Lecture III: Compactness, EVT, Correspondences

Contents

1. Compactness

So I'm gonna say, straight up, that the first time I encountered compactness (way back in undergrad real analysis) it seemed like an inscrutable concept. If you get it right away, then awesome, but if you don't then no worries! It can take a little bit for this to sink in.

1.1. Introduction.

Definition 1. A class $\mathcal{F} = \{\mathcal{F}_\omega\}_{\omega \in \Omega}$ is said to **cover** a set S if $S \subseteq \cup_{\omega \in \Omega} \mathcal{F}_\omega$. If all members of the class $\mathcal{F}$ are open, we say it is an **open cover**.

Definition 2. A set S is **compact** if every open cover of S has a **finite subcover** of S.

Some examples of sets that are and are not compact:

- $S = (0, 1)$ is not compact. $\mathcal{F} = \{(1/n, 1) : n \in \mathbb{N}\}$ covers S. However, there is no finite sub-cover: Any finite sub-cover gives the interval $(1/N, 1)$; take $z = (1/2N)$ and $z \in (0, 1)$ but $z \notin (1/N, 1)$.

- $S = [0, \infty)$ is not compact. $\mathcal{F} = \{(-1, n) : n \in \mathbb{N}\}$ covers S. However, there is no finite sub-cover: Any finite sub-cover gives the interval $(-1, N)$; take $z = N + 1$ and $z \in [0, \infty)$ but $z \notin (-1, N)$.

- $[0, 1]$ *is* compact. Compactness is really trying to get to a notion of "finiteness," and there is a sense in which intervals that are open or not bounded are not finite. Of course, compactness is more general than that, but at least in $\mathbb{R}^N$ we will get a more intuitive definition of compactness.

Remark 1. Any finite set A is compact. Take any open cover $\mathcal{F} = \{\mathcal{F}_\omega\}_{\omega \in \Omega}$. For $x \in A$, $x \in \mathcal{F}_\omega$ for some $\omega \in \Omega$ (there may be several, and a single $\mathcal{F}_\omega$ may contain many $x \in A$). Name this ω_x for each x; since A is finite, $\{\omega_x\}_{x \in A}$ is a finite. Hence

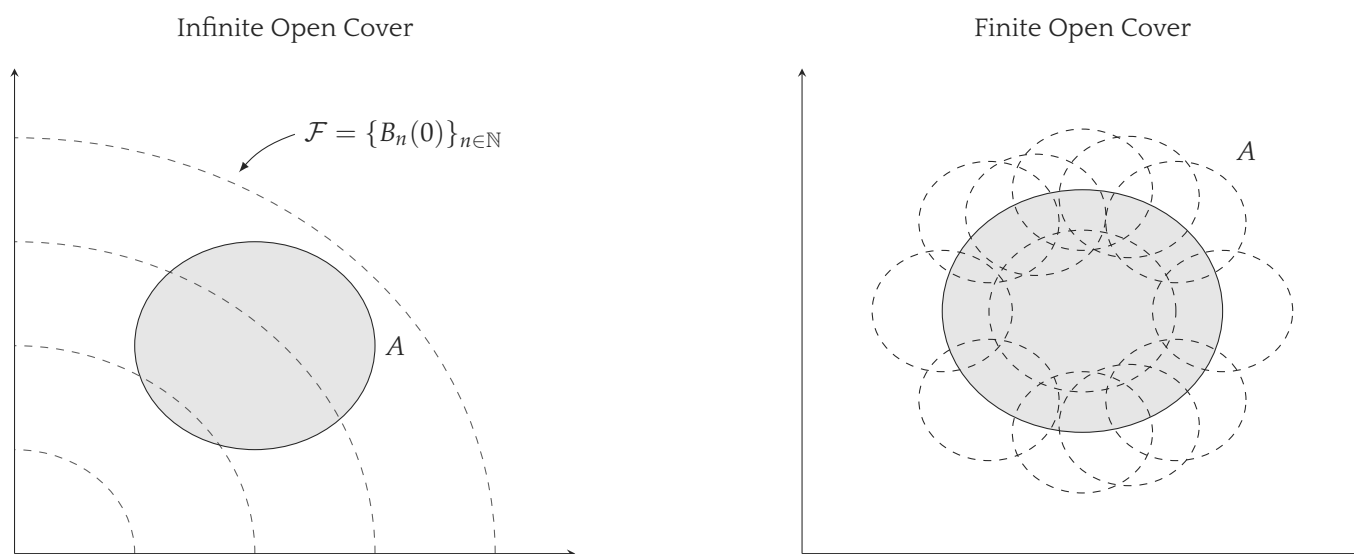$$\mathcal{F}_A = \{\mathcal{F}_{\omega_x}\}_{x \in A}$$

is a finite sub-cover of A. $\square$

Figure 1: Examples of Open Covers in $\mathbb{R}^2$

Definition 3. A set S **sequentially compact** if every sequence in S has a sub-sequence that converges to a point in S ($\forall (x_m) \in S \ \exists (x_{m_k})$ s.t. $x_{m_k} \to x \in S$).

Theorem 1. *A set S is compact $\iff$ S is sequentially compact.*

1.2. Heine–Borel and Other Theorems.

Theorem 2 (Heine–Borel)**.** *For any finite N, $S \subseteq \mathbb{R}^N$ is compact iff S is closed and bounded.*

Proof. We show that compactness $\implies$ closed and bounded. The converse is a bit more involved. Let S be a compact set; first we show it is bounded. Take

$$\mathcal{F} = \{\mathcal{F}_n = B_n(0)\}_{n \in \mathbb{N}}$$

$\mathcal{F}$ is an open cover for $\mathbb{R}^N$, and hence an open cover for $S \subseteq \mathbb{R}^N$. Since S is compact, it admits a finite subcover. Since $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$,

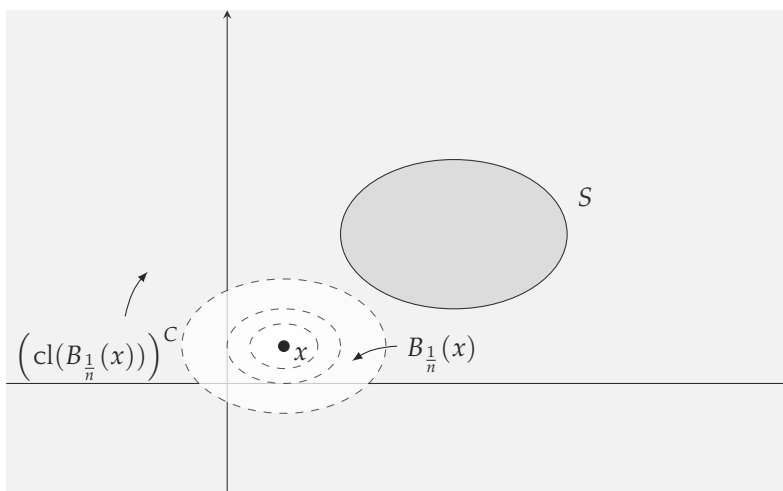$$\bigcup_{n \leq N} \mathcal{F}_n = \mathcal{F}_N = B_N(0)$$

