Numerical Assignment

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1 Defining Shallow Water Equations

The shallow water equations are

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -2\Omega \times \mathbf{u} - g\nabla(h + h_0) \tag{1}$$

$$\frac{\partial h}{\partial t} + \mathbf{u} \cdot \nabla h = -h\nabla \cdot \mathbf{u} \tag{2}$$

where $\mathbf{u}$ is the velocity of the flow, Ω is the angular velocity of the rotating frame of reference, g is the gravitational acceleration constant, h is the depth of the fluid and h_0 represents the underlying shape that the fluid is flowing over (as defined in [1]).

The first equation (1) represents the conservation of momentum and the second equation (2) represents the conservation of mass. Both equations have a $u \cdot \nabla$ which represents advection.

In order to solve these equations numerically, they first need to be linearised about the state $u = 0$ and $h = H$ ie.

$$\begin{aligned} \mathbf{u} &= \quad \hat{\mathbf{u}} \\ h &= \quad H + \hat{h}. \end{aligned}$$

where H is the average fluid depth. If we further assume that h_0 is constant, this gives

$$\frac{\partial \hat{\mathbf{u}}}{\partial t} = 2\Omega \times \hat{\mathbf{u}} - g\nabla\hat{h} \tag{3}$$

$$\frac{\partial \hat{h}}{\partial t} = -H\nabla \cdot \hat{\mathbf{u}} \tag{4}$$

These equations can be simplified further by assuming there is no rotation and taking the one-dimensional form. Dropping the ˆ notation this gives

$$\frac{\partial u}{\partial t} = -g\frac{\partial h}{\partial x} \tag{5}$$

$$\frac{\partial h}{\partial t} = -H\frac{\partial u}{\partial x}. \tag{6}$$

This report will seek to solve these one-dimensional lineriased shallow water equations numerically. Note that throughout this report we will assume for simplicity that u and h have periodic boundary conditions.

2 Numerical Methods

The first numerical method we use to attempt to solve (5) and (6) is a simple colocated forward-backward in time and centred in space scheme. As in [1], we consider the scheme to be forward in u and backward in h. Co-located means we define the velocity and the height at the same location in space on the meshgrid and these schemes are also referred to as A-grid or unstaggered schemes.

This scheme can be written as

$$\frac{u_j^{(n+1)} - u_j^{(n)}}{\Delta t} = -g\frac{h_{j+1}^{(n)} - h_{j-1}^{(n)}}{2\Delta x} \tag{7}$$

$$\frac{h_j^{(n+1)} - h_j^{(n)}}{\Delta t} = -H\frac{u_{j+1}^{(n+1)} - u_{j-1}^{(n+1)}}{2\Delta x} \tag{8}$$

where $h_j^{(n)} = h(x_j, t^{(n)})$, $u_j^{(n)} = h(u_j, t^{(n)})$, $x_j = j\Delta x$ and $t^{(n)} = n\Delta t$.

We can determine the order of accuracy of this scheme, by using Taylor series. We note the following Taylor series expansions:

$$u_j^{(n+1)} = u_j^{(n)} + \Delta t\frac{\partial u_j^{(n)}}{\partial t} + \frac{(\Delta t)^2}{2}\frac{\partial^2 u_j^{(n)}}{\partial t^2} + O((\Delta t)^3) \tag{9}$$

$$h_{j\pm1}^{(n)} = h_j^{(n)} \pm \Delta x\frac{\partial h_j^{(n)}}{\partial x} + \frac{(\Delta x)^2}{2}\frac{\partial^2 h_j^{(n)}}{\partial x^2} \pm \frac{(\Delta x)^3}{6}\frac{\partial^3 h_j^{(n)}}{\partial x^3} + O((\Delta x)^4) \tag{10}$$

Substuting these expansions into the scheme (7), we find that

$$\frac{\partial u_j^{(n)}}{\partial t} + O(\Delta t) = -g\frac{\partial h_j^{(n)}}{\partial x} + O((\Delta x)^2) \tag{11}$$

and therefore the scheme is first order accurate in time and second order accurate in space. (Note the same order of accuracy is obtained by performing the same analysis with (8)).

In order to find when this scheme is stable, we use a von-neumann stability analysis. As in [1], we assume that u and h have wave-like solutions with an amplification factor A and wavenumber k:

$$u = \mathbb{U}A^n e^{ikj\Delta x} \tag{12}$$

$$h = \mathbb{H}A^n e^{ikj\Delta x} \tag{13}$$

for constant $\mathbb{U}$ and $\mathbb{H}$.

