

# Lecture II: Sequences, Continuity

## Contents

1	Sequences	1
1.1	Convergence	2
1.2	Sets and Sequences (Visualization)	3
1.3	Properties of Convergent Sequences	4
1.4	Bolzano–Weierstrass	5
1.5	Cauchy Sequences	6
2	Continuous Functions	7
3	Sequential and Open Set Characterizations	9
3.1	Sets and Continuity (Visualization)	9
4	Intermediate Value Theorem (IVT)	14

## 1. Sequences

A sequence is a collection of elements of a set indexed by the natural numbers. We will not be terribly precise with notation,<sup>1</sup> and denote sequences with elements in  $S$  as  $(x_m) \in S$ .

**Definition 1.** Let  $(x_m) \in S$  be a sequence:

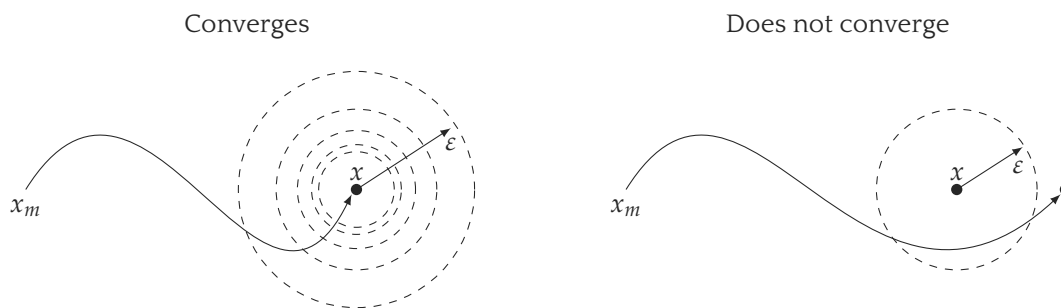
- a)  $(x_m)$  is **increasing** if  $\forall m$  we have  $x_m \leq x_{m+1}$ ; it is *strictly* increasing if  $x_m < x_{m+1}$ .
- b)  $(x_m)$  is **decreasing** if  $\forall m$  we have  $x_m \geq x_{m+1}$ ; it is *strictly* decreasing if  $x_m > x_{m+1}$ .
- c)  $(x_m)$  is (strictly) **monotonic** if it is (strictly) increasing or decreasing.
- d)  $(y_k)$  is a **subsequence** of  $(x_m)$  if  $\exists$  some strictly increasing sequence  $(n_k) \in \mathbb{N}$  s.t.  $y_k = x_{n_k}$ .
- e)  $(x_m)$  is bounded above, below, or bounded if  $\{x_m\}_{m \in \mathbb{N}}$  is bounded above, below, or bounded (resp).
- f)  $(x_m) \rightarrow +\infty$  if  $\forall N > 0 \exists M$  s.t.  $m \geq M \implies x_m \geq N$ .  $(x_m) \rightarrow -\infty$  if  $\forall N < 0 \exists M$  s.t.  $m \geq M \implies x_m \leq N$ . If either property holds, we say the sequence  $(x_m)$  **diverges**.

<sup>1</sup>Formally, a sequence is an element of the infinite Cartesian product of  $\mathbb{R}^N \times \dots \times \mathbb{R}^N$ . While tempting to define a sequence as a countable or finite subset of  $\mathbb{R}^N$ , sets have no notion of order; further, sequences can have repeated elements.

### 1.1. Convergence.

**Definition 2.** A sequence  $x_m$  **converges** to  $x$  if  $\forall \varepsilon > 0 \exists M \in \mathbb{N}$  s.t.  $d(x_m, x) < \varepsilon$  whenever  $m \geq M$ . We denote this as  $x_m \rightarrow x$  or  $\lim_{m \rightarrow \infty} x_m = x$ .

We can visualize this idea in the figure below:  $x_m$  is eventually contained within any  $\varepsilon$ -ball if  $x$ .<sup>2</sup>



**Claim 1.** A sequence converges to at most one limit.

*Proof.* This is a consequence of the fact  $d(x, y) \iff x = y$ . Suppose  $x_m \rightarrow x$  and  $x_m \rightarrow y$ . If  $x \neq y$ , then  $d(x, y) > 0$ . Take  $\varepsilon = d(x, y)/4$ ; we know by the definition of convergence that there is some  $M_x, M_y$  s.t. for  $m \geq M = \max\{M_x, M_y\}$  we get

$$d(x_m, x) < \varepsilon/2 \quad \text{and} \quad d(x_m, y) < \varepsilon/2$$

Now the triangle inequality gives

$$d(x, y) \leq d(x_m, x) + d(x_m, y) < \varepsilon/4 + \varepsilon/4 = \varepsilon/2 < d(x, y)$$

contradiction. Hence  $d(x, y) = 0$ , or  $x = y$ . □

**Theorem 1.**  $S$  is closed  $\iff$  for any  $(x_m) \in S$  s.t.  $x_m \rightarrow x$  for some  $x \in \mathbb{R}^N$ , we have that  $x \in S$ .

This is an equivalent definition of closedness: A set that “contains all its limits.”

*Proof.* See [Subsection 1.2](#). □

**Theorem 2.** Take any set  $S \subseteq \mathbb{R}^N$ ; the following are equivalent:

- a)  $x \in cl(S)$ .
- b)  $\forall \varepsilon > 0, B_\varepsilon(x) \cap S \neq \emptyset$ .
- c)  $\exists (x_m) \in S$  s.t.  $x_m \rightarrow x$ .

*Proof.* We will cycle through the proofs. First we show  $b \implies c$ : Take any  $x$  s.t.  $b$  holds. Let  $1/m = \varepsilon_m$  for each  $m = 1, 2, 3, \dots$ ; we know that  $\exists x_m$  s.t.  $x_m \in B_{\varepsilon_m}(x)$ . Now take an arbitrary  $\varepsilon > 0$ ; there is some  $M \in \mathbb{N}$  s.t.  $1/M < \varepsilon$ , so  $m \geq M$  gives  $x_m \in B_\varepsilon(x)$ . Thus  $x_m \rightarrow x$  with  $(x_m) \in S$ .

<sup>2</sup>Recall  $\varepsilon$ -ball is the equivalent of a neighborhood in Euclidean space, even though here in two dimensions it's technically a  $\varepsilon$ -circle.