But by definition any $x \in B_N(0)$ is s.t. $d(x, 0) < N$. Since $B_N(0)$ is bounded and $S \subseteq B_N(0)$ then S is also bounded. Visually,
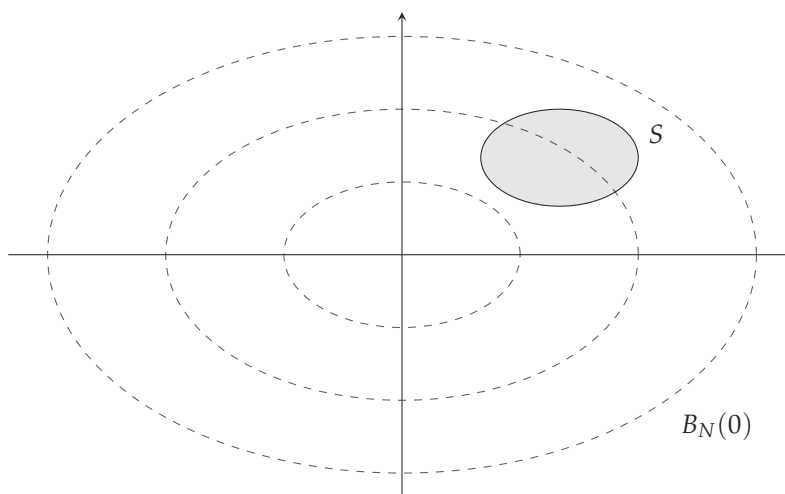
Now we show that it is closed. Take any $x \in S^C = \mathbb{R}^N \setminus S$, the complement of S in $\mathbb{R}^N$. Define the collection

$$\mathcal{F} = \left\{ \mathcal{F}_n = \left(\text{cl}(B_{\frac{1}{n}}(x)) \right)^C \right\}_{n \in \mathbb{N}}$$

that is, the **complement** of the collection of closed balls of radius $\dfrac{1}{n}$ around x. A graphical example in $\mathbb{R}^2$:

Note we need the complements of the *closed* balls because we want the sets in the collection to be open.

Now $\text{cl}(B_{\frac{1}{n}}(x)) \to \{x\}$, so

$$\bigcup_{n \in \mathbb{N}} \left(\text{cl}(B_{\frac{1}{n}}(x)) \right)^C = \mathbb{R}^N \setminus \{x\}$$

That is, the union of the complements converges to the entire space *except* for x. Since $S \subseteq \mathbb{R}^N \setminus \{x\}$ ($S \subseteq \mathbb{R}^N$ and $x \in S^C \implies x \notin S$), then $\mathcal{F}$ is an open cover of S. By compactness of S, we know that it admits a finite sub-cover. Note

$$B_{\frac{1}{n+1}}(x) \subseteq B_{\frac{1}{n}}(x) \implies \left(\text{cl}(B_{\frac{1}{n}}(x)) \right)^C \subseteq \left(\text{cl}(B_{\frac{1}{n+1}}(x)) \right)^C$$
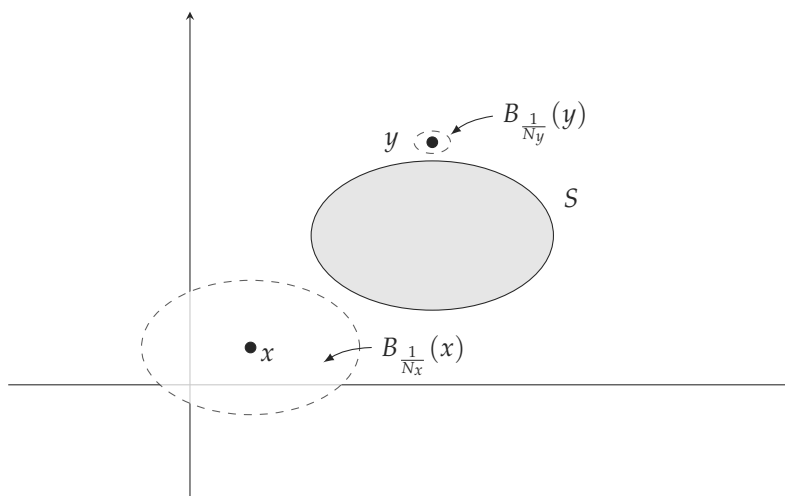
Hence any finite union gives

$$\bigcup_{n \leq N} \left(B_{\frac{1}{n}}(x) \right)^C = \left(\text{cl}(B_{\frac{1}{N}}(x)) \right)^C$$

Since $S \subseteq \left(\text{cl}(B_{\frac{1}{N}}(x)) \right)^C$, it must be that $B_{\frac{1}{N}}(x) \subseteq S^C$. Finally, we can say $\forall x \in S^C \; \exists \varepsilon > 0$ (any $\varepsilon < 1/N$) s.t.

$$B_\varepsilon(x) \subseteq S^C$$

which is the definition of an open set. This shows S^C is open, so S is closed. Graphically, we see that at any point x outside of the set we can construct a ball of radius $1/N_x$ for some N_x that is entirely outside of S:


$\square$

The other direction follows from Theorems 3 and 4 below. If a set S is bounded in $\mathbb{R}^N$ then it is the subset of some N-dimensional cube. Once we show the cube is compact, Theorem 4 gives that S is compact. For the curious, I sketch the proof here.

Theorem 3. $\forall -\infty < a < b < \infty$, the N-dimensional cube $[a, b]^N$ is compact.

Proof. Suppose $-\infty < a < b < \infty$ and let $S = [a, b]^N$ be such that for some open cover $\mathcal{F}$ of S there does not exist a finite sub-cover. Bisect S into 2^N equal closed hypercubes with planes parallel to the faces of S (by the way, this is why I call this proof a "sketch," as I have not defined hypercube, plane, parallel, or face). While that sounds fairly complicated, we can visualize it in 3-dimensional space:
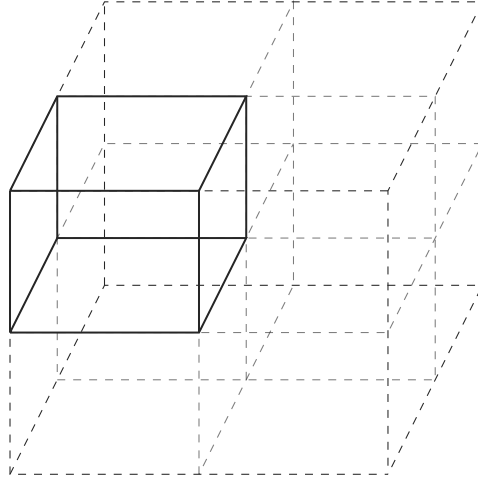
Note that at least one of the hypercubes cannot admit a finite sub-cover (maybe none of them admit a finite sub-cover, but we only need one). Let this cube be C_1. We proceed by induction, and now take C_m to be our main cube, and construct C_{m+1} by partitioning C_m into 2^N equal closed cubes where at least one non-empty cube does not admit a finite-sub-cover.

1. C_m are closed.

2. C_m are non-empty. If at any point all sub-hypercubes of C_m are empty, then C_m itself is empty, a contradiction (since we started with a non-empty hypercube).

3. $C_{m+1} \subset C_m$

4. Let δ be the maximum distance between any two points in S. The maximum distance in C_m is $\dfrac{\delta}{2^m}$.

5. C_m is not covered by any finite sub-cover of $\mathcal{F}$. Since $\mathcal{F}$ covers C_m, if all sub-hypercubes of C_m admit a finite sub-cover, the union of these will be a finite sub-cover of C_m, a contradiction.

