Lecture II: Sequences, Continuity

Contents

1. Sequences

A sequence is a collection of elements of a set indexed by the natural numbers. We will not be terribly precise with notation,[1] and denote sequences with elements in S as $(x_m) \in S$.

Definition 1. Let $(x_m) \in S$ be a sequence:

a) (x_m) is ***increasing*** if $\forall m$ we have $x_m \leq x_{m+1}$; it is *strictly* increasing if $x_m < x_{m+1}$.

b) (x_m) is ***decreasing*** if $\forall m$ we have $x_m \leq x_{m+1}$; it is *strictly* decreasing if $x_m < x_{m+1}$.

c) (x_m) is (strictly) ***monotonic*** if it is (strictly) increasing or decreasing .

d) (y_k) is a ***subsequence*** of (x_m) if $\exists$ some strictly increasing sequence $(n_k) \in \mathbb{N}$ s.t. $y_k = x_{m_k}$.

e) (x_m) is bounded above, below, or bounded if $\{x_m\}_{m \in \mathbb{N}}$ is bounded above, below, or bounded (resp).

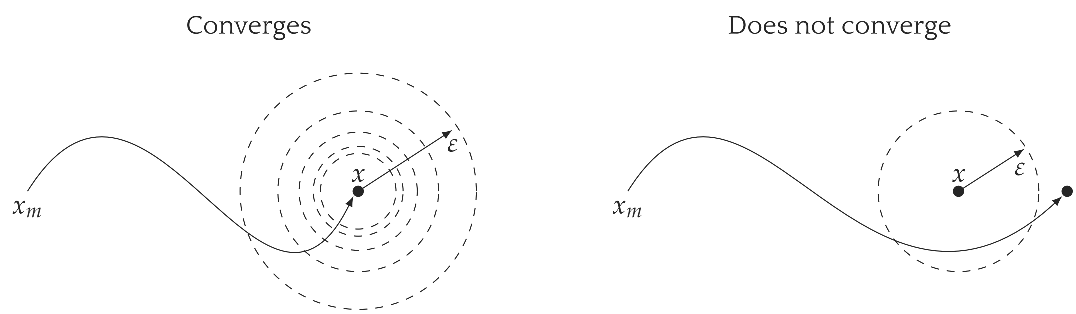
f) $(x_m) \to +\infty$ if $\forall N > 0 \ \exists M$ s.t. $m \geq M \implies x_m \geq M$. $(x_m) \to -\infty$ if $\forall N < 0 \ \exists M$ s.t. $m \geq M \implies x_m \leq M$. If either property holds, we say the sequence (x_m) ***diverges***.

[1] Formally, a sequence is an element of the infinite Cartesian product of $\mathbb{R}^N \times \ldots \times \mathbb{R}^N$. While tempting to define a sequence as a countable or finite subset of $\mathbb{R}^N$, sets have no notion of order; further, sequences can have repeated elements.

1.1. Convergence.

Definition 2. A sequence x_m ***converges*** to x if $\forall \varepsilon > 0 \ \exists M \in \mathbb{N}$ s.t. $d(x_m, x) < \varepsilon$ whenever $m \geq M$. We denote this as $x_m \to x$ or $\lim_{m\to\infty} x_m = x$.

We can visualize this idea in the figure below: x_m is eventually contained within any ε-ball if x.[2]



Claim 1. *A sequence converges to at most one limit.*

Proof. This is a consequence of the fact $d(x, y) \iff x = y$. Suppose $x_m \to x$ and $x_m \to y$. If $x \neq y$, then $d(x, y) > 0$. Take $\varepsilon = d(x, y)/4$; we know by the definition of convergence that there is some M_x, M_y s.t. for $m \geq M = \max\{M_x, M_y\}$ we get

$$d(x_m, x) < \varepsilon/2 \quad \text{and} \quad d(x_m, y) < \varepsilon/2$$

Now the triangle inequality gives

$$d(x, y) \leq d(x_m, x) + d(x_m, y) < \varepsilon/4 + \varepsilon/4 = \varepsilon/2 < d(x, y)$$

contradiction. Hence $d(x, y) = 0$, or $x = y$. □

Theorem 1. *S is closed $\iff$ for any $(x_m) \in S$ s.t. $x_m \to x$ for some $x \in \mathbb{R}^N$, we have that $x \in S$.*

This is an equivalent definition of closedness: A set that "contains all its limits."

Proof. See Subsection 1.2. □

Theorem 2. *Take any set $S \subseteq \mathbb{R}^N$; the following are equivalent:*

a) $x \in cl(S)$.

b) $\forall \varepsilon > 0, B_\varepsilon(x) \cap S \neq \varnothing$.

c) $\exists (x_m) \in S$ s.t. $x_m \to x$.

Proof. We will cycle through the proofs. First we show $b \implies c$: Take any x s.t. b holds. Let $1/m = \varepsilon_m$ for each $m = 1, 2, 3, \ldots$; we know that $\exists x_m$ s.t. $x_m \in B_{\varepsilon_m}(x)$. Now take an arbitrary $\varepsilon > 0$; there is some $M \in \mathbb{N}$ s.t. $1/M < \varepsilon$, so $m \geq M$ gives $x_m \in B_\varepsilon(x)$. Thus $x_m \to x$ with $(x_m) \in S$.

[2] Recall ε-ball is the equivalent of a neighborhood in Euclidean space, even through here in two dimensions it's technically a ε-circle.

Now we show that $c \implies a$. Note that for any closed T s.t. $S \subseteq T$, we have that $(x_m) \in S \implies (x_m) \in T$. Since T is closed and $x_m \to x$, $x \in T$ by Theorem 1. Since $S \subseteq \text{cl}(S)$ and $\text{cl}(S)$ is closed, both by definition, this proves that $c \implies a$.

