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Orbital Debris

Abstract

We study orbital debris. We derive a model which computes the trajectory of objects in orbit, taking into account orbital altitudes as well as forces which may be exerted to change a trajectory. We fit atmospheric data to a model to calculate how long different objects stay in orbit. We then derive a method to remove space debris using a spacecraft while optimising fuel efficiency and calculating how much fuel is required to reach our target.

1 Introduction

Whenever something is discarded in orbit, whether it be a defunct spacecraft or a tiny speck of paint, it becomes space debris. This is a major problem for space agencies due to the immense potential damage that could be caused if the debris were to collide with a satellite. To illustrate the scale of this issue, the ISS is only designed to withstand the impact of debris as large as $1cm$ in diameter, but there are an estimated 500,000 particles with diameter between 1 and $10cm$ in orbit[1].

In this essay, we will explore the dynamics of objects in orbit and derive a model to compute their trajectories. We will calculate how long it takes for orbital debris to re-enter the atmosphere naturally and then investigate how we could go about removing space debris manually by modelling the techniques necessary to alter a spacecraft's trajectory in order to intercept a piece of debris. We will aim to discover the methods required to reach our target and refine our trajectories in order to use the smallest amount of fuel possible. However, this is not as easy as it may appear. As a result of the orbital dynamics involved, simply thrusting in the direction of our target becomes counter-productive. Let us now investigate the dynamics of objects in orbit.

2 Orbital Dynamics

We can describe the dynamics of an object orbiting the Earth using Newton's equations:

$$\begin{aligned} m\ddot{x} &= -\frac{GM_E m}{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}} + F_x(t) \\ m\ddot{y} &= -\frac{GM_E m}{x^2+y^2}\frac{y}{\sqrt{x^2+y^2}} + F_y(t), \end{aligned} \tag{1}$$

where G is Newton's constant, M_E is the mass of the Earth and m is the mass of the spacecraft. Forces exerted on the object, such as from the thrusters of a spacecraft or friction, are represented by F_x and F_y for forces exerted along the x and y axis respectively. In order to solve these equations, it is easier if we first convert them to polar co-ordinates:

$$x = r\sin(\theta), \qquad y = r\cos(\theta). \tag{2}$$

We first differentiate equations 2 twice with respect to t to obtain $\ddot{x}$ and $\ddot{y}$ in polar co-ordinates:

$$\begin{aligned} \dot{x} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \\ \dot{y} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \\ \ddot{x} &= \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta + r\ddot{\theta}\cos\theta \\ \ddot{y} &= \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta - r\ddot{\theta}\sin\theta \end{aligned} \tag{3}$$

Substituting 3 and 2 into 1, we see for $m\ddot{x}$ and $m\ddot{y}$ respectively:

$$\begin{aligned} m(\ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta + r\ddot{\theta}\cos\theta) &= -\frac{GM_E m}{r^2(\sin^2\theta+\cos^2\theta)}\frac{r\sin\theta}{\sqrt{r^2(\sin^2\theta+\cos^2\theta)}} + F_x \\ m(\ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta - r\ddot{\theta}\sin\theta) &= \frac{GM_E m}{r^2(\sin^2\theta+\cos^2\theta)}\frac{r\cos\theta}{\sqrt{r^2(\sin^2\theta+\cos^2\theta)}} + F_y. \end{aligned}$$

Then using $\sin^2\theta + \cos^2\theta = 1$ and simplifying, we obtain for $m\ddot{x}$ and $m\ddot{y}$ respectively:

$$\begin{aligned} m(\ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta + r\ddot{\theta}\cos\theta) &= -\frac{GM_E m}{r^2}\frac{r\sin\theta}{\sqrt{r^2}} + F_x \\ &= -\frac{GM_E m}{r^2}\sin\theta + F_x \end{aligned} \tag{4}$$

$$\begin{aligned} m(\ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta - r\ddot{\theta}\sin\theta) &= -\frac{GM_E m}{r^2}\frac{r\cos\theta}{\sqrt{r^2}} + F_y \\ &= -\frac{GM_E m}{r^2}\cos\theta + F_y. \end{aligned} \tag{5}$$

Rearranging 4 and 5 respectively for $\ddot{r}$, we obtain

$$\begin{aligned} \ddot{r} &= -\frac{GM_E}{r^2} + r\dot{\theta}^2 + \frac{F_x}{m\sin\theta} - \frac{\cos\theta}{\sin\theta}(2\dot{r}\dot{\theta} + r\ddot{\theta}) \\ \ddot{r} &= -\frac{GM_E}{r^2} + r\dot{\theta}^2 + \frac{F_y}{m\cos\theta} + \frac{\sin\theta}{\cos\theta}(2\dot{r}\dot{\theta} + r\ddot{\theta}). \end{aligned} \tag{6}$$

Equating the two equations for $\ddot{r}$ in 6, we have

$$-\frac{GM_E}{r^2} + r\dot{\theta}^2 + \frac{F_x}{m\sin\theta} - \frac{\cos\theta}{\sin\theta}(2\dot{r}\dot{\theta} + r\ddot{\theta}) = -\frac{GM_E}{r^2} + r\dot{\theta}^2 + \frac{F_y}{m\cos\theta} + \frac{\sin\theta}{\cos\theta}(2\dot{r}\dot{\theta} + r\ddot{\theta}), \tag{7}$$

which implies

$$\frac{F_x}{m\sin\theta} - \frac{\cos\theta}{\sin\theta}(2\dot{r}\dot{\theta} + r\ddot{\theta}) = \frac{F_y}{m\cos\theta} + \frac{\sin\theta}{\cos\theta}(2\dot{r}\dot{\theta} + r\ddot{\theta}), \tag{8}$$

which can be rearranged to obtain

$$\frac{F_x}{m\sin\theta} - \frac{F_y}{m\cos\theta} = \frac{\cos\theta}{\sin\theta}(2\dot{r}\dot{\theta} + r\ddot{\theta}) + \frac{\sin\theta}{\cos\theta}(2\dot{r}\dot{\theta} + r\ddot{\theta}), \tag{9}$$

which we can simplify to

$$\frac{F_x\cos\theta - F_y\sin\theta}{m\sin\cos\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}(2\dot{r}\dot{\theta} + r\ddot{\theta}). \tag{10}$$