For each m, let x_m be any element of C_m. This sequence is Cauchy: For any $\varepsilon > 0$,

$$k, l > M > \frac{\log(\delta) - \log(\varepsilon)}{\log(2)} \implies d(x_k, x_l) < \frac{\delta}{2^M} < \varepsilon$$

Because $\mathbb{R}^N$ is complete, we know that $x_m \to x$ for some $x \in \mathbb{R}^N$. Therefore,[1]

$$x \in C_m \quad \forall m$$

Since $\mathcal{F}$ has an infinite sub-cover of C_m (which might be comprised of all the sets in $\mathcal{F}$), $x \in \mathcal{F}_\omega$ for some set in that infinite sub-cover. $\mathcal{F}_\omega$ is open, so for some $\varepsilon > 0$,

$$d(y, x) < \varepsilon \implies y \in \mathcal{F}_\omega$$

But $x \in C_m$ for any m, and the maximum distance between any two points in C_m is $\delta/2^m$. Hence for M s.t.

$$M > \frac{\log(\delta) - \log(\varepsilon)}{\log(2)} \implies d(x, y) < \varepsilon \ \forall y \in C_M \implies y \in \mathcal{F}_\omega \implies C_M \subseteq \mathcal{F}_\omega$$

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[1] This uses the fact that each C_m is closed and $x_{m+n} \in C_m$ for all n; hence the limit is also in C_m.

$\{\mathcal{F}_\omega\}$ is a finite sub-cover of C_M, a contradiction. Therefore $[a,b]^N$ is compact. ☐

Theorem 4. *Any closed subset of a compact set is compact.*

Proof. X be compact and S be a closed subset of X. Let $\mathcal{F}$ be any open cover of S and consider

$$\mathcal{G} = \mathcal{F} \cup \mathbb{R}^N \setminus S$$

Since S is closed, $\mathbb{R}^N \setminus S$ is open. Since $\mathcal{F}$ is an open cover of S, $\mathcal{G}$ is therefore an open cover of $S \cup (\mathbb{R}^N \setminus S) = \mathbb{R}^N \supseteq X$. Since X is compact, $\mathcal{G}$ has a finite sub-cover $\{\mathcal{G}_n : n \in \mathbb{N}, n \leq N\}$ s.t.

$$S \subseteq X \subseteq \bigcup_{n=1}^{N} \mathcal{G}_n$$

The only set in $\mathcal{G}$ that is not in $\mathcal{F}$ is $\mathbb{R}^N \setminus S$, but by definition $S \not\subseteq \mathbb{R}^N \setminus S$. Hence $S \subseteq \left(\cup_{n=1}^{N} \mathcal{G}_n\right) \setminus \{\mathbb{R}^N \setminus S\}$, which means $\{\mathcal{G}_n : n \in \mathbb{N}, n \leq N\} \setminus \{\mathbb{R}^N \setminus S\} \subseteq \mathcal{F}$ is a finite sub-cover of S. ☐

1.3. Weierstrass Extreme Value Theorem (EVT).

Theorem 5. *Let $S \subseteq \mathbb{R}$ with $S \neq \varnothing$ be a compact set; then S has a minimum and a maximum.*

Proof. Since S is compact, it is closed and bounded. Since it is bounded, $\sup S$ exists. Suppose $\sup S \notin S$. Since S is closed, we can find some $\varepsilon > 0$ s.t. $B_\varepsilon(\sup S) \cap S = \varnothing$. However, any such $y \in B_\varepsilon(\sup S)$ has the property that

$$\sup S - \varepsilon < y < \sup S + \varepsilon$$

Further, we know that $x \in S \implies x \notin B_\varepsilon(\sup S)$. Hence

$$x < \sup S - \varepsilon < y < \sup S < \sup S + \varepsilon$$

for every $x \in S$ and some $y \in B_\varepsilon(\sup S)$, which means that y is an upper bound for S that is smaller than $\sup S$, a contradiction. Thus $\max S = \sup S$; the proof for $\min S = \inf S$ is analogous. ☐

Theorem 6. *Let $f : S \to T$ be a continuous function. If S is compact, then $f(S)$ is compact.*

Proof. Take any open cover of $f(S)$:

$$\mathcal{F} = \{\mathcal{F}_\omega : \omega \in \Omega\} \quad \text{with} \quad f(S) \subseteq \bigcup_{\omega \in \Omega} \mathcal{F}_\omega$$

Consider the inverse-image of each set in the open cover:

$$f^{-1}(\mathcal{F}) = \{f^{-1}(\mathcal{F}_\omega) : \omega \in \Omega\}$$

For each $s \in S$, we know $f(s) \in f(S)$, and in turn for each $f(s) \in S$ there is some ω s.t. $f(s) \in \mathcal{F}_\omega$, which implies $s \in f^{-1}(\mathcal{F}_\omega)$. In other words $f^{-1}(\mathcal{F})$ covers S. Since f is continuous, we know the pre-image of open sets is open, meaning this is an open cover. Since S is compact, it admits a finite sub-cover:

$$\mathcal{G} = \{f^{-1}(\mathcal{F}_{\omega_i}) : i = 1, \ldots, N\} \quad \text{with} \quad S \subseteq \bigcup_{i=1}^{N} f^{-1}(\mathcal{F}_{\omega_i})$$

The image of a finite union of sets is just the union of their individual images. Hence

$$f(S) \subseteq f\left(\bigcup_{i=1}^{N} f^{-1}(\mathcal{F}_{\omega_i})\right) = \bigcup_{i=1}^{N} f\left(f^{-1}(\mathcal{F}_{\omega_i})\right) = \bigcup_{i=1}^{N} \mathcal{F}_{\omega_i} \cap f(S) \subseteq \bigcup_{i=1}^{N} \mathcal{F}_{\omega_i}$$

$\mathcal{F}$ was arbitrary and we found a finite sub-cover $\{\mathcal{F}_{\omega_i} : i = 1, \ldots, N\}$. By definition $f(S)$ is compact. (We remark that we need to write $\mathcal{F}_{\omega_i} \cap f(S)$ because the image of the pre-image of an arbitrary set need not be the set itself! Even if, in this case, we know the pre-image is non–empty, there is no reason why every element in the set will map to an element in S or even in S. For example, let $f(x) = x$ with $S = (0, 3)$. Note $f([1, 2]) = [1, 2] \subseteq (0, 4)$, but $f(f^{-1}((0, 4))) = f((0, 3)) = (0, 3) \neq (0, 4)$.) $\qquad\square$

Theorem 7 (Weierstrass' EVT). *If S is a compact set $\varphi : S \to \mathbb{R}$ is continuous then $\exists x, y$ s.t. $\varphi(x) = \sup \varphi(S)$ and $\varphi(y) = \inf \varphi(S)$.*

Proof. The **EVT** follows directly from other theorems in this section. Since S is compact and φ continuous, $\varphi(S)$ is compact. Since $\varphi(S) \subseteq \mathbb{R}$ is compact, it has a minimum and a maximum. $\qquad\square$

Application to Economics   Consider a standard utility maximization problem

$$\max_{x \in B(p,w)} u(x)$$

with $B(p, w) = \{x : p \cdot x \leq w\}$ and $x, p \in \mathbb{R}_+^N$. $B(p, w)$ is closed and bounded, so if $u(x)$ is continuous the maximum exist and the problem has a solution at somee $x^* \in B(p, w)$.

