Optimal sensing of stationary mirror displacement in an interferometric arm

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A note
dated December 22, 2017

1 Theoretical part

Let us consider the full single mode Hamiltonian as given below (full derivation can be found in Refs. [2, 4]):

$$\widehat{H} = \hbar\omega_0 \hat{a}^\dagger \hat{a} + \hbar\omega_m \hat{b}^\dagger \hat{b} - \hbar g \hat{a}^\dagger \hat{a}(\hat{b}^\dagger + \hat{b}), \tag{1}$$

where the terms correspond to the photon energy, mirror energy and radiation pressure (with $g = \frac{\omega_0}{l_0}\sqrt{\frac{\hbar}{2m\omega_m}}$), respectively. Here ω_0 is the light frequency, ω_m and m is the frequency and mass of the mirror (which we treat as a quantum harmonic oscillator), $\hat{a}$ and $\hat{b}$ are the annihilation operators of the light and mirror, l_0 is the length of the interferometric mirror. Let us note that this model does not correspond to any particular interferometric configuration (such as Michelson-Morley, Mach-Zehnder etc.), but just to one interferometric arm in which the light interacts with the mirror. We will still define the total number of phonons N, though, and will interchangeably use the notation $|n\rangle$ and $|n\rangle \otimes |N-n\rangle$ for the state of light. Such state denotes n photons in the interferometric arm under consideration and remaining $N-n$ outside it (we do not explicitly specify what happens with them as this is irrelevant for the conclusions that can be drawn from our model).

In the model of Eq. (1) there is no external force acting on the system , while our main goal is to detect gravitational waves. Therefore, to include the gravitational waves in the simplest possible way, let us consider the following situation: first, the interferometer is subject to an external force which results in a constant mirror displacement and after that, the light is injected and interaction as governed by the Hamiltonian of Eq. (1) starts. Let us denote the mirror length change due to the external force by x', so that the new length is $l_0 + x'$, $x'/l_0 \ll 1$. Therefore, in the Hamiltonian of Eq. (1), the following changes occur:

$$\omega_0 \to \frac{\omega_0}{1+f} \approx \omega_0(1-f), \quad g \to \frac{g}{(1+f)^2} \approx g(1-2f), \quad \hat{b} \to \hat{b}+f/2, \quad \hat{b}^\dagger \to \hat{b}^\dagger + f/2, \tag{2}$$

where we have denoted $f := x/l_0$ and $\hat{b}$, $\hat{b}^\dagger$ are displaced by $f/2$ in order for the position operator to be displaced by f. We may now write the Hamiltonian as:

$$\widehat{H} = \hbar\omega_0(1-f)\hat{a}^\dagger \hat{a} + \hbar\omega_m \hat{b}'^\dagger \hat{b}' - \hbar g(1-2f)\hat{a}^\dagger \hat{a}(\hat{b}'^\dagger + \hat{b}'), \tag{3}$$

where $\hat{b}'$ and $\hat{b}'^\dagger$ denote the displaced annihilation and creation operators of the mirror: $\hat{b}' = \hat{b} + f/2$, $\hat{b}'^\dagger = \hat{b}^\dagger + f/2$.

The time evolution operator corresponding to the Hamiltonian of Eq. (3) has been derived in Ref. [1]:

$$\begin{aligned}&\hat{U}(t) = e^{-i\hat{H}t/\hbar} = \\ &\exp\left[-i\omega_0(1-f)t\hat{a}^\dagger\hat{a}\right]\exp\left[i\epsilon(t)^2\left(\frac{g(1-2f)}{\omega_m}\hat{a}^\dagger\hat{a}\right)^2\right]\times \\ &\times\exp\left[\left(\frac{g(1-2f)}{\omega_m}\hat{a}^\dagger\hat{a}\right)\left(\eta(t)\hat{b}'^\dagger-\eta^*(t)\hat{b}'\right)\right]\exp\left[-i\omega_m t\hat{b}'^\dagger\hat{b}'\right],\end{aligned} \tag{4}$$

where

$$\eta(t) = 1-e^{-i\omega_m t}, \quad \epsilon(t) = \omega_m t - \sin(\omega_m t). \tag{5}$$

