Problem Set 3

1. *Take a collection of functions with $f_i : \Omega \to \mathbb{R}^N$, $\Omega \subseteq \mathbb{R}^M$, $i \in \mathbb{N}$. The collection $\{f_i\}$ defines a sequence of funtions, and for each $x \in \Omega$ we have a possibly different sequence $\{f_i(x)\}$ in $\mathbb{R}^N$.*

   Let $\{f_i\}$ be a sequence of functions, with $f_i : \Omega \to \mathbb{R}^N$ and $\Omega \subseteq \mathbb{R}^M$. We say that $\{f_i\}$ ***point-wise convrges*** *to $f : \Omega_0 \to \mathbb{R}^N$ if $x \in \Omega_0 \implies f_i(x) \to f(x)$.*

   Let $\{f_i\}$ be a sequence of functions, with $f_i : \Omega \to \mathbb{R}^N$ and $\Omega \subseteq \mathbb{R}^M$. We say that $\{f_i\}$ ***uniformly convrges*** *to $f : \Omega_0 \to \mathbb{R}^N$ if $\forall \varepsilon > 0\ \exists I_0(\varepsilon)$ s.t. for $i > I_0(\varepsilon)$ we have $\|f_i(x) - f(x)\| < \varepsilon$.*

   a) *Let $f_i(x) = x/i$ and $f(x) = 0$. Check that $f_i \to f$ point-wise.*

   b) *Show f_i defined above does not converge uniformly to f.*

   c) *Show uniform convergence implies point-wise convergence.*

2. *Let $A \subseteq \mathbb{R}^N$ be a convex set. $f : A \to \mathbb{R}^N$ is quasiconcave if for any $x, y \in A$ and $\alpha \in [0, 1]$ we have*

$$f(\alpha x + (1-\alpha)y) \geq \min\{f(x), f(y)\}$$

   and strictly quasiconcave if the above holds strictly. Show if f is quasiconcave then $\operatorname{argmax}_{x \in A} f(x)$ is a convex set (recall the empty set is convex by vacuity). Further show that if f is strictly quasiconcave then $\operatorname{argmax}_{x \in A} f(x)$ is a singleton or empty.

3. *Consider a continuous function $f : \mathbb{R}^N \to \mathbb{R}$. Show*

   a) *If f is differentiable and $x^* \in \mathbb{R}^N$ is a local maximizer or minimizer of f, then $\nabla f(x^*) = 0$.*

   b) *If f is twice continuously differentiable and $x^* \in \mathbb{R}^N$ is s.t. $\nabla f(x^*) = 0$, then if x^* is a local maximizer the symmetric $N \times N$ Hessian $D^2 f(x^*)$ is negative semidefinite. Extra credit: If $D^2 f(x^*)$ is negative definite then x^* is a strict local maximizer. (Hint: I used a Taylor expansion without the explicit remainder formula. For the extra-credit, I additionally leveraged the fact a matrix is ND iff it has all strictly negative eigenvalues, but there may be a way to do it without that.)*

   c) *If f is concave then $f(x + z) \leq f(x) + z^T Df(x)$ for any x, z.*

   d) *If f is concave then any critical point (i.e. x s.t. $Df(x) = 0$) is a global maximizer.*

4. *Define the set $\Delta = \{p \in \mathbb{R}^L_+ : \sum_l p_l = 1\}$ and the function z^+ on Δ as $z^+_l(p) = \max\{z_l(p), 0\}$, where $z(p) = \{z_1(p), z_2(p), \ldots, z_L(p)\}$ is a continuous function, homogeneous of degree 0, and satisfying $p \cdot z(p) = 0$ for all $p \in \mathbb{R}^L$. Denote $\alpha(p) = \sum_l \left[p_l + z^+_l\right]$.*

   a) *Show that Δ is a non-empty compact and convex set.*

   b) *Show that $f : \Delta \to \Delta$ is continuous in p.*

$$f(p) = \frac{1}{\alpha(p)}\left(p + z^+(p)\right)$$

   c) *Prove that f has a fixed point. (Hint: You can use existing theorems!)*

d) *Use the fact f has a fixed point and the properties of z to argue that $\exists p^*$ s.t. $z^+(p^*) \cdot z(p^*) = 0$. (Hint: Use the fact $p^* \cdot z(p^*) = 0$.)*

e) *Conclude thet $z(p^*) \leq 0$.*

Remark 1. If for consumer i we define the excess demand function $z_i(p) = x_i(p, \omega_i) - \omega_i$ for wealth ω_i and prices p. One way to define general equilibrium is vector of prices s.t. $\sum_i z_i(p) \leq 0$ for all i (i.e. there is no aggregate excess demand). You have just shown that under some conditions such a price vector always exists. $\square$

5. *Use the chain rule and the FTC to prove the Leibniz rule:*

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}$$