1.4. Using Sequential Definitions. The idea here is to show examples of how to construct sequences in a way that helps when doing proofs. We saw these proofs already without using sequences; however, we we have seen that various definitions often have a sequential version, so let us see how they might help.

1. Let us show if $f : S \to T$ is continuous function, then S is compact implies $f(S)$ compact.

    Proof. • Take any sequence $y_m \in f(S)$; we know $\forall m \; \exists x_m \in S$ s.t. $f(x_m) = y_m$.

    • S is compact, so it is sequentially compact; that is, $\exists x_{m_k} \to x$ for some $x \in S$.

    • f is continuous, so $y_{m_k} = f(x_{m_k}) \to f(x) \in f(S)$. Let $y \equiv f(x)$.

    • Hence $\forall y_m \in f(S) \; \exists y_{m_k} \to y$ for some $y \in S$.

    By definition, $f(S)$ is sequentially compact, which means it is compact. $\qquad\square$

2. Let us show Weierstrass' EVT:

    Proof. • Since S is compact and φ is continuous, $\varphi(S)$ is compact.

    • $\varphi(S)$ is compact, so it is closed and bounded.

    • $\varphi(S)$ bounded means $-\infty < \inf \varphi(S) \leq \sup \varphi(S) < \infty$.

    • By definition of $\sup \varphi(S)$, $\forall \varepsilon_m = 1/m \; \exists z_m \in \varphi(S)$ s.t. $\sup \varphi(S) - \varepsilon < z_m \leq \sup \varphi(S)$ (if not, then $\sup \varphi(S) - \varepsilon$ would be the sup, contradiction). Note $z_m \to \sup \varphi(S)$.

    • $\varphi(S)$ closed means it has all its limits, so $\sup \varphi(S) \in \varphi(S)$. Hence $\exists x \in S$ s.t. $x = \sup \varphi(S)$.

- For the inf, construct a sequence $z_m \in \varphi(S)$ s.t. $\inf \varphi(S) \le z_m < \inf \varphi(S) + \varepsilon$. $z_m \to \inf \varphi(S)$ so $\inf \varphi(S) \in \varphi(S)$, and $\exists y \in S$ s.t. $\varphi(y) = \inf \varphi(S)$.

Therefore φ attains its $\sup$ and its $\inf$. $\square$

1.5. Review. I think focusing on the *properties* of compactness can be more important than all the proofs above. Further, since we'll typically work with the reals, I think the intuition of compactness as equivalent to closed and bounded is fine (certainly for this course).

Table 1: Compactness! What is it good for? Actually, quite a bit.

S is compact:	
Definition	For any **open cover** there exists a **finite sub-cover**.
	$\forall \mathcal{O} = \{O_\omega : \omega \in \Omega\}$ open cover $\exists W \subseteq \Omega$ s.t. W finite and $S = \bigcup_{\omega \in W} O_\omega$
Characterization	$\iff$ **sequentially compact**: Any sequence has a convergent subsequence.
	$\forall (x_m) \in S \;\; \exists x \in S$ and (x_{m_k}) s.t. $x_{m_k} \to x$.
Implications	$\implies$ S is **closed and bounded**.
	$\implies$ any **closed subset** of S is compact.
	$\implies$ $f(S)$ is compact for any continuous f.
	$\implies$ $f(S)$ has a min and a max for any continuous f (**EVT**).
Heine-Borel	**In Euclidean space only** $(\mathbb{R}^N)$: $\iff$ S is **closed and bounded**.

- If S is compact, then I can construct an **arbitrary collection of open sets** that contains S, and I know I will get **something finite** out of it.

- If S is compact, then I can construct an **arbitrary sequence** in S, and I know I will get something **convergent** out of it.

Finally, I wanted to make a note about why continuity is additionally required to get maxima and minima. It's easiest to visualize with real functions: Consider $f(x) = 1/x$ if $x \ne 0$ and $f(x) = 0$ if $x = 0$. This is not continuous, and does not have a min or a max on, say, $[-1, 1]$, which is a compact set. Visually:
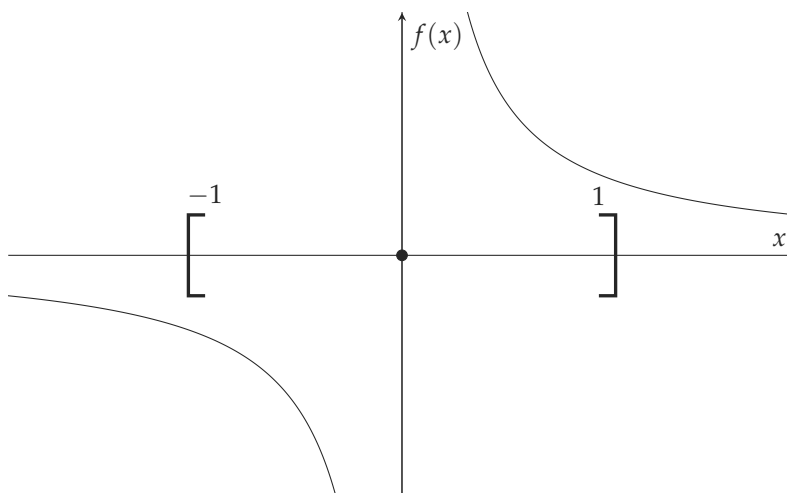
The set is compact, but the function diverges to ∞ as it approaches 0 from the right, and to $-\infty$ as it approaches 0 from the left.

2. Correspondences

2.1. "Set–Valued Functions". A correspondence, denoted $\Gamma : X \rightrightarrows Y$, assigns a non–empty subset of Y to each point in X. In a sense, a correspondence is a "set-valued function" with "input" $x \in X$ and "output" is $\Gamma(x) \subseteq Y$.[2] Some terminology is completely analogous relative to when we were working with functions:

- X is the **domain** and Y is the **co–domain**.

- $\forall S \subseteq X$ let $\Gamma(S) \equiv \bigcup \{\Gamma(x) : x \in S\}$ is the **image** of S.

- $\Gamma(X)$ is the range, and if $\Gamma(X) = Y$ we say Γ is **surjective**.

Here's the first roadblock: What would it mean for a correspondence to be *injective*? For functions, we want to capture the idea of *one-to-one*. A correspondence, however, starts from the premise that a mapping can be one to many. Is there an analogous idea that we *should* try to capture? We leave this question unanswered as an example of why we need to be specially careful when dealing with correspondences.

Example 1. If $\Gamma(X) \subseteq X$ then we say it's a "self-correspondence." An example of such a correspondence is the choice correspondence from utility maximization.

$$\operatorname*{argmax}_{\mathbb{R}^N_+} u(x) \quad \text{s.t.} \quad p \cdot x \leq w$$

We can go a step further and also define the problem over correspondences. Let

$$\Gamma(p, w) = \{x \in \mathbb{R}^+ : p \cdot x \leq w\}$$

be the budget correspondence. Then we can define the argmax correspondence as

$$\operatorname*{argmax} u(x) \quad \text{s.t.} \quad x \in \Gamma(p, w)$$

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[2]Conversely, functions are "singleton–valued correspondences," where $f(x)$ is equivalent to the correspondence $\Gamma(x) = \{f(x)\}$.

