ} = 0$ as well. From now on, we only use single subscripts for the metric components: $g_1, \dots, g_N$.

We first note that when g is diagonal, we have

$$\Sigma = \sum_{k=1}^{n} g^{kk} g^{k+n,k+n} = \sum_{k=1}^{n} \frac{1}{g_k g_{k+n}}. \tag{3.5}$$

Then we have

$$\begin{aligned} R_{ij} &= \begin{cases} -\frac{1}{2} g^{i+n} g_N & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \\ R_{i,j+n} &= 0, \\ R_{iN} &= 0, \\ R_{i+n,j+n} &= \begin{cases} -\frac{1}{2} g^{i} g_N & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \\ R_{i+n,N} &= 0, \\ R_{NN} &= \frac{1}{2} g_N^2 \Sigma. \end{aligned}$$

This means that the natural basis for $\mathfrak{h}_N\mathbb{R}$ (or the frame for $H_N\mathbb{R}$ from Lemma 3.4) is stably Ricci-diagonal. In other words, the Ricci tensor stays diagonal under the flow, and we have some hope to understand the behavior of the Ricci flow system:

$$\frac{d}{dt}g_i = \frac{g_N}{g_{i+n}} \tag{3.6}$$

$$\frac{d}{dt}g_{i+n} = \frac{g_N}{g_i} \tag{3.7}$$

$$\frac{d}{dt}g_N = -g_N^2\sum_{k=1}^{n}\frac{1}{g_kg_{k+n}} = -g_N^2\Sigma \tag{3.8}$$

for $i = 1, \ldots, n$ and $N = 2n + 1$.

Remark 3.9. It is possible to find an explicit Ricci flow solution in some cases. For example, make following ansatz. Let $X(t) = t + K$, where K is some constant depending on the initial data. This means $X(0) = K$ and $X'(t) = 1$. Then we look for solutions of the form

$$g_i(t) = \gamma_i X^{1/n+2}, \quad g_\alpha(t) = \gamma_\alpha X^{1/n+2}, \quad g_N(t) = \gamma_N X^{-n/n+2}.$$

However, when solving for K, constraints on the initial data appear. In particular, a solution of this form requires initial data to come from an $(n + 1)$-paramater family of diagonal metrics.

Here we again note that there is indeed a Ricci soliton on $H_N\mathbb{R}$. This follows from a theorem of Lauret.

Theorem 3.10 ([24])**.** *A homogeneous nilmanifold (N, g) with corresponding metric Lie algebra $(\mathfrak{n}, \langle\cdot,\cdot\rangle_\mathfrak{n})$ is a Ricci soliton if and only if $(\mathfrak{n}, \langle\cdot,\cdot\rangle_\mathfrak{n})$ admits a metric solvable extension $(\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}, \langle\cdot,\cdot\rangle_\mathfrak{s})$, with $\mathfrak{a}$ Abelian, whose corresponding solvmanifold $(S, \tilde{g})$ is Einstein.*

The simply connected Lie group corresponding to the Lie algebra $\mathfrak{s} = \mathfrak{h}_N\mathbb{R} \oplus \mathbb{R}$ is an example of a *Damek-Ricci space*. These are known to be Einstein manifolds, and their Lie algebras are metric solvable extensions as in the theorem; see [3] for details. Therefore, there is a left-invariant metric g such that $(H_N\mathbb{R}, g)$ is a Ricci nilsoliton.

As mentioned in the introduction, this metric is unique up to scaling and isometry. We will describe it explicitly.

3.4. Asymptotics of general solutions. Now we consider the behavior of arbitrary diagonal solutions of Ricci flow. From this we will obtain the nilsoliton using the blowdown method.

Assume that $g_1, \ldots, g_{2n}, g_N$ solve the Ricci flow. As diagonal components of a metric, they are positive functions of t. We can use (3.6), (3.7), and (3.8) to see that

$$\frac{d}{dt}\frac{g_i}{g_{i+n}} = \frac{d}{dt}\frac{g_{i+n}}{g_i} = 0,$$

$$\frac{d}{dt}g_1 \cdots g_n g_N = \frac{d}{dt}g_{1+n} \ldots g_{2n} g_N = 0.$$

This means these quantities are conserved (i.e., constant), so we set

$$\begin{aligned} A_i &= \frac{g_i}{g_{i+n}} = \frac{g_i(0)}{g_{i+n}(0)}, \\ B_{i+n} &= \frac{g_{i+n}}{g_i} = \frac{g_{i+n}(0)}{g_i(0)}, \\ C_1 &= g_1 \cdots g_n g_N = g_1(0) \cdots g_n(0) g_N(0), \\ C_2 &= g_{1+n} \ldots g_{2n} g_N = g_{1+n}(0) \ldots g_{2n}(0) g_N(0), \end{aligned}$$

where $1 \leq i \leq n$. Note that $A_i B_{i+n} = 1$, and that

$$\frac{C_1}{A_1 \cdots A_n} = C_2, \quad \frac{C_2}{B_{1+n} \cdots B_{2n}} = C_1. \tag{3.11}$$

We rewrite the Ricci flow equation for g_i:

$$\frac{d}{dt} g_i = \frac{g_N}{g_{i+n}} \frac{g_i}{g_i} = A_i \frac{g_N}{g_i}, \tag{3.12}$$

and similarly

$$\frac{d}{dt} g_i^2 = 2 g_i \frac{g_N}{g_{i+n}} = 2 A_i g_N, \tag{3.13}$$

which can be solved by integrating.

Note that (3.6) implies that g_i is an increasing function, so Σ is positive and decreasing by (3.5). Then equation (3.8) implies that g_N is a decreasing function, and since it is positive we have

$$\frac{d}{dt} g_N = -g_N^2 \Sigma \geq -\Sigma(0) g_N^2,$$

and this implies

$$g_N(t) \geq \frac{1}{g_N(0)^{-1} + \Sigma(0) t}. \tag{3.14}$$

If we set $G_N(t) = \int_1^t g_N(r)\, dr$, then this is a positive, increasing function. By (3.14), we see that

$$\lim_{t \to \infty} G_N(t) = \lim_{t \to \infty} \int_1^t g_N(r)\, dr \geq \lim_{t \to \infty} \int_1^t \frac{dr}{g_N(0)^{-1} + \Sigma(0) r} = \infty,$$

so $G_N(t) \to \infty$ as $t \to \infty$. Using (3.13) we have

$$g_i(t)^2 = g_i(0)^2 + 2 A_i G_N(t).$$

If $1 \leq i \neq j \leq n$, we use this to obtain

$$\lim_{t \to \infty} \frac{g_i^2}{g_j^2} = \lim_{t \to \infty} \frac{g_i(0)^2 + 2 A_i G_N(t)}{g_i(0)^2 + 2 A_j G_N(t)} = \frac{A_i}{A_j}.$$

This implies that

$$g_i \sim \sqrt{\frac{A_i}{A_j}} g_j, \quad g_{i+n} \sim \sqrt{\frac{B_{i+n}}{B_{j+n}}} g_{j+n}. \tag{3.15}$$

Since C_1 is conserved, for each fixed $1 \leq i \leq n$, we have

$$\begin{aligned} g_N &= \frac{C_1}{g_1 \cdots g_n} \\ &\sim \frac{1}{\sqrt{\frac{A_1}{A_i}} g_i \cdots \sqrt{\frac{A_i}{A_i}} g_i \cdots \sqrt{\frac{A_n}{A_i}} g_i} \\ &= \sqrt{\frac{A_i^n}{A_1 \cdots A_n}} g_i^{-n} \\ &= \sqrt{A_i^n C_1 C_2} g_i^{-n}, \end{aligned} \tag{3.16}$$

by (3.15). With reference to (3.12), this gives

$$A_i \frac{g_N}{g_i} \sim \sqrt{A_i^{n+2} C_1 C_2} g_i^{-(n+1)}.$$

We would like to see that the solution $\tilde{g}_i$ to the equation

$$\frac{d}{dt} \tilde{g}_i = \sqrt{A_i^{n+2} C_1 C_2} \tilde{g}_i^{-(n+1)} \tag{3.17}$$

is asymptotically equivalent to g_i. For this we need a basic lemma.

Lemma 3.18. *Suppose that $u(t)$ is a solution to the ordinary differential equation*

$$\frac{d}{dt} u = c, \quad u(0) = u_0,$$

where $c > 0$, and that $v(t)$ is a solution to the asymptotic equation

$$\frac{d}{dt} v = c(1 + \epsilon(t)), \quad v(0) = v_0,$$

where $\epsilon(t) \to 0$ as $t \to \infty$. Then $u/v \to 1$ as $t \to \infty$. That is, $u \sim v$.

Proof. We can solve both equations by integrating:

$$u(t) = u_0 + c \int_0^t ds = t \left(\frac{u_0}{t} + c \right),$$

$$v(t) = v_0 + c \int_0^t (1 + \epsilon(s))\, ds = t \left(\frac{u_0}{t} + c + \frac{c}{t} \int_0^t \epsilon(s)\, ds \right).$$

To analyze the ratio u/v, we must know the behavior of the integral term in v. Note that, as a positive increasing function,

$$\int_0^t |\epsilon(s)|\, ds \longrightarrow L \in (0, \infty]$$

as $t \to \infty$. If $L < \infty$, then

$$\left| \lim_{t\to\infty} \frac{c}{t} \int_0^t \epsilon(s)\, ds \right| \leq c \lim_{t\to\infty} \frac{1}{t} \int_0^t |\epsilon(s)|\, ds \leq cL \lim_{t\to\infty} \frac{1}{t} = 0.$$

On the other hand, if $L = \infty$, then

$$\left| \lim_{t\to\infty} \frac{c}{t} \int_0^t \epsilon(s)\, ds \right| \leq c \lim_{t\to\infty} \frac{1}{t} \int_0^t |\epsilon(s)|\, ds \overset{LH}{=} c \lim_{t\to\infty} \frac{|\epsilon(t)|}{1} = 0.$$

This means

$$\lim_{t\to\infty} \frac{u}{v} = \lim_{t\to\infty} \frac{\frac{u_0}{t} + c}{\frac{u_0}{t} + c + \frac{c}{t} \int_0^t \epsilon(s)\, ds} = 1,$$

by the squeeze theorem. $\square$

Note that equation (3.17) is equivalent to

$$\frac{d}{dt}\tilde{g}_i^{n+2} = (n+2)\sqrt{A_i^{n+2} C_1 C_2},$$

and equation (3.12) is equivalent to

$$\frac{d}{dt}g_i^{n+2} = (n+2)A_i g_i^n g_N.$$

By (3.16), the right sides of these equations are asymptotically equivalent. Now, taking $u = \tilde{g}_i^{n+2}$ and $v = g_i^{n+2}$ in the Lemma, we see that $g_i \sim \tilde{g}_i$.

Equation (3.17) has an explicit solution:

$$\begin{aligned}\tilde{g}_i &= \left((n+2)\sqrt{A_i^{n+2} C_1 C_2}\right)^{1/n+2} t^{1/n+2} \\ &= (n+2)^{1/n+2}\sqrt{A_i}(C_1C_2)^{1/2(n+2)} t^{1/n+2}. \qquad (3.19)\end{aligned}$$

We can plug this into (3.16) to obtain

$$\begin{aligned}g_N &= \frac{C_1}{g_1 \cdots g_n} \\ &\sim \frac{C_1}{(n+2)^{n/n+2}\sqrt{A_1\cdots A_n}(C_1C_2)^{n/2(n+2)}} t^{-n/n+2} \\ &= (n+2)^{-n/n+2}(C_1C_2)^{1/n+2} t^{-n/n+2}, \qquad (3.20)\end{aligned}$$

which we call $\tilde{g}_N$. Note that this is independent of i.

We can repeat these calculations starting with $g_N = C_2/g_{n+1}\cdots g_{2n}$ to obtain

$$\tilde{g}_{i+n} = (n+2)^{1/n+2}\sqrt{B_{i+n}}(C_1C_2)^{1/2(n+2)} t^{1/n+2}.$$

If we plug this back into $g_N = C_2/g_{1+n}\cdots g_{2n}$, then we get the same result for g_N that we found in equation (3.20). Putting everything together, we have the following result.

Theorem 1 (1.1(a))**.** *If g_0 is a diagonal left-invariant metric on $H_N\mathbb{R}$, then the solution $g(t)$ of Ricci flow, with $g(0) = g_0$, has the following asymptotic behavior:*

$$\begin{aligned}g_i &\sim (n+2)^{1/n+2}\sqrt{A_i}C^{1/2(n+2)} t^{1/n+2}, \\ g_{i+n} &\sim (n+2)^{1/n+2}\sqrt{B_{i+n}}C^{1/2(n+2)} t^{1/n+2}, \\ g_N &\sim (n+2)^{-n/n+2}C^{1/n+2} t^{-n/n+2},\end{aligned}$$

where

$$A_i = \frac{g_i(0)}{g_{i+n}(0)}, \quad B_{i+n} = \frac{g_{i+n}(0)}{g_i(0)}, \quad C = g_1(0)\cdots g_{2n}(0) g_N(0)^2.$$

Remark 3.21. These asymptotics coincide with the case $n = 1$ from Example 2.1.

3.5. The nilsoliton. Writing $g(t)$ for the asymptotic solution of Theorem 1.1(a), we now use the blowdown procedure of Section 2 to obtain the soliton metric. The components of $g_s(t)$ are

$$\begin{aligned}(g_s(t))_i &= (n+2)^{1/n+2}A_i^{1/2}C^{1/2(n+2)}s^{-(n+1)/n+2}t^{1/n+2}, \\ (g_s(t))_{i+n} &= (n+2)^{1/n+2}B_{i+n}^{1/2}C^{1/2(n+2)}s^{-(n+1)/n+2}t^{1/n+2}, \\ (g_s(t))_N &= (n+2)^{-n/n+2}C^{1/n+2}t^{-n/n+2}.\end{aligned}$$

Using the coordinates and coframe from Subsection 3.1, we seek diffeomorphisms ϕ_s such that $\phi_s^* g_s(t)$ is a metric for all s, and

$$\lim_{s\to\infty} \phi_s^* g_s(t)$$

exists. Suppose that ϕ_s is of the form

$$\phi_s(x^1, \dots, x^N) = \big(\alpha^1(s)x^1, \dots, \alpha^N(s)x^N\big)$$

for some functions $\alpha^i(s)$. Then we have $\phi_s^*\theta^i = \alpha^i(s)\,\theta^i$ for $1 \le i \le 2n$. When $1 \le i \le n$, setting

$$\alpha^i(s) = (n+2)^{-1/2(n+2)} A_i^{-1/4} C^{-1/4(n+2)} s^{(n+1)/2(n+2)}$$

gives, for fixed i,

$$\phi_s^* \left((g_s(t))_i\, \theta^i \otimes \theta^i\right) = \alpha^i(s)^2 (g_s(t))_i\, \theta^i \otimes \theta^i = t^{1/n+2}\, \theta^i \otimes \theta^i.$$

Next, note that

$$\alpha^i(s)\alpha^{i+n}(s) = (n+2)^{-1/n+2} C^{-1/2(n+2)} s^{n+1/n+2},$$

and this does not depend on i. Therefore, if we set $\alpha^N(s) = \alpha^i(s)\alpha^{i+n}(s)$, then we have

$$\begin{aligned}\phi_s^*\theta^N &= \alpha^i(s)\alpha^{i+n}(s)\, dx^N - \sum_{i=1}^n \alpha^i(s)\alpha^{i+n}(s) x^i\, dx^{i+n} \\ &= \alpha^N(s)\, \theta^N,\end{aligned}$$

and so

$$\phi_s^* \left((g_s(t))_N\, \theta^N \otimes \theta^N\right) = \frac{1}{n+2} t^{-n/n+2}\, \theta^N \otimes \theta^N.$$

We have a limit metric

$$g_\infty(t) = \phi_s^* g_s(t) = t^{1/n+2}\Big(\theta^1 \otimes \theta^1 + \cdots + \theta^{2n} \otimes \theta^{2n}\Big) + \frac{1}{n+2} t^{-n/n+2}\, \theta^N \otimes \theta^N,$$

and to verify that it is a soliton, we seek diffeomorphisms $\{\eta_t\}$ such that $g_\infty(t)$ satisfies

$$g_\infty(t) = t\eta_t^* g_\infty(1).$$

For some numbers a and b, suppose that the diffeomorphisms are of the form

$$\eta_t(x^1, \dots, x^{2n}, x^N) = (t^a x^1, \dots, t^a x^{2n}, t^b x^N).$$

Then for $1 \le i \le 2n$, we have $\eta_t^*\theta^i = t^a\theta^i$ and, if $b = 2a$, $\eta_t^*\theta^N = t^b\theta^N$. This means

$$t\eta_t^* g(1) = t^{2a+1}\Big(\theta^1 \otimes \theta^1 + \cdots + \theta^{2n} \otimes \theta^{2n}\Big) + \frac{1}{n+2} t^{2b+1}\, \theta^N \otimes \theta^N.$$

For this to equal $g(t)$, we must have

$$\frac{1}{n+2} = 2a+1, \qquad -\frac{n}{n+2} = 2b+1,$$

which implies

$$a = -\frac{1}{2}\frac{n+1}{n+2}, \qquad b = 2a = -\frac{n+1}{n+2}.$$

Thus, $g(t)$ is an expanding Ricci soliton with respect to the diffeomorphisms

$$\eta_t(x^1, \dots, x^{2n}, x^N) = (t^{-\frac{1}{2}\frac{n+1}{n+2}} x^1, \dots, t^{-\frac{1}{2}\frac{n+1}{n+2}} x^{2n}, t^{-\frac{n+1}{n+2}} x^N).$$

To summarize, we have another result.

Theorem 2 (1.1 (b))**.** *Let $H_N\mathbb{R}$ have coordinates (x^i) as in (3.3) and coframe as in Lemma 3.4. Let $g(t)$ be any solution to Ricci flow on $H_N\mathbb{R}$ with diagonal initial data. For the diffeomorphisms $\{\phi_s\}$ defined as above, we have*

$$\begin{aligned}\lim_{s\to\infty}\frac{1}{s}\phi_s^* g(st) &= t^{1/n+2}\Big(\theta^1\otimes\theta^1+\cdots+\theta^{2n}\otimes\theta^{2n}\Big)+\frac{1}{n+2}t^{-n/n+2}\,\theta^N\otimes\theta^N\\ &= g_\infty(t).\end{aligned}$$

The metric $g_\infty(1)$ is a nilsoliton with respect to the diffeomorphisms

$$\eta_t(x^1,\ldots,x^{2n},x^N)=(t^{-\frac{1}{2}\frac{n+1}{n+2}}x^1,\ldots,t^{-\frac{1}{2}\frac{n+1}{n+2}}x^{2n},t^{-\frac{n+1}{n+2}}x^N).$$

The behavior here is analagous to the “pancake” effect mentioned in Example 2.1. The first $2n$ directions become more and more spread out, while the last direction collapses. More precisely, there is Gromov-Hausdorff convergence to $(\mathbb{R}^{2n}, g_{\mathrm{can}})$.

Remark 3.22. The diffeomorphisms ϕ_s and η_t here, and those in Example 2.1, are actually group automorphisms. Compare with [30], Remark 3.1 and Section 4.

Remark 3.23. Looking at the three-dimensional nilsoliton, one can extrapolate with the following ansatz:

$$g_i = g_{i+n} = t^a, \quad g_N = ct^b,$$

for some numbers a, b and c. Using the Ricci flow equations, it is easy to obtain

$$a=\frac{1}{n+2},\quad b=\frac{-n}{n+2},\quad c=\frac{1}{n+2}.$$

Thus,

$$g_i(t)=g_{i+n}(t)=t^{1/n+2},\quad g_N(t)=\frac{1}{n+2}t^{-n/n+2},$$

which is the nilsoliton g_∞ above. This does not provide any information about behavior of general solutions, however.

3.6. The groupoid interpretation. In [30] and [31], Lott initiated the use of Riemannian groupoids in understanding the notion of convergence under Ricci flow. One motivating issue is that, as in the case of Nil^3, the limit of a Ricci flow solution $(M, g(t))$ as $t\to\infty$ may not be an object of the same dimension (i.e., it may collapse). This means some data has been lost in the process of taking the limit. The groupoid formalism provides a way to keep track of all such data (e.g., the limiting object has the same dimension as M), and to provide a picture of the limiting behavior that is similar to, but more convenient than, the usual Gromov-Hausdorff notion of convergence. One may consult [30] and [13] for background on Riemannian groupoids, or the books [32], [33] for a more general introduction to groupoids.

Our analysis here follows the examples found in [13], which give concrete pictures of collapse. Here is the basic idea, tailored to our present context. In order to understand the collapse under Ricci flow of certain compact, locally homogenous manifolds arising as quotients of $H_N\mathbb{R}$, we replace such a manifold $(M = H_N\mathbb{R}/\Gamma, g)$ by its representation as a Riemannian “action” groupoid, $(H_N\mathbb{R}\rtimes\Gamma,\tilde{g})$. Also called a “cross-product” groupoid, this is an object whose orbit space is M. Here,

$$\pi\colon (H_N\mathbb{R},\tilde{g})\longrightarrow (M,g)$$

is the universal cover with induced metric, and $\Gamma \subset H_N\mathbb{R}$ is a discrete, cocompact subgroup that can be interpreted in several ways. It is the fundamental group $\pi_1(M, m_0)$, the group of deck transformation of the cover, or a group of isometries acting transitively on $(H_N\mathbb{R}, \tilde{g})$. In any case, it acts by left translation on $H_N\mathbb{R}$.

If $g(t)$ is a Ricci flow solution on M, then we are considering a solution $\tilde{g}(t)$ on $H_N\mathbb{R}$. By the prevous section, the blowdown technique provides a sequence $\phi_s\tilde{g}_s(t)$ of metrics converging to a metric $\tilde{g}_\infty(t)$, where $\tilde{g}_\infty(1)$ is a soliton. To understand the limiting behavior as $s \to \infty$, we now consider

$$(H_N\mathbb{R} \rtimes \Gamma_s, \phi_s\tilde{g}_s(t)).$$

Note that the subgroup Γ_s acting on $H_N\mathbb{R}$ depends on s, since the metric is changing. If, in the limit, this sequence of discrete subgroups converges to a continuous subgroup, then there is collapse. Therefore, we must understand how these subgroups evolve.

Recall that the blowdown metrics $\tilde{g}_s(t)$ are obtained using diffeomorphisms

$$\phi_s(x^1, \dots, x^N) = (\alpha^1(s)x^1, \dots, \alpha^N(s)x^N).$$

(The explicit forms of the α's are not imporant here.) Then the limit is

$$\tilde{g}_\infty(t) = \phi_s^* g_s(t) = t^{1/n+2}\Big(\theta^1 \otimes \theta^1 + \cdots + \theta^{2n} \otimes \theta^{2n}\Big) + \frac{1}{n+2} t^{-n/n+2}\theta^N \otimes \theta^N.$$

Without loss of generality, after change of coordinates we can take Γ_s to be an integer lattice. Therefore, write elements of Γ_s as

$$h_z(s) = h_{z^1(s),\dots,z^N(s)} = \left(z^1(s), \dots, z^N(s)\right),$$

with $z^i(s) \in \mathbb{Z}$. These isometries act on $(H_N\mathbb{R}, \tilde{g}_s(t))$ by left translation and, as deck transformations, they pull back by conjugation. Therefore,

$$\begin{aligned}
&\phi_x^* h_z(x^1, \dots, x^{2n}, x^N) \\
&= \phi_x^{-1} h_z \phi_s(x^1, \dots, x^{2n}, x^N) \\
&= \left(x^1 + \frac{z^1(s)}{\alpha^1(s)}, \dots, x^{2n} + \frac{z^{2n}(s)}{\alpha^{2n}(s)}, x^N + \frac{z^N(s)}{\alpha^N(s)} + \frac{z^1(s)}{\alpha^1(s)}x^{n+1} + \cdots + \frac{z^n(s)}{\alpha^n(s)}x^{2n}\right)
\end{aligned}$$

using the component-wise form of the group multiplication.

It is a basic fact that, given any strictly increasing sequence $\{\sigma_j\}$ with $\sigma_j \to \infty$ as $j \to \infty$, and any $u \in \mathbb{R}$, there is some sequence of integers $\{\tau_j\}$ such that $\tau_j/\sigma_j \to u$. Indeed, take $\tau_j = \lfloor \sigma_j u \rfloor$.

Therfore, consider any strictly increasing sequence $\{s_j\}$ with $s_j \to \infty$ as $j \to \infty$. The sequences $\{\alpha^I(s_j)\}$ are also strictly increasing. Then given any real numbers $u^1, \dots, u^N$, we may choose $z^i(s_j) \in \Gamma_{s_j}$ such that

$$\lim_{j\to\infty} \frac{z^i(s_j)}{\alpha^i(s_j)} = u^i, \quad \lim_{j\to\infty} \frac{z^{i+n}(s_j)}{\alpha^{i+n}(s_j)} = u^{i+n}, \quad \lim_{j\to\infty} \frac{z^N(s_j)}{\alpha^N(s_j)} = u^N.$$

This means that as $j \to \infty$, the isometries $\phi_{s_j}^* h_z$ converge to isometries h_u of $\tilde{g}_\infty(t)$ that act on $H_N\mathbb{R}$ as follows:

$$h_u(x^1, \dots, x^{2n}, x^N) = (x^1 + u^1, \dots, x^{2n} + u^{2n}, x^N + u^N + u^1x^{n+1} + \cdots + u^nx^{2n}).$$

The u^i were arbitary real numbers, so every element of $H_N\mathbb{R}$ is attained this way. This means Γ_{s_j} converges to a continuous group: the entire group $H_N\mathbb{R}$.