Multiplying through both sides by $\sin\theta\cos\theta$, and using the fact that $\sin^2\theta + \cos^2\theta = 1$, we find

$$\frac{F_x\cos\theta - F_y\sin\theta}{m} = 2\dot{r}\dot{\theta} + r\ddot{\theta}. \tag{11}$$

Substituting 11 into our second equation for $\ddot{r}$ in 6, we obtain

$$\begin{aligned} \ddot{r} &= -\frac{GM_E}{r^2} + r\dot{\theta}^2 + \frac{F_y}{m\cos\theta} + \frac{\sin\theta}{\cos\theta}(\frac{F_x\cos\theta - F_y\sin\theta}{m}) \\ &= -\frac{GM_E}{r^2} + r\dot{\theta}^2 + \frac{F_y + F_x\sin\theta\cos\theta - F_y\sin^2\theta}{m\cos\theta} \\ &= -\frac{GM_E}{r^2} + r\dot{\theta}^2 + \frac{F_x\sin\theta\cos\theta + F_y(1-\sin^2\theta)}{m\cos\theta} \\ &= -\frac{GM_E}{r^2} + r\dot{\theta}^2 + \frac{F_x\sin\theta\cos\theta + F_y(\cos^2\theta)}{m\cos\theta} \\ &= -\frac{GM_E}{r^2} + r\dot{\theta}^2 + \frac{F_x\sin\theta + F_y\cos\theta}{m}. \end{aligned} \tag{12}$$

Now defining $F_r = F_x \sin\theta + F_y \cos\theta$, we obtain

$$\ddot{r} \;=\; -\frac{GM_E}{r^2} + r\dot{\theta}^2 + \frac{F_r}{m}. \tag{13}$$

Now rearranging 11 for $\ddot{\theta}$, we obtain:

$$\ddot{\theta} \;=\; -2\frac{\dot{r}\dot{\theta}}{r} + \frac{F_x\cos\theta - F_y\sin\theta}{mr}, \tag{14}$$

and defining $F_\theta = F_x\cos\theta - F_y\sin\theta$, we obtain

$$\ddot{\theta} \;=\; -2\frac{\dot{r}\dot{\theta}}{r} + \frac{F_\theta}{mr}. \tag{15}$$

Notice that if we take $F_r = F_\theta = 0$, then the simplest solution is given by

$$r = r_0, \qquad \theta = \omega_0 t + \theta_0, \tag{16}$$

which corresponds to a circular orbit of period

$$T_0 = \frac{2\pi}{\omega_0}, \qquad \omega_0 = \sqrt{\frac{GM_E}{r_0^3}}, \tag{17}$$

where r_0 and ω_0 are both constant.

We can now derive the expression for the speed of a piece of debris as a function of G, M_E and r_0 (assuming it follows a circular orbit). We know that the distance travelled in one period is the circumference of the orbit, $2\pi r_0$. Substituting this and our T_0 from 17 into the definition of speed, we have:

$$\begin{aligned} speed &= \frac{2\pi r_0}{T_0} \\ &= 2\pi r_0 \div \frac{2\pi}{\sqrt{\frac{GM_E}{r_0^3}}} \\ &= r_0\sqrt{\frac{GM_E}{r_0^3}} \\ &= \sqrt{\frac{GM_E r_0^2}{r_0^3}} \\ &= \sqrt{\frac{GM_E}{r_0}} \end{aligned} \tag{18}$$

We can use the expression for T in 17 to compute the time it takes a piece of debris to go around the earth in a circular orbit. As an example, if we take the lower and higher orbits of the International Space Station, $330km$ and $435km$ respectively, we find:

For $h = 330km$ (where h is the altitude of the orbit above the surface of the Earth in m):

$$\begin{aligned} T &= \frac{2\pi}{\sqrt{\frac{GM_E}{(h+R_E)^3}}} \\ T &= \frac{2\pi}{\sqrt{\frac{GM_E}{(330000+6370000)^3}}} \\ &= 5459.4s \\ &\approx 91mins. \end{aligned}$$

For $h = 435km$:

$$
\begin{aligned}
T &= \frac{2\pi}{\sqrt{\frac{GM_E}{(h+R_E)^3}}} \\
T &= \frac{2\pi}{\sqrt{\frac{GM_E}{(435000+6370000)^3}}} \\
&= 5588.2s \\
&\approx 93mins.
\end{aligned}
$$

3 Atmospheric capture of debris

Debris in orbit travel at very high speeds and as a result are progressively slowed down by the friction force exerted by the atmosphere. Eventually, they are slowed enough that they reach the denser part of the atmosphere where they burn out. The time that this takes depends on the orbit of the debris as well as its shape and size. We will try to estimate the friction force and the how this affects the altitude of the debris. Since the friction force is expected to be small, we can assume that the altitude will decrease very slowly, and we will also assume the debris is in a circular orbit for simplicity. We have already derived the speed of a piece of debris on a circular orbit in 18, so calling $r_h = R_E + h$ the radius of the debris' orbit, where R_E is the radius of the Earth and h is the altitude of the orbit above the Earth's surface, we can compute the energy of the debris as the sum of its gravitational energy and kinetic energy.

