1 Super Linear Algebra

If R is a set (actually, a ring whose structure we will use later), an *R-graded vector space* is a vector space V with a decomposition

$$V = \bigoplus_{r \in R} V^r.$$

Given $r \in R$, an element of V^r is called *homogeneous of degree* r. The degree of a vector v is denoted $\deg v$, and each subspace V^r is a *homogeneous subspace*.

A *super vector space* is a $\mathbb{Z}_2$-graded vector space, where $\mathbb{Z}_2$ is the ring with elements 0 and 1. Thus if V is a super vector space, we can write

$$V = V^0 \oplus V^1,$$

and we say V^0 is the subspace of *even* (or degree 0) elements, and V^1 is the subspace of *odd* (or degree 1) elements. The homogenenous subspaces have an associated involution

$$\begin{aligned} \epsilon : V^i &\longrightarrow V^i \\ v &\longmapsto (-1)^{\deg v} v \end{aligned}$$

(In general, an *involution* is a map T satisfying $T^2 = I$.)

If V is a super vector space, consider $E = \mathrm{End} V$. This is an algebra with usual addition and composition of maps, but we can give it a *super algebra* structure. This means that it has a decomposition

$$E = E^0 \oplus E^1$$

into even and odd subspaces, and additionally

$$E^i E^j \subseteq E^{i+j} \quad \text{for } i, j \in \mathbb{Z}_2.$$

We define the even and odd subspaces as

$$E^0 = \{T \in E : T(V^0) \subseteq V^0, T(V^1) \subseteq V^1\},$$

$$E^1 = \{T \in E : T(V^0) \subseteq V^1, T(V^1) \subseteq V^0\}.$$

Clearly $E^0 \cap E^1 = \{0\}$. We also have $E^0 + E^1 = E$ since if we select a basis for E, the matrix corresponding to any map $X \colon V \to V$ can be written in block form

$$X = \begin{pmatrix} X_{00} & X_{01} \\ X_{10} & X_{11} \end{pmatrix},$$

where

$$X_{00} : V^0 \longrightarrow V^0 \qquad\qquad X_{01} : V^0 \longrightarrow V^1$$
$$X_{10} : V^1 \longrightarrow V^0 \qquad\qquad X_{11} : V^1 \longrightarrow V^1$$

Now, we can write

$$\begin{pmatrix} X_{00} & X_{01} \\ X_{10} & X_{11} \end{pmatrix} = \begin{pmatrix} X_{00} & 0 \\ 0 & X_{11} \end{pmatrix} + \begin{pmatrix} 0 & X_{01} \\ X_{10} & 0 \end{pmatrix},$$

which is a sum of even and odd maps. The super structure (i.e., the $\mathbb{Z}_2$ grading) is obtained from the following properties of matrix multiplication.

- even $\times$ even $=$ even $\quad$ (or $E^0E^0 \subseteq E^0$)

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} AC & 0 \\ 0 & BD \end{pmatrix}$$

- odd $\times$ odd $=$ even $\quad$ (or $E^1E^1 \subseteq E^0$)

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} = \begin{pmatrix} AD & 0 \\ 0 & BC \end{pmatrix}$$

- even $\times$ odd $=$ odd $\times$ even $=$ odd $\quad$ (or $E^0E^1, E^1E^0 \subseteq E^1$)

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} 0 & AD \\ BC & 0 \end{pmatrix}$$

To summarize, an even endomorhpism respects the super structure of V, while and odd endomorphism reverses it.

Let $X \in E^0$, and let $v_0 \in V^0, v_1 \in V^1$. Then let $w_0 = Xv_0 \in V^0, w_1 = Xv_1 \in V^1$. We see that

$$\epsilon X(v_0 + v_1) = \epsilon(w_0 + w_1) = w_0 - w_1 = X(v_0 - v_1) = X\epsilon(v_0 + v_1),$$

so ϵ commutes with even endomorphisms. On the other hand, let $X \in E^1$, and let $v_0 \in V^0, v_1 \in V^1$. Then let $w_1 = Xv_0 \in V^1, w_0 = Xv_1 \in V^0$. We see that

$$\epsilon X(v_0 + v_1) = \epsilon(w_1 + w_0) = -w_1 + w_0 = X(-v_0 + v_1) = -X\epsilon(v_0 + v_1),$$

so ϵ anti-commutes with odd endomorphisms.