We conclude that

$$\lim_{j\to\infty}(H_N \rtimes \Gamma_{s_j}, \phi_{s_j}^* \tilde{g}_{s_j}(t)) = (H_N\mathbb{R} \rtimes H_N\mathbb{R}, \tilde{g}_\infty(t))$$

as Riemannian groupoids. There is maximal collapsing, as the orbit space of the groupoid $H_N\mathbb{R} \rtimes H_N\mathbb{R}$ is a point. This is the same behavior seen in the three-dimensional case.

Remark 3.24. Note that this is a different description than the "pancake" model described earlier, which occurs as $t \to \infty$. The model here illustrates collapse as the metrics converge to the actual soliton metric.

4. Nilsolitons on spaces of unitriangular matrices

4.1. Unitriangular matrices. Let $\mathrm{UT}_n\,\mathbb{R} \subset \mathrm{SL}_n\,\mathbb{R}$ denote the collection of real, unitriangular $n \times n$ matrices under matrix multiplication. These are matrices with 1 on the diagonal and 0 below. This is a Lie group of dimension $N = \binom{n}{2} = n(n-1)/2$, and $\mathrm{UT}_n\,\mathbb{R} \cong \mathbb{R}^N$. These groups are nilpotent, and are in some sense "model" nilpotent Lie groups. Indeed, it is a consequence of Engel's theorem that every simply connected nilpotent Lie group is a subgroup of $\mathrm{UT}_n\,\mathbb{R}$ for some n.

The Lie algebra $\mathfrak{ut}_n\mathbb{R}$ of $\mathrm{UT}_n\,\mathbb{R}$ consists of upper-triangular matrices with 0 on the diagonal. It has a basis

$$\mathcal{B}_n = \{B_{ij}\}_{1\le i<j\le n},$$

where B_{ij} is the $n \times n$ matrix such that that

$$(B_{ij})_{pq} = \delta_{ip}\delta_{jq}.$$

In other words, B_{ij} is the matrix with 1 in the (i,j) component, and zero elsewhere.

This Lie algebra inherits the Lie bracket from $\mathfrak{gl}_n\mathbb{R}$. To describe the bracket, note that if $i < j$ and $k < l$, then

$$(B_{ij}B_{kl})_{pq} = \sum_r (B_{ij})_{pr}(B_{kl})_{rq} = \sum_r \delta_{ip}\delta_{jr}\delta_{kr}\delta_{lq} = \delta_{ip}\delta_{lq}\delta_{jk} = \delta_{jk}(B_{il})_{pq},$$

which implies

$$[B_{ij}, B_{kl}] = \delta_{jk}B_{il} - \delta_{il}B_{kj},$$

and so the structure constants are

$$c_{ij,kl}^{pq} = \delta_{ip}\delta_{lq}\delta_{jk} - \delta_{kp}\delta_{jq}\delta_{il}. \tag{4.1}$$

Any diffeomorphism $\mathrm{UT}_n\,\mathbb{R} \cong \mathbb{R}^N$ gives us coordinates, so let us take

$$\begin{pmatrix} 1 & x^{12} & x^{13} & \cdots & x^{1,n-1} & x^{1n} \\ 0 & 1 & x^{23} & \cdots & x^{2,n-1} & x^{2n} \\ 0 & 0 & 1 & \cdots & x^{3,n-1} & x^{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & x^{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \longmapsto (x^{12}, x^{13}, \ldots, x^{n-1,n}). \tag{4.2}$$

With respect to these coordinates, we can find a left-invariant frame with the same bracket relations as those above, and then find its coframe. If $a = (a_{ij})$ and $b = (b_{ij})$ are elements of $\mathrm{UT}_n\,\mathbb{R}$, the multiplication rule is

$$x^{ij}(a \cdot b) = x^{ij}(a) + x^{ij}(b) + \sum_{i<k<j} x^{ik}(a)x^{kj}(b).$$

Lemma 4.3. *With respect to the coordinates from (4.2), the space* $\mathrm{UT}_n\,\mathbb{R}$ *has the following left-invariant frame* $\{F_{ij}\}$ *and dual coframe* $\{\theta^{ij}\}$*:*

$$F_i = \partial_{ij} + \sum_{k<i} x^{ki}\partial_{kj},$$
$$\theta^{ij} = dx^{ij} - \sum_{i<p<j} (2x^{ip} - \Theta^{ij}_p)\, dx^{pj},$$

where

$$\Theta^{ij}_p = \sum_{k=0}^{p} \sum_{i<r_1<\cdots<r_k<p} x^{ir_1}x^{r_1r_2}\cdots x^{r_kp},$$

and the inner sum ranges over all ordered subsets of $\{i+1, i+2\ldots, p-1\}$ *of size* k*. The frame* $\{F_{ij}\}$ *satisfies the same bracket relations as the basis* $\{B_{ij}\}$ *above.*

4.2. Computing the Ricci tensor. Our goal is to analyze solutions of Ricci flow on $\mathrm{UT}_n\,\mathbb{R}$. By Lemma 4.3, such metrics $g(t)$ can be written as

$$g(t) = g_{ij,kl}(t)\,\theta^{ij}\otimes\theta^{kl}.$$

Once again, our analysis requires us to understand the Ricci tensor, and equation (0.23) still applies. In terms of $\mathrm{UT}_n\,\mathbb{R}$, where double indices are needed, we can rewrite it as

$$\begin{aligned}
&4R_{ij,kl} = \\
&\quad\underbrace{\left[2c^{tu}_{pq,ij}c^{vw}_{kl,rs} + c^{tu}_{pq,kl}c^{vw}_{ij,rs} - c^{tu}_{pq,rs}c^{vw}_{ij,kl}\right] g^{pq,rs}g_{tu,vw}}_{\langle 1\rangle} \\
&+ \underbrace{\left[c^{tu}_{kl,rs}c^{vw}_{tu,ij}g_{vw,pq} - c^{tu}_{kl,rs}c^{vw}_{tu,pq}g_{vw,ij} + c^{tu}_{pq,ij}c^{vw}_{tu,rs}g_{vw,kl} - c^{tu}_{pq,ij}c^{vw}_{tu,kl}g_{vw,rs}\right] g^{pq,rs}}_{\langle 2\rangle} \\
&+ \underbrace{\left[(a^{tu}_{pq,kl} + a^{tu}_{kl,pq})(a^{vw}_{ij,rs} + a^{vw}_{rs,ij}) - (a^{tu}_{pq,rs} + a^{tu}_{rs,pq})(a^{vw}_{ij,kl} + a^{vw}_{kl,ij})\right] g^{pq,rs}g_{tu,vw}}_{\langle 3\rangle},
\end{aligned} \tag{4.4}$$

where $1 \le p < q \le n, 1 \le r < s \le n, 1 \le t < u \le n, 1 \le v < w \le n$.

With the help of a computer algebra system, we can substitute (4.1) and a double-indexed version of (0.24) into this rather unwieldy formula to obtain the following enormous expressions.

$$4R_{ij,kl}\langle 1\rangle = \sum_{\substack{1\le p<q\le n\\ 1\le r<s\le n\\ 1\le t<u\le n\\ 1\le v<w\le n}} \left\{\begin{array}{l}
-g_{tu,vw}g^{pq,rs}\delta_{il}\delta_{jw}\delta_{kv}\delta_{ps}\delta_{qu}\delta_{rt}\\
+g_{tu,vw}g^{pq,rs}\delta_{iv}\delta_{jk}\delta_{lw}\delta_{ps}\delta_{qu}\delta_{rt}\\
-g_{tu,vw}g^{pq,rs}\delta_{is}\delta_{jw}\delta_{kq}\delta_{lu}\delta_{pt}\delta_{rv}\\
-2g_{tu,vw}g^{pq,rs}\delta_{iq}\delta_{ju}\delta_{ks}\delta_{lw}\delta_{pt}\delta_{rv}\\
+g_{tu,vw}g^{pq,rs}\delta_{is}\delta_{jw}\delta_{kt}\delta_{lp}\delta_{qu}\delta_{rv}\\
+2g_{tu,vw}g^{pq,rs}\delta_{it}\delta_{jp}\delta_{ks}\delta_{lw}\delta_{qu}\delta_{rv}\\
+g_{tu,vw}g^{pq,rs}\delta_{il}\delta_{jw}\delta_{kv}\delta_{pt}\delta_{qr}\delta_{su}\\
-g_{tu,vw}g^{pq,rs}\delta_{iv}\delta_{jk}\delta_{lw}\delta_{pt}\delta_{qr}\delta_{su}\\
+2g_{tu,vw}g^{pq,rs}\delta_{iq}\delta_{ju}\delta_{kv}\delta_{lr}\delta_{pt}\delta_{sw}\\
+g_{tu,vw}g^{pq,rs}\delta_{iv}\delta_{jr}\delta_{kq}\delta_{lu}\delta_{pt}\delta_{sw}\\
-g_{tu,vw}g^{pq,rs}\delta_{iv}\delta_{jr}\delta_{kt}\delta_{lp}\delta_{qu}\delta_{sw}\\
-2g_{tu,vw}g^{pq,rs}\delta_{it}\delta_{jp}\delta_{kv}\delta_{lr}\delta_{qu}\delta_{sw}
\end{array}\right.$$

$$4R_{ij,kl}\langle 2\rangle = \sum_{\substack{1\le p<q\le n\\1\le r<s\le n\\1\le t<u\le n\\1\le v<w\le n}} \left\{ \begin{array}{l} -g_{vw,rs}g^{pq,rs}\delta_{iq}\delta_{ju}\delta_{ku}\delta_{lw}\delta_{pt}\delta_{tv} \\ +g_{vw,rs}g^{pq,rs}\delta_{it}\delta_{jp}\delta_{ku}\delta_{lw}\delta_{qu}\delta_{tv} \\ -g_{vw,pq}g^{pq,rs}\delta_{iu}\delta_{jw}\delta_{ks}\delta_{lu}\delta_{rt}\delta_{tv} \\ +g_{vw,ij}g^{pq,rs}\delta_{ks}\delta_{lu}\delta_{pu}\delta_{qw}\delta_{rt}\delta_{tv} \\ +g_{vw,pq}g^{pq,rs}\delta_{iu}\delta_{jw}\delta_{kt}\delta_{lr}\delta_{su}\delta_{tv} \\ -g_{vw,ij}g^{pq,rs}\delta_{kt}\delta_{lr}\delta_{pu}\delta_{qw}\delta_{su}\delta_{tv} \\ +g_{vw,kl}g^{pq,rs}\delta_{iq}\delta_{ju}\delta_{pt}\delta_{ru}\delta_{sw}\delta_{tv} \\ -g_{vw,kl}g^{pq,rs}\delta_{it}\delta_{jp}\delta_{qu}\delta_{ru}\delta_{sw}\delta_{tv} \\ +g_{vw,rs}g^{pq,rs}\delta_{iq}\delta_{ju}\delta_{kv}\delta_{lt}\delta_{pt}\delta_{uw} \\ -g_{vw,rs}g^{pq,rs}\delta_{it}\delta_{jp}\delta_{kv}\delta_{lt}\delta_{qu}\delta_{uw} \\ +g_{vw,pq}g^{pq,rs}\delta_{iv}\delta_{jt}\delta_{ks}\delta_{lu}\delta_{rt}\delta_{uw} \\ -g_{vw,ij}g^{pq,rs}\delta_{ks}\delta_{lu}\delta_{pv}\delta_{qt}\delta_{rt}\delta_{uw} \\ -g_{vw,kl}g^{pq,rs}\delta_{iq}\delta_{ju}\delta_{pt}\delta_{rv}\delta_{st}\delta_{uw} \\ +g_{vw,kl}g^{pq,rs}\delta_{it}\delta_{jp}\delta_{qu}\delta_{rv}\delta_{st}\delta_{uw} \\ -g_{vw,pq}g^{pq,rs}\delta_{iv}\delta_{jt}\delta_{kt}\delta_{lr}\delta_{su}\delta_{uw} \\ +g_{vw,ij}g^{pq,rs}\delta_{kt}\delta_{lr}\delta_{pv}\delta_{qt}\delta_{su}\delta_{uw} \end{array} \right.$$

$$4R_{ij,kl}\langle 3\rangle = \sum_{\substack{1\le a<b\le n\\ 1\le c<d\le n\\ 1\le p<q\le n\\ 1\le r<s\le n\\ 1\le t<u\le n\\ 1\le v<w\le n}} \left\{ \begin{array}{l}
-g_{mn,ab}g_{pq,ef}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{av}\delta_{bl}\delta_{cj}\delta_{df}\delta_{ei}\delta_{kw}\\
-g_{ij,ef}g_{mn,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{av}\delta_{bl}\delta_{cq}\delta_{df}\delta_{ep}\delta_{kw}\\
+g_{mn,ab}g_{pq,ef}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{av}\delta_{bl}\delta_{ce}\delta_{di}\delta_{fj}\delta_{kw}\\
+g_{ij,ef}g_{mn,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{av}\delta_{bl}\delta_{ce}\delta_{dp}\delta_{fq}\delta_{kw}\\
+g_{mn,ab}g_{pq,ef}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{ak}\delta_{bw}\delta_{cj}\delta_{df}\delta_{ei}\delta_{lv}\\
+g_{ij,ef}g_{mn,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{ak}\delta_{bw}\delta_{cq}\delta_{df}\delta_{ep}\delta_{lv}\\
-g_{mn,ab}g_{pq,ef}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{ak}\delta_{bw}\delta_{ce}\delta_{di}\delta_{fj}\delta_{lv}\\
-g_{ij,ef}g_{mn,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{ak}\delta_{bw}\delta_{ce}\delta_{dp}\delta_{fq}\delta_{lv}\\
+g_{kl,ef}g_{pq,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{av}\delta_{bn}\delta_{cj}\delta_{df}\delta_{ei}\delta_{mw}\\
-g_{kl,ab}g_{pq,ef}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{av}\delta_{bn}\delta_{cj}\delta_{df}\delta_{ei}\delta_{mw}\\
+g_{ij,ef}g_{pq,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{av}\delta_{bn}\delta_{cl}\delta_{df}\delta_{ek}\delta_{mw}\\
-g_{ij,ef}g_{kl,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{av}\delta_{bn}\delta_{cq}\delta_{df}\delta_{ep}\delta_{mw}\\
-g_{kl,ef}g_{pq,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{av}\delta_{bn}\delta_{ce}\delta_{di}\delta_{fj}\delta_{mw}\\
+g_{kl,ab}g_{pq,ef}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{av}\delta_{bn}\delta_{ce}\delta_{di}\delta_{fj}\delta_{mw}\\
-g_{ij,ef}g_{pq,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{av}\delta_{bn}\delta_{ce}\delta_{dk}\delta_{fl}\delta_{mw}\\
+g_{ij,ef}g_{kl,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{av}\delta_{bn}\delta_{ce}\delta_{dp}\delta_{fq}\delta_{mw}\\
-g_{kl,ef}g_{pq,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{am}\delta_{bw}\delta_{cj}\delta_{df}\delta_{ei}\delta_{nv}\\
+g_{kl,ab}g_{pq,ef}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{am}\delta_{bw}\delta_{cj}\delta_{df}\delta_{ei}\delta_{nv}\\
-g_{ij,ef}g_{pq,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{am}\delta_{bw}\delta_{cl}\delta_{df}\delta_{ek}\delta_{nv}\\
+g_{ij,ef}g_{kl,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{am}\delta_{bw}\delta_{cq}\delta_{df}\delta_{ep}\delta_{nv}\\
+g_{kl,ef}g_{pq,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{am}\delta_{bw}\delta_{ce}\delta_{di}\delta_{fj}\delta_{nv}\\
-g_{kl,ab}g_{pq,ef}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{am}\delta_{bw}\delta_{ce}\delta_{di}\delta_{fj}\delta_{nv}\\
+g_{ij,ef}g_{pq,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{am}\delta_{bw}\delta_{ce}\delta_{dk}\delta_{fl}\delta_{nv}\\
-g_{ij,ef}g_{kl,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{am}\delta_{bw}\delta_{ce}\delta_{dp}\delta_{fq}\delta_{nv}\\
+g_{kl,ef}g_{mn,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{av}\delta_{bq}\delta_{cj}\delta_{df}\delta_{ei}\delta_{pw}\\
+g_{ij,ef}g_{mn,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{av}\delta_{bq}\delta_{cl}\delta_{df}\delta_{ek}\delta_{pw}\\
-g_{kl,ef}g_{mn,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{av}\delta_{bq}\delta_{ce}\delta_{di}\delta_{fj}\delta_{pw}\\
-g_{ij,ef}g_{mn,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{av}\delta_{bq}\delta_{ce}\delta_{dk}\delta_{fl}\delta_{pw}\\
-g_{kl,ef}g_{mn,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{ap}\delta_{bw}\delta_{cj}\delta_{df}\delta_{ei}\delta_{qv}\\
-g_{ij,ef}g_{mn,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{ap}\delta_{bw}\delta_{cl}\delta_{df}\delta_{ek}\delta_{qv}\\
+g_{kl,ef}g_{mn,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{ap}\delta_{bw}\delta_{ce}\delta_{di}\delta_{fj}\delta_{qv}\\
+g_{ij,ef}g_{mn,ab}g_{rs,tu}g^{mn,pq}g^{rs,vw}g^{tu,cd}\delta_{ap}\delta_{bw}\delta_{ce}\delta_{dk}\delta_{fl}\delta_{qv}
\end{array}\right.$$

Let us describe how obtain something usable from this. First, the expressions simplify somewhat, due to the presence of myriad Kronecker deltas. For example,

$$\sum_{\substack{1\le p<q\le n\\ 1\le r<s\le n\\ 1\le t<u\le n\\ 1\le v<w\le n}} g_{tu,vw}g^{pq,rs}\delta_{il}\delta_{jw}\delta_{kv}\delta_{ps}\delta_{qu}\delta_{rt} = \delta_{il}\sum_{\substack{1\le p<q\le n\\ 1\le r<p\le n}} g_{rq,kj}g^{pq,rp}.$$

Even after simplifying each term in this way, the result is still hopelessly complicated. So, in order to analyze it effectively we must assume that the inital metric g is diagonal, and then show that the Ricci tensor stays diagonal. Then the above expressions can be simplified once again by dropping any terms that vanish due to off-diagonal metric factors. For example,

$$\delta_{il}\sum_{\substack{1\le p<q\le n\\ 1\le r<p\le n}} g_{rq,kj}g^{pq,rp} = \delta_{il}g_{kj,kj}g^{kj,kk}.$$

Again, we do this for each term, and we must show that the off-diagonal terms of the Ricci tensor vanish. The indices for such terms satisfy $i \neq k$ or $j \neq l$. There are several index cases that force terms to vanish:

(1) impossible indexing situations, e.g. $g_{ii,kl}$;
(2) both $i = k$ and $j = l$ appear in a factor, or both $i = l$ and $j = k$ appear in a factor;
(3) all four indices appear in one metric/metric inverse component.

Using this list, one can then see by inspection that $R_{ij,kl}\langle 1\rangle$, $R_{ij,kl}\langle 2\rangle$, and $R_{ij,kl}\langle 3\rangle$ vanish, so the natural basis for $\mathfrak{ut}_n\mathbb{R}$ (or the frame for $\mathrm{UT}_n\,\mathbb{R}$ from Lemma 4.3) is stably Ricci-diagonal. As before, this means g stays diagonal under the flow. To see what these diagonal terms are, just replace k with i and l with j. Most of the resulting terms contain factors that obviously vanish, according to the list above, and others can be combined. Once this is done, we drop back to two indices. That is, write g_{ij} and R_{ij} to mean $g_{ij,ij}$ and $R_{ij,ij}$, respectively. This yields

$$4R_{ij} = -2 \sum_{1\le p<i} \frac{g_{pj}}{g_{pi}} + 2g_{ij}^2 \sum_{i<q<j} \frac{1}{g_{iq}g_{qj}} - 2 \sum_{j<r\le n} \frac{g_{ir}}{g_{jr}}.$$

4.3. Ricci flow and the nilsoliton. With this part of the calculation complete, we see that the Ricci flow on $\mathrm{UT}_n\,\mathbb{R}$ is the system

$$\frac{d}{dt} g_{ij} = \sum_{1\le p<i} \frac{g_{pj}}{g_{pi}} - g_{ij}^2 \sum_{i<q<j} \frac{1}{g_{iq}g_{qj}} + \sum_{j<r\le n} \frac{g_{ir}}{g_{jr}}, \tag{4.5}$$

for $1 \le i < j \le n$.

Example 4.6. When $n = 3$, $\mathrm{UT}_3\,\mathbb{R}$ is the familiar Heisenberg group, Nil^3. The Ricci flow is then the system:

$$\frac{d}{dt} g_{12} = \frac{g_{13}}{g_{23}}, \quad \frac{d}{dt} g_{13} = -\frac{g_{13}^2}{g_{12}g_{23}}, \quad \frac{d}{dt} g_{23} = \frac{g_{13}}{g_{12}}.$$

If we set $A = g_{12}, B = g_{23}$, and $C = g_{13}$, then this becomes

$$\frac{d}{dt} A = \frac{C}{B}, \quad \frac{d}{dt} B = \frac{C}{A}, \quad \frac{d}{dt} C = -\frac{C^2}{AB},$$

which agrees with the equations from Example 2.1. (Those equations were ordered differently to agree with the pattern in Section 3.)

The goal is now to construct a nilsoliton on each space $\mathrm{UT}_n\,\mathbb{R}$. These exist by Lauret's theorem, 3.10 above. The Iwasawa decomposition of the general linear group is $\mathrm{GL}_n\,\mathbb{R} = KAN$, where $K = \mathrm{O}_n\,\mathbb{R}$, A is the abelian subgroup of diagonal matrices, and $N = \mathrm{UT}_n\,\mathbb{R}$. The quotient G/K is an irreducible symmetric space of non-compact type, and such spaces are all Einstein. But $G/K \cong AN$, whose Lie algebra is a metric solvable extension of $\mathfrak{ut}_n\mathbb{R}$. Thus, Lauret's theorem applies.

Now, due to the complexity of the system (4.5), we are unable to determine the asymptotics of an arbitrary diagonal solution. Thus, we cannot use the blowdown method of Section 2. Instead, we must make a suitable ansatz.

If we picture a diagonal metric as an upper triangular matrix with zeros on the diagonal (which is natural, given the indices), and extrapolate from low-dimensional cases, we might suspect that metric components along diagonals of the matrix have "the same" behavior, and that this behavior (with respect to time) changes in fixed increments from diagonal to diagonal. The components g_{ij} along any diagonal have the property that the quantity $j-i$ is constant. This means there should be $n-1$ "different" types of behavior.

We make the ansatz that the components of the solution corresponding to the soliton are of the form

$$g_{ij}(t) = a_{j-i}t^{1-2(j-i)/n},$$

for some constants a_{j-i} to be determined shortly. Then the right side of (4.5) becomes

$$\begin{aligned}
&\sum_{1\leq p<i} \frac{g_{pj}}{g_{pi}} - \sum_{i<q<j} \frac{g_{ij}^2}{g_{iq}g_{qj}} + \sum_{j<r\leq n} \frac{g_{ir}}{g_{jr}} \\
&= \sum_{1\leq p<i} \frac{a_{j-p}t^{1-2(j-p)/n}}{a_{i-p}t^{1-2(i-p)/n}} - \sum_{i<q<j} \frac{a_{j-i}^2 t^{2-4(j-i)/n}}{a_{q-i}t^{1-2(q-i)/n}a_{j-q}t^{1-2(j-q)/n}} \\
&\qquad + \sum_{j<r\leq n} \frac{a_{r-i}t^{1-2(r-i)/n}}{a_{r-j}t^{1-2(r-j)/n}} \\
&= \sum_{1\leq p<i} \frac{a_{j-p}}{a_{i-p}} t^{-2(i-p)/n} - \sum_{i<q<j} \frac{a_{j-i}^2}{a_{q-i}a_{j-q}} t^{-2(j-q)/n} + \sum_{j<r\leq n} \frac{a_{r-i}}{a_{r-j}} t^{-2(r-j)/n} \\
&= t^{-2(i-p)/2} \left(\sum_{1\leq p<i} \frac{a_{j-p}}{a_{i-p}} - \sum_{i<q<j} \frac{a_{j-i}^2}{a_{q-i}a_{j-q}} + \sum_{j<r\leq n} \frac{a_{r-i}}{a_{r-j}} \right).
\end{aligned}$$

The left side is

$$\frac{d}{dt} a_{j-i} t^{1-2(j-i)/2} = a_{j-i}\left(1 - \frac{2(j-i)}{n}\right) t^{-2(j-i)/2}.$$

The powers of t cancel, and so we must find a_{j-i} such that

$$a_{j-i}\left(1 - \frac{2(j-i)}{n}\right) = \sum_{1\leq p<i} \frac{a_{j-p}}{a_{i-p}} - \sum_{i<q<j} \frac{a_{j-i}^2}{a_{q-i}a_{j-q}} + \sum_{j<r\leq n} \frac{a_{r-i}}{a_{r-j}}.$$

For some $A > 0$, set

$$a_{j-i} = \frac{A^{j-i}}{n^{j-i-1}}.$$

Then the right side becomes

$$\begin{aligned}
&\sum_{1\leq p<i} \frac{a_{j-p}}{a_{i-p}} - \sum_{i<q<j} \frac{a_{j-i}^2}{a_{q-i}a_{j-q}} + \sum_{j<r\leq n} \frac{a_{r-i}}{a_{r-j}} \\
&= \sum_{1\leq p<i} \frac{A^{j-p}}{n^{j-p-1}} \frac{n^{i-p-1}}{A^{i-p}} - \sum_{i<q<j} \frac{A^{2j-2i}}{n^{2j-2i-2}} \frac{A^{q-i}A^{j-q}}{n^{q-i-1}n^{j-q-1}} + \sum_{j<r\leq n} \frac{A^{r-i}}{n^{r-i-1}} \frac{n^{r-j-1}}{A^{r-j}} \\
&= \sum_{1\leq p<i} \frac{A^{j-i}}{n^{j-i}} - \sum_{i<q<j} \frac{A^{j-i}}{n^{j-i}} + \sum_{j<r\leq n} \frac{A^{j-i}}{n^{j-i}} \\
&= \left(\frac{A}{n}\right)^{j-i} \left(\sum_{1\leq p<i} 1 - \sum_{i<q<j} 1 + \sum_{j<r\leq n} 1 \right) \\
&= \left(\frac{A}{n}\right)^{j-i} \big(n - 2(j-i)\big).
\end{aligned}$$

The left side is

$$a_{j-i}\left(1-\frac{2(j-i)}{n}\right)=\frac{A^{j-i}}{n^{j-i-1}}\left(1-\frac{2(j-i)}{n}\right)=\left(\frac{A}{n}\right)^{j-i}(n-2(j-i)),$$

as desired.