To finish, we show that $a \implies b$. Suppose that $x \in \text{cl}(S)$ but for some $\varepsilon > 0$, $B_\varepsilon(x) \cap S = \varnothing$. This means that $B_\varepsilon(x) \subset \mathbb{R}^N \setminus S$. Since $B_\varepsilon(x)$ is open, $\text{cl}(S) \setminus B_\varepsilon(x)$ is closed. Further, since $x \notin S$ (because $x \in B_\varepsilon(x)$ and $B_\varepsilon(x) \cap S = \varnothing$), we have that $S \subseteq \text{cl}(S) \setminus B_\varepsilon(x)$. Since $x \in \text{cl}(S)$, this implies $\text{cl}(S) \subset \text{cl}(S) \setminus B_\varepsilon(x)$. At the same time, $\text{cl}(S)$ is the intersection of *every* closed set containing S, so $\text{cl}(S) \setminus B_\varepsilon(x) \subseteq \text{cl}(S)$. Thus $\text{cl}(S) \subset \text{cl}(S)$, a contradiction. □

Definition 3. Let (x_m) be any sequence. We define

$$\limsup_{n\to\infty} = \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right)$$

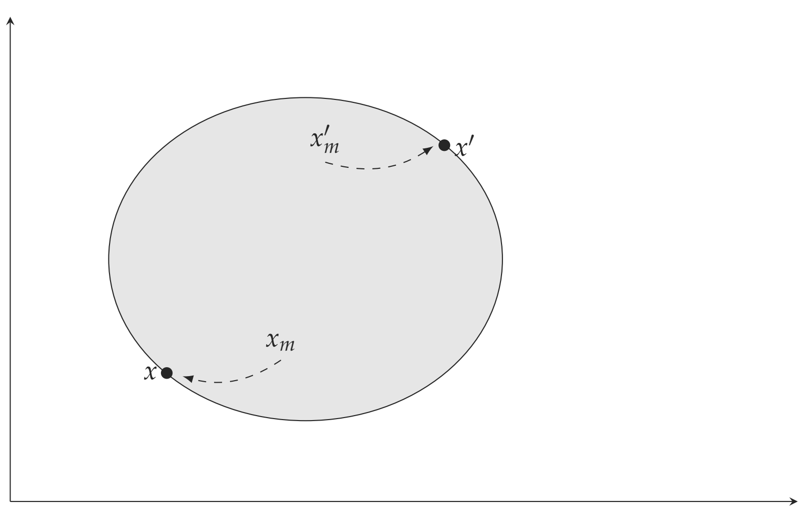
and

$$\liminf_{n\to\infty} = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right)$$

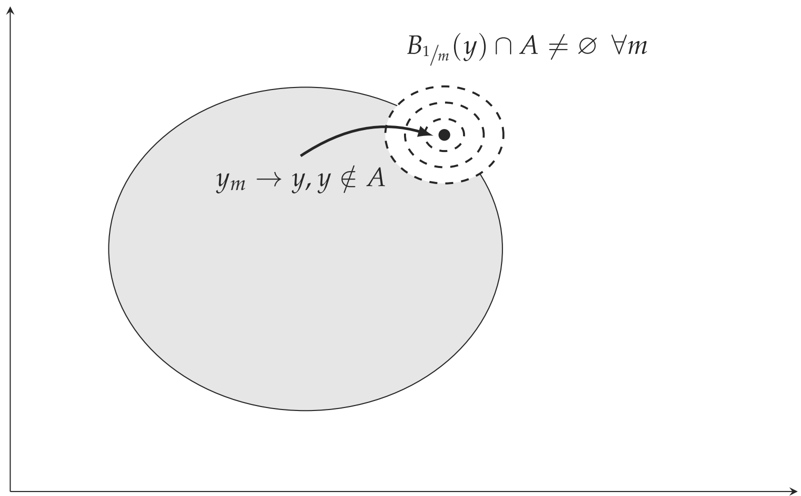
The $\limsup$ and $\liminf$ do not require convergence. Take, for instance, $(x_m) = 0, 1, 0, 1 \ldots$. Clearly $\limsup = 1$ and $\liminf = 0$ but the limit does not exist.

1.2. Sets and Sequences (Visualization). The mathy sequel to Dungeons & Dragons. In this section we will try to visualize the proof of Theorem 1. This should give some intuition for why convergence is related to sets being closed beyond the formality of the theorem.

- If a sequence x_m converges to x, then it becomes arbitrarily close to a point. If for every $(x_m) \in A$ s.t. $x_m \to x$ we also have $x \in A$, that means that no sequence can ever "escape" outside of A:

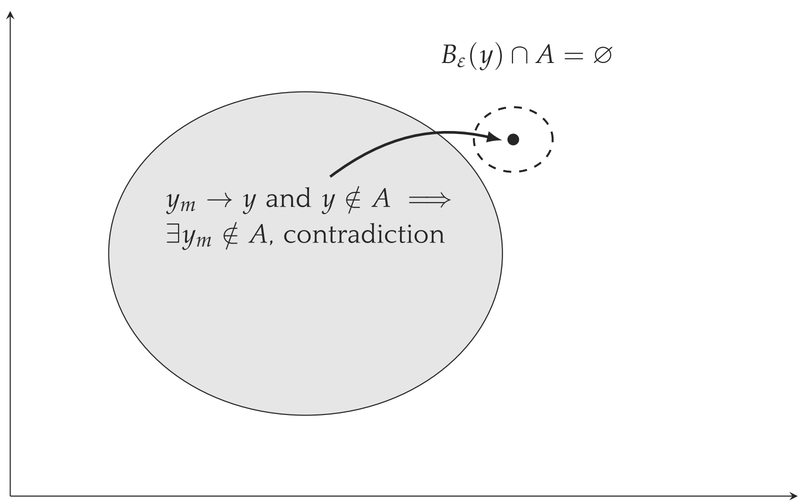


  By contrapositive, if A is not closed, its complement is not open, so $\exists y \in \mathbb{R}^N \setminus A$ that cannot be enclosed in an ε-ball inside of $\mathbb{R}^N \setminus A$. In other words, there is some y s.t. for each $\varepsilon_m = 1/m$ we can find a corresponding $y_m \in A$. The resulting $(y_m) \in A$ converges to $y \notin A$.



In other words, if A is not closed, we can find a sequence that "escapes" A, which by contrapositive proves that if every sequence in A that converges does so to a point in A, the set is closed.