A map $X \in E$ has *supertrace*

$$\mathrm{tr}_s X = \mathrm{tr} \epsilon X.$$

If X is even then we can write

$$X = \begin{pmatrix} X_{00} & 0 \\ 0 & X_{11} \end{pmatrix} \quad \text{and} \quad \epsilon X = X \epsilon = \begin{pmatrix} X_{00} & 0 \\ 0 & -X_{11} \end{pmatrix},$$

so

$$\mathrm{tr}_s X = \mathrm{tr} X_{00} - \mathrm{tr} X_{11},$$

the difference of the normal traces of the diagonal blocks. If X is odd then we can write

$$X = \begin{pmatrix} 0 & X_{01} \\ X_{10} & 0 \end{pmatrix}$$

and so the supertrace is obviously zero. The supertrace also satisfies the following symmetry property:

$$\mathrm{tr}_s XY = (-1)^{\deg X \deg Y} \mathrm{tr}_s YX.$$

This is seen by looking at the cases for multiplication above. For example, if $X, Y \in E^0$, then

$$\mathrm{tr}_s XY = \mathrm{tr} X_{00} Y_{00} - \mathrm{tr} X_{11} Y_{11} = \mathrm{tr}_s YX = (-1)^{0 \cdot 0} \mathrm{tr}_s YX.$$

This symmetry property implies that supertrace vanishes on *supercommutators*:

$$[X, Y] = XY - (-1)^{\deg X \deg Y} YX,$$

$$\mathrm{tr}_s [X, Y] = 0.$$

2 Vector Bundles

A *vector bundle of rank k* over a manifold M is consists of a manifold E (called the *total space*) together with a continuous surjection $\pi \colon E \to M$ (called the *projection*), satisfying the following conditions:

(i) For each $p \in M$, each *fiber* $E_p = \pi^{-1}(p)$ is a vector space, and is isomorphic to $\mathbb{R}^k$ (or $\mathbb{C}^k$ in the complex case);

(ii) Each $p \in M$ has a neighborhood U and homeomorphism ϕ_U (called a *local trivialization*) such that the following diagram commutes:

$$\begin{tikzcd}
E_U = \pi^{-1}(U) \arrow[rr, "\phi_U"] \arrow[dr, "\pi"'] & & U \times \mathbb{C}^m \arrow[dl, "\mathrm{proj}_1"] \\
& U &
\end{tikzcd}$$

(iii) Given $U \in M$ and a local trivialization ϕ_U,

$$\phi_U|_{E_q} \colon E_q \longrightarrow \{q\} \times \mathbb{C}^k$$

is a linear isomorphism for each $q \in U$.

We will write $E \xrightarrow{\pi} M$ to indicate a vector bundle.

One may think of a vector bundle as a collection of vector spaces, parametrized by M:

$$E = \bigsqcup_{p \in M} E_p.$$

If π is smooth and each ϕ_U is a diffeomorphism, then we call E a *smooth* vector bundle.

Let neighborhoods $U, V \subseteq M$ with $U \cap V \neq \varnothing$ have trivializations ϕ_U and ϕ_V:

$$\begin{tikzcd}
(U \cap V) \times \mathbb{C}^k \arrow[rr, "\phi_U \circ \phi_V^{-1}"] \arrow[dr, "\mathrm{proj}_1"'] & E_{U \cap V} \arrow[l, "\phi_V"'] \arrow[r, "\phi_U"] \arrow[d, "\pi"] & (U \cap V) \times \mathbb{C}^k \arrow[dl, "\mathrm{proj}_1"] \\
& U \cap V &
\end{tikzcd}$$

