Lecture VI: Linear Algebra, ODE and Dynamic Systems

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1. Linear Algebra

In economics, it is very common to have ***systems of equations***,

$$\begin{aligned} Q_d &= a_0 + a_1 P \\ Q_s &= b_0 + b_1 P \\ Q_d - Q_s &= 0 \end{aligned} \tag{1}$$

We have supply, demand, and equilibrium. This is oozing economics, right? This problem can be expressed in general as

$$Ax = d$$

with A some $N \times M$ matrix times a column vector $x \in \mathbb{R}^M$ equal to some other vector $d \in \mathbb{R}^N$. The above is matrix notation for the system

$$\begin{aligned} a_{11}x_1 = a_{12}x_2 + \ldots + a_{1M}x_M &= d_1 \\ &\vdots \\ a_{N1}x_1 = a_{N2}x_2 + \ldots + a_{NM}x_M &= d_N \end{aligned}$$

With some wrangling, we can see that Equation (1) can be expressed in this way:

$$\underbrace{\begin{bmatrix} a_0 \\ b_0 \\ 0 \end{bmatrix}}_{d} = \underbrace{\begin{bmatrix} -a_1 & 1 & 0 \\ -b_1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} P \\ Q_d \\ Q_s \end{bmatrix}}_{x}$$

In this case, A is 3×3 and "invertible," so we can explicitly solve for equilibrium price and quantity.

$$\begin{bmatrix} P \\ Q_d \\ Q_s \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 \\ -b_1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} a_0 \\ b_0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} P \\ Q_d \\ Q_s \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} 1 & -1 & -1 \\ b_1 & -a_1 & -a_1 \\ b_1 & -a_1 & -b_1 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} P \\ Q_d \\ Q_s \end{bmatrix} = \frac{1}{b_1 - a_1} \begin{bmatrix} a_0 - b_0 \\ a_0 b_1 - a_1 b_0 \\ a_0 b_1 - a_1 b_0 \end{bmatrix}$$

We can check this is the same answer we'd get if we'd solve this by leveraging equilibrium:

$$a_0 + a_1 P = b_0 + b_1 P \implies P = \frac{a_0 - b_0}{b_1 - a_1}$$

Then for quantity,

$$Q^s = Q^d = a_0 + a_1 \frac{a_0 - b_0}{b_1 - a_1} = \frac{a_0 b_1 - a_0 a_1 + a_1 a_0 - a_1 b_0}{b_1 - a_1} = \frac{a_0 b_1 - a_1 b_0}{b_1 - a_1}$$

In general, we find that

- If A is invertible[1] (more on invertible matrices below) then there exists a ***unique*** solution given by $x^* = A^{-1} d$.
- If $M < N$ then in general there no solutions (you have more equations than parameters; in general it is not possible to satisfy all the equalities).
- If $N < M$ then in general there are infinitely many solutions (you have more parameters than equations; the additional parameters give you additional degrees of freedom).

[1] Note in this case $N = M$ is necessary, but not sufficient.

Hence it will be useful to study the properties of matrices and vectors, which are the building blocks of these types of systems.

1.1. Vector Spaces.

Definition 1. A ***vector space*** V is a collection of objects called vectors endowed with

1. Addition.
2. Multiplication by scale.

Some additive properties of vector spaces:

1. ***Commutativity***: $v + w = w + v$ for any $v, w \in V$.
2. ***Associativity***: $(v + w) + u = v + (w + u)$ for any $v, w \in V$.
3. ***Additive identity***: $\exists 0 \in V$ s.t. $v + 0 = v$ for any $v \in V$ (the 0 vector or the origin).
4. ***Additive inverse***: $\forall v \in V \;\; \exists - v \in V$ s.t. $v + (-v) = 0$.

Some multiplicative properties of vector spaces:

1. ***Unit rule*** or ***multiplicative identity***: $1v = v$ for any $v \in V$.
2. Multiplicative ***associativity***: $(\alpha\beta)v = \alpha(\beta v)$ for any $v \in V$ and $\alpha, \beta \in \mathbb{R}$.
3. ***Distributivity***: For any $\alpha, \beta \in \mathbb{R}$, $v, w \in V$, we have
$$\alpha(u + v) = \alpha u + \alpha v \quad \text{and} \quad (\alpha + \beta)v = \alpha v + \beta v$$

Remark 1. If you are like me, you might think it's a bit odd that we are making a big deal of associativity, commutativity, and distributivity (like we did back in primary school). The reason is that a more formal treatment of linear spaces would not take any of these properties for granted, and would be very careful in discussing everything in this section (and henceforth) using just the definitions.

And if you are like me you might also wonder, why don't we do that? We've done a lot of definition-theorem-proof style lectures in this class. It is a math class after all. So why not here as well? The reason is that it turns out that for all intents and purposes, vector spaces are equivalent to $\mathbb{R}^N$. There is a formal way of defining what that means and of showing it. While I invite the curious to read Appendix A, for this class I will simply discuss $\mathbb{R}^N$ and I will take it to have all the properties we expect. □

Definition 2. Let V be a vector space and $\{v_1, \ldots v_N\}$ s.t. $v_i \in V$. A ***linear combination*** of v_i is given by
$$\sum_{i=1}^{N} \alpha_i v_i = \alpha_1 v_1 + \ldots + \alpha_N v_N$$
for some arbitrary set of coefficients $\{\alpha_1, \ldots, \alpha_N\}$ s.t. $\alpha_i \in \mathbb{R}^N$.