Now we show that  $c \implies a$ . Note that for any closed  $T$  s.t.  $S \subseteq T$ , we have that  $(x_m) \in S \implies (x_m) \in T$ . Since  $T$  is closed and  $x_m \rightarrow x$ ,  $x \in T$  by [Theorem 1](#). Since  $S \subseteq \text{cl}(S)$  and  $\text{cl}(S)$  is closed, both by definition, this proves that  $c \implies a$ .

To finish, we show that  $a \implies b$ . Suppose that  $x \in \text{cl}(S)$  but for some  $\varepsilon > 0$ ,  $B_\varepsilon(x) \cap S = \emptyset$ . This means that  $B_\varepsilon(x) \subset \mathbb{R}^N \setminus S$ . Since  $B_\varepsilon(x)$  is open,  $\text{cl}(S) \setminus B_\varepsilon(x)$  is closed. Further, since  $x \notin S$  (because  $x \in B_\varepsilon(x)$  and  $B_\varepsilon(x) \cap S = \emptyset$ ), we have that  $S \subseteq \text{cl}(S) \setminus B_\varepsilon(x)$ . Since  $x \in \text{cl}(S)$ , this implies  $\text{cl}(S) \subset \text{cl}(S) \setminus B_\varepsilon(x)$ . At the same time,  $\text{cl}(S)$  is the intersection of every closed set containing  $S$ , so  $\text{cl}(S) \setminus B_\varepsilon(x) \subseteq \text{cl}(S)$ . Thus  $\text{cl}(S) \subset \text{cl}(S)$ , a contradiction.  $\square$

**Definition 3.** Let  $(x_m)$  be any sequence. We define

$$\limsup = \lim_{m \rightarrow \infty} \left( \sup_{k \geq m} x_k \right)$$

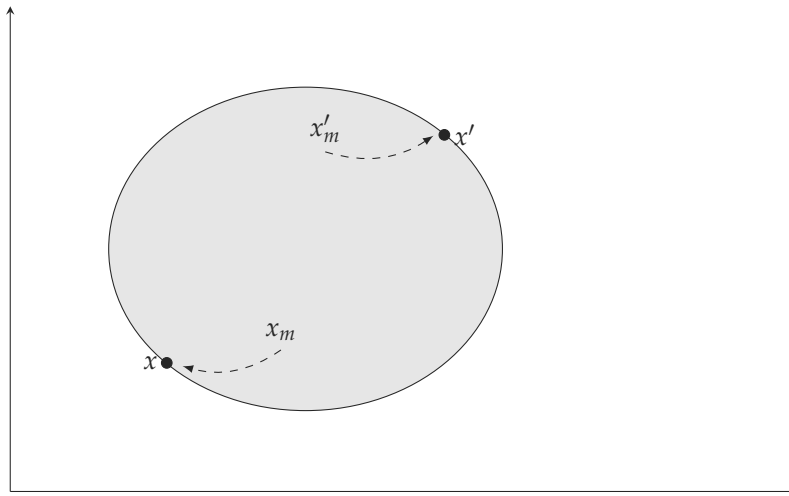
and

$$\liminf = \lim_{m \rightarrow \infty} \left( \inf_{k \geq m} x_k \right)$$

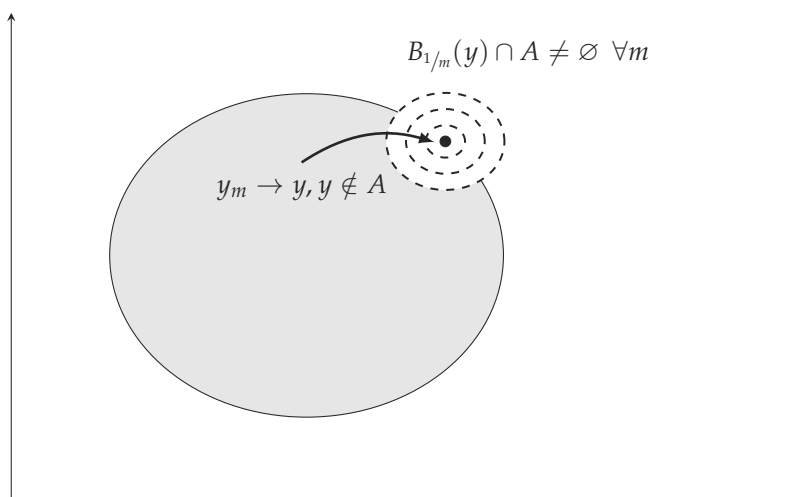
The  $\limsup$  and  $\liminf$  do not require convergence. Take, for instance,  $(x_m) = 0, 1, 0, 1, \dots$ . Clearly  $\limsup = 1$  and  $\liminf = 0$  but the limit does not exist.

**1.2. Sets and Sequences (Visualization).** The mathy sequel to Dungeons & Dragons. In this section we will try to visualize the proof of [Theorem 1](#). This should give some intuition for why convergence is related to sets being closed beyond the formality of the theorem.

- If a sequence  $x_m$  converges to  $x$ , then it becomes arbitrarily close to a point. If for every  $(x_m) \in A$  s.t.  $x_m \rightarrow x$  we also have  $x \in A$ , that means that no sequence can ever “escape” outside of  $A$ :

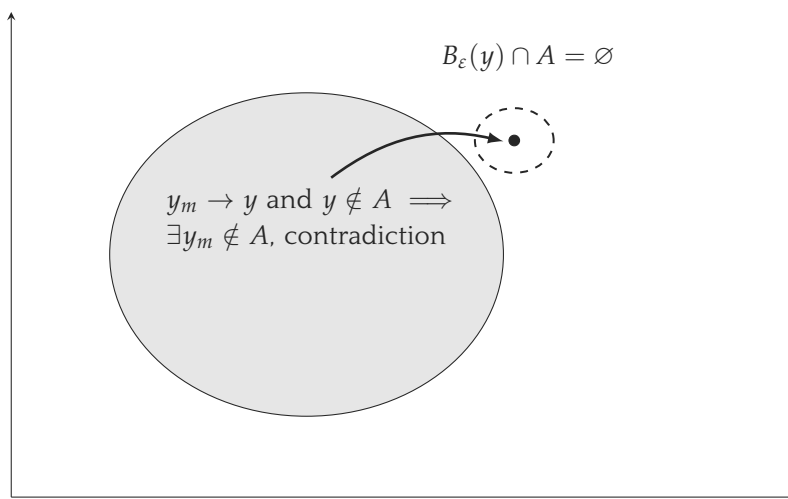


By contrapositive, if  $A$  is not closed, its complement is not open, so  $\exists y \in \mathbb{R}^N \setminus A$  that cannot be enclosed in an  $\varepsilon$ -ball inside of  $\mathbb{R}^N \setminus A$ . In other words, there is some  $y$  s.t. for each  $\varepsilon_m = 1/m$  we can find a corresponding  $y_m \in A$ . The resulting  $(y_m) \in A$  converges to  $y \notin A$ .



