AU412
Applied Control Systems

Part 01

Introduction – Modeling – Analysis
linear/nonlinear, discrete/continuous systems

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1

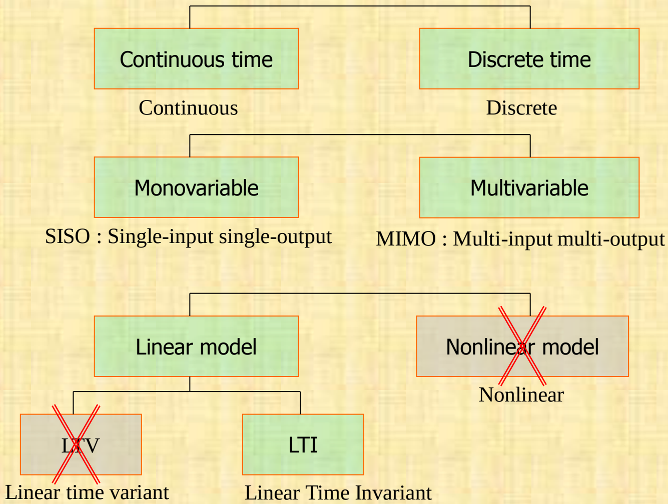
Summary of AU47

part 1
- Introduction, Internal/external model, Discrete/continuous transfer function, Discrete/continuous state space representation, SISO/MIMO, Conversion from continuous to discrete, Stability analysis…

part 2
- Sensors / Actuators :
  Modeling temperature, position, speed, acceleration, IMU sensors
  Hydraulic, pneumatic, electrical actuators, PWM drivers

part 3
- Practical aspects of control systems (PID, lead/lag; filtering, windup, …),
  Converting continuous controller into discrete controller,
  Control with state space representation …

Classification of systems

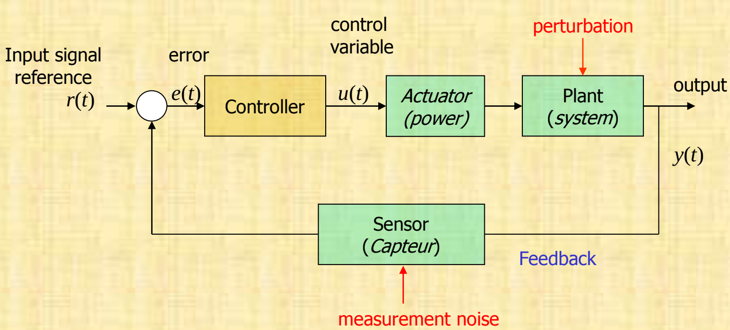


SISO or **MIMO** / **Continuous** or **Discrete**

Easier to control LTI systems

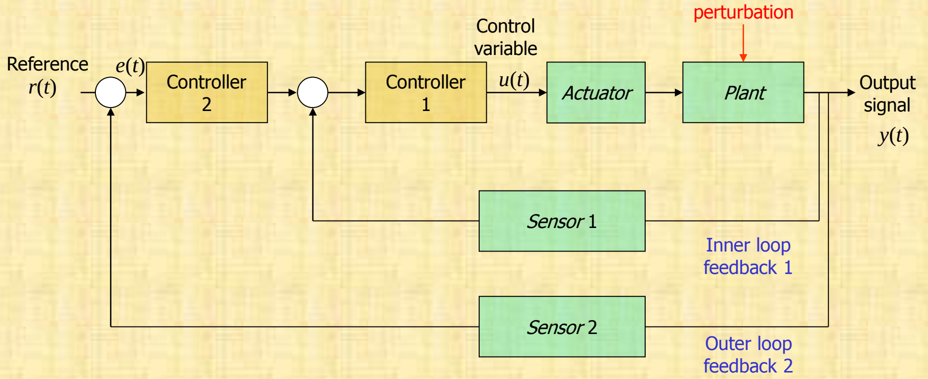
A classical control scheme

- An Example of Closed loop control



A control scheme with two loops

- Examples of Closed loop control with two loops

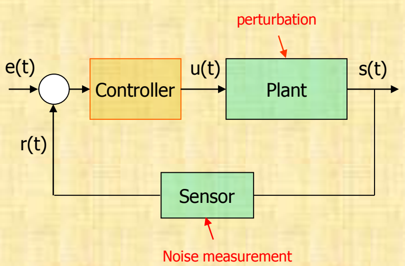


Example 1: Control of DC motor
Loop 1 : to control current
Loop 2 : to control motor speed
output of controller 2 = speed reference

