

1 Introduction

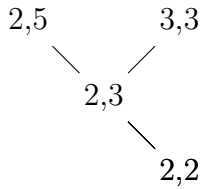
Proposition 1.1 (Goldbach Conjecture). *Every even integer greater than 2 is the sum of two primes.*

The Goldbach Conjecture is equivalent to the following.

Proposition 1.2. *Every integer $n \geq 2$ is the average of two primes.*

A natural extension of the Goldbach conjecture is the following.

Proposition 1.3. *If a prime p divides an integer $n \geq 2$, then n is the average of p primes.*



Another equivalent statement due to Sebastian Martín Ruiz is as follows.

Proposition 1.4. *For all even integers $n \geq 4$, there exists an integer $k \in [n - 1]$ such that*

$$\varphi(n^2 - k^2) = (n - 1)^2 - k^2.$$

Proposition 1.5. *Proposition 1.4 is equivalent to the Goldbach Conjecture.*

Proof. Suppose $e \geq 4$ is an even integer.

If the Goldbach Conjecture is true, then there exist two primes p and q such that $e = p + q$, and without loss of generality, we can suppose $p > q$. Let $n := (p + q)/2$ and $k := (p - q)/2$.

Now,

$$\begin{aligned}
 \varphi(n^2 - k^2) &= \varphi((n + k)(n - k)) \\
 &= \varphi(pq) \\
 &= \varphi(p)\varphi(q) \\
 &= (p - 1)(q - 1) \\
 &= ((n + k) - 1)((n - k) - 1) \\
 &= ((n - 1) + k)((n - 1) - k) \\
 &= (n - 1)^2 - k^2
 \end{aligned}$$

Conversely, if 1.4 holds, then there exists an integer $k \in [n - 1]$ such that

$$\varphi(n^2 - k^2) = (n - 1)^2 - k^2.$$

Let $n := e/2$. If $n + k$ and $n - k$ are both prime, then we are done by setting $p := n + k$ and $q := n - k$. Otherwise, at least one of $n + k$ or $n - k$ is composite. Let $d := \gcd(n + k, n - k)$. Then we arrive at the contradiction $\varphi(n^2 - k^2) < (n - 1)^2 - k^2$:

$$\begin{aligned}
\varphi(n^2 - k^2) &= \varphi((n + k)(n - k)) \\
&= \varphi\left(d^2 \frac{(n + k)}{d} \frac{(n - k)}{d}\right) \\
&= \varphi(d^2) \varphi\left(\frac{n + k}{d} \frac{n - k}{d}\right) \\
&= d \varphi(d) \varphi\left(\frac{n + k}{d}\right) \varphi\left(\frac{n - k}{d}\right) \\
&= d \varphi\left(\frac{n + k}{d}\right) \varphi(d) \varphi\left(\frac{n - k}{d}\right) \\
&\leq \varphi\left(d \frac{n + k}{d}\right) \varphi\left(d \frac{n - k}{d}\right) \quad (\text{equality if } d = 1) \\
&= \varphi(n + k) \varphi(n - k) \\
&< (n + k - 1)(n - k - 1) \\
&= (n - 1)^2 - k^2.
\end{aligned}$$

□

2 References

1. ...