In other words, if  $A$  is not closed, we can find a sequence that “escapes”  $A$ , which by contrapositive proves that if every sequence in  $A$  that converges does so to a point in  $A$ , the set is closed.

- On the other hand, if  $A$  is closed and we have a sequence  $(y_m) \in A$  s.t.  $y_m \rightarrow y$  with  $y \notin A$ , then we have a sequence that “escaped”  $A$ . However,  $A$  closed implies  $X \setminus A$  is open, and  $\exists B_\epsilon(y) \subseteq \mathbb{R}^N \setminus A$ . Since  $y_m$  will get arbitrarily close to  $y$ ,  $\exists y_m \in B_\epsilon(y) \subseteq X \setminus A$ . Since  $y_m \in A$  by premise, this is a contradiction.



I think in general it's very useful to visualize proofs in  $\mathbb{R}^2$  (making drawings, as above):

- $\mathbb{R}$  is not enough: A ton of things will hold in one dimension that won't in general, and one-dimensional intuition can end up being misleading.
- $\mathbb{R}^3$  can be too much: I cannot draw 3D very easily and it is harder to visualize clearly (certainly  $\mathbb{R}^N$  would be too many dimensions).
- $\mathbb{R}^2$  is a nice trade-off between rigor and intuition. (I know we discuss properties in more general terms than in  $\mathbb{R}^2$ , but for intuition I think it's a great benchmark.)

### 1.3. Properties of Convergent Sequences.

**Theorem 3.** Take any sequence  $(x_m) \in \mathbb{R}^N$ :

- a)  $x_m \rightarrow x \implies x_{m_k} \rightarrow x$  for all subsequences  $(x_{m_k})$  of  $(x_m)$ .
- b)  $x_m \rightarrow x \implies (x_m)$  is bounded. (Is the converse true? Can you prove your answer?<sup>3</sup>)
- c) If  $x_m \leq y_m \leq z_m$  and  $x_m, z_m \rightarrow x$  then  $y_m \rightarrow x$ .
- d) If  $x_m \rightarrow 0$  and  $(y_m)$  is bounded then  $x_m \cdot y_m \rightarrow 0$ .

Let  $x_m \rightarrow x$  and  $y_m \rightarrow y$ :

- e)  $c \cdot x_m \rightarrow c \cdot x$  for any  $c \in \mathbb{R}$ .
- f)  $x_m \pm y_m \rightarrow x \pm y$ .
- g)  $x_m \cdot y_m \rightarrow x \cdot y$ .
- h)  $x_m/y_m \rightarrow x/y$  if  $y \neq 0$ .

#### 1.4. Bolzano-Weierstrass.

**Theorem 4.** Let  $(x_m) \in \mathbb{R}$ . If  $(x_m)$  is bounded and monotonic then  $(x_m)$  converges.

**Theorem 5** (Nested Intervals Theorem). Let  $I_m = [a_m, b_m]$  s.t.  $I_{m+1} \subset I_m$ .

- a)  $\bigcap_{m \in \mathbb{N}} I_m \neq \emptyset$
- b) If  $b_m - a_m \rightarrow 0$  then  $\bigcap_{m \in \mathbb{N}} I_m$  is a singleton.

*Proof.* If  $I_{m+1} \subseteq I_m$ , then  $I_m \subseteq I_1$  for all  $m$ . Hence

$$a_1 \leq a_m \leq a_{m+1} \leq b_{m+1} \leq b_m \leq b_1$$

for all  $m$ . In other words,  $a_m$  and  $b_m$  are bounded and monotonic, so  $a_m \rightarrow a$  and  $b_m \rightarrow b$  for some  $a, b$  by [Theorem 4](#). Further, since  $a_m \leq b_m$  for all  $m$ , we have that  $a \leq b$  (do you see why?<sup>4</sup>) and  $I_m \rightarrow [a, b] \neq \emptyset$ . If  $b_m - a_m \rightarrow 0$ , that means  $b - a = 0$  so the interval  $[a, b]$  is just the singleton  $a = b$ .  $\square$

**Theorem 6** (Bolzano-Weierstrass). Every bounded sequence in  $\mathbb{R}$  admits a convergent sub-sequence.

*Proof.* This follows from the [Nested Intervals Theorem](#) above—the trick is to construct the nested intervals, which we can do for bounded sequences. If  $(x_m)$  is bounded, first define

$$I_1 = [L, U] = [a_1, b_1]$$

<sup>3</sup>A counterexample is sufficient:  $x_m = (-1)^m$  is bounded above by 1 and below by  $-1$  but does not converge.

<sup>4</sup>If  $a > b$  then we will have that  $a_m > b_m$  for some  $m$ . That is, pick  $0 < \varepsilon < a - b$ ; we can then find  $M_a, M_b$  s.t.

$$a - \varepsilon/2 < a_m < a + \varepsilon/2 \quad b - \varepsilon/2 < b_m < a + \varepsilon/2$$

whenever  $m > \max\{M_a, M_b\}$ . However,  $\varepsilon < a - b$  gives

$$b_m < b + \varepsilon/2 < a - \varepsilon/2 < a_m$$

Since  $b_m \geq a_m$  for all  $m$  we have a contradiction.

the interval with endpoints equal to the lower and upper bounds of  $x_m$ . Let  $x_{m_1}$  the first element of the sub-sequence be any term of  $x_m \in I_1$ . Now take

$$I_1^- = \left[ a_1, \frac{a_1 + b_1}{2} \right] \quad I_1^+ = \left[ \frac{a_1 + b_1}{2}, b_1 \right]$$

and let  $I_2 = I_1^-$  if  $\{x_m : m \in \mathbb{N}\} \cap I_1^+$  is non-finite and  $I_2 = I_1^+$  otherwise. Let  $a_2, b_2$  be the endpoints of  $I_2$  and  $x_{m_2}$  be any term of  $x_m \in I_2$ . We iterate this process: In general

$$I_k^- = \left[ a_k, \frac{a_k + b_k}{2} \right] \quad I_k^+ = \left[ \frac{a_k + b_k}{2}, b_k \right]$$

and  $I_{k+1} = I_k^-$  if  $\{x_m : m \in \mathbb{N}\} \cap I_k^+$  is non-finite and  $I_{k+1} = I_k^+$  otherwise, with  $x_{m_k}$  any element of  $x_m \in I_k$ . (What if both halves have a finite intersection?<sup>5</sup>) We have that

$$a_1 \leq a_{k-1} \leq a_k \leq x_{m_k} \leq b_k \leq b_{k-1} \leq b_1$$

Furthermore,  $b_n - a_n = \frac{b_1 - a_1}{2^{n-1}} \rightarrow 0$ . Hence we can apply the [Nested Intervals Theorem](#):  $a_n \rightarrow a$  and  $b_n \rightarrow b$  with  $a = b$  implies  $x_{m_k} \rightarrow a = b$ .  $\square$