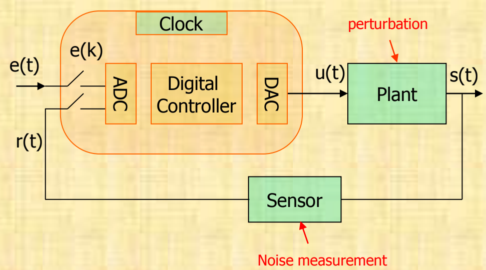
Exemple 2: Control of Quadrotor (roll)
Loop 1 : roll control
Loop 2 : y (translation) control
output of controller 2 = roll reference

Continuous vs Discrete

- Continuous control vs. Discrete control



Continuous time control system
(Analog control)



Discrete time control system
(Digital control)

Continuous-time Representation of systems

7

LTI SISO systems : continuous case

- For continuous linear SISO (single input single output) systems

Differential Equation (external model)

$$\ddot{y} + 3\dot{y} + 2y = 2\dot{u} + u$$

Transfer Function LTI (linear time invariant)

$$\frac{Y(s)}{U(s)} = G(s) = \frac{2s+1}{s^2+3s+2}$$

$$Y(s) = G(s)U(s)$$

Impulse Response $u(t) = \delta(t);\ \ U(s) = 1$

$$g(t) = \mathcal{L}^{-1}\{G(s)\}$$

$$y(t) = g(t) \otimes u(t) = g(t)$$

State Space Representation

$$\dot{X} = f(t, X, u) = AX + Bu$$
$$y = h(t, X, u) = CX + Du$$

A, B, C, D : state matrix, input (control) matrix, output (observation) matrix, feedtrough (transfer) matrix

$X: 1 \times n$, $A: n \times n$, $B: n \times 1$, $u: 1 \times 1\ (one\ input)$
$y: 1 \times 1\ (one\ output)$, $C: 1 \times n$, $D: 1 \times 1$

From SS to TF:

$$G(s) = C(sI - A)^{-1}B + D$$

Proof!!!

Give the state space representation of the following linear systems:

$\ddot{y} + 3\dot{y} + 2y = 2u \quad ; \qquad \ddot{y} + 3\dot{y} + 2y = 2\dot{u} + u$

What are the TF and SS representation of the LTV (time varying) system: $\ddot{y} + 3t\,\dot{y} + (2+t)y = 2u$

8

Nonlinear SISO systems

- Continuous nonlinear SISO systems

External representation:	State space representation:
$y\ddot{y} + \dot{y} + cos(y) = u^2$ What is the TF of this system?	$\dot{X} = f(t, X, u)$ $y = h(t, X, u)$ $X{:}\, n \times 1 \;\;,\;\; f{:}\, n \times 1 \;\; (vector\ of\ n\ functions)$ $y{:}\, 1 \times 1 \;\; (1\ output)\,,\;\; h{:}\, 1 \times 1\ one\ function$ 2nd form, particular case $\dot{X} = f(X) + \; g(X)u$ $y = h(X)$ $f{:}\, n \times 1\,, g(X){:}\, n \times 1 \;\; (vectors\ of\ n\ functions)$

Give the state space representation of the following nonlinear systems:

$\ddot{y} + \dot{y}^2 + cos(y) = 3u^2 \quad ; \quad \ddot{y} + y\dot{y} + cos(y) = u^2 \cos(y)$

MIMO systems

- Linear/nonlinear MIMO (multi-input multi-output)

Continuous linear system: external model	Cont. Nonlinear system: external model
$\ddot{y}_1 + 3y_1 + \dot{y}_2 + y_2 = u_1 + u_2$ $\dot{y}_2 - y_1 + 3y_2 = 3u_1 + u_3$	$\ddot{y}_1 + 3y_1y_2 + cos(y_1) = u_1^2 + u_2$ $\dot{y}_2 - y_1 + 3y_2 = 3u_1 + u_3$
Continuous transfer function $\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} G_{11(s)} & G_{12(s)} & G_{13(s)} \\ G_{21(s)} & G_{22(s)} & G_{23(s)} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \\ U_3(s) \end{bmatrix}$ G_{ij} : transfer function that links the output i to the input j (assuming the other inputs null)	**Cont. Nonlinear system: state space** $\dot{X} = f(t, X, u)$ $y = h(t, X, u)$ $X{:}\, n \times 1 \;\;,\;\; f{:}\, n \times p \;\; (vector\ of\ n\ functions)$ $y{:}\, q \times 1\, (q\ outputs)\,,$ $h{:}\, q \times 1\ vector\ of\ q\ functions$
Continuous state space $\dot{X} = f(t, X, u) = AX + Bu$ $y = h(t, X, u) = CX + Du$ $p\ inputs\ et\ q\ outputs$ $X{:}\, 1 \times n \;\;,\;\; A{:}\, n \times n \;\;,\;\; B{:}\, n \times p \;\;,\;\; u{:}\, p \times 1$ $y{:}\, q \times 1 \;\;,\;\; C{:}\, q \times n \;\;,\;\; D{:}\, q \times p$	**A particular form :** $\dot{X} = f(X) + \; g(X)u$ $y = h(X)$ $f{:}\, n \times 1\,, g(X){:}\, n \times p \;\; (matrix\ of\ n \times p\ fcts)$

Compute the transfer function matrix of the two systems described by the above external model.