- On the other hand, if A is closed and we have a sequence $(y_m) \in A$ s.t. $y_m \to y$ with $y \notin A$, then we have a sequence that "escaped" A. However, A closed implies $X \setminus A$ is open, and $\exists B_\varepsilon(y) \subseteq \mathbb{R}^N \setminus A$. Since y_m will get arbitrarily close to y, $\exists y_m \in B_\varepsilon(y) \subseteq X \setminus A$. Since $y_m \in A$ by premise, this is a contradiction.



I think in general it's very useful to visualize proofs in $\mathbb{R}^2$ (making drawings, as above):

- $\mathbb{R}$ is not enough: A ton of things will hold in one dimension that won't in general, and one-dimensional intuition can end up being misleading.
- $\mathbb{R}^3$ can be too much: I cannot draw 3D very easily and it is harder to visualie clearly (certainly $\mathbb{R}^N$ would be too many dimensions).
- $\mathbb{R}^2$ is a nice trade-off between rigor an intuition. (I know we discuss properties in more general terms than in $\mathbb{R}^2$, but for intuition I think it's a great benchmark.)

1.3. Properties of Convergent Sequences.

Theorem 3. *Take any sequence $(x_m) \in \mathbb{R}^N$:*

a) $x_m \to x \implies x_{m_k} \to x$ *for all subsequences* (x_{m_k}) *of* (x_m).

b) $x_m \to x \implies (x_m)$ *is bounded. (Is the converse true? Can you prove your answer?*[3]*)*

c) *If* $x_m \leq y_m \leq z_m$ *and* $x_m, z_m \to x$ *then* $y_m \to x$.

d) *If* $x_m \to 0$ *and* (y_m) *is bounded then* $x_m \cdot y_m \to 0$.

Let $x_m \to x$ *and* $y_m \to y$:

e) $c \cdot x_m \to c \cdot x$ *for any* $c \in \mathbb{R}$.

f) $x_m \pm y_m \to x \pm y$.

g) $x_m \cdot y_m \to x \cdot y$.

h) $x_m / y_m \to x/y$ *if* $y \neq 0$.

1.4. Bolzano-Weierstrass.

Theorem 4. *Let* $(x_m) \in \mathbb{R}$. *If* (x_m) *is bounded and monotonic then* (x_m) *converges.*

Theorem 5 (Nested Intervals Theorem)**.** *Let* $I_m = [a_m, b_m]$ *s.t.* $I_{m+1} \subset I_m$.

a) $\cap_{m \in \mathbb{N}} I_m \neq \varnothing$

b) *If* $b_m - a_m \to 0$ *then* $\cap_{m \in \mathbb{N}}$ *is a singleton.*

Proof. If $I_{m+1} \subseteq I_m$, then $I_m \subseteq I_1$ for all m. Hence

$$a_1 \leq a_m \leq a_{m+1} \leq b_{m+1} \leq b_m \leq b_1$$

for all m. In other words, a_m and b_m are bounded and monotonic, so $a_m \to a$ and $b_m \to b$ for some a, b by Theorem 4. Further, since $a_m \leq b_m$ for all m, we have that $a \leq b$ (do you see why?[4]) and $I_m \to [a, b] \neq \varnothing$. If $b_m - a_m \to 0$, that means $b - a = 0$ so the interval $[a, b]$ is just the singleton $a = b$. □

Theorem 6 (Bolzano-Weierstrass)**.** *Every bounded sequence in* $\mathbb{R}$ *admits a convergent sub-sequence.*

Proof. This follows from the Nested Intervals Theorem above—the trick is to construct the nested intervals, which we can do for bounded sequences. If (x_m) is bounded, first define

$$I_1 = [L, U] = [a_1, b_1]$$

[3] A counterexample is sufficient: $x_m = (-1)^m$ is bounded above by 1 and below by -1 but does not converge.

[4] If $a > b$ then we will have that $a_m > b_m$ for some m. That is, pick $0 < \varepsilon < a - b$; we can then find M_a, M_b s.t.

$$a - \varepsilon/2 < a_m < a + \varepsilon/2 \qquad b - \varepsilon/2 < b_m < a + \varepsilon/2$$

whenever $m > \max\{M_a, M_b\}$. However, $\varepsilon < a - b$ gives

$$b_m < b + \varepsilon/2 < a - \varepsilon/2 < a_m$$

Since $b_m \geq a_m$ for all m we have a contradiction.

the interval with enpoints equal to the lower and upper bounds of x_m. Let x_{m_1} the first element of the sub-sequence be any term of $x_m \in I_1$. Now take

$$I_1^- = \left[a_1, \frac{a_1+b_1}{2}\right] \qquad I_1^+ = \left[\frac{a_1+b_1}{2}, b_1\right]$$

and let $I_2 = I_1^-$ if $\{x_m : m \in \mathbb{N}\} \cap I_1^+$ is non-finite and $I_2 = I_1^+$ otherwise. Let a_2, b_2 be the endpoints of I_2 and x_{m_2} be any term of $x_m \in I_2$. We iterate this process: In general

$$I_k^- = \left[a_k, \frac{a_k+b_k}{2}\right] \qquad I_k^+ = \left[\frac{a_k+b_k}{2}, b_k\right]$$

and $I_{k+1} = I_k^-$ if $\{x_m : m \in \mathbb{N}\} \cap I_k^+$ is non-finite and $I_{n+1} = I_k^+$ otherwise, with x_{m_k} any element of $x_m \in I_k$. (What if both halves have a finite intersection?[5]) We have that

$$a_1 \leq a_{k-1} \leq a_k \leq x_{m_k} \leq b_k \leq b_{k-1} \leq b_1$$

Furthermore, $b_n - a_n = \dfrac{b_1 - a_1}{2^{n-1}} \to 0$. Hence we can apply the Nested Intervals Theorem: $a_n \to a$ and $b_n \to b$ with $a = b$ implies $x_{m_k} \to a = b$. □

1.5. Cauchy Sequences.

Definition 4. A sequence is ***Cauchy*** if $\forall \varepsilon > 0 \ \exists M$ s.t.

$$m, n > M \implies d(x_m, x_n) < \varepsilon$$

Theorem 7. *If (x_m) converges, then it is Cauchy.*

Proof. Suppose $x_m \to x$; by the triangle inequality

$$d(x_m, x_n) \leq d(x_m, x) + d(x_n, x)$$

Now take any $\varepsilon > 0$; for $\varepsilon/2$ we have that for some M, $m, n > M$ gives

$$d(x_m, x) < \frac{\varepsilon}{2} \qquad d(x_n, x) < \frac{\varepsilon}{2}$$

Hence

$$d(x_m, x_n) \leq d(x_m, x) + d(x_n, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which shows (x_m) is Cauchy. □

While the converse of the theorem above is also true in $\mathbb{R}^N$, during your math course you will probably encounter the fact that in general metric spaces, Cauchy sequences needn't converge. (The reason is, again, this property of Euclidean space called "completeness.")