### 1.5. Cauchy Sequences.

**Definition 4.** A sequence is **Cauchy** if  $\forall \varepsilon > 0 \exists M$  s.t.

$$m, n > M \implies d(x_m, x_n) < \varepsilon$$

**Theorem 7.** If  $(x_m)$  converges, then it is Cauchy.

*Proof.* Suppose  $x_m \rightarrow x$ ; by the triangle inequality

$$d(x_m, x_n) \leq d(x_m, x) + d(x_n, x)$$

Now take any  $\varepsilon > 0$ ; for  $\varepsilon/2$  we have that for some  $M, m, n > M$  gives

$$d(x_m, x) < \frac{\varepsilon}{2} \quad d(x_n, x) < \frac{\varepsilon}{2}$$

Hence

$$d(x_m, x_n) \leq d(x_m, x) + d(x_n, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which shows  $(x_m)$  is Cauchy.  $\square$

While the converse of the theorem above is also true in  $\mathbb{R}^N$ , during your math course you will probably encounter the fact that in general metric spaces, Cauchy sequences needn't converge. (The reason is, again, this property of Euclidean space called “completeness.”)

**Theorem 8.** If  $(x_m)$  is a Cauchy sequence in  $\mathbb{R}$  then  $(x_m)$  converges.

<sup>5</sup>Note that for  $k > 1$ , it must always be that either  $I_k^+$  or  $I_k^-$  have a non-finite intersection with  $\{x_m : m \in \mathbb{N}\}$ , since we chose  $I_k$  to have a non-finite intersection. The only way both halves will have a finite intersection is if  $I_1$  is finite to begin with. However, this means that some  $M, x_n = x_m$  whenever  $n, m > M$ , which means we have a convergent sub-sequence  $x_{m_k} = x_{M+1}$  with  $m_k = M + k$ .

*Proof.* I will show this in  $\mathbb{R}$ : Cauchy sequences are bounded (why?), which means that there exist some sub-sequence  $x_{m_k}$  that converges to some  $x$ . We show that this is also the limit for the sequence  $x_m$ . By the triangle inequality,

$$d(x_m, x) \leq d(x_m, x_{m_k}) + d(x_{m_k}, x)$$

Take any  $\varepsilon > 0$ , then for  $\varepsilon/2$  we can find  $K$  s.t.  $k > K$  gives

$$d(x_{m_k}, x) < \varepsilon/2$$

because  $x_{m_k} \rightarrow x$ , and  $M$  s.t.  $n, m_k > M$  gives

$$d(x_m, x_{m_k}) < \varepsilon/2$$

because  $x_m$  is Cauchy. Take  $k > K$  s.t.  $m_k \geq M$ . Then for  $n > M$  we have

$$d(x_m, x) \leq d(x_m, x_{m_k}) + d(x_{m_k}, x) < \varepsilon/2 + \varepsilon/2 < \varepsilon$$

which is what we wanted to show. This proof should generalize to  $\mathbb{R}^N$  if you argue that along each coordinate,  $(x_m)$  is bounded and then use that to find a candidate limit  $x$ . Then the identical argument goes through (note it used generic properties of the distance rather than anything specific to  $\mathbb{R}$ ).  $\square$

**Remark 1.** The sticking point about “completeness” being required has to do with the fact the candidate limit  $x$  needn’t be in the space (e.g. take some sequence in  $\mathbb{Q}$  that “converges” to  $\pi \notin \mathbb{Q}$ ).  $\square$

## 2. Continuous Functions

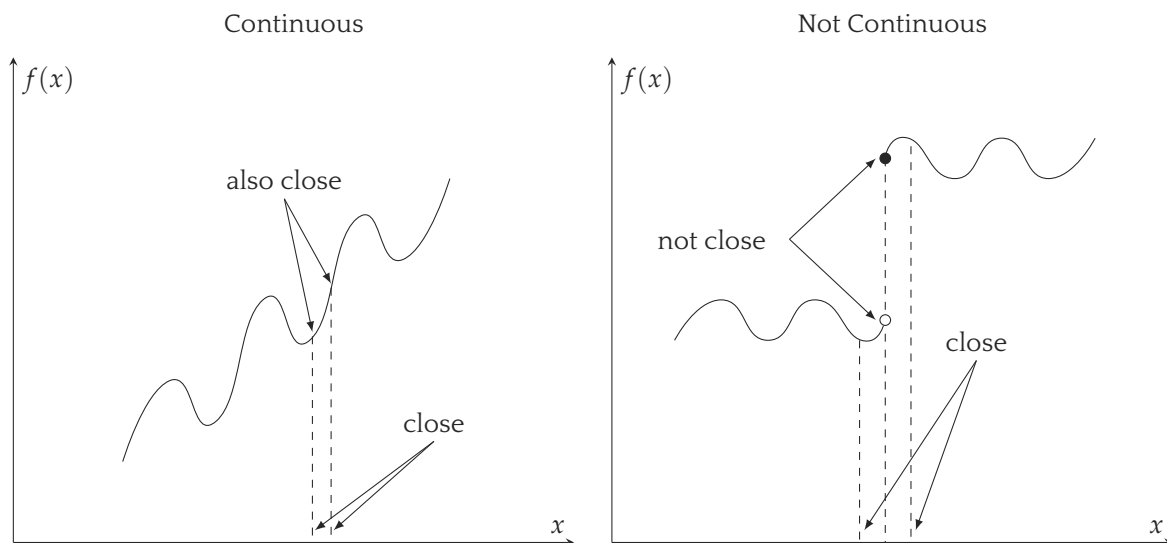
The intuition for continuity is that “you can draw the function without picking up your pencil.” (What about asymptotes like  $1/x$  at 0?<sup>6</sup>)

More precisely, the idea is that if two points in the domain are close, then the corresponding points in the co-domain must also be close. Put another way, a small neighborhood in the domain maps to a small neighborhood in the co-domain. (Note that the converse is not true! Take  $f(x) = x^2$ ;  $f(-x) = f(x) = x^2$ , so the points are close in the image—they are identical—but as  $x$  gets large,  $x, -x$  get farther apart.)