Compute the state space representation of these two systems.

From nonlinear to linear : Linearization

- Linearisation of a nonlinear system around an operating point x_m

$$\left.\begin{aligned}\dot{x} &= f(x,u,v)\\ y &= h(x,u,w)\end{aligned}\right\}$$

$$\boxed{\begin{aligned}\dot{\tilde{x}} &= A\tilde{x} + B\tilde{u} + B_v\tilde{v}\\ \tilde{y} &= C\tilde{x} + D\tilde{u} + D_w\tilde{w}\end{aligned}}$$

$$A = \frac{\delta f}{\delta x}\ ,\ B = \frac{\delta f}{\delta u}\ ,\ B_v = \frac{\delta f}{\delta v}$$
$$C = \frac{\delta h}{\delta x}\ ,\ D = \frac{\delta h}{\delta u}\ ,\ D_w = \frac{\delta h}{\delta w}$$

Change variable

$$x = x_m + \tilde{x}$$
$$u = u_m + \tilde{u}$$
$$y = y_m + \tilde{y}$$
$$v = v_m + \tilde{v}$$
$$w = w_m + \tilde{w}$$

$\tilde{\boldsymbol{x}}$: small variation of $\boldsymbol{x}$ around the operating point $\boldsymbol{x_m}$

If f and h are functions that depend explicitly on time t, then A, B, C, D, ... will be functions of time (→ Linear Time Variant System).

From linear to linear: Bilinear transformation

- Variable change (by a bilinear transformation)

For either discrete or continuous linear system

1st form	Transformation	2nd form
$\dot{x} = Ax + Bu$ $y = Cx + Du$	$z = Px$ $x = P^{-1}z$	$\dot{z} = A_2 z + B_2 u$ $y = C_2 z + D_2 u$ $\dot{z} = PAP^{-1}\, z + PB\, u$ $y = CP^{-1}\, z\ + D\, u$

Both of the above forms describe the same linear system.

Bilinear transformation : Some forms

Canonical modal form:

$$A_2 = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

The obtained system is decoupled.
λ_1, λ_2 ... are the **eigenvalues of A_2**
Transformation:
$P = M^{-1}$, $avec : M = [v_1, v_2, \dots, v_n]$
M : matrix of eigenvalues of A_2

In Matlab:

```
[lambda,M]=eig(A);   P = inv(M);
or
 sys2 = canon(sys, 'modal')
```

Canonical controllable form:

$$A_2 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}$$

$a_0, \dots, a_n$: are the coefficients of the **characteristic equation** of A_2.

In Matlab :

```
EC = poly(A); % EC = det(λI − A)
or
sys2 = canon(sys, 'companion')
```


with Matlab

- Some useful commands in Matlab

`tf, zpk, ss`: create TF (transfer function), ZPK (zeros/poles/gain) or SS state space objects.

`sys1 = ss(G)` : state space representation (from any other representation: tf or zpk)

`G = tf(sys1)` : transfer function (from any other representation: zpk or ss)

`c2d, d2c` : convert LTI model from continuous to discrete or vice-versa.

`ss2ss` : bilinear transformation z = Px

`sys2 = canon(sys1,'modal')` : transforms sys1 into a canonical model form

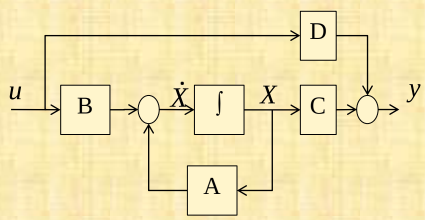
`sys2 = canon(sys1,'companion')` : transforms sys1 into a canonical companion form

`EC = poly(A)` : determines the characteristic polynomial of the matrix A. EC=det(λI-A)

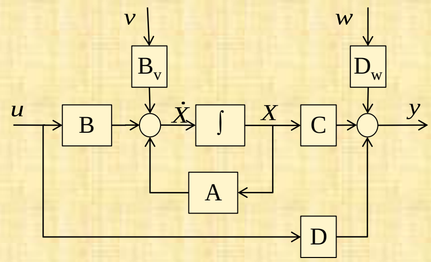
`poly(r)` : if *r* is a vector, returns the polynomial whose roots are the elements of *r*.