Further if we define the courant number

$$c = \frac{\sqrt{gH}\Delta t}{\Delta x} \tag{14}$$

then substituting these solutions into (7) and (8) gives

$$A = 1 - \frac{c^2}{2}\sin^2(k\Delta x) \pm \frac{ic}{2}\sin(k\Delta x)\sqrt{4 - c^2\sin^2 k\Delta x}. \tag{15}$$

Hence as found in [1], when $|c| \leq 2$, $|A|^2 = 1$ and the scheme is stable, but when $|c| > 2$, the scheme is unstable.

We can also find the dispersion relation, because the frequency of the numerical method is

$$\omega = \pm\frac{1}{\Delta t}\arctan\left(\frac{Im(A)}{Re(A)}\right) \tag{16}$$

Using the result from [1],

$$\omega_n \Delta x = \pm\frac{2}{c}\sin^{-1}\left(\frac{c}{2}\sin(k\Delta x)\right) \tag{17}$$

assuming $\sqrt{gH} = 1$. The positive branch of this dispersion relation is plotted in Figure 1 and compared with the dispersion relation of the analytical solution. (The dispersion relation of the analytical solution $\omega = k\sqrt{gH}$ is found by substituting the wave-like solutions (12) and (13) into (5) and (6)).

However, we would like to have a method that was stable for all courant numbers. Therefore another method we can use is an implicit method on a colocated grid. As in [1], we look at the forward in time for both u and h and centred in space colocated scheme:

$$\frac{u_j^{(n+1)} - u_j^{(n)}}{\Delta t} = -g\frac{h_{j+1}^{(n+1)} - h_{j-1}^{(n+1)}}{2\Delta x} \tag{18}$$

$$\frac{h_j^{(n+1)} - h_j^{(n)}}{\Delta t} = -H\frac{u_{j+1}^{(n+1)} - u_{j-1}^{(n+1)}}{2\Delta x}. \tag{19}$$

In order to find the values of h_j^n and u_j^n at the next time iteration for all j, we consider the matrix:

$$A = \begin{pmatrix}
1 + \frac{c^2}{2} & 0 & -\frac{c^2}{4} & 0 & 0 & & 0 & 0 & 0 & -\frac{c^2}{4} & 0 \\
0 & 1 + \frac{c^2}{2} & 0 & -\frac{c^2}{4} & 0 & \vdots & 0 & 0 & 0 & 0 & -\frac{c^2}{4} \\
-\frac{c^2}{4} & 0 & 1 + \frac{c^2}{2} & 0 & -\frac{c^2}{4} & & \vdots & & \vdots & & \vdots \\
\vdots & & \vdots & & \vdots & \ddots & -\frac{c^2}{4} & 0 & 1 + \frac{c^2}{2} & 0 & -\frac{c^2}{4} \\
-\frac{c^2}{4} & 0 & 0 & 0 & 0 & \vdots & 0 & -\frac{c^2}{4} & 0 & 1 + \frac{c^2}{2} & 0 \\
0 & -\frac{c^2}{4} & 0 & 0 & 0 & & 0 & 0 & -\frac{c^2}{4} & 0 & 1 + \frac{c^2}{2}
\end{pmatrix}$$

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where c is the courant number as before.

Note that here and throughout this report we are assuming periodic boundary conditions and hence that $u_0^{(n)} = u_N^{(n)}$ and $h_0^{(n)} = h_N^{(n)}$ where $N\Delta x$ is the right hand boundary of the x-domain. Hence the matrix A has dimension $N \times N$ and we consider the first row to be $j = 0$.

We then rewrite the scheme (18) and (19) as the matrix equation $A\mathbf{x} = \mathbf{b}$ where

$$x_j = h_j^{(n+1)} \text{ and } b_j = h_j^{(n)} - \frac{c}{2}\sqrt{\frac{H}{g}}(u_{j+1}^{(n)} - u_{j-1}^{(n)}) \qquad \forall j \in [0, N-1] \qquad (20)$$

if solving for h and

$$x_j = u_j^{(n+1)} \text{ and } b_j = u_j^{(n)} - \frac{c}{2}\sqrt{\frac{g}{H}}(h_{j+1}^{(n)} - h_{j-1}^{(n)}) \qquad \forall j \in [0, N-1] \qquad (21)$$

if solving for u.

We can determine the order of accuracy of this scheme, as before by using the Taylor series expansions (9) and

$$h_{j\pm1}^{(n+1)} = h_j^{(n)} \pm \Delta x \frac{\partial h_j^{(n)}}{\partial x} + \frac{(\Delta x)^2}{2}\frac{\partial^2 h_j^{(n)}}{\partial x^2} \pm \Delta x \Delta t \frac{\partial^2 h_j^{(n)}}{\partial x \partial t} + \frac{(\Delta t)^2}{2}\frac{\partial^2 h_j^{(n)}}{\partial t^2} + O((\Delta x)^3, (\Delta t)^3). \qquad (22)$$

Substituting this into equation (18) we find that

$$\frac{\partial u_j^{(n)}}{\partial t} + O(\Delta t) = -g\frac{\partial h_j^{(n)}}{\partial x} + O((\Delta x)^2). \qquad (23)$$

Therefore the scheme is first order accurate in time and second order accurate in space as with the explicit method on the colocated grid. (As before a similar result can be obtained by substituting in to (19)).