Using its definition, we can calculate the gravitational energy of the debris to be

$$
P_h = -\frac{GmM_E}{r_h}, \tag{19}
$$

while the kinetic energy is given by

$$
K_h = \frac{1}{2}mv^2. \tag{20}
$$

Substituting in our expression for v from 18 (using $r_0 = R_E + h = r_h$), we obtain

$$
\begin{aligned}
K_h &= \frac{m}{2}\sqrt{\frac{GM_E}{r_h}}^2 \\
&= \frac{GmM_E}{2r_h}.
\end{aligned} \tag{21}
$$

Adding 19 and 21, we obtain the total energy:

$$
\begin{aligned}
E_h &= E_h = P_h + K_h \\
&= -\frac{GmM_E}{r_h} + \frac{GmM_E}{2r_h} \\
&= -\frac{GmM_E}{2r_h}.
\end{aligned} \tag{22}
$$

To estimate the friction force on a piece of debris we will assume that the atmospheric molecules are at rest and that their relative speed with the piece of debris is the orbital speed of the piece of debris. The debris will collide with $v_h A \rho_{At}$ kg of molecules where ρ_{At} is the density of the atmosphere and A is the average area cross-section of the piece of debris. When the piece of debris hits a molecule, the molecule will bounce from it with a speed equal to the initial relative speed between the two objects but with the opposite sign, as the molecules are much lighter than debris. The moment gained by the molecules is equal to the moment lost by the debris and the resulting force on the pieces of debris is given by

$$F_{fr} = 2A\rho_{At}v_h^2. \tag{23}$$

The energy loss due to this force is equal to the force multiplied by the distance travelled, so the power dissipation is then given by the force multiplied by the speed of the piece of debris:

$$\frac{dE_h}{dt} = -2A\rho_{At}v_h^3. \tag{24}$$

The average density of the high atmosphere which can be found in [2] can be fitted reasonably well with the function

$$\rho_{At}(h) = ae^{-hl} + B(\frac{h}{h_0})^{-\sigma}, \tag{25}$$

where h is in km, ρ in kg/m^3, and

$$B = 8.82 \times 10^7 kg/m3, \quad h_0 = 1km \tag{26}$$

We will write a python code to fit the atmospheric density data to our function for the parameters a, l and σ, fixing $B = 8.82 \times 10^7$. This produces the graph 1 which shows us how the density of the atmosphere changes with altitude.

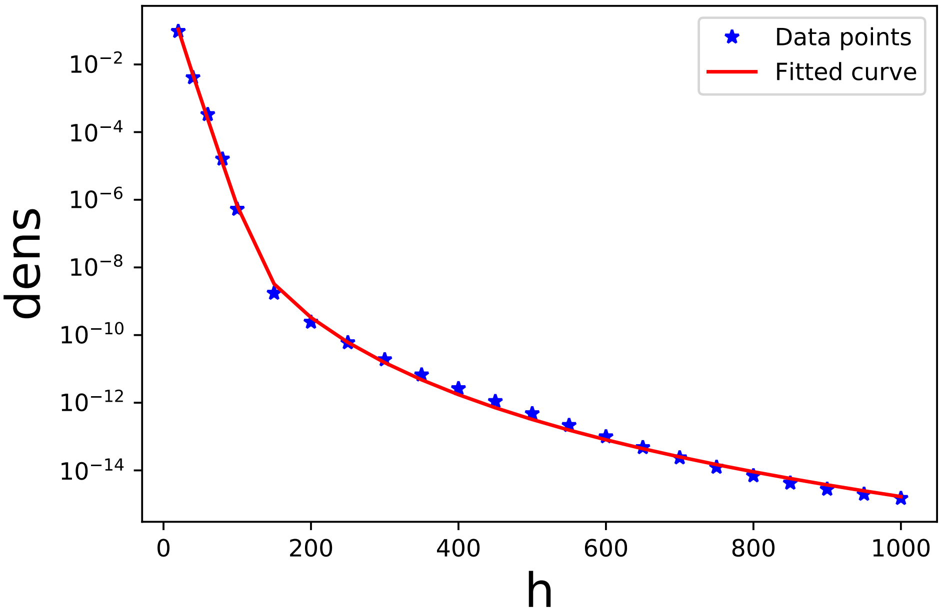


Figure 1: Fitting atmospheric density data with our equation

The program returns us values of

$$a = 1.946kg/m^3, \quad l = 0.15/km, \quad \sigma = 7.57. \tag{27}$$

We can convert 26 and 27 so that is h is measured in metres, so we have

$$a = 1.946kg/m^3, \quad l = 1.5 \times 10^{-4}, \quad B = 4.63 \times 10^30 kg/m3, \quad \sigma = 7.57, \quad h_0 = 1m. \tag{28}$$

We can use 22 and our expression for v from 18 to express v as a function of E_h. Rearranging 22 for GM_E:

$$GM_E = -\frac{2E_h r_h}{m}, \tag{29}$$

and then substituting this into our expression for v, using $r_0 = r_h$, we obtain

$$\begin{aligned} v &= \sqrt{\frac{GM_E}{r_h}} \\ &= \sqrt{-\frac{2E_h r_h}{m r_h}} \\ &= \sqrt{-\frac{2E_h}{m}}, \end{aligned} \tag{30}$$

We can also use 22 to compute dE/dt as a function of dr_h/dt using the chain rule:

$$\begin{aligned} E &= -\frac{GmM_E}{2r_h} \\ \frac{dE}{dt} &= \frac{dE}{dr_h}\frac{dr_h}{dt} \\ &= \frac{GmM_E}{2r_h^2}\frac{dr_h}{dt}. \end{aligned} \tag{31}$$