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<sup>6</sup>The intuition should still hold because the function is not defined at 0; with infinitely long paper you needn’t pick up your pencil.

Figure 1: Intuition for Continuity



**Definition 5.** A function  $f : X \rightarrow Y$  is **continuous** at  $x \in X$  if for every  $\varepsilon > 0$  there exist a  $\delta > 0$  s.t.

$$d(z, x) < \delta \implies d(f(z), f(x)) < \varepsilon$$

Put another way,  $z \in B_\delta(x) \implies f(z) \in B_\varepsilon(f(x))$ , or

$$f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$$

If  $f$  is continuous at every  $x \in X$  we say it is continuous.

**Proposition 1.** Let  $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}$ . If  $\varphi$  is continuous then the sets  $\{x \in \mathbb{R}^N : \varphi(x) \geq \alpha\}$  and  $\{x \in \mathbb{R}^N : \varphi(x) \leq \alpha\}$  are closed for all  $\alpha \in \mathbb{R}$ .

*Proof.* Let  $A = \{x \in \mathbb{R}^N : \varphi(x) \geq \alpha\}$  and  $B = \mathbb{R}^N \setminus A$ . If  $B$  is open then  $A$  is closed. Note

$$B = \{x \in \mathbb{R}^N : \varphi(x) < \alpha\}$$

Pick any  $x \in B$  and let  $0 < \varepsilon < \alpha - \varphi(x)$ . Then since  $\varphi$  is continuous we know there is some  $\delta > 0$  s.t.

$$z \in B_\delta(x) \implies \varphi(x) - \varepsilon < \varphi(z) < \varphi(x) + \varepsilon < \alpha \implies z \in B$$

therefore  $B$  is open (for any point  $\exists \delta$ -ball that is entirely in  $B$ ). Thus  $\mathbb{R}^N \setminus B = A$  is closed. The proof for the lower sets is analogous.  $\square$

**Theorem 9.** Let  $f : X \rightarrow Y$  and  $g : f(X) \subseteq Y \rightarrow Z$  with  $X, Y, Z \subseteq \mathbb{R}^N$ . If  $f, g$  are continuous then  $(g \circ f) : X \rightarrow Z$  is continuous.

**Theorem 10.** Let  $f : X \rightarrow \mathbb{R}$  and  $g : X \rightarrow \mathbb{R}$  be continuous functions. Then

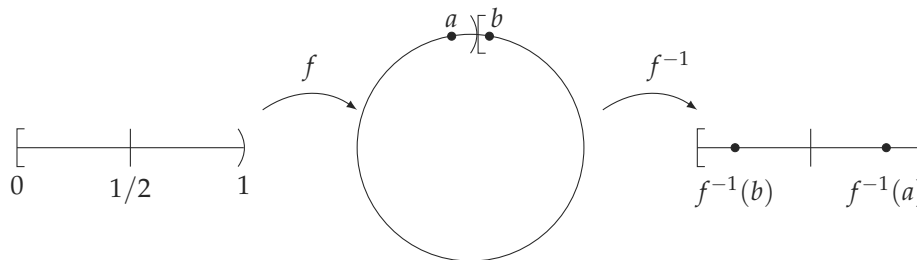
1.  $h(x) = f(x) \pm g(x)$  are continuous functions.
2.  $h(x) = f(x) \cdot g(x)$  is continuous.
3.  $h(x) = f(x)/g(x)$  is continuous whenever  $g(x) \neq 0$ .



### 3. Sequential and Open Set Characterizations

**Remark 2.** Continuous functions don't map open sets to open sets! Consider  $f(x) = x^2$ . The image of  $(-2, 2)$  is  $[0, 2)$ , which is not open. Further, even if all maps of closed sets are closed, the function might not be continuous. For example, Consider a function that is 0 if  $x < 0$  and 1 if  $x \geq 0$ . This has a jump at 0, but  $f([a, b])$  is either  $\{0\}$ ,  $\{1\}$ , or  $\{0, 1\}$ , which are closed.

Last, if a function is continuous the inverse image need not be. Consider a function  $f : [0, 1] \rightarrow \mathbb{R}^2$  that maps the line into a circle:



We can see that  $a, b$  are closed in the image, but not in the inverse image. □

The remarks above all get to the same idea: Continuity states that points in the image are close if the points in the domain are close, not the converse. Hence we have the following characterization in terms of the *inverse image* and *converging sequences*.

**Theorem 11.** The following are equivalent for any  $f : \mathbb{R}^N \rightarrow \mathbb{R}^M$ :

1.  $f$  is continuous.
2. If  $O \subseteq \mathbb{R}^M$  is open,  $f^{-1}(O)$  is open.
3. If  $S \subseteq \mathbb{R}^M$  is closed,  $f^{-1}(S)$  is closed.
4. For every  $(x_m) \in \mathbb{R}^N$  s.t.  $x_m \rightarrow x$  for some  $x \in \mathbb{R}^N$ ,  $f(x_m) \rightarrow f(x)$ .