This means

$$g(t)=\frac{A^{j-i}}{n^{j-i-1}}t^{1-2(j-i)/n}\,\theta^{ij}\otimes\theta^{ij}$$

is a Ricci flow solution on $\mathrm{UT}_n\,\mathbb{R}$. To see that $g(1)$ is a soliton, we need to find diffeomorphisms η_t of $\mathrm{UT}_n\,\mathbb{R}$ such that

$$g(t)=t\eta_t^*g(1)$$

is also a Ricci flow solution. In something of a *deus ex machina*, we claim that these diffeomorphisms are of the form

$$(\eta_t(x))^{ij}=t^{-(j-i)/n}x^{ij},$$

for $x\in\mathrm{UT}_n\,\mathbb{R}$. Considering the coframe from Lemma 4.3, we see that

$$\begin{aligned}
\eta_t^*\Theta_p^{ij}&=\sum_{k=0}^{p}\sum_{i<r_1<\cdots<r_k<p}(x^{ir_1}\circ\eta_t)(x^{r_1r_2}\circ\eta_t)\cdots(x^{r_kp}\circ\eta_t)\\
&=\sum_{k=0}^{p}\sum_{i<r_1<\cdots<r_k<p}t^{-(r_1-i)/n}x^{ir_1}t^{-(r_2-r_1)/n}x^{r_1r_2}\cdots t^{-(p-r_k)/n}x^{r_kp}\\
&=\sum_{k=0}^{p}\sum_{i<r_1<\cdots<r_k<p}t^{-(p-r_k+\cdots-r_1+r_1-i)/n}x^{ir_1}x^{r_1r_2}\cdots x^{r_kp}\\
&=t^{-(p-i)/n}\Theta_p^{ij},
\end{aligned}$$

and so

$$\begin{aligned}
\eta_t^*\theta^{ij}&=d(x^{ij}\circ\eta_t)-\sum_{i<p<j}\left(2(x^{ip}\circ\eta_t)-(\Theta_p^{ij}\circ\eta_t)\right)d(x^{pj}\circ\eta_t)\\
&=t^{-(j-i)/n}dx^{ij}-\sum_{i<p<j}\left(2t^{-(p-i)/n}x^{pi}-t^{-(p-i)/n}\Theta_p^{ij}\right)t^{-(j-p)/n}dx^{pj}\\
&=t^{-(j-i)/n}\theta^{ij}.
\end{aligned}$$

Now we have

$$\begin{aligned}
t\eta_t^*g(1)&=t\eta_t^*(g(1))\,\eta_t^*\theta^{ij}\otimes\eta_t^*\theta^{ij}\\
&=\frac{A^{j-i}}{n^{j-i-1}}t^{1-2(j-i)/n}\,\theta^{ij}\otimes\theta^{ij}\\
&=g(t)
\end{aligned}$$

as required. Thus, $g(t)$ is an expanding Ricci soliton with respect to the diffeomorphisms $\{\eta_t\}$ just described.

Set $A=1$. To conclude, we have another theorem.

Theorem 3 (1.2)**.** *Let $\mathrm{UT}_n\,\mathbb{R}$ be the Lie group of real $n\times n$ unitriangular matrices, with coordinates as in (4.2) and coframe $\{\theta^{ij}\}$ as in Lemma 4.3. Then the family of metrics $g(t)=g_{ij,ij}(t)\,\theta^{ij}\otimes\theta^{ij}$, where*

$$g_{ij,ij}(t)=\frac{1}{n^{j-i-1}}t^{1-2(j-i)/n},$$

is a Ricci flow solution on $\mathrm{UT}_n\,\mathbb{R}$*. The metric* $g(1)$ *is a nilsoliton with respect to the diffeomorphisms* η_t*, where*

$$(\eta_t(x))^{ij} = t^{-(j-i)/n} x^{ij}.$$

Remark 4.7. It would appear that we have constructed a family of soliton metrics $\{g_A\}$ depending on the parameter A, but it is easy to see that there is a Lie algebra automorphism[3] Φ_A, such that $g_1(t) = \Phi_A \cdot g_A(1)$, which means they are equivalent as required by Theorem 3.5 in [24].

[3] An automorphism Φ acts on a left-invariant metric g by $\Phi \cdot g = g(\Phi^{-1}\cdot, \Phi^{-1}\cdot)$.

CHAPTER 3

Solsolitons derived from Heisenberg groups

1. Introduction

A common goal in the study of solitons is to find new examples –especially in families– and to describe them explicitly. For example, Payne has constructed continuous families of nilsolitons [35], and Kadioglu and Payne have made progress towards describing all algebras admitting nilsolitons with simple derivations in dimensions seven and eight [20]. Building on earlier work of Lauret and Will [28], Lafuente has constructed families of solsolitons associated with certain graphs [23]. Will has also used the results of Lauret to classify all solsolitons in dimensions less than seven [43]. In Chapter 2 we considered nilsolitons on the classical Heisenberg groups [44], and in this chapter we use the results of Lauret to describe solsoliton extensions of those nilsolitons.

1.1. Characterization of solsolitons. Lauret has obtained several strong results regarding the structure and uniqueness of solsolitons [27], and we review those results here.

If $\mathfrak{n}$ is a Lie algebra and $\mathfrak{a} \subset \mathrm{Der}(\mathfrak{n})$ is an abelian subalgebra, recall that the Lie bracket $[\cdot,\cdot]$ on the *semi-direct product* $\mathfrak{s} = \mathfrak{n} \rtimes \mathfrak{a}$ is

$$[X,Y] = [X,Y]_{\mathfrak{n}}, \quad [A,B] = [A,B]_{\mathfrak{a}} = 0, \quad [X,A] = A(X), \tag{1.1}$$

for all $X, Y \in \mathfrak{n}$, $A, B \in \mathfrak{a}$. If $\mathfrak{s}$ has an inner product $\langle\cdot,\cdot\rangle$, also recall that the *mean curvature vector* of $\mathfrak{s}$ is the unique $H \in \mathfrak{a}$ such that $\langle H, A\rangle = \operatorname{tr}\operatorname{ad} H$ for all $A \in \mathfrak{a}$.

Theorem 1.2 (The structure and uniqueness of solsolitons [27])**.**

(a) *Let $(\mathfrak{n}, \langle\cdot,\cdot\rangle_{\mathfrak{n}})$ be a nilsoliton with Ricci operator $\mathrm{Ric}_{\mathfrak{n}} = cI + D_{\mathfrak{n}}$, $c < 0$, and $D_{\mathfrak{n}} \in \mathrm{Der}(\mathfrak{n})$. Consider an abelian Lie algebra $\mathfrak{a}$ of $\langle\cdot,\cdot\rangle_{\mathfrak{n}}$-symmetric derivations of $\mathfrak{n}$. Then the solvmanifold S with Lie algebra $\mathfrak{s} = \mathfrak{n} \rtimes \mathfrak{a}$ and inner product*

$$\langle\cdot,\cdot\rangle_{\mathfrak{s}} = \langle\cdot,\cdot\rangle_{\mathfrak{n}} + \langle\cdot,\cdot\rangle_{\mathfrak{a}}, \quad \textit{where } \langle A,B\rangle_{\mathfrak{a}} = -\tfrac{1}{c}\operatorname{tr}(AB),$$

is a solsoliton with $\mathrm{Ric} = cI + D$. Here, $D \in \mathrm{Der}(\mathfrak{s})$ is defined by $D|_{\mathfrak{a}} = 0$, $D|_{\mathfrak{n}} = D_{\mathfrak{n}} - \operatorname{ad} H|_{\mathfrak{n}}$, and H is the mean curvature vector of S. Furthermore, S is Einstein if and only if $D_1 \in \mathfrak{a}$.

(b) *All solsolitons are of the form described in (a).*

(c) *Let S and S' be two solsolitons which are isomorphic as Lie groups. Then S is isometric to S' up to scaling.*

(d) *Let $(\mathfrak{n}, \langle\cdot,\cdot\rangle_{\mathfrak{n}})$ be a nilsoliton and consider two solsolitons S and S', constructed as in (a) from abelian Lie algebras $\mathfrak{a}$ and $\mathfrak{a}'$ of $\langle\cdot,\cdot\rangle_{\mathfrak{n}}$-symmetric derivations. Then S is isometric to S' if and only if there exists $h \in \mathrm{Aut}(\mathfrak{n}) \cap \mathrm{O}(\mathfrak{n}, \langle\cdot,\cdot\rangle_{\mathfrak{n}})$ such that $\mathfrak{a}' = h\mathfrak{a}h^{-1}$.*

Using these results, Will has described the spaces of solitons, up to scaling and isometry, on Lie algebras of low dimension. We quickly discuss a general phenomenon discussed in [43, Section 3]. Let $(\mathbb{R}^n, \mu)$ be a nilpotent Lie algebra with inner product $\langle \cdot, \cdot \rangle$. Let

$$\begin{aligned} G_\mu &= \{g \in \mathrm{Aut}(\mu) \mid g^t \in \mathrm{Aut}(\mu)\} = K_\mu \exp(\mathfrak{p}_\mu), \\ \mathfrak{g}_\mu &= \{A \in \mathrm{Der}(\mu) \mid A^t \in \mathrm{Der}(\mu)\} = \mathfrak{k}_\mu \oplus \mathfrak{p}_\mu. \end{aligned}$$

The group G_μ is real reductive, and the above decompositions are the Cartan decompositions, where[1]

$$\begin{aligned} K_\mu &= G_\mu \cap \mathrm{O}(n) = \mathrm{Aut}(\mu) \cap \mathrm{O}(n), \\ \mathfrak{k}_\mu &= \mathfrak{g}_\mu \cap \mathfrak{so}(n) = \mathrm{Der}(\mu) \cap \mathfrak{so}(n), \\ \mathfrak{p}_\mu &= \mathfrak{g}_\mu \cap \mathrm{sym}(n) = \mathrm{Der}(\mu) \cap \mathrm{sym}(n). \end{aligned}$$

Assume that $\langle \cdot, \cdot \rangle$ is a nilsoliton. By Lauret's work, solsoliton extensions of this nilsoliton are parametrized by K_μ-conjugacy classes of abelian subspaces of $\mathfrak{p}_\mu$. Let $\mathfrak{a}_\mu \subseteq \mathfrak{p}_\mu$ be a maximal abelian subalgebra. Define the *rank* of the nilpotent Lie algebra to be $\mathrm{rank}(\mu) = \dim(\mathfrak{a}_\mu)$ and let $0 < k \leq \mathrm{rank}(\mu)$.

Proposition 1.3 ([43]). *The set of $(n + k)$-dimensional solsoliton extensions of $(\mathbb{R}^n, \mu, \langle \cdot, \cdot \rangle)$, up to isometry and scaling, is parametrized by*

$$\mathrm{Gr}_k(\mathfrak{a}_\mu)/W_\mu,$$

where W_μ is the Weyl group of G_μ.

Here

$$\begin{aligned} N_{K_\mu}(\mathfrak{a}_\mu) &= \{g \in K_\mu \mid \mathrm{Ad}(g)\mathfrak{a}_\mu \subseteq \mathfrak{a}_\mu\}, \\ Z_{K_\mu}(\mathfrak{a}_\mu) &= \{g \in K_\mu \mid \mathrm{Ad}(g)A = A \text{ for all } A \in \mathfrak{a}_\mu\}, \\ W_\mu &= N_{K_\mu}(\mathfrak{a}_\mu)/Z_{K_\mu}(\mathfrak{a}_\mu). \end{aligned}$$

1.2. Summary of results. The goal of this chapter is to use the results of Lauret and Will to describe all the solsolitons that are constructed from nilsolitons on Heisenberg algebras. We also explicitly give many-parameter families of these solsolitons.

Theorem 1.4. *Let $N = 2n + 1$, and consider the Heisenberg algebra $\mathfrak{h}_N\mathbb{R}$ of dimension N with its unique nilsoliton. Up to isometry and scaling, the space of $(N + k)$-dimensional solsolitons constructed from $\mathfrak{h}_N\mathbb{R}$ according to Theorem 1.2 is given by*

$$\mathrm{Gr}_k \mathbb{R}^{n+1} \Big/ S_n \ltimes (\mathbb{Z}/2)^n,$$

where $1 \leq k \leq n + 1 = \mathrm{rank}(\mathfrak{h}_N\mathbb{R})$.

Corollary 1.5. *For fixed $n \geq 1$, there is a space, depending on*

$$\sum_{k=1}^{n} \dim \mathrm{Gr}_k \mathbb{R}^{n+1} = \frac{1}{6} n(n+1)(n+2)$$

parameters, of solsoliton extensions of $\mathfrak{h}_N\mathbb{R}$ with dimensions $N + k$, for $1 \leq k \leq n$.

---

[1]We write $\mathrm{O}(n), \mathfrak{so}(n), \mathrm{sym}(n)$ to refer to the orthogonal, skew-symmetric, and symmetric linear maps on $\mathbb{R}^n$, with respect to $\langle \cdot, \cdot \rangle$.

Theorem 1.6. *Given k parameters, for $1 \leq k \leq n$, it is possible to explicitly describe many of the solsolitons of Theorem 1.4, as well as their related geometric quantities. When $k = n+1$, there is a unique solsoliton, which is Einstein.*

Remark 1.7. Solsolitons arising from Heisenberg algebras have been previously explored in low dimensions. The graphs in [23] generate certain nilpotent Lie algebras, and the three-dimensional Heisenberg algebra is such an algebra. Also, the classification of solsolitons in low dimensions [43] includes the study of the three- and five-dimensional Heisenberg algebras.

2. Proof of the main theorem

In this section we give descriptions of the automorphisms groups and derivation algebras associated with the Heisenberg algebras. With this information, we compute the Weyl group mentioned in Proposition 1.3, which proves Theorem 1.4.

Recall the construction of the Heisenberg algebras $\mathfrak{h}_N\mathbb{R} = V \oplus Z$ from Section 3.1. For $V = W \oplus W$, let W have an inner product (written $\cdot$) that induces an inner product on V. Furthermore, suppose that $\mathfrak{h}_N\mathbb{R}$ has an inner product $\langle \cdot, \cdot \rangle$ whose restriction to V is the one just described (and not necessarily the nilsoliton inner product from Chapter 2), and for which $V \perp Z$.

2.1. Automorphisms. The condition for a map $\tau \in \mathrm{End}(\mathfrak{h}_N\mathbb{R})$ to be an automorphism is

$$\tau[\xi, \eta] = [\tau\xi, \tau\eta] \tag{2.1}$$

for all $\xi, \eta \in \mathfrak{h}_N\mathbb{R}$. *A priori*, we can write $\tau \in \mathrm{Aut}(\mathfrak{h}_N\mathbb{R})$ as

$$\tau = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \text{where} \quad \begin{aligned} \alpha &\colon V \longrightarrow V, & \beta &\colon Z \longrightarrow V, \\ \gamma &\colon V \longrightarrow Z, & \delta &\colon Z \longrightarrow Z. \end{aligned}$$

Since τ satisfies (2.1), and since Z is 1-dimensional, we know that τ must preserve Z. Therefore $\beta = 0$, and

$$\tau\xi = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} v \\ x \end{pmatrix} = \begin{pmatrix} \alpha(v) \\ \gamma(v) + \delta(x) \end{pmatrix}.$$

We must also have $\alpha \in \mathrm{GL}(V)$ and $\delta \in \mathrm{GL}(Z)$, since $\det(\tau) = \det(\alpha)\det(\delta) \neq 0$. We can say much more about elements of $\mathrm{Aut}(\mathfrak{h}_N\mathbb{R})$, however.

Theorem 2.2. *Each element of* $\mathrm{Aut}(\mathfrak{h}_N\mathbb{R})$ *can be written uniquely as*

$$\tau = \tau_1\tau_2\tau_3\tau_4,$$

where the τ_i are of the following form:

$$\tau_1 \in \{g_S \mid S \in \mathrm{Sp}(V, \omega)\}, \qquad \textit{where } g_S = \begin{pmatrix} S & 0 \\ 0 & \mathrm{id}_Z \end{pmatrix}$$

$$\tau_2 \in \{g_\gamma \mid \gamma \in \mathrm{Hom}(V, Z)\}, \qquad \textit{where } g_\gamma = \begin{pmatrix} \mathrm{id}_V & 0 \\ \gamma & \mathrm{id}_Z \end{pmatrix},$$

$$\tau_3 \in \{g_r \mid r > 0\}, \qquad \textit{where } g_r = \begin{pmatrix} r\,\mathrm{id}_V & 0 \\ 0 & r^2\,\mathrm{id}_Z \end{pmatrix},$$

$$\tau_4 \in \{\mathrm{id}, g_\iota\}, \qquad \textit{where } g_\iota = \begin{pmatrix} \iota & 0 \\ 0 & -\mathrm{id}_Z \end{pmatrix}.$$

The proof is a combination of [12, Theorem 1.22] and the fact that $\mathrm{Aut}(H_N\mathbb{R}) = \mathrm{Aut}(\mathfrak{h}_N\mathbb{R})$ (which is true for simply connected Lie groups and their Lie algebras). Next we consider the compact subgroup of orthogonal automorphisms.

Theorem 2.3. *Each $\tau \in \mathrm{Aut}(\mathfrak{h}_N\mathbb{R}) \cap \mathrm{O}(N)$ can be written uniquely as*

$$\tau = g_U g,$$

where $U \in \mathrm{U}(n)$ and $g \in \{\mathrm{id}, g_\iota\}$.

Proof. We take automorphisms of each type and see what happens when they are assumed to be orthogonal. First, suppose that for some $\tau \in \mathrm{Aut}(\mathfrak{h}_N\mathbb{R})$, we have

$$\langle \tau\xi, \tau\eta \rangle = \langle \xi, \eta \rangle \tag{2.4}$$

for all $\xi, \eta \in \mathfrak{h}_N\mathbb{R}$.

For $\tau = g_S$ with $S \in \mathrm{Sp}(V, \omega)$, (2.4) implies that

$$\langle Sv, Sv' \rangle_V = \langle v, v' \rangle_V \,,$$

which means $S \in \mathrm{O}(V) \cap \mathrm{Sp}(V, \omega) \cong \mathrm{U}(n)$.

For $\tau = g_\gamma$ with $\gamma \in \mathrm{Hom}(V, Z)$, (2.4) implies that

$$\langle \gamma(v), \gamma(v') \rangle_Z + \langle \gamma(v), z' \rangle_Z + \langle z, \gamma(v') \rangle_Z = 0.$$

Taking $z = 0$ gives $\gamma = 0$. Hence, the only orthogonal automorphism of the form g_γ is $g_0 = \mathrm{id}_{\mathfrak{h}_N\mathbb{R}}$.

For $\tau = g_r$ with $r > 0$, (2.4) implies that

$$(r^2 - 1) \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \right\rangle_V + (r^4 - 1)\langle z, z' \rangle_Z = 0.$$

If we take $x = x' \neq 0$, $y = y' = 0$, $z = z' \neq 0$, then we have a polynomial equation in r:

$$|z|^2 r^4 + |X_1|^2 r^2 - |x|^2 - |z|^2 = 0.$$

For (2.4) to hold, the polynomial equation must hold for all possible $|x|^2, |z|^2 \in \mathbb{R}^+$, which is clearly impossible. Hence, the only orthogonal automorphisms of the form g_r is is $g_1 = \mathrm{id}_{\mathfrak{h}_N\mathbb{R}}$.

Finally, the identity map is clearly orthogonal, and it is simple to check that g_ι is as well. □

Since $g_\iota^2 = \mathrm{id}$, we have $\{\mathrm{id}, g_\iota\} \cong \mathbb{Z}/2$. Identifying $\mathrm{U}(n)$ with the subgroup

$$\{g_U \mid U \in \mathrm{U}(n)\} \subset \mathrm{Aut}(\mathfrak{h}_N\mathbb{R}) \cap \mathrm{O}(N),$$

we can informally write

$$\mathrm{Aut}(\mathfrak{h}_N\mathbb{R}) \cap \mathrm{O}(N) \cong \mathrm{U}(n)\mathbb{Z}/2.$$

2.2. Derivations. The derivation algebra of $\mathfrak{h}_N\mathbb{R}$ is the Lie algebra of $\mathrm{Aut}(\mathfrak{h}_N\mathbb{R})$. We will need an explicit description of this space, so we think of a derivation $\phi \in \mathrm{Der}(\mathfrak{h}_N\mathbb{R})$ as an $N \times N$ matrix relative to the basis $\mathcal{E}_N$ that satisfies

$$\phi[E_I, E_J] = [\phi E_I, E_J] + [E_I, \phi E_J]. \tag{2.5}$$

If we let $\{E^I\}$ denote the dual basis, then we can write $\phi = \phi^I_J\, E_I \otimes E^J$. The division of $\mathcal{E}_N$ into three parts as in (3.1) gives ϕ the following structure:

$$\phi = \left(\begin{array}{c:c:c} \phi^i_j & \phi^i_{j+n} & \phi^i_N \\ \hdashline \phi^{i+n}_j & \phi^{i+n}_{j+n} & \phi^{i+n}_N \\ \hdashline \phi^N_j & \phi^N_{j+n} & \phi^N_N \end{array}\right).$$

Now (2.5) applied to each pair of basis elements gives us constraints on the matrix components,

$$\begin{aligned} \phi^{i+n}_j &= \phi^{j+n}_i, \\ \phi^j_{i+n} &= \phi^i_{j+n}, \\ \phi^{i+n}_{j+n} &= -\phi^j_i \\ \phi^N_N &= \phi^j_i + \phi^{i+n}_{j+n}, \\ 0 &= \phi^i_N = \phi^{i+j}_N. \end{aligned}$$

From this, we see that

$$\operatorname{Der}(\mathfrak{h}_N\mathbb{R}) \cong \left\{ \left(\begin{array}{c:c:c} A & B & \\ \hdashline C & cI_n - A^T & \\ \hdashline \vec{a}^T & \vec{b}^T & c \end{array}\right) \;\middle|\; \begin{array}{c} \vec{a}, \vec{b} \in \mathbb{R}^n, c \in \mathbb{R}, \\ A \in M_n\mathbb{R}, \\ B, C \in \operatorname{sym}(n) \end{array} \right\}.$$

The space has dimension $2n^2 + 3n + 1$.

We consider two subspaces of this algebra. First, the subalgebra of symmetric derivations is

$$\operatorname{Der}(\mathfrak{h}_N\mathbb{R}) \cap \operatorname{sym}(N) \cong \left\{ \left(\begin{array}{c:c:c} A & B & \\ \hdashline B & cI_n - A & \\ \hdashline & & c \end{array}\right) \;\middle|\; c \in \mathbb{R}, A, B \in \operatorname{sym}(n) \right\},$$

and it has dimension n^2+n+1. Second, the abelian algebra of diagonal derivations is

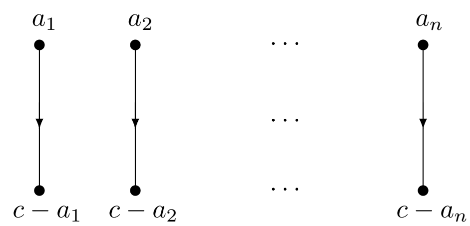
$$\begin{aligned} \mathfrak{d}(\mathfrak{h}_N\mathbb{R}) &= \left\{ \left(\begin{array}{c:c:c} A & & \\ \hdashline & cI_n - A & \\ \hdashline & & c \end{array}\right) \;\middle|\; c \in \mathbb{R}, A \text{ diagonal} \right\} \\ &= \{d(a_1, \dots, a_n, c - a_1, \dots, c - a_n, c) \mid a_i, c \in \mathbb{R}\} \\ &\subset \operatorname{Der}(\mathfrak{h}_N\mathbb{R}) \cap \operatorname{sym}(N), \end{aligned}$$

where d refers to the diagonal matrix with given diagonal entries. This is is a maximal abelian subalgebra of $\operatorname{Der}(\mathfrak{h}_N\mathbb{R})$, and therefore $\operatorname{rank}(\mathfrak{h}_N\mathbb{R}) = n + 1$.

