1 Introduction

Proposition 1.1 (Goldbach Conjecture)**.** *Every even integer greater than 2 is the sum of two primes.*

The Goldbach Conjecture is equivalent to the following.

Proposition 1.2. *Every integer $n \geq 2$ is the average of two primes.*

A natural extension of the Goldbach conjecture is the following.

Proposition 1.3. *If a prime p divides an integer $n \geq 2$, then n is the average of p primes.*

$$\begin{tikzcd}[arrows=dash]
2,5 \arrow[dr] & & 3,3 \arrow[dl] \\
& 2,3 \arrow[dr] & \\
& & 2,2
\end{tikzcd}$$

Another equivalent statement due to Sebastian Martín Ruiz is as follows.

Proposition 1.4. *For all even integers $n \geq 4$, there exists an integer $k \in [n-1]$ such that*

$$\varphi(n^2 - k^2) = (n-1)^2 - k^2.$$

Proposition 1.5. *Proposition 1.4 is equivalent to the Goldbach Conjecture.*

Proof. Suppose $e \geq 4$ is an even integer.

If the Goldbach Conjecture is true, then there exist two primes p and q such that $e = p + q$, and without loss of generality, we can suppose $p > q$. Let $n := (p+q)/2$ and $k := (p-q)/2$.

Now,

$$\begin{aligned}
\varphi(n^2 - k^2) &= \varphi((n+k)(n-k)) \\
&= \varphi(pq) \\
&= \varphi(p)\varphi(q) \\
&= (p-1)(q-1) \\
&= ((n+k)-1)((n-k)-1) \\
&= ((n-1)+k)((n-1)-k) \\
&= (n-1)^2 - k^2
\end{aligned}$$

Conversely, if 1.4 holds, then there exists an integer $k \in [n-1]$ such that

$$\varphi(n^2 - k^2) = (n-1)^2 - k^2.$$

Let $n := e/2$. If $n+k$ and $n-k$ are both prime, then we are done by setting $p := n+k$ and $q := n-k$. Otherwise, at least one of $n+k$ or $n-k$ is composite. Let $d := gcd(n+k, n-k)$. Then we arrive at the contradiction $\varphi(n^2-k^2) < (n-1)^2 - k^2$:

$$\begin{aligned}
\varphi(n^2-k^2) &= \varphi((n+k)(n-k)) \\
&= \varphi\Big(d^2 \frac{(n+k)}{d} \frac{(n-k)}{d}\Big) \\
&= \varphi(d^2)\varphi\Big(\frac{n+k}{d}\frac{n-k}{d}\Big) \\
&= d\varphi(d)\varphi\Big(\frac{n+k}{d}\Big)\varphi\Big(\frac{n-k}{d}\Big) \\
&= d\varphi\Big(\frac{n+k}{d}\Big)\varphi(d)\varphi\Big(\frac{n-k}{d}\Big) \\
&\leq \varphi\Big(d\frac{n+k}{d}\Big)\varphi\Big(d\frac{n-k}{d}\Big) && \text{(equality if } d=1\text{)} \\
&= \varphi(n+k)\varphi(n-k) \\
&< (n+k-1)(n-k-1) \\
&= (n-1)^2 - k^2.
\end{aligned}$$

$\square$

2 References

1. ...