

## Suggested Solutions 3

1. Take a collection of functions with  $f_i : \Omega \rightarrow \mathbb{R}^N$ ,  $\Omega \subseteq \mathbb{R}^M$ ,  $i \in \mathbb{N}$ . The collection  $\{f_i\}$  defines a sequence of functions, and for each  $x \in \Omega$  we have a possibly different sequence  $\{f_i(x)\}$  in  $\mathbb{R}^N$ .

Let  $\{f_i\}$  be a sequence of functions, with  $f_i : \Omega \rightarrow \mathbb{R}^N$  and  $\Omega \subseteq \mathbb{R}^M$ . We say that  $\{f_i\}$  **point-wise converges** to  $f : \Omega \rightarrow \mathbb{R}^N$  if  $x \in \Omega \implies f_i(x) \rightarrow f(x)$ .

Let  $\{f_i\}$  be a sequence of functions, with  $f_i : \Omega \rightarrow \mathbb{R}^N$  and  $\Omega \subseteq \mathbb{R}^M$ . We say that  $\{f_i\}$  **uniformly converges** to  $f : \Omega \rightarrow \mathbb{R}^N$  if  $\forall \varepsilon > 0 \exists I_0(\varepsilon)$  s.t. for  $i > I_0(\varepsilon)$  we have  $\|f_i(x) - f(x)\| < \varepsilon$ .

- a) Let  $f_i(x) = x/i$  and  $f(x) = 0$ . Check that  $f_i \rightarrow f$  point-wise.

**Solution.** Pick an arbitrary  $x$  and  $\varepsilon > 0$ . We have the sequence  $x_m = x/m$ , and for  $M = |x|/\varepsilon$  we have that

$$|x/m| < \varepsilon = |x/M|$$

whenever  $m > M$ . Hence for any given  $x$ ,  $f_i(x) \rightarrow f(x)$ .

- b) Show  $f_i$  defined above does not converge uniformly to  $f$ .

**Solution.** Crucially, the number  $I_0(\varepsilon)$  we choose in this case cannot depend on  $x$ , only on  $\varepsilon$ . Suppose that it does converge uniformly, then for any  $\varepsilon > 0$  we find  $I_0(\varepsilon)$  s.t.  $i > I_0(\varepsilon)$  gives

$$\|f_i(x) - f(x)\| < \varepsilon \iff \frac{1}{i} \|x\| < \varepsilon \iff \|x\| < \varepsilon i$$

However,  $f_i$  are defined on  $\mathbb{R}$ , for any fixed  $\varepsilon$  and  $I_0(\varepsilon)$ , take any given  $i > I_0(\varepsilon)$ , for instance  $i = I_0(\varepsilon) + 1$ , and  $x = \varepsilon i + 1 \in \mathbb{R}$  will violate the equation above, a contradiction.

- c) Show uniform convergence implies point-wise convergence.

**Solution.** This follows directly from the definition. Pick an  $x' \in \Omega$ ;  $f_i \rightarrow f$  point-wise if the sequence  $x'_i = f_i(x') \rightarrow f(x')$ . This is true if for every  $\varepsilon > 0$  there exists some  $I$  s.t.  $i > I$  gives

$$\|f_i(x') - f(x')\| < \varepsilon$$

If  $f_i \rightarrow f$  uniformly, then we know that  $I_0(\varepsilon)$  exists s.t.

$$\|f_i(x) - f(x)\| < \varepsilon$$

for every  $x \in \Omega$ , which means it must also be true of  $x'$ . Let  $I = I_0(\varepsilon)$  and we are done.

2. Let  $A \subseteq \mathbb{R}^N$  be a convex set.  $f : A \rightarrow \mathbb{R}$  is quasiconcave if for any  $x, y \in A$  and  $\alpha \in [0, 1]$  we have

$$f(\alpha x + (1 - \alpha)y) \geq \min\{f(x), f(y)\}$$

and strictly quasiconcave if the above holds strictly. Show if  $f$  is quasiconcave then  $\operatorname{argmax}_{x \in A} f(x)$  is a convex set (recall the empty set is convex by vacuity). Further show that if  $f$  is strictly quasiconcave then  $\operatorname{argmax}_{x \in A} f(x)$  is a singleton or empty.

