A Solution for Superposition of Multiple Wheel Live Loads Through Earth Fill Using the Gauss Circle Problem

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ABSTRACT

Buried structures such as pipes and reinforced concrete boxes are subject to a combination of multiple wheel live loads, whose imparted vertical stresses must be accounted for in a structural analysis. An accurate estimation of these stresses can be a daunting task. An equation is presented based on the solution to the Gauss Circle Problem, to calculate the magnitude of stress at any depth due to multiple lanes of typical, code-specified, design vehicle wheel loads. The result provides excellent comparison to a systematic calculation of the expected demand, as well as field data on distributed wheel pressures. The approach is extended to apply to horizontal surcharge pressures on buried structures, such as retaining walls and reinforced concrete boxes.

KEYWORDS
Buried structures, live load, distribution through fill, vertical earth pressures, lateral earth pressures, Gauss Circle Problem, Culverts, Comparative studies, Numerical analysis

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INTRODUCTION

Buried structures are typically designed for a combination of vertical earth pressure and vertical live load pressure resulting from moving live wheel loads on highway or local street lanes. A point or wheel load, P, at the surface is assumed to provide a pressure over an area larger than the initial applied surface area; this area increases linearly with depth. AASHTO [1] specifies in Section 3.6.1.2.6 – Distribution of Wheel Loads Through Earth Fills,

"*…where the depth of fill is 2.0 ft. or greater, wheel loads may be considered to be uniformly distributed over a rectangular area with sides equal to the dimension of the tire contact area,… and increased by either 1.15 times the depth of the fill in select granular backfill, or the depth of the fill in all other cases.*"

In equation form, this may be represented as:

$$\sigma = \frac{P}{\left(B+{}^{2\cdot Z}/_{V}\right)\cdot\left(L+{}^{2\cdot Z}/_{V}\right)} \quad (1)$$

where
P = Wheel Load
Z = Depth from Ground Surface
B, L = Dimensions of Tire Contact Area
V = Slope of Pressure Distribution

This calculation is straightforward. However, when one begins to consider overlapping wheel loads due to multiple lanes and multiple vehicles, the calculation is cumbersome and prone to error.

Solutions to the Gauss Circle Problem lend much time savings to this dilemma. The Gauss Circle Problem considers how many points, arranged in an array of constant dimensions, fall within a circle of radius "r." This paper will step through a graphical solution of the problem, followed by analytical solutions using exact and approximate solutions of the Gauss Circle Problem.

GRAPHICAL SOLUTION

The calculation of pressure is straightforward for a single wheel load, as shown in Equation 1. However, the typical design truck has 6 or 14 wheels whose influence may overlap at a given depth for the HL-93 and Tandem design vehicles, respectively. Design axle loads and spacing are shown in Figures 1 and 2 for HL-93 and Tandem vehicles respectively. When one considers that there is usually more than one lane and typically a train of trucks is not unreasonable to occur, the amount of overlapping wheel loads is an incredible bookkeeping effort.

This bookkeeping effort exists because (a) the spacing of axles varies from four to approximately 18 feet, (b) wheels between adjacent lanes are separated by 6 feet, and (c) any number of lanes may occur. Therefore, in order to achieve a closed form solution of the vertical stress with depth, the following ***simplifying assumptions*** are made:

(i) there is a single constant spacing, S, between every wheel, and
(ii) there is an infinite array of wheels.