Block diagram

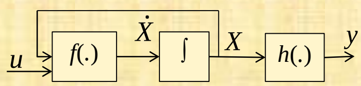
- Block diagram (schéma fonctionnel, schéma bloc) from state space representation



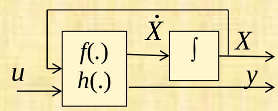
Deterministic linear system



Linear system with noise



Nonlinear system



Nonlinear model
(**more general form**)

Discretization

Review : Discretization

- Digital signal processing

$e(t) \rightarrow$ Discretization $\xrightarrow{e_1, e_2, \ldots, e_k, \ldots}$ Digital processing $\xrightarrow{u_1, u_2, \ldots, u_k, \ldots}$

Discretization $\mapsto$ Sampling + Quantization

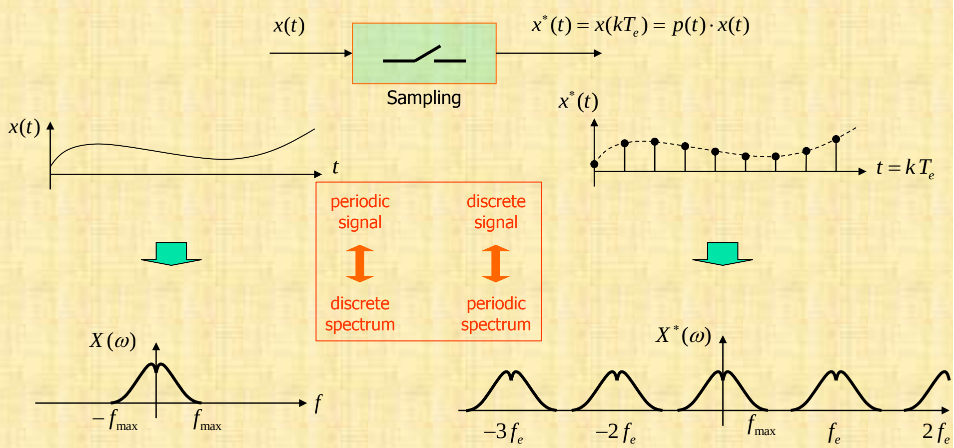
$e(t) \rightarrow$ Sampling $\xrightarrow[e(kT_e)]{e^*(t)}$ ADC $\xrightarrow{e_1, e_2, \ldots, e_k, \ldots}$

Sampling: extracting values of a continuous signal spaced at intervals of Ts (at rate Fs = 1/Ts) (time discretization)

Quantization: converting the signal to digital values (n bits) by an ADC (analog to digital converter) (discretization in amplitude)

Review : Spectrum – Shannon's theorem

- Spectrum of a continuous / discrete signal

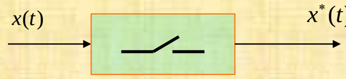


$f_e > 2 f_{max}$ $\qquad$ $T_e < \dfrac{1}{2 f_{max}}$

The sampling rate f_s must be higher that twice the maximal rate f_{max} of the signal

Review : Laplace transform / Z transform

- Z transform principle



continuous signal $x(t)$ $\quad \boxed{X(s) = \int_0^\infty x(t)\,e^{-st}\,dt}$

discrete signal $x(k) \equiv x(kT_e)$ $\quad \boxed{X(z) = \sum_{k=0}^{\infty} x(k)\,z^{-k}}$

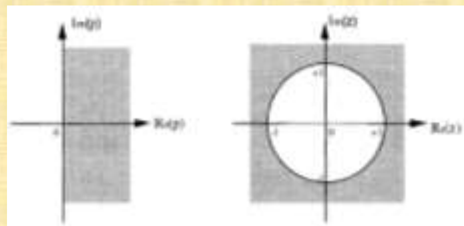
Z transform ≡ **Laplace** transform with :

$$z = e^{T_e s}$$

- Stability analysis: Z vs Laplace

$$\boxed{s = \sigma + j\omega \quad \Leftrightarrow \quad z = e^{T_e s} = e^{T_e\sigma}\cdot e^{j\omega T_e}}$$

$$\boxed{\begin{array}{l}\mathrm{Re}(s) = \sigma \\ \mathrm{Im}(s) = \omega\end{array}} \quad \Leftrightarrow \quad \boxed{\begin{array}{l}|z| = e^{\sigma T_e} \\ \arg z = \omega T_e\end{array}}$$



$$\boxed{\sigma \le 0 \quad \Leftrightarrow \quad |z| \le 1}$$

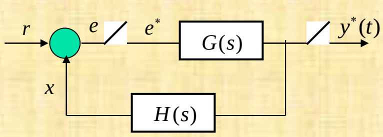
$$\omega_{\max} < \omega_e/2 \quad \Leftrightarrow \quad \omega T < \pi$$