Now using 24, as well as 30 and the fact that $h = r_h - R_E$, we can express dE/dt as a function of A, ρ_{At} and R_E. Substituting 30 into 24, we obtain

$$\frac{dE}{dt} = -2A\rho_{At}\sqrt{-\frac{2E}{m}}^3$$

Then substituting in 22, we obtain

$$\begin{aligned} \frac{dE}{dt} &= -2A\rho_{At}\sqrt{\frac{2\frac{GmM_E}{2r_h}}{m}}^3 \\ &= -2A\rho_{At}\sqrt{\frac{GM_E}{r_h}}^3, \end{aligned} \tag{32}$$

and finally substituting $h = r_h - R_E$:

$$\frac{dE}{dt} = -2A\rho_{At}\sqrt{\frac{GM_E}{R_E + h}}^3. \tag{33}$$

Now using 22 to rearrange for r_h,

$$r_h = -\frac{GmM_E}{2E_h}, \tag{34}$$

we can use the chain rule to express dr_h/dt in terms of dE/dt

$$\begin{aligned} \frac{dr_h}{dt} &= \frac{dr_h}{dE_h}\frac{dE_h}{dt} \\ &= \frac{GmM_E}{2E_h^2}\frac{dE_h}{dt}. \end{aligned} \tag{35}$$

Noting that since R_E is a constant, $\frac{dr_h}{dt} = \frac{d}{dt}(h + R_E) = \frac{dh}{dt}$, and then substituting 22 and 33 into 35, we have

$$\begin{aligned}\frac{dh}{dt} = \frac{dr_h}{dt} &= \frac{GmM_E}{2(-\frac{GmM_E}{2r_h})^2}(-2A\rho_{At}\sqrt{\frac{GM_E}{R_E+h}}^3) \\ &= \frac{2r_h^2}{GmM_E}(-2A\rho_{At}\sqrt{\frac{GM_E}{R_E+h}}^3). \end{aligned} \tag{36}$$

Substituting $r_h = R_E + h$, we obtain

$$\begin{aligned}\frac{dh}{dt} &= \frac{-4(R_E+h)^2A\rho_{At}}{GmM_E}\sqrt{\frac{GM_E}{R_E+h}}^3 \\ &= \frac{-4(R_E+h)A\rho_{At}}{m}\sqrt{\frac{GM_E}{R_E+h}} \\ &= \frac{-4A\rho_{At}}{m}\sqrt{GM_E(R_E+h)} \end{aligned} \tag{37}$$

This gives a separable first order differentiable equation for h(t). To integrate 37, we will use the approximations $r_h \approx R_E$ and $\rho_{At}(h) \approx Bh^{-\sigma}$, so we have

$$\frac{dh}{dt} = \frac{-4ABh^{-\sigma}}{m}\sqrt{GM_ER_E}. \tag{38}$$

Approximating ρ_{At} in this way results in underestimating the density of the atmosphere below altitudes of $150km$ and consequently overestimating the time of re-entry by a very small amount.

Since 38 is a first order separable differential equation, we can write

$$\int \frac{1}{H(h)}dh = \int dt, \tag{39}$$

where $H(h) = \frac{-4ABh^{-\sigma}}{m}\sqrt{GM_ER_E}$. Then integrating 38 from $h = h_0$, the initial altitude, to $h = 0$, the Earth's surface, we obtain

$$\int_{h_0}^{0} -\frac{m\sqrt{GM_ER_E}}{4AB}h^{\sigma}\, dh = \int dt, \tag{40}$$

which implies

$$\begin{aligned}\left[-\frac{m\sqrt{GM_ER_E}}{4AB}\frac{h^{\sigma+1}}{\sigma+1}\right]_{h_0}^{0} &= t \\ 0 - (-\frac{m\sqrt{GM_ER_E}}{4AB}\frac{h_0^{\sigma+1}}{\sigma+1}) &= t \\ \frac{m\sqrt{GM_ER_E}}{4AB}\frac{h_0^{\sigma+1}}{\sigma+1} &= t, \end{aligned} \tag{41}$$

which can be rearranged to give

$$t = \frac{m}{A}\frac{\sqrt{GM_ER_E}}{4B(\sigma+1)}h_0^{\sigma+1}. \tag{42}$$

This gives us an estimate for the time of re-entry of a piece of debris in the atmosphere as a function of its initial orbital altitude and the ratio m/A.

Let us now consider the following types of debris:

- An aluminium bolt we will model as a plain cube of density $\rho = 2700kg/m^3$ and edge length $L \approx 1cm$ with cross-sectional area $A = L^2$. Then $m = L^3\rho = 0.0027kg$ so we have $m/A = 27$.

- A rod of length L and square cross-section l and density ρ. The average cross-sectional area is $A \approx L \times l/2$, with $m = l^2 L\rho$ and hence $m/A = 2\rho l$. So for aluminium with $L = 10cm$, $l = 1cm$, we have $m/A \approx 50$

- A square aluminium plate of length L and thickness l with an average area $A \approx L^2/2$. Taking $l = 1mm$, we have $m = \rho L^2 l$ so $m/A = 2\rho l = 5.4$.

- Gemini spacecraft with mass $m = 3850kg$ and a larger diameter of $3m$. Then if we model the spacecraft as a sphere of diameter $3m$, we have the average cross-sectional area $A = \pi(1.5)^2$, so $m/A = 15400/9\pi$.

We can write a python program to plot a graph of the time of re-entry, t for the 4 different values of m/A above, shown in Figure 2.