It is not hard to see that we have

$$\begin{aligned} \phi_U \circ \phi_V^{-1} : (U \cap V) \times \mathbb{C}^k &\longrightarrow (U \cap V) \times \mathbb{C}^k \\ (p, v) &\longmapsto (p, g_{UV}(p)v) \end{aligned}$$

for some *transition map*

$$g_{UV} \colon U \cap V \longrightarrow \mathrm{GL}(k, \mathbb{C}).$$

The action in the second component of the image is the standard action of a matrix on a vector. The transition maps necessarily satisfy the following conditions

$$g_{UV}(p) \cdot g_{VU}(p) = I \tag{1}$$

$$g_{UV}(p) \cdot g_{VW}(p) \cdot g_{WU}(p) = I \tag{2}$$

for neighborhoods $U, V, W \subset M$.

Conversely, it is a standard fact that given a cover $\{U_\alpha\}$ of M and a collection of transition maps

$$\{g_{\alpha\beta} \colon U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(k, \mathbb{C})\}$$

that satisfy (1) and (2), then $E \xrightarrow{\pi} M$ is a vector bundle with transition maps $\{g_{\alpha\beta}\}$. The total space is union of each set $U_\alpha \times \mathbb{C}^k$, with the identifications $(x, \lambda) \sim (x, g_{\alpha\beta}(x)\lambda)$, for $(x, \lambda) \in U_\beta \times \mathbb{C}^k, (x, g_{\alpha\beta}(x)\lambda) \in U_\alpha \times \mathbb{C}^k$. Thus, a vector bundle is determined by its transition maps, which tell us how all the fibers are glued together.

If $E \xrightarrow{\pi} M$ is a vector bundle, a *subbundle* F is a submanifold of E such that for each $p \in M$, $F_p \subset E_p$ is a vector subspace, and that for each $p \in M$, there exists a neighborhood U of p with trivialization ϕ_U such that

$$\phi_U|_{F_U} \colon F_U \longrightarrow U \times \mathbb{C}^l \subset U \times \mathbb{C}^k.$$

If $\{g_{UV}\}$ are the transition maps of E, then with respect to a subbundle F we can write

$$g_{UV}(p) = \begin{pmatrix} h_{UV}(p) & k_{UV}(p) \\ 0 & j_{UV}(p) \end{pmatrix},$$

where $\{h_{UV}\}$ are the transition maps of F.

Most operations that one can perform on vector spaces carry over to vector bundles, and are most easily described in terms of transition maps. For example, we consider the direct sum. If $E \xrightarrow{\pi} M$ is a rank k vector bundle with transition maps $\{g_{\alpha\beta}\}$ and $F \xrightarrow{\pi'} M$ is a rank l vector bundle with transition maps $\{h_{\alpha\beta}\}$, then the bundle $E \oplus F$ is determined by the transition maps

$$j_{\alpha\beta}(p) = \begin{pmatrix} g_{\alpha\beta}(p) & 0 \\ 0 & h_{\alpha\beta}(p) \end{pmatrix} \in \mathrm{GL}(\mathbb{C}^k \oplus \mathbb{C}^l).$$

A *super vector bundle* is of the form $E = E^0 \oplus E^1$.

Similarly, the tensor product vector bundle has transition maps

$$j_{\alpha\beta} = g_{\alpha\beta}(p) \otimes h_{\alpha\beta}(p) \in \mathrm{GL}(\mathbb{C}^k \otimes \mathbb{C}^l).$$

If $E \xrightarrow{\pi} M$ and $F \xrightarrow{\pi'} M$ are bundles, then a *smooth bundle map over* M is a smooth map $u\colon E \to F$ such that $\pi' \circ u = \pi$,

$$\begin{tikzcd} E \arrow[rr, "u"] \arrow[dr, "\pi"'] & & F \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

and whose restriction to each fiber is linear:

$$u|_{E_p}\colon E_p \longrightarrow F_{u(p)}.$$

A *smooth section* of a vector bundle $E \xrightarrow{\pi} M$ is a smooth map $\sigma\colon M \to E$ such that $\pi \circ \sigma = Id_M$. Sections can be added pointwise and multiplied by smooth function on M, so the space of smooth sections of E is a $C^\infty(M)$-module, denoted $\Gamma(E)$.