Discrete Closed Loop Transfer Function

- Closed Loop Transfer Function of a discrete system

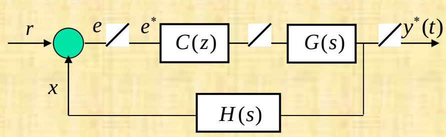
$$e = r - x$$
$$e^* = (r - x)^*$$

$$e^* = r^* - x^*$$



$$OLTF(z) = GH(z)$$

$$CLTF(z) = \frac{G(z)}{1 + (GH)(z)}$$

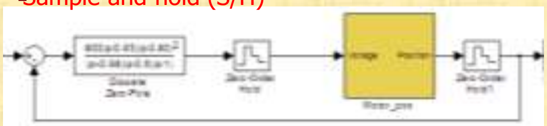


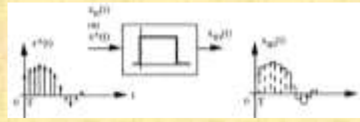
Open Loop TF : $OL(z) = C(z)\,GH(z)$

Closed Loop TF : $CL(z) = \dfrac{C(z)G(z)}{1 + C(z)\,GH(z)}$

D/A Converter modeling

- Sample and hold (S/H)





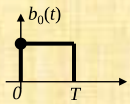
The sampled signal is zero between sample times!!!! But the physical plant input is analog.
After digital to analog conversion (**DAC**), the control output is *continuous*.
So, in modeling, we add a hold block after sampling (for example a zero order hold).

- Modeling a zero-order hold (bloqueur d'ordre 0)

Zero order hold (zoh) converts a discrete-time signal to a continuous-time signal by holding each sample value for one sample interval.

Impulse response $b_0(t)$:

$$\delta(k) \longrightarrow \boxed{B_0(s)} \longrightarrow$$



$$b_0(t) = u(t) - u(t - T_e)$$

Transfer function = LT{$b_0(t)$}

$$B_0(s) = L\{b_0(t)\} = \frac{1}{s} - \frac{1}{s}e^{-T_e s}$$

$$\boxed{B_0(s) = \frac{1 - e^{-T_e s}}{s}}$$

Conversion From Continuous to Discrete

Review: Continuous → Discrete

- Continuous → Discrete Conversion

Definition: Representing a signal x(t) in the complex space (Laplace domain)	Definition: Representing a signal x(k) in the complex space (Z domain)
$X(s) = \int_0^{\infty} x(t)e^{-st}\,dt$	$X(z) = \sum_{k:0}^{\infty} x_k\, z^{-k}$ Notation: $x_k = x(kT)$ T : being the sample time

$$s = \sigma + j\omega \;\rightarrow\; \boxed{z = e^{Ts}}$$

$$|z| = e^{\sigma T} \;\; ; \qquad \arg(z) = \omega\, T$$

Continuous → Discrete conversion

- From Ordinary Differential Equation to Recurrence Equation

$$G(s) \;\Rightarrow\; G(z)$$

$$a_n y^{(n)}(t) + ... + a_0 y(t) = b_m u^{(m)}(t) + ... + b_0 u(t) \;\overset{?}{\Rightarrow}\; \boxed{a_n y(k-n) + ... + a_0 y(k) = b_m u(t-m) + ... + b_0 u(k)}$$

1) First approximation :

Approximate all the derivatives by:

$$y(t) = y(k) \quad ; \quad t = kT$$

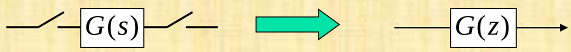
$$\boxed{\dot{y}(k) \approx \frac{y(k) - y(k-1)}{T_e}}$$

$$y^{(2)}(k) \approx \frac{\dot{y}(k) - \dot{y}(k-1)}{T_e} \approx \frac{y(k) - 2y(k-1) + y(k-2)}{T_e^2}$$

...

Continuous → Discrete (TF)

- From continuous transfer function to discrete transfer function



1) First approximation :

or simply replace in the continuous transfer function $G(s)$ the variable s by:

$$\boxed{s \approx \frac{1}{T_e}\left(1-z^{-1}\right)=\frac{1}{T_e}\left(\frac{z-1}{z}\right)}$$

Here the same transfer function as above is obtained.

