Suggested Solutions 1

1. *Using the fact that every Cauchy sequence converges, prove the Bolzano-Weierstrass theorem.*

   Solution. The Bolzano–Weierstrass theorem states every bounded sequence has a convergent sub-sequence. Our strategy to prove it using Cauchy sequences will be:

   - Show that every bounded sequence admits a monotonic sub-sequence.

   - Show every bounded monotonic sequence is Cauchy.

   For the first part, the claim is, formally,

   - For each m there exists some $n > m$ s.t. $x_m \geq x_n$

   - *Or* for each k there exists some $n > k$ s.t. $x_k \leq x_n$.

   Maybe both are true, but if at least one is true then we have the first step. By contradiction, suppose this claim is false, meaning both statements must be false. That is,

   - $\exists m_1$ s.t. $\forall n > m_1$ we have $x_{m_1} < x_n$

   - and $\exists k_1$ s.t. $\forall n > k_1$ we have $x_{k_1} > x_n$.

   In other words, for $n \geq M_1 \equiv \max\{m_1, k_1\}$ we have $x_n \in (x_{m_1}, x_{k_1})$. Now we can iterate the claim: Suppose there is no monotonic sub-sequence of x_m for $m \geq M_1$. Then

   - $\exists m_2$ s.t. $\forall n > m_2$ we have $x_{m_2} < x_n$

   - and $\exists k_2$ s.t. $\forall n > k_2$ we have $x_{k_2} > x_n$.

   Note m_2, k_2 must each be larger than M_1, so $x_{m_2}, x_{k_2} \in (x_{m_1}, x_{k_1})$, and for $n \geq M_2 \equiv \max\{m_2, k_2\}$ we have $x_n \in (x_{m_2}, x_{k_2}) \subset (x_{m_1}, x_{k_1})$. We can proceed by induction, and construct a sequence (two sequences, actually) x_{m_l} strictly increasing and x_{k_l} strictly decreasing. These are strictly monotonic sub-sequences of x_m, which is a contradiction.

   For the second step, we again by contradiction suppose that monotonic sequences are not Cauchy. Remember the definition of Cauchy is for every $\varepsilon > 0$ there exists some M s.t. $d(x_k, x_n) < \varepsilon$ whenever $n, k \geq M$. If a sequence is not Cauchy, then $\exists \varepsilon > 0$ s.t. for each k, n we have

   $$d(x_k, x_n) \geq \varepsilon$$

   Consider x_{m_k} any monotonic sub-sequence of x_m and suppose it is increasing. We know that sequential elements of this sub-sequence are at least ε away from each other, since this is true of any two elements of x_m. Hence we can write

   $$x_{m_1} + \varepsilon \leq x_{m_2} \quad x_{m_2} + \varepsilon \leq x_{m_3} \quad x_{m_1} + 2\varepsilon \leq x_{m_3} \quad \cdots$$

   In general, we can find that

   $$x_{m_1} + (K - 1)\varepsilon \leq x_{m_K}$$

for any K. Since $\varepsilon > 0$, take any upper bound of the sequence U. We have that

$$K > \frac{U - x_{m_1}}{\varepsilon} + 1 \implies x_{m_1} + (K-1)\varepsilon > U \implies x_{m_K} > U$$

meaning U is not an upper bound, contradiction. The proof for a decreasing sub-sequence is analogous. Thus we have shown every bounded sequence admits a monotonic sub-sequence; this sub-sequence is Cauchy, and since Cauchy sequences converge this sub-sequence converges.

2. *Let (x_m) be any sequence. We define*

$$\limsup_{m \to \infty} = \lim_{m \to \infty} \left(\sup_{k \geq m} x_k \right)$$

and

$$\liminf_{m \to \infty} = \lim_{m \to \infty} \left(\inf_{k \geq m} x_k \right)$$

Consider $(x_m) \in \mathbb{R}$ bounded. Show $x_m \to x \iff \limsup x_m = \liminf x_m = x$.