Theorem 8. *If (x_m) is a Cauchy sequence in $\mathbb{R}$ then (x_m) converges.*

[5]Note that for $k > 1$, it must always be that either I_k^+ or I_k^- have a non-finite intersection with $\{x_m : m \in \mathbb{N}\}$, since we chose I_k to have a non-finite intersection. The only way both halves will have a finite intersection is if I_1 is finite to begin with. However, this means that some M, $x_n = x_m$ whenever $n, m > M$, which means we have a convergent sub-sequence $x_{m_k} = x_{M+1}$ with $m_k = M + k$.

Proof. I will show this in $\mathbb{R}$: Cauchy sequences are bounded (why?), which means that there exist some sub-sequence x_{m_k} that converges to some x. We show that this is also the limit for the sequence x_m. By the triangle inequality,

$$d(x_m, x) \leq d(x_m, x_{m_k}) + d(x_{m_k}, x)$$

Take any $\varepsilon > 0$, then for $\varepsilon/2$ we can find K s.t. $k > K$ gives

$$d(x_{m_k}, x) < \varepsilon/2$$

because $x_{m_k} \to x$, and M s.t. $n, m_k > M$ gives

$$d(x_m, x_{m_k}) < \varepsilon/2$$

because x_m is Cauchy. Take $k > K$ s.t. $m_k \geq M$. Then for $n > M$ we have

$$d(x_m, x) \leq d(x_m, x_{m_k}) + d(x_{m_k}, x) < \varepsilon/2 + \varepsilon/2 < \varepsilon$$

which is what we wanted to show. This proof should generalize to $\mathbb{R}^N$ if you argue that along each coordinate, (x_m) is bounded and then use that to find a candidate limit x. Then the identical argument goes through (note it used generic properties of the distance rather than anything specific to $\mathbb{R}$). □

Remark 1. The sticking point about "completeness" being required has to do with the fact the candidate limit x needn't be in the space (e.g. take some sequence in $\mathbb{Q}$ that "converges" to $\pi \notin \mathbb{Q}$). □

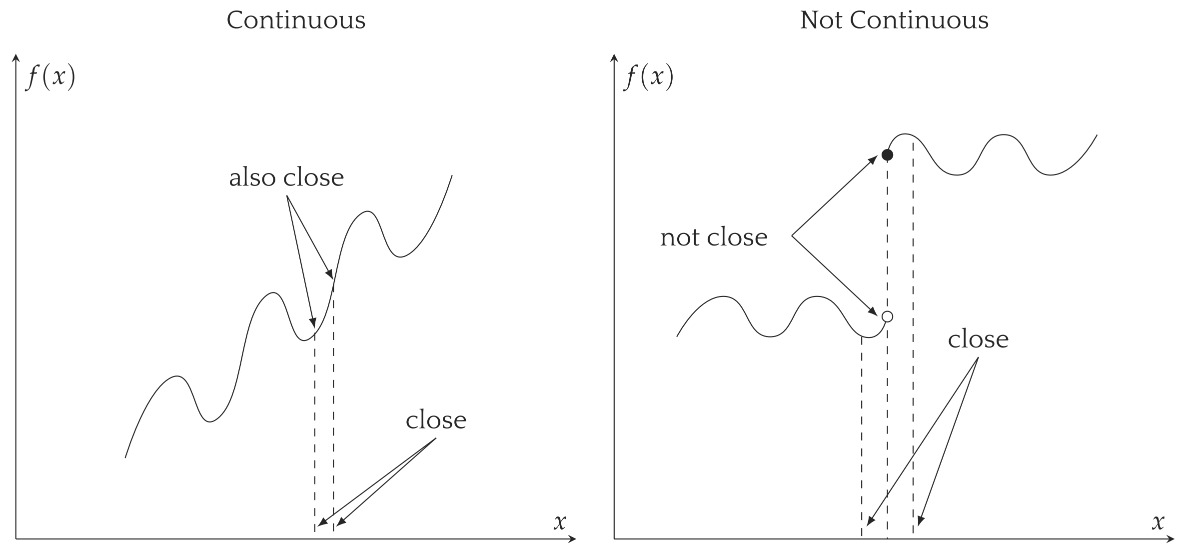
2. Continuous Functions

The intuition for continuity is that "you can draw the function without picking up your pencil." (What about asymptotes like $1/x$ at 0?[6])

More precisely, the idea is that if two points in the domain are close, then the corresponding points in the co-domain must also be close. Put another way, a small neighborhood in the domain maps to a small neighborhood in the co-comain. (Note that the converse is not true! Take $f(x) = x^2$; $f(-x) = f(x) = x^2$, so the points are close in the image—they are identical—but as x gets large, $x, -x$ get farther appart.)

[6]The intuition should still hold because the function is not defined at 0; with infinitely long paper you needn't pick up your pencil.

Figure 1: Intuition for Continuity



Definition 5. A function $f : X \to Y$ is ***continuous*** at $x \in X$ if for every $\varepsilon > 0$ there exist a $\delta > 0$ s.t.

$$d(z, x) < \delta \implies d(f(z), f(x)) < \varepsilon$$

Put another way, $z \in B_\delta(x) \implies f(z) \in B_\varepsilon(f(x))$, or

$$f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$$

If f is continuous at every $x \in X$ we say it is continuous.