In order to find where this method is stable, we use von-neumann stability analysis and assume u and h have wave-like solutions (12) and (13). Substituting these solutions into (18) and (19) we find

$$A = \frac{1 \pm ic\sin(k\Delta x)}{1 + c^2\sin^2(k\Delta x)} \qquad (24)$$

Therefore

$$|A|^2 = \frac{1}{1 + c^2\sin^2(k\Delta x)} < 1 \qquad \forall k, \Delta x \qquad (25)$$

and the system is stable everywhere.

Using (16) we can find the dispersion relation

$$\omega_n \Delta x = \pm\frac{\Delta x}{\Delta t}\arctan(c\sin(k\Delta x)) = \frac{1}{c}\arctan(c\sin(k\Delta x)) \qquad (26)$$

assuming $\sqrt{gH} = 1$. The positive branch of this dispersion relation is again plotted in Figure 1.

We can see from this Figure 1 that the analytic solution and numerical solution do not propagate at the same rate. The numeric solution disperses too slowly and in fact at $k = \pi$, the wave is stationary.

Therefore we seek a scheme which propagates at a more similar speed to the analytic solution. Following [1], we use a staggered grid (sometimes known as a C-grid) instead of a co-located grid. For a staggered grid, we shift u so that it is defined at $x_j + \frac{\Delta x}{2}$ and h remains defined at x_j (where $x_j = j\Delta x$) ie. where u and h are defined alternates in space.

As in [1], we take again the forward-backward in time and centred in space scheme where the scheme is forward in u and backward in h, but this time on a staggered grid. This gives us the following scheme:

$$\frac{u_{j+\frac{1}{2}}^{(n+1)} - u_{j+\frac{1}{2}}^{(n)}}{\Delta t} = -g\frac{h_{j+1}^{(n)} - h_j^{(n)}}{\Delta x} \tag{27}$$

$$\frac{h_j^{(n+1)} - h_j^{(n)}}{\Delta t} = -H\frac{u_{j+\frac{1}{2}}^{(n+1)} - u_{j-\frac{1}{2}}^{(n+1)}}{\Delta x} \tag{28}$$

We can determine the order of accuracy of this scheme, as before, by using the Taylor series expansions (10) and

$$u_{j\pm\frac{1}{2}}^{(n)} = u_j^{(n)} \pm \frac{\Delta x}{2}\frac{\partial u_j^{(n)}}{\partial x} + \frac{(\Delta x)^2}{8}\frac{\partial^2 u_j^n}{\partial x^2} + O((\Delta x)^3). \tag{29}$$

$$u_{j\pm\frac{1}{2}}^{(n+1)} = u_j^{(n)} \pm \frac{\Delta x}{2}\frac{\partial u_j^{(n)}}{\partial x} + \Delta t\frac{\partial u_j^{(n)}}{\partial t} + \frac{(\Delta x)^2}{8}\frac{\partial^2 u_j^n}{\partial x^2} \pm \frac{\Delta t\Delta x}{2}\frac{\partial^2 u_j^{(n)}}{\partial x\partial t} + \frac{(\Delta t)^2}{2}\frac{\partial^2 u_j^{(n)}}{\partial t^2} + O((\Delta x)^3, (\Delta t)^3) \tag{30}$$

Substituting these into (27), we find that

$$\frac{\partial u_j^{(n)}}{\partial t} + O(\Delta t) = -g\frac{\partial h_j^{(n)}}{\partial x} + O((\Delta x)^2) \tag{31}$$

and therefore the scheme is first order accurate in time and second order accurate in space. (Note as before the same order of accuracy is obtained by performing the same analysis with (28)).

In order to find where this method is stable, we use von-neumann stability analysis and assume u and h have wave-like solutions (12) and (13). Substituting these solutions into (27) and (28) we find

$$A = 1 - 2c^2\sin^2\left(\frac{k\Delta x}{2}\right) \pm 2ic\sin\left(\frac{k\Delta x}{2}\right)\sqrt{1 - c^2\sin^2(\frac{k\Delta x}{2})} \tag{32}$$

Therefore if $|c| \leq 1$, this scheme is stable, but if $|c| > 1$ the scheme is unstable.