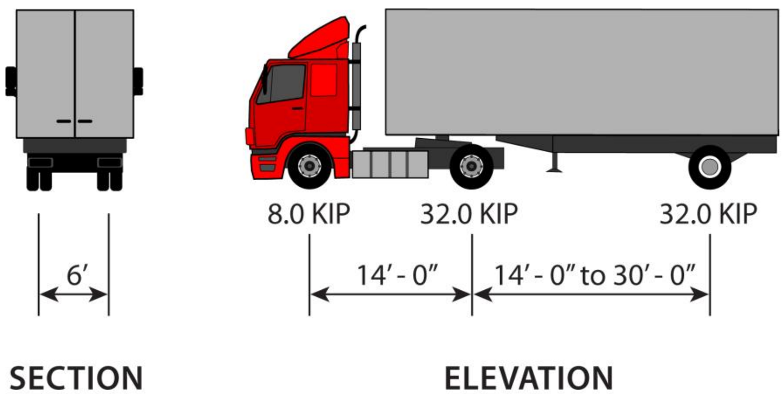

Figure 1: Design HL-93 vehicle

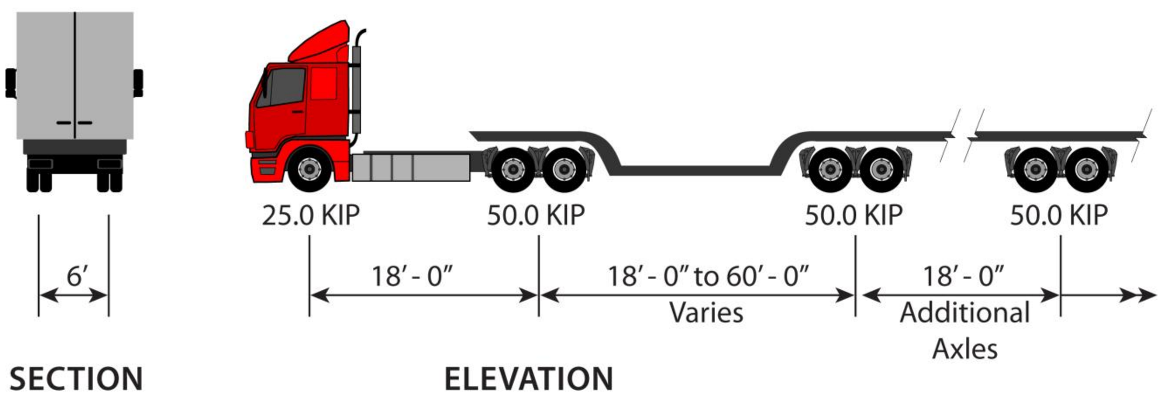

Figure 2: Design Tandem vehicle

Based on the assumptions stated above, a graphical solution as shown in Figure 3 and Figure 4 is first derived for each instance the wheel loads, distributing in increasing circular area, begin to overlap. The number of wheels which overlap at each instance is defined in this paper as the variable G_r.

Presented in Figure 5, a cross section through a line of wheel loads shows there is a solution to specific configurations "*r*", or instances, where the wheel loads overlap.

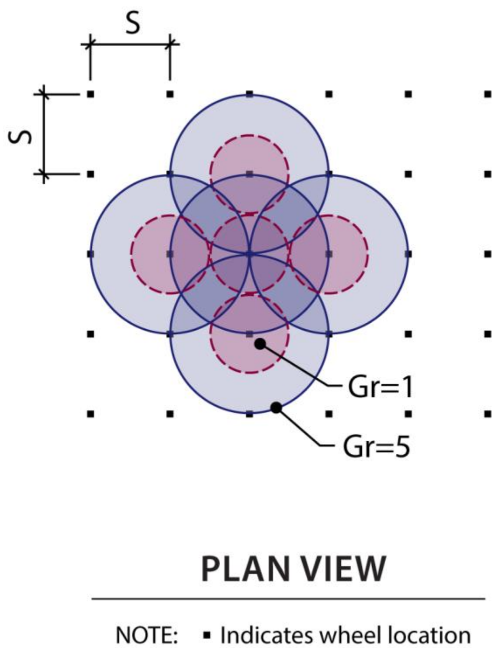

Figure 3: Design HL-93 vehicle, G_r = 1, 5

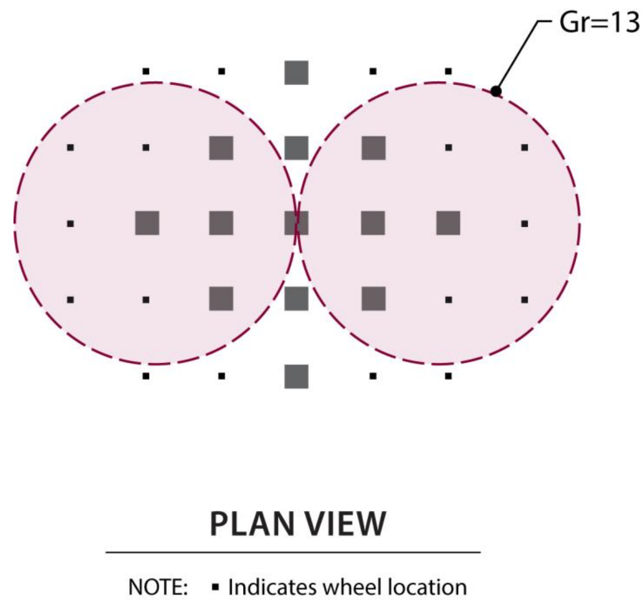

Figure 4: Design HL-93 vehicle, G_r = 13 (Note, only the influence of the extreme tire loads is shown for clarity)

