

## Problem Set 2

1. Using the fact that every Cauchy sequence converges, prove the Bolzano-Weierstrass theorem. (Hint: First show every bounded sequence admits a monotonic sub-sequence. Second, show bounded monotonic sequences are Cauchy.)

2. Let  $(x_m)$  be any sequence. We define

$$\limsup_{m \rightarrow \infty} = \lim_{m \rightarrow \infty} \left( \sup_{k \geq m} x_k \right)$$

and

$$\liminf_{m \rightarrow \infty} = \lim_{m \rightarrow \infty} \left( \inf_{k \geq m} x_k \right)$$

Consider  $(x_m) \in \mathbb{R}$  bounded. Show  $x_m \rightarrow x \iff \limsup x_m = \liminf x_m = x$ .

3. Given a set  $A \subseteq \mathbb{R}$  we say that a function  $f : S \rightarrow \mathbb{R}$  is **uniformly continuous** if  $\forall \varepsilon > 0 \exists \delta > 0$  s.t.  $\forall x, y \in S$

$$|x - y| < \varepsilon \implies |f(x) - f(y)| < \varepsilon$$

The main difference between this and the usual definition of continuity, is that the latter can have different values of  $\delta$  given  $x, y \in S$ . If a function is uniformly continuous, we need  $\delta$  picked for any  $x, y$  (although it might depend on  $\varepsilon$ .)

- Take  $f(x) = 1/x$  for  $f : (0, 1) \rightarrow \mathbb{R}$ . Show  $f$  is continuous.
  - Show that if for two sequences  $(x_m), (y_m) \subseteq S$  with  $\lim_{m \rightarrow \infty} (x_m - y_m) = 0 \exists \varepsilon > 0$  s.t.  $|g(x_m) - g(y_m)| > \varepsilon \forall m \in \mathbb{N}$ , then  $g : S \rightarrow \mathbb{R}$  is not uniformly continuous.
  - Use the result above to check that  $f$  defined in (a) is not uniformly continuous.
  - Show that if  $(x_n) \in S$  is Cauchy, then  $(y_m)$  defined by  $y_m = h(x_m)$  is also Cauchy when  $h : S \rightarrow \mathbb{R}$  is uniformly continuous.
  - Check  $x_m = 1/m$  is Cauchy but  $x_m = m$  is not.
  - Use the sequences in (e) and the result in (d) to give an alternative proof that the function defined in (a) is not uniformly continuous.
4. A function  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  is homogeneous of degree  $r \in \mathbb{Z}$  if  $\forall t > 0, x \in \mathbb{R}^N$  we have

$$f(tx) = t^r f(x)$$

- Show if  $f(x)$  is homogeneous of degree  $r$  the partial derivative is homogeneous of degree  $r - 1$ .
- Show that if  $f(x)$  is homogeneous of degree  $r$  and differentiable then for any  $\tilde{x}$  we have

$$\sum_{n=1}^N \frac{\partial f(\tilde{x})}{\partial x_n} \tilde{x}_n = r f(\tilde{x})$$

that is,  $\nabla f(\tilde{x}) \cdot \tilde{x} = rf(\tilde{x})$ .

5. Take the simplified IS-LM system of equations

$$Y - C(Y - T) - I(r) = G \quad M^D(Y, r) = M^S$$

Suppose that  $0 < C'(x) < 1$ ,  $I'(r) < 0$ ,  $\frac{\partial M}{\partial Y} > 0$  and  $\frac{\partial M}{\partial r} < 0$ .

- Use the IFT to check that one can represent the endogenous variables  $Y, r$  as a function of the exogenous variables  $G, M^S, T$
- Check the sign of the effect of an infinitesimal increase in government spending  $G$  on  $r$  and  $Y$  keeping  $M^S$  and  $T$  fixed.

6. Take the correspondence  $f : \mathbb{R}_{++}^3 \rightarrow \mathbb{R}_{++}^2$  (i.e. with strictly positive arguments) defined by

$$f(p_1, p_2, w) = \begin{cases} \left( \frac{w}{2p_1}, \frac{w}{2p_2} \right) & \text{if } \frac{w^2}{4p_1p_2} \neq 1 \\ \emptyset & \text{if } \frac{w^2}{4p_1p_2} = 1 \end{cases}$$

(a) Is  $f$  upper hemicontinuous?

(b) Is  $f$  lower hemicontinuous?

(Hint: Use sequential definitions.)