Definition 3. Let V be a vector space; $\{v_i\}$ s.t. $v_i \in V$ is a ***basis*** of V if for every $u \in V$ there exists a unique linear combination of $\{v_i\}$ s.t.

$$u = \sum_{i=1}^{N} \alpha_i v_i$$

The coefficients $\{\alpha_i\}$ are called the ***coordinates*** of u in V with respect to the basis $\{v_i\}$.

The easiest example of a basis is the standard basis in $\mathbb{R}^N$:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \dots \quad e_N = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Naturally any $u \in \mathbb{R}^N$ can be uniquely represented as a linear combination of the standard basis if we take $\{\alpha_i\} = \{u_i\}$ the coefficients equal each of the entries in u.

Definition 4. Let V be a vector space; we say that $\{v_i\}$ s.t. $v_i \in V$ is a ***spanning set*** of V if for every $u \in V$ there exists a linear combination of $\{v_i\}$ s.t.

$$u = \sum_{i=1}^{N} \alpha_i v_i$$

Note that in the definition above, the linear combination does not have to be unique. Hence every basis of a vector space V is a spanning set of V, but not every spanning set of V can be a basis. Hence the question: When is a spanning set a basis?

Definition 5. A linear combination $\{\alpha_i\}$ is called ***trivial*** if $\alpha_i = 0$ for every i.

Definition 6. Let V be a vector space; we say that $\{v_i\}$ s.t. $v_i \in V$ are ***linearly independent*** if the only linear combination of v_i that is equal to 0 is trivial. That is

$$\sum_i \alpha_i v_i = 0 \implies \alpha_i = 0$$

Definition 7. Let V be a vector space; if $\{v_i\}$ s.t. $v_i \in V$ are not linearly independent, we say they are ***linearly dependent***.

Theorem 1. *Let V be a vector space; $\{v_i\}$ s.t. $v_i \in V$ are linearly dependent if for some j and some linear combination $\{\beta_{-j}\}$ we have that*

$$v_j = \sum_{i \neq j} \beta_i v_i$$

Proof. Take the linear combination $\{\alpha_i\}$ s.t. $\alpha_i = \beta_i$ for $i \neq j$ and $\alpha_j = -1$. Then

$$\sum_i \alpha_i v_i = \sum_{i \neq j} \beta_i v_i - v_j = 0$$

Since $\alpha_j \neq 0$, $\{\alpha_i\}$ is non-trivial, and so $\{v_i\}$ are not linearly independent. $\square$

Theorem 2. *Let V be a vector space; $\{v_i\}$ s.t. $v_i \in V$ are a basis for $V \iff \{v_i\}$ are a linearly independent spanning set of V.*

Claim 1. *Let $\{v_i\}, \{u_i\}$ be any basis for a vector space V. Then $|\{v_i\}| = |\{u_i\}|$.*

That is, basis for a vector space have the same number of elements. This leads to the following:

Definition 8. A vector space V has ***dimension*** $\dim(V)$ equal to the number of elements in any basis.

1.2. Linear Transformations.

Definition 9. Let V, W be vector spaces. $T : V \to W$ is a ***linear transformation*** if

$$T(\alpha v + w) = T(v) + T(w)$$

for any $v \in V, w \in W, \alpha \in \mathbb{R}$.

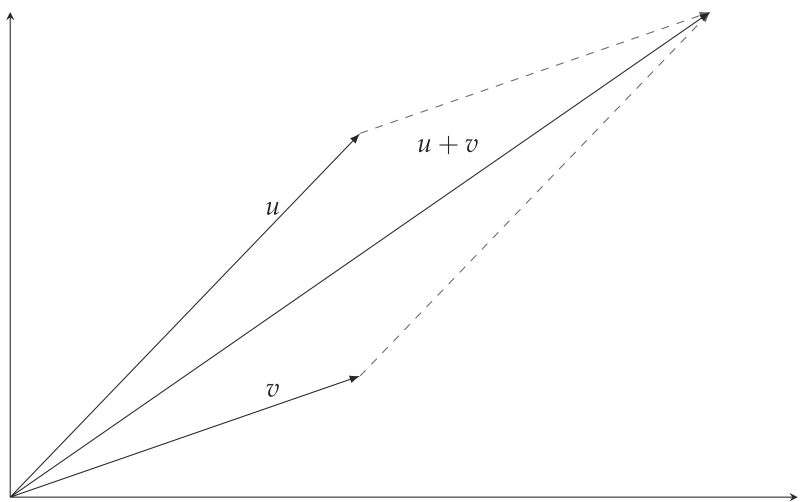


Figure 1: Example of a Linear Transformation

Theorem 3. *Let V, W be vector spaces, $T : V \to W$ a linear transformation, and $\{v_i\}$ a basis for V. Then*

$$T(v) = \sum_i \alpha_i T(v_i)$$

for $v = \sum_i \alpha_i v_i$.