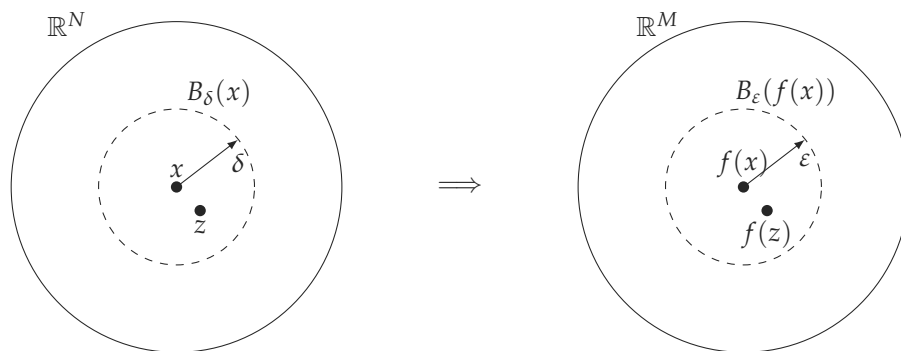
*Proof.* See [Subsection 3.1](#). □

**3.1. Sets and Continuity (Visualization).** We show [Theorem 11](#) using, again, drawings in  $\mathbb{R}^2$ .

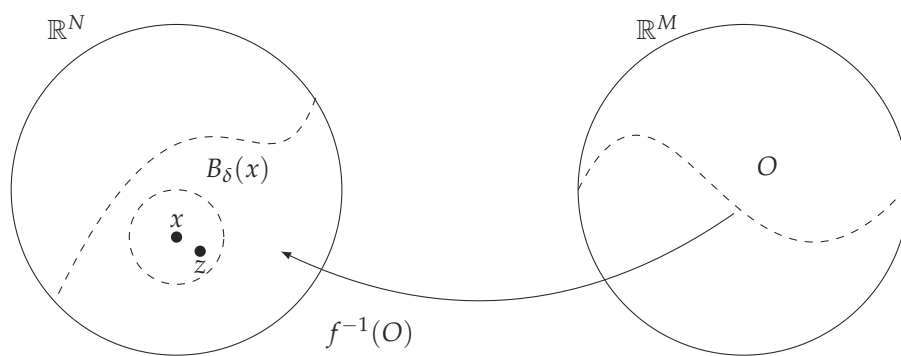
*Proof.* We cycle through the statements. If we show  $a \implies b \implies c \implies d \implies a$  then we have shown they are equivalent.

a.  $a \implies b$ . First, it is a good idea to write down what you have and what you want to show:

- Continuity means that  $\forall \epsilon > 0 \exists \delta > 0$  s.t.  $z \in B_\delta(x) \implies f(z) \in B_\epsilon(f(x))$ .



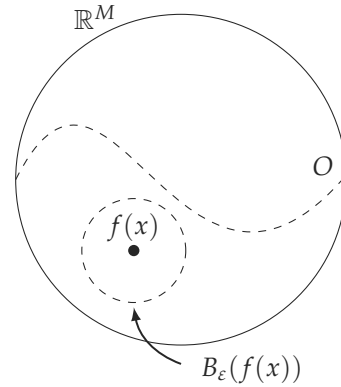
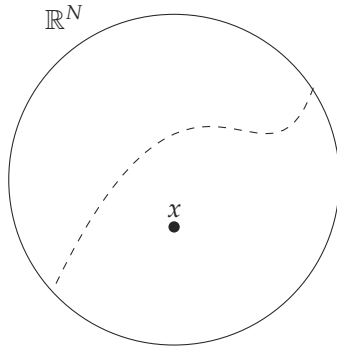
We want to show that if  $O \subseteq \mathbb{R}^M$  is open, then  $f^{-1}(O)$  is open. That is,  $\forall x \in f^{-1}(O) \exists \delta > 0$  s.t.  $z \in B_\delta(x) \implies z \in f^{-1}(O)$ . You will note this is a very similar statement!



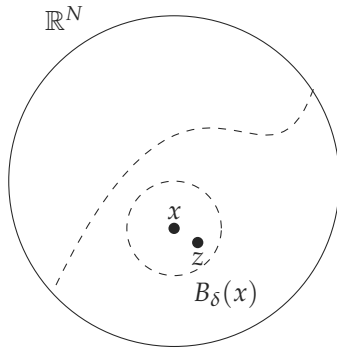
- It's basically the same picture: All we are missing is  $B_\epsilon(f(x))$  and it looks like we're done. How do we get it? We use the fact that  $O$  is open in  $\mathbb{R}^M$ .

Pick any  $x \in f^{-1}(O)$

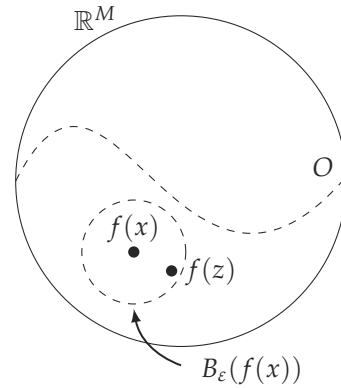
$\exists \varepsilon > 0$  s.t.  $B_\varepsilon(f(x)) \subseteq O$



$\exists \delta$  s.t.  $z \in B_\delta(x) \implies f(z) \in B_\varepsilon(f(x))$   
(by the continuity assumption!)



$\implies$

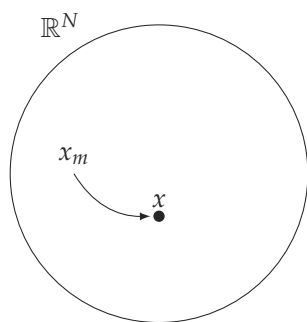


Note  $f(z) \in B_\varepsilon(f(x)) \subseteq O$ , so  $z \in f^{-1}(O)$ . This statement is the heart of the proof! It is not obvious that  $B_\delta(x)$  will be contained in  $f^{-1}(O)$ , so we need the link with  $B_\varepsilon(f(x))$  we drew above. Only then can we say that for arbitrary  $x$  we found  $\delta > 0$  s.t.  $z \in B_\delta(x) \implies z \in f^{-1}(O)$ ; by definition that means the set is open.

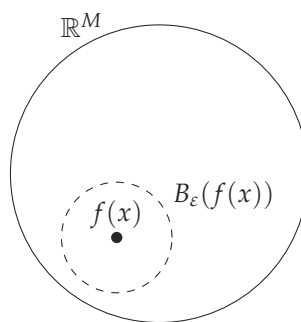
- b. We show  $c \iff b$ . First, consider any closed set  $S \in \mathbb{R}^M$ , so  $\mathbb{R}^M \setminus S$  is open; by premise,  $f^{-1}(\mathbb{R}^M \setminus S)$  is also open, which means  $\mathbb{R}^N \setminus f^{-1}(\mathbb{R}^M \setminus S) = f^{-1}(\mathbb{R}^N \setminus (\mathbb{R}^M \setminus S)) = f^{-1}(S)$  is closed. (The only sticking point here would be to show that in general  $f^{-1}(\mathbb{R}^M \setminus O) = \mathbb{R}^N \setminus f^{-1}(O)$ , which I trust you can do.<sup>7</sup>) Now consider open set  $O \in \mathbb{R}^M$ , so  $\mathbb{R}^M \setminus O$  is open; by premise,  $f^{-1}(\mathbb{R}^M \setminus O)$  is also closed, which means  $\mathbb{R}^N \setminus f^{-1}(\mathbb{R}^M \setminus O) = f^{-1}(O)$  is open. You will notice this is an entirely analogous argument.
- c. For this one it is easier to show that  $b \implies d$  (noting we already argued  $c \implies b$ ).
- $\forall O \subseteq \mathbb{R}^M$ , if  $O$  open then  $f^{-1}(O)$  open. This means that  $\forall x \in f^{-1}(O) \exists \delta$  s.t.  $B_\delta(x) \subseteq f^{-1}(O)$ .
  - We WTS  $x_m \rightarrow x \implies f(x_m) \rightarrow f(x)$ ; i.e.  $\forall \varepsilon > 0 \exists M$  s.t.  $m \geq M \implies f(x_m) \in B_\varepsilon(f(x))$ .