**Solution.** We have that for any pair  $x, y \in A$ , and  $\alpha \in [0, 1]$

$$f(\alpha x + (1 - \alpha)y) \geq \min\{f(x), f(y)\}$$

Furthermore,

$$\operatorname{argmax}_{x \in A} f(x) = \{x \in A : f(x) \geq f(y) \ \forall y \in A\}$$

We want to show that for any two points,  $x^*, y^* \in \operatorname{argmax}_{x \in A} f(x)$ , the following holds:

$$\alpha x^* + (1 - \alpha)y^* \in \operatorname{argmax}_{x \in A} f(x)$$

Note that since  $A$  is convex and  $x^*, y^* \in A$ , we have  $\alpha x^* + (1 - \alpha)y^* \in A$ .

If there is no maximum the above is true by vacuity: For any two points in the empty set, their combination is also in the empty set. This statement is true by vacuity, but perhaps the contrapositive makes it more clear: If the convex combination of any two points is not on the empty set for some  $\alpha \in [0, 1]$  (true because there are no points in the empty set), then the two points are not in the set (also true because there are no points in the empty set).

So let us consider the non-trivial case.  $x^*, y^* \in \operatorname{argmax}$  gives that  $f(x^*) \geq f(x) \ \forall x \in A$  and  $f(y^*) \geq f(x) \ \forall x \in A$ . Since  $f$  is quasiconcave,

$$f(\alpha x^* + (1 - \alpha)y^*) \geq \min\{f(x^*), f(y^*)\} \geq f(x) \ \forall x \in A$$

which is exactly what we wanted to show, so  $\operatorname{argmax}$  is a convex set. If we have *strict* quasiconcavity and the maximum does not exist, then the  $\operatorname{argmax}$  is empty. So, again, let us consider the non-trivial case: Suppose that  $\operatorname{argmax}$  is not a singleton. We know that it must be convex, so for  $x^*, y^* \in \operatorname{argmax}$  with  $x^* \neq y^*$  we have

$$f(\alpha x^* + (1 - \alpha)y^*) \in \operatorname{argmax}_{x \in A} f(x)$$

However, by strict quasiconcavity we also have

$$f(\alpha x^* + (1 - \alpha)y^*) > \min\{f(x^*), f(y^*)\} \geq f(x) \ \forall x \in A$$

Since  $A$  is convex and  $x^*, y^* \in A$ , we have that  $\alpha x^* + (1 - \alpha)y^* \in A$ , and thus

$$f(\alpha x^* + (1 - \alpha)y^*) > f(\alpha x^* + (1 - \alpha)y^*)$$

a contradiction. So if the  $\operatorname{argmax}$  is not empty, it is a singleton.

3. Consider a continuous function  $f : \mathbb{R}^N \rightarrow \mathbb{R}$ . Show

a) If  $f$  is differentiable and  $x^* \in \mathbb{R}^N$  is a local maximizer of  $f$  then  $\nabla f(x^*) = 0$ .

**Solution.** Suppose  $x^*$  a local maximizer of  $f$ . Then  $\exists \varepsilon > 0$  s.t.

$$x \in B_\varepsilon(x^*) \implies f(x^*) \geq f(x)$$

Since  $f$  is differentiable, we know that  $\frac{\partial}{\partial x_j} f(x^*)$  exists for each  $j$ . Hence

$$\frac{\partial}{\partial x_j} f(x^*) = \lim_{t \rightarrow 0} \frac{f(x^* + te_j) - f(x^*)}{t} = \lim_{t \rightarrow 0+} \frac{f(x^* + te_j) - f(x^*)}{t} = \lim_{t \rightarrow 0-} \frac{f(x^* + te_j) - f(x^*)}{t}$$

where  $\{e_j\}$  are the standard basis for  $\mathbb{R}^N$ . Further, for  $t < \varepsilon$ , we have that  $\|x^* + te_j - x^*\| = t < \varepsilon$ .

Hence  $x^* + te_j \in B_\varepsilon(x)$  and

$$\begin{aligned} f(x^* + te_j) \leq f(x^*) &\implies f(x^* + te_j) - f(x^*) \leq 0 \\ (1) \quad t \geq 0 &\implies \frac{f(x^* + te_j) - f(x^*)}{t} \leq 0 \\ t \leq 0 &\implies \frac{f(x^* + te_j) - f(x^*)}{t} \geq 0 \end{aligned}$$

This means that as  $t \rightarrow 0^+$  or  $t \rightarrow 0^-$  we get

$$\begin{aligned} \frac{\partial}{\partial x_j} f(x^*) &= \lim_{t \rightarrow 0^+} \frac{f(x^* + te_j) - f(x^*)}{t} \leq 0 \\ \frac{\partial}{\partial x_j} f(x^*) &= \lim_{t \rightarrow 0^-} \frac{f(x^* + te_j) - f(x^*)}{t} \geq 0 \\ \implies \frac{\partial}{\partial x_j} f(x^*) &= 0 \end{aligned}$$

- b) If  $f$  is twice continuously differentiable and  $x^* \in \mathbb{R}^N$  is s.t.  $\nabla f(x^*) = 0$ , then if  $x^*$  is a local maximizer the symmetric  $N \times N$  Hessian  $D^2 f(x^*)$  is negative semidefinite. (Hint: Use a Taylor expansion and assume the  $D^3 f(x)$  is bounded.)