Proof. We have that

$$T(v) = T\left(\sum_i \alpha_i v_i\right)$$

We can then proceed by induction, using the properties of a linear transformation we saw above. Since V is a linear space, it has all its linear combinations. Thus $\alpha_1 v_1 \in V$ and $\sum_{i>1} \alpha_i v_i \in V$. Applying the definition,

$$T(v) = T\left(\alpha_1 v_1 + \sum_{i>1} \alpha_i v_i\right) = \alpha_1 T(v_1) + T\left(\sum_{i>1} \alpha_i v_i\right)$$

Iterating the above:

$$T(v) = \sum_{i<k} \alpha_i T(v_i) + T\left(\alpha_k v_k + \sum_{i>k} \alpha_k v_k\right) = \sum_{i<k} \alpha_i T(v_i) + \alpha_k T(v_k) + T\left(\sum_{i>k} \alpha_k v_k\right)$$

Therefore we can simply write

$$T(v) = \sum_i \alpha_i T(v_i)$$

□

Theorem 4. *Any linear transformation $T : V \to W$ can be represented by a matrix.*

Proof. Let $\dim(V) = M \leq N$ so $V \subseteq \mathbb{R}^N$. A matrix A is a collection of vectors $\{a_1, \ldots, a_M\}$ with $a_i \in W$. Now consider the standard basis $\{e_i\}$ for $\mathbb{R}^N$ and take the subset of the standard basis that spans V. WLOG take these to be the first M elements of the sandard basis. For any vector $x \in V$ we find

$$T(x) = T\left(\sum_i x_i e_i\right) = \sum_i x_i T(e_i)$$

Let $a_i \equiv T(e_i)$ and we can see that

$$T(x) = Ax = \sum_i a_i x_i$$

Hence $T : V \to W$ can be equivalently represented by $A = [a_1 \ \cdots \ a_M]$ with $a_i \in W$. □

Example 1. Consider the following operation

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 6 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 8 \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Which we express as a linear transform

$$T(l_1) = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \qquad T(l_2) = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \qquad T(l_3) = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

That is,

$$T\begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} = 4T(l_1) + 6T(l_2) + 8T(l_3)$$

□

In general we can define matrix multiplication as linear operations on vector spaces:

$$AB = A[b_1 \ldots b_N] = [Ab_1 \ldots Ab_N]$$

Some properties of ***matrix multiplication***:

1. ***Associativity***: $(AB)C = A(BC)$
2. ***Distributivity***: $A(B + C) = AB + AC$

3. ***Associativity***: $A + B = B + A$

However, in general ***matrix multiplication is not commutative***. That is, $AB \neq BA$ in general (in fact, BA may not even be well-defined).

1.3. Matrix Inverse, Rank, and Determinant.

Definition 10. The vector space spanned by the columns of a matrix A is the ***column space*** of A. The ***rank*** of a matrix, $\text{rank}(A)$, is the dimension of the column space (the maximum number of linearly independent columns).[2] A square matrix $N \times N$ is called ***full-rank*** if $\text{rank}(A) = N$ and ***rank-deficient*** if $\text{rank}(A) < N$.

Definition 11. A $N \times N$ matrix A is called ***diagonal*** if all its non-diagonal entries are 0.

Definition 12. The ***identity matrix*** is an $N \times N$ diagonal matrix with all diagonal entries equal to 1 (and non-diagonal entries equal to 0). For any $N \times N$ matrix A, $AI = IA = A$.

$$I = \begin{bmatrix} 1 & 0 & 0 & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Definition 13. The $N \times N$ matrix A is said to be ***left invertible*** if $\exists C_L$ s.t. $AC_L = I$ the identity. We say it is ***right invertible*** if $\exists C_R$ s.t. $C_R A = I$. If A is left and right invertible with $C_L = C_R$, we say that A is ***invertible***, and we denote $C_R = C_L = A^{-1}$ the ***inverse*** of A.

Definition 14. If A is not invertible, we say that A is ***singular***.

Theorem 5. *Let A be a $N \times N$ matrix. A is invertible $\iff$ $\text{rank}(A) = N$.*

Some other properties of the rank:

- $\text{rank}(A) = \text{rank}(A') = \text{rank}(AA') = \text{rank}(A'A)$.
- $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$.
- $\text{rank}(CAB) = \text{rank}(A)$ if C, B are non-singular.