Solution. First we do $\implies$, that is, $x_m \to x$. We want to show that $\limsup x_m = \liminf x_m = x$. By definition,

$$\limsup_{m \to \infty} x_m = \lim_{m \to \infty} \sup_{k \geq m} x_k$$

We want to show that for every $\varepsilon > 0$, there is some M s.t. $m > M$ gives

$$\left| \sup_{k \geq m} x_k - x \right| < \varepsilon \qquad x - \varepsilon < \sup_{k \geq m} x_k < x + \varepsilon$$

Suppose not; since $x_m \to x$, where is some M s.t. $|x_m - x| < \varepsilon$. Hence for $m > N, m > M$, we have

$$\sup_{k \geq m} x_k < x - \varepsilon < x_m \text{ or } x_m < x + \varepsilon < \sup_{k \geq m} x_k$$

The first statement is not true, or $\sup_{k \geq m} x_k$ cannot be the sup, and the second statement cannot be true, or $x + \varepsilon$ would instead be the sup. Hence $\left| \sup_{k \geq m} x_k - x \right| < \varepsilon$ for $m > \max\{M, N\}$. The argument for inf is completely analogous; if we assume the inf does not converge to x, then we find $\tilde{\varepsilon} > 0, \tilde{M}, \tilde{N}$ s.t.

$$\inf_{k \geq m} x_k < x - \tilde{\varepsilon} < x_m \text{ or } x_m < x + \tilde{\varepsilon} < \inf_{k \geq m} x_k$$

for $m > \max\{\tilde{M}, \tilde{N}\}$. Both statements are, again, false: The first because it means $x - \tilde{\varepsilon}$ would tbe the inf, and the second because the inf cannot be greater than x_m.

Now we show $\impliedby$. We want to show that for every $\varepsilon > 0$, there is some M s.t.

$$m > M \implies |x_m - x| < \varepsilon$$

We have that

$$\sup_{k \geq m} x_k \to x \text{ and } \inf_{k \geq m} x_k \to x$$

So for $\varepsilon > 0$, there is some M_1, M_2 s.t.

$$m > M_1 \implies \left| \sup_{k \geq m} x_k - x \right| < \varepsilon \text{ and } m > M_2 \implies \left| \inf_{k \geq m} x_k - x \right| < \varepsilon$$

Take $M > \max\{M_1, M_2\}$. Then we have

$$x - \varepsilon < \inf_{k \geq m} x_k < x + \varepsilon \text{ and } x - \varepsilon < \sup_{k \geq m} x_k < x + \varepsilon$$

To finish, we simply need to note that

$$\inf_{k \geq m} x_k \leq x_m \leq \sup_{k \geq m} x_k$$

by definition of $\inf$ and $\sup$. Hence we have

$$x - \varepsilon < \inf_{k \geq m} x_k \leq x_m \leq \sup_{k \geq m} x_k < x + \varepsilon \implies x - \varepsilon < x_m < x + \varepsilon$$

which is what we wanted.

3. *Given a set $A \subseteq \mathbb{R}$ we say that a function $f : S \to \mathbb{R}$ is **uniformly continuous** if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall x, y \in S$*

$$|x - y| < \varepsilon \implies |f(x) - f(y)| < \varepsilon$$

   The main difference between this and the usual definition of continuity, is that the latter can have different values of δ given $x, y \in S$. If a function is uniformly continuous, we need δ picked for any x, y (although it might depend on ε.)

   a) *Take $f(x) = 1/x$ for $f : (0, 1) \to \mathbb{R}$. Show f is continuous.*

   Solution. First, note for any $x \in (0, 1)$ and $\delta > 0$ small enough we have

   $$|x - a| < \delta \implies \frac{1}{a} \in \left(\frac{1}{x + \delta}, \frac{1}{x - \delta} \right)$$

   $$\implies \left| \frac{1}{x} - \frac{1}{a} \right| = \left| \frac{x - a}{xa} \right| < \left| \frac{1}{x} - \frac{1}{a} \right| < \left| \frac{\delta}{x(x - \delta)} \right|$$

   Let $\delta \equiv \varepsilon x^2 / (1 + \varepsilon x)$ (noting $x \in (0, 1)$). We have that

   $$|x - a| < \delta \implies \left| \frac{1}{x} - \frac{1}{a} \right| < \left| \frac{\delta}{x(x - \delta)} \right| = \frac{\dfrac{\varepsilon x^2}{1 + \varepsilon x}}{x \left(x - \dfrac{\varepsilon x^2}{1 + \varepsilon x} \right)} = \varepsilon$$

   which shows that $1/x$ is continuous.

   b) *Show that if for two sequences $(x_m), (y_m) \subseteq S$ with $\lim_{m \to \infty}(x_m - y_m) = 0 \ \exists \varepsilon > 0$ s.t. $|g(x_m) - g(y_m)| > \varepsilon \ \forall m \in \mathbb{N}$, then $g : S \to \mathbb{R}$ is not uniformly continuous.*

   Solution. First, note every ε_1 there is some M s.t.