Proposition 1. *Let $\varphi : \mathbb{R}^N \to \mathbb{R}$. If φ is continuous then the sets $\{x \in \mathbb{R}^N : \varphi(x) \geq \alpha\}$ and $\{x \in \mathbb{R}^N : \varphi(x) \leq \alpha\}$ are closed for all $\alpha \in \mathbb{R}$.*

Proof. Let $A = \{x \in \mathbb{R}^N : \varphi(x) \geq \alpha\}$ and $B = \mathbb{R}^N \setminus A$. If B is open then A is closed. Note

$$B = \left\{x \in \mathbb{R}^N : \varphi(x) < \alpha\right\}$$

Pick any $x \in B$ and let $0 < \varepsilon < \alpha - \varphi(x)$. Then since φ is continuous we know there is some $\delta > 0$ s.t.

$$z \in B_\delta(x) \implies \varphi(x) - \varepsilon < \varphi(z) < \varphi(x) + \varepsilon < \alpha \implies z \in B$$

therefore B is open (for any point $\exists\delta$-ball that is entirely in B). Thus $\mathbb{R}^N \setminus B = A$ is closed. The proof for the lower sets is analogous. □

Theorem 9. *Let $f : X \to Y$ and $g : f(X) \subseteq Y \to Z$ with $X, Y, Z \subseteq \mathbb{R}^N$. If f, g are continuous then $(g \circ f) : X \to Z$ is continuous.*

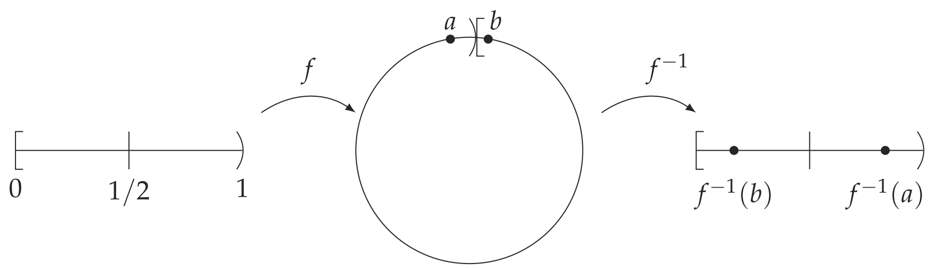
Theorem 10. *Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be continuous functions. Then*

1. *$h(x) = f(x) \pm g(x)$ are continuous functions.*
2. *$h(x) = f(x) \cdot g(x)$ is continuous.*
3. *$h(x) = f(x)/g(x)$ is continuous whenever $g(x) \neq 0$.*

3. Sequential and Open Set Characterizations

Remark 2. Continuous functions don't map open sets to open sets! Consider $f(x) = x^2$. The image of $(-2, 2)$ is $[0, 2)$, which is not open. Further, evern if all maps of closed sets are closed, the function might not be continuous. For example, Consider a function that is 0 if $x < 0$ and 1 if $x \geq 1$. This has a jump at 1, but $f([a, b])$ is either $\{0\}, \{1\}$, or $\{0, 1\}$, which are closed.

Last, if a function is continuous the inverse image need not be. Consider a function $f : [0, 1) \to \mathbb{R}^2$ that maps the line into a circle:



We can see that a, b are closed in the image, but not in the inverse image. □

The remarks above all get to the same idea: Continuity states that points in the image are close if the points in the domain are close, not the converse. Hence we have the following characterization in terms of the *inverse image* and *converging sequences*.

Theorem 11. *The following are equivalent for any $f : \mathbb{R}^N \to \mathbb{R}^M$:*

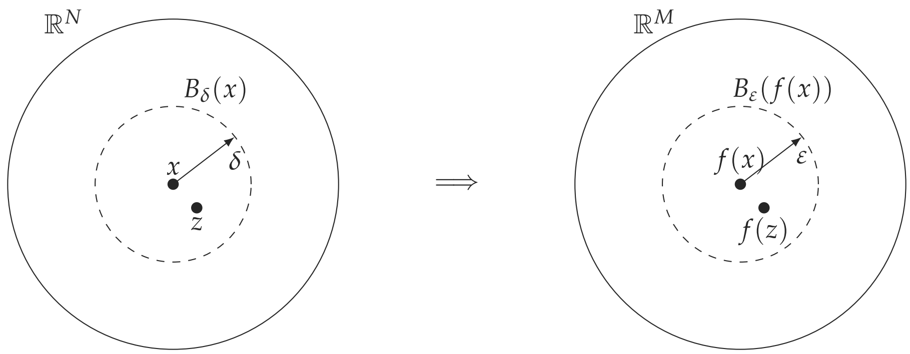
1. *f is continuous.*
2. *If $O \subseteq \mathbb{R}^M$ is open, $f^{-1}(O)$ is open.*
3. *If $S \subseteq \mathbb{R}^M$ is closed, $f^{-1}(S)$ is closed.*
4. *For every $(x_m) \in \mathbb{R}^N$ s.t. $x_m \to x$ for some $x \in \mathbb{R}^N$, $f(x_m) \to f(x)$.*

Proof. See Subsection 3.1. □

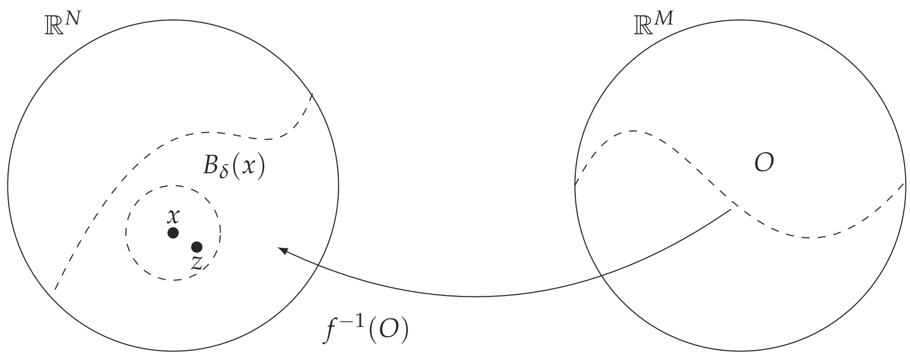
3.1. Sets and Continuity (Visualization). We show Theorem 11 using, again, drawings in $\mathbb{R}^2$.

