General Strategy for Proofs: Before delving into the solutions, I wanted to give a simple algorithm for how to approach proofs *when you don't know where to start*. If you have an idea of what to do, go for it! However, if you are struggling or a bit lost, this is a simple algorithm I like to follow to get going:

1. ***List*** the assumptions given by the problem. (*Literally* make a bullet list.)
2. State what you ***WTS*** (want to show) and what it ***means***. (Often what you are asked to show should be stated precisely so that your goal is clear.)
3. Start walking through the ***implications*** of the list of assumptions.

Spelling out every detail can take a long time, so I don't necessarily recommend this if you already know where to start. However, when I am lost I find this structured way of approaching problems helps. (Though keep in mind that this is just my personal suggestion; if you are looking for something more polished, my math professor recommended *How to Solve It* by G. Polya. I haven't read it but I hear good things.) In this set of solutions, I solve the last problem in tedious detail following the algorithm above.

Suggested Solutions 1

1. *Show, by induction, the Bernoulli inequaliity:* $x > -1 \implies (1+x)^n \geq 1 + nx \; \forall n \in \mathbb{N}$

   Solution. The base case is $n = 1$. So we have

   $$1 + x \geq 1 + x$$

   which holds with equality, so the base case is true. Now the inductive step:

   $$(1+x)^n \geq 1 + nx$$

   If $x > -1$, then $1 + x > 0$ and $(1+x)^n(1+x) \geq (1+nx)(1+x)$. Now

   $$\begin{aligned}(1+x)^{n+1} &\geq 1 + nx + x + nx^2 \\ &\geq 1 + nx + x \\ &= 1 + (n+1)x\end{aligned}$$

   Hence $(1+x)^{n+1} \geq 1 + (n+1)x$ which is what we wanted to show.

2. *Show, by contradiction, that the set of prime numbers is infinite.*

   Solution. Suppose not, that is, that the finite set $\mathcal{P} = \{p_1, \ldots, p_N\}$ contains all prime numbers. Define

   $$\widetilde{p}_N = 1 + \prod_{i=1}^{N} p_i$$

   Note $\widetilde{p}_N > p_i$ for any i, so $\widetilde{p}_N \notin \mathcal{P}$. Further, since p_i is prime for all i, $\widetilde{p}_N$ cannot be divided by any p_i. Hence $\widetilde{p}_N$ is prime and not in the set of all primes, a contradiction. Now we only need to show that the product of N primes plus 1 is not divisible by any of them. Suppose that it is, then we can write

   $$1 + \prod_{i=1}^{N} p_i = p_j \cdot K$$

   for some $j \leq N$ and some $K \in \mathbb{N}$ (since $p_i > 1$ for all i). However,

   $$p_j \cdot K = 1 + p_j \cdot \prod_{i=1, i \neq j}^{N} p_i$$

   Therefore

   $$1 = p_j \left(K - \prod_{i=1, i \neq j}^{N} p_i \right) = p_j \cdot L$$

   Note $L > 0$ (can you see why? Alternatively, I suppose you can remark $L \leq 0$ means $1 \leq 0$, contradiction); hence p_j divides 1, contradiction. Thus it cannot be the case that $\widetilde{p}_N$ is divisible by any p_j.

3. *Show the supremum of a set of real numbers is unique.*

   Solution. Take a set $S \subseteq \mathbb{R}$. If it is not bounded above, the supremum does not exist.[1] Hence S is

[1] Suppose the supremum $\sup S$ is finite; then since S is not bounded, $\exists \widetilde{s} \in S$ that is greater than $\sup S$, which means $\sup S$ is not

bounded above: Take a supremum of the set and call it s, and by contradiction suppose $\tilde{s} \neq s$ is also a supremum of S. If $s > \tilde{s}$ then s is not a supremum because there is a smaller number that also bounds S; similarly if $s < \tilde{s}$ then $\tilde{s}$ is not a supremum because there is a smaller number that also bounds S. Either way we have a contradiction.

4. *Let A and B be non-empty real-valued sets bounded above. Let $C = \{a + b : a \in A, b \in B\}$. Show* $\sup C = \sup A + \sup B$

   Solution. We have that $\sup C \geq c \ \ \forall c \in C$, hence $\sup C \geq a + b \ \ \forall a \in A, b \in B$. Since $\sup A \geq a \ \ \forall a \in A$ and $\sup B \geq b \ \ \forall b \in B$, $\sup A + \sup B \geq a + b \ \ \forall a \in A, b \in B$.

   Importantly, $\nexists \tilde{c} \geq c \ \ \forall c \in C$ s.t. $\tilde{c} < \sup C$, by definition of the supremum. Similarly for $\sup A$ and $\sup B$. Hence it cannot be that $\sup C > \sup A + \sup B$, or $\sup A + \sup B$ would be the supremum of C instead of $\sup C$, a contradiction.