Example 2. Can we find the inverse of the matrix

$$B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

[2] It turns out that the row-rank of a matrix, the maximum number of linearly independent rows, is the same as the column-rank of a matrix. Hence we can just talk about the rank without clarifying row or column.

using elementary row operations?

$$\begin{bmatrix} 2 & 1 & \rightarrow & 1 & 0 \\ 1 & 1 & & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1/2 & \rightarrow & 1/2 & 0 \\ 1 & 1 & & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1/2 & \rightarrow & 1/2 & 0 \\ 0 & -1/2 & & 1/2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \rightarrow & 1 & -1 \\ 0 & -1/2 & & 1/2 & -1 \end{bmatrix}$$

In general, however, we have a shortcut for 2×2 matrices:

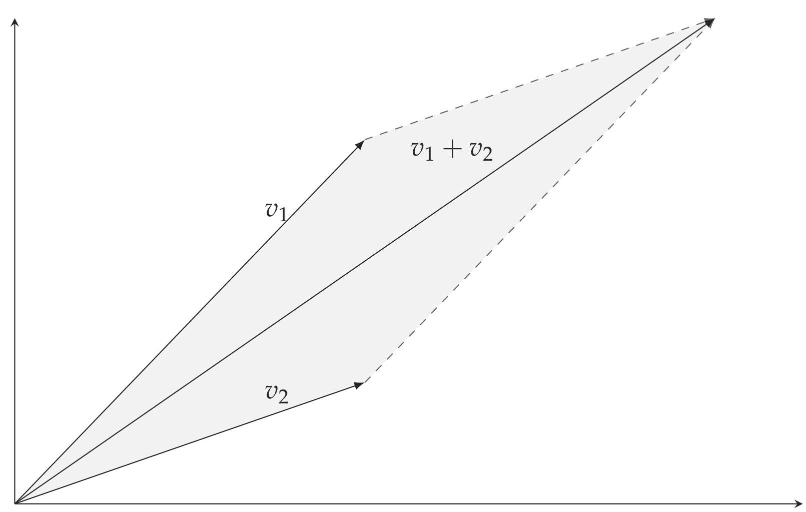
$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad B^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

□

Definition 15. The ***determinant*** of a $N \times N$ square matrix A is

$$\det(A) = \sum_{j=1}^{N} (-1)^{i+j} a_{ij} \det(A_{-i,-j})$$

Consider this picture:



The set

$$\left\{ v : v = \sum_{i=1}^{N} t_i v_i \quad t_i \in [0, 1] \right\}$$

will just be the area of the parallelogram. Take a square matrix A

1. If A has a column of row that is all 0, then $\det(A) = 0$
2. If A has two columns or rows that are equal, then $\det(A) = 0$
3. If A has a columns is a multiple of another column, then $\det(A) = 0$
4. If A has a columns or rows that are linearly dependent, then $\det(A) = 0$

Theorem 6. *A is non-invertible* $\iff$ $\det(A) = 0$

Some other properties of the determinant:

- $\det(I) = 1$.
- $\det(A) = \prod_i a_{ii}$ if A is diagonal.
- $\det(\alpha A) = \alpha^N \det(A)$ for any $\alpha \in \mathbb{R}$.
- $\det(AB) = \det(A)\det(B)$.
- $\det(A^{-1}) = \det(A)^{-1}$.

Definition 16. For a $N \times N$ matrix A, the M_{ij} ***minor*** is the $(N-1) \times (N-1)$ sub-matrix obtained by deleting the ith row and jth column of A.

Definition 17. The i, j ***cofactor*** of the square $N \times N$ matrix A is given by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Definition 18. The ***adjoint*** of an $N \times N$ matrix A is given by

$$\text{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{N1} \\ \vdots & & \ddots & C_{N1} \\ C_{N1} & C_{N2} & \cdots & C_{NN} \end{bmatrix}$$

Theorem 7. *If A is a non-singular $N \times N$ matrix then*

$$A^{-1} = \frac{1}{\det(A)}\, adj(A)$$

For instance, for a 2×2 matrix we have

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Applying the theorem,

$$B^{-1} = \frac{1}{\det(B)} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem 8 (Cramer's Rule). *Let A be a $N \times N$ non-singular matrix s.t. $Ax = b$. Then*

$$x_i = \frac{\det(A_i)}{\det(A)}$$

for

$$A_i = \begin{bmatrix} a_1 & \cdots & a_{i-1} & b & a_{i+1} & \cdots & a_N \end{bmatrix}$$

That is, the matrix obtained by replacing the ith column of A with b.

Definition 19. The ***trace*** of an $N \times N$ matrix A is the sum of its diagonal elements,

$$tr(A) = \sum_{i=1}^{N} a_{ii}$$

Some properties of the trace:

- $\text{trace}(A + B) = \text{trace}(A) + \text{trace}(B)$
- $\text{trace}(AB) = \text{trace}(BA)$ if both products exist.
- $\text{trace}(\alpha A) = \alpha \, \text{trace}(A)$ for any $\alpha \in \mathbb{R}$.

An example which comes up in econometrics is that for any $N \times K$ matrix X s.t. $(X'X)^{-1}$ exists, we have

$$\text{trace}\left(\underbrace{X}_{A} \overbrace{(X'X)^{-1}X'}^{B} \right) = \text{trace}((X'X)^{-1}X'X) = \text{trace}(I_{K\times K}) = K$$

Further,

$$\text{trace}\left(I_{N\times N} - X(X'X)^{-1}X' \right) = \text{trace}\left(I_{N\times N} \right) - \text{trace}(X(X'X)^{-1}X'\Big) = N - K$$

1.4. Eigenvalues and Eigenvectors. Take

$$T(v) = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Note that for the standard basis,

$$\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} l_1 = 3 l_1 \qquad \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} l_2 = -l_2$$

For this particular example, we can see the standard basis is exactly scaled by 3 and -1.