Proof. We cycle through the statements. If we show $a \implies b \implies c \implies d \implies a$ then we have shown they are equivalent.

a. $a \implies b$. First, it is a good idea to write down what you have and what you want to show:
   - Continuity means that $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $z \in B_\delta(x) \implies f(z) \in B_\varepsilon(f(x))$.

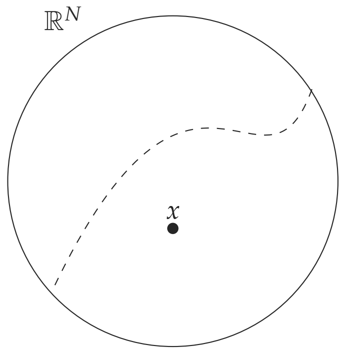


We want to show that if $O \subseteq \mathbb{R}^M$ is open, then $f^{-1}(O)$ is open. That is, $\forall x \in f^{-1}(O)\ \exists \delta > 0$ s.t. $z \in B_\delta(x) \implies z \in f^{-1}(O)$. You will note this is a very similar statement!

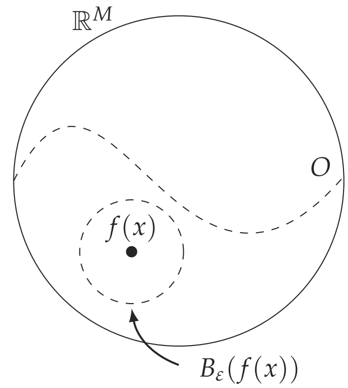


- It's basically the same picture: All we are missing is $B_\varepsilon(f(x))$ and it looks like we're done. How do we get it? We use the fact that O is open in $\mathbb{R}^M$.

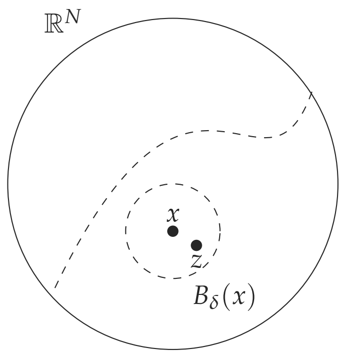
Pick any $x \in f^{-1}(O)$



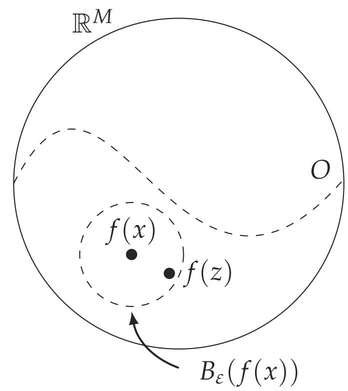
$\exists \varepsilon > 0$ s.t. $B_\varepsilon(f(x)) \subseteq O$



$\exists \delta$ s.t. $z \in B_\delta(x) \implies f(z) \in B_\varepsilon(f(x))$
(by the continuity assumption!)



$\Longrightarrow$



Note $f(z) \in B_\varepsilon(f(x)) \subseteq O$, so $z \in f^{-1}(O)$. This statement is the heart of the proof! It is not obvious that $B_\delta(x)$ will be contained in $f^{-1}(O)$, so we need the link with $B_\varepsilon(f(x))$ we drew above. Only then can we say that for arbitrary x we found $\delta > 0$ s.t. $z \in B_\delta(x) \implies z \in f^{-1}(O)$; by definition that means the set is open.

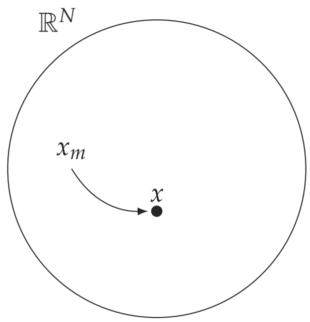
b. We show $c \iff b$. First, consider any closed set $S \in \mathbb{R}^M$, so $\mathbb{R}^M \setminus S$ is open; by premise, $f^{-1}(\mathbb{R}^M \setminus S)$ is also open, which means $\mathbb{R}^N \setminus f^{-1}(\mathbb{R}^M \setminus S) = f^{-1}(\mathbb{R}^N \setminus (\mathbb{R}^M \setminus S)) = f^{-1}(S)$ is closed. (The only sticking point here would be to show that in general $f^{-1}(\mathbb{R}^M \setminus O) = \mathbb{R}^N \setminus f^{-1}(O)$, which I trust you can do.[7]) Now consider open set $O \in \mathbb{R}^M$, so $\mathbb{R}^M \setminus O$ is open; by premise, $f^{-1}(\mathbb{R}^M \setminus O)$ is also closed, which means $\mathbb{R}^N \setminus f^{-1}(\mathbb{R}^M \setminus O) = f^{-1}(O)$ is open. You will notice this is an entirely analogous argument.

c. For this one it is easier to show that $b \implies d$ (noting we already argued $c \implies b$).

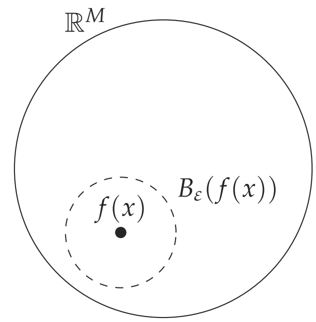
- $\forall O \subseteq \mathbb{R}^M$, if O open then $f^{-1}(O)$ open. This means that $\forall x \in f^{-1}(O)\ \exists \delta$ s.t. $B_\delta(x) \subseteq f^{-1}(O)$.
- We WTS $x_m \to x \implies f(x_m) \to f(x)$; i.e. $\forall \varepsilon > 0\ \exists M$ s.t. $m \geq M \implies f(x_m) \in B_\varepsilon(f(x))$.