Let us also note that the third exponential in the above Eq. (4) is a displacement operator, where the displacement magnitude depends on the number of photons hitting the mirror. Now let us act with the time evolution operator of Eq. (4) on the following state:

$$|\psi(t=0)\rangle = \sum_{n=0}^{N} c_n\,|n\rangle_l\,|0'\rangle_m\,, \tag{6}$$

where we assume that the mirror was prepared in ground state. This ground state $|0'\rangle_m$ refers to the displaced operators $\hat{b}'$ and $\hat{b}'^\dagger$, hence implicitly taking into account the external force that had acted on the interferometer prior to light injection. The result is:

$$\begin{aligned}&|\psi(t)\rangle = \hat{U}(t)\,|\psi(t=0)\rangle = \\ &\sum_{n=0}^{N} c_n e^{-in\omega_0(1-f)t}e^{-i\omega_m t/2}e^{i(\frac{g(1-2f)}{\omega_m}n)^2(\omega_m t-\sin(\omega_m t))^2}\,|n\rangle_l\left|\eta(t)\frac{g(1-2f)n}{\omega_m}\right\rangle_m.\end{aligned} \tag{7}$$

In order to provide a deeper understanding the physical meaning of the above Eq. (7) let us consider the limit $m\to\infty$. In this regime we obtain the following scalings:

$$\omega_m, \eta(t) \propto \frac{1}{\sqrt{m}}, \quad \omega_m t - \sin(\omega_m t) \propto \left(\frac{1}{\sqrt{m}}\right)^3, \quad g\propto m^{-1/4}, \tag{8}$$

resulting in:

$$\lim_{m\to\infty}|\psi(t)\rangle = \sum_{n=0}^{N} c_n e^{-in\omega_0(1-f)t}\,|n\rangle_l\,|0\rangle_m\,. \tag{9}$$

This limit physically corresponds to complete lack of radiation pressure effects (as the mirror is very heavy) and an additional phase shift $e^{in\omega_0 ft}$ corresponding to the fact that the mirror had been displaced by an external force prior to light injection into the interferometer.

Now we can calculate the reduced density matrix of light:

$$\begin{aligned}&\hat{\rho}_l = \mathrm{Tr}_m[|\psi(t)\rangle\langle\psi(t)|] = \\ &\sum_{n,n'=0}^{N} c_n c_n' e^{-i(n-n')\omega_0(1-f)t}e^{i(\frac{g(1-2f)}{\omega_m}n)^2(\omega_m t-\sin(\omega_m t))^2}e^{-i(\frac{g(1-2f)}{\omega_m}n')^2(\omega_m t-\sin(\omega_m t))^2}\,|n\rangle\langle n'|_l\times \\ &\times\mathrm{Tr}\left[\left|\eta(t)\frac{g(1-2f)n}{\omega_m}\right\rangle\left\langle\eta(t)\frac{g(1-2f)n'}{\omega_m}\right|\right]\end{aligned} \tag{10}$$

and the Quantum Fisher Information can be calculated at this point with respect to the external force parameter f. Such calculation for arbitrary time, though, requires numerical techniques.

The state of the light, however, simplifies drastically when time is a multiple of the mirror's own period: $\omega_m t = 2\pi M$, $M \in \mathbb{Z}$. The mirror state goes back to vacuum and $\hat{\rho}_l$ becomes a pure state denoted by $|\psi\rangle_l$:

$$|\psi\rangle_l = \sum_{n=0}^{N} c_n e^{-in\omega_0(1-f)2\pi M/\omega_m}(-1)^M e^{i(\frac{g(1-2f)}{\omega_m}n)^2(2\pi M)^2} |n\rangle_l \,. \tag{11}$$