Definition 20. Let A be a $N \times N$ square matrix. The $N \times 1$ vector $v \neq 0$ is an ***eigenvector*** (or characteristic vector) of A with corresponding ***eigenvalue*** or (characteristic root) λ if

$$Av = \lambda v$$

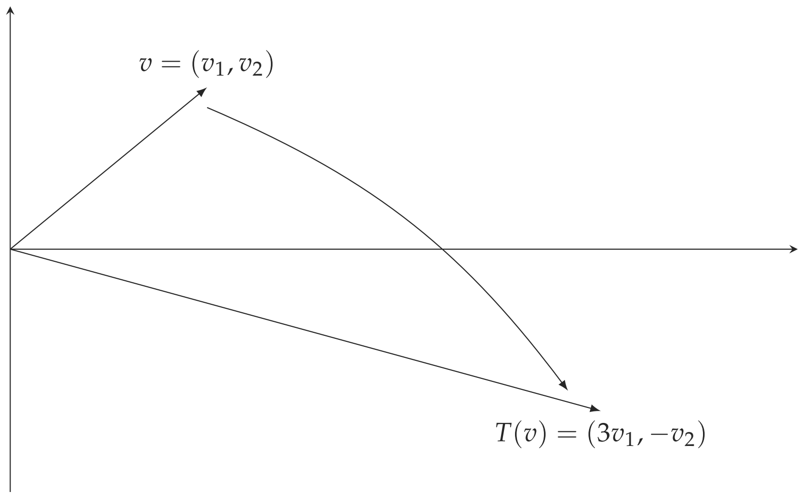


Figure 2: Visualization of a Transformation

In general we want to find a basis $\{v_i\}$ of (the column space of) A s.t.

$$Av_i = \lambda_i v_i$$

that is, a basis of eigenvectors. Note that

$$Av = \lambda v \iff (A - \lambda I)v = 0$$

This means that for any $v \neq 0$, we have that

$$\det(A - \lambda I) = 0$$

Example 3. Consider

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad \det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = 0 \iff (1-\lambda)^2 - 4 = 0 \iff (\lambda+1)(\lambda-3) = 0$$

Hence $\lambda = \{-1, 3\}$. These are the ***eigenvalues***. Now we find the ***eigenvectors***. For $\lambda = -1$

$$(A - \lambda I)v = 0 \iff \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} v = 0$$

Which gives that v must be s.t. $v_1 = -v_2$. For $\lambda = 3$

$$(A - \lambda I)v = 0 \iff \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} v = 0$$

Which gives that v must be s.t. $v_1 = v_2$. It is often useful for the norm of the basis to be 1. Hence the eigenvectors are given by

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

□

Definition 21. A an $N \times N$ matrix is ***diagonalizable*** if $\exists P$ and diagonal matrix Λ s.t.

$$P^{-1}AP = \Lambda$$

Note we can equivalently write $A = P\Lambda P^{-1}$.

Claim 2. *If A is diagonalizable then P is a matrix of eigenvectors and Λ is a matrix of eigenvalues.*

Proof. If A is diagonalizable then

$$AP = P\Lambda$$

Since the kth column of P is s.t. $Av_k = \lambda_k v_k$ we can see v_k is an eigenvector and λ_k an eigenvalue. □

Theorem 9. *Let A be a $N \times N$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_N$.*

- *$trace(A) = \sum_{i=1}^N \lambda_i$*
- $\det(A) = \prod_{i=1}^N \lambda_i$.

Proof. Let us show this assuming A is diagonalizable.[3]

$$\text{trace}(A) = \text{trace}(P^{-1}\Lambda P) = \text{trace}(\Lambda P P^{-1}) = \text{trace}(\Lambda) = \sum_i \lambda_i$$
$$\det(A) = \det(P^{-1}\Lambda P) = \det(P^{-1})\det(\Lambda)\det(P) = \det(\Lambda) = \prod_i \lambda_i$$

□

Theorem 10. *If A is symmetric then A is diagonalizable and P can be chosen to be orthonormal ($P^{-1} = P^T$).*

EigenExample

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

We can find that $\det(A - \lambda I) = 0$ gives

$$(1-\lambda)^2 + 4 = 0 \qquad \lambda = \frac{2 \pm \sqrt{4 - 4 \cdot 5}}{2}$$

[3]For the general proof, recall $\lambda_1, \ldots, \lambda_N$ are the roots of the polynomial given by

$$\det(A - \lambda I) = (\lambda_1 - \lambda) \times (\lambda_2 - \lambda) \times \ldots \times (\lambda_N - \lambda)$$

For the determinant, set $\lambda = 0$ and find $\det(A) = \prod_{i=1}^N \lambda_i$. For the trace, we need to leverage something known as "Vieta's formulas." The relevant result is that for a polynomial of order N, the $N-1$ coefficient is the sum of the roots of the polynomial (this sounds esoteric, but think about how you learned to expand formulas like $(\lambda_1 - \lambda)(\lambda_2 - \lambda)$; the "middle" term is $-(\lambda_1 + \lambda_2)$, and this is just the generalization). If we can show trace(A) is the $N-1$ coefficient of the characteristic polynomial we'd be done. To see it, note that λ^{N-1} will only appear if all the diagonal elements are multiplied (any other permutation will give at most a polynomial of order $N-2$). Hence λ^{N-1} only appears as part of the term

$$\prod_i (a_{ii} - \lambda)$$

Here the roots are a_{ii}, so the coefficient on λ^{N-1} is $\sum_i a_{ii} = \text{trace}(A)$.