[7] The way to prove two sets are equal is to show either set conains the other. Take any $x \in f^{-1}(\mathbb{R}^M \setminus O) \subseteq \mathbb{R}^N$, so $f(x) \in \mathbb{R}^M \setminus O$. If $x \in f^{-1}(O)$ then $f(x) \in O$, contradiction. Hence $x \in \mathbb{R}^N$ and $x \notin O$, so $x \in \mathbb{R}^N \setminus f^{-1}(O)$. Pick $z \in \mathbb{R}^N \setminus f^{-1}(O)$. If $f(z) \in O$ then $z \in f^{-1}(O)$, contradiction. Hence $f(z) \in \mathbb{R}^M \setminus O$, which means $z \in f^{-1}(\mathbb{R}^M \setminus O)$.

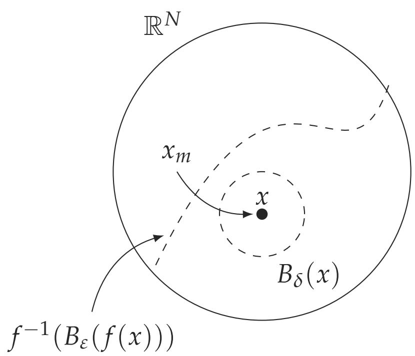
Start with any sequence $x_m \to x$



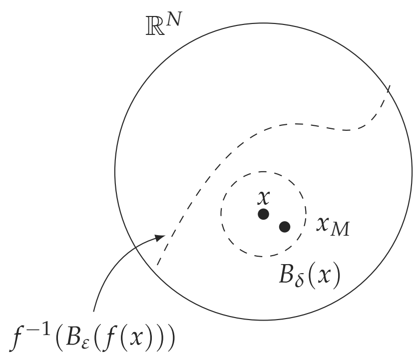
Pick any $\varepsilon > 0$ and note $B_\varepsilon(f(x))$ is open.



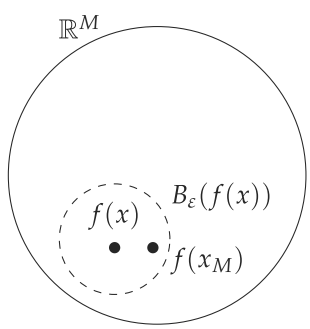
Hence $f^{-1}(B_\varepsilon(f(x)))$ is open, and
$\exists \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$



$x_m \to x$ so $\exists M$ s.t.
$m \geq M \implies x_m \in B_\delta(x)$



$$x_m \in B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x))) \implies f(x_m) \in B_\varepsilon(f(x))$$



Hence for any $x_m \to x$, for any $\varepsilon > 0$ we found M s.t.

$$m \geq M \implies x_m \in f^{-1}(B_\varepsilon(f(x))) \implies f(x_m) \in B_\varepsilon(f(x))$$

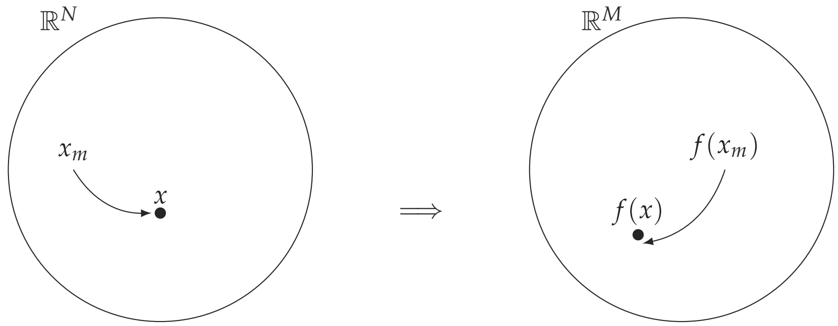
which by definition means $f(x_m) \to f(x)$. The tricky step here was that $f^{-1}(B_\varepsilon(f(x)))$ does not need to be a nice set. We need the premise that the inverse image of open sets is open so that we can fit a neighborhood inside of it, and *then* use the fact $x_m \to x$.

d. Finally, we show that $d \implies a$. We do this by contradiction.

- It is not clear why contradiction is the way to go; it boils down to the fact I think it's easier, but I don't think that's obvious. In general, if unsure how to start a proof, one strategy is to try to make progress with a direct proof, and if you get stuck, switch to contradiction or contrapositive to see if it helps.

- First, we have $x_m \to x \implies f(x_m) \to f(x)$.

If we had a sequence $x_m \to x$ Then we'd know $f(x_m) \to f(x)$.



- We want to show that $\forall x \in \mathbb{R}^N \ \forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $z \in B_\delta(x) \implies f(z) \in B_\varepsilon(f(x))$.

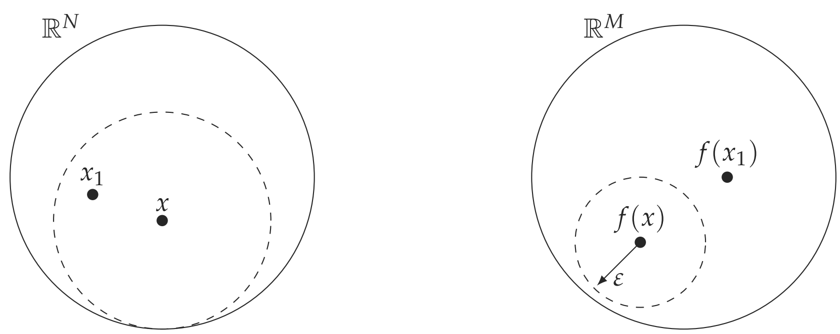
- A great starting point is to construct a sequence in $\mathbb{R}^N$ that converges to x, because the premise here is a statement about sequences. I don't see an obvious way to do this directly, but if we think about doing contradiction, we can negate the previous bullet point:

$$\exists x \in \mathbb{R}^N \ \exists \varepsilon > 0 \ \text{ s.t. } \ \forall \delta > 0 \ \exists z \in B_\delta(x) \ \text{ and } \ f(z) \notin B_\varepsilon(f(x))$$