Why go through the trouble? The idea is that if we can prove enough theorems and properties of correspondences, re-expressing some problems we're familiar with in terms of correspondences might make how to solve them and what their properties are more transparent. □

Remark 2. Since correspondences map points to sets, it is typical to refer to correspondences as [property]-valued, where [property] is any property of a set. For example, they can be closed-valued, compact-valued, convex-valued, and so on. □

2.2. Inverse Images. With functions we had a nice characterizations of continuity: f^{-1} the inverse image maps open sets to open sets. What is the analogue for Γ^{-1}?

- $f^{-1}(O) = \{x \in X : \{f(x)\} \subseteq O\}$.

- $f^{-1}(O) = \{x \in X : \{f(x)\} \cap O \neq \varnothing\}$.

These are analogous for functions, but for correspondences it defines two distinct sets, the *upper* inverse image and the *lower* inverse image, which will give rise to two different notions of continuity:

Definition 4. Given a correspondence $\Gamma : X \rightrightarrows Y$

- $\Gamma^{-1}(O) \equiv \{x \in X : \Gamma(x) \subseteq O\}$ is the **upper inverse image**.

- $\Gamma_{-1}(O) \equiv \{x \in X : \Gamma(x) \cap O \neq \varnothing\}$ is the **lower inverse image**.

Note that $\Gamma(x) \neq \varnothing$ and $\Gamma(x) \subseteq O \implies \Gamma(x) \cap O \neq \varnothing$. So necessarily $\Gamma^{-1}(O) \subseteq \Gamma_{-1}(O)$.

Figure 2: Visualizing the Upper and Lower Inverse Image
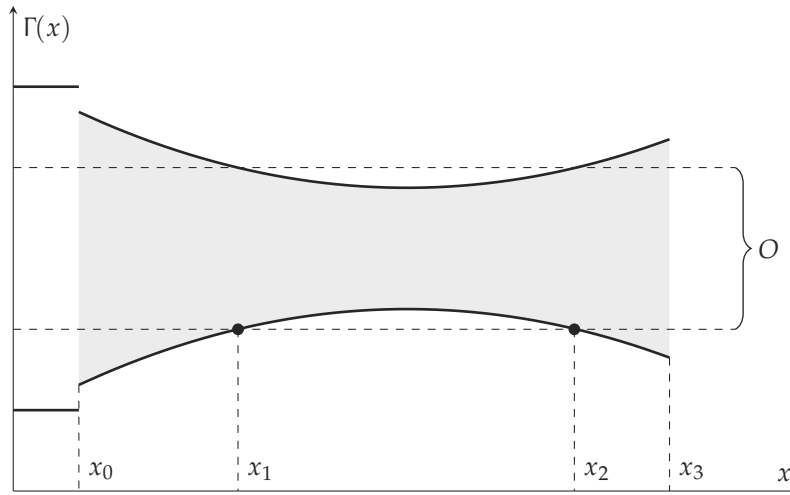
In the image above, *every* point $x \in [x_0, x_3]$ is s.t. $\Gamma(x) \cap O \neq \varnothing$, so $\Gamma_{-1}(O) = [x_0, x_3]$; however, not every point is s.t. $\Gamma(x) \subseteq O$. In this case, only points $y \in (x_1, x_2)$ are s.t. $\Gamma(y) \subseteq O$, so $\Gamma^{-1}(O) = (x_1, x_2)$. Last, if $z \in [0, x_0)$ then $\Gamma(z)$ is neither contained in nor intersects with 0.

2.3. Hemicontinuity. We present two distinct definitions of continuity. If we use the *upper* inverse image:

Definition 5. $\Gamma : X \rightrightarrows Y$ is **upper hemi-continuous** (uhc) if whenever $O \subseteq Y$ is open, $\Gamma^{-1}(O)$ is also open.

If $\Gamma(x) \subseteq O$ then $x \in \Gamma^{-1}(O)$; if $\Gamma^{-1}(O)$ is open $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq \Gamma^{-1}(O)$, so $\Gamma(B_\delta(x)) \subseteq O$. Therefore we have the following equivalent definition of uhc.

Definition 6. $\Gamma: X \rightrightarrows Y$ is uhc iff $\forall O \subseteq Y$ open with $\Gamma(x) \subseteq O$ $\exists \delta > 0$ s.t. $\Gamma(B_\delta(x)) \subseteq O$.

We can similarly define continuity in terms of the *lower* inverse image instead:

Definition 7. $\Gamma: X \rightrightarrows Y$ is ***lower hemi–continuous*** (lhc) if whenever $O \subseteq Y$ is open, $\Gamma_{-1}(O)$ is also open.

If $\Gamma(x) \cap O \neq \varnothing$ then $x \in \Gamma_{-1}(O)$; if $\Gamma_{-1}(O)$ is open $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq \Gamma_{-1}(O)$, so $z \in B_\delta(x) \implies \Gamma(z) \cap O \neq \varnothing$. Therefore we can equivalently write the following definition:

Definition 8. $\Gamma: X \rightrightarrows Y$ is lhc iff $\forall O \subseteq Y$ open with $\Gamma(x) \cap O \neq \varnothing$ $\exists \delta > 0$ s.t. $\Gamma(z) \cap O \neq \varnothing$ $\forall z \in B_\delta(x)$.

- Intuitively, if Γ is uhc at x and z is "close" to x, every point in $\Gamma(z)$ will be "close" to some point in $\Gamma(x)$.

  If there is some neighborhood around x s.t. every open set *containing* $\Gamma(x)$ also contains $\Gamma(z)$ for z in the neighborhood, then nothing in $\Gamma(z)$ can be suddenly "far" from the all values of x.

- By contrast, if Γ is lhc at x and z "close" to x, each point in $\Gamma(x)$ will be "close" to some point in $\Gamma(z)$.

  Intersections, unlike containment, can happen at *any* point. Hence lhc *does not* require every point in $\Gamma(z)$ to always be close to $\Gamma(x)$; rather, it requires *every* point in $\Gamma(x)$ to be close to *some* point in $\Gamma(z)$.

We will formalize the intuition above when we discuss the sequential definition of uhc and lhc.