Using (16) we can find the dispersion relation

$$\omega_n \Delta x = \pm \frac{2\Delta x}{\Delta t} \arcsin\left(c\sin\left(\frac{k\Delta x}{2}\right)\right) = \frac{2}{c}\arcsin\left(c\sin\left(\frac{k\Delta x}{2}\right)\right) \tag{33}$$

assuming $\sqrt{gH} = 1$. The positive branch of this dispersion relation is again plotted in Figure 1.

Although this solution is still dispersive we can see from the Figure that ω of this numeric scheme is much closer to ω of the analytic solution.

However we again have the problem that the solution is unstable. Therefore the final numerical scheme we will look at is the semi-implicit scheme on a staggered grid outlined in [2].

The scheme used in [2] is the theta-method:

$$\frac{u_{j+\frac{1}{2}}^{(n+1)} - u_{j+\frac{1}{2}}^{(n)}}{\Delta t} = -\frac{g}{\Delta x}\left(\theta(h_{j+1}^{(n+1)} - h_j^{(n+1)}) + (1-\theta)(h_{j+1}^{(n)} - h_j^{(n)})\right) \tag{34}$$

$$\frac{h_j^{(n+1)} - h_j^{(n)}}{\Delta t} = -\frac{H}{\Delta x}\left(\theta(u_{j+\frac{1}{2}}^{(n+1)} - u_{j-\frac{1}{2}}^{(n+1)}) + (1-\theta)(u_{j+\frac{1}{2}}^{(n)} - u_{j-\frac{1}{2}}^{(n)})\right) \tag{35}$$

For simplicity we have taken $\theta = \frac{1}{2}$ and are therefore using the Crank-Nicholson method centred in space on a staggered grid:

$$\frac{u_{j+\frac{1}{2}}^{(n+1)} - u_{j+\frac{1}{2}}^{(n)}}{\Delta t} = -\frac{g}{2\Delta x}\left((h_{j+1}^{(n+1)} - h_j^{(n+1)}) + (h_{j+1}^{(n)} - h_j^{(n)})\right) \tag{36}$$

$$\frac{h_j^{(n+1)} - h_j^{(n)}}{\Delta t} = -\frac{H}{2\Delta x}\left((u_{j+\frac{1}{2}}^{(n+1)} - u_{j-\frac{1}{2}}^{(n+1)}) + (u_{j+\frac{1}{2}}^{(n)} - u_{j-\frac{1}{2}}^{(n)})\right) \tag{37}$$

In order to find the values of h_j^n and u_j^n at the next time iteration for all j, we consider the matrix:

$$A = \begin{pmatrix} 1 + \frac{c^2}{2} & -\frac{c^2}{4} & 0 & 0 & 0 & -\frac{c^2}{4} \\ -\frac{c^2}{4} & 1 + \frac{c^2}{2} & -\frac{c^2}{4} & \vdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & -\frac{c^2}{4} & 1 + \frac{c^2}{2} & -\frac{c^2}{4} \\ -\frac{c^2}{4} & 0 & 0 & 0 & -\frac{c^2}{4} & 1 + \frac{c^2}{2} \end{pmatrix}$$

where c is the courant number as before.

We then rewrite the scheme (36) and (37) as the matrix equation $A\mathbf{x} = \mathbf{b}$.

$$x_j = h_j^{(n+1)} \text{ and } b_j = -c\sqrt{\frac{g}{H}}(h_{j+1}^{(n)}-h_j^n)+\frac{c^2}{4}u_{j+\frac{3}{2}}^{(n)}+(1-\frac{c^2}{2})u_{j+\frac{1}{2}}^{(n)}+\frac{c^2}{4}u_{j-\frac{1}{2}}^{(n)} \qquad \forall j \in [0, N-1] \tag{38}$$

if solving for h and

$$x_j = u_{j+\frac{1}{2}}^{(n+1)} \text{ and } b_j = -c\sqrt{\frac{H}{g}}(u_{j+\frac{1}{2}}^{(n)}-u_{j-\frac{1}{2}}^n)+\frac{c^2}{4}h_{j+1}^{(n)}+(1-\frac{c^2}{2})h_j^{(n)}+\frac{c^2}{4}h_{j-1}^{(n)} \qquad \forall j \in [0, N-1] \tag{39}$$

if solving for u.

Note that the matrix A has dimension $N \times N$ and we consider the first row to be $j = 0$.

Furthermore as before we have assumed periodic boundary conditions and hence that $u_{\frac{1}{2}}^{(n)} = u_{N+\frac{1}{2}}^{(n)}$ and $h_0^{(n)} = h_N^{(n)}$ where $N\Delta x$ is the right hand boundary of the x-domain.