2.3. The Weyl groups. In this section we compute the Weyl groups of $G_\mu \subset \operatorname{Aut}(\mathfrak{h}_N\mathbb{R})$, which we can think of as a subgroup of $\operatorname{Aut}(\mathfrak{h}_N\mathbb{R}) \cap \mathrm{O}(N)$. This will prove Theorem 1.4. One way to characterize the Weyl group is by its action as permutations, in this case on the elements of $\mathfrak{d}(\mathfrak{h}_N\mathbb{R})$. In other words, it consists of those orthogonal automorphisms that permute the entries of the diagonal derivations while perserving their overall structure. Consider one such derivation,

$$d(a_1, \ldots, a_n, c - a_1, \ldots, c - a_n, c) \in \mathfrak{d}(\mathfrak{h}_N\mathbb{R}).$$

To understand *all* possible structure-preserving permutations, we consider the equivalent problem of determining the automorphism group of the following graph $\mathcal{G}$.



We see that $\mathrm{Aut}(\mathcal{G})$ is generated by maps that exchange a_i and $c - a_i$ (i.e., reverse the orientation of an edge), and those that exchange a_i with a_j and $c - a_i$ with $c - a_j$ (i.e., tranpose two connected components). Thus $\mathrm{Aut}(\mathcal{G}) \cong S_n \ltimes (\mathbb{Z}/2)^n$, where S_n acts on $(\mathbb{Z}/2)^n$ by permuting indices. This automorphism group is of course a subgroup of S_{2n}, which can be thought of as permuting the $2n$ vertices of $\mathcal{G}$ (or the first $2n$ diagonal entries of elements of $\mathfrak{d}(\mathfrak{h}_N\mathbb{R})$). Writing transpositions as $\sigma_{i,j}$, we can describe the generators of $\mathrm{Aut}(\mathcal{G})$ described above as

$$\sigma_{i,n+i}, \quad \sigma_{i,j} \cdot \sigma_{n+i,n+j},$$

for $1 \leq i, j, \leq n$.

If we can show that the these generators can all be realized as elements of $\mathrm{U}(n)$, thought of as a subgroup of $\mathrm{Aut}(\mathfrak{h}_N\mathbb{R}) \cap \mathrm{O}(N)$, then this will prove that the Weyl group is indeed isomorphic to $S_n \ltimes (\mathbb{Z}/2)^n$. All of the generators can be realized as $2n \times 2n$ permutation matrices, but we must check that such matrices are actually unitary[2].

First, consider the generator $\sigma_{i,n+i}$ and its corresponding permutation matrix

$$P_{i,n+i} = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}.$$

Here, X is the identity matrix, except with $(i,i) = 0$, and Y is the zero matrix, except with $(i,i) = 1$. It is easy to see that

$$X^T X + Y^T Y = I, \quad X^T Y = Y^T X,$$

so $P_{i,n+i}$ is unitary.

Next, consider the $n \times n$ permutation matrix $P_{i,j}$ corresponding to $\sigma_{i,j}$. This is the identity matrix, except with $(i,i) = (j,j) = 0$ and $(i,j) = (j,i) = 1$. Then the permutation matrix corresponding to the generator $\sigma_{i,j} \cdot \sigma_{n+i,n+j}$ is

$$U_{i,j} = \begin{pmatrix} P_{i,j} & 0 \\ 0 & P_{i,j} \end{pmatrix}.$$

This is unitary since as $P_{i,j}$ is orthogonal.

[2] Recall that $\mathrm{U}(n) \cong \left\{ \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \;\middle|\; X, Y \in M_n\mathbb{R}, \begin{array}{c} X^T X + Y^T Y = I \\ X^T Y = Y^T X \end{array} \right\} \subset M_{2n}\mathbb{R}.$

Hence, the proof of Theorem 1.4 follows. Corollary 1.5 follows by recalling that $\dim \mathrm{Gr}_k \mathbb{R}^{n+1} = k(n+1-k)$. We have ignored the $k = n+1$ case, since it results in a single (Einstein) solsoliton. See [43, Section 3.2].

3. Constructing explicit solsolitons

In this section, we construct the many-parameter families of the solsolitons described above, as in Theorem 1.6. We exhibit the solsoliton inner products, mean curvature vectors, solsoliton derivations, Ricci endomorphisms, and scalar curvatures. For ease of notation, we now write Q for inner products, and Q_i for components with respect to a basis.

3.1. The nilsolitons. The Heisenberg algebras $\mathfrak{h}_N\mathbb{R}$ all admit nilsolitons, which were studied for arbitrary n in [44]. Here we recall those results. With respect to the basis $\mathcal{E}_N$, let us choose a diagonal inner product, which we can represent as

$$Q = \left(\begin{array}{c:c:c} Q_i & & \\ \hdashline & Q_{n+i} & \\ \hdashline & & Q_N \end{array}\right).$$

The corresponding Ricci $(2,0)$-tensor is

$$\mathrm{ric}_Q = -\frac{1}{2}\left(\begin{array}{c:c:c} Q^{n+i}Q_N & & \\ \hdashline & Q^i Q_N & \\ \hdashline & & -Q_N^2\Sigma \end{array}\right),$$

where

$$\Sigma = \sum_{k=1}^{n} \frac{1}{Q_k Q_{k+n}}. \tag{3.1}$$

The next proposition was originally phrased in terms of left-invariant metrics on the Heisenberg groups, but these are equivalent to inner products on the corresponding Lie algebras.

Proposition 3.2 ([44]). *Let $Q(t)$ be a Ricci flow solution on $\mathfrak{h}_N\mathbb{R}$, starting at a diagonal inner product Q_0.*

(a) *The solution $Q(t)$ has the following asymptotic behavior:*

$$Q_i \sim \gamma_i t^{1/n+2}$$
$$Q_N \sim \gamma_N t^{-n/n+2}$$

where $i = 1, \dots, 2n$, and the γs are constants depending only on Q_0 and n.

(b) *The solution $Q(t)$ converges, after pullback by automorphisms, to the solution $Q_\infty(t)$ corresponding to the inner product*

$$Q_\infty = \left(\begin{array}{c:c} I_{2n} & \\ \hdashline & \frac{1}{n+2} \end{array}\right),$$

which is a nilsoliton.

Remark 3.3. We could use the standard inner product here, since it is equivalent by automorphisms to Q_∞, but we would only save a constant factor by doing so.

The Ricci tensor corresponding to Q_∞ is

$$\operatorname{ric}_\infty = -\frac{1}{2}\frac{1}{n+2}\left(\begin{array}{c:c} I_{2n} & \\ \hdashline & -\frac{n}{n+2} \end{array}\right),$$

and the Ricci endomorphism is

$$\operatorname{Ric}_\infty = Q_\infty^{-1}\operatorname{ric}_\infty = -\frac{1}{2}\frac{1}{n+2}\left(\begin{array}{c:c} I_{2n} & \\ \hdashline & -n \end{array}\right).$$

3.2. The nilsoliton derivations. Recall that the algebraic soliton condition for a metric Lie algebras $(\mathfrak{g}, Q)$ is

$$\operatorname{Ric}_Q = cI + D,$$

for some $c \in \mathbb{R}$ and $D \in \operatorname{Der}(\mathfrak{g})$. Let $\{E_i\}$ be a basis for $\mathfrak{g}$. In components, this condition becomes

$$D_i^i = R_i^i - c = \frac{1}{Q_i}\operatorname{ric}_i - c.$$

For an endomorphism $D\colon \mathfrak{g} \to \mathfrak{g}$ to be a derivation, it must satisfy (2.5) for all elements of $\mathfrak{g}$. Since $D \in \operatorname{End}(\mathfrak{g}) \cong \mathfrak{g} \otimes \mathfrak{g}^*$, we have

$$D = D_j^i\, E_i \otimes E^j.$$

Using the basis and structure constants, we can translate the derivation condition into a system of equations involving the components of D:

$$\begin{aligned} D[E_i, E_j] &= [DE_i, E_j] + [E_i, DE_j] \\ \implies c_{ij}^k D_k^l E_l &= [D_i^k E_k, E_j] + [E_i, D_j^k, E_k] \\ &= c_{kj}^l D_i^k E_l + c_{ik}^l D_j^k E_l \\ \implies c_{ij}^k D_k^l &= c_{kj}^l D_i^k + c_{ik}^l D_j^k \end{aligned}$$

for all i, j, l. If we assume that D is diagonal, then this becomes

$$c_{ij}^k (D_i^i + D_j^j - D_k^k) = 0 \tag{3.4}$$

for all i, j, k.

For $\mathfrak{h}_N\mathbb{R}$, we have the following potential derivation:

$$D = \operatorname{Ric}_\infty - cI = \left(\begin{array}{c:c} \left(-\frac{1}{2}\frac{1}{n+2} - c\right) I_{2n} & \\ \hdashline & \frac{1}{2}\frac{n}{n+2} - c \end{array}\right).$$

Using (3.2) and (3.4), we require that

$$\begin{aligned} 0 &= c_{i,i+n}^N (D_i^i + D_{i+n}^{i+n} - D_N^N) \\ &= -\tfrac{1}{2}\tfrac{1}{n+2} - c - \tfrac{1}{2}\tfrac{1}{n+2} - c - \tfrac{1}{2}\tfrac{n}{n+2} + c \\ &= -\tfrac{1}{2} - c, \end{aligned}$$

so we must have $c = -1/2$ for D to be a derivation. Thus,

$$D = \frac{1}{2}\frac{n+1}{n+2}\left(\begin{array}{c:c} I_{2n} & \\ \hdashline & 2 \end{array}\right).$$

3.3. Subalgebras of $\mathfrak{d}(\mathfrak{h}_N\mathbb{R})$. To form solsoliton extensions of $\mathfrak{h}_N\mathbb{R}$, we need to consider abelian subalgebras of $\mathrm{Der}(\mathfrak{h}_N\mathbb{R}) \cap \mathrm{sym}(n)$. We saw that it was enough to consider subalgebras of the maximal abelian subalgebra of diagonal derivations,

$$\mathfrak{d}(\mathfrak{h}_N\mathbb{R}) = \{d(a_1, \ldots, a_n, c - a_1, \ldots, c - a_n, c) \mid a_i, c \in \mathbb{R}\}.$$

There is a basis $\mathcal{A}_n = \{A_1, \ldots, A_{n+1}\}$ for $\mathfrak{d}(\mathfrak{h}_N\mathbb{R})$, where if $\{C_{ij}\}$ is the standard basis of $M_n\mathbb{R}$, then

$$A_i = \left(\begin{array}{c:c:c} C_{ii} & & \\ \hdashline & -C_{ii} & \\ \hdashline & & 0 \end{array}\right), \qquad A_{n+1} = \left(\begin{array}{c:c:c} 0_n & & \\ \hdashline & I_n & \\ \hdashline & & 1 \end{array}\right).$$

Note that $\operatorname{tr} A_i = 0$ for $i = 1, \ldots, n$ and $\operatorname{tr} A_{n+1} = n + 1$.

We want subalgebras of $\mathfrak{d}(\mathfrak{h}_N\mathbb{R})$ that can be conveniently described in terms of this basis. There are natural subalgebras of dimension $1, \ldots, n + 1$ obtained by taking the span of subsets of $\mathcal{A}$. They are trivially isomorphic as Lie algebras when the dimensions are the same, and the inclusion of the vector A_{n+1} makes the resulting semi-direct product with $\mathfrak{h}_N\mathbb{R}$ non-unimodular. For $0 \le k \le n$, we will consider subalgebras spanned by vectors

$$A_1, \ldots, A_k, A_{n+1}.$$

In order to obtain families of solsolitons for each choice of spanning vectors, select parameters $r = (r_1, \ldots, r_{k+1}) \in \mathbb{R}^{k+1}$ with $r_i \neq 0$ for all i and let

$$\mathfrak{a}(r) = \mathrm{span}\{B_1 = t_1 A_1, \ldots, B_r = r_k A_k, B_{k+1} = r_{k+1} A_{n+1}\} \subseteq \mathfrak{d}(\mathfrak{h}_N\mathbb{R}).$$

When $k = n$, $\mathfrak{a}(r)$ is just $\mathfrak{d}(\mathfrak{h}_N\mathbb{R})$.

The inner product on $\mathfrak{a}(r)$ required by Theorem 1.2 is $Q_{\mathfrak{a}(r)} = 2\operatorname{tr}(AB)$, and it has components

$$\begin{aligned} Q_{\mathfrak{a}(r}(B_i, B_j) &= 4r_i r_j \delta_{ij}, \\ Q_{\mathfrak{a}(r)}(B_i, B_{n+1}) &= -2r_i r_{k+1}, \\ Q_{\mathfrak{a}(r)}(B_{n+1}, B_{n+1}) &= 2r_{k+1}^2(n+1). \end{aligned}$$

Set $E_{N+j} = B_j$ for $j = 1, \ldots, r + 1$. Then

$$\mathfrak{s}(r) = \mathfrak{h}_N\mathbb{R} \rtimes \mathfrak{a}(r)$$

has basis

$$\underbrace{E_1, \ldots, E_n}_{E_i}, \underbrace{E_{n+1}, \ldots, E_{2n}}_{E_{n+i}}, E_N, \underbrace{E_{N+1}, \ldots, E_{N+k}}_{E_{N+j}}, E_{N+k+1}$$

and the solsoliton inner product is

$$Q_{\mathfrak{s}(r)} = \left(\begin{array}{c:c:ccc:c} I_{2n} & & & & & \\ \hdashline & \frac{1}{n+2} & & & & \\ \hdashline & & 4r_1^2 & & & -2r_1r_{k+1} \\ & & & \ddots & & \vdots \\ & & & & 4r_k^2 & -2r_kr_{k+1} \\ \hdashline & & -2r_1r_{k+1} & \cdots & -2r_kr_{k+1} & 2r_{k+1}^2(n+1) \end{array}\right).$$

The Lie bracket on $\mathfrak{s}(r)$ is described in equation (1.1), and the non-zero structure constants are

$$\begin{aligned} c_{i,n+i}^{N} &= 1, \\ c_{j,N+j}^{j} &= r_j, \\ c_{n+j,N+j}^{n+j} &= -r_j, \\ c_{n+j,N+k+1}^{n+j} &= r_j, \\ c_{N,N+k+1}^{N} &= r_{k+1}. \end{aligned}$$

3.4. The mean curvature vectors. The mean curvature vector is $H \in \mathfrak{a}(r)$ such that

$$Q_{\mathfrak{s}(r)}(H, B) = \operatorname{tr}(\operatorname{ad} B)$$

for all $B \in \mathfrak{a}(r)$. Write $H = \sum_j H^j B_j$ and fix some B_l. Then we have

$$\begin{aligned} Q_{\mathfrak{s}(r)}(H, B_l) &= \sum_j H^j Q_{\mathfrak{s}(r)}(B_j, B_l) \\ &= \begin{cases} 2r_l(2r_lH^j - r_{k+1}H^{k+1}) & \text{if } l = 1, \dots, k \\ -2r_{k+1}\sum_j r_jH^j + 2(n+1)r_{k+1}^2H^{k+1} & \text{if } l = k+1 \end{cases}. \end{aligned}$$

Next, for $B \in \mathfrak{a}(r) \subset \operatorname{Der}(\mathfrak{h}_N\mathbb{R})$, we have $\operatorname{ad} B|_{\mathfrak{h}_N\mathbb{R}} = B$. Then

$$\operatorname{tr}\operatorname{ad} B_j = \begin{cases} 0 & \text{if } \beta = 1, \dots, k \\ (n+1)r_{k+1} & \text{if } \beta = k+1 \end{cases}.$$

To determine the components of H, we see that $Q_{\mathfrak{s}(r)}(H, B_l) = \operatorname{tr}(\operatorname{ad} B_l)$ gives

$$\begin{aligned} 2r_l(2r_lH^j - r_{k+1}H^{k+1}) = 0, \quad \forall \beta = 1, \dots, n, \\ -2r_{k+1}(r_1H^1 + \cdots + r_kH^k) + 2(n+1)r_{k+1}^2H^{k+1} = (n+1)r_{k+1}. \end{aligned}$$

Solving these equations gives

$$\begin{aligned} H^j &= \frac{n+1}{2(n+1)-k}\frac{1}{2r_j}, \quad \forall j = 1, \dots, k, \\ H^{n+1} &= \frac{n+1}{2(n+1)-k}\frac{1}{r_{k+1}}. \end{aligned}$$

Since it occurs frequently, we set

$$\eta_{n,k} = \frac{n+1}{2(n+1)-k}.$$

As an $N \times N$ matrix,

$$H = \eta_{n,k} \left(\begin{array}{ccc|c|c}
\ddots & & & & \\
 & \frac{1}{2r_j} & & & \\
 & & \ddots & & \\
\hline
 & & & 0_{N-k-1} & \\
\hline
 & & & & \frac{1}{r_{k+1}}
\end{array}\right) \in \mathfrak{a}(r).$$

The upper left block is $k \times k$.

3.5. The Ricci endomorphisms. By Theorem 1.2 (a), we have

$$\mathrm{Ric}_{\mathfrak{s}(r)} = -\tfrac{1}{2} I + D,$$

where $D \in \mathrm{Der}(\mathfrak{s}(r))$ satisfies

$$D|_{\mathfrak{a}(r)} = 0, \quad D|_{\mathfrak{h}_N\mathbb{R}} = D_{\mathfrak{h}_N\mathbb{R}} - \mathrm{ad}\, H|_{\mathfrak{h}_N\mathbb{R}}.$$

This gives

$$D = \left(\begin{array}{ccc|c|c|c}
\ddots & & & & & \\
 & \frac{1}{2}\frac{n+1}{n+2} - \frac{\eta_{n,k}}{2r_j} & & & & \\
 & & \ddots & & & \\
\hline
 & & & \frac{1}{2}\frac{n+1}{n+2} I_{N-k-1} & & \\
\hline
 & & & & \frac{n+1}{n+1} - \frac{\eta_{n,k}}{r_{k+1}} & \\
\hline
 & & & & & 0_{k+1}
\end{array}\right).$$

Note that

$$\mathrm{tr}\, D = \frac{N}{2}\frac{n+1}{n+2} - \eta_{n,k}\left(\sum_{j=1}^{k} \frac{1}{2r_j} + \frac{1}{r_{k+1}}\right).$$

The Ricci endomorphism is

$$\mathrm{Ric}_{\mathfrak{s}(r)} = \left(\begin{array}{ccc|c|c|c}
\ddots & & & & & \\
 & -\frac{1}{2(n+2)} - \frac{\eta_{n,k}}{2r_j} & & & & \\
 & & \ddots & & & \\
\hline
 & & & -\frac{1}{2(n+2)} I_{N-k-1} & & \\
\hline
 & & & & \frac{n}{2(n+2)} - \frac{\eta_{n,k}}{r_{k+1}} & \\
\hline
 & & & & & -\frac{1}{2} I_{k+1}
\end{array}\right).$$

The scalar curvature is the trace of this matrix. It is

$$\begin{aligned}\mathrm{scal}_{\mathfrak{s}(r)} &= \mathrm{tr}(-\tfrac{1}{2}I_{N+k+1} + D)\\ &= -\frac{1}{2}(N+k+1) + \frac{N}{2}\frac{n+1}{n+2} - \eta_{n,k}\sum_{j=1}^{n}\frac{1}{2r_j} + \frac{\eta_{n,k}}{r_{k+1}}\\ &= -\frac{k}{2} - \frac{3}{2}\frac{n+1}{n+2} - \eta_{n,k}\sum_{j=1}^{n}\frac{1}{2r_j} + \frac{\eta_{n,k}}{r_{k+1}}.\end{aligned}$$

3.6. Distinguishing the solsolitons. We normalize such that the scalar curvature equals -1 by choosing r_{k+1} appropriately. Namely,

$$r_{k+1} = -\frac{\eta_{n,k}}{\dfrac{k}{2} + \dfrac{3}{2}\dfrac{n+1}{n+2} + \eta_{n,k}\displaystyle\sum_{j=1}^{n}\frac{1}{2r_j} - 1}.$$

Of course, we must also select the r_j such that the denominator is nonzero, but this is an open condition. Hence we only lose one degree of freedom in this process, which is why we originally chose $k+1$ parameters.

Let us now consider the eigenvalues of Ric:

Eigenvalue	multiplicity
λ_j	1
$\frac{1}{2}\frac{n+1}{n+2}$	$N-k-1$
$\frac{n+1}{n+2} - \frac{\eta_{n,k}}{n+2}$	1
$-\frac{1}{2}$	$k+1$

where

$$\lambda_j = -\frac{1}{2}\frac{1}{n+2} - \frac{\eta_{n,k}}{2r_j}, \qquad j = 1, \dots, k.$$

The eigenvalues of the Ricci endomorphism are isometric invariants. Since there is a $(k+1)$-parameter family of such eigenvalues, there is a $(k+1)$-parameter family of non-isometric solsoliton inner products. Once the scalar curvature is fixed, at the cost of 1 parameter, the inner products do not differ by a scalar factor. This allows us to use the contrapositive of Theorem 1.2 (c). That is, if the solsoliton inner products are not isomorphic up to scaling, then the Lie algebras on which they live are not isomorphic.

We summarize what we have shown in this section.

Theorem 3.5. *For each $k = 1, \dots, n$, we have described a k-parameter family of distinct solsoliton inner products on the Lie algebras $\mathfrak{s}(r) = \mathfrak{h}_N\mathbb{R} \rtimes \mathfrak{a}(r)$, where $r \in \mathbb{R}^{k+1}$ is chosen so that*

- *$\mathfrak{a}(r)$ is a k-dimensional abelian subalgebra of $\mathrm{Der}(\mathfrak{h}_N\mathbb{R}) \cap \mathrm{sym}(n)$,*
- *$\mathrm{scal}_{\mathfrak{s}(r)} = -1$, and*
- *the eigenvalues of $\mathrm{Ric}_{\mathfrak{s}(r)}$ that depend on r are distinct.*

Since the solsolitons are Einstein if and only if the nilsoliton derivation is contained in the abelian subalgebra of derivations, in our construction Einstein solsolitons occur only when $k = n$, in which case there is one and it is unique.

CHAPTER 4

Results on coupled Ricci and harmonic map flows

1. Introduction and summary of results

As described in Section 3, in this chapter we study the coupling of Ricci flow and the harmonic map flow. There are two main setting in which this occurs. First is in the context of locally $\mathbb{R}^N$-invariant Ricci flow solutions on twisted principal bundles, where Ricci flow turns out to be equivalent to a coupled flow. Second is the general coupling of the two flows. Here we give precise statements of the results of this chapter.

In [21], Knopf proved convergence and stability of certain Ricci flow solutions in Lott's twisted principal bundle context, where the dimension of the total space is three. In Section 2, we review the necessary constructions and extend the result to arbitrary dimension in some cases. See Equation (2.2) for the locally $\mathbb{R}^N$-invariant Ricci flow system.

Theorem 1.1. *Let $\mathbf{g} = (g, A, G)$ be a locally $\mathbb{R}^N$-invariant metric on a product $\mathbb{R}^N \times \mathcal{B}$, where $\mathcal{B}$ is compact and orientable. Suppose that A vanishes and G is constant, and that either*

(i) *g has constant sectional curvature $-1/2(n-1)$, or*
(ii) *$\mathcal{B} = \mathcal{S}^2$ and g has constant positive sectional curvature.*

Then for any $\rho \in (0,1)$, there exists $\theta \in (\rho, 1)$ such that the following holds.

There exists a $(1+\theta)$ little-Hölder neighborhood $\mathcal{U}$ of $\mathbf{g}$ such that for all initial data $\widetilde{\mathbf{g}}(0) \in \mathcal{U}$, the unique solution $\widetilde{\mathbf{g}}(t)$ of rescaled locally $\mathbb{R}^N$-invariant Ricci flow exists for all $t \geq 0$ and converges exponentially fast in the $(2+\rho)$-Hölder norm to a limit metric $\mathbf{g}_\infty = (g_\infty, A_\infty, G_\infty)$ such that A_∞ vanishes, G_∞ is constant, and

in case (i), g_∞ is hyperbolic, and
in case (ii), g_∞ has constant positive sectional curvature.

In Section 3, we consider the general coupling of Ricci and harmonic map flows, and prove a version of Hamilton's compactness theorem in this context. We also prove a version of the theorem for étale Riemannian groupoids, a setting which is well-suited for discussion of convergence of solutions under the flow, especially when collapse is involved. See Equation (3.2) for the $(RH)_c$ flow.

Theorem 1.2. *Let $\{(\mathcal{M}_k^n, g_k(t), \phi_k(t), O_k)\}$ be a sequence of complete, pointed $(RH)_c$ flow solutions, with $0, t \in (\alpha, \omega)$, $c(t)$ non-increasing, and $\phi_k(t)$ mapping $\mathcal{M}_k$ into a closed Riemannian manifold $(\mathcal{N}, h)$, such that*

(a) *the geometry is uniformly bounded: for all k,*

$$\sup_{(x,t) \in \mathcal{M}_k \times (\alpha,\omega)} |\operatorname{Rm}_k|_k \leq C_1$$

for some C_1 independent of k;

(b) *the initial injectivity radii are uniformly bounded below: for all k,*

$$\mathrm{inj}_{g_k(0)}(O_k) \geq \iota_0 > 0,$$

for some ι_0 independent of k.

Then there is a subsequence such that

$$\big(\mathcal{M}_k, g_k(t), \phi_k(t), O_k\big) \longrightarrow \big(\mathcal{M}_\infty, g_\infty(t), \phi_\infty(t), O_\infty\big),$$

where the limit is also a pointed, complete, $(RH)_c$ flow solution.

If we do not assume any injectivity radius bound, then we have convergence to

$$\big(\mathcal{G}_\infty, g_\infty(t), \phi_\infty(t), O_\infty\big),$$

a complete, pointed, n-dimensional, étale Riemannian groupoid with map ϕ_∞ on the base.

We conclude with a detailed example of $(RH)_c$ solutions on the Lie group Nil^3, where the metrics are left-invariant and the map is a harmonic real-valued function. The behavior of these solutions depends strongly on the coupling function, although it is similar to that of Ricci flow solutions if the function decays fast enough as $t \to \infty$.