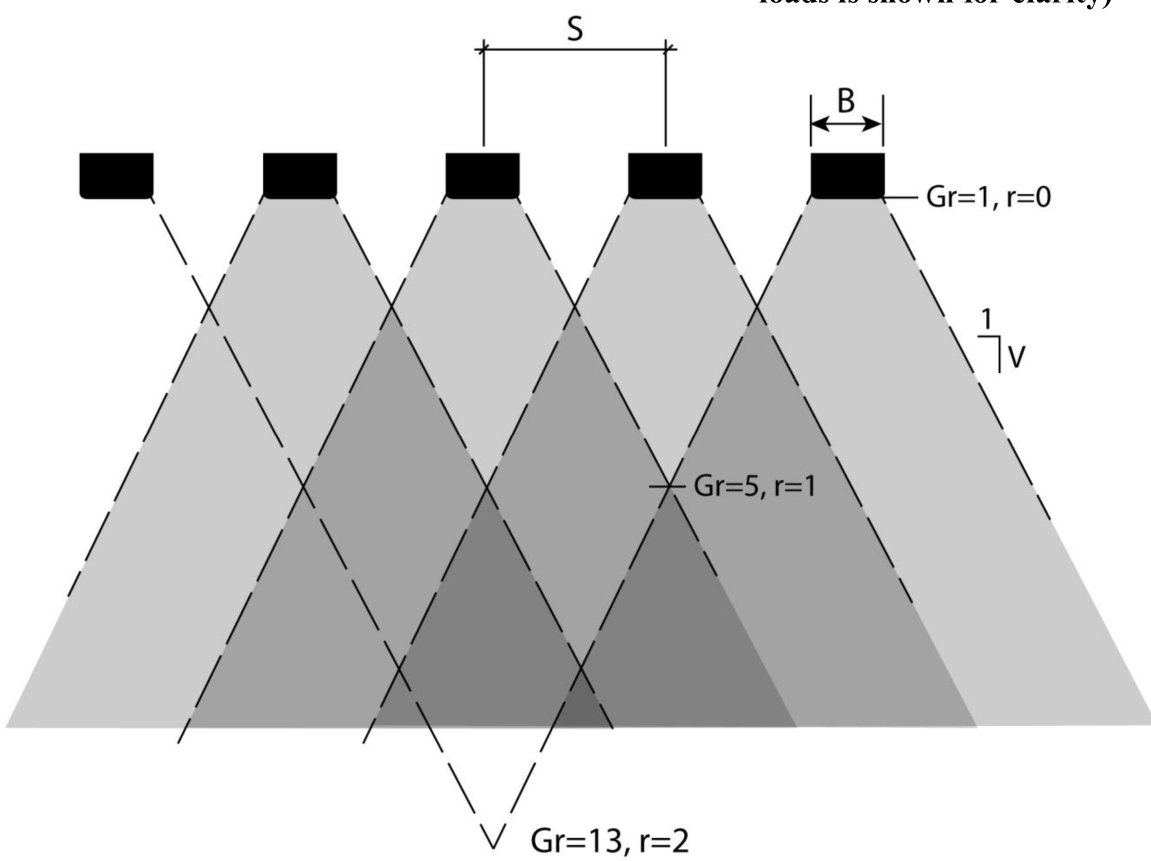

Figure 5: Section showing distribution of stress

The equations which describe the general cross section of Figure 5 can be determined by inspection of the geometry and each instance where loads overlap. Several cases are listed in Table 1. Additional variables listed include “r,” which indicates a specific number in the sequence to be used later in this article at each depth of interest.

Table 1. Geometric Properties of Figure 5

r	G_r	Z
0	1	$Z = V \cdot 0 \cdot S$
1	5	$Z = V \cdot 1 \cdot S$
2	13	$Z = V \cdot 2 \cdot S$
r	*f(r)*	$Z = V \cdot r \cdot S$

From Table 1, the depth Z is defined as follows:

$$Z = V \cdot r \cdot S \tag{2}$$

Solving for *r*,

$$r = Z \;/\; (V \cdot S) \tag{3}$$

Where Z = depth below grade, V = slope of wheel distribution through fill, and S = typical wheel spacing. Equation 3 is necessary as one considers use of various solutions to the Gauss Circle Problem, following.