Hence $\lambda = 1 \pm 2i$. Solving for v, we get

$$v_1 = (\alpha, \alpha i) v_2 = (\alpha, -\alpha i)$$

Let $\alpha = 1$. Then

$$P = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}$$

Hence

$$A = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} \begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}$$

2. Introduction to Ordinary Differential Equations (ODE)

2.1. Dynamic Equations. The basic idea is that we want to understand how a variable changes over time. If time is continuous (like it is IRL), then we say that

$$\dot{y} \equiv \frac{dy}{dt}$$

In general, we have an nth order linear differential equation as

$$y^{(n)} + \alpha_{n-1}(t) y^{(n-1)} + \ldots + \alpha_0(t) y = f(t)$$

where $\alpha_i(t)$ are constants that depend only on t and $y^{(n)}$ is the nth derivative of y with respect to t. We say it is ***homogeneous*** if $f(t) = 0$.

The equation is called ***autonomous*** if t only enters through $y(t)$; that is, $\alpha_i(t) \equiv \alpha_i$. For example,

$$\dot{y} = ky \iff \dot{y} - ky = 0$$

is an autonomous *and* homogeneous equation. By contrast,

$$\dot{y} = ty \iff \dot{y} - ty = 0$$

is homogeneous *but not* autonomous.

Example 4. Consider the autonomous and homogeneous system

$$\dot{y} = ky$$

This translates to

$$\frac{dy}{dt} = ky$$

which means that

$$\int \frac{dy}{dt} \frac{1}{y} dt = \int k dt \iff \log(y) + C_1 = kt + C_2$$

Let $C_3 \equiv \exp(C_2 - C_1)$, and we have that

$$y = C_3 e^{kt}$$

This equation is not quite defined, since the behavior of y depends on the constant C_3. Hence we need some initial conditions to pin down the path of y with respect to t. For example,

$$y(0) = 5 \qquad 5 = C_3 e^0 \implies C_3 = 5 \implies y(t) = 5e^{kt}$$

□

Example 5. Consider the homogeneous but not autonomous system

$$\dot{y} = ty$$

Again,

$$\frac{dy}{dt} = ty$$

which means that

$$\int \frac{dy}{dt}\frac{1}{y}dt = \int t dt \iff \log(y) + C_1 = \frac{t^2}{2} + C_2$$

Let $C_3 \equiv \exp(C_2 - C_1)$, and we have that

$$y = C_3 e^{\frac{t^2}{2}}$$

Note that $y(0) = C_3$. Hence we can write

$$y(t) = y(0)e^{\frac{t^2}{2}}$$

□

We will not always use the separation of variables method. In general, if we have an equation of the form

$$\dot{y} + p(t)y = q(t)$$

we use the ***integrating factor***. Multiply each term by $\exp(\int p(t)dt)$:

$$\exp\left\{\int p(t)dt\right\}\left(\dot{y} + p(t)y\right) = \exp\left\{\int p(t)dt\right\} q(t)$$

Why? Because we have that

$$\frac{d}{dt}\left(\exp\left\{\int p(t)dt\right\} y\right) = \exp\left\{\int p(t)dt\right\}\dot{y} + \exp\left\{\int p(t)dt\right\} p(t)y$$

Hence

$$\int \left[\frac{d}{dt}\left(\exp\left\{\int p(t)dt\right\} y\right)\right] dt = \int \left[\exp\left\{\int p(t)dt\right\} q(t)\right] dt$$

gives

$$y(t) = \frac{\int \left[\exp\left\{\int p(t)dt\right\} q(t)\right] dt + C}{\exp\left\{\int p(t)dt\right\}}$$

for some constant C. Let's see an example:

Example 6. Consider the system

$$\dot{y} + t^2 y = t^2$$

How to proceed in this case?

$$\dot{y} = ay + b$$

$p(t) = -a$ and $q(t) = b$. The integrating factor is

$$\exp\left\{\int p(t)dt\right\} = \exp\left\{\int -adt\right\} = \exp\{-at\}$$

Plugging the formula,

$$y(t) = \frac{\int \exp\{-at\} bdt + C}{\exp\{-at\}} = -\frac{b}{a} + C\exp\{at\}$$