**Solution.** Take any  $z$  and the 2nd-order Taylor expansion of  $g(\varepsilon) = f(x^* + \varepsilon v)$  around  $\varepsilon = 0$ .

$$g(\varepsilon) = g(0) + g'(0)\varepsilon + \frac{1}{2}g''(0)\varepsilon^2 + \frac{1}{6}g'''(c)\varepsilon^3$$

for some  $c \in (0, \varepsilon)$ . By the chain rule,

$$g'(0) = v^T D_x f(x^*) \quad \text{and} \quad g''(0) = v^T D_x^2 f(x^*) v$$

However, from the FOC we know that  $D_x f(x^*) = 0$ . Therefore we find

$$\begin{aligned} f(x^* + \varepsilon v) - f(x^*) &= \frac{1}{2}v^T D_x^2 f(x^*) v \varepsilon^2 + \frac{1}{6}g'''(c)\varepsilon^3 \\ \frac{2}{\varepsilon^2} (f(x^* + \varepsilon v) - f(x^*)) &= v^T D_x^2 f(x^*) v + \frac{1}{3}g'''(c)\varepsilon \end{aligned}$$

We want to show that  $v^T D_x^2 f(x^*) v \leq 0$ . Suppose by contradiction  $\kappa_v \equiv v^T D_x^2 f(x^*) v > 0$ . Since  $x^*$  is a maximizer, we know that  $f(x^* + \varepsilon v) \leq f(x^*)$ . Hence

$$0 \geq v^T D_x^2 f(x^*) v + \frac{1}{3}g'''(c)\varepsilon \implies \frac{1}{3}g'''(c)\varepsilon \leq -\kappa_v < 0$$

If  $D^3 f$  is bounded then  $g'''$  is bounded, which means  $\exists \varepsilon$  s.t.  $|g'''(c)/3|\varepsilon < \kappa_v$ . Therefore

$$\frac{1}{3}g'''(c)\varepsilon > -\kappa_v$$

contradiction. This means  $v^T D_x^2 f(x^*) v \leq 0$  for any  $v$ , so the Hessian is negative semi-definite.

- c) If  $f$  is concave then  $f(x + z) \leq f(x) + Df(x)z$  for any  $x, z$ .

**Solution.** Take any  $x, y, \alpha \in [0, 1]$  and let  $z = y - x$ . We have

$$f(\alpha y + (1 - \alpha)x) \geq \alpha f(y) + (1 - \alpha)f(x) = \alpha(f(y) - f(x)) + f(x)$$

Plugging in  $z$ , we have

$$f(\alpha y + (1 - \alpha)x) = f(x + \alpha z)$$

$$\begin{aligned}
& \alpha(f(y) - f(x)) + f(x)\alpha(f(x+z) - f(x)) + f(x) \\
& f(x + \alpha z) \geq \alpha(f(x+z) - f(x)) + f(x) \\
& f(x+z) \leq \frac{f(x + \alpha z) - f(x)}{\alpha} + f(x)
\end{aligned}$$

holds for every  $\alpha$ . Let  $g(\alpha) \equiv f(x + \alpha z)$  and note

$$g'(0) = \lim_{\alpha \rightarrow 0} \frac{g(\alpha) - g(0)}{\alpha} = \lim_{\alpha \rightarrow 0} \frac{f(x + \alpha z) - f(x)}{\alpha}$$

However, by the chain rule, we know that

$$g'(0) = D_x f(x) \cdot z$$

Plugging back in, we get

$$f(x+z) \leq Df(x) \cdot z + f(x)$$

which is what we wanted to show.

- d) If  $f$  is concave then any critical point (i.e.  $x$  s.t.  $Df(x) = 0$ ) is a global maximizer.