   Now suppose that $\sup C < \sup A + \sup B$, which implies $\sup C - \sup B < \sup A$. By definition of the sup, this implies $\exists a \in A$ s.t. $\sup C - \sup B < a \leq \sup A$, or $\sup C - a < \sup B$. Again by definition of the sup, $\exists b \in B$ s.t. $\sup C - a < b \leq \sup B$, or $\sup C < a + b$. However, one last time, $a + b \leq \sup C$ by definition of the sup, contradiction.

5. *Given a real sequence (a_j), define*

$$b_m = \sum_{j=1}^{m} a_j \qquad c_m = \sum_{j=1}^{m} |a_j|$$

   Show (b_m) converges if (c_m) converges. Give an example of (a_j) to show the converse may not hold.

   Solution. Suppose $c_m \to c$ and define

$$b_m^+ = \sum_{j:a_j \geq 0}^{\infty} a_j \qquad b_m^- = \sum_{j:a_j < 0}^{\infty} a_j$$

   b_m^+ is increasing since $b_{m+1}^+ = b_m^+$ or $b_m^+ + a_{m+1} \geq b_m^+$. Similarly, we have that b_m^- is decreasing. Since $0 \leq a_j \leq |a_j|$, we further have that

$$0 \leq b_m^+ \leq c_m \leq c \qquad 0 \geq b_m^- \geq -c_m \geq -c$$

   Since c_m is increasing and $c_m \to c$, we know that $c_m \leq c$. Further, we know from class that a bounded increasing sequence converges, and a bounded decreasing sequence also converges. Since $b_m = b_m^+ + b_m^-$ for all m, and $b_m^+ \to b^+, b_m^- \to b^-$ for some b^+, b^-, it must be that $b_m \to b^+ + b^-$. For the converse, take for instance

$$a_j = \{-1, 1, -1/2, 1/2, \ldots, (-1^m)/m\}$$

   The sum converges to 0. If m is odd, then $b_m = -1/m$; if m is even, then $b_m = 0$. However, c_m diverges, since it becomes 2 times the sum of the harmonic series, which diverges.

6. *Show if (x_m) is a bounded and monotonic real sequence then (x_m) converges.*

   Solution. Here I want to try something different: The idea is to give you a detailed outline of how I would work through a proof if I wasn't sure where to start, spelling out my reasoning and each step in overly verbose detail. (It really is way more than you'd write for a normal proof, but it might be useful to read through something like this.)

greater than every element in S, a contradiction.

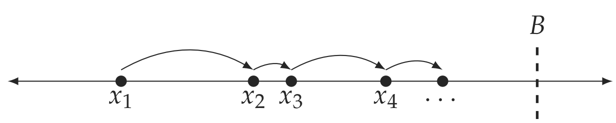
Solution for Problem 6. I will prove the last exercise in two ways, so I will offer different proofs for the case when it is decreasing and the case when it is increasing. This is way more than you would ever need to write down, but hopefully you get something out of it.

1. First, we make a ***list*** of what the problem gives us:
   a) (x_m) is a bounded increasing or decreasing sequence. Let us consider (x_m) increasing first.
   b) (x_m) bounded means $\exists A, B$ s.t. $A \leq x_m \leq B$ for all m.
   c) (x_m) increasing means $m > n \iff x_m \geq x_n$.

   The problem asks us to show two things: If (x_m) is bounded and increasing *or* decreasing then it converges. Often I find it useful to show each part in turn, so I consider the increasing case first.[2]

2. We ***WTS*** that (x_m) converges. That is, $\forall \varepsilon > 0 \ \exists M$ s.t. $m \geq M \implies d(x, x_m) < \varepsilon$ for some x.

3. Let us go through the proof and try to use the ***implications*** of the problem statements as we do it.
   - If (x_m) is bounded, then it is bounded above.
   - If (x_m) is increasing, then it is getting progressively closer to any upper bound.
   - Therefore it seems reasonable to suspect it will converge to one of its upper bounds. It can never get past it (that is what an upper bound is) and it is always getting closer (that is what increasing means):



   - The issue is that it is clearly not getting *arbitrarily* closer to any bound. Any number C bigger than B will be an upper bound as well but it has no chance of being the limit because the distance will always be at least $C - B > 0$.
   - The turning point of the proof is to realize that you want the *smallest* upper bound you can get away with, otherwise known as the sup.

$$\overline{x} \equiv \sup \{x_m\}$$

     Since x_m is a bounded subset of $\mathbb{R}$, we know from class the sup exists.
   - Now leverage the fact $\overline{x}$ is the *smallest* upper bound. In particular, for *any* $\widetilde{x} < \overline{x}$, there is some element of x_m that is strictly greater than $\widetilde{x}$ (if not, then $\widetilde{x}$ is an upper bound of $\overline{x}$ that is smaller than $\overline{x}$, contradiction because $\overline{x}$ is the smallest upper bound).