We can determine the order of accuracy of this scheme, as before by using the Taylor series expansions (29) and (30) and

$$h_j^{(n+1)} = h_j^{(n)} + \Delta t \frac{\partial h_j^{(n)}}{\partial t} + \frac{(\Delta t)^2}{2}\frac{\partial^2 h_j^{(n)}}{\partial t^2} + O((\Delta t)^3). \tag{40}$$

Substituting these into (37) we find that

$$\frac{\partial h_j^{(n)}}{\partial t} + O((\Delta t)^2) = -H\frac{\partial u_j^{(n)}}{\partial x} + O((\Delta x)^2) \tag{41}$$

and therefore the scheme is second order accurate in time and second order accurate in space which is better than any of the other schemes we have looked at so far. (Note as before the same order of accuracy is obtained by performing the same analysis with (36)).

As before in order to find where this method is stable, we use von-neumann stability analysis and assume u and h have wave-like solutions (12) and (13). Substituting these solutions into (36) and (37) we find:

$$A = \frac{2 - 2c^2\sin^2(\frac{k\Delta x}{2}) \pm 4ic\sin(\frac{k\Delta x}{2})}{2 + 2c^2\sin^2(\frac{k\Delta x}{2})} \tag{42}$$

$|A|^2 = 1$ and therefore the scheme is stable and undamping $\forall k$.

Using (16) we can find the dispersion relation

$$\omega_n\Delta x = \pm\frac{\Delta x}{\Delta t}\arctan\left(\frac{2c\sin(\frac{k\Delta x}{2})}{1 - c^2\sin^2(\frac{k\Delta x}{2})}\right) = \pm\frac{1}{c}\arctan\left(\frac{2c\sin(\frac{k\Delta x}{2})}{1 - c^2\sin^2(\frac{k\Delta x}{2})}\right) \tag{43}$$
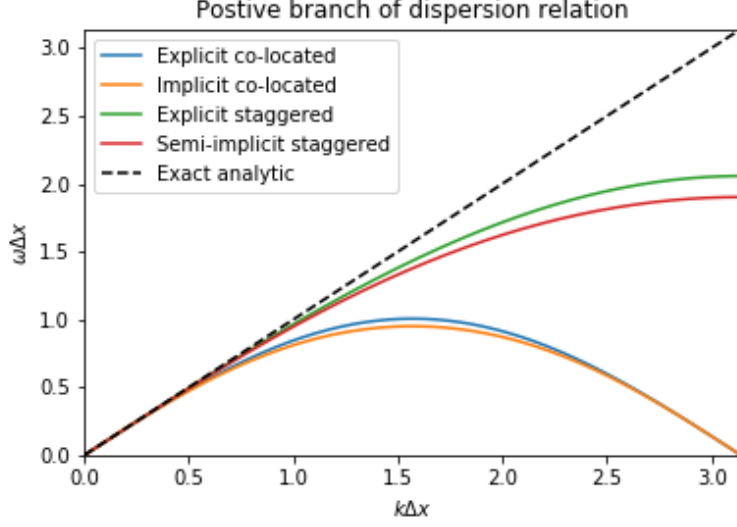
Figure 1: Postive branch of dispersion relation ω for analytic solution and numerical schemes. Note that as in [1], we have used $c = 0.4$ and $\sqrt{gH} = 1$.

assuming $\sqrt{gH} = 1$. The positive branch of this dispersion relation is again plotted in Figure 1.

As with the previous staggered grid scheme, this solution is still dispersive. The dispersion relation for the semi-implicit staggered scheme diverges more from the analytic solution than the explicit staggered scheme but it is still significantly better than either of the colocated schemes.

3 Methodology to test Numerical Methods

In order to test the properties of the Numerical methods outlined in the above section we have devised a series of tests.

3.1 Test 1

To begin with, we attempt to solve the shallow water equations using a simple initial condition (shown in Figure 2a) with the colocated forward-backward explicit scheme.