**Solution.** Following the previous result, for any  $x, z$  we have

$$f(x+z) \leq f(x) + \nabla f(x) \cdot z$$

Let  $x^*$  be a critical point,  $y$  be any other point, and  $z \equiv y - x$ . Since  $Df(x^*) = 0$ ,

$$f(y) = f(x^* + (y - x^*)) \leq f(x^*) + Df(x^*) \cdot (y - x^*) = f(x^*) + 0 \cdot (y - x^*) = f(x^*)$$

Since  $f(x^*) \geq f(y)$  for every  $y \in \mathbb{R}^N$ ,  $x^*$  is a global maximizer.

**Remark 1.** We focused on maxima but the proofs for minima are analogous. □

4. Define the set  $\Delta = \{p \in \mathbb{R}_+^L : \sum_l p_l = 1\}$  and the function  $z^+$  on  $\Delta$  as  $z_l^+(p) = \max\{z_l(p), 0\}$ , where  $z(p) = \{z_1(p), z_2(p), \dots, z_L(p)\}$  is a continuous function, homogeneous of degree 0, and satisfying  $p \cdot z(p) = 0$  for all  $p \in \mathbb{R}^L$ . Denote  $\alpha(p) = \sum_l [p_l + z_l^+]$ .

- a) Show that  $\Delta$  is a non-empty compact and convex set.

**Solution.** For non-emptiness, note

$$p_l = \frac{1}{L} \implies \sum_l p_l = \sum_l \frac{1}{L} = 1$$

The set is bounded below by 0, since  $p_l \geq 0$  for each  $l$ , and above by 1, since  $p_j \leq p_j + \sum_{l \neq j} p_l \leq 1$  (note the sum is positive since  $p_l \geq 0$  for each  $l$ ). If we can show that  $\Delta$  is closed, then we can show that it is compact. Take any  $q \notin \Delta$ . We know that

$$\sum_l q \neq 1 \implies \sum_l q < 1 \quad \text{or} \quad \sum_l q > 1$$

We want to show that for any such  $q$  there is some  $\varepsilon$  s.t. every element  $r \in B_\varepsilon(q)$  is not in  $\Delta$ . Suppose  $\sum_l q < 1$ . It must be that  $0 < \delta < 1 - \sum_l q_l$  for some  $\delta$ . Take  $0 < \varepsilon < \delta/L$  and  $r \in B_\varepsilon(q)$ . We have that  $0 \leq r_l \leq q_l + \varepsilon < q_l + \delta/L$  for each  $l$ , and so

$$\sum_l r_l \leq \sum_l [q_l + \delta/L] = \sum_l q_l + \delta < 1$$

Suppose  $\sum_l q > 1$ . Then  $0 < \delta < \sum_l q - 1$  for some  $\delta$ . Take  $0 < \varepsilon < \delta/L$  and  $r \in B_\varepsilon(q)$ . We have that

$q_l - \delta/L \leq q_l - \varepsilon \leq r_l$  for each  $l$ , and so

$$\sum_l r_l \geq \sum_l [q_l - \delta/L] = \sum_l q_l - \delta > 1$$

Hence if  $\sum_l q_l \neq 1$ , there is some  $\varepsilon > 0$  s.t. for any  $r \in B_\varepsilon(q)$ ,  $r \notin \Delta$ , which means that  $\Delta^C$  is open, and thus  $\Delta$  is closed. Since  $\Delta$  is closed and bounded, it is compact.

Last, we show that it is convex. Take any  $p, q \in \Delta$  and let  $r = \alpha p + (1 - \alpha)q$  for any  $\alpha \in [0, 1]$ . We can see that

$$\sum_l r_l = \sum_l [\alpha p_l + (1 - \alpha)q_l] = \alpha \sum_l p_l + (1 - \alpha) \sum_l q_l = \alpha + (1 - \alpha) = 1$$

Hence  $r \in \Delta$ , and  $\Delta$  is convex.

b) Show that  $f : \Delta \rightarrow \Delta$  is continuous in  $p$ .

$$f(p) = \frac{1}{\alpha(p)} (p + z^+(p))$$

**Solution.** We will use the result that if  $g : A \rightarrow B$  is continuous and  $h : B \rightarrow C$  is continuous then the composition  $h \circ g$  is continuous. Clearly  $f_1(x) = x$  the identity function is continuous. Since  $f_2(x) = \max\{x, 0\}$  is continuous and  $z_l(p)$  is continuous,  $z_l^+(p) = f_2(z_l(p))$  is continuous. Hence  $p_l + z_l^+(p)$  is continuous for each  $l$ , which means  $p \mapsto p + z^+(p)$  will be continuous.