     Formally, $\forall \widetilde{x} < \overline{x}$ there exists some M s.t.

$$x_M > \widetilde{x}$$

     Let $\varepsilon \equiv \overline{x} - \widetilde{x}$, so $\forall \varepsilon > 0 \ \exists M$ s.t.

$$x_M > \overline{x} - \varepsilon \iff \varepsilon > \overline{x} - x_M = d(\overline{x}, x_M)$$

[2] If you are careful, you can get away with showing just one of the two and claiming WLOG or that the steps for the other are analogous (since the proofs can be basically the same for the two cases).

- This is very close to the definition of convergence! What is missing? Take any $m \geq M$; since x_m is increasing, we know $x_m \geq x_M$. Therefore $\forall \varepsilon > 0$ we have

$$m \geq M \implies x_m \geq x_M \implies \overline{x} - x_m \leq \overline{x} - x_M < \varepsilon \implies d(\overline{x}, x_m) < \varepsilon$$

which means $x_m \to \overline{x}$ by definition.

There is a completely analogous proof for (x_m) decreasing (or it would suffice to show $-x_m \to x \implies x_m$ converges). In general there is no need to do two different proofs when taking a shortcut would suffice; I only offer a different proofs below for illustrative purposes.

4. Let us try a proof by contradiction to exhibit how one can use a theorem instead of the definition of convergence (though we will use other definitions). Let (x_m) be decreasing.

- Suppose (x_m) does not converge. What do we know about sequences that do not converge? A promising avenue is to consider the contrapositive of statements that give us convergent sequences. The one that will help us here is the fact that Cauchy sequences converge in $\mathbb{R}$.

- If the real sequence (x_m) does not converge it is not Cauchy.

- The definition of Cauchy is that $\forall \varepsilon > 0 \;\; \exists M$ s.t. $\forall k, l \geq M$ we have $d(x_k, x_l) \leq \varepsilon$.

- The negation of Cauchy is that $\exists \varepsilon > 0$ s.t. $\forall M \;\; \exists k, l \geq M$ s.t. $d(x_k, x_l) > \varepsilon$.

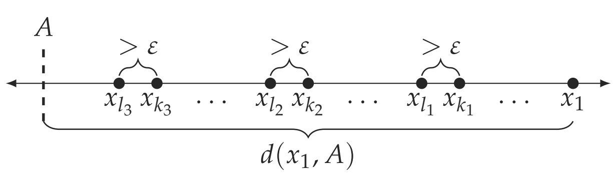
- The sequence is decreasing, so for such an $\varepsilon > 0$ and for $M_1 = 1$ we know $\exists k_1, l_1 \geq M_1$ s.t.

$$d(x_{k_1}, d_{l_1}) > \varepsilon$$

Let us iterate, for $M_n > \max\{k_{n-1}, l_{n-1}\}$ we know $\exists k_n, l_n \geq M_n$ s.t.

$$d(x_{k_n}, x_{l_n}) > \varepsilon$$

- Intuitively, this means that we can always eventually find two elements of the sequence that are ε way from each other. Hence there are infinitely many elements of the sequence with a distance of ε. However, A is a lower bound, and $d(x_1, A)$ is finite because x_1 is a *given number*.



The sequence is decreasing, so there should be at most $(x_1 - A)/\varepsilon$ elements with a distance as big as ε, not infinitely many. How can we get the contradiction formally?

- WLOG suppose $x_{k_n} > x_{l_n}$ (note they cannot be equal at any n or $d(x_{k_n}, x_{l_n})$ would be $0 < \varepsilon$). This is WLOG because at any n we can just denote the smaller element of the pair to be x_{l_n}. Hence

$$d(x_{k_n}, x_{l_n}) = x_{k_n} - x_{l_n} > \varepsilon$$

- Note $k_n, l_n \geq M_n > \max\{k_{n-1}, l_{n-1}\}$, *both* k_n, l_n are greater than *either* k_{n-1}, l_{n-1}. Since (x_m) is decreasing, $k_n > l_{n-1} \implies x_{k_n} \leq x_{l_{n-1}}$. Now pick any n:

$$x_{l_n} - A < x_{k_n} - A - \varepsilon \leq x_{l_{n-1}} - A - \varepsilon < x_{k_{n-1}} - A - 2\varepsilon < \ldots \leq x_{k_1} - A - n\varepsilon$$

We can see that

$$\frac{x_{l_1} - A}{\varepsilon} < n \implies x_{l_1} - A - n\varepsilon < 0 \implies x_{l_n} - A < 0$$

so $x_{l_n} < A$, meaning A is not a lower bound, contradiction.

- Hence (x_m) converges.