□

2.2. Dynamic Systems. Consider an equation in discrete time

$$y_t = \alpha y_{t-1} + x_t$$

An nth order ***linear difference*** equation is

$$y_{t-n} + \alpha_{(n-1)}(t) y_{t-(n-1)} + \ldots + \alpha_0(t) y_t = f(t)$$

We say that this is ***homogeneous*** if $f(t) = 0$ and ***autonomous*** if $\alpha_i(t) \equiv \alpha_i$ do not depend on t. The way to solve this is by recursive computation:

$$y_t = \frac{1}{k} y_{t-1}$$

In general,

$$y_t = \frac{1}{k^2} y_{t-2} = \frac{1}{k^3} y_{t-3} = \ldots = \frac{1}{k^t} y_0$$

For example take the 2nd order homogeneous and autonomous difference equation

$$y_{t-2} + \alpha y_{t-1} + \beta y_t = 0$$

We assume that $y_t = Ca^t$, and we check against that. So

$$Ca^{t-2} + \alpha Ca^{t-1} + \beta Ca^t = 0 \iff a^{-2} + \alpha a^{-1} + \beta = 0 \iff \beta a^2 + \alpha a + 1 = 0$$

Hence

$$a = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2\beta}$$

Considera first-order ***dynamic system***:

$$x_t = Ax_{t-1}$$

If A is diagonalizable, then

$$x_t = P^{-1}\Lambda P\left(P^{-1}\Lambda P x_{t-2}\right) = P^{-1}\Lambda^2 P x_{t-2} = \ldots = P^{-1}\Lambda^t P x_0$$

Hence the dynamics of the system can be determined by analyzing the matrix of eigenvectors Λ.

- If all eigenvalues have absolute value < 1 then the system is globally stable ($\Lambda^t \to 0$).
- If all eigenvalues have absolute value > 1 then the system is globally unstable ($\Lambda^t \to \pm\infty$)..

In continuous time, consider

$$\dot{x} = Ax$$

Again, assuming A id diagonalizable,

$$P\dot{x} = \Lambda P x$$

Let $y = Px$, so $\dot{y} = P\dot{x}$, and we have

$$\dot{y} = \Lambda y$$

with Λ diagonal. In this case, the solution is given by

$$y = \exp(\Lambda t)C$$

for some constant vector C and $\exp(\Lambda t)$ denoting the diagonal matrix diag $\{\lambda_1 t, \ldots, \lambda_N t\}$. Note

$$\dot{y} = \Lambda \underbrace{\exp(\Lambda t)C}_{y} = \Lambda y$$

Appendix A. Linear Algebra in the Abstract

Only for the very curious, I will exhibit here an entirely formal treatment of linear spaces and build up to the theorem I mentioned wherein we see that any linear space (under a very mild condition; namely that it is finite-dimensional) is equivalent to $\mathbb{R}^N$. Proofs for this appendix are available on request, but again this is just here to show you there is a very abstract and formal way of defining everything, even if it leads to the exact place you would expect (the real numbers).

Definition 22. A ***binary operator*** on X is a map $f : X \times X \to X$. We denote $f(x, y) \equiv xfy$.

Definition 23. Let X be a non-empty set and $+$ a binary operator. $(X, +)$ is a ***group*** if the following hold:

1. ***Associativity***: $+(+(x, y), z) = +(x, +(y, z))$; that is, $(x + y) + z = x + (y + z)$.
2. Existence of an ***additive identity***: $\exists e \in X$ s.t. $e + x = x = x + e$. We denote $e \equiv 0$, i.e. $0 + x = x = x + 0$.
3. Existence of an ***additive inverse***: $\forall x \in X \;\; \exists y$ s.t. $x + y = y + x = 0$. We denote $y \equiv -x$.

Note we are using $+$ to denote addition (because, seriously, what else would we use?)

Definition 24. A group $(X, +)$ is an ***Abelian group*** if the following also holds:

4. ***Commutativity***: $+(x, y) = +(y, x)$; that is, $x + y = y + x$.

An easy example of an Abelian group is $\mathbb{R}$ with addition. A more complex example:

Example 7. Consider a Rubiks cube and the set of all possible permutations of the faces. This is a group:

- There is an identity element, namely the solved cube.
- There is an inverse element: Given any permutation, the set of operations that solves the cube would give the inverse element from the solved cube.
- It is associative: Given any sequence of permutations, as long as they are applied in the same sequence, whether we "group" any set of operations together doesn't matter.

However, this is ***not*** an Abelian group because operations on the cube are not commutative—order matters. □

Claim 3. *Let $(X, +)$ be a group:*

- *The identity element is unique.*
- *The inverse element is unique.*

Definition 25. $(X, +, \cdot)$ is a ***linear space*** (or ***vector space***) if $(X, +)$ is an Abelian group and $\cdot : \mathbb{R} \times X \to X$ satisfies, for any $\alpha, \lambda \in \mathbb{R}$ and $\vec{x}, \vec{y} \in X$,

1. ***Associativity***: $\cdot(\alpha, \cdot(\lambda, \vec{x})) = \cdot(\alpha\lambda, \vec{x})$; that is, $\alpha \cdot (\lambda \cdot \vec{x}) = (\alpha\lambda) \cdot \vec{x}$.
2. ***Distributivity***: In $\mathbb{R}$, $\cdot(\alpha + \lambda, \vec{x}) = +(\cdot(\alpha, \vec{x}), \cdot(\lambda, \vec{x}))$; i.e. $(\alpha + \lambda) \cdot \vec{x} = \alpha \cdot \vec{x} + \lambda \cdot \vec{x}$.
   in X, $\cdot(\lambda, +(\vec{x}, \vec{y})) = +(\cdot(\lambda, \vec{x}), \cdot(\lambda, \vec{y}))$; i.e. $\lambda \cdot (\vec{x} + \vec{y}) = \lambda \cdot \vec{x} + \lambda \cdot \vec{y}$.