3.2 Test 2

We found using Von-Neumann stability analysis the colocated forward-backward explicit scheme is unstable for courant numbers higher than 2. Therefore we test to see if this is in fact the case.
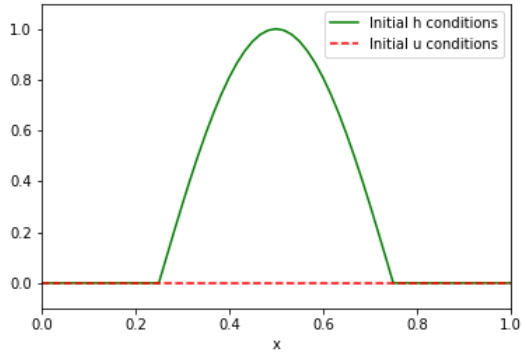
Figure 2a: Initial condition such that u is zero everywhere and h has a bump in the centre and is surrounded by zero either side
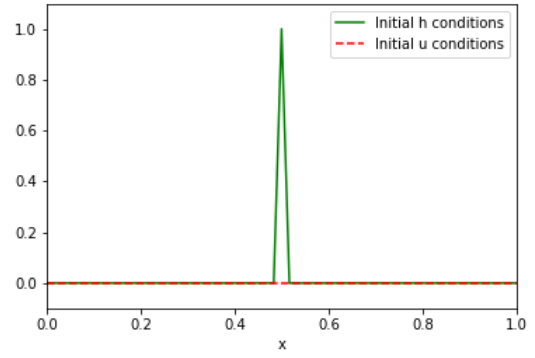
Figure 2b: Initial condition such that u is zero everywhere and h is zero everywhere apart from one point at the centre where it is one
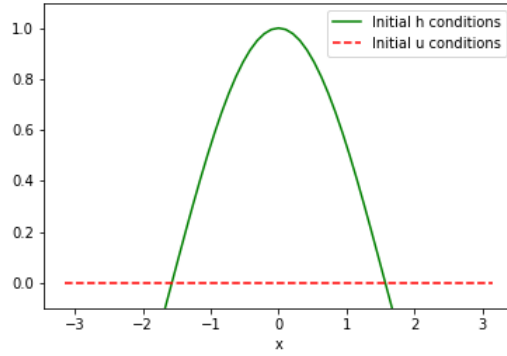
Figure 2c: Initial condition such that u is zero everywhere and h is $\cos(x)$

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3.3   Test 3

We found using Von-Neumann stability analysis the colocated forward-backward implicit scheme is stable for all courant numbers. Therefore we test to see if this is in fact the case.

3.4   Test 4

When looking at the dispersion relation for the colocated grid it seems that the results produced by the scheme may be unphysical. We test this hypothesis by taking a different initial condition (shown in Figure 2b) and plotting the solutions of u and h at multiple time steps for the colocated forward-backward explicit scheme and the colocated forward-backward implicit scheme.

3.5   Test 5

Our analysis of the dispersion relations suggests that on a staggered grid the solutions for u and h should be more physical. Therefore we repeat test 3 with the same initial conditions (shown in Figure 2b), but instead using the staggered explicit scheme.

3.6   Test 6

So far all the comparisons have been about the properties of the schemes themselves. If the initial condition shown in Figure 2c is chosen it is possible to find an analytic solution of the shallow water equations

$$u = \sin(x)\sin(t) \tag{44}$$

$$h = \cos(x)\cos(t). \tag{45}$$

If we choose the interval $[-\pi, \pi]$ then these solutions are periodic.

This last test will look at the error between the solutions produced by the numerical scheme and the analytic solution. It will also compare these errors to Δx and Δt to see if the error relationships found by the Taylor series expansions are correct.

4 Results

4.1 Test 1
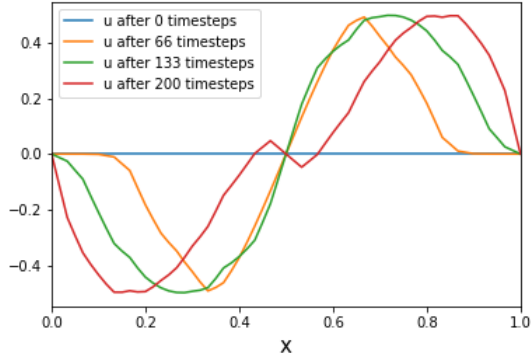
Figure 3a: CAPTION
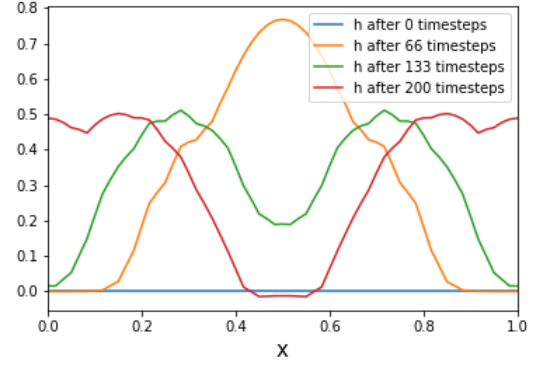
Figure 3b: CAPTION