Analytical Solution Using Superposition and the Gauss Sequence

The graphical solution provides a basis and contribution for a more refined analytical solution. It is observed that the number of wheels which overlap at each instance, *r*, is predicted by a number, G_r, in the Hilbert and Cohn-Vossen sequence, or “Gauss Sequence” (as termed in this paper):

$$G_r = G(r) = 1 + \sum_{i=0}^{\infty}\left(floor\left({r^2}/_{4i+1} \right) - floor\left({r^2}/_{4i+3} \right) \right) \tag{4}$$

Or,

$$G_r = G(r) = 1 + \sum_{i=0}^{\infty}\left(\left\lfloor {r^2}/_{4i+1} \right\rfloor - \left\lfloor {r^2}/_{4i+3} \right\rfloor \right) \tag{5}$$

Equations 4 and 5 above are the exact solution of the Gauss circle problem given in series form [2]. Gauss provides an approximate solution as a function of the area of a circle of radius "r", with an additional error term as follows [3],

$$G_r = G(r) = \pi r^2 + 2\sqrt{2} \cdot \pi r \tag{6}$$

Replacing the approximate error term of $2\sqrt{2} \cdot \pi r$ with $\alpha \cdot \pi r$, the result is,

$$G_r = G(r) = \pi r^2 + \alpha \cdot \pi r \tag{7}$$

Equation 6 will be used later in the paper to provide the engineer with an easier means of estimating earth pressures due to overlapping wheel loads.

Finally, since the projected influence of wheels overlap by the number G_r, the actual stress at any depth is equal to the superposition, or addition, of these wheel pressures as follows:

$$\sigma = \frac{P \cdot G_r}{\left(B + {}^{2\cdot Z}/_{V}\right) \cdot \left(L + {}^{2\cdot Z}/_{V}\right)} \tag{8}$$

where
Z = Depth from Ground Surface
B, L = Dimensions of Tire Contact Area
V = Slope of Pressure Distribution
$r = Z \,/\, (V \cdot S)$
$G_r = \pi r^2 + \alpha \cdot \pi r$

The method of calculating live load pressures using Equation 8 will be referred to in the remainder of this article as "Superposition using the Gauss Sequence," or the acronym "SGS."

Comparison to Systematic Calculation of Wheel Pressures

For comparison to a rigorous addition of overlapping wheel pressures, the expected magnitude of wheel pressures at any depth is calculated based on five lanes of traffic, with trains of both HL-93 and Tandem vehicles. Dimensions of B = 10 inches and L = 20 inches for wheel contact area are used in the comparison, according to AASHTO [1] Section 3.6.1.2.5. Figures 6 and 7 show the comparison of superimposed stresses by both the "systematic" and "SGS" approaches, for HL-93 and Tandem vehicles, respectively. Here "SGS" refers to the approach suggested in this paper using superposition and the Gauss Sequence, while "systematic" refers to the rigorous method of superimposing multiple wheel loads over multiple lanes of traffic. For each figure, two plots are indicated for the simplified approach.

The first plot (Curve 1) uses Equation 5 (Exact Solution) with wheel spacing varied such that the area under the curve, with respect to the "z" axis, matches the area of the systematic curve. Figure 6 shows the SGS approach well predicts the shape of the systematic plot.

The second plot (Curve 2) uses Equation 7 (Simplified Solution) with "α" coefficient set equal to $2\sqrt{2}$ and wheel spacing varied also such that the area under the curve (with respect to the y-axis) matches the area defined by the **peak points** on the systematic plot.

The resulting wheel spacings and "α" coefficients are summarized in Table 2, for use by the Engineer who will not know the correct variables "a priori."

On Figure 6, points are presented for comparison from Abel-Karim et al. [2], who measured stresses in the field on a buried double-cell concrete box culvert. Vehicle loading included a traditional HS-20 (HL-93) truck.

On Figure 7, points are presented for comparison from Acharya et al. [5], James and Brown [6], and McGrath et al. [7]. These studies provide measured stresses in the field on buried structures from Tandem vehicles.

Table 2. Calculated Wheel Spacing for Simplified and Envelope Solution

Design Vehicle	Axle Spacing per AASHTO [1]		SGS Curve 1 (Exact) *(Uses exact Gauss Solution)*	SGS Curve 2 (Approximate) *(Uses Approximate Gauss Solution)*	
	Minimum[1]	Maximum	Wheel Spacing (ft)	Wheel Spacing (ft)	α
HL-93	14.0 ft	30.0 ft	4.5 ft	5.2 ft	$2\sqrt{2}$
Tandem	18.0 ft	60.0 ft	4.5 ft	5.6 ft	$2\sqrt{2}$

[1] Note: Spacing between wheels of adjacent lanes equal to six feet.