We can calculate the derivative of the above state with respect to the parameter f:

$$\begin{aligned}\left|\dot{\psi}\right\rangle_l = \sum_{n=0}^{N} c_n \left(in\omega_0 2\pi M/\omega_m - 4i(1-2f)(\frac{g}{\omega_m}n)^2(2\pi M)^2\right) \times \\ \times\, e^{-in\omega_0(1-f)2\pi M/\omega_m}(-1)^M e^{i(\frac{g(1-2f)}{\omega_m}n)^2(2\pi M)^2} |n\rangle_l\end{aligned} \tag{12}$$

and the Quantum Fisher Information can be readily obtained:

$$\begin{aligned}F_Q &= 4\left(\left\langle\dot{\psi}\middle|\dot{\psi}\right\rangle_l - \left|\left\langle\dot{\psi}\middle|\psi\right\rangle_l\right|^2\right) = \\ &= 4\left(\sum_{n=0}^{N}|c_n|^2\left(n\omega_{k0}2\pi M/\omega_m - 2(1-f)(\frac{g}{\omega_m}n)^2(2\pi M)^2\right)^2\right. \\ &\left. - \left|\sum_{n=0}^{N} c_n\left(-in\omega_{k0}2\pi M/\omega_m + 2i(1-f)(\frac{g}{\omega_m}n)^2(2\pi M)^2\right)\right|^2\right).\end{aligned} \tag{13}$$

This result paves the way for further research, such as determining optimal state (set of coefficients $\{c_n\}$), which maximizes the Quantum Fisher Information (covered in Sec. 2) . Furthermore, in the future one could potentially generalize these considerations and treat the mirror as an open quantum system, hence enabling to study not only the decoherence resulting from the light-mirror interaction but also the decoherence caused by the mirror-environment interaction.

2 Numerical part

2.1 Full optimization over state coefficients

Here we use the algorithm described in [3] in its general form, which is mentioned in Sec. IV (mainly Eq. 11) of the article [3]. The aim is to obtain the optimum (i.e. the one that maximizes the Quantum Fisher Information) state (set of coefficients c_n, $n = 0, \dots, N$) of light prior to the injection to the interferometer. The quantum channel Λ, which is the key construct used in the algorithm, follows directly from Eq. 10:

$$\Lambda(\rho) = \Lambda\left(\begin{bmatrix} c_{00} & c_{01} & \cdots & c_{0N} \\ c_{10} & c_{11} & \cdots & c_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ c_{N0} & c_{N1} & \cdots & c_{NN}\end{bmatrix}\right) = \begin{bmatrix} c_{00}\xi_{00} & c_{01}\xi_{01} & \cdots & c_{0N}\xi_{0N} \\ c_{10}\xi_{10} & c_{11}\xi_{11} & \cdots & c_{1N}\xi_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ c_{N0}\xi_{N0} & c_{N1}\xi_{N1} & \cdots & c_{NN}\xi_{NN}\end{bmatrix}, \tag{14}$$

where:

$$\xi_{nn'} = e^{-i(n-n')\omega_0(1-f)t} e^{i(\frac{g(1-2f)}{\omega_m}n)^2(\omega_m t-\sin(\omega_m t))^2} e^{-i(\frac{g(1-2f)}{\omega_m}n')^2(\omega_m t-\sin(\omega_m t))^2} |n\rangle\langle n'|_l \times$$
$$\times \mathrm{Tr}\left[\left|\eta(t)\frac{g(1-2f)n}{\omega_m}\right\rangle\left\langle\eta(t)\frac{g(1-2f)n'}{\omega_m}\right|\right] \tag{15}$$

The channel $\Lambda'(\rho)$ is simply implemented by taking derivative of $\xi_{nn'}$ with respect to f, while $\Lambda^{\dagger}(\rho)$ and $\Lambda^{\dagger'}(\rho)$ are implemented as $(\Lambda(\rho))^{\dagger}$ and $(\Lambda'(\rho))^{\dagger}$, respectively. We finish the algorithm when the difference between QFI obtained in two consecutive steps is lower than a certain threshold (in this work, we used 10^{-7}).

Below in Figs. 1, 2 we present exemplary results of the optimization algorithm obtained for $N = 1$ and $N = 4$. We used the following parameter values: $\omega_0 = 10$, $\omega_m = 0.1$, $f = 0.001$, $g = 0.03$. Let us note that as for now, these parameters do not correspond to typical values for real interferometric arms, but were rather chosen to qualitatively uncover the physics of the system under consideration without generating extremely large numbers within the numerical procedures.