<sup>7</sup>The way to prove two sets are equal is to show either set contains the other. Take any  $x \in f^{-1}(\mathbb{R}^M \setminus O) \subseteq \mathbb{R}^N$ , so  $f(x) \in \mathbb{R}^M \setminus O$ . If  $x \in f^{-1}(O)$  then  $f(x) \in O$ , contradiction. Hence  $x \in \mathbb{R}^N$  and  $x \notin O$ , so  $x \in \mathbb{R}^N \setminus f^{-1}(O)$ . Pick  $z \in \mathbb{R}^N \setminus f^{-1}(O)$ . If  $f(z) \in O$  then  $z \in f^{-1}(O)$ , contradiction. Hence  $f(z) \in \mathbb{R}^M \setminus O$ , which means  $z \in f^{-1}(\mathbb{R}^M \setminus O)$ .

Start with any sequence  $x_m \rightarrow x$

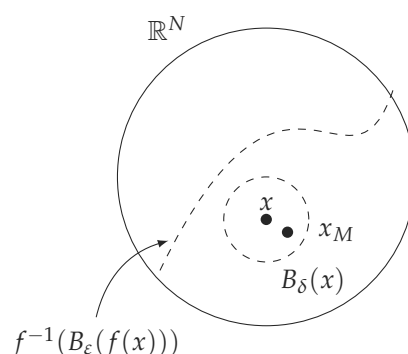
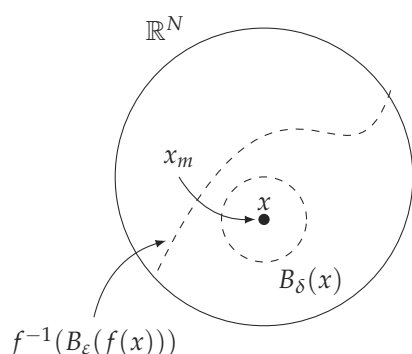


Pick any  $\varepsilon > 0$  and note  $B_\varepsilon(f(x))$  is open.

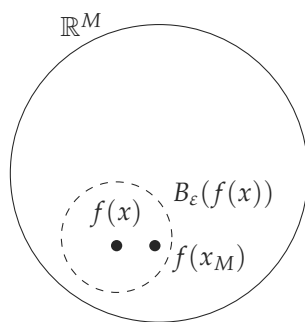


Hence  $f^{-1}(B_\varepsilon(f(x)))$  is open, and  
 $\exists \delta > 0$  s.t.  $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$

$x_m \rightarrow x$  so  $\exists M$  s.t.  
 $m \geq M \implies x_m \in B_\delta(x)$



$$x_m \in B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x))) \implies f(x_m) \in B_\varepsilon(f(x))$$



Hence for any  $x_m \rightarrow x$ , for any  $\varepsilon > 0$  we found  $M$  s.t.

$$m \geq M \implies x_m \in f^{-1}(B_\varepsilon(f(x))) \implies f(x_m) \in B_\varepsilon(f(x))$$

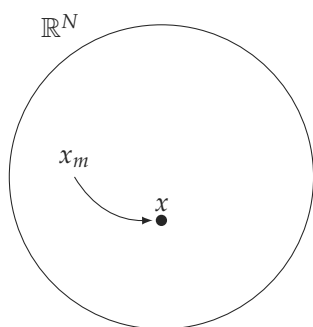
which by definition means  $f(x_m) \rightarrow f(x)$ . The tricky step here was that  $f^{-1}(B_\varepsilon(f(x)))$  does not need to be a nice set. We need the premise that the inverse image of open sets is open so that we can fit a neighborhood inside of it, and *then* use the fact  $x_m \rightarrow x$ .

d. Finally, we show that  $d \implies a$ . We do this by contradiction.

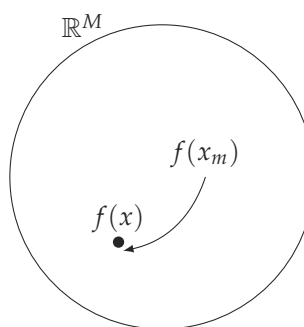
- It is not clear why contradiction is the way to go; it boils down to the fact I think it's easier, but I don't think that's obvious. In general, if unsure how to start a proof, one strategy is to try to make progress with a direct proof, and if you get stuck, switch to contradiction or contrapositive to see if it helps.

- First, we have  $x_m \rightarrow x \implies f(x_m) \rightarrow f(x)$ .

If we had a sequence  $x_m \rightarrow x$



Then we'd know  $f(x_m) \rightarrow f(x)$ .



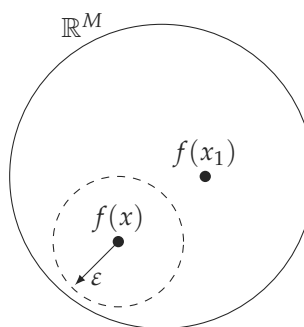
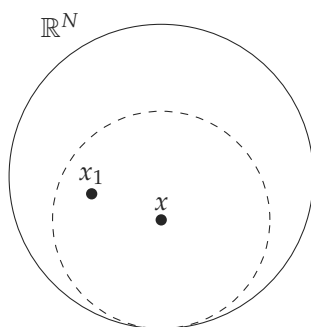
- We want to show that  $\forall x \in \mathbb{R}^N \quad \forall \varepsilon > 0 \quad \exists \delta > 0 \text{ s.t. } z \in B_\delta(x) \implies f(z) \in B_\varepsilon(f(x))$ .
- A great starting point is to construct a sequence in  $\mathbb{R}^N$  that converges to  $x$ , because the premise here is a statement about sequences. I don't see an obvious way to do this directly, but if we think about doing contradiction, we can negate the previous bullet point:

$$\exists x \in \mathbb{R}^N \quad \exists \varepsilon > 0 \text{ s.t. } \forall \delta > 0 \quad \exists z \in B_\delta(x) \text{ and } f(z) \notin B_\varepsilon(f(x))$$