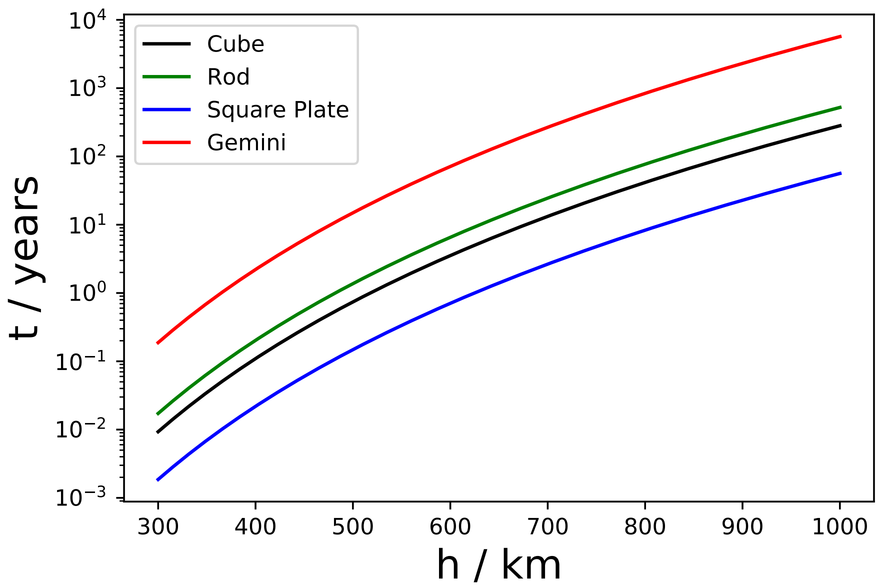


Figure 2: Graph showing how time of re-entry varies with altitude for different objects

According to NASA, [1], space debris at altitudes of below $600km$ usually fall back to Earth within several years. At $800km$, the time before re-entry is often measured in decades, and above $1000km$, debris can remain in orbit for over a century. Comparing Figure 2 with these statistics, our estimates seem to be fairly accurate.

4 Orbital Rendezvouz

As we have shown, debris can take a very long time to come out of orbit. As a result, it may be necessary to perform rendez-vouz missions using spacecraft to clear the debris.

During a rendez-vouz, the cleaning spacecraft starts the manoeuvre close to the target, only a few kilometres apart. This causes a problem to arise with our equations 13 and 15 as the radius, r of the altitudes will be several thousand kilometres and the angle θ covers the full circle while the angular separation between the 2 spacecraft during the manoeuvre will be a small fraction of a degree. This could not only lead to numerical inaccuracies, but also make it more difficult than necessary to track the relative position between the two objects. In order to solve this issue, we will assume that the target piece of debris is on a circular orbit of radius $r_0 = R_E + h$, where h is the altitude of the orbit above the Earth surface, and travelling at the angular speed ω_0 given in 17. Thus the trajectory of the debris will be

$$r = r_0, \quad \theta = \omega_0 t. \tag{43}$$

We will refer to this as the reference trajectory. The intercepting spacecraft will then have the following relative co-ordinates:

$$r = r_0 + z(t), \quad \theta = \omega_0 t + \phi(t). \tag{44}$$

Inserting 44 into 13 and 15, we obtain

$$\begin{aligned} \ddot{z} &= -\frac{GM_E}{(r_0+z)^2} + (r_0+z)(\omega_0+\dot{\phi})^2 + \frac{F_r}{m} \\ \ddot{\phi} &= -2\frac{(\omega_0+\dot{\phi})\dot{z}}{r_0+z} + \frac{F_\theta}{m(r_0+z)}. \end{aligned} \tag{45}$$

To make the pair of second order ordinary differential equations 45 easier to solve, we can convert them into a system of 4 first order ordinary differential equations by defining:

$$g_z(t) = \dot{z}, \qquad g_\phi(t) = \dot{\phi}. \tag{46}$$

We can then substitute 46 into 45 to obtain:

$$\begin{aligned} \dot{z} &= g_z \\ \dot{\phi} &= g_\phi \\ \dot{g}_z &= -\frac{GM_E}{(r_0+z)^2} + (r_0+z)(\omega_0+g_\phi)^2 + \frac{F_r}{m} \\ \dot{g}_\phi &= -2\frac{(\omega_0+g_\phi)g_z}{(r_0+z)} + \frac{F_\theta}{m(r_0+z)}. \end{aligned} \tag{47}$$

To derive the distance, d, between the spacecraft 44 and reference trajectory 43, we can picture d as the third side of a triangle with one side the radius of reference orbit and the other the radius of the spacecraft orbit. The angle between the orbits is ϕ. Then using the Law of Cosines:

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)},$$

where $r_1 = r_0$, $\theta_1 = \omega_0 t$, (corresponding to the orbit of the debris) and $r_2 = r_0 + z$, $\theta_2 = \omega_0 t + \phi$ (the orbit of the spacecraft). So substituting in these values, we obtain

$$\begin{aligned} d &= \sqrt{r_0^2 + (r_0+z)^2 - 2r_0(r_0+z)\cos((\omega_0 t+\phi) - \omega_0 t)} \\ &= \sqrt{r_0^2 + (r_0+z)^2 - 2r_0(r_0+z)\cos(\phi)}. \end{aligned} \tag{48}$$

To check this expression, we can check the following cases where d can easily be computed:

- Taking $z = 0, \phi = 0$, we would of course expect the distance to be 0 since the spacecraft would be in the same location as the reference. Indeed, using 48 we find

$$\begin{aligned} d &= \sqrt{r_0^2 + (r_0)^2 - 2r_0(r_0)\cos(0)} \\ &= \sqrt{2r_0^2 - 2r_0^2} \\ &= 0. \end{aligned}$$

- Taking $z = a, \phi = 0$, we would expect $d = a$ as spacecraft would be directly above the piece of debris. Checking with 48 we find

$$\begin{aligned} d &= \sqrt{r_0^2 + (r_0+a)^2 - 2r_0(r_0+a)\cos(0)} \\ &= \sqrt{(2r_0^2 + 2r_0a + a^2) - (2r_0^2 + 2r_0a)} \\ &= \sqrt{a} = a \end{aligned} \tag{49}$$

- Now taking $z = 0, \phi = a$, if we assume a to be very small, then we can estimate d by calculating the arc length between the two points (since $z = 0$, the points lie on an orbit of the same radius). The ratio of the arc length to the total orbit is $a/2\pi$, so multiplying this by the circumference of the orbit, $2\pi r_0$, we have

$$d = \frac{a}{2\pi} \times 2\pi r_0 = r_0 a \tag{50}$$

Now checking with our expression 48,

$$\begin{aligned} d &= \sqrt{r_0^2 + (r_0)^2 - 2r_0(r_0)\cos(a)} \\ &= \sqrt{2r_0^2(1-\cos(a))}, \end{aligned}$$