4.2 Test 2
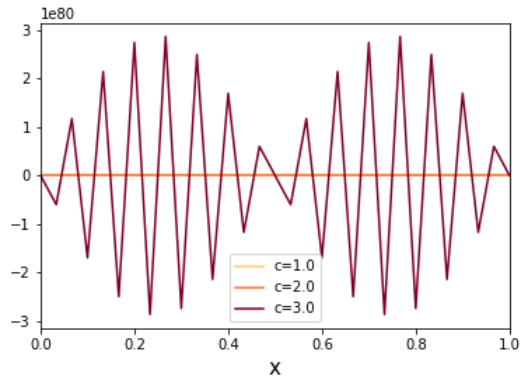
Figure 4a: CAPTION
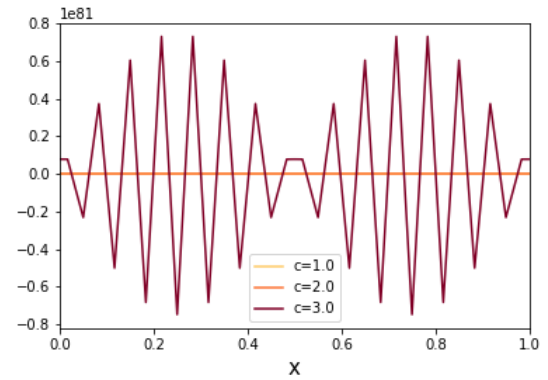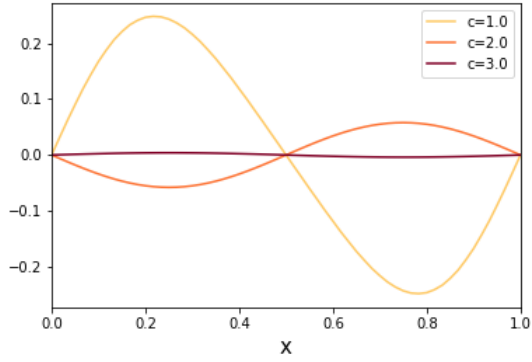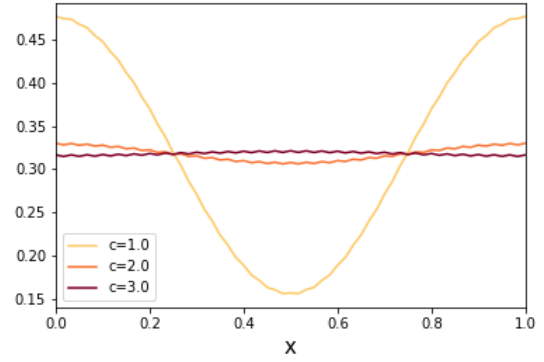