   $$m > M \implies |x_m - y_m| < \varepsilon_1$$

   $g : A \to \mathbb{R}$ is uniformly continuous if for every ε_2 there is some δ such that for all $x, y \in A$,

   $$|x - y| < \delta \implies |g(x) - g(y)| < \varepsilon_2$$

   Suppose $g(\cdot)$ is uniformly continuous and let $\varepsilon_1 = \delta, \varepsilon_2 = \varepsilon$. Then we know there is some M s.t. $m > M$ gives $|x_m - y_m| < \delta$, which in turn would imply $|g(x_m) - g(u_m)| < \varepsilon$. However, we assumed $\varepsilon < |g(x_m) - g(u_m)|$, which is a contradiction.

   c) *Use the result above to check that f defined in (a) is not uniformly continuous.*

   Solution. Let $x_m = 1/m$ and $y_m = 1/(m + 1)$. Clearly,

   $$\frac{1}{m} - \frac{1}{m + 1} = \frac{m + 1 - m}{m(m + 1)} = \frac{1}{m(m + 1)} \to 0$$

However,

$$\left| f\left(\frac{1}{m}\right) - f\left(\frac{1}{m+1}\right) \right| = \left| \frac{1}{1/m} - \frac{1}{1/(m+1)} \right| = |m - (m+1)| = 1 > \varepsilon$$

for, say, $\varepsilon = 1/2 > 0$. By (b) we know then that f is not uniformly continuous.

d) *Show that if $(x_m) \in S$ is Cauchy, then (y_m) defined by $y_m = h(x_m)$ is also Cauchy when $h : S \to \mathbb{R}$ is uniformly continuous.*

Solution. If $h(\cdot)$ uniformly continuous, we know for every $\varepsilon > 0$ there is some $\delta > 0$ s.t.

$$|x - y| < \delta \implies |h(x) - h(y)| < \varepsilon$$

If x_m is Cauchy, we know that for this $\delta > 0$, we can find some M s.t.

$$m > M \implies |x_m - x_{m+1}| < \delta$$

Hence for $m > M$,

$$|x_m - x_{m+1}| < \delta$$

Since $h()$ is uniformly continuous, this in turn implies

$$|h(x_m) - h(x_{m+1})| < \varepsilon$$

Which means that for any ε we can find some M s.t.

$$|y_m - y_{m+1}| < \varepsilon$$

e) *Check $x_m = 1/m$ is Cauchy but $x_m = m$ is not.*

Solution. $x_m = m$ and $x_{m+1} = m + 1$ gives $|x_m - x_{m+1}| = |m - (m+1)| = |1| = 1 > \varepsilon$ for all $0 < \varepsilon < 1$, so it cannot be Cauchy. Now

$$\left| \frac{1}{m} - \frac{1}{m+1} \right| = \left| \frac{m+1-m}{m(m+1)} \right| = \left| \frac{1}{m(m+1)} \right|$$

For any $\varepsilon > 0$, let

$$M = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1 > \frac{1}{\varepsilon} \implies \varepsilon > \frac{1}{M} \text{ and } \varepsilon > \frac{1}{M+1}$$

Hence for any $m > M$,

$$\left| \frac{1}{m(m+1)} \right| < \varepsilon^2 \leq \varepsilon$$

given $\varepsilon \leq 1$ (and if $\varepsilon > 1$ then the inequality holds for all m).

f) *Use the sequences in (e) and the result in (d) to give an alternative proof that the function defined in (a) is not uniformly continuous.*

Solution. If $f(\cdot)$ is uniformly continuous, then $y_m \equiv f(x_m)$ is Cauchy for any x_m that is Cauchy. However, we just proved that $x_m = 1/m$ is Cauchy while $y_m = 1/(1/m) = m$ is not Cauchy, which is a contradiction. Hence $f(\cdot)$ cannot be uniformly continuous.

4. *A function $f : \mathbb{R}^N \to \mathbb{R}$ is homogeneous of degree $r \in \mathbb{Z}$ if $\forall t > 0, x \in \mathbb{R}^N$ we have*

$$f(tx) = t^r f(x)$$

a) *Show if $f(x)$ is homogeneous of degree r the partial derivative is homogeneous of degree $r - 1$.*

Solution. If we can assume that the partial derivative exists, then we have

$$\frac{\partial}{\partial x_j} f(tx) = t^r \frac{\partial}{\partial x_j} f(x)$$

$$t \frac{\partial}{\partial x_j} f(tx) = t^r \frac{\partial}{\partial x_j} f(x)$$

$$\frac{\partial}{\partial x_j} f(tx) = t^{r-1} \frac{\partial}{\partial x_j} f(x)$$

which is the definition of homogeneity of degree $r - 1$.

b) *Show that if $f(x)$ is homogeneous of degree r and differentiable then for any $\tilde{x}$ we have*

$$\sum_{n=1}^{N} \frac{\partial f(\tilde{x})}{\partial x_n} \tilde{x}_n = rf(\tilde{x})$$

that is, $\nabla f(\tilde{x}) \cdot \tilde{x} = rf(\tilde{x})$.