Since a is very small, we can use a second order Taylor expansion about 0 to approximate $\cos(a)$:

$$\begin{aligned} \cos(a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n &= \cos(0) - \frac{\sin(0)}{1}a - \frac{\cos(0)}{2!}a^2 + \mathcal{O}(a^3) \\ &= 1 - 0 - \frac{1}{2}a^2 + \mathcal{O}(a^4) \\ &\approx 1 - \frac{a^2}{2}. \end{aligned} \tag{51}$$

Then using 51 to substitute $\cos(a) = 1 - \frac{a^2}{2}$, we indeed obtain

$$\begin{aligned} d &= \sqrt{2r_0^2(1-(1-\frac{a^2}{2}))} \\ &= \sqrt{2r_0^2(\frac{a^2}{2})} \\ &= \sqrt{r_0^2a^2} \\ &= r_0 a, \end{aligned} \tag{52}$$

as expected.

- Taking $z = 0, \phi = \frac{\pi}{2}$, we can use Pythagorus' Theorem to calculate the distance to be $\sqrt{r_0^2 + r_0^2} = \sqrt{2r_0^2}$. Indeed, using our expression 48 we see

$$\begin{aligned} d &= \sqrt{r_0^2 + (r_0)^2 - 2r_0(r_0)\cos(\frac{\pi}{2})} \\ &= \sqrt{2r_0^2 - 2r_0^2(0)} \\ &= \sqrt{2r_0^2} \end{aligned} \tag{53}$$

- Taking $z = a, \phi = \pi$, we would expect the distance to be the diameter of the debris' orbit plus a, so $d = 2r_0 + a$. Substituting our values into 48, we confirm:

$$\begin{aligned} d &= \sqrt{r_0^2 + (r_0+a)^2 - 2r_0(r_0+a)\cos(\pi)} \\ &= \sqrt{r_0^2 + (r_0+a)^2 + 2r_0(r_0+a)} \\ &= \sqrt{4r_0^2 + 4ar_0 + a^2} \\ &= \sqrt{(2r_0+a)^2} \\ &= 2r_0 + a \end{aligned} \tag{54}$$

5 Modelling our Trajectories

Let our cleaning spacecraft have a mass of $m = 4000kg$, with 4 thrusters it can use to perform manoeuvres (we shall consider that the spacecraft and piece of debris are confined to the same plane). Two of the thrusters will exert a force parallel to the spacecraft's orbit, corresponding to F_θ in 15, while the other two push the spacecraft radially; these correspond to F_r in 13. A positive value for F_r pushes the spacecraft into a higher orbit while a negative one pushes it downwards (using a different thruster). Similarly, a positive value for F_θ pushes the spacecraft forward while a negative one pushes backwards. We will assume that each thruster can exert a maximum force of 100N and that the orientation of the spacecraft will not change.

Initially, the spacecraft will be on a circular orbit $1km$ lower and $2km$ behind the piece of debris, which will be on the reference circular orbit at an altitude of $h = 400km$. Thus our initial values for the spacecraft will be:

$$\begin{aligned} z &= -1000, \\ \dot{z} &= 0, \\ \phi &= -\frac{2000}{r_0}, \\ \dot{\phi} &= \sqrt{\frac{GM_E}{(r_0+z)^3}} - \omega_0. \end{aligned} \tag{55}$$

We will write a python program to solve the system of equations 45. It will also compute the distance 48 at each integration step and find the minimum distance from the piece of debris that the spacecraft achieves, along with the time at which this distance is reached. We will then further use python to compute the trajectory of the spacecraft in order to determine how we can fire the thrusters to bring the spacecraft within $1m$ of the piece of debris to complete a rendez-vouz. To simplify the problem, we will assume the thrusters fire only from $t = 0$ to $t = t_{thrust}$. F_r and F_θ will each have a fixed value, and start and stop at the same time. We can also calculate the total amount of fuel that is used: $Fuel = (|F_r| + |F_\theta|)t_{thrust}$, measured in kgm/s.

We will first try to solve the problem of manoeuvring the spacecraft so that it reaches the debris by aiming the spacecraft directly at the piece of debris and giving it a steady push for a finite amount of time. If we have $F_r = 50N, F_\theta = 100N$, we will use our python program to see how close we can get the spacecraft within 15 minutes after thrusting. We see that as we increase t_{thrust}, the minimum distance to the piece of debris decreases. However, this is only up to a point; after $t_{thrust} = 350$, the minimum distance given by our code remains at $220.6m$ no matter how much we increase t_{thrust}. This is because the minimum distance within this timeframe is achieved by having the thrusters fire for the entire duration of the journey to this minimum distance. Firing our thrusters after this point no longer decreases our minimum distance as we are now firing away from our target. So our minimum distance for these values

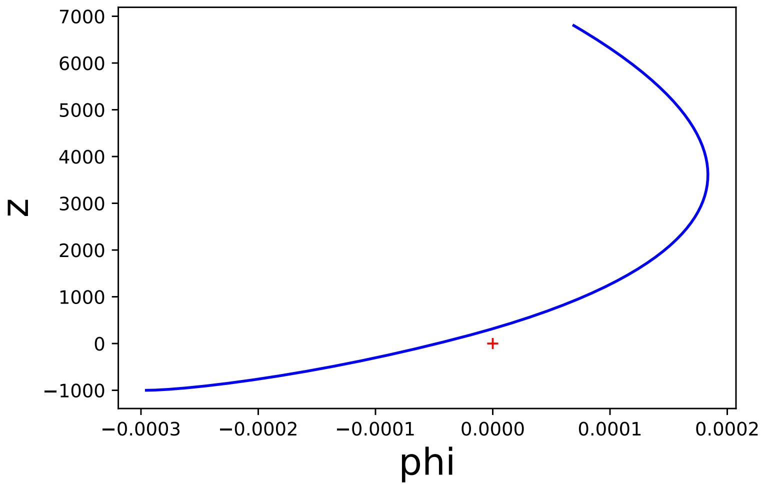


Figure 3: Closest possible approach for $F_r = 50$, $F_\theta = 100$ within 15 minutes ($t_{thrust} = 350$)