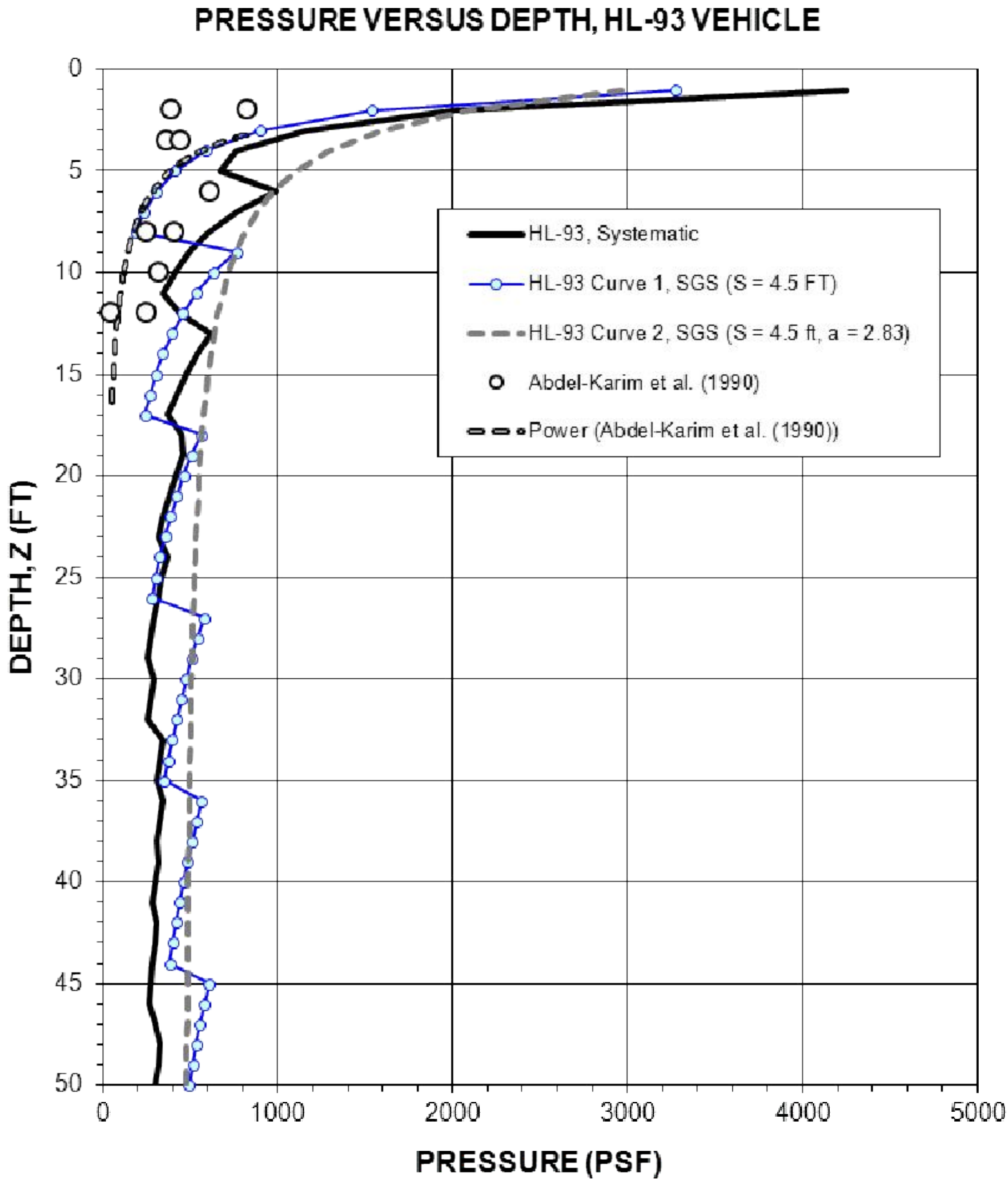

Figure 6: Pressure versus depth for AASHTO HL-93 Live Load.