Solution. Take derivatives with respect to t, then we have

$$\frac{d}{dt} f(tx) = \frac{d}{dt} t^r f(x) \qquad \sum_{n=1}^{N} \left(\frac{\partial}{\partial x_n} f(tx) \right) \underbrace{\left(\frac{d}{dt}(tx_n) \right)}_{x_n} = rt^{r-1} f(x)$$

is true for every t. So simply choose $t = 1$, and we get

$$\sum_{n=1}^{N} \frac{\partial}{\partial x_n} f(x) x_n = rf(x)$$

which is what we wanted.[1]

5. *Take the simplified IS-LM system of equations*

$$Y - C(Y - T) - I(r) = G \qquad M^D(Y, r) = M^S$$

Suppose that $0 < C'(x) < 1, I'(r) < 0, \frac{\partial M}{\partial Y} > 0$ and $\frac{\partial M}{\partial r} < 0$.

a) *Use the IFT to check that one can represent the endogenous variables Y, r as a function of the exogenous variables G, M^S, T.*

Solution. Note that

$$Y - C(Y - T) - I(r) - G = 0$$
$$M^S - M^D(Y, r) = 0$$

with endogenous variables $x \equiv (Y, r)$, national income and the interest rate, and exogenous variables $\theta \equiv (M^S, G, T)$, money supply, government spending, and taxes.

(1)
$$f(\theta, x) = \begin{bmatrix} f_1(\theta, x) \\ f_2(\theta, x) \end{bmatrix} = \begin{bmatrix} Y - C(Y - T) - I(r) - G \\ M^S - M^D(Y, r) \end{bmatrix} = 0$$

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[1]I think the only sticky point here is to show that

$$\frac{d}{dt} f(tx) = \nabla f(tx) \cdot \frac{d}{dt} tx$$

which is true by the chain rule.

is the exact type of equation where IFT applies. If D_x is non-singular, then for some open neighborhood of θ and some h we can write

(2)
$$h(\theta) = \begin{bmatrix} Y(M^S, G, T) \\ r(M^S, G, T) \end{bmatrix}$$

$$D_\theta f(\theta, h(\theta)) + D_x f(\theta, h(\theta)) D_\theta h(\theta) = 0$$

We find that

$$D_\theta f(\theta, h(\theta)) = \begin{bmatrix} 0 & -1 & C'(Y(\cdot) - T) \\ 1 & 0 & 0 \end{bmatrix}$$

$$D_x f(\theta, h(\theta)) = \begin{bmatrix} \dfrac{\partial f_1}{\partial Y} = 1 - C'(Y(\cdot) - T) & \dfrac{\partial f_1}{\partial r} = -I'(r(\cdot)) \\ \dfrac{\partial f_2}{\partial Y} = -\dfrac{\partial M^D}{\partial Y} & \dfrac{\partial f_2}{\partial r} = -\dfrac{\partial M^D}{\partial r} \end{bmatrix}$$

$$[D_x f(\theta, h(\theta))]^{-1} = \frac{1}{D} \begin{bmatrix} -\dfrac{\partial M^D}{\partial r} & I'(r(\cdot)) \\ \dfrac{\partial M^D}{\partial Y} & 1 - C'(Y(\cdot) - T) \end{bmatrix}$$

$$D = - \overbrace{\frac{\partial M^D}{\partial r}}^{<0} \underbrace{\left(1 - C'(Y(\cdot) - T)\right)}_{>0} - \overbrace{I'(r(\cdot))}^{<0} \underbrace{\frac{\partial M^D}{\partial Y}}_{>0} \implies D > 0$$

$$\underbrace{\phantom{-\frac{\partial M^D}{\partial r}\left(1 - C'(Y(\cdot) - T)\right)}}_{>0} \underbrace{\phantom{-I'(r(\cdot))\frac{\partial M^D}{\partial Y}}}_{>0}$$

A non-zero determinant $\implies$ the inverse exists, and we can use Equation (2) to characterize $D_\theta h(\theta)$

b) *Check the sign of the effect of an infinitesimal increase in government spending G on r and Y keeping M^S and T fixed.*