3. ***Unit rule***: $\cdot(1, \vec{x}) = \vec{x}$; that is, $1 \cdot \vec{x} = \vec{x}$.

Did you catch the sleight of hand in this definition?[4]

Claim 4. *Let $(X, +, \cdot)$ be a linear space. Then*

- $\lambda \cdot \vec{0} = \vec{0}$.
- $\lambda = 0 \implies \lambda\vec{x} = \vec{0}$.
- *If $\vec{x} \neq \vec{0}$ then $\lambda\vec{x} = \vec{0} \implies \lambda = 0$.*
- $-(\lambda \cdot \vec{x}) = (-\lambda) \cdot \vec{x}$.
- *If $\vec{x} \neq 0$ then $\alpha\vec{x} = \beta\vec{x} \iff \alpha = \beta$.*

All the above might seem obvious, but try to prove them *only using the definitions* to see why they are not!

Definition 26. For any $m \in \mathbb{N}$, a ***linear combination*** of $\vec{x}_1, \ldots, \vec{x}_m$ is a vector

$$\sum_{i=1}^{m} \lambda_i \vec{x}_i$$

with $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ the ***coefficients*** of the linear combination.

Theorem 11. *S is a linear space iff it contains all linear combinations of any set of vectors in S.*

Definition 27. The set of all linear combinations of finitely many elements of a non-empty set S of is called the ***span*** of S, defined as the set

$$\operatorname{span}(S) \equiv \Big\{ \sum_{i=1}^{m} \lambda_i \vec{x}_i : m \in \mathbb{N}, (\vec{x}_i, \lambda_i) \in S \times \mathbb{R} \;\; \forall i = 1, \ldots, m \Big\}$$

By convention, $\operatorname{span}(\varnothing) = \{\vec{0}\}$.

Definition 28. A ***basis*** for a linear space X is a $\supseteq$-minimal subset of X that spans X, i.e. S is a basis for X iff

1. $X = \operatorname{span}(S)$
2. $X = \operatorname{span}(T) \implies T \not\subset S$.

Definition 29. If a linear space X has a finite basis then it is finite-dimensional with ***dimension*** $\dim(X)$ the cardinality of any of its bases. If X does not have a finite basis, then it is ∞-dimensional and $\dim(X) = \infty$.

Definition 30. A subset S of a linear space X is ***linearly dependent*** in X if it either equals $\{\vec{0}\}$ or if $\exists \vec{x} \in S$ s.t. $\vec{x}$ can be expressed as a linear combination of finitely many vectors in $S \setminus \{\vec{x}\}$. Any finite set of vectors is called linearly dependent if $\{x_1, \ldots, x_m\}$ is linearly dependent.

Definition 31. A subset S of a linear space X is called ***linearly independent*** if no finite subset is linearly dependent. By convention, we say $\varnothing$ is linearly independent.

[4] $\mathbb{R}$ with addition and multiplication as normally defined is what is known as a ***field*** (another example of a field are the rationals, $\mathbb{Q}$). I will not delve into that here, but the reason linear spaces end up being equivalent fo $\mathbb{R}^N$ is because the field we use to define them is $\mathbb{R}$! This does mean there is a second level of abstraction where if we used some arbitrary field $\mathcal{F}$ then those more abstract linear spaces would be equivalent to $\mathcal{F}^N$.

Proposition 1. *A subset S of a linear space X is a basis for X iff S is linearly independent and span$(S) = X$.*

Corollary 1. *Any two bases of a linear space X have the same cardinality.*

Definition 32. Let X, Y be two linear spaces. A function $L : X \to Y$ is a ***linear operator*** or a ***linear transformation*** if

$$L(\alpha\vec{x} + \vec{y}) = \alpha L(\vec{x}) + L(\vec{y}) \qquad \forall \vec{x}, \vec{y} \in X \quad \text{and} \quad \alpha \in \mathbb{R}$$

Definition 33. Let X, Y be two linear spaces and L a linear operator $L : X \to Y$. If L is a bijection it is called a ***linear isomorphism*** between X and Y. If such a bijection exists we say X and Y are ***isomorphic***.

Proposition 2. *Two finite-dimensional linear spaces X, Y are isomorphic iff they have the same dimension.*

Corollary 2. *Every non-trivial finite-dimensional linear space X (i.e. $X \neq \{\vec{0}\}$) is isomorphic to $\mathbb{R}^{\dim(X)}$.*

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