2. Locally $\mathbb{R}^N$-invariant Ricci flow

2.1. Setup. The manifolds that we will consider in this section have a special bundle structure. Let $\mathcal{B}$ be a connected, oriented, compact manifold, and let $\mathcal{E} \xrightarrow{p} \mathcal{B}$ be a flat $\mathbb{R}^N$-vector bundle. We consider $\mathcal{M}$ to be a principal $\mathbb{R}^N$-bundle over $\mathcal{B}$, twisted by $\mathcal{E}$. That is, there exists a smooth map

$$\mathcal{E} \times_{\mathcal{B}} \mathcal{M} = \bigcup_{b \in \mathcal{B}} \mathcal{E}_b \times \mathcal{M}_b \longrightarrow \mathcal{M}$$

that, over each point $b \in \mathcal{B}$, gives a free and transitive action that is consistent with the flat connection on $\mathcal{E}$. This means that if $\mathcal{U} \subset \mathcal{B}$ is such that $\mathcal{E}_{\mathcal{U}} \to \mathcal{U}$ is trivializiable, then $\pi^{-1}(\mathcal{U})$ has a free $\mathbb{R}^N$ action. Let $\mathcal{M}$ have a connection A such that $A|_{\pi^{-1}(U)}$ is an $\mathbb{R}^N$-valued connection. If we assume that $\mathcal{M}$ also has a flat connection itself, then A is an $\mathbb{R}^N$-valued 1-form.

We will use this bundle structure to describe local coordinates for $\mathcal{M}$. Let $\mathcal{U} \subseteq \mathcal{B}$ be an open set such that $\mathcal{E}_{\mathcal{U}} \to \mathcal{U}$ is trivializable and has a local section $\sigma \colon \mathcal{U} \to \pi^{-1}(\mathcal{U})$. Additonally, let $\rho \colon \mathbb{R}^n \to \mathcal{U}$ be a parametrization of $\mathcal{U}$, with coordinates x^α, and let e_i be a basis for $\mathbb{R}^N$. Then we obtain coordinates (x^α, x^i) on $\pi^{-1}(\mathcal{U})$ via

$$\begin{aligned} \mathbb{R}^n \times \mathbb{R}^N &\longrightarrow \pi^{-1}(\mathcal{U}) \\ (x^\alpha, x^i) &\longmapsto (x^i e_i) \cdot \sigma\big(\rho(x^\alpha)\big) \end{aligned}$$

where $\cdot$ denotes the free $\mathbb{R}^N$-action described above.

Let $\mathbf{g}$ be a Riemannian metric on $\mathcal{M}$ such that the $\mathbb{R}^N$-action is a local isometry. With respect to the coordinates above, one may write

$$\begin{aligned} \mathbf{g} &= \sum_{\alpha,\beta=1}^{n} g_{\alpha\beta}\, dx^\alpha\, dx^\beta + \sum_{i,j=1}^{N} G_{ij}\big(dx^i + \textstyle\sum_{\alpha=1}^n A^i_\alpha\, dx^\alpha\big)\big(dx^j + \textstyle\sum_{\beta=1}^n A^j_\beta\ , dx^\beta\big) \\ &= g_{\alpha\beta}\, dx^\alpha\, dx^\beta + G_{ij}(dx^i + A^i_\alpha\, dx^\alpha)(dx^j + A^j_\beta\ , dx^\beta). \end{aligned} \tag{2.1}$$

We will write this informally as $\mathbf{g} = (g, A, G)$, where $g(b) = g_{\alpha\beta}(b)\, dx^\alpha\, dx^\beta$ is locally a Riemannian metric on $\mathcal{U} \subset \mathcal{B}$, $A(b) = A^i_\alpha(b)\, dx^\alpha$ is locally the pullback by σ of a connection on $\pi^{-1}(\mathcal{U}) \to \mathcal{U}$, and $G(b) = G_{ij}(b)\, dx^i\, dx^j$ is an inner product on the fiber $\mathcal{M}_b$.

2.2. The rescaled flow. In [31], Lott considered metrics of the form (2.1) that evolve under Ricci flow, which are called *locally* $\mathbb{R}^N$*-invariant* solutions. He showed that the Ricci flow equation for $(M, \mathbf{g})$ becomes three equations: one for each of g, A, and G (see [31, Equation (4.10)]). To study the asymptotic stabilty of this system, Knopf transformed it into an equivalent one that has legitimate fixed points (see [21, Equation (1.3)]). Let $s(t)$ be a function and c a constant. Then the transformed system is [1]

(2.2a)
$$\frac{\partial}{\partial t} g_{\alpha\beta} = -2R_{\alpha\beta} + \frac{1}{2} G^{ik} G^{j\ell} \nabla_\alpha G_{ij} \nabla_\beta G_{k\ell} + g^{\gamma\delta} G_{ij} (dA)^i_{\alpha\gamma} (dA)^j_{\beta\delta} - s g_{\alpha\beta},$$
(2.2b)
$$\frac{\partial}{\partial t} A^i_\alpha = -(\delta dA)^i_\alpha + g^{\beta\gamma} G^{ij} \nabla^\gamma G_{jk} (dA)^k_{\beta\alpha} - \frac{1+c}{2} s A^i_\alpha,$$
(2.2c)
$$\frac{\partial}{\partial t} G_{ij} = \Delta G_{ij} - g^{\alpha\beta} G^{k\ell} \nabla_\alpha G_{ik} \nabla^\beta G_{\ell j} - \frac{1}{2} g^{\alpha\gamma} g^{\beta\delta} G_{ik} G_{j\ell} (dA)^k_{\alpha\beta} (dA)^\ell_{\gamma\delta} + cs G_{ij}.$$

We call this system a *rescaled locally* $\mathbb{R}^N$*-invariant Ricci flow.*

The case where the bundle connection is flat (i.e., A vanishes) was studied in [30], in the context of structures that arise from certain expanding Ricci solitons on low-dimensional manifolds. There and in the more general setting, certain Ricci flow solutions give rise to a (twisted) harmonic map $G\colon \mathcal{B} \to \mathrm{SL}(N, \mathbb{R})/\mathrm{SO}(N)$ (the target being the space of symmetric positive-definite bilinear forms of fixed determinant) together with a “soliton-like” equation relating the metrics g and G. These are the *harmonic-Einstein equations.*

We will need a byproduct of this fact. Write $\mathcal{S}_N = \mathrm{SL}(N, \mathbb{R})/\mathrm{SO}(N)$. The tangent space $T_G\mathcal{S}_N$ at $G \in \mathcal{S}_N$ consists of symmetric bilinear forms with no trace. There is a Riemannian metric on $T_G\mathcal{S}_N$ defined by

$$\bar{g}_G(X, Y) = \mathrm{tr}(G^{-1} X G^{-1} Y) = G^{ij} X_{jk} G^{k\ell} Y_{\ell i}. \tag{2.3}$$

The tension field of $G\colon \mathcal{B} \to \mathcal{S}_N$, with respect to the metrics g and $\bar{g}$, has components

$$\tau_{g,\bar{g}}(G)_{ij} = \Delta G_{ij} + g^{\alpha\beta} \sum_{\substack{p<q \\ r<s}}^{N} ({}^{\mathcal{S}_N}\Gamma \circ G)^{ij}_{pq,rs} \nabla_\alpha G_{pq} \nabla_\beta G_{rs}. \tag{2.4}$$

The reader is invited to compare this definition to the general formulation in (3.1) below.

Proposition 2.5. *The evolution equation for G from* (2.2) *is a modified harmonic map flow for $G\colon \mathcal{B} \to \mathcal{S}_N$. More precisely,*
$$\frac{\partial}{\partial t} G_{ij} = \tau_{g(t),\bar{g}}(G)_{ij} - \frac{1}{2} g^{\alpha\gamma} g^{\beta\delta} G_{ik} G_{j\ell} (dA)^k_{\alpha\beta} (dA)^\ell_{\gamma\delta} + cs G_{ij}.$$

[1] Here $\nabla_\alpha = \frac{\partial}{\partial x^\alpha}$, $(dA)^i_{\alpha\beta} = \nabla_\alpha A^i_\beta - \nabla_\beta A^i_\alpha$, $(\delta dA)^i_\alpha = -g^{\beta\gamma} \nabla^\gamma (dA)^i_{\beta\alpha}$, and $\Delta G_{ij} = g^{\alpha\beta} \nabla_\alpha \nabla_\beta G_{ij} = g^{\alpha\beta}(\frac{\partial^2}{\partial x^\alpha \partial x^\beta} G_{ij} - \Gamma^\gamma_{\alpha\beta} \frac{\partial}{\partial x^\gamma} G_{ij})$, where Γ represents the Christoffel symbols of g.

PROOF. What we are really claiming is that

$$\Delta G_{ij} - g^{\alpha\beta} G^{k\ell} \nabla_\alpha G_{ik} \nabla_\beta G_{\ell j} = \tau_{g,\overline{g}}(G)_{ij}. \tag{2.6}$$

The map G has energy

$$E(G) = \frac{1}{2} \int_B g^{\alpha\beta} \operatorname{tr}(G^{-1} \nabla_\alpha G^{-1} \nabla_\beta G)\, dV.$$

In [30, Proposition 4.17] it is shown that the variational equation of this energy is precisely

$$\Delta G_{ij} - g^{\alpha\beta} G^{k\ell} \nabla_\alpha G_{ik} \nabla_\beta G_{\ell j} = 0.$$

It follows from general harmonic map theory that, as it came from the variation of the energy of a map, the quantity on the left must be the tension field $\tau_{g(t),\overline{g}}(G)$ from (2.4). $\square$

Remark 2.7. It is possible to verify equation (2.6) directly with a (lengthy) computation.

Now, we want to describe the fixed points of (2.2), with the proper choices of s and c. There are two cases that we will consider.

Lemma 2.8. *Let $(\mathbb{R}^N \times \mathcal{B}, \mathbf{g}(t))$ be a Riemannian product that solves (2.2), such that (B, g) is nonflat and Einstein. Choose coordinates so that G is constant, $A = 0$, and $g(t) = -Kt\, g(-K)$, where $K = \pm 1$ is the Einstein constant such that $2\operatorname{Rc}[g(-K)] = Kg(-K)$. (Note that $g(t)$ exists for $t < 0$ if $K = 1$ and for $t > 0$ if $K = -1$.) The choices $s = -K$ and $c = 0$ make $\mathbf{g}(0)$ a stationary solution for (2.2).*

With these choices, we call (2.2) the *K-rescaled locally $\mathbb{R}^N$-invariant Ricci flow* system.

Next, given any smooth function $f\colon \mathcal{B} \to \mathbb{R}$, we define

$$\fint_{\mathcal{B}} f \, \mathrm{d}\mu = \frac{\int_{\mathcal{B}} f \, \mathrm{d}\mu}{\int_{\mathcal{B}} \mathrm{d}\mu}.$$

Let $V(t)$ denote the volume of $(\mathcal{B}, g(t))$ and define

$$r = R - \frac{1}{4} |\nabla G|^2 - \frac{1}{2} |dA|^2 ,$$

where everything is computed with respect to g. Because

$$\frac{dV}{dt} = -\int_{\mathcal{B}} r \, \mathrm{d}\mu - \frac{n}{2} s V(t),$$

it follows that V is fixed if and only if

$$s = -\frac{2}{n} \fint_{\mathcal{B}} r \, \mathrm{d}\mu. \tag{2.9}$$

Lemma 2.10. *Let $(\mathbb{R}^N \times \mathcal{B}, \mathbf{g}(t))$ be a Riemannian product that solves (2.2), such that (B, g) is Einstein. Choose coordinates such that G is constant and $A = 0$. For any t_0 in its time domain of existence, taking $c = 0$ and s as in (2.9) makes $\mathbf{g}(0) = (\sigma^{-1}(t_0) g(t_0), 0, G)$ into a stationary solution of (2.2) for any choice of positive antiderivative $\sigma(t)$ of s.*

With these choices, we call (2.2) the *volume-rescaled locally* $\mathbb{R}^N$*-invariant Ricci flow* system.

In this section, we prove the following results that imply convergence in the little-Hölder spaces as defined in [21]. It is a generalization of [21, Theorems 1 & 2].

Theorem 2.11. *Let* $\mathbf{g} = (g, A, G)$ *be a locally* $\mathbb{R}^N$*-invariant metric of the form (2.1) on a product* $\mathbb{R}^N \times \mathcal{B}$*, where* $\mathcal{B}$ *is compact and orientable. Suppose that* A *vanishes and* G *is constant, and that either*

(i) *g has constant sectional curvature* $-1/2(n-1)$*, or*
(ii) $\mathcal{B} = \mathcal{S}^2$ *and g has constant positive sectional curvature.*

Then for any $\rho \in (0,1)$*, there exists* $\theta \in (\rho, 1)$ *such that the following holds.*

There exists a $(1+\theta)$ *little-Hölder neighborhood* $\mathcal{U}$ *of* $\mathbf{g}$ *such that for all initial data* $\widetilde{\mathbf{g}}(0) \in \mathcal{U}$*, the unique solution* $\widetilde{\mathbf{g}}(\tau)$ *of (2.2) exists for all* $t \geq 0$ *and converges exponentially fast in the* $(2+\rho)$*-Hölder norm to a limit metric* $\mathbf{g}_\infty = (g_\infty, A_\infty, G_\infty)$ *such that* A_∞ *vanishes,* G_∞ *is constant, and*

in case (i), with choices of c and s as in Lemma 2.8, g_∞ *is hyperbolic, and*

in case (ii), with choices of c and s as in Lemma 2.10, g_∞ *has constant positive sectional curvature.*

2.3. Linearization at a stationary solution of rescaled flow. Consider a fixed point of the flow (2.2) on a Riemannian product $(\mathbb{R}^N \times \mathcal{B}, \mathbf{g})$. From Lemmas (2.8) and (2.10), we can assume that g is Einstein with $2\operatorname{Rc}(g) = Kg$, A is identically zero, and G is constant. Also, $c = 0$ and s is a (known) function.

To analyze the stability near a fixed point, we must compute the linearization of the flow. Write $\mathbf{g}_0 = (g_0, 0, G_0)$ for such a fixed point. Let

$$\tilde{\mathbf{g}}(\epsilon) = \big(\tilde{g}(\epsilon), \tilde{A}(\epsilon), \tilde{G}(\epsilon)\big) \tag{2.12}$$

be a variation of $\mathbf{g}$ such that

$$\tilde{\mathbf{g}}(0) = \mathbf{g}_0, \quad \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \tilde{\mathbf{g}} = \mathbf{h} = (h, B, F). \tag{2.13}$$

Let Δ_ℓ denote the Lichnerowicz Laplacian acting on symmetric $(2,0)$-tensor fields. In coordinates,

$$\Delta_\ell h_{ij} = \Delta h_{ij} + 2R_{ipqj}h^{pq} - R_i^k h_{kj} - R_j^k h_{ik}. \tag{2.14}$$

Also, let $H = \operatorname{tr}_g h$ and $H_0 = \operatorname{tr}_{g_0} h$.

Lemma 2.15. *The linearization of (2.2) at a fixed point* $\mathbf{g}_0 = (g_0, 0, G_0)$ *with* $2\operatorname{Rc} = Kg_0$ *and* G_0 *constant acts on* $\mathbf{h} = (h, B, F)$ *by*

$$\frac{\partial}{\partial t} h_{\alpha\beta} = \Delta_\ell h_{\alpha\beta} + \nabla_\alpha(\delta h)_\beta + \nabla_\beta(\delta h)_\alpha + \nabla_\alpha \nabla_\beta H_0 + X, \tag{2.16a}$$

$$\frac{\partial}{\partial t} B^i_\alpha = -(\delta d B)^i_\alpha + k B^i_\alpha, \tag{2.16b}$$

$$\frac{\partial}{\partial t} F_{ij} = \Delta F_{ij}, \tag{2.16c}$$

where

$$X = \begin{cases} K h_{\alpha\beta} & \text{in case (i)} \\ 2K\big(h_{\alpha\beta} - \frac{1}{n}(g_0)_{\alpha\beta} \oint_B H_0 \,\mathrm{d}\mu\big) & \text{in case (ii)} \end{cases}$$

and

$$k = \begin{cases} \frac{K}{2} & \textit{in case (i)} \\ K & \textit{in case (ii)} \end{cases}.$$

PROOF. With a variation of $\mathbf{g}$ as in (2.12) and (2.13), we must compute

$$\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \left(\frac{\partial}{\partial t} \tilde{\mathbf{g}}(\epsilon) \right).$$

Here and in the next lemma, we will use standard variational formulas for geometric objects like g^{-1}, Γ, Rc, R, $\mathrm{d}\mu$, and $\oint R\,\mathrm{d}\mu$. See [9, Section 3.1], for example.

Considering the first equation, we have

$$\frac{\partial}{\partial t} h_{\alpha\beta} = \Delta_\ell h_{\alpha\beta} + \nabla_\alpha (\delta h)_\beta + \nabla_\beta (\delta h)_\alpha + \nabla_\alpha \nabla_\beta H_0 - \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} (s\tilde{g}_{\alpha\beta}).$$

In case (i),

$$\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} (s\tilde{g}_{\alpha\beta}) = K h_{\alpha\beta},$$

and in case (ii),

$$\begin{aligned} \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} (s\tilde{g}_{\alpha\beta}) &= \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \left(\frac{2}{n} \tilde{g}_{\alpha\beta} \oint_B \tilde{r}\,\mathrm{d}\tilde{\mu} \right) \\ &= 2K \left(h_{\alpha\beta} - \frac{1}{n} (g_0)_{\alpha\beta} \oint_B H_0\,\mathrm{d}\mu_0 \right). \end{aligned}$$

For the second equation, we have

$$\frac{\partial}{\partial t} B^i_\alpha = -(\delta dB)^i_\alpha + \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \left(\frac{1}{2} s\tilde{A}^i_\alpha \right).$$

In case (i),

$$\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \left(\frac{1}{2} s\tilde{A}^i_\alpha \right) = \frac{K}{2} B^i_\alpha,$$

and in case (ii),

$$\begin{aligned} \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \left(\frac{1}{2} s\tilde{A}^i_\alpha \right) &= \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \left(\frac{1}{n} \tilde{A}^i_\alpha \oint_B \tilde{r}\,\mathrm{d}\tilde{\mu} \right) \\ &= K B^i_\alpha. \end{aligned}$$

Here, we use that $r = nK$ when g_0 is Einstein.

For the third equation, we use Proposition 2.5 to write

$$\frac{\partial}{\partial \tau} G_{ij} = \tau_{g(t),\overline{g}}(G)_{ij} - \frac{1}{2} g^{\alpha\gamma} g^{\beta\delta} G_{ik} G_{j\ell} (dA)^k_{\alpha\beta} (dA)^\ell_{\gamma\delta},$$

where the first term is the tension field from (2.4). Then it is easy to see that

$$\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} F_{ij} = \Delta F_{ij},$$

as desired. □

As in [21], we use the DeTurck trick to make the linear (2.2) system strictly parabolic. That is, to each equation in (2.2) we add a term consisting of the Lie derivative of the metric with respect to a carefully chosen family of vector fields $W(t)$. To this end, fix a background connection $\underline{\Gamma}$ and define

$$W^\gamma = g^{\alpha\beta}(\Gamma^\gamma_{\alpha\beta} - \underline{\Gamma}^\gamma_{\alpha\beta}), \qquad \gamma = 1, \ldots, n \tag{2.17a}$$

$$(W_\flat)_k = (\delta A)_k, \qquad k = 1, \ldots, N. \tag{2.17b}$$

Let ψ_t be diffeomorphisms generated by $W(t)$, with initial condition $\psi_0 = \mathrm{id}$. The one-parameter family of metrics $\psi_t^* \mathbf{g}(t)$ is the solution of the *rescaled* $\mathbb{R}^N$-*invariant Ricci–DeTurck flow*. We now take $\underline{\Gamma}$ to be the Levi-Civita connection of the stationary solution around which we linearize. Observe that a stationary solution $\mathbf{g}_0 = (g_0, 0, G_0)$ of (2.2) with $2\,\mathrm{Rc} = K g_0$ and G_0 constant is then also a stationary solution of the rescaled Ricci–DeTurck flow.

Lemma 2.18. *The linearization of (2.2) at a fixed point* $\mathbf{g}_0 = (g_0, 0, G_0)$ *with* $2\,\mathrm{Rc} = K g_0$ *is the autonomous, self-adjoint, strictly parabolic system*

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ B \\ F \end{pmatrix} = \mathbf{L}\begin{pmatrix} h \\ B \\ F \end{pmatrix} = \begin{pmatrix} \mathbf{L}_2 h \\ \mathbf{L}_1 B \\ \mathbf{L}_0 H \end{pmatrix},$$

where

$$\mathbf{L}_0 = \Delta, \tag{2.19}$$

$$\mathbf{L}_1 = \begin{cases} \Delta_1 + \frac{K}{2}\,\mathrm{id} & \text{in case (i)} \\ \Delta_1 + K\,\mathrm{id} & \text{in case (ii)} \end{cases}. \tag{2.20}$$

$$\mathbf{L}_2 = \begin{cases} \Delta_\ell + K\,\mathrm{id} & \text{in case (i)} \\ \Delta_\ell + \Phi & \text{in case (ii)} \end{cases} \tag{2.21}$$

Here $-\Delta_1 = d\delta + \delta d$ *denotes the Hodge–de Rham Laplacian acting on* 1*-forms, and*

$$\Phi(h) = 2K\left(h - \frac{1}{n} g_0 \fint_B H_0 \,\mathrm{d}\mu\right).$$

Proof. The normalized Ricci-DeTurck flow is obtained by subtracting a Lie derivative from the right side of (2.2):

$$\frac{\partial}{\partial t}\mathbf{g} = -2\,\mathrm{Rc}[\mathbf{g}] - \mathcal{L}_W \mathbf{g},$$

so we must compute the linearization of this Lie derivative, as in Lemma 2.15.

First, since W^γ is defined the same way as it was in [21, Equations (3.6), (4.4)], the components of the form $(\mathcal{L}_W \mathbf{g})_{\alpha\beta}$ are unchanged from that paper. This means that when computing its ϵ-derivative, we get

$$\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} (\mathcal{L}_W \mathbf{g})_{\alpha\beta} = \Delta_\ell h_{\alpha\beta} + \nabla_\alpha(\delta h)_\beta + \nabla_\beta(\delta h)_\alpha + \nabla_\alpha\nabla_\beta H_0.$$

Subtracting this from (2.16a) gives (2.21).

Next, we have

$$(\mathcal{L}_W \mathbf{g})_{\alpha i} = (d\delta A)^i_\alpha,$$

and so

$$\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} (\mathcal{L}_W \tilde{\mathbf{g}})_{\alpha i} = (d\delta B)^i_\alpha.$$

Subtracting this from (2.16b) gives (2.20).

Finally, we have

$$(\mathcal{L}_W \mathbf{g})_{ij} = 0,$$

and so

$$\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} (\mathcal{L}_W \tilde{\mathbf{g}})_{ij} = 0.$$

Subtracting this from (2.16c) gives (2.19). □

Now assume that $\mathbf{g}_0 = (g_0, 0, G_0)$ is a fixed point of the rescaled $\mathbb{R}^N$-invariant Ricci–DeTurck flow with G_0 constant and g_0 a metric of constant sectional curvature. In case (i), $\mathrm{sect}[g_0] = -1/2(n-1) < 0$, and in case (ii), $\mathrm{sect}[g_0] = (n-1)k > 0$. In case (ii), by passing to a covering space if necessary, we may assume that $\mathcal{B}^n$ is the round n-sphere of radius $\sqrt{1/k}$.

Recall that a linear operator L is *weakly (strictly) stable* if its spectrum is confined to the half plane $\mathrm{Re}\, z \leq 0$ (and is uniformly bounded away from the imaginary axis).

Because $\mathbf{L}$ is diagonal, we can determine its stability by examining its component operators. The conclusions we obtain here will hold below when we extend $\mathbf{L}$ to a complex-valued operator on a larger domain in which smooth representatives are dense.

Lemma 2.22. *Let $\mathbf{g}_0 = (g_0, 0, G_0)$ be a metric of the form (2.1) such that G_0 is constant and g_0 has constant sectional curvature. Then the linear system* (2.19)-(2.21) *has the following stability properties:*

The operator $\mathbf{L}_0$ is weakly stable.

The operator $\mathbf{L}_1$ is strictly stable.

If $n = 2$, then the operator $\mathbf{L}_2$ is weakly stable.

Let $n \geq 3$. In case (i), the operator $\mathbf{L}_2$ is strictly stable. In case (ii), the operator $\mathbf{L}_2$ is unstable.

Proof. It is well-known that $\mathbf{L}_0 = \Delta$, the Laplacian acting on $(2,0)$-forms, is weakly stable. The statements about $\mathbf{L}_2$ and $\mathbf{L}_3$ carry over directly from Lemmas 5 and 7 in [21]. □

We now turn to the proof of the the main theorem. See [21, Section 2] for summary of the machinery that is used in the proof.

Proof of Theorem 2.11. Following [21], the proof consists of four step. First, one must show that the complexified operator is sectorial. This depends only on Lemma 2.22, which has the same conclusion as [21, Lemmas 5 & 7]. Therefore, there is no modification to this step.

Second, one applies Simonett's theorem from [41]. This is valid by Step 1, and [21, Lemma 2]. Since that Lemma was stated in the full generality of our context, there is no modification to this step.

Third, one proves the uniqueness of a smooth center manifold consisting of fixed points of the flow (2.2). Since fixed points of this flow still coincide with those of the rescaled Ricci-DeTurck flow, there is no modification.