Proof :

$$\dot{y}(k) \approx \frac{y(k)-y(k-1)}{T_e}$$

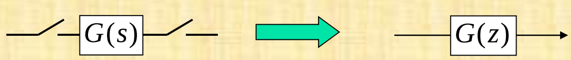
$$TZ\{\dot{y}(k)\} \approx \frac{Y(z)-z^{-1}Y(z)}{T_e}=\frac{1-z^{-1}}{T_e}Y(z)$$

$$TL\{\dot{y}(t)\} \approx s\ Y(s)$$

$$\Rightarrow \quad s \approx \frac{1-z^{-1}}{T_e}$$

Continuous → Discrete (Tustin)

- From continuous transfer function to discrete transfer function



2) Second approximation :

replace in the continuous transfer function $G(s)$ the complex variable s by:

$$\boxed{s \approx \frac{2}{T_e}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)=\frac{2}{T_e}\left(\frac{z-1}{z+1}\right)}$$

This approximation is called **bilinear transformation** (Tustin transform)

Proof:

$$\frac{y(k)-y(k-1)}{T_e} \approx \text{moyenne de } \dot{y}(t) \text{ aux instants } k-1 \text{ et } k$$

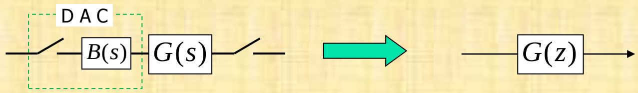
$$\frac{y(k)-y(k-1)}{T_e} \approx \frac{\dot{y}(k)+\dot{y}(k-1)}{2}$$

$$\frac{1-z^{-1}}{T_e}Y(z)=\frac{1+z^{-1}}{2}Z\{\dot{y}(k)\}$$

$$\Rightarrow \quad s \approx \frac{2}{T_e}\frac{1-z^{-1}}{1+z^{-1}}$$

Continuous → Discrete (zero order hold)

- From continuous transfer function to discrete transfer function



3) Third approximation : by using the table of ZT of usual signals

The table you have (in TD1) includes: continuous **signal** / its **LT** / discrete **signal** / its **ZT**.

♦ Add a hold block (zoh for example) to model the Digital to Analog Converter

the input of the plant G(s) is continuous but not a sample signal

Do not confuse the discrete transfer function $G(z)$ with the Z transform of a signal G(s)

$$\boxed{G(z) = Z\{B_0(s)G(s)\}} \qquad \neq Z\{G(s)\}$$

$$G(z) = Z\{B_0(s)G(s)\} = Z\left\{\frac{1-e^{-T_e s}}{s}G(s)\right\} \quad \Rightarrow \quad \boxed{G(z) = (1-z^{-1})Z\left\{\frac{G(s)}{s}\right\} = \frac{z-1}{z}Z\left\{\frac{G(s)}{s}\right\}}$$

Continuous → Discrete (Summary)

- Continu → Discret

Differential Equations --> Recurrence Equations. Substitute the derivatives (in the diff. equ.) with the approximations:

$$\dot{x}(t) \approx \frac{x_k - x_{k-1}}{T} \quad , \quad \ddot{x}(t) \approx \frac{\dot{x}_k - \dot{x}_{k-1}}{T} = \frac{x_k - 2x_{k-1} + x_{k-2}}{T^2}, \ldots$$

First order method. Substitute the s operator (in the TF) with the approximation:

$$s \approx \frac{1}{T}(1 - z^{-1}) = \frac{1}{T}\frac{z-1}{z}$$

Bilinear (Tustin) method. Substitute the s operator (in the TF) with the approximation:

$$s \approx \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

Zero order hold method. Take into account the DAC model and use the Z transform table of usual signals:

$$G(z) = Z\{B_0(s)G(s)\} = (1 - z^{-1})Z\left\{\frac{G(s)}{s}\right\}$$

Continuous → Discrete (Comparison)

- Comparison between three discrete transfer functions of a linear system

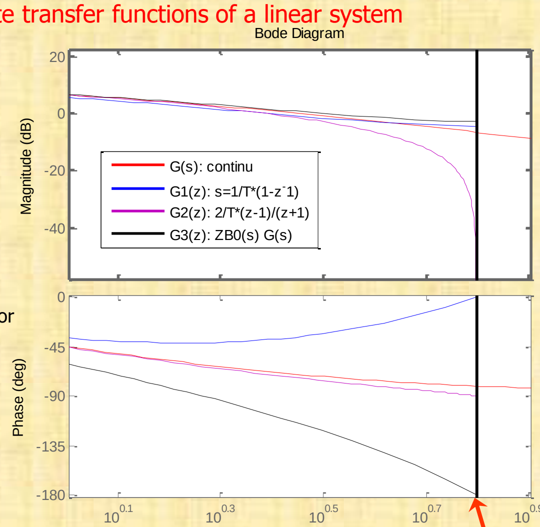
$$G(s)=\frac{3}{s+1} \quad ; \quad T_e = 0.5\,s$$

Note:
Bode diagrams are drawn by considering
s = jω for G(s) and z = exp(jωT) for G(z).