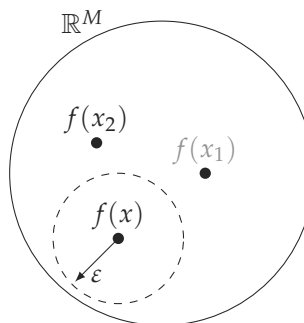
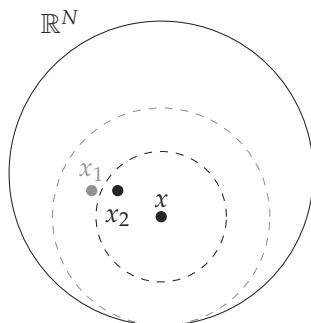
Note that  $x$  and  $\varepsilon$  here are fixed, and that we don't get to choose  $z$ —all we know is one such a  $z$  exists. However,  $\delta$  is a free parameter here, because this must be true for any  $\delta$ .

- If we pick  $\delta = 1/m$  then we can construct a sequence  $x_m \rightarrow x$  s.t.  $f(x_m) \notin B_\varepsilon(f(x))$ :

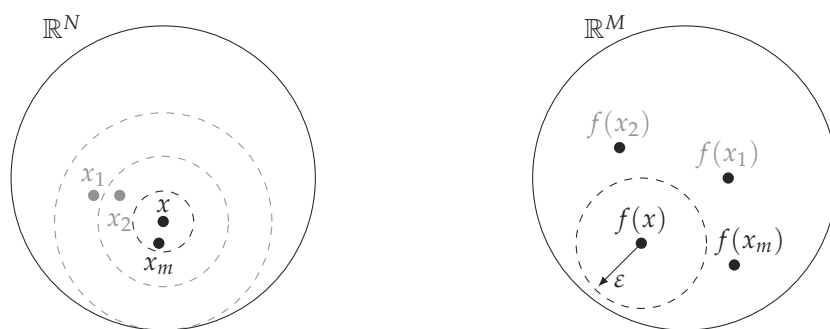
For  $m = 1 \quad \exists x_1 \in B_1(x)$  s.t.  $f(x_1) \notin B_\varepsilon(f(x))$



For  $m = 2 \quad \exists x_2 \in B_{1/2}(x)$  s.t.  $f(x_2) \notin B_\varepsilon(f(x))$



$$\forall m \exists x_m \in B_{1/m}(x) \text{ s.t. } f(x_m) \notin B_\varepsilon(f(x))$$



We can see as  $x_m$  becomes increasingly closer to  $x$ ,  $f(x_m)$  is always at least  $\varepsilon$  away from  $f(x)$ . In other words, we have constructed a sequence  $x_m \rightarrow x$  were  $f(x_m) \not\rightarrow f(x)$ , contradiction.  $\square$

## 4. Intermediate Value Theorem (IVT)

**Theorem 12** (Intermediate Value Theorem (IVT)). *If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous in  $[a, b]$  then*

$$\forall L : f(a) < L < f(b) \exists c \in [a, b] \text{ s.t. } f(c) = L$$

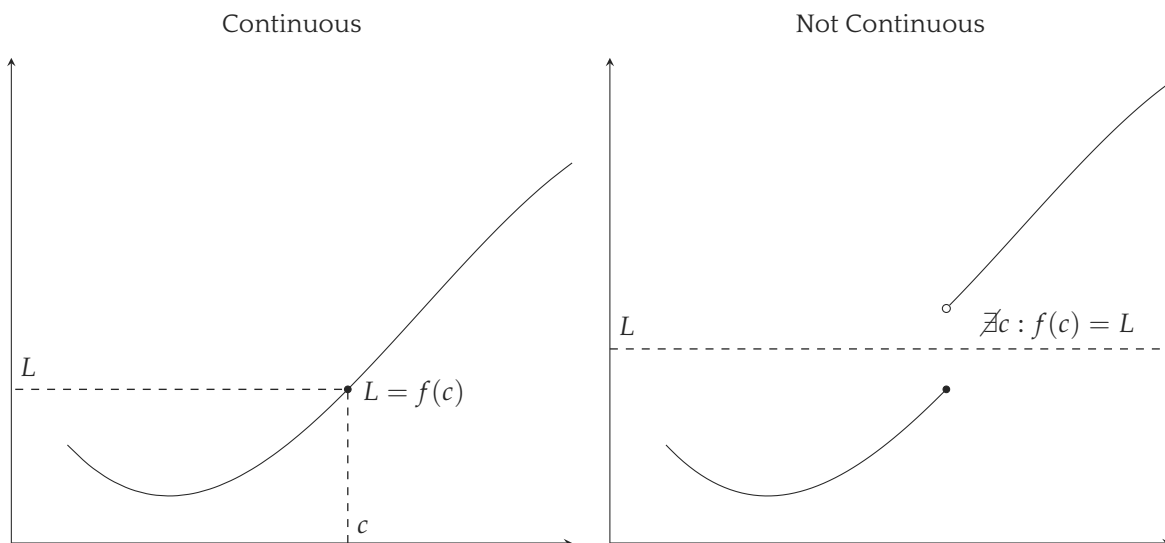


Figure 2: Intermediate Value Theorem (IVT)

*Proof.* Take  $A = \{x \in [a, b] : f(x) < L\}$ .  $A$  is bounded so the sup exists; let  $c = \sup A$ . For  $\varepsilon_m = 1/m$  take  $x_m \in (c - \varepsilon_m, c] \cap A$ , so  $x_m \rightarrow c$  and  $f(x_m) \rightarrow f(c)$  (by continuity). Since  $x_m \in A$ ,  $f(x_m) < L$  and so  $f(c) \leq L$ . If  $f(c) = L$  we are done; if  $f(c) < L$  then, by continuity, for  $\varepsilon : f(c) - \varepsilon < f(c) + \varepsilon < L \exists \delta$  s.t.

$$x : c - \delta < x < c + \delta \implies |f(x) - f(c)| < \varepsilon \implies f(a) < f(x) < L \implies x \in A$$

However,  $x \leq \sup A = c$  for any such  $x$ , contradiction. Hence  $f(c) = L$ .  $\square$

# Keywords

cauchy, [6](#)  
continuous, [8](#)  
converges, [2](#)  
decreasing, [1](#)  
diverges, [1](#)

increasing, [1](#)  
monotonic, [1](#)  
subsequence, [1](#)