Problem Set 2

1. *Using the fact that every Cauchy sequence converges, prove the Bolzano-Weierstrass theorem. (Hint: First show every bounded sequence admits a monotonic sub-sequence. Second, show bounded monotonic sequences are Cauchy.)*

2. *Let (x_m) be any sequence. We define*

$$\limsup_{m\to\infty} = \lim_{m\to\infty}\left(\sup_{k\geq m} x_k\right)$$

and

$$\liminf_{m\to\infty} = \lim_{m\to\infty}\left(\inf_{k\geq m} x_k\right)$$

Consider $(x_m) \in \mathbb{R}$ bounded. Show $x_m \to x \iff \limsup x_m = \liminf x_m = x$.

3. *Given a set $A \subseteq \mathbb{R}$ we say that a function $f : S \to \mathbb{R}$ is* ***uniformly continuous*** *if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x, y \in S$*

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

The main difference between this and the usual definition of continuity, is that the latter can have different values of δ given $x, y \in S$. If a function is uniformly continuous, we need δ picked for any x, y (although it might depend on ε.)

   a) *Take $f(x) = 1/x$ for $f : (0, 1) \to \mathbb{R}$. Show f is continuous.*

   b) *Show that for any two sequences $(x_m), (y_m) \in S$ with $\lim_{m\to\infty}(x_m - y_m) = 0$ and s.t. $\exists \varepsilon > 0$ with $|g(x_m) - g(y_m)| > \varepsilon\ \ \forall m \in \mathbb{N}$, we have $g : S \to \mathbb{R}$ is not uniformly continuous.*

   c) *Use the result above to check that f defined in (a) is not uniformly continuous.*

   d) *Show that if $(x_n) \in S$ is Cauchy, then (y_m) defined by $y_m = h(x_m)$ is also Cauchy when $h : S \to \mathbb{R}$ is uniformly continuous.*

   e) *Check $x_m = 1/m$ is Cauchy but $x_m = m$ is not.*

   f) *Use the sequences in (e) and the result in (d) to give an alternative proof that the function defined in (a) is not uniformly continuous.*

4. *A function $f : \mathbb{R}^N \to \mathbb{R}$ is homogeneous of degree $r \in \mathbb{Z}$ if $\forall t > 0, x \in \mathbb{R}^N$ we have*

$$f(tx) = t^r f(x)$$

   a) *Show if $f(x)$ is homogeneous of degree r the partial derivative is homogeneous of degree $r - 1$.*

   b) *Show that if $f(x)$ is homogeneous of degree r and differentiable then for any $\widetilde{x}$ we have*

$$\sum_{n=1}^{N} \frac{\partial f(\widetilde{x})}{\partial x_n}\widetilde{x}_n = r f(\widetilde{x})$$

that is, $\nabla f(\tilde{x}) \cdot \tilde{x} = r f(\tilde{x})$.

5. *Take the simplified IS-LM system of equations*

$$Y - C(Y - T) - I(r) = G \qquad M^D(Y, r) = M^S$$

Suppose that $0 < C'(x) < 1$, $I'(r) < 0$, $\dfrac{\partial M}{\partial Y} > 0$ *and* $\dfrac{\partial M}{\partial r} < 0$.

a) *Use the IFT to check that one can represent the endogenous variables* Y, r *as a function of the exogenous variables* G, M^S, T

b) *Check the sign of the effect of an infinitesimal increase in government spending* G *on* r *and* Y *keeping* M^S *and* T *fixed.*

6. *Take the correspondence* $f : \mathbb{R}^3_{++} \to \mathbb{R}^2_{++}$ *(i.e. with strictly posisive arguments) defined by*

$$f(p_1, p_2, w) = \begin{cases} \left(\dfrac{w}{2p_1}, \dfrac{w}{2p_2}\right) & \text{if } \dfrac{w^2}{4p_1 p_2} \neq 1 \\ \varnothing & \text{if } \dfrac{w^2}{4p_1 p_2} = 1 \end{cases}$$

(a) *Is* f *upper hemicontinuous?*

(b) *Is* f *lower hemicontinuous?*

(Hint: Use sequential definitions.)