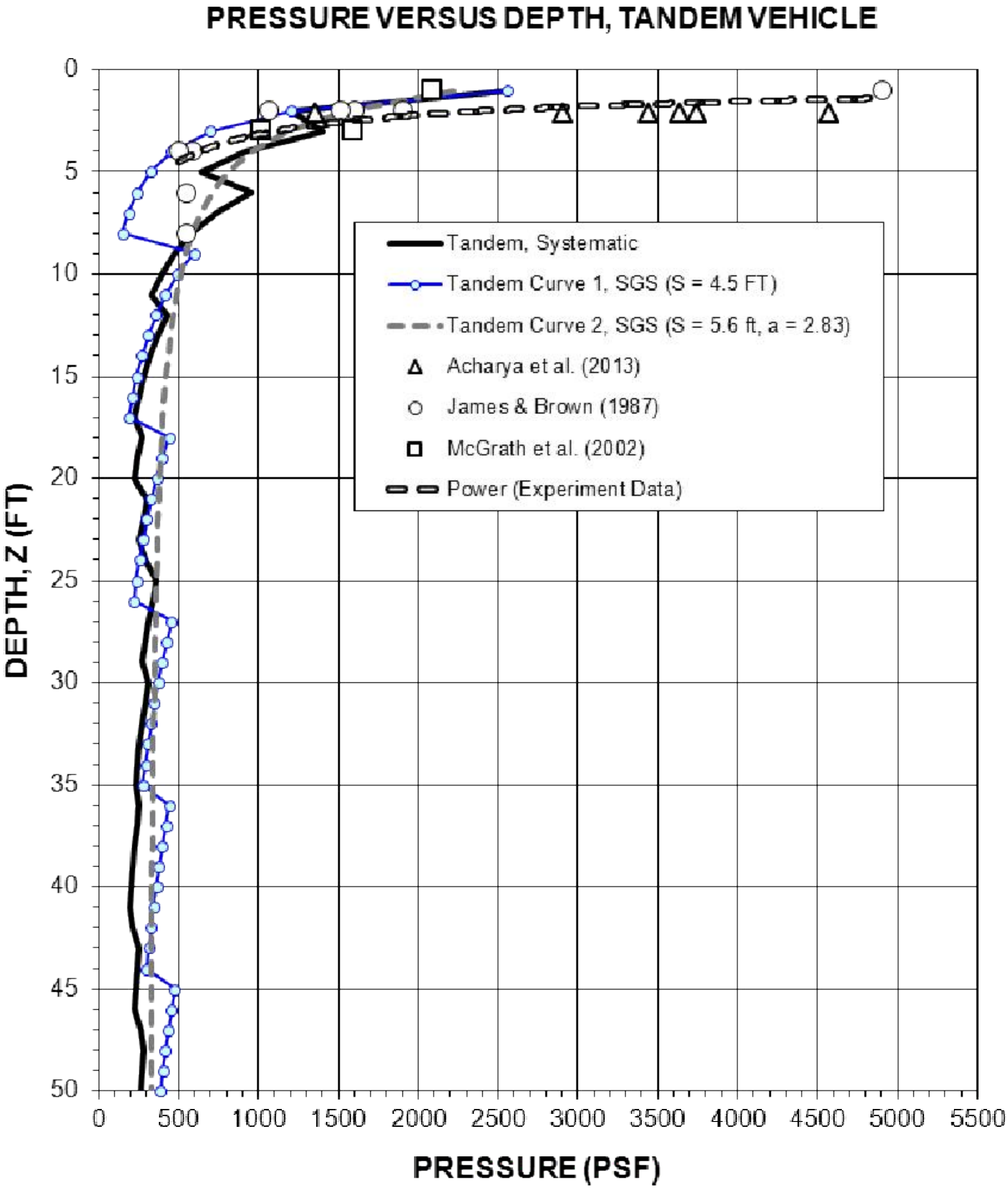

Figure 7: Vertical Pressure versus depth for AASHTO Tandem Live Load.

As shown in Figure 6 for the HL-93 and Figre 7 for the Tandem vehicle, the shape of the systematic plot compares well with the first SGS curve, and well-enveloped by the second SGS curve.

SURCHARGE LOAD

The approach is now extended to the apply to horizontal stresses on buried structures such as retaining walls and culverts, again in the interest of saving computation and programming time for the engineer. The conventional approach for horizontal loads on buried structures is to use the Boussinesq Method. Again, with multiple wheel loads this can be a cumbersome task. Note that the pressure due to a single point load using the Boussinesq method is satisfactory or conservative at shallow depths, but not considering multiple wheel loads at greater depths would

The horizontal pressure Δ_{ph} in ksf, on a wall resulting from a point load according to the Boussinesq Method (AASHTO [1], Equation 3.11.6.2-2) is given as:

$$\Delta_{ph} = \frac{P}{\pi R^2}\left[\frac{3ZX^2}{R^3} - \frac{R(1-2\nu)}{R+Z}\right] \tag{9}$$

Where:

P = point load (kip)
R = racial distance from point of load application to a point on the wall as specified in Figure 3.11.6.2-2 where $R^2 = (x^2 + y^2 + z^2)^{0.5}$ (ft)
X = horizontal distance from back of wall to point of load application (ft)
Y = horizontal distance from point of wall under consideration to a plane, which is perpendicular to the wall and passes through the point of load application measured along the wall (ft)
Z = vertical distance from point of load application to the elevation of a point on the wall under consideration (ft)
ν = Poisson's ratio (dim)