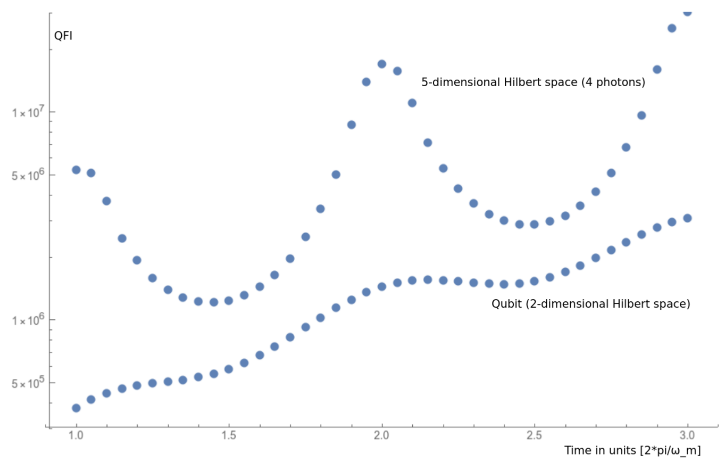

Figure 1: Quantum Fisher Information vs. time (in units $2\pi/\omega_m$) for two-dimensional Hilbert space ($N = 1$) and five-dimensional Hilbert space ($N = 4$). At each point, the state was optimized using the procedure outlined in the text. The optimal state for $N = 1$ was always the equal superposition state $(|0\rangle|1\rangle + |1\rangle|0\rangle)/\sqrt{2}$, while for $N = 4$ we illustrate state coefficients in Fig. 2 for selected moments of time.

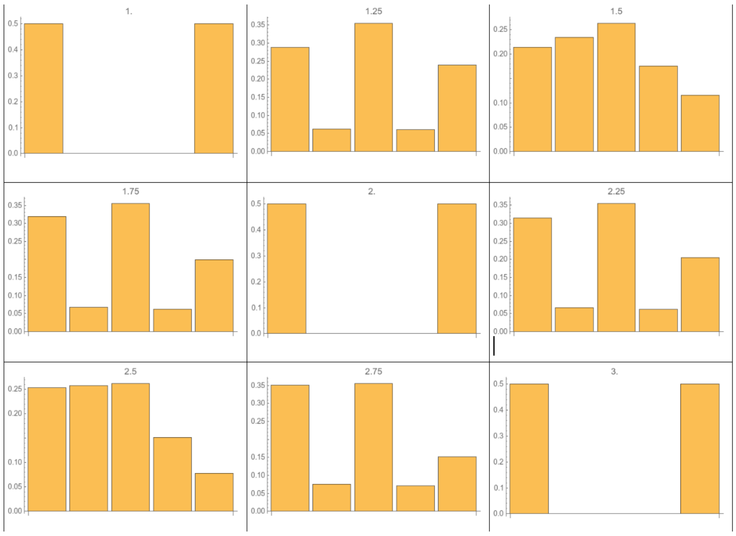

Figure 2: Moduli squared of the state coefficients obtained for $N = 4$ for selected moments of time $1, 1.25, 1.5, \ldots, 2.75, 3$ (in units $2\pi/\omega_m$). Let us note that throughout all numerical procedures, we did not observe any dependence of the results on the phases of state coefficient. Hence, we might restrict ourselves to positive real state coefficients c_n. The Quantum Fisher Information calculated with respect to these states is the optimal QFI shown in Fig. 1.

The time points $t = 1 * 2\pi/\omega_m$, $t = 2 * 2\pi/\omega_m$, and $t = 3 * 2\pi/\omega_m$ correspond to moments when the quantum state of the light in the interferometer becomes pure. The effects resulting from radiation pressure do not influence the system at these moments. Hence, the QFI has local maxima and the optimal states are $N00N$ states known to maximize the QFI in ideal scenario (i.e. without radiation pressure). Between the "purification" moments, the QFI experiences drops due to detrimental effects of radiation pressure and the states containing non-zero middle coefficients start to be optimal.

Then we study the dependence of optimal state on the coupling g, which is shown in Fig. 3.

Figure 3: Moduli of the state coefficients obtained for $N = 4$ for selected values of coupling g at time $t = 1.5 * 2\pi/\omega_m$.