Figure 3: Upper and Lower Hemicontinuity (Inverse Images)

Lower but not upper hemicontinuous

Upper but not lower hemicontinuous
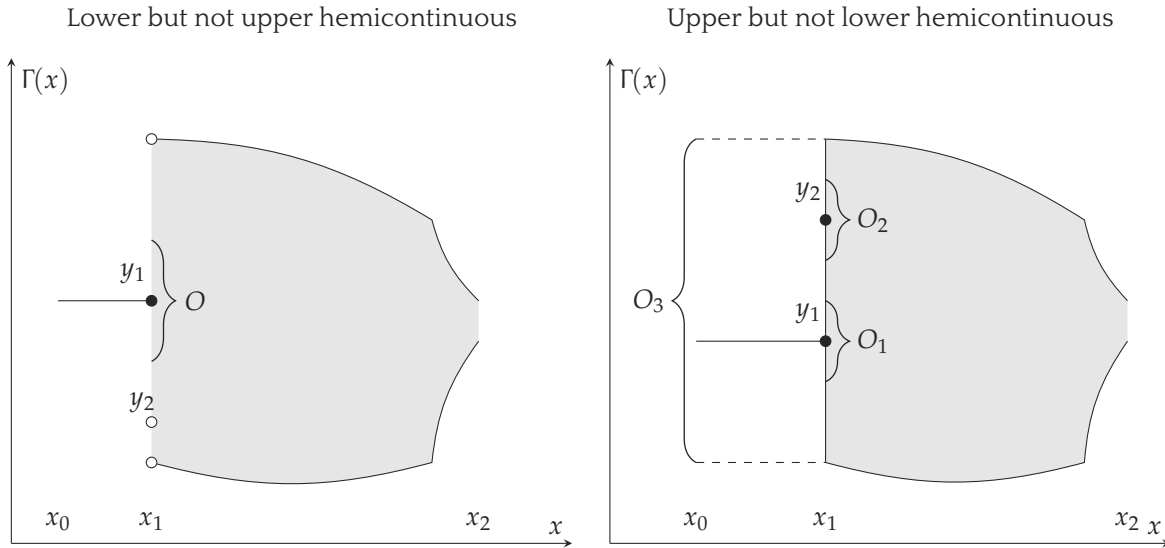
Let $\Gamma: [x_0, x_2] \rightrightarrows \mathbb{R}$ be as depicted above:

- In the left figure, $\Gamma(x_1) = y_1 \subseteq O$; however, no matter how small the δ $y_2 \in \Gamma(x_1 + \delta) \notin O$, meaning the set values of points near x_1 always have elements far away from $\Gamma(x_1)$. Hence it cannot be uhc. Formally, $\Gamma^{-1}(O) = [x_0, x_1]$, which is not open.

  But it is lhc: The set values of every point near x_1 will intersect O, and generally every set intersecting $\Gamma(x_1)$. Formally, for any such set $\Gamma_{-1}(O) = [x_0, x_2]$ (note $X = [x_0, x_2]$, and X is open relative to X).

- In the right figure, $y_1 \in \Gamma(x_1) \cap O_1 \neq \emptyset$, and every point around x_1 will also intersect O_1. *However*, $y_2 \in \Gamma(x_1) \cap O_2 \neq \emptyset$, but no matter how small the δ, $\Gamma(x_1 - \delta) \cap O_2 = \emptyset$, meaning not every element in $\Gamma(x_1)$ is near the set-values of points near x_1. Formally, $\Gamma_{-1}(O_2) = [x_1, x_2]$, which is not open in $[x_0, x_2]$.

  But it is uhc: O_3, and generally any set containning all of x_1, will also contain the set value of every point around x_1. Formally, $\Gamma^{-1}(O_3)$, or any such set, is $[x_0, x_2]$, the space itself, which is open.

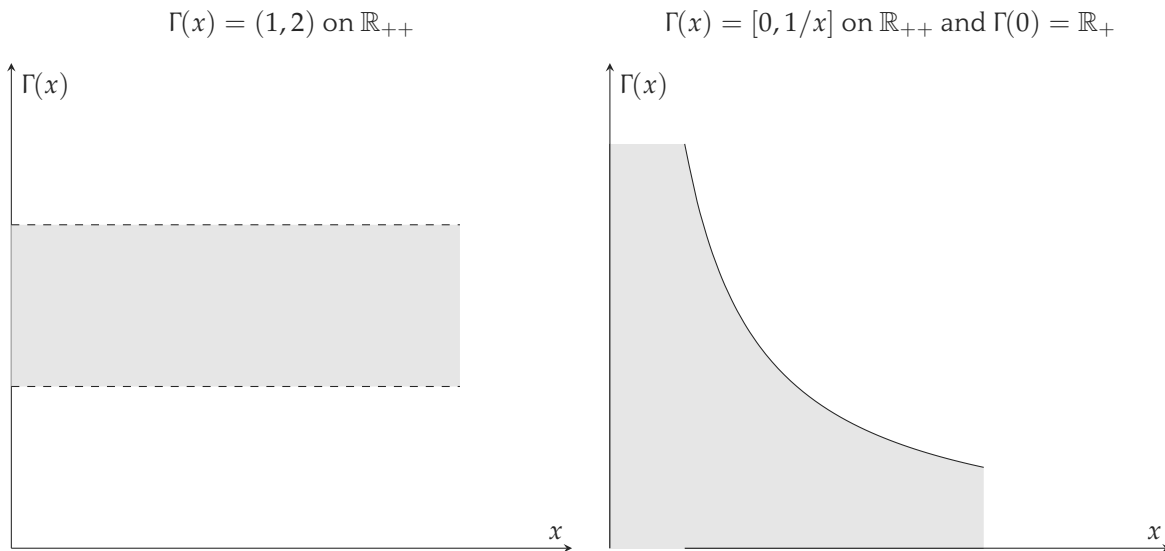### 2.4. Sequential Characterization of Hemicontinuity.

Remark 3. It not uncommon to encounter the sequential characterizations as the definition (in fact the very first time I learned what a correspondence was, I only encountered the sequential characterization of hemicontinuity).   □

Theorem 8. $\Gamma: X \rightrightarrows Y$. *If* $\forall (x_m) \in X, (y_m) \in Y$ *s.t.* $x_m \to x, y_m \in \Gamma(x_m)$ $\exists y_{m_k} \to y$ *for some* $y \in \Gamma(x)$, *then* Γ *is uhc at* x. *If* Γ *is compact-valued, the converse is also true.*

Some remarks:

- The definition says that for every sequence converging to x and every sequence in the set-values of x_m, $y_m \in \Gamma(x_m)$, there is a convergent sub-sequence to an element $y \in \Gamma(x)$.

  Recall our intuition for uhc: Every point in $\Gamma(z)$, for z sufficiently "close" to x, is also "close" to *some* point of $\Gamma(x)$. This closely mirrors the sequential definition: If every time we get arbitrarily close to x (that is, $x_m \to x$) *every* point in those set values (arbitrary $y_m \in \Gamma(x_m)$) will be arbitrarily close to *some* point in $\Gamma(x)$ (there is some sub-sequence $y_{m_k} \to y \in \Gamma(x)$, which means that *every* sequence in $\Gamma(x_m)$ has infinitely many points near *some* value of $\Gamma(x)$).

- So why doesn't the converse hold? The above sequential definition requires the function to be uhc, but uhc correspondences don't have to be closed *or* bounded:

<div align="center">Figure 4</div>

<div align="center">$\Gamma(x) = (1, 2)$ on $\mathbb{R}_{++}$ $\Gamma(x) = [0, 1/x]$ on $\mathbb{R}_{++}$ and $\Gamma(0) = \mathbb{R}_+$</div>

  ○ The figure on the left is uhc: Since the function is constant, if $\Gamma(x) \subseteq O$ for any O, then $\Gamma(y) = \Gamma(x) \subseteq O$ for every $y \in (0, \infty)$. Thus $\Gamma^{-1}(O) = \mathbb{R}_{++}$. However, for $1/n \to 0$ and $1 + 1/n \in \Gamma(1/n)$, $1 + 1/n \to 1 \notin \Gamma(0) = (1, 2)$.