Figure 4b: CAPTION

4.3   Test 3

Figure 5a: CAPTION

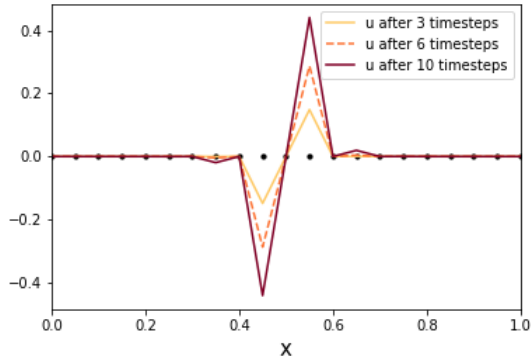Figure 5b: CAPTION

4.4   Test 4
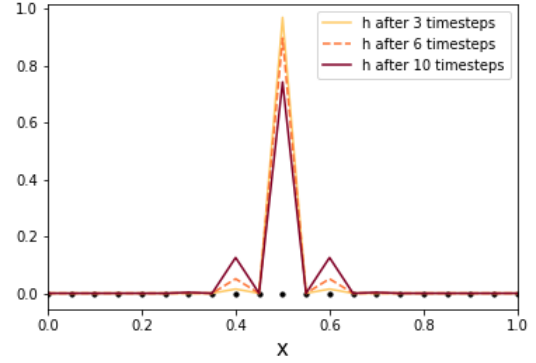

Figure 6a: CAPTION

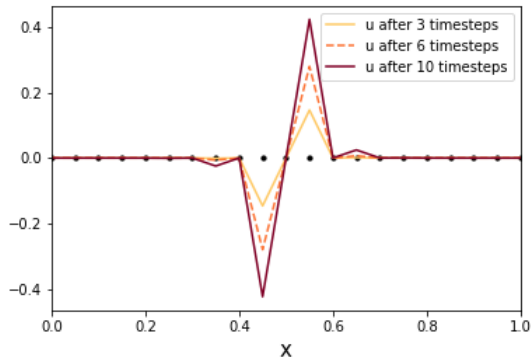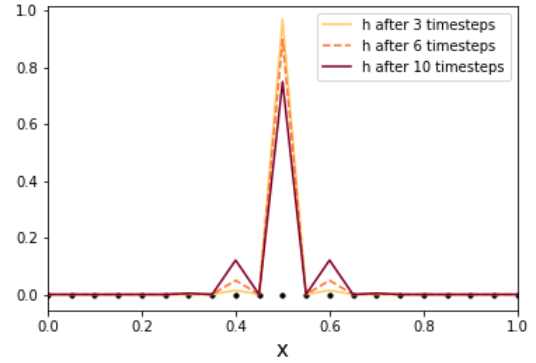

Figure 6b: CAPTION

Figure 7a: CAPTION

Figure 7b: CAPTION
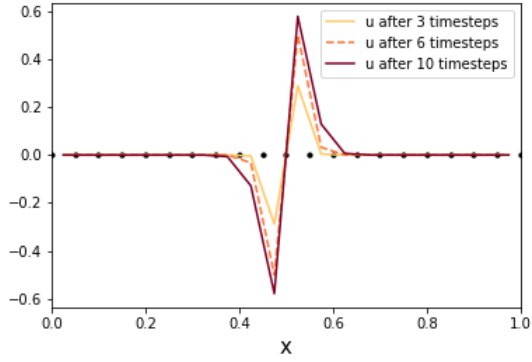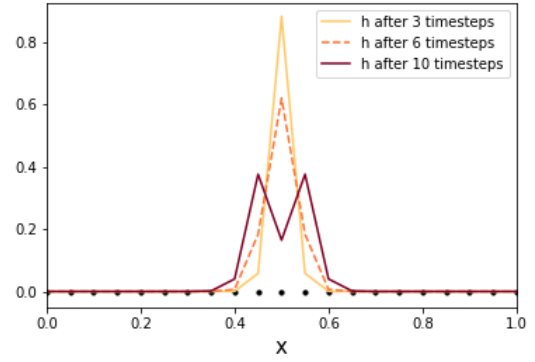
4.5   Test 5

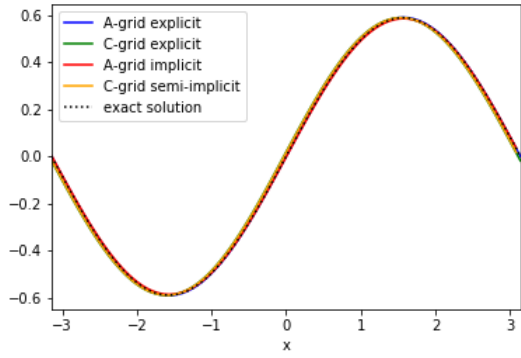Figure 8a: CAPTION

Figure 8b: CAPTION
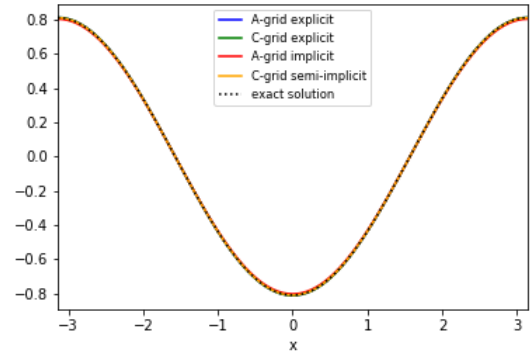
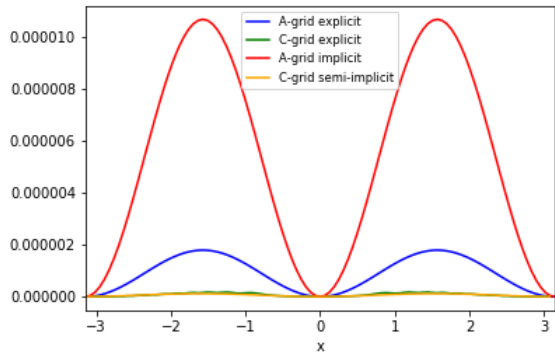## 4.6   Test 6

Figure 9a: CAPTION
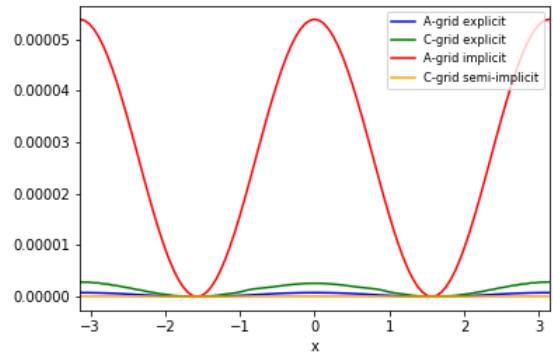

Figure 9b: CAPTION

Figure 10a: CAPTION

Figure 10b: CAPTION

5 Conclusions

References

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