We observe that for small coupling the optimum state is again $N00N$ as the effects of radiation pressure are very small. Then, as the radiation pressure increases, we see the advantage of middle state coefficients that lower the variance of the photon number. Let us also note that at the time when the light state is pure (we tested for $t = 2 * 2\pi/\omega_m$), regardless of g, the optimal state was always $N00N$, hence we do not show a figure analogous to Fig. 3 for $t = 2 * 2\pi/\omega_m$.

2.2 Study of spin-squeezed states

In this part, we do not fully optimize state over all possible coefficients – we study the dependence of Quantum Fisher information over squeezing parameter (squeezing "force") and squeezing angle for a class of states called spin-squeezed states. We start with a state $|0\rangle \otimes |N\rangle$, and then create a spin-squeezed state $|\psi_{a,\theta}\rangle$ (with a – squeezing parameter and θ – squeezing angle) as follows:

$$|\psi_{a,\theta}\rangle = \exp\Big(i\hat{J}_y\pi/2\Big)\exp\Big(-i\theta\hat{J}_z\Big)\exp\Big(a(\hat{J}_+^2 - \hat{J}_-^2)\Big)|0\rangle \otimes |N\rangle \tag{16}$$

Here, we use the angular momentum notation for the states $|n\rangle \otimes |N-n\rangle$. The total angular momentum is equal to $N/2$ and the state $|n\rangle \otimes |N-n\rangle$ can be associated with angular momentum state $|J = N/2, J_z = n - N/2\rangle$.

The meaning of every operator acting on the state in Eq. 16 is as follows:

1. $\exp\Big(a(\hat{J}_+^2 - \hat{J}_-^2)\Big)$ is a two-axis twisting operation with strength a
2. $\exp\Big(-i\theta\hat{J}_z\Big)$ is a rotation with respect to z-axis, which introduces the squeezing angle θ
3. $\exp\Big(i\hat{J}_y\pi/2\Big)$ is a rotation with respect to y-axis by 90 degrees which moves the state from the pole to the equator of the Bloch sphere

Then we calculate Quantum Fisher Information for the state $|\psi_{a,\theta}\rangle$ fed into the interferometer, i.e. for $\Lambda(|\psi_{a,\theta}\rangle\langle\psi_{a,\theta}|)$ (note that for the spin-squeezed states we no longer use the global optimization algorithm). Below in Fig. 4 we plot the squeezing angle for which the QFI is maximal. Again, we use the following parameter values: $\omega_0 = 10$, $\omega_m = 0.1$, $f = 0.001$

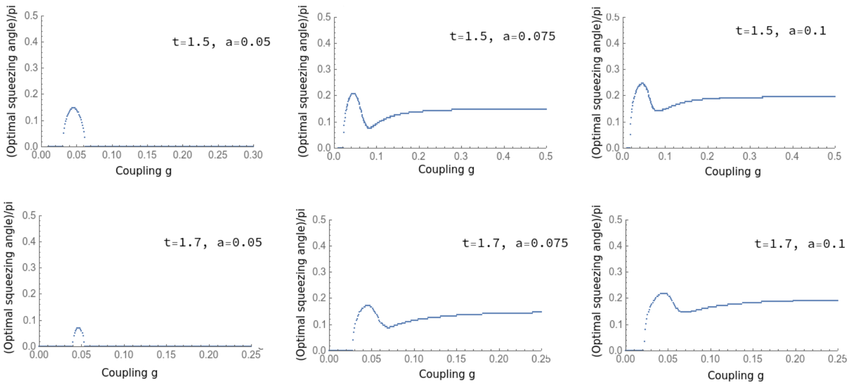

Figure 4: Optimal squeezing angle vs. the light-mirror coupling g for selected values of time t (measured in units of $2\pi/\omega_m$) and squeezing parameter a.

We see that for small coupling g (i.e. large mirror mass, small effect of radiation pressure) the optimal squeezing angle is zero – as there is no radiation pressure, there is no need to squeeze the light. For moderate ranges of g the optimal angle is nonzero (and different depending on a and t) – this illustrates the advantage of squeezing to counteract the detrimental effects of radiation pressure. For large g, depending on a, the optimal angle may go back to zero (this we preliminarily account for radiation pressure so large that squeezing loses its advantage) or still be finite.

References

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