Since  $p_l + z_l^+(p)$  is continuous,  $\alpha(p) = \sum_l p_l + z_l^+(p)$ , the linear combination of  $L$  continuous functions, is continuous. To finish, we have that  $f_3(x) = 1/x$  is continuous when  $x \neq 0$ , so  $f_3(\alpha(p))$  will be continuous whenever  $\alpha(p) \neq 0$ . However, we have that

$$\alpha(p) = \sum_l p_l + \sum_l z_l^+(p) \quad \sum_l p_l = 1 \quad z_l^+(p) = \max\{z_l(p), 0\} \geq 0$$

Hence  $\alpha(p) \geq 1 > 0$ . This means that  $f(p)$  is the composition of continuous functions, and thus itself is continuous.

c) Prove that  $f$  has a fixed point. (Hint: You can use existing theorems!)

**Solution.** This follows directly from Brouwer's fixed point theorem. Any continuous  $f : \Delta \rightarrow \Delta$  with  $\Delta \subseteq \mathbb{R}^N$  non-empty, convex, and compact has a fixed point. I think we just need to show that the co-domain of  $f$  is, indeed,  $\Delta$ . Here note that for any  $q = f(p)$ , we have

$$\sum_l q_l = \frac{1}{\sum_l [p_l + z_l^+]} \sum_l [p_l + z_l^+] = 1$$

d) Use the fact  $f$  has a fixed point and the properties of  $z$  to argue that  $\exists p^*$  s.t.  $z^+(p^*) \cdot z(p^*) = 0$ . (Hint: Use the fact  $p^* \cdot z(p^*) = 0$ .)

**Solution.** For  $p^* \in \Delta$  a fixed point of  $f$  we have

$$\begin{aligned} p^* &= f(p^*) = \frac{1}{\alpha(p^*)} (p^* + z^+(p^*)) \\ p^* \cdot z(p^*) &= \frac{1}{1 + \sum_l z_l^+(p^*)} (p^* + z^+(p^*)) \cdot z(p^*) \\ 0 &= \frac{1}{1 + \sum_l z_l^+(p^*)} (p^* \cdot z(p^*) + z^+(p^*) \cdot z(p^*)) \\ 0 &= z^+(p^*) \cdot z(p^*) \end{aligned}$$

e) Conclude that  $z(p^*) \leq 0$ .

**Solution.**

$$z^+(p^*) \cdot z(p^*) = \sum_l z_l(p^*) \max\{z_l(p^*), 0\}$$

If  $z_l(p^*) \leq 0$  for any  $l$ , then  $z_l(p^*) \max\{z_l(p^*), 0\} = 0$ . Hence

$$z^+(p^*) \cdot z(p^*) = \sum_{l: z_l(p^*) > 0} z_l(p^*) \max\{z_l(p^*), 0\}$$

But if  $z_l > 0$  for any  $l$ , then  $z_l(p^*) \max\{z_l(p^*), 0\} = z_l(p^*)^2 > 0$ , which means

$$z^+(p^*) \cdot z(p^*) > 0$$

a contradiction. Thus  $z_l(p^*) \leq 0$ .

**Remark 2.** If for consumer  $i$  we define the excess demand function  $z_i(p) = x_i(p, \omega_i) - \omega_i$  for wealth  $\omega_i$  and prices  $p$ . One way to define general equilibrium is vector of prices  $p$  s.t.  $\sum_i z_i(p) \leq 0$  for all  $i$  (i.e. there is no aggregate excess demand). You have just shown that under some conditions such a price vector always exists.  $\square$

5. Use the chain rule and the FTC to prove the Leibniz rule:

$$\frac{d}{dx} \int_{v(x)}^{u(x)} f(t) dt = f(v(x)) \frac{dv}{dx} - f(u(x)) \frac{du}{dx}$$

**Solution.** The FTC gives that if  $f$  is integrable on  $[a, b]$  then for any  $x \in [a, b]$

$$F(x) = \int_x^a f(t) dt$$

is continuously differentiable on  $(a, b)$  and

$$\frac{d}{dx} F(x) = f(x)$$

Further,

$$F(b) - F(a) = \int_b^a f(t) dt$$

If  $f$  is integrable on  $[v(x), u(x)]$ , then we have that

$$\int_{v(x)}^{u(x)} f(t) dt = F(v(x)) - F(u(x))$$

Now we combine the FTC and the chain rule to see that

$$\frac{d}{dx} \int_{v(x)}^{u(x)} f(t) dt = \frac{d}{dx} F(v(x)) - \frac{d}{dx} F(u(x)) = f(v(x)) \frac{dv}{dx} - f(u(x)) \frac{du}{dx}$$

which is what we wanted to show.