Fourth, one proves convergence of the metric. In both cases (i) and (ii), the arguments involved do not depend on the dimension N, so there is no modification. □

3. A compactness theorem

In this section, we consider the Ricci flow coupled with the harmonic map flow, or the $(RH)_c$ flow. We prove a version of Hamilton's Compactness Theorem for a class of such flows. This is first done in the category of smooth manifolds, where we assume uniform bounds on the curvatures and injectivity radii. We also prove a version in the category of étale Riemannian groupoids, where no information about the injectivity radii is needed. The compactness theorems presented here (like Hamilton's) provide subsequential convergence in general; in cases where stability theorems like those above apply, this can be improved to genuine asymptotic convergence. Let us recall the setup for the coupled flow in question.

3.1. Definitions. Let $(\mathcal{M}, g)$ be a closed Riemannian manifold, with $(\mathcal{N}, h)$ a closed target manifold. Let $\phi\colon \mathcal{M} \to \mathcal{N}$ be a smooth map. The Levi-Civita covariant derivative $\nabla^{T\mathcal{M}}$ of the metric g on $\mathcal{M}$ induces a covariant derivative $\nabla^{T^*\mathcal{M}}$ on the cotangent bundle, which satisfies

$$\nabla^{T^*\mathcal{M}}_X \omega(Y) = X\big(\omega(Y)\big) - \omega\left(\nabla^{T\mathcal{M}}_X Y\right).$$

By requiring a product rule and compatibility with the metric, we also have convariant derivatives on all tensor bundles

$$T^p_q(\mathcal{M}) = (T\mathcal{M})^{\otimes p} \otimes (T^*\mathcal{M})^{\otimes q}.$$

The Levi-Civita covariant derivative $\nabla^{T\mathcal{N}}$ of the metric h on N induces a covariant derivative $\nabla^{\phi^*T\mathcal{N}}$ on the pull-back bundle $\phi^*T\mathcal{N} \to \mathcal{M}$, given by

$$\nabla^{\phi^*T\mathcal{N}}_X \phi^* Y = \phi^*\left(\nabla^{T\mathcal{N}}_{\phi_* X} Y\right),$$

for $X \in \mathcal{T}(\mathcal{M})$ and $Y \in \mathcal{T}(\mathcal{N})$. As before, we get a covariant derivative on all tensor bundles over $\mathcal{M}$ of the form

$$T^p_q(\mathcal{M}) \otimes T^r_s(\phi^*\mathcal{N}) = (T\mathcal{M})^{\otimes p} \otimes (T^*\mathcal{M})^{\otimes q} \otimes (\phi^*T\mathcal{N})^{\otimes r} \otimes (\phi^*T^*\mathcal{N})^{\otimes s}.$$

We refer to them simply as ∇. In local coordinates,

$$\nabla\phi = \phi_* = \partial_i \phi^\lambda \, dx^i \otimes \partial_\lambda|_\phi \in \Gamma(T^*\mathcal{M} \otimes \phi^*T\mathcal{N}).$$

Similarly, if we write ${}^{\mathcal{N}}\nabla$ for $\nabla^{T\mathcal{N}}$, we have

$$\begin{aligned}\nabla^2\phi &= \left(\partial_i\partial_j\phi^\lambda - \Gamma^k_{ij}\partial_k\phi^\lambda + ({}^{\mathcal{N}}\Gamma \circ \phi)^\lambda_{\mu\nu}\partial_i\phi^\mu\phi^\nu_j\right) dx^i \otimes dx^j \otimes \partial_\lambda|_\phi \\ &\in \Gamma(T^*\mathcal{M} \otimes T^*\mathcal{M} \otimes \phi^*T\mathcal{N}).\end{aligned}$$

Additionally

$$\nabla\phi \otimes \nabla\phi = h_{\lambda\mu}\partial_i\phi^\lambda\partial_j\phi^\mu \, dx^i \otimes dx^j$$

and is a symmetric $(2,0)$-tensor on $\mathcal{M}$, and we define

$$\mathrm{S} = \mathrm{Rc} - c\nabla\phi \otimes \nabla\phi$$

where $c = c(t) \geq 0$ is a coupling function. Finally, the *tension field* of ϕ with respect to g and h was defined in (2.4), but the general description is

$$\tau_{g,h}\phi = \mathrm{tr}_g \nabla^2\phi. \tag{3.1}$$

Now, the flow for initial data $(\mathcal{M}, g_0, \phi_0)$ is the system

$$\begin{aligned}\frac{\partial}{\partial t} g &= -2\mathrm{S} = -2\,\mathrm{Rc} + 2c\nabla\phi \otimes \nabla\phi \\ \frac{\partial}{\partial t}\phi &= \tau_{g,h}\phi \\ \big(g(0), \phi(0)\big) &= (g_0, \phi_0)\end{aligned} \tag{3.2}$$

For short, call this the $(RH)_c$ flow. We will assume that $c(t)$ is non-increasing.

As written here, this flow was introduced in [34] and is a generalization of one studied in [29]. Indeed, the latter considers the case when ϕ is a real-valued function.

Definition 3.3. A family $\{(\mathcal{M}^n, g(t), \phi(t), O)\}$ of complete, pointed Riemannian manifolds with maps

$$\phi(t)\colon \mathcal{M} \longrightarrow \mathcal{N}$$

that solves the system (3.2) with coupling function $c(t)$, for $t \in (\alpha, \omega)$, is a *complete, pointed* $(RH)_c$ *flow solution*.

Example 3.4. Consider the special case of the twisted bundle construction, seen in [30]. Let $\mathcal{M}$ be an $\mathbb{R}^N$-vector bundle with flat connection, flat metric G on the fibers, and Riemannian base $(\mathcal{B}, g)$. In the notion of Section 2, write the metric on $\mathcal{M}$ as $\mathrm{g} = (g, 0, G)$. Then the fiber metrics constitute a map

$$G\colon \mathcal{B} \longrightarrow \mathrm{SL}(N, \mathbb{R}) / \mathrm{SO}(N).$$

From [30, Equation (4.10)], Ricci flow on $\mathcal{M}$ becomes the pair of equations

$$\begin{aligned} \frac{\partial}{\partial t} g_{\alpha\beta} &= -2R_{\alpha\beta} + \frac{1}{2} G^{ij} \nabla_\alpha G_{jk} G^{k\ell} \nabla_\beta G_{\ell i} \\ \frac{\partial}{\partial t} G_{ij} &= g^{\alpha\beta} \nabla_\alpha \nabla_\beta G_{ij} - g^{\alpha\beta} \nabla_\alpha G_{ik} G^{kl} \nabla_\beta G_{\ell i}. \end{aligned}$$

But with the metric $\overline{g}$ on $\mathrm{SL}(N, \mathbb{R}) / \mathrm{SO}(N)$ as in (2.3), we see that

$$\frac{1}{2} G^{ij} \nabla_\alpha G_{jk} G^{k\ell} \nabla_\beta G_{\ell i} = \frac{1}{4} (\nabla G \otimes \nabla G)_{\alpha\beta},$$

and Proposition 2.5 says that

$$g^{\alpha\beta} \nabla_\alpha \nabla_\beta G_{ij} - g^{\alpha\beta} \nabla_\alpha G_{ik} G^{kl} \nabla_\beta G_{\ell i} = \tau_{g,\overline{g}} G.$$

This means Ricci flow on $\mathcal{M}$ is precisely $(RH)_c$ flow on $\mathcal{B}$, with target manifold $(\mathrm{SL}(N, \mathbb{R}) / \mathrm{SO}(N), \overline{g})$, maps G, and $c = 1/4$.

This gives many examples of $(RH)_c$ flow solutions. For instance, the homogeneous spaces in [30] that admit expanding Ricci solitons all have the bundle structure just described, so those Ricci flow solutions are $(RH)_c$ solutions.

Definition 3.5. A sequence $\{\mathcal{M}^n_k, g_k(t), \phi_k(t), O_k)\}$ of complete, pointed $(RH)_c$ flow solutions *converges* to $(\mathcal{M}^n_\infty, g_\infty(t), \phi_\infty(t), O_\infty)$ for $t \in (\alpha, \omega)$ if there exists

- an exhaustion $\{\mathcal{U}_k\}$ of $\mathcal{M}_\infty$ by open sets with $O_\infty \in \mathcal{U}_k$ for all k, and
- a family of diffeomorphisms $\{\Psi_k \colon \mathcal{U}_k \to \mathcal{V}_k \subset \mathcal{M}_k\}$ with $\Psi_k(O_\infty) = O_k$

such that

$$\left(\mathcal{U}_k \times (\alpha, \omega), \Psi_k^* \left(g_k(t)|_{\mathcal{V}_k} + dt^2\right), \Psi_k^* \phi_k|_{\mathcal{V}_k}\right)$$

converges uniformly in C^∞ on compact sets to

$$\left(\mathcal{M}^n_\infty \times (\alpha, \omega), g_\infty(t) + dt^2, \phi_\infty(t)\right).$$

Here, dt^2 is the standard metric on $(\alpha, \omega) \subset \mathbb{R}$.

We mention that we will use abbreviated notation for geometric objects associated with metric $g_k(t)$. For example, $\mathrm{Rc}_k(t)$ means $\mathrm{Rc}[g_k(t)]$, and ∇_k refers to the Levi-Civita covariant derivative corresponding to the metric $g_k(t)$. Also, an undecorated ∇ will refer to the Levi-Civita covariant derivative corresponding to a background metric.

3.2. Statement of the theorem. The original version of this theorem appears in [16] in the context Ricci flow, and was generalized to the groupoid setting in [30]. Such theorems are crucial in the study of geometric flows, especially regarding singularity models. For example, one often wishes to construct sequences of rescaled solutions to investigate the behavior at a singular time (possibly $T = +\infty$), and it is helpful to be able to extract convergent subsequences.

Theorem 3.6. *Let $\{(\mathcal{M}_k^n, g_k(t), \phi_k(t), O_k)\}$ be a sequence of complete, pointed $(RH)_c$ flow solutions, with $0, t \in (\alpha, \omega)$ and $c(t)$ non-increasing, such that*

(a) *the geometry is uniformly bounded: for all k,*
$$\sup_{(x,t)\in\mathcal{M}_k\times(\alpha,\omega)} |\operatorname{Rm}_k|_k \leq C_1$$
for some C_1 independent of k;

(b) *the initial injectivity radii are uniformly bounded below: for all k,*
$$\operatorname{inj}_{g_k(0)}(O_k) \geq \iota_0 > 0,$$
for some ι_0 independent of k.

Then there is a subsequence such that
$$\big(\mathcal{M}_k, g_k(t), \phi_k(t), O_k\big) \longrightarrow \big(\mathcal{M}_\infty, g_\infty(t), \phi_\infty(t), O_\infty\big),$$
where the limit is also a pointed, complete, $(RH)_c$ flow solution.

If we do not assume a bound on he injectivity radius bound, then we have convergence to
$$\big(\mathcal{G}_\infty, g_\infty(t), \phi_\infty(t), O_\infty\big),$$
a complete, pointed, n-dimensional, étale Riemannian groupoid with map ϕ_∞ on the base.

The idea of the proof is the same as in [16], and subsequently [29], although we follow the exposition found in [8, Chapter 3]. Briefly, the main ingredients are derivative estimates to bound the curvature and the derivatives of the map ϕ, a general compactness theorem of Hamilton, a technical lemma, and corollary of the Arzela-Ascoli theorem. Of course, many facts about the $(RH)_c$ flow, found in [34], are used along the way.

Example 3.7. Here is a way to obtain sequences of $(RH)_c$ flow solutions like those considered in the compactness theorem. To be clear about the dependence on the coupling constant, let us write a solution of the $(RH)_c$ flow as a triple $\big(g(t), \phi(t), c(t)\big)$. We can obtain a family of $(RH)_c$ flow solutions by performing a *blowdown*, a technique used extensively in [30] and [31]. For $s \in (0, \infty)$, define
$$\big(g_s(t), \phi_s(t), c_s(t)\big) = \left(\frac{1}{s}g(st), \frac{1}{s}\phi(st), s^2 c(st)\right).$$
Now we see that
$$\frac{\partial}{\partial t}g_s(t) = \left(\frac{\partial}{\partial t}g\right)(st) \quad \text{and} \quad \frac{\partial}{\partial t}\phi_s(t) = \left(\frac{\partial}{\partial t}\phi\right)(st),$$
and
$$\begin{aligned} \mathrm{S}[g_s(t), \phi_s(t), c_s(t)] &= -2\operatorname{Rc}[g_s(t)] + 2c_s(t)\nabla\phi_s(t)\otimes\nabla\phi_s(t) \\ &= -2\operatorname{Rc}[g(st)] + 2c(st)\nabla\phi(st)\otimes\nabla\phi(st) \\ &= \mathrm{S}[g(st), \phi(st), c(st)], \end{aligned}$$

$$\begin{aligned}\tau_{g_s(t),h}\phi_s(t) &= \operatorname{tr}_{g_s(t)} \nabla^2\phi_s(t)\\ &= \operatorname{tr}_{g(st)} \nabla^2\phi(st)\\ &= \tau_{g(st),h}\phi(st).\end{aligned}$$

Therefore, for each s, the blowdown gives another $(RH)_c$ solution. It is common to replace the continuous parameter s with a sequence $\{s_j\}$ converging to infinity.

3.3. Two lemmas. In this section we prove two lemmas that will be used in the proof of Theorem 3.6. The first is an analogue of [16, Lemma 2.4], [8, Lemma 3.11], and [29, Lemma 7.6].

Lemma 3.8. *Let $(\mathcal{M}^n, g)$ be a Riemannian manifold, with $\mathcal{K} \subset \mathcal{M}$ compact. Let $\{(g_k(t), \phi_k(t))\}$ be a sequence of solutions to the $(RH)_c$ flow, defined on $\mathcal{K} \times [\beta, \psi]$, where $t_0 \in [\beta, \psi]$. Suppose the following hold.*

The metrics $g_k(t_0)$ are uniformly equivalent to g on $\mathcal{K}$. That is, for all $x \in \mathcal{K}$, $V \in T_x\mathcal{M}$, and k, there is $C < \infty$ such that

$$C^{-1}g(V,V) \le g_k(t_0)(V,V) \le Cg(V,V). \tag{3.9}$$

The covariant derivatives of $g_k(t_0)$ and ϕ_k with respect to g are uniformly bounded on K. That is, for all $p \ge 0$, there exist C_p, C'_p such that

$$\max_{x\in\mathcal{K}} |\nabla^{p+1} g_k(t_0)| \le C_p < \infty, \tag{3.10}$$

$$\max_{x\in\mathcal{K}} |\nabla^{p} \phi_k(t_0)| \le C'_p < \infty. \tag{3.11}$$

The covariant derivatives of Rm_k and ϕ_k with respect to $g_k(t)$ are uniformly bounded on $K \times [\beta, \psi]$. That is, for all $p \ge 0$, there exist C''_p, C'''_p such that

$$\max_{x\in\mathcal{K}} |\nabla^{p}_k \operatorname{Rm}_k|_k \le C''_p < \infty, \tag{3.12}$$

$$\max_{x\in\mathcal{K}} |\nabla^{p}_k \phi_k|_k \le C'''_p < \infty. \tag{3.13}$$

Then the following hold.

The metrics $g_k(t)$ are uniformly equivalent to g on $\mathcal{K} \times [\beta, \psi]$. That is, for all $x \in \mathcal{K}$, $V \in T_x\mathcal{M}$, k, there exists $B > 0$ such that

$$B^{-1}g(V,V) \le g_k(t)(V,V) \le Bg(V,V). \tag{3.14}$$

The time and covariant derivatives with respect to g of $g_k(t)$ and $\phi_k(t)$ are uniformly bounded on $\mathcal{K} \times [\beta, \psi]$. That is, for all p and q, there exist $\tilde{C}_{p,q}, \tilde{D}_{p,q}$ such that

$$\max_{x\in\mathcal{K}} \left| \frac{\partial^q}{\partial t^q} \nabla^p g_k(t) \right| \le \tilde{C}_{p,q} < \infty, \tag{3.15}$$

$$\max_{x\in\mathcal{K}} \left| \frac{\partial^q}{\partial t^q} \nabla^p \phi_k(t) \right| \le \tilde{D}_{p,q} < \infty. \tag{3.16}$$

PROOF. First, note that throughout the proof we will follow standard practice in not indexing constants, and will often use the same symbol (e.g., C) for different constants within a sequence of inequalities.

To prove (a), we have

$$\frac{\partial}{\partial t} g_k(t) = -2\mathrm{S}_k(t) = -2\,\mathrm{Rc}_k(t) + 2c(t)\nabla_k\phi_k(t)\otimes\nabla_k\phi_k(t),$$

so that for $V \in T\mathcal{M}$,

$$\begin{aligned}\left|\frac{\partial}{\partial t} g_k(t)(V,V)\right| &= |-2\,\mathrm{Rc}_k(t)(V,V) + 2c(t)\nabla_k\phi_k(t)\otimes\nabla_k\phi_k(t)(V,V)| \\ &\le 2|\,\mathrm{Rc}_k(t)||V|_k^2 + 2|c(t)||\nabla_k\phi_k(t)|^2|V|_k^2 \\ &\le C'|V|_k^2 \\ &= C' g_k(t)(V,V).\end{aligned}$$

This implies

$$|\partial_t \log g_k(t)(V,V)| = \left|\frac{\partial_t g_k(t)(V,V)}{g_k(t)(V,V)}\right| \le C',$$

and thus for any $t_1 \in [\beta,\psi]$, we have

$$\int_{t_0}^{t_1} |\partial_t \log g_k(t)(V,V)|\,dt \le C'|t_1 - t_0|.$$

This gives

$$\begin{aligned}C'|t_1 - t_0| &\ge \int_{t_0}^{t_1} |\partial_t \log g_k(t)(V,V)|\,dt \\ &\ge \left|\int_{t_0}^{t_1} \partial_t \log g_k(t)(V,V)\,dt\right| \\ &= \left|\log\frac{g_k(t_1)(V,V)}{g_k(t_0)(V,V)}\right|.\end{aligned}$$

Expanding this gives

$$-C'|t_1 - t_0| \le \log\frac{g_k(t_1)(V,V)}{g_k(t_0)(V,V)} \le C'|t_1 - t_0|,$$

and exponentiating gives

$$\exp(-C'|t_1 - t_0|)g_k(t_0)(V,V) \le g_k(t_1) \le \exp(C'|t_1 - t_0|)g_k(t_0)(V,V).$$

Combining this with the original hypotheses, we get

$$C^{-1}\exp(-C'|t_1 - t_0|)g(V,V) \le g_k(t_1) \le C\exp(C'|t_1 - t_0|)g(V,V).$$

Since t_1 was arbitrary, and since $C\exp(C'|t_1 - t_0|) \le C\exp(C'|\psi - \beta|) = B$, this completes the proof of (a).

Next, we prove (3.15) and (3.16). Observe that

$$\left|\frac{\partial^q}{\partial t^q}\nabla^p g_k(t)\right| = \left|\nabla^p \frac{\partial^{q-1}}{\partial t^{q-1}}\frac{\partial}{\partial t} g_k(t)\right| = 2\left|\nabla^p \frac{\partial^{q-1}}{\partial t^{q-1}}\mathcal{S}_k(t)\right|, \tag{3.17}$$

$$\left|\frac{\partial^q}{\partial t^q}\nabla^p \phi_k(t)\right| = \left|\nabla^p \frac{\partial^{q-1}}{\partial t^{q-1}}\frac{\partial}{\partial t}\phi_k(t)\right| = \left|\nabla^p \frac{\partial^{q-1}}{\partial t^{q-1}}\tau_{g_k}\phi_k(t)\right|. \tag{3.18}$$

Recall that ∇ is the Levi-Civita covariant derivative corresponding to the background metric g. In general,

$$\begin{aligned} S_{ij} &= R_{ij} - c\nabla_i\phi\nabla_j\phi, \\ (\tau_{g,h}\phi)^\lambda &= g^{ij}\nabla_i\nabla_j\phi^\lambda, \end{aligned}$$

and we have the following evolution equations for $\mathcal{S}$ and $\tau_{g,h}\phi$:

$$\begin{aligned} \frac{\partial}{\partial t}S_{ij} &= \Delta_\ell S_{ij} + 2c\,\tau_{g,h}\phi\,\nabla_i\nabla_j\phi - \dot{c}\nabla_i\phi\nabla_j\phi \\ \frac{\partial}{\partial t}\tau_{g,h}\phi &= -g^{ik}g^{jl}S_{kl}(\nabla_i\nabla_j\phi) + g^{ij}\Big(\Delta(\nabla_i\phi\nabla_j\phi) - 2\nabla_p\nabla_i\phi\nabla_p\nabla_j\phi \\ &\qquad - R_{ip}\nabla_p\phi\nabla_j\phi - R_{jp}\nabla_p\phi\nabla_i\phi + 2\left\langle{}^{\mathcal{N}}\mathrm{Rm}(\nabla_i\phi,\nabla_p\phi)\nabla_p\phi,\nabla_j\phi\right\rangle\Big). \end{aligned}$$

To bound (3.17) and (3.18), we need to consider the evolution equations for all quantities involved, which appear in [34]:

$$\frac{\partial}{\partial t}\Gamma^p_{ij} = -g^{pq}(\nabla_i R_{jq} + \nabla_j R_{iq} - \nabla_q R_{ij} - 2c\nabla_i\nabla_j\phi\nabla_q\phi)$$

$$\begin{aligned} \frac{\partial}{\partial t}R_{ijk\ell} &= \nabla_i\nabla_k R_{j\ell} - \nabla_i\nabla_\ell R_{jk} - \nabla_j\nabla_k R_{i\ell} + \nabla_j\nabla_\ell R_{ik} - R_{ijq\ell}R_{kq} - R_{ijkq}R_{\ell q} \\ &\quad + 2c\big(\nabla_i\nabla_k\phi\nabla_j\nabla_\ell\phi - \nabla_i\nabla_\ell\phi\nabla_j\nabla_k\phi - \left\langle{}^{\mathcal{N}}\mathrm{Rm}(\nabla_i\phi,\nabla_j\phi)\nabla_k\phi,\nabla_\ell\phi\right\rangle\big). \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial t}R_{ij} &= \Delta_\ell R_{ij} - 2R_{iq}R_{jq} + 2R_{ipjq}R_{pq} + 2c\,\tau_{g,h}\phi\nabla_i\nabla_j\phi - 2c\nabla_p\nabla_i\phi\nabla_p\nabla_j\phi \\ &\quad + 2cR_{pijq}\nabla_p\phi\nabla_q\phi + 2c\left\langle{}^{\mathcal{N}}\mathrm{Rm}(\nabla_i\phi,\nabla_p\phi)\nabla_p\phi,\nabla_j\phi\right\rangle. \end{aligned}$$

In these equations, we used

$$\left\langle{}^{\mathcal{N}}\mathrm{Rm}(\nabla_i\phi,\nabla_j\phi)\nabla_j\phi,\nabla_i\phi\right\rangle := {}^{\mathcal{N}}R_{\kappa\mu\lambda\nu}\nabla_i\phi^\kappa\nabla_j\phi^\mu\nabla_i\phi^\lambda\nabla_j\phi^\nu,$$

and k was a coordinate index, not a sequence index.

The types of terms that will appear in the expansions of (3.17) and (3.18) therefore involve factors containing

$$\mathrm{S}_k, \mathrm{Rc}_k, \mathrm{Rm}_k, \nabla_k\phi_k, \tau_{g_k,h}\phi_k, {}^{\mathcal{N}}\mathrm{Rm}, \tag{3.19}$$

as well as time and covariant derivatives, whose norms we must show are bounded. Note that we can ingore the geometric factors coming from the manifold $\mathcal{N}$, since those quantities are bounded by compactness of $\mathcal{N}$ and by the chain rule.

Now, let us consider the case $p = 1, q = 0$ for (3.15) and (3.16). As in the proof of Lemma 3.11 in [8], we have

$$\frac{1}{2}|\nabla g_k(t)|_k \le |\Gamma_k - \Gamma|_k \le \frac{3}{2}|\nabla g_k(t)|_k. \tag{3.20}$$

That is, up to lowering/raising indices, the tensors $\nabla g_k(t)$ and $\Gamma_k - \Gamma$ are equivalent. Using the evolution of the Christoffel symbols, an estimation in normal coodinates gives

$$\left|\frac{\partial}{\partial t}(\Gamma_k - \Gamma)\right|^2_k \le 12|\nabla_k \mathrm{Rc}_k|^2_k + 8\bar{c}|\nabla^2_k\phi_k|^2_k|\nabla_k\phi_k|^2_k \le C.$$

We can show that $|\Gamma_k - \Gamma|_k$ is bounded by integrating the above inequality:

$$\begin{aligned} C|t_1 - t_0| &\geq \int_{t_0}^{t_1} |\partial_t(\Gamma_k(t) - \Gamma)|_k \, dt \\ &\geq \left| \int_{t_0}^{t_1} \partial_t(\Gamma_k(t) - \Gamma) \, dt \right|_k \\ &\geq |\Gamma_k(t_1) - \Gamma|_k - |\Gamma_k(t_0) - \Gamma|_k. \end{aligned}$$

Since t_1 is arbitrary, we see that

$$\begin{aligned} |\Gamma_k(t) - \Gamma|_k &\leq C|t - t_0| + |\Gamma_k(t_0) - \Gamma|_k \\ &\leq C|t - t_0| + \frac{3}{2}|\nabla g_k(t_0)|_k \\ &\leq C|t - t_0| + \frac{3}{2}B|\nabla g_k(t_0)| \\ &\leq C. \end{aligned}$$

From this and (3.14) it follows that

$$\begin{aligned} |\nabla g_k(t)| &\leq C|\nabla g_k(t)|_k \\ &\leq C|\Gamma_k(t) - \Gamma|_k \\ &\leq C, \end{aligned}$$

and we also have

$$\begin{aligned} |\nabla \phi_k(t)| &\leq C|\nabla \phi_k(t)|_k \\ &\leq C\left(|(\nabla - \nabla_k)\phi_k(t)|_k + |\nabla_k \phi_k(t)|_k\right) \\ &\leq C\left(|\Gamma_k(t) - \Gamma|_k |\phi_k(t)| + |\nabla_k \phi_k(t)|_k\right) \\ &\leq C. \end{aligned}$$

This completes the case for $p = 1$, $q = 0$.