An alternative approach to the calculation above may combine the SGS solution, with the following equation from AASHTO [1],

$$\Delta_p = k_s \cdot q_s \tag{10}$$

Where:

Δ = constant horizontal earth pressure due to uniform surcharge (ksf)
k_s = coefficient of earth pressure due to surcharge
q_s = uniform surcharge applied to the support surface of the active earth wedge (ksf)

It is posited that if one substitutes Equation 8 for vertical earth pressure in place of q_s, and the active earth pressure coefficient in place of k_s, a similar result to Equation 9 will be achieved. The resulting, simplified equation is therefore:

$$\sigma_H = \frac{P \cdot G_r}{(B+V\cdot Z)\cdot(L+V\cdot Z)} \cdot K_A \tag{11}$$

where
Z = Depth from Ground Surface
B, L = Dimensions of Tire Contact Area
V = Slope of Pressure Distribution
$r = Z \,/\, (V \cdot S)$
$G_r = \pi r^2 + \alpha \cdot \pi r$
$K_A = \tan(45deg - \frac{\emptyset}{2})^2$

In order to compare the conventional Boussinesq approach with the proposed SGS approach of Equation 11, it is necessary to equate the soil friction angle soil friction angle, $\emptyset$, and Poisson's ratio, u. Bowles [5] provides the coefficient of earth pressure at rest in terms of soil friction angle, $\emptyset$, and also in terms of Poisson's ratio, u' (plane strain), as a starting point in the derivation.

$$\mathrm{K_o} = 1 - \sin(\emptyset) \qquad (12)$$

$$\mathrm{K_o} = \frac{\mathrm{u'}}{1-\mathrm{u'}} \qquad (13)$$

Setting these terms equal, and solving for u' (plane strain),

$$1 - \sin(\emptyset) = \frac{\mathrm{u'}}{1-\mathrm{u'}} \qquad (14)$$

$$\mathrm{u'} = \frac{1-\sin(\emptyset)}{\sin(\emptyset)} \qquad (15)$$

Poisson's ratio u is related to u' (plane strain), also from Bowles [5],

$$\mathrm{u'} = \frac{\mathrm{u}}{1-\mathrm{u}} \qquad (16)$$

Therefore,

$$\mathrm{u} = \frac{\mathrm{u'}}{1-\mathrm{u'}} \qquad (17)$$

Finally, substituting Equation 15 achieves:

$$\mathrm{u} = \frac{1-\sin(\emptyset)}{2\sin(\emptyset)-1} \qquad (18)$$

Table 3 lists the resulting values of the active earth pressure coefficient and Poisson's ratio for various design friction angles.

Table 3. Calculated Geotechnical Design Factors for Comparing the Boussinesq Method to the Proposed Simplified Method

Design Friction Angle, ϕ, (deg)	Poisson's Ratio (Plane Strain), μ'	Poisson's Ratio, μ	Coefficient of Active Earth Pressure, K_A
26	1.281	0.562	0.39
28	1.130	0.531	0.36
30	1.000	0.500	0.33
32	0.887	0.470	0.31
34	0.788	0.441	0.28
36	0.701	0.412	0.26
38	0.624	0.384	0.24
40	0.556	0.357	0.22

Figure 8 compares the Boussinesq method, with stresses from a single wheel load as well as the sum of multiple super-imposed vehicle loads, to the SGS solution of Equation 11. The graph shows very good comparison, which is consistent across multiple design friction angles.

It is interesting to note that at the top two feet below ground surface, all three curves are very similar. This suggests that point loads further away from the common reference point have less influence at shallow depths. However, it is clear that multiple point loads begin to overlap at deeper depths, and discounting this overlap would not be conservative. In the author's opinion, the method presented is much easier to calculate and implement for deeper depths and multiple point loads, compared to the Boussinesq method.