Solution. Using the results from the previous section, we have that

$$D_\theta h(\theta) = \begin{bmatrix} \dfrac{\partial Y}{\partial M^S} & \dfrac{\partial Y}{\partial G} & \dfrac{\partial Y}{\partial T} \\ \dfrac{\partial r}{\partial M^S} & \dfrac{\partial r}{\partial G} & \dfrac{\partial r}{\partial T} \end{bmatrix}$$

$$= -\frac{1}{D} \begin{bmatrix} -\dfrac{\partial M^D}{\partial r} & I'(r(\cdot)) \\ \dfrac{\partial M^D}{\partial Y} & 1 - C'(Y(\cdot) - T) \end{bmatrix} \begin{bmatrix} 0 & -1 & C'(Y(\cdot) - T) \\ 1 & 0 & 0 \end{bmatrix}$$

$$= -\frac{1}{D} \begin{bmatrix} I'(r(\cdot)) & \dfrac{\partial M^D}{\partial r} & -\dfrac{\partial M^D}{\partial r} C'(Y(\cdot) - T) \\ 1 - C'(Y(\cdot) - T) & -\dfrac{\partial M^D}{\partial Y} & \dfrac{\partial M^D}{\partial Y} C'(Y(\cdot) - T) \end{bmatrix}$$

$$= -\frac{1}{D}\begin{bmatrix} <0 & <0 & >0 \\ >0 & <0 & >0 \end{bmatrix} = \frac{1}{D}\begin{bmatrix} >0 & >0 & <0 \\ <0 & >0 & <0 \end{bmatrix}$$

Which means that for some $x = (Y, r), \theta = (M^S, G, T)$ that satisfies Equation (1) there is some local neighborhood around θ where we can characterize the behavior of (Y, r) with respect to each of the variables in θ. In particular, income reacts positively to increases money supply or government spending but negatively to taxes, while the interest rate goes down with increases in the money supply or taxes but goes up with increases in government spending.

6. *Take the correspondence $f : \mathbb{R}^3_{++} \to \mathbb{R}^2_{++}$ (i.e. with strictly posisive arguments) defined by*

$$f(p_1, p_2, w) = \begin{cases} \left(\dfrac{w}{2p_1}, \dfrac{w}{2p_2}\right) & \text{if } \dfrac{w^2}{4p_1p_2} \neq 1 \\ \varnothing & \text{if } \dfrac{w^2}{4p_1p_2} = 1 \end{cases}$$

(a) *Is f upper hemicontinuous?*

Solution. No. Take

$$p_1^m, p_2^m = \left(1 - \frac{1}{m}, 1 - \frac{1}{m}\right) \qquad w_m = 2$$

Hence $(p_1^m, p_2^m, w^m) \to (1, 1, 2)$, and $f(1, 1, 2) = \varnothing$ because $2^2/4 = 1$. However

$$y_m \equiv f(p_1^m, p_2^m, w^m) = \left(\frac{2}{2(1 - 1/m)}, \frac{2}{2(1 - 1/m)}\right) = \left(\frac{m}{m-1}, \frac{m}{m-1}\right) \to (1, 1)$$

This means that $y_m \to 1$ and there is no sub-sequence of y_m that converges to a point in the empty set.

(b) *Is f lower hemicontinuous?*

Solution. Yes. Take any $(p_1^m, p_2^m, w^m) \to (p_1, p_2, w)$. We want to show for each $y \in f(p_1, p_2, w)$ then $\exists y_m$ s.t. $y_m \to y$ with $y_m \in f(p_1^m, p_2^m, w^m)$. The trick here is to note

$$\frac{w}{2p_1} < y < \frac{w}{2p_2}$$

means that for some $\varepsilon > 0$ we have

$$\frac{w}{2p_1} + \varepsilon < y < \frac{w}{2p_2} - \varepsilon$$

(in particular this works for any $\varepsilon < \min\{y - (w/2p_1), (w/2p_2) - y\}$). By convergence, $\exists M_1$ s.t.

$$\frac{w^m}{2p_1^m} < \frac{w}{2p_1} + \varepsilon < y < \frac{w}{2p_2} - \varepsilon < \frac{w^m}{2p_2^m}$$

and by the Archimedean principle (!) there is some M_2 s.t. $1/M_2 < \varepsilon$. Let $M \equiv \max\{M_1, M_2\}$ and we have that the sequence $y_m = y + \frac{1}{m}$ is s.t.

$$\frac{w^m}{2p_1^m} << y_m < \frac{w^m}{2p_2^m}$$

for any $m \geq M$. Since $1/m \to 0$ we have $y_m \to y$, which shows that f is lhc. (Note there is no sticking point if $(p_1^m, p_2^m, w^m) \to (p_1, p_2, w)$ s.t. $w^2/4p_1p_2 = 1$ because then the correspondence is empty.)