The general case will follow once we bound the norms of the quantities listed in (3.19) and their deriviatives. For this we need several preliminary bounds:

$$|\nabla^p \mathrm{S}_k(t)| \leq C|\nabla^p g_k(t)| + C', \tag{3.21}$$

$$|\nabla^p \phi_k(t)| \leq C'', \tag{3.22}$$

$$|\nabla^p g_k(t)| \leq C'''. \tag{3.23}$$

We prove these by induction. Consider (3.21). Since $\mathcal{S} = \operatorname{Rc} - c\nabla\phi \otimes \nabla\phi$, we have

$$\begin{aligned} |\mathrm{S}_k|_k &= |\operatorname{Rc}_k - c\nabla_k \phi_k \otimes \nabla_k \phi_k|_k \\ &\leq |\operatorname{Rc}_k|_k + \overline{c}|\nabla_k \phi_k|_k^2 \\ &\leq C, \end{aligned}$$

and

$$\begin{aligned} |\nabla_k \mathrm{S}_k|_k &= |\nabla_k \operatorname{Rc}_k - \nabla_k(c\nabla_k \phi_k \otimes \nabla_k \phi_k)|_k \\ &= |\nabla_k \operatorname{Rc}_k - \dot{c}\nabla_k \phi_k \otimes \nabla_k \phi_k - c\nabla_k(\nabla_k \phi_k \otimes \nabla_k \phi_k)|_k \\ &\leq |\nabla_k \operatorname{Rc}_k|_k + |\dot{c}||\nabla_k \phi_k \otimes \nabla_k \phi_k|_k + 2\overline{c}|\nabla_k^2 \phi_k \otimes \nabla_k \phi_k|_k \\ &\leq C_1' + |\dot{c}||\nabla_k \phi_k|_k^2 + 2\overline{c}|\nabla_k^2 \phi_k|_k |\nabla_k \phi_k|_k \\ &\leq C. \end{aligned}$$

Now, we can use this to see that

$$\begin{aligned}|\nabla \mathrm{S}_k| &\leq C|\nabla \mathcal{S}_k|_k\\ &\leq C|(\nabla - \nabla_k)\mathcal{S}_k|_k + B^{3/2}|\nabla_k \mathcal{S}_k|_k\\ &\leq C|\Gamma_k - \Gamma|_k|\mathcal{S}_k|_k + B^{3/2}|\nabla_k \mathcal{S}_k|_k\\ &\leq C,\end{aligned}$$

so the base case is complete.

Assume that (3.21) holds for all $p < N$, and then consider $p = N$, for $N \geq 2$. Using the difference of powers formula, we have

$$\begin{aligned}|\nabla^N \mathrm{S}_k| &= \left|\sum_{i=1}^{N} \nabla^{N-i}(\nabla - \nabla_k)\nabla_k^{i-1}\mathrm{S}_k + \nabla_k^N \mathrm{S}_k\right|\\ &\leq \sum_{i=1}^{N} |\nabla^{N-i}(\nabla - \nabla_k)\nabla_k^{i-1}\mathrm{S}_k| + |\nabla_k^N \mathrm{S}_k|.\end{aligned}$$

The goal now is to show that we can bound $|\nabla^{N-i}(\nabla - \nabla_k)\nabla_k^{i-1}\mathcal{S}_k|$. Recall that $\nabla - \nabla_k = \Gamma - \Gamma_k$ is a sum of terms of the form ∇g_k. In what follows, we will informally write this as $\sum \nabla g_k$.

Now, suppose $i = 1$. Then using the product rule repeatedly, we have

$$\begin{aligned}|\nabla^{N-1}(\nabla - \nabla_k)\mathrm{S}_k| &= |\nabla^{N-1}(\textstyle\sum \nabla g_k)\mathrm{S}_k|\\ &= \left|\sum_{j=0}^{N-1} \binom{N-1}{j} \nabla^{N-1-j}(\textstyle\sum \nabla g_k)\nabla^j \mathrm{S}_k\right|\\ &\leq \sum_{j=0}^{N-1} \binom{N-1}{j} \sum |\nabla^{N-j} g_k||\nabla^j \mathrm{S}_k|.\end{aligned}$$

Each term here is bounded by inductive hypothsis.

Similarly, for $2 \leq i \leq N$, we have

$$|\nabla^{N-i}(\nabla - \nabla_k)\nabla_k^{i-1}\mathrm{S}_k| \leq \sum_{j=0}^{N-i} \binom{N-i}{j} \sum |\nabla^{N-i-j+1} g_k||\nabla^j \nabla_k^{i-1}\mathrm{S}_k|.$$

We need to estimate the last factor. In general we have

$$\begin{aligned}|\nabla^j \nabla_k^i \mathrm{S}_k| &= |[(\nabla - \nabla_k) + \nabla_k]^j \nabla_k^i \mathrm{S}_k|\\ &= \left|\sum_{l=0}^{j} \binom{j}{l} (\nabla - \nabla_k)^{j-l} \nabla_k^l \nabla_k^i \mathrm{S}_k\right|\\ &\leq \sum_{l=0}^{j} \binom{j}{l} |\nabla - \nabla_k|^{j-l} |\nabla_k^{l+i}\mathrm{S}_k|\\ &\leq \sum_{l=0}^{j} \binom{j}{l} \sum |\nabla g_k|^{j-l} |\nabla_k^{l+i}\mathrm{S}_k|.\end{aligned}$$

This is also bounded by inductive hypothesis. Putting it all together (the assumptions of the lemma, the inductive hypotheses, equivalence of the norms) we have the desired bounds.

The same method can be used to verify (3.22).

For (3.23), we have

$$\frac{\partial}{\partial t}\nabla^N g_k(t) = \nabla^N \frac{\partial}{\partial t} g_k(t) = -2\nabla^N \mathcal{S}_k(t).$$

This implies

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla^N g_k|^2 &= 2\left\langle \frac{\partial}{\partial t}\nabla^N g_k, \nabla^N g_k \right\rangle \\
&\leq \left|\frac{\partial}{\partial t}\nabla^N g_k(t)\right|^2 + |\nabla^N g_k(t)|^2 \\
&= 4|\nabla^N \mathcal{S}_k|^2 + |\nabla^N g_k(t)|^2 \\
&\leq C|\nabla^N g_k|^2 + D
\end{aligned}$$

We can integrate this differential inequality to get

$$|\nabla^N g_k(t)|^2 \leq C,$$

as desired.

Using the arguments above, one can show that

$$|\nabla^p \nabla_k^q \operatorname{Rc}_k|, |\nabla^p \nabla_k^q \operatorname{Rm}_k|, |\nabla^p \nabla_k^q R_k|, |\nabla^p \nabla_k^q \mathrm{S}_k|, |\nabla^p \nabla_k^q \phi_k|$$

are bounded, independent of k.

Finally, we note that $\tau_{g_k,h}\phi_k$ and its derivatives have bounded norm. This follows from $\tau_{g,h}\phi = g^{ij}\nabla_i\nabla_j\phi$.

All terms are thus bounded, and we conclude that (3.17) and (3.18) are as well. $\square$

The second lemma, which is a corollary of the Arzela-Ascoli theorem is a modification of [8, Corollary 3.15].

Lemma 3.24. *Let $(\mathcal{M}^n, g)$ be a Riemannian manifold, with $\mathcal{K} \subset \mathcal{M}$ compact and $p \in \mathbb{Z}^{\geq 0}$. Suppose $\{(g_k, \phi_k)\}$ is a sequence of Riemannian metrics on $\mathcal{K}$ and maps $\mathcal{K} \to \mathcal{N}$, where $\mathcal{N}$ is some fixed target manifold, such that*

$$\sup_{0\leq\alpha\leq p+1} \max_{x\in\mathcal{K}} |\nabla^\alpha g_k| \leq C_1 < \infty,$$

$$\sup_{0\leq\alpha\leq p+1} \max_{x\in\mathcal{K}} |\nabla^\alpha \phi_k| \leq C_2 < \infty.$$

Addionally, suppose that there exists $\delta > 0$ such that $|V|_k \geq \delta|V|$ for all $V \in T\mathcal{M}$. Then there exists a subsequence $\{(g_{k_j}, \phi_{k_j})\}$, a Riemannian metric g_∞ on $\mathcal{K}$, and a smooth map $\phi_\infty\colon \mathcal{K} \to \mathcal{N}$ such that $(g_{k_j}, \phi_{k_j}) \to (g_\infty, \phi_\infty)$ in C^p as $k \to \infty$.

Proof. The existence of the subsequence will follow from the Arzela-Ascoli theorem, so we need to show that the collection of component functions $\{(g_k)_{ab}\} \cup \{(\phi_k)^\lambda\}$ is an equibounded and equicontinuous family. Equiboundedness follows from the hypotheses.

Now, in a fixed coordinate chart, by writing

$$\nabla_a (g_k)_{bc} = \partial_a (g_k)_{bc} - \Gamma^d_{ab}(g_k)_{dc} - \Gamma^d_{ac}(g_k)_{bd}$$

we see that bounds on $|\nabla g_k|$ give bounds on $|\partial_a (g_k)_{bc}|$. Similarly,

$$|\nabla_a (\phi_k)^\lambda| = |\partial_a (\phi_k)^\lambda|$$

is assumed to be bounded. Now, the mean value theorem for functions of several variables implies that

$$|(g_k)_{bc}(y) - (g_k)_{bc}(x)| \le C_1 \operatorname{diam}(K),$$

for all $x, y \in K$ and all indices b, c, and similarly for components of ϕ_k. This means the family $\{(g_k)_{ab}\} \cup \{(\phi_k)^\lambda\}$ is equicontinuous in the chart. Since K is compact, we can take finitely many charts to see that there is a finite uniform bound. Now apply the Arzela-Ascoli theorem to obtain the limits g_∞ and ϕ_∞. The bounds on the metrics imply that g_∞ is also a metric, and clearly ϕ_∞ is smooth.

We have only demonstrated subsequential convergence in C^0. For C^p convergence, repeat the same arguments starting with covariant derivatives of g_k and ϕ_k, obtaining bounds on the higher partial derivatives. □

3.4. The proof of the theorem. We will need a result of Hamilton, Theorem 2.3 in [16], which he used to prove the original compactness theorem for Ricci flow.

Theorem 3.25. *Let $\{(\mathcal{M}_k^n, g_k, O_k)\}$ be a sequence of pointed, complete, Riemannian manifolds such that*

(a) *the geometry is uniformly bounded:*

$$|\nabla_k^p \operatorname{Rm}_k|_k \le C_p$$

on $\mathcal{M}_k$, for all $p \ge 0$, all k, for C_p independent of k;

(b) *the injectivity radii are uniformly bounded below:*

$$\operatorname{inj}_k(O_k) \ge \iota_0\, 0,$$

for some ι_0 independent of k.

Then there is a subsequence such that

$$(\mathcal{M}_k, g_k, O_k) \longrightarrow (\mathcal{M}_\infty, g_\infty, O_\infty),$$

where the limit is also a pointed, complete, Riemannian manifold.

We will also need the derivative estimate for the curvature and the map, Theorem 6.10 in [34]. This is a version of the Bernstein-Bando-Shi estimates for Ricci flow (see [9, Section 7.1] for exposition).

Theorem 3.26. *Let $(\mathcal{M}^n, g(t), \phi(t))$ solve the $(RH)_c$ flow for $t \in [0, \omega)$ and $c(t)$ non-increasing. Assume $0 < \underline{c} \le c(t) \le \overline{c} < \infty$ for all t, and that $\omega < \infty$. Suppose that the curvature is uniformly bounded:*

$$\sup_{\mathcal{M} \times [0,\omega)} |\operatorname{Rm}| \le R_0.$$

Then there exists a constant $C = C(\underline{c}, \overline{c}, R_0, T, m, N) < \infty$ such that

$$\sup_{\mathcal{M} \times (0,\omega)} |\nabla \phi|^2 \le \frac{C}{t},$$

$$\sup_{M \times (0,\omega)} \left(|\operatorname{Rm}|^2 + |\nabla^2 \phi|^2 \right) \le \frac{C^2}{t^2}.$$

Moreover, there exist constants C_p depending on $p, \overline{c}$, m and N such that

$$\sup_{M \times (0,\omega)} \left(|\nabla^p \operatorname{Rm}|^2 + |\nabla^{p+2} \phi|^2 \right) \le C_p \left(\frac{C}{t} \right)^{p+2}.$$

Now we prove the theorem, in the presense of a bound on the injectivity radius. The proof of the groupoid statement will appear in the next subsection.

PROOF OF THEOREM 3.6. First, note that we may use a diagonalization argument, as in [16, Section 2], to show that we can assume that the interval of existence of the solutions is finite in length, that is,

$$-\infty < \alpha < \omega < \infty.$$

Since we are assuming that the curvatures are uniformly bounded, Theorem 3.26 applies to give uniform bounds on the derivatives of the curvatures and on the derivatives of the maps ϕ_k. With the former, and with the injectivity radius bound, we can use Theorem 3.25 to get pointed subsequential convergence of the metrics at a single time, say $0 \in (\alpha, \omega)$:

$$\big(\mathcal{M}_k, g_k(0), O_k\big) \to \big(\mathcal{M}_\infty, g_\infty, O_\infty\big).$$

The limit is a complete, pointed Riemannian manifold.

Unpacking this convergence, we have the existence of

- an exhaustion $\{\mathcal{U}_k\}$ of $\mathcal{M}_\infty$ by open sets with $O_\infty \in \mathcal{U}_k$ for all k, and
- a family of diffeomorphisms $\{\Psi_k : \mathcal{U}_k \to \mathcal{V}_k \subset \mathcal{M}_k\}$ with $\Psi_k(O_\infty) = O_k$

such that

$$\big(\mathcal{U}_k, \Psi_k^* g_k(0)|_{\mathcal{V}_k}\big) \longrightarrow (\mathcal{M}^n_\infty, g_\infty)$$

uniformly in C^∞ on compact sets.

The metrics and maps we are now interested in are $\bar{g}_k(t) = \Psi_k^* g_k(t)$ and $\bar{\phi}_k(t) = \Psi_k^* \phi_k(t)$.

Now we see that the hypotheses of the Lemma 3.8 are satisfied. For any compact $\mathcal{K} \subset \mathcal{M}_\infty$ and $[\beta, \psi] \subset [\alpha, \omega]$ containing 0, the collection $\{(\bar{g}_k(t), \bar{\phi}_k(t))\}$ is a sequence of $(RH)_c$ solutions on $K \times [\beta, \psi]$. Let g_∞ be the background metric and $t_0 = 0$.

The uniform convergence implies that the $\bar{g}_k(0)$ are uniformly equivalent to g_∞, and that the needed bounds hold. For example, using the equivalence of metrics and convergence at one time, we see that

$$\begin{aligned}
|\nabla^p_\infty \bar{\phi}_k(0)|_\infty &\le C |\nabla^p_\infty \bar{\phi}_k(0)|_{\bar{g}_k(0)} \\
&\le C |\nabla^p_{\bar{g}_k(0)} \bar{\phi}_k(0)|_{\bar{g}_k(0)} \\
&\le C |\nabla^p_{g_k(0)} \phi_k(0)|_{g_k(0)} \\
&\le C,
\end{aligned}$$

for large enough k.

By the lemma, we conclude that $\bar{g}_k(t)$ are uniformly equivalent to g_∞ on $\mathcal{K} \times [\beta, \psi]$, and that the time and space derivatives of $\bar{g}_k(t)$ and $\bar{\phi}_k(t)$ are uniformly bounded with respect to g_∞.

Now, the conditions of Lemma 3.24 are exactly satisfied by the implications of Lemma 3.8, so we have the desired subsequential convergence. Our limit solution is defined by

$$g_\infty(t) = \lim_{k \to \infty} \bar{g}_k(t), \quad \phi_\infty(t) = \lim_{k \to \infty} \bar{\phi}_k(t).$$

Finally, since all derivatives of the metric and the of map converge, the appropriate tensors converge, so that the limit is a metric/smooth map solving $(RH)_c$ flow. $\square$

3.5. The flow on groupoids. In [30] and [31], Lott initiated the use of Riemannian groupoids in understanding the convergence of Ricci flow solutions, especially in the presense of collapsing. This idea has also been used in [13]. We will not review groupoid theory, as the souces above do this well. We will, however, mention two other general references. A comprehensive guide to the subject, with an emphasis on differential geometry, is a book by Mackenzie [32]. A more concise introduction, with an emphasis on foliation theory, is a book by Moerdijk and Mrčun [33].

A Lie groupoid $\mathcal{G} \rightrightarrows \mathcal{B}$ is *Riemannian* if the base $\mathcal{B}$ has a $\mathcal{G}$-invariant metric g. That is, if $\mathcal{U} \subset \mathcal{B}$ is open, $\sigma \colon \mathcal{U} \to \mathcal{G}$ is any local bisection, and $t \colon \mathcal{G} \to \mathcal{B}$ is the target map, then $(t \circ \sigma)^* g = g$. From this, we can construct the Ricci tensor $\operatorname{Rc}[g]$, which is a symmetric $(2,0)$-tensor on $\mathcal{B}$, and which is $\mathcal{G}$-invariant in the same sense as g. Therefore it makes sense to consider the Ricci flow on this groupoid:

$$\frac{\partial}{\partial t} g = -2 \operatorname{Rc}.$$

Let $(\mathcal{N}, h)$ be another Riemannian manifold, thought of as a trivial groupoid, and consider $\phi \colon \mathcal{B} \to \mathcal{N}$ such that $\nabla\phi \otimes \nabla\phi$ is a $\mathcal{G}$-invariant $(2,0)$-tensor on $\mathcal{B}$. Additionally, the tension field $\tau_{g,h}\phi$ of ϕ is well-defined in the usual Riemannian manifold sense. Therefore, we have a well-defined coupling of Ricci flow and harmonic map flow:

$$\frac{\partial}{\partial t} g = -2 \operatorname{Rc} + 2c \nabla\phi \otimes \nabla\phi$$
$$\frac{\partial}{\partial t} \phi = \tau_{g,h}\phi$$

where $c(t)$ is a non-negative coupling function.

To use this approach to understand limits and convergence of $(RH)_c$ flow on Riemannian manifolds, we show how this groupoid setting can arise from the manifold setting. Let $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ be complete Riemannian manifolds, and $\phi \colon \mathcal{M} \to \mathcal{N}$ a smooth map. Select $\{p_i\}_{i \in I} \subset \mathcal{M}$ such that $\mathscr{U} = \{\mathcal{U}_i\}_{i \in I}$ is an open cover of $\mathcal{M}$, where the $\mathcal{U}_i$ are such that $\exp_{p_i}(0) = p_i \in \mathcal{U}_i$, and

$$\exp_{p_i}|_{B_{r_i}(0)} \colon B_{r_i}(0) \longrightarrow \mathcal{U}_i$$

is a diffeomorphism, for some sufficiently small $r_i > 0$. Put the metric $(\exp_{p_i})^* g$ on each $B_{r_i}(0)$. Call $\mathscr{U}$ an *open exponential cover* of $\mathcal{M}$.

As in [30, Example 5.7], from this we form a Riemannian groupoid $\mathcal{G}^{\mathscr{U}} \rightrightarrows \mathcal{B}^{\mathscr{U}}$, which is isometrically equivalent to the trivial groupoid $(\mathcal{M}, g)$. Set

$$\mathcal{B}^{\mathscr{U}} = \bigsqcup_{i \in I} B_{r_i}(0) = \{(i, v) \mid i \in I, v \in B_{r_i}(0)\},$$

$$\mathcal{G}^{\mathscr{U}} = \bigsqcup_{i,j \in I} \{(v_i, v_j) \in B_{r_i}(0) \times B_{r_j}(0) \mid \exp_{p_i}(v_i) = \exp_{p_j}(v_j)\}.$$

We will write elements of $\mathcal{B}^{\mathscr{U}}$ as $v_i = (i, v)$ and arrows as (v_i, v_j). Note that we always have $v_i = \exp_{p_i}^{-1}(x)$ for some $x \in U_i$.

The structure maps of this groupoid are defined as follows:

- source: $s(v_i, v_j) = v_i$
- target: $t(v_i, v_j) = v_j$
- unit: $u(v_i) = (v_i, v_i)$
- inverse: $(v_i, v_j)^{-1} = (v_j, v_i)$

- composition: $(v_j, v_k) \cdot (v_i, v_j) = (v_i, v_k)$

Call the étale Riemannian groupoid $\mathcal{G}^{\mathscr{U}} \rightrightarrows \mathcal{B}^{\mathscr{U}}$ the *Riemannian exponential groupoid* with respect to the open cover $\mathscr{U}$ of $\mathcal{M}$.

Proposition 3.27. *The $(RH)_c$ flow on a manifold $(\mathcal{M}, g, \phi)$ and target manifold $(\mathcal{N}, h)$ becomes $(RH)_c$ flow on the n-dimensional Riemannian exponential groupoid $(\mathcal{G}^{\mathscr{U}} \rightrightarrows \mathcal{B}^{\mathscr{U}}, g, \phi)$ associated to an open exponential cover $\mathscr{U}$ of $\mathcal{M}$.*

Proof. The map $\phi\colon \mathcal{M} \to \mathcal{N}$ induces a Lie groupoid morphism $\phi = (\phi_0, \phi_1)$ from $\mathcal{G}^{\mathscr{U}} \rightrightarrows \mathcal{B}^{\mathscr{U}}$ to the trivial groupoid $\mathcal{N} \rightrightarrows \mathcal{N}$. It is defined by

$$\begin{aligned} \phi_0(v_i) &= \phi(\exp_{p_i}(v_i)), \\ \phi_1(v_i, v_j) &= \phi(\exp_{p_i}(v_i)) = \phi(\exp_{p_j}(v_j)). \end{aligned}$$

Thus we can write $\phi_0 = \phi_1 = \exp^* \phi$ for these induced maps. Note also that we could have defined them as

$$\begin{aligned} \phi_0(v_i) &= \phi_0(\exp_{p_i}^{-1}(x)) = \phi(x), \\ \phi_1(v_i, v_j) &= \phi_1(\exp_{p_i}^{-1}(x), \exp_{p_j}^{-1}(x)) = \phi(x). \end{aligned}$$

It is easy to check that these maps are compatible with the structure maps of both groupoids. That is, the following diagram is commutative.

The main question is the $\mathcal{G}$-invariance of $\nabla\phi_0 \otimes \nabla\phi_0$. Let $\mathcal{U}_i \subset \mathcal{M}$ have coordinates (x^i), and let a neighborhood $\mathcal{V}_i$ of $\phi(p_i)$ have coordinates (y^α). Then $B_{r_i}(0) \subset \mathcal{B}^{\mathscr{U}}$ has coordinates (z^i), where

$$z^i = \exp_{p_i}^* x^i = x^i \circ \exp_{p_i},$$

and a coframe on $TB_{R_i}(0)$ is dz^i, where

$$dz^i = \exp_{p_i}^* dx^i = d(x^i \circ \exp_{p_i}).$$

To understand invariance, we must understand bisections of $\mathcal{G}^{\mathscr{U}} \rightrightarrows \mathcal{B}^{\mathscr{U}}$. Let σ be a bisection, say

$$\begin{aligned} \sigma\colon B_{r_i}(0) &\longrightarrow \mathcal{G}^{\mathscr{U}} \\ v_i &\longmapsto \big(\sigma_1(v_i), \sigma_2(v_i)\big) \end{aligned}$$

Since it is a bisection, we have $s \circ \sigma = \mathrm{id}_{\mathcal{B}^{\mathscr{U}}}$, and this implies $\sigma_1 = \mathrm{id}_{\mathcal{B}^{\mathscr{U}}}$. Therefore we write

$$\sigma(v_i) = \big(v_i, \tilde{\sigma}(v_i)\big),$$

where $\tilde{\sigma}(v_i)$ satisfies

$$\exp_{p_i} \tilde{\sigma}(v_i) = \exp_{p_i}(v_i).$$

Now we see that

$$(t \circ \sigma)(v_i) = t\big(v_i, \tilde{\sigma}(v_i)\big) = \tilde{\sigma}(v_i),$$

or $t \circ \sigma = \tilde{\sigma}$.

