

Problem Set 3

1. Take a collection of functions with $f_i : \Omega \rightarrow \mathbb{R}^N$, $\Omega \subseteq \mathbb{R}^M$, $i \in \mathbb{N}$. The collection $\{f_i\}$ defines a sequence of functions, and for each $x \in \Omega$ we have a possibly different sequence $\{f_i(x)\}$ in \mathbb{R}^N .

Let $\{f_i\}$ be a sequence of functions, with $f_i : \Omega \rightarrow \mathbb{R}^N$ and $\Omega \subseteq \mathbb{R}^M$. We say that $\{f_i\}$ **point-wise converges** to $f : \Omega \rightarrow \mathbb{R}^N$ if $x \in \Omega \implies f_i(x) \rightarrow f(x)$.

Let $\{f_i\}$ be a sequence of functions, with $f_i : \Omega \rightarrow \mathbb{R}^N$ and $\Omega \subseteq \mathbb{R}^M$. We say that $\{f_i\}$ **uniformly converges** to $f : \Omega \rightarrow \mathbb{R}^N$ if $\forall \varepsilon > 0 \exists I_0(\varepsilon)$ s.t. for $i > I_0(\varepsilon)$ we have $\|f_i(x) - f(x)\| < \varepsilon$.

- a) Let $f_i(x) = x/i$ and $f(x) = 0$. Check that $f_i \rightarrow f$ point-wise.
- b) Show f_i defined above does not converge uniformly to f .
- c) Show uniform convergence implies point-wise convergence.
2. Let $A \subseteq \mathbb{R}^N$ be a convex set. $f : A \rightarrow \mathbb{R}^N$ is quasiconcave if for any $x, y \in A$ and $\alpha \in [0, 1]$ we have

$$f(\alpha x + (1 - \alpha)y) \geq \min\{f(x), f(y)\}$$

and strictly quasiconcave if the above holds strictly. Show if f is quasiconcave then $\operatorname{argmax}_{x \in A} f(x)$ is a convex set (recall the empty set is convex by vacuity). Further show that if f is strictly quasiconcave then $\operatorname{argmax}_{x \in A} f(x)$ is a singleton or empty.

3. Consider a continuous function $f : \mathbb{R}^N \rightarrow \mathbb{R}$. Show

- a) If f is differentiable and $x^* \in \mathbb{R}^N$ is a local maximizer or minimizer of f , then $\nabla f(x^*) = 0$.
- b) If f is twice continuously differentiable and $x^* \in \mathbb{R}^N$ is s.t. $\nabla f(x^*) = 0$, then if x^* is a local maximizer the symmetric $N \times N$ Hessian $D^2 f(x^*)$ is negative semidefinite. Extra credit: If $D^2 f(x^*)$ is negative definite then x^* is a strict local maximizer. (Hint: I used a Taylor expansion without the explicit remainder formula. For the extra-credit, I additionally leveraged the fact a matrix is ND iff it has all strictly negative eigenvalues, but there may be a way to do it without that.)
- c) If f is concave then $f(x + z) \leq f(x) + z^T Df(x)$ for any x, z .
- d) If f is concave then any critical point (i.e. x s.t. $Df(x) = 0$) is a global maximizer.
4. Define the set $\Delta = \{p \in \mathbb{R}_+^L : \sum_l p_l = 1\}$ and the function z^+ on Δ as $z_l^+(p) = \max\{z_l(p), 0\}$, where $z(p) = \{z_1(p), z_2(p), \dots, z_L(p)\}$ is a continuous function, homogeneous of degree 0, and satisfying $p \cdot z(p) = 0$ for all $p \in \mathbb{R}^L$. Denote $\alpha(p) = \sum_l [p_l + z_l^+]$.

- a) Show that Δ is a non-empty compact and convex set.
- b) Show that $f : \Delta \rightarrow \Delta$ is continuous in p .

$$f(p) = \frac{1}{\alpha(p)} (p + z^+(p))$$

- c) Prove that f has a fixed point. (Hint: You can use existing theorems!)

d) Use the fact f has a fixed point and the properties of z to argue that $\exists p^*$ s.t. $z^+(p^*) \cdot z(p^*) = 0$. (Hint: Use the fact $p^* \cdot z(p^*) = 0$.)

e) Conclude that $z(p^*) \leq 0$.

Remark 1. If for consumer i we define the excess demand function $z_i(p) = x_i(p, \omega_i) - \omega_i$ for wealth ω_i and prices p . One way to define general equilibrium is vector of prices s.t. $\sum_i z_i(p) \leq 0$ for all i (i.e. there is no aggregate excess demand). You have just shown that under some conditions such a price vector always exists. \square

5. Use the chain rule and the FTC to prove the Leibniz rule:

$$\frac{d}{dx} \int_{v(x)}^{u(x)} f(t) dt = f(v(x)) \frac{dv}{dx} - f(u(x)) \frac{du}{dx}$$