Bode diagrams are drawn from 0 to $\omega_e/2$

The three discrete TF approximate well G(s) for low frequencies $\omega \ll \omega_e/2$.

Which approximation is better?



$$\frac{\omega_e}{2}=\frac{\pi}{T_e}\approx 10^{0.8}$$

Continuous → Discrete (State Space)

- State space representation

From continuous state space --> Discrete state space representation

Differential equation	**Recurrence Equation**
$\dot{x}(t)=f\big(t,x(t),u(t),v(t)\big)$ $y(t)=h\big(t,x(t),u(t),w(t)\big)$	$x(k+1)=f_d\big(kT,x(k),u(k),v(k)\big)$ $y(k)=h\big(kT,x(k),u(k),w(k)\big)$ $t=kT$, T : sampling (échantillonnage) period f_d is different if the the sampling period changes $h(.)$ remains the same as is continuous case
	Notations : $X(kT)\equiv X(k)\equiv X_k$ $x_{k+1}=f_d(kT,x_k,u_k,v_k)$ $y_k=h(kT,x_k,u_k,w_k)$

Continuous → Discrete (Linear State Space)

- State space representation

Linear case: $\dot{X} = AX + Bu$ $y = CX + Du$	$X_{k+1} = F\,X_k + G\,u_k$ $y_k = CX_k + Du_k$ **Exact form exacte of *F* and *G* :** $\boldsymbol{F} = \boldsymbol{e}^{\boldsymbol{A}\,\boldsymbol{T}} \cong I + A\,T + A^2\frac{T^2}{2!} + \cdots$ $\boldsymbol{G} = \int_0^T e^{A\,t}\,B\,dt$ F = expm(A*Ts) : matrix exponential of A.Ts
	1st order approximation $\dot{\boldsymbol{X}} \cong (\boldsymbol{X}_{k+1} - \boldsymbol{X}_k)/\boldsymbol{T}$ $X_{k+1} = (\boldsymbol{I} + \boldsymbol{A}\,\boldsymbol{T})\,X_k + \boldsymbol{B}\,\boldsymbol{T}\,u_k$ $y_k = C\,X_k + D\,u_k$
Nonlinear case: $\dot{x}(t) = f\big(x(t), u(t), v(t)\big)$ $y(t) = h\big(x(t), u(t), v(t)\big)$	**1st order approximation** $x_{k+1} = x_k + T\,f(x_k, u_k, v_k) = f_d(x_k, u_k, v_k)$ $y_k = h(x_k, u_k, v_k)$

34

Discrete-time representation of systems

35

Discrete-time LTI SISO systems (Summary)

- For discrete SISO (single input single output) systems

Recurrence Equation	State space representation (représentation d'état)
$y_{k+2} - 1.72\, y_{k+1} + 0.74\, y_k = 0.17\, u_{k+1} + u_k$	$X_{k+1} = FX_k + Gu_k$ $y_k = CX_k + Du_k$
Discrete Transfer Function *G(z)*	A, B, C, D : state matrix, input (contro) matrix, output (observation) matrix, feedforward (transfer) matrix
$\frac{Y(z)}{U(z)} = G(z) = \frac{0.17\, z + 1}{z^2 - 1.72\, z + 0.74}$ $Y(z) = G(z)U(z)$	$X_k: 1 \times n\ ,\ F: n \times n\ ,\ G: n \times 1\ ,\ u_k: 1 \times 1\ (one\ input)$ $y_k: 1 \times 1\ (one\ output)\ ,\ C: 1 \times n\ ,\ D: 1 \times 1$
Impulse Response *g(k)*	Write the transfer function as a function of *F, G, C, D*
$g(k) = \mathcal{Z}^{-1}\{G(z)\}$ $y(k) = g(k) \otimes u(k)$	F, G, C, D

36

Discrete-time LTI MIMO systems (Summary)