of F_r and F_θ is $d \approx 220m$, and the trajectory is shown in Figure 3.
In fact, it is not possible to reach the piece of debris by aiming directly at it because as the speed of the spacecraft increases, its altitude also increases in line with our equation for speed that we derived in equation 18. As a result, the total altitude gain will be more than what we would expect, so the spacecraft will end up above the piece of debris, as demonstrated in Figure 3. The higher altitude also increases the orbital period of the spacecraft, hence the spacecraft actually ends up further behind the piece of debris than we would expect if we were to ignore orbital mechanics.

Now keeping $F_\theta = 100$, but adjusting F_r and t_{thrust}, we will try to reach the piece of debris using as little fuel as possible. We will assume NASA has the precision to fire the thrusters to the accuracy of tenths of a second and tenths of a Newton. Figure 4 is a graph of various trajectories; $F_r = 32$, $t_{thrust} = 266$ (shown in blue) requires $35112kgm/s$ of fuel, $F_r = 16$, $t_{thrust} = 17$ (yellow) requires $1972/s$ of fuel, $F_r = 25$, $t_{thrust} = 15$ (black) requires $1875kgm/s$ of fuel, $F_r = 8$, $t_{thrust} = 12$ (green) requires $1296kgm/s$ of fuel. Our best trajectory, shown in red, with $F_r = 0$, $t_{thrust} = 12.2$ requires just $1220kgm/s$ of fuel, a fraction of the amount needed by the trajectory shown in blue.

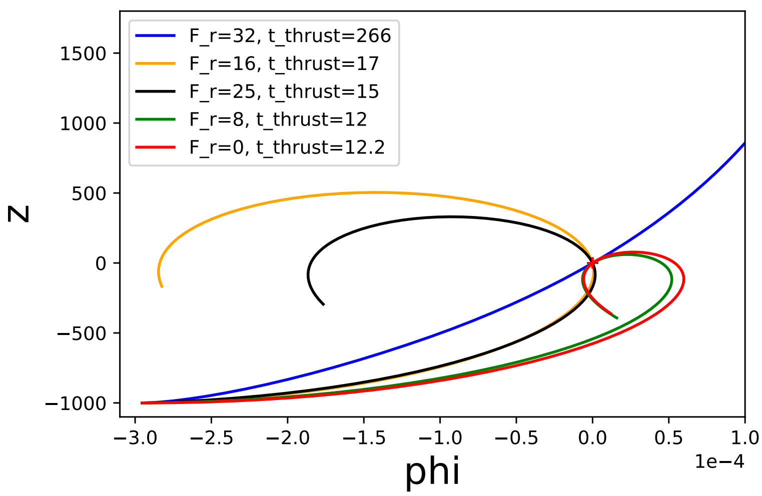
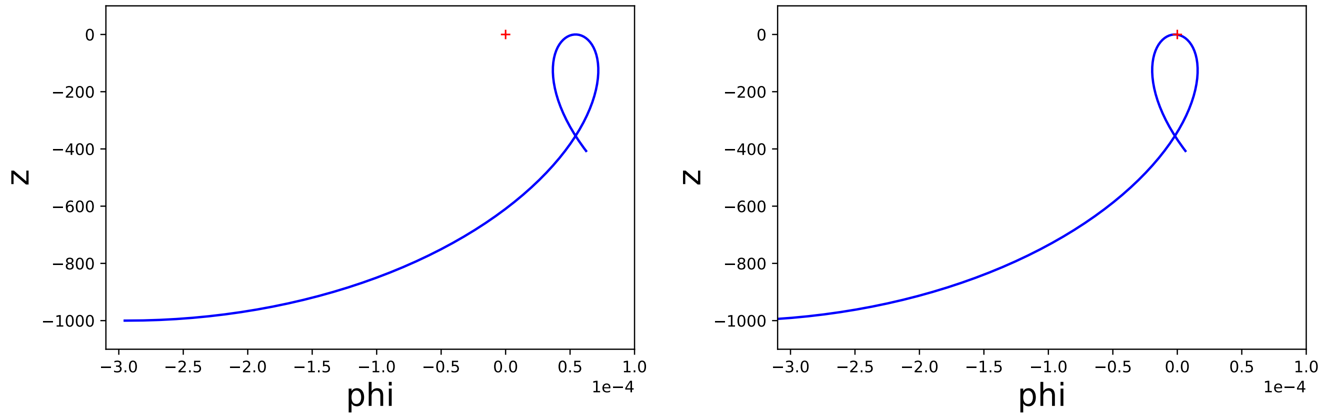


Figure 4: Successful trajectories for $F_\theta = 100N$

Based on the results shown in Figure 4, we can determine that it is actually more fuel efficient to use F_θ to reach the target altitude than F_r. Hence ideally we would fire only F_θ. The best strategy would be to fire F_θ such that just enough force is exerted for the spacecraft to reach the target altitude. We are able to do this whilst minimising fuel use with values of $F_r = 0, F_\theta = 85.1$, and $t_{thrust} = 13.3$. For these data values, $1132kgm/s$ of fuel is used. The trajectory is shown in Figure 5(a).

However, in order for the spacecraft to reach the debris, the manoeuvre must be carried out at a specific distance behind the debris such that the target altitude is reached at the exact same time that the angular difference between the two objects is zero, otherwise the rendez-vouz will not have been completed. We find that this perfect window occurs when the spacecraft is $2380m$ behind the piece of debris. The trajectory is shown in Figure 5(b).