Now, the induced map $\phi_0\colon \mathcal{B}^{\mathscr{U}} \to \mathcal{N}$ has pushforward

$$(\phi_0)_* \in \Gamma(T^*\mathcal{B}^{\mathscr{U}} \otimes (\phi_0)^* T\mathcal{N}),$$

so

$$(\phi_0)_* = \frac{\partial \phi_0^\alpha}{\partial x^i}\, dx^i \otimes \left(\frac{\partial}{\partial y^\alpha}\right)_{\phi_0} = d\phi_0^\alpha \otimes \left(\frac{\partial}{\partial y^\alpha}\right)_{\phi_0}.$$

In any $B_{r_i}(0)$, we have

$$\begin{aligned}(t\circ\sigma)^* d\phi_0^\alpha &= d(\phi_0^\alpha \circ t\circ\sigma)\\ &= d(\phi^\alpha\circ \exp_{p_i}\circ\tilde{\sigma})\\ &= d(\phi^\alpha\circ\exp_{p_i})\\ &= d\phi_0^\alpha.\end{aligned}$$

If $f_0\colon B_{r_i}(0)\to\mathbb{R}$ is smooth, locally it is of the form $f_0 = f\circ\exp_{p_i}$ for some $f\colon\mathcal{U}_i\to\mathbb{R}$. Then

$$\begin{aligned}(t\circ\sigma)_*\left(\frac{\partial}{\partial y^\alpha}\right)_{\phi_0} f_0 &= \tilde{\sigma}_*\left(\frac{\partial}{\partial y^\alpha}\right)_{\phi_0} f_0\\ &= \partial_\alpha(f_0\circ\tilde{\sigma})\\ &= \partial_\alpha(f\circ\exp_{p_i}\circ\tilde{\sigma})\\ &= \partial_\alpha(f\circ\exp_{p_i})\\ &= \left(\frac{\partial}{\partial y^\alpha}\right)_{\phi_0} f_0.\end{aligned}$$

From this, we conclude that $\nabla\phi_0(\phi_0)_*$ is a G_M-invariant tensor.

In general, a metric h on $T\mathcal{N}$ induces a metric h_ϕ on the pull-back bundle $\phi^*T\mathcal{N}$, given by

$$h_\phi(\xi,\eta) = h(\phi_*\xi,\phi_*\eta),$$

for all $\xi,\eta\in T\mathcal{M}$. In this way, we get a metric on $(\phi_0)^*T\mathcal{N}$, and it is $\mathcal{G}^{\mathscr{U}}$-invariant:

$$(t\circ\sigma)^* h_{\phi_0}(\xi,\eta) = h_{\phi_0}(\xi,\eta).$$

Thus $\nabla\phi_0\otimes\nabla\phi_0$ is a $(2,0)$-tensor on $\mathcal{B}^{\mathscr{U}}$:

$$\nabla\phi_0\otimes\nabla\phi_0 = (h_{\phi_0})_{\lambda\mu}\partial_i\phi_0^\lambda\partial_j\phi_0^\mu\, dz^i\otimes dz^j.$$

It is therefore $\mathcal{G}^{\mathscr{U}}$-invariant, and the $(RH)_c$ flow makes sense on $\mathcal{G}^{\mathscr{U}}\rightrightarrows\mathcal{B}^{\mathscr{U}}$. $\square$

This proposition shows that this framework is at least non-vacuous. Before completing the proof of Theorem 3.6, we need a definition and a result of Lott.

Definition 3.28. Let $\{(\mathcal{G}_k\rightrightarrows\mathcal{B}_k, g_k,\phi_k, O_{x_k})\}$ be a sequence of pointed, n-dimensional Riemannian groupoids with maps into some fixed Riemannian manifold $(\mathcal{N},h)$. Let $\{(\mathcal{G}_\infty\rightrightarrows\mathcal{B}_\infty, g_\infty,\phi_\infty, O_{x_\infty})\}$ be a pointed Riemannian groupoid with map $\phi_\infty\colon\mathcal{B}_\infty\to\mathcal{N}$. Let J_1 be the groupoid of 1-jets of local diffeomorphisms of $\mathcal{B}_\infty$. We say that

$$(\mathcal{G}_k\rightrightarrows\mathcal{B}_k, g_k,\phi_k, O_{x_k})\longrightarrow(\mathcal{G}_\infty\rightrightarrows\mathcal{B}_\infty, g_\infty,\phi_\infty, O_{x_\infty})$$

in the *pointed smooth topology* if for all $R>0$, the following hold.

- There are pointed diffeomorphisms $\Psi_{k,R}\colon B_R(O_{x_\infty})\to B_R(O_{x_k})$, defined for large k, so that

$$\Psi_{k,R}^* g_k|_{B_R(O_{x_i})}\longrightarrow g_\infty|_{B_R(O_{x_\infty})}.$$

$$\Psi_{k,R}^*\phi_k|_{B_R(O_{x_i})}\longrightarrow \phi_\infty|_{B_R(O_{x_\infty})}.$$

- After conjugating by $\Psi_{k,R}$, the images of
$$s_k^{-1}\left(\overline{B_{R/2}(O_{x_k})}\right) \cap t_k^{-1}\left(\overline{B_{R/2}(O_{x_k})}\right)$$
converge in J_1 in the Hausdorff sense to the image of
$$s_\infty^{-1}\left(\overline{B_{R/2}(O_{x_\infty})}\right) \cap t_\infty^{-1}\left(\overline{B_{R/2}(O_{x_\infty})}\right)$$
in J_1.

The following is [30, Proposition 5.8].

Theorem 3.29. *Let $\{(\mathcal{M}_k, g_k, O_k)\}$ be a sequence of pointed complete n-dimensional Riemannian manifolds. Suppose that for each $p \geq 0$ and $r > 0$, there is some $C_{p,r} < \infty$ such that for all k,*
$$\max_{B_R(O_i)} |\nabla^p \operatorname{Rm}_k|_\infty \leq C_{p,r}.$$
Then there is a subsequence of $\{(M_k, O_k)\}$ that converges to some pointed n-dimensional Riemannian groupoid $(G_\infty \rightrightarrows B_\infty, g_\infty, O_{x_\infty})$ in the pointed smooth topology.

Now we can complete the proof.

Proof of Theorem 3.6. As Lott mentions, there is very little difference between the proofs of Hamilton's original theorem and [30, Theorem 5.12]. The same is true here. Namely, using Theorem 3.26, we obtain uniform bounds on the derivatives of the curvatures, which allow us to use Theorem 3.29. This is a version of Theorem 3.25 for groupoids, and gives subsequential convergence at one time to a pointed Riemannian groupoid.

To extend this to the whole time interval, we apply Lemma 3.8 and a version of 3.24, which gives another convergent subsequence. Hence we get a limiting metric and map, which together solve $(RH)_c$ flow on $\mathcal{M}$. $\square$

Remark 3.30. As in [30, Section 5], Theorem 3.6 implies that the space of pointed n-dimensional $(RH)_c$ flow solutions with $\sup_t t|\operatorname{Rm}[g(t)]|_\infty < C$ is relatively compact among all $(RH)_c$ solutions on étale Riemannian groupoids. Let $\mathscr{S}_{n,C}$ be the closure of this space. It is easy to see that the blowdown procedure from Example 3.7 defines an $\mathbb{R}^+$-action on the compact space $\mathscr{S}_{n,C}$.

4. A detailed example of $(RH)_c$ flow

We conclude with an example of $(RH)_c$ flow on the three-dimensional nilpotent Lie group Nil^3, and compare the asympotics of the solutions with those for Ricci flow.

Consider
$$\mathrm{Nil}^3 \cong \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\} \subset \mathrm{SL}_3 \mathbb{R}.$$
The obvious diffeomorphism with $\mathbb{R}^3$ provides global coordinates (x, y, z) in which the group multiplication is
$$(x, y, z) \cdot (z', y', z') = (x + x', y + y', z + z' + xy').$$
There is a frame of left-invariant vector fields,
$$F_1 = \frac{\partial}{\partial x}, \quad F_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad F_3 = \frac{\partial}{\partial z},$$

and the only nontrivial Lie bracket relation is

$$[F_1, F_2] = F_3.$$

The dual coframe is

$$\theta^1 = dx, \quad \theta^2 = dy, \quad \theta^3 = dz - x dy.$$

A family of left-invariant metrics on Nil^3 is given by

$$g(t) = A(t)\,\theta^1 \otimes \theta^1 + B(t)\,\theta^2 \otimes \theta^2 + C(t)\,\theta^3 \otimes \theta^3, \tag{4.1}$$

and the corresponding Ricci $(2,0)$-tensors satisfy

$$-2\,\mathrm{Rc}(g(t)) = \frac{C}{B}\,\theta^1 \otimes \theta^1 + \frac{C}{A}\,\theta^2 \otimes \theta^2 - \frac{C^2}{AB}\,\theta^3 \otimes \theta^3.$$

Proposition 4.2. *Solutions of Ricci flow on* Nil^3 *of the form* (4.1) *have the following asympototics:*

$$\begin{aligned} A(t) &\sim A_0 K^{-1/3} t^{1/3}, \\ B(t) &\sim B_0 K^{-1/3} t^{1/3}, \\ C(t) &\sim C_0 K^{1/3} t^{-1/3}, \end{aligned} \tag{4.3}$$

for the constant $K = A_0 B_0 / 3C_0$.

Ricci flow on Nil^3 been studied extensively. See, for example, [18], [22], [1], [30], [13], [36], [44]. We want to study $(RH)_c$ flow on Nil^3. Consider a function

$$\phi\colon \big(\mathrm{Nil}^3, g(t)\big) \longrightarrow (\mathbb{R}, g_{\mathrm{can}}),$$

and let $c = c(t) \geq 0$ be a non-increasing function. For the resulting $(RH)_c$ flow system to remain a system of ordinary differential equations for the metric, we need ϕ to be harmonic and $\nabla\phi \otimes \nabla\phi = d\phi \otimes d\phi$ to be a diagonal left-invariant tensor. It is not hard to see that the latter condition requires that

$$\phi(x, y, z) = ax + by,$$

for some $a, b \in \mathbb{R}$. Note that such a function is also a group homomorphism, and that

$$\tau_{g, g_{\mathrm{can}}} \phi = g^{ij} (\partial_i \partial_j \phi - \Gamma_{ij}^k \partial_k \phi) = 0,$$

so it is harmonic. Then

$$d\phi \otimes d\phi = a^2\, dx \otimes dx + ab(dx \otimes dy + dy \otimes dx) + b^2\, dy \otimes dy.$$

To keep the system diagonal, take $b = 0$, so that $d\phi \otimes d\phi = a^2\, \theta^1 \otimes \theta^1$. The $(RH)_c$ flow system is

$$\begin{aligned} \frac{d}{dt} A &= \frac{C}{B} + 2a^2 c, \\ \frac{d}{dt} B &= \frac{C}{A}, \\ \frac{d}{dt} C &= -\frac{C^2}{AB}. \end{aligned} \tag{4.4}$$

Let us first make a few general observations about the long-time behavior of A, B, and C. Set $f(t) = 2a^2c(t)$ for simplicity. Note that $\Phi = BC = B_0C_0$ is conserved, A and B are increasing, and C is decreasing. This implies

$$C' = -\frac{C^2}{AB} \geq -\frac{1}{A_0B_0}C^2,$$

and integrating tells us that

$$0 < \frac{A_0B_0C_0}{A_0B_0 + C_0t} \leq C(t) \leq C_0,$$

for $t \geq 0$. We conclude that $C(t) \to C_\infty \in [0, C_0)$ as $t \to \infty$. Similarly, we see that

$$C' = -\frac{C^3}{\Phi A} \geq -\frac{1}{\Phi A_0}C^3,$$

which implies

$$0 < \frac{A_0B_0C_0^2}{A_0B_0 + 2C_0^2t} \leq C(t)^2 \leq C_0^2.$$

This gives

$$\int_0^t C(s)^2\, ds \geq \int_0^t \frac{A_0B_0C_0^2}{A_0B_0 + 2C_0s} ds \longrightarrow \infty \tag{4.5}$$

as $t \to \infty$.

Next we use Φ to see that

$$A' = \frac{C^2}{\Phi} + f = \frac{\Phi}{B^2} + f, \tag{4.6}$$

which we integrate to obtain

$$A(t) = A_0 + \frac{1}{\Phi}\int_0^t C(s)^2\, ds + \int_0^t f(s)\, ds. \tag{4.7}$$

By (4.5), we have $A(t) \to \infty$ as $t \to \infty$, and we have a bound on the growth of A:

$$A(t) \leq A_0 + \frac{C_0^2}{\Phi}\int_0^t ds + f_0 \int_0^t ds \leq A_0 + \left(\frac{C_0}{B_0} + f_0\right)t. \tag{4.8}$$

This implies

$$\int_0^t \frac{ds}{A(s)} \geq \int_0^t \frac{ds}{A_0 + \left(\frac{C_0}{B_0} + f_0\right)s} \longrightarrow \infty \tag{4.9}$$

as $t \to \infty$.

Finally,

$$(B^2)' = \frac{2\Phi}{A},$$

which implies

$$B(t)^2 = B_0^2 + 2\Phi \int_0^t \frac{ds}{A(s)}, \tag{4.10}$$

so, by (4.9), $B(t) \to \infty$ and $C(t) = \Phi/B(t) \to 0$ as $t \to \infty$.

4.1. Constant coupling function. Let us now consider the case when c (and therefore f, which we write as f_0) is a constant. From (4.6) we compute that

$$\lim_{t\to\infty} \frac{A(t)}{f_0 t} \overset{LH}{=} \lim_{t\to\infty} \frac{A'(t)}{f_0} = \lim_{t\to\infty} \left(\frac{C^2}{f_0 \Phi} + 1 \right) = 1,$$

so $A(t) \sim f_0 t$. Using (4.10) and $A \sim f_0 t \sim f_0(t+1)$, we have

$$B^2 \sim 2\Phi \int_0^t \frac{ds}{A(s)} \sim \frac{2\Phi}{f_0} \int_0^t \frac{ds}{s+1} \sim \frac{2\Phi}{f_0} \log t.$$

This gives the following.

Proposition 4.11. *Solutions of $(RH)_c$ flow on* Nil^3 *of the form* (4.1) *with map* $\phi(x,y,z) = ax$ *and* $c > 0$ *constant have the following asympototics:*

$$\begin{aligned} A(t) &\sim 2a^2 ct, \\ B(t) &\sim \sqrt{\frac{B_0 C_0}{a^2 c} \log t}, \\ C(t) &\sim 2\sqrt{\frac{a^2 c B_0 C_0}{\log t}}. \end{aligned} \tag{4.12}$$

Note that if we attempt to take a limit of these solutions as $f_0 \to 0$, they *do not* converge in a naive sense to the solutions of Ricci flow from (4.3). To explain this, we examine certain coupling functions that decay as $t \to \infty$, and which yield behavior similar to that for Ricci flow.

4.2. Nonconstant coupling function. Now consider a coupling function such that

$$c(t) \sim \frac{1}{t^r},$$

where $r \geq 1$. We make the ansatz that $A(t) \sim \alpha t^p$, for some $\alpha, p > 0$ to be determined. From (4.8), it is consistent to assume that $0 < p \leq 1$. Then using (4.7),

$$\lim_{t\to\infty} \frac{A(t)}{\alpha t^p} \overset{LH}{=} \lim_{t\to\infty} \frac{A'(t)}{p\alpha t^{p-1}} = \lim_{t\to\infty} \frac{1}{p\alpha t^{p-1}} \left(\frac{\Phi}{B^2} + \frac{2a^2}{t^r} \right). \tag{4.13}$$

Finding this limit comes down to analyzing two limits:

$$\lim_{t\to\infty} \frac{1}{B^2 t^{p-1}}, \tag{4.14}$$

$$\lim_{t\to\infty} \frac{1}{t^{r+p-1}}. \tag{4.15}$$

Since $r \geq 1$ implies that (4.15) is zero for any $p > 0$, we need that

$$B \sim \beta t^{\frac{1-p}{2}}$$

for some $\beta > 0$. To find β, consider

$$B^2 \sim 2\Phi \int_0^t \frac{ds}{A(s)} \sim \frac{2\Phi}{\alpha} \int_0^t \frac{ds}{(s+1)^p} \sim \frac{2\Phi}{\alpha(1-p)} t^{1-p}.$$

This now implies

$$\lim_{t\to\infty} \frac{1}{B^2 t^{p-1}} = \frac{\alpha(1-p)}{2\Phi},$$

and so

$$1 \overset{?}{=} \lim_{t\to\infty} \frac{A(t)}{\alpha t^p} = \frac{\Phi}{p\alpha} \lim_{t\to\infty} \frac{1}{B^2 t^{p-1}} = \frac{\Phi}{p\alpha} \frac{\alpha(1-p)}{2\Phi} = \frac{1-p}{2p}.$$

For $A(t) \sim \alpha t^p$ we therefore need $p = 1/3$. From here, we obtain the asymptotic behavior. Modulo constants, it is that of the Ricci flow solutions (4.3).

Proposition 4.16. *Solutions of* $(RH)_c$ *flow on* Nil^3 *of the form* (4.1) *with map* $\phi(x, y, z) = ax$ *and* $c \sim 1/t^r$, $r \geq 1$, *have the following asympototics:*

$$\begin{aligned} A &\sim \alpha t^{1/3}, \\ B &\sim \sqrt{\frac{3\Phi}{\alpha}} t^{1/3}, \\ C &\sim \sqrt{\frac{\alpha\Phi}{3}} t^{-1/3}, \end{aligned} \tag{4.17}$$

for some constant α *depending on* r *and the initial data.*

The reason that the limit as $f_0 \to 0$ of the solutions (4.12) is not the Ricci flow solutions (4.3) lies in the integrability of the coupling function. Informally, the lack of such a limit results from

$$0 = \lim_{t\to\infty} \lim_{f_0\to 0} \int_0^t f_0 \, ds \neq \lim_{f_0\to 0} \lim_{t\to\infty} \int_0^t f_0 \, ds = \infty.$$

To be more precise, consider $A(t)$ as given in (4.7). When $r > 1$, $f(t)$ is integrable, allowing

$$\int_0^t C(s)^2 \, ds$$

to dominate and produce growth like $t^{1/3}$. When $r < 1$, $f(t)$ is not integrable, and

$$\int_0^t f(s) \, ds$$

dominates to produce linear growth.

In numerical simulations, $0 < r < 1$ appears to be a transitionary region where solutions have properties of both (4.12) and (4.17). We were unable to obtain the precise asymptotics, but we expect that letting $r \to 0$ should recover (4.12) and letting $r \to \infty$ should recover (4.3).

APPENDIX A

Curvature of Lie groups

In this section, we recall some general facts about the geometry of Lie groups with left-invariant metrics, and derive the formula for the Ricci tensor that was used above.

Let $\langle \cdot, \cdot \rangle$ be a left-invariant metric on a Lie group G, which is equivalent to an inner product on the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Let ∇ denote the Levi-Civita connection for the metric, and Let $X, Y, Z, W \in \mathfrak{g}$. Recall that $\mathrm{ad}_X = [X, \cdot]$, and its adjoint with respect to $\langle \cdot, \cdot \rangle$ is defined by

$$\langle (\mathrm{ad}_X)^* Y, Z \rangle = \langle Y, \mathrm{ad}_X Z \rangle.$$

Remark 0.18. Formulas like those in the following propositions appear throughout the literature (e.g., [4] and [7]). Most of these, however, are derived with the goal of expressing the various related curvatures with respect to a fixed orthonormal basis. As we are working with evolving metrics, with no initial assumptions on orthonormality, it is more convenient to have curvature formulas that do not depend on an orthonormal basis.

Proposition 0.19. *We have the following formulas for ∇ and the Riemannian curvature tensor:*

(a) $\nabla_X Y = \dfrac{1}{2}\big(\mathrm{ad}_X Y - (\mathrm{ad}_X)^* Y - (\mathrm{ad}_Y)^* X \big)$,

(b) $\langle R(X,Y)Z, W \rangle = \langle \nabla_X Z, \nabla_Y W \rangle - \langle \nabla_Y Z, \nabla_X W \rangle - \langle \nabla_{[X,Y]} Z, W \rangle$.

This result is standard, so we omit the proof. Now, the maps $(X, Y) \mapsto \mathrm{ad}_X Y$ and $(X, Y) \mapsto (\mathrm{ad}_X)^* Y$ are bilinear maps $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Define

$$\begin{aligned} U \colon \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathfrak{g} \\ (X, Y) &\longmapsto -\frac{1}{2}\Big((\mathrm{ad}_X)^* Y + (\mathrm{ad}_Y)^* X \Big) \end{aligned}$$

This is symmetric, bilinear, and $U(X, X) = -(\mathrm{ad}_X)^* X$. It is useful in computing the Riemannian curvature tensor, as we shall see.

Proposition 0.20. *The Riemannian curvature* $(4, 0)$*-tensor on G is given by*

$$\begin{aligned} &4\langle R(X,Y)Z, W \rangle \\ &= 2\langle [X,Y], [Z,W] \rangle + \langle [X,Z], [Y,W] \rangle - \langle [X,W], [Y,Z] \rangle \\ &\quad - \langle [[X,Y],Z], W \rangle + \langle [[X,Y],W], Z \rangle - \langle [[Z,W],X], Y \rangle + \langle [[Z,W],Y], X \rangle \\ &\quad + 4\langle U(X,Z), U(Y,W) \rangle - 4\langle U(X,W), U(Y,Z) \rangle. \end{aligned}$$

As a special case,

$$\begin{aligned}\langle R(X,Y)Y,X\rangle = \frac{1}{4}\|(\mathrm{ad}_X)^*Y + (\mathrm{ad}_Y)^*X\|^2 - \langle(\mathrm{ad}_X)^*X, (\mathrm{ad}_Y)^*Y\rangle \\ - \frac{3}{4}\|[X,Y]\|^2 - \frac{1}{2}\langle[[X,Y],Y],X\rangle - \frac{1}{2}\langle[[Y,X],X],Y\rangle,\end{aligned}$$

which is the sectional curvature $K(X\wedge Y)$ if X and Y are orthonormal.

The derivation of the first formula is straight-forward, relying mainly on Proposition 0.19 and various Lie bracket manipulations. The second formula follows immediately from the first.

Let $\{e_i\}$ be a basis for $\mathfrak{g}$. Then we write

$$\mathrm{ad}_{e_i} e_j = c_{ij}^k e_k, \quad (\mathrm{ad}_{e_i})^* e_j = a_{ij}^k e_k, \quad \langle e_i, e_j\rangle = g_{ij}.$$

We can use this to write the above formulas in terms of components.

Corollary 0.21. (a) *If $\nabla_{e_i} e_j = \gamma_{ij}^k$, then*

$$\gamma_{ij}^k = \frac{1}{2} g^{kl}\big(c_{ij}^m g_{lm} - c_{il}^m g_{jm} - c_{jl}^m g_{im}\big);$$

(b) *The components of the Riemann curvature $(4,0)$-tensor satisfy*

$$\begin{aligned}4R_{ijkl} &= 2c_{ij}^p c_{kl}^q g_{pq} + c_{ik}^p c_{jl}^q g_{pq} - c_{il}^p c_{jk}^q g_{pq} \\ &\quad - c_{ij}^p c_{pk}^q g_{ql} + c_{ij}^p c_{pl}^q g_{qk} - c_{kl}^p c_{pi}^q g_{qj} + c_{kl}^p c_{pj}^q g_{qi} \\ &\quad + (a_{ik}^p + a_{ki}^p)(a_{jl}^q + a_{lj}^q) g_{pq} - (a_{il}^p + a_{li}^p)(a_{jk}^q + a_{kj}^q) g_{pq}.\end{aligned} \tag{0.22}$$

(c) *The components of the Ricci curvature $(2,0)$-tensor satisfy*

$$\begin{aligned}4R_{ij} &= \big(2c_{ki}^p c_{jm}^q g_{pq} + c_{kj}^p c_{im}^q g_{pq} - c_{km}^p c_{ij}^q g_{pq} \\ &\quad - c_{ki}^p c_{pj}^q g_{qm} + c_{ki}^p c_{pm}^q g_{qj} - c_{jm}^p c_{pk}^q g_{qi} + c_{jm}^p c_{pi}^q g_{qk} \\ &\quad + (a_{kj}^p + a_{jk}^p)(a_{im}^q + a_{mi}^q) g_{pq} - (a_{km}^p + a_{mk}^p)(a_{ij}^q + a_{ji}^q) g_{pq}\big) g^{km}.\end{aligned} \tag{0.23}$$

(d) *The sectional curvature $K(e_i \wedge e_j)$ satisfies*

$$\begin{aligned}4K_{ij} &= \Big(3c_{ij}^p c_{ji}^q g_{pq} - c_{ij}^p c_{pj}^q g_{qi} + c_{ij}^p c_{pi}^q g_{qj} - c_{ji}^p c_{pi}^q g_{qj} + c_{ji}^p c_{pj}^q g_{qi} \\ &\quad + (a_{ij}^p + a_{ji}^p)(a_{ji}^q + a_{ij}^q) g_{pq} - (a_{ii}^p + a_{ii}^p)(a_{jj}^q + a_{jj}^q) g_{pq}\Big)/(g_{ii}g_{jj} - g_{ij}^2).\end{aligned}$$

(e) *The scalar curvature satisfies*

$$\begin{aligned}4S &= \big(2c_{ki}^p c_{jm}^q g_{pq} + c_{kj}^p c_{im}^q g_{pq} - c_{km}^p c_{ij}^q g_{pq} \\ &\quad - c_{ki}^p c_{pj}^q g_{qm} + c_{ki}^p c_{pm}^q g_{qj} - c_{jm}^p c_{pk}^q g_{qi} + c_{jm}^p c_{pi}^q g_{qk} \\ &\quad + (a_{kj}^p + a_{jk}^p)(a_{im}^q + a_{mi}^q) g_{pq} - (a_{km}^p + a_{mk}^p)(a_{ij}^q + a_{ji}^q) g_{pq}\big) g^{ij} g^{km}.\end{aligned}$$

We finally note that the "adjoint structure constants" a_{ij}^k can be expressed in terms of c_{ij}^k and g_{ij}, by using the definition of ad^*:

$$a_{ij}^k = c_{il}^m g_{jm} g^{kl}. \tag{0.24}$$

This formula makes it possible to eliminate the a_{ij}^k from the curvature formulas.

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