- ○ The figure on the right is also uhc: Take any open $O \subset \mathbb{R}_+$ s.t. $\Gamma(x) \subseteq O$ and we have $\Gamma^{-1}(O) = (x, \infty)$. Last, if $\Gamma(0) \subseteq O \subseteq \mathbb{R}$ then $O = \mathbb{R}_+$, and $\Gamma^{-1}(\mathbb{R}_+) = [0, 1] = X$ (and, again, X is open relative to itself). However, take $1/n \to 0$ and $n \in \Gamma(1/n); n \to \infty \notin \mathbb{R}_+$.

- Therefore uhc is not enough to guarantee that a sequence always exists. More precisely, it's not so much that we need the set-values of Γ to be closed and bounded: We need them to be compact, and that will guarantee the existence of a sub-sequence. (Recall here the link between compactness and sequential compactness, which we will use in our proof to guarantee the existence of a convergent sub-sequence; being closed and bounded, however, is only equivalent to compactness in Euclidean space. The latter is nice for exposition but the actual technical requirement is compactness.)

Theorem 9. $\Gamma : X \rightrightarrows Y$ *is lhc at* $x \in X \iff \forall (x_m) \in X$ *s.t.* $x_m \to x \in X$ *and* $\forall y \in \Gamma(x) \; \exists (y_m) \in Y$ *s.t.* $y_m \to y$ *and* $y_m \in \Gamma(x_m)$.

Given both sequential definitions of uhc and lhc, we repeat our intuition:

1. If Γ is uhc at x then every point $y \in \Gamma(z)$, for z arbitrarily close to x, is itself close to some point in $\Gamma(x)$. The limit definition follows this closely: As $x_m \to x$, every sequence $y_m \in \Gamma(x_m)$ will have inifinitely many elements arbitrarily close to *some* point in $\Gamma(x)$, so there will be *some* sub-sequence $y_{m_k} \to y \in \Gamma(x)$ (with the caveat that Γ is compact-valued).

2. If Γ is lhc at x and z is arbitrarily close to x, then every point $y \in \Gamma(x)$ must be arbitrarily close to *some* point in $\Gamma(z)$. Again, the limit definition follows this: As $x_m \to x$, every point $y \in \Gamma(x)$ will be arbitrarily close to *some* point in $\Gamma(x_m)$, so there will exist a sequence $y_m \in \Gamma(x_m)$ s.t. $y_m \to y$.

Figure 5: Upper and Lower Hemicontinuity (Sequences)

Lower but not upper hemicontinuous   Upper but not lower hemicontinuous
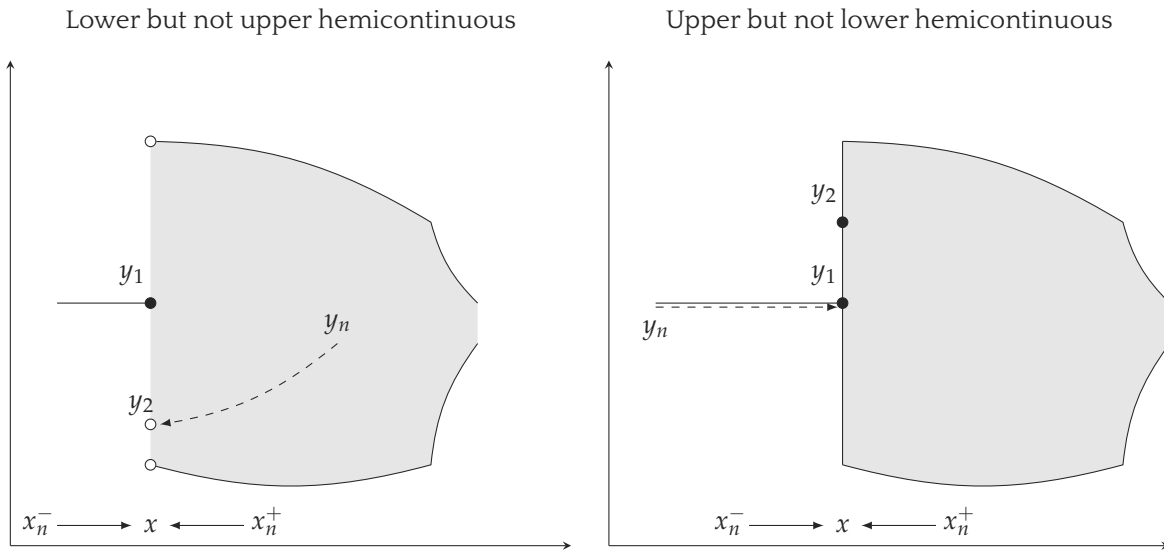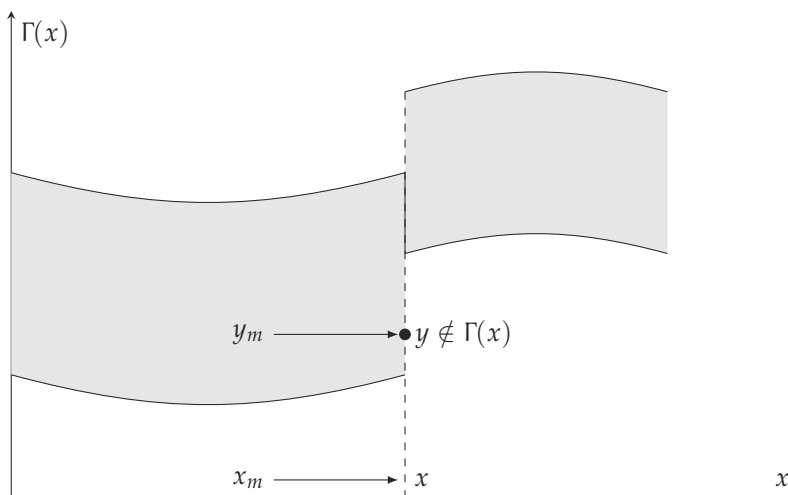
2.5. Closed Graph.

Definition 9. The ***graph*** of a correspondence $\Gamma : X \rightrightarrows Y$, denoted $Gr(\Gamma)$, is

$$Gr(\Gamma) \equiv \{(x, y) \in X \times Y : y \in \Gamma(x)\}$$

Γ has a closed graph if …its graph is closed.

Definition 10. Γ is **closed** at $x \in X$ if $\forall (x_m) \in X^\infty, (y_m) \in Y^\infty$ s.t. $x_m \to x \in X, y_m \to y \in Y$, and $y_m \in \Gamma(x_m)$ we also have $y \in \Gamma(x)$. Γ has a **closed graph** if it is closed at every $x \in X$.

Figure 6: Not Closed Graph

The idea is that *every* series in $\Gamma(x_m)$ that converges will converge to a point in $\Gamma(x)$ (if $x_m \to x$).

Remark 4. Closed graph is **not** the same as closed-valued!

The example in Figure 6 above is closed-valued but not closed graph, so the former need not imply the latter. Further, a closed graph is not the same as uhc: A uhc correspondence doesn't have to be closed-valued, which would mean that it would not have a closed graph (if $\Gamma(x)$ is not closed-valued, then there is some sequence in $\Gamma(x)$ that converges to a point outside of $\Gamma(x)$, contradicting the definition of closed graph).