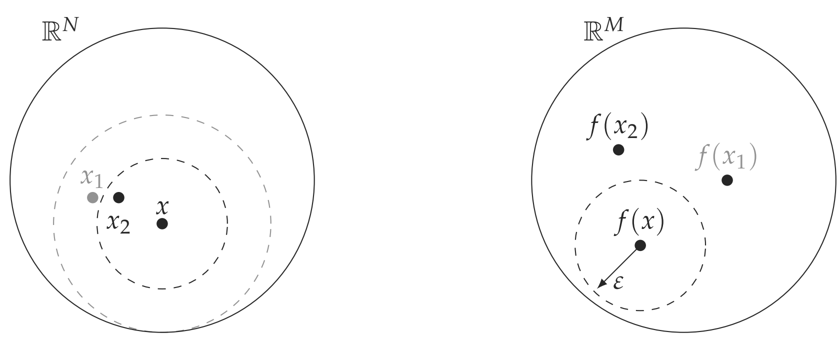
  Note that x and ε here are fixed, and that we don't get to choose z—all we know is one such a z exists. However, δ is a free parameter here, because this must be true for any δ.

- If we pick $\delta = 1/m$ then we can construct a sequence $x_m \to x$ s.t. $f(x_m) \notin B_\varepsilon(f(x))$:

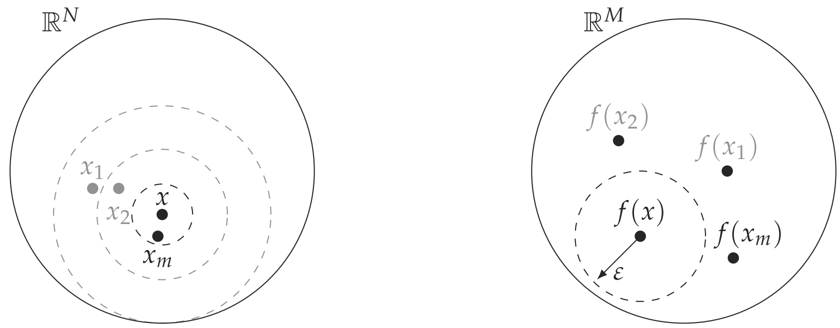
For $m = 1$ $\exists x_1 \in B_1(x)$ s.t. $f(x_1) \notin B_\varepsilon(f(x))$



For $m = 2$ $\exists x_2 \in B_{1/2}(x)$ s.t. $f(x_2) \notin B_\varepsilon(f(x))$



$$\forall m \;\; \exists x_m \in B_{1/m}(x) \text{ s.t. } f(x_m) \notin B_\varepsilon(f(x))$$



We can see as x_m becomes increasingly closer to x, $f(x_m)$ is always at least ε away from $f(x)$. In other words, we have constructed a sequence $x_m \to x$ were $f(x_m) \not\to f(x)$, contradiction. □

4. Intermediate Value Theorem (IVT)

Theorem 12 (Intermediate Value Theorem (IVT)). *If $f : [a, b] \to \mathbb{R}$ is continuous in $[a, b]$ then*

$$\forall L : f(a) < L < f(b) \;\; \exists c \in [a, b] \quad \textit{s.t.} \quad f(c) = L$$

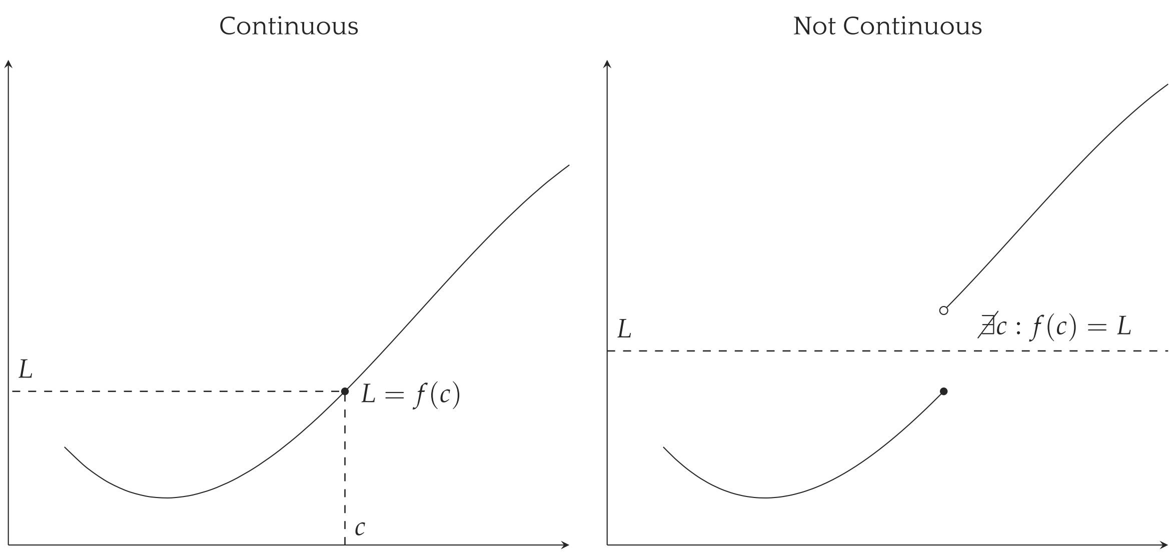


Figure 2: Intermediate Value Theorem (IVT)

Proof. Take $A = \{x \in [a, b] : f(x) < L\}$. A is bounded so the sup exists; let $c = \sup A$. For $\varepsilon_m = 1/m$ take $x_m \in (c - \varepsilon_m, c] \cap A$, so $x_m \to c$ and $f(x_m) \to f(c)$ (by continuity). Since $x_m \in A$, $f(x_m) < L$ and so $f(c) \leq L$. If $f(c) = L$ we are done; if $f(c) < L$ then, by continuity, for $\varepsilon : f(a) < f(c) - \varepsilon < f(c) + \varepsilon < L \;\; \exists \delta$ s.t.

$$x : c - \delta < c < x < c + \delta \implies |f(x) - f(c)| < \varepsilon \implies f(a) < f(x) < L \implies x \in A$$

However, $x \leq \sup A = c$ for any such x, contradiction. Hence $f(c) = L$. □

Keywords