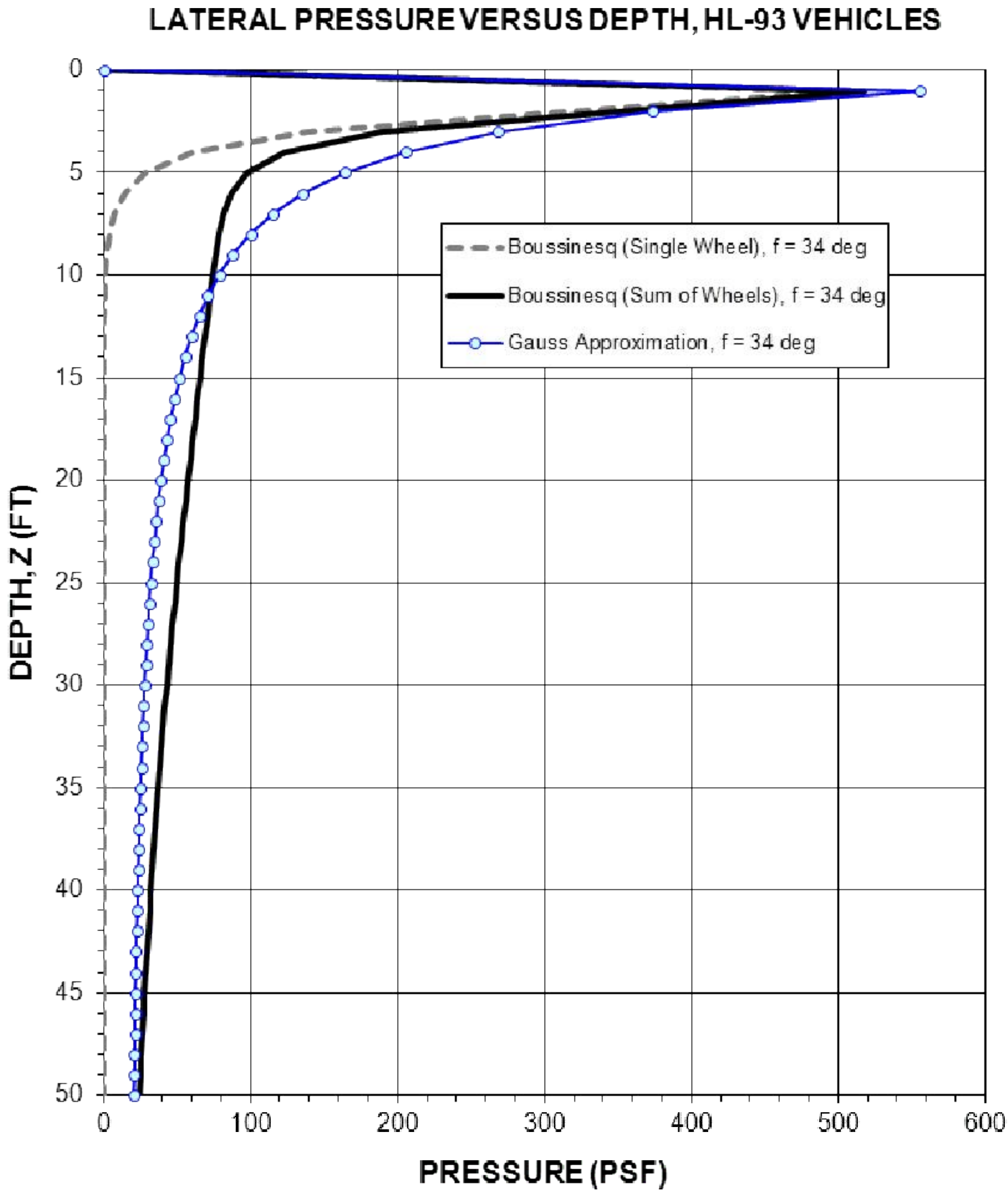

Figure 8: Lateral Pressure versus depth for AASHTO HL-93 Live Load.

CONCLUSIONS

A solution for the magnitude of vertical and horizontal stress at any depth is presented to account for overlapping stresses due to multiple lanes of wheel loads. The graphical solution of the number of overlapping wheel loads is predicted by use of the solution to the Gauss Circle Problem. The solution compares well with a systematic approach of calculated vertical stresses, and will provide the Engineer with an alternative and efficient means for this calculation in structural analysis.

In summary, the suggested equation for calculating the effect of multiple, overlapping wheel loads with depth is,

$$\sigma = \frac{P \cdot G_r}{\left(B + {}^{2 \cdot Z}/_{V}\right) \cdot \left(L + {}^{2 \cdot Z}/_{V}\right)} \tag{19}$$

And for horizontal stresses due to the surcharge of the vertical load,

$$\sigma_H = \frac{P \cdot G_r}{\left(B + {}^{2 \cdot Z}/_{V}\right) \cdot \left(L + {}^{2 \cdot Z}/_{V}\right)} \cdot K_A \tag{20}$$

where
Z = Depth from Ground Surface
B, L = Dimensions of Tire Contact Area
V = Slope of Pressure Distribution
$r = Z \,/\, (V \cdot S)$
$G_r = \pi r^2 + 2\sqrt{2} \cdot \pi r$
$K_A = \tan(45deg - \frac{\emptyset}{2})^2$

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