- For linear MIMO (multi-input multi-output) systems

Continuous linear system: external model	Discrete linear system: external model
$\ddot{y}_1 + 3y_1 + \dot{y}_2 + y_2 = u_1 + u_2$ $\dot{y}_2 - y_1 + 3y_2 = 3u_1 + u_3$	$y_1(k) + 0.1y_1(k-1) + y_2(k-2) = u_1(k) + u_2(k)$ $y_2(k) + 0.2y_1(k-1) - y_2(k-2) = 2u_2(k)$
Continuous transfer function	**Discrete transfer function**
$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} G_{11(s)} & G_{12(s)} & G_{13(s)} \\ G_{21(s)} & G_{22(s)} & G_{23(s)} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \\ U_3(s) \end{bmatrix}$	$\begin{bmatrix} Y_1(z) \\ Y_2(z) \end{bmatrix} = \begin{bmatrix} G_{11(z)} & G_{12(z)} \\ G_{21(z)} & G_{22(z)} \end{bmatrix} \begin{bmatrix} U_1(z) \\ U_2(z) \end{bmatrix}$
G_{ij} : transfer function that links the output i to the input j (assuming the other inputs null)	
Continuous state space	**Discrete state space**
$\dot{X} = AX + Bu$ $y = CX + Du$	$X_{k+1} = FX_k + Gu_k$ $y_k = CX_k + Du_k$
p entrées et q sorties $X: 1 \times n\ ,\ A: n \times n\ ,\ B: n \times p\ ,\ u: p \times 1$ $y: q \times 1\ ,\ C: q \times n\ ,\ D: q \times p$	*p entrées et q sorties* $X_k: 1 \times n\ ,\ F: n \times n\ ,\ G: n \times p\ ,\ u_k: p \times 1$ $y_k: q \times 1\ ,\ C: q \times n\ ,\ D: q \times p$

Compute the transfer function matrix of the two systems described by the above external model.

Compute the state space representation of these two systems.

37

Stability/performance analysis

of a controlled system

38

Stability : Definitions

- Standard definition
  - A LTI system is stable if its impulse response $y_\delta(t)$ is bounded
    It is asymptotically stable if $y_\delta(t) \to 0$ when $t \to \infty$
    It is marginally stable if it is not asymptotically stable but $y_\delta(t)$ is bounded

- Relative stability
  - Relative stability defines the stability degree of the system.
    How stable is the system (does it become more or less stable than before)?

    Numerically measure of the stability is needed.
    Two known measures of relative stability: phase margin and gain margin.

Stability of LTI systems

- By studying the poles (or eigen values)

A linear system : is stable if its impulse response is bounded.

Continuous Case:	Discrete case:
• In time domain: $g(t) = \mathcal{L}^{-1}\{FTBF(s)\}$: is bounded For MIMO systems : The impulse responses of all pairs (I/O) are bounded • In complex domain (S plane): $Real(poles) \leq 0$ • In continuous state space: $\dot{x} = Ax + Bu$ $Real(\lambda_i) \leq 0$ $eigen\ values\ of\ A$: $\lambda_i \equiv poles\ of\ CLTF(s)^*$ CE**: $det(sI - A) \equiv denominator of\ CLTF(s)$	• In time domaine: $g(k) = Z^{-1}\{FTBF(z)\}$: is bounded For MIMO systems : The impulse responses of all pairs (I/O) are bounded • In complex domain (Z plane): $Module(poles) \leq 1$ • In discrete state space: $x_k = Fx_{k-1} + G\,u_{k-1}$ $Module(\lambda_i) \leq 1$ $eigen\ values\ of\ F$: $\lambda_i \equiv poles\ of\ CLTF(z)$ CE: $det(zI - F) \equiv denominator\ of\ CLTF(z)$

*CLTF : closed loop transfer function
**CE : caracteristic equation

Performance analysis

- Performance analysis

Temporal analysis (peak response, settling time, rise time, overshoot, steady state error, minimal phase, …)	
Frequency analysis (Bode/Nyquist diagram …) If the input of a linear system is sinusoidal, then the output is also **sinusoidal**, with the **same frequency**. It is possible to plot the Bode diagram.	For nonlinear systems, the output is not necessary sinusoidal. - No Bode diagram - No Laplace/Z transforms - No transfer function
	Analysis by phase plane (if *n* = 2)
Stability : If a linear system is stable, then the impulse response is bounded **for all initial conditions**.	For nonlinear systems, there is a region of stability. If the system is initially outside this region, the stability is not guaranteed.

Controller design

- Criterion for choosing a regulator

- Compromise between stability / precision
- Good temporal behavior (acceptable errors, …)
- Stable for all initial conditions in the operation region
- Robust control according to parameter uncertainties, modeling uncertainties and external disturbances