Conversely, a correspondence can have a closed graph with a discontinuity. $\Gamma(x) = \{1/x\}$ if $x > 0$ and $\Gamma(x) = \{0\}$ if $x = 0$. Note that any sequence in $\Gamma(x)$ as $x \to 0^+$ will diverge, so there is no contradiction of the closed graph property. However, there is clearly a discontinuity at 0. □

We can, however, say that closed graph, closed-valued, and uhc are related:

Claim 1. 1. *If $\Gamma : X \rightrightarrows Y$ has a closed graph and Y is compact then Γ is uhc.*

 2. *If Γ is uhc and closed-valued then it has a closed-graph.*

3. Fixed Point Theorems

Definition 11. A self-map $f : S \to S$ has a **fixed point** if $\exists x^* \in S$ s.t. $x^* = f(x^*)$.

Theorem 10 (Brouwer's FPT). *Take any $S \subseteq \mathbb{R}$ compact, convex, and non-empty. If $f : S \to S$ is continuous then it has a fixed point.*

Definition 12. A self-map correspondence $\Gamma : S \rightrightarrows S$ has a **fixed point** if $\exists x^* \in S$ s.t. $x^* \in \Gamma(x^*)$.

Definition 13. A set S is ***convex*** if $\forall x, y \in S$ and $\forall \alpha \in [0, 1]$ we have $\alpha x + (1 - \alpha)y \in S$.

Theorem 11 (Kakutani's FPT). *Take any $S \subseteq \mathbb{R}^N$ compact, convex-valued, and non-empty. If a correspondence $\Gamma : S \rightrightarrows S$ is upper hemicontinuous then it has a fixed point.*
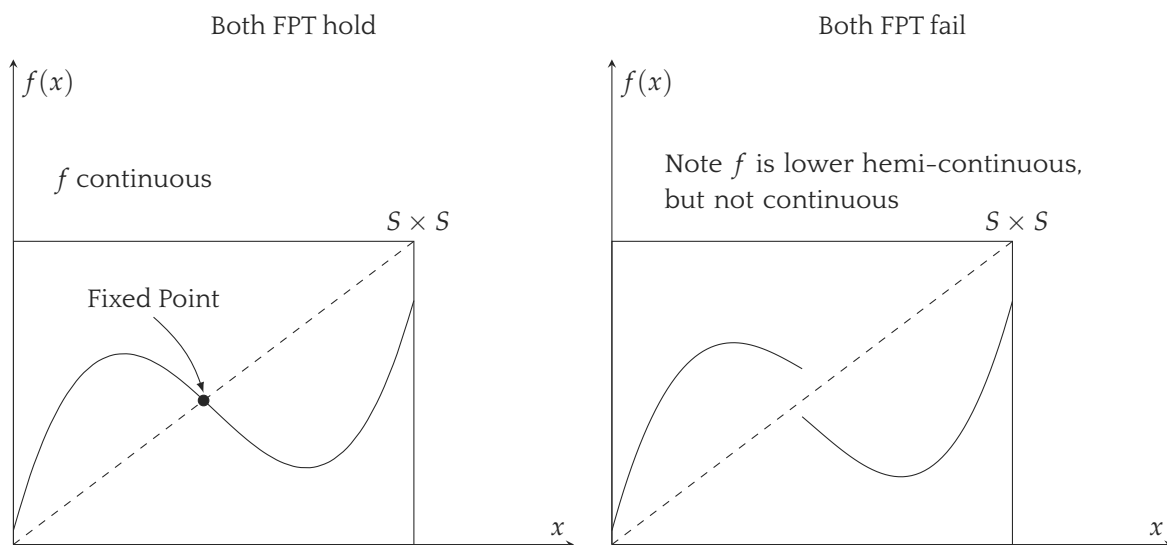

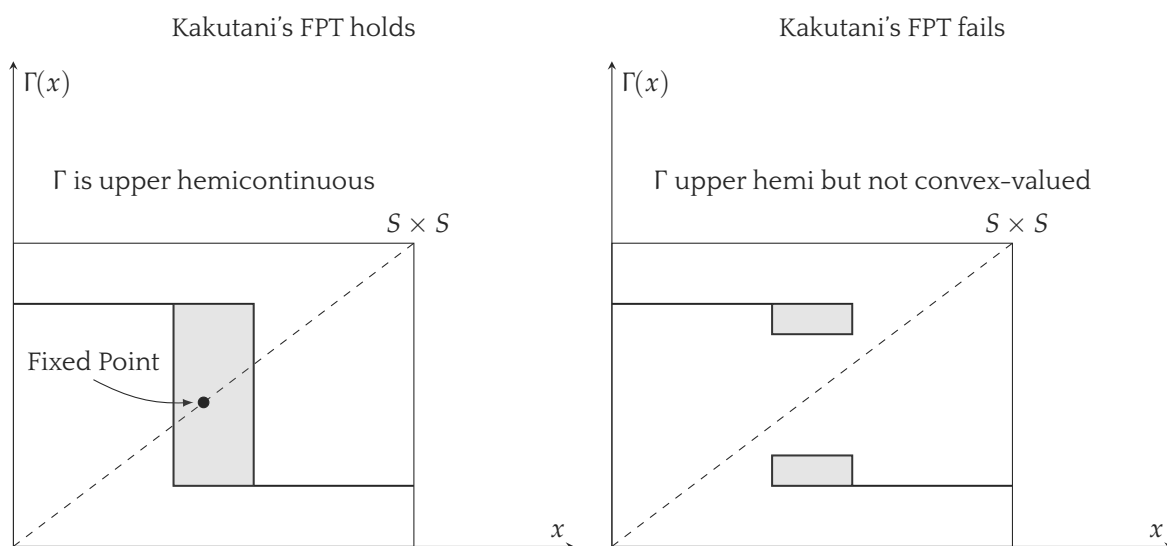Figure 7: Examples of when f does or not have a fixed point

Figure 8: Examples of when Γ does or not have a fixed point

Application to Economics   The proof of the existence of a Nash Equilibrium in game theory is an application of Kakutani's fixed point theorem.

- We have N players, $1, 2, \ldots, N$ and the corresponding strategy sets, $S_1, \ldots, S_N$.

- Let $s = (s_1, \ldots, s_N)$ be any collection of strategies from all players, with $s_i \in S_i$.

- Let $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N)$ the collection of strategies from all players other than i.

- For each player i we can define a *best-response* function to the strategies of other players,

$$b_i(s_{-i}) = \underset{s_i \in S_i}{\operatorname{argmax}}\, u_i(s_i; s_{-i})$$

the utility-maximizing strategy for i given the other player's strategies.

- Let $b(s) = (b_1(s_{-1}), \ldots, b_N(s_{-N}))$ be the collection of best-response strategies from all players.

A Nash Equilibrium is then defined as a set of strategies such that no player has an incentive to deviate. That is, the strategy s_i^* chosen by player i is in the set of best responses to all the other strategies, s_{-i}^*, or $s_i^* \in b_i(s_{-i}^*)\ \forall i$. We can hence express a Nash Equilibrium s^* as a fixed point of b,

$$s^* \in b(s^*)$$

If S_i are compact, non-empty, and convex-valued, and u_i are continuous and quasiconcave, then we will be able to apply Kakutani's fixed point theorem to show that $s^* \in b(s^*)$ for some $s^* \in S = \prod_{i=1}^{N} S_i$ (with some work in between to show that $b : S \to S$ is upper hemicontinuous).

Keywords