(a) Starting $2km$ behind debris

(b) Starting $2.38km$ behind debris

Figure 5: Trajectory for $F_r = 0$, $F_\theta = 85.1$, $t_{thrust} = 13.3$

Now let us investigate trying to reach the target only after a full orbit has taken place. This may be useful if we first want to get close to the piece of debris to perform reconnaissance before capturing it on the next orbit, but we still want to perform just one initial thrust. We can calculate the orbital period using 17 with $h = 400km$ to be

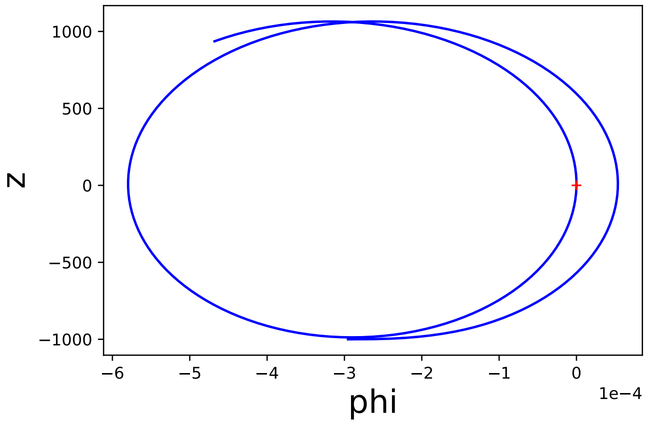
$$\begin{aligned} T &= 2\pi \div \sqrt{\frac{GM_E}{r_h^3}} \\ &\approx 5500s. \end{aligned} \tag{56}$$

We will specify in our python program to only return our closest distance to the debris once $5500s$ has passed. Figure 6(a) shows a successful rendezvous after a full orbit completion, with values of $F_r = 0$, $F_\theta = 5$, and $t_{thrust} = 470.7$

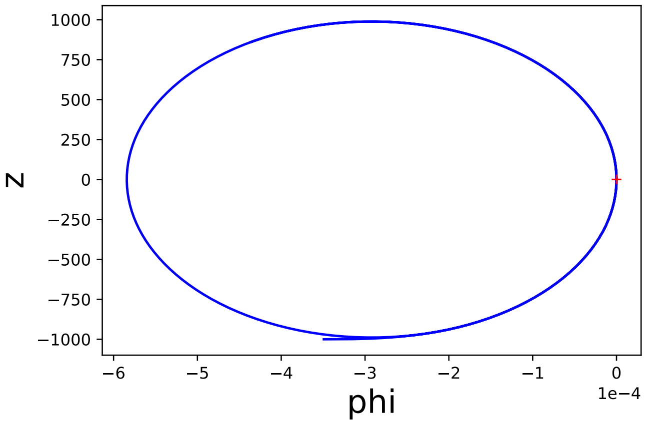
We now consider the possibility of making contact with the target twice; once on the first orbit after thrusting and then a second time an orbit later. We find that if we start $2364m$ behind the target, this is possible with values of $F_r = 0$, $F_\theta = 5$, and $t_{thrust} = 453.2$. The trajectory is shown in 6(b)

6 Orbital phasing

Now let us consider that the spacecraft is orbiting at the same altitude as the piece of debris, but 2km behind it. We cannot simply increase our velocity to catch up with the debris as we have shown that this will increase altitude whilst simultaneously moving the spacecraft further behind the piece of debris. Instead, we must undertake a process called orbital phasing [3]. This



(a) $t_{thrust} = 470.7$, starting $2000m$ behind



(b) $t_{thrust} = 453.2$, starting $2364m$ behind

Figure 6: Trajectory for $F_r = 0$, $F_\theta = 5$

is the adjustment of a spacecraft to a different position within the same orbit. The spacecraft must slow down in order to enter a lower, shorter orbit so that it can catch up with the piece of debris. The orbit will be elliptical such that the spacecraft returns to its initial altitude just as $\phi = 0$. The trajectory for $F_r = 0$, $F_\theta = -48$, $t_{thrust} = 10$ is shown in Figure 7. This trajectory uses just $480kgm/s$ of fuel.

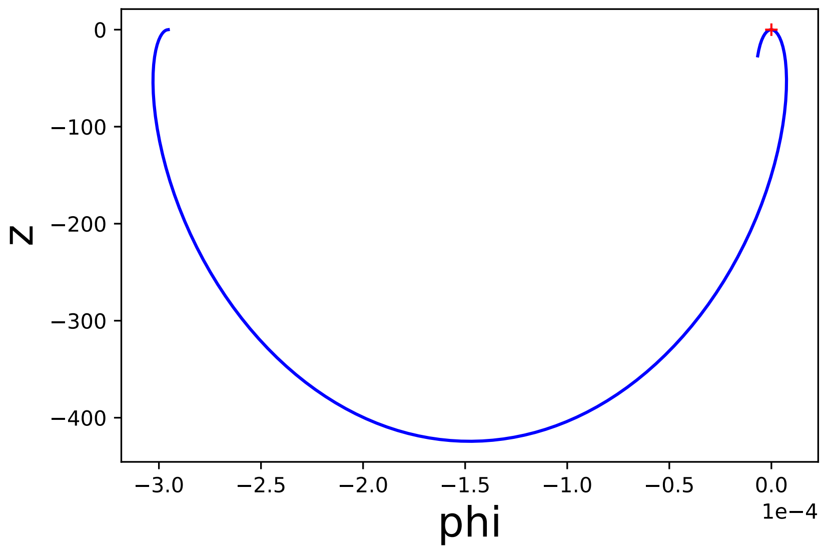


Figure 7: Trajectory for$F_r = 0$, $F_\theta = -48$, $t_{thrust} = 10$

7 Conclusion

In this essay, we have derived the equations involved in orbital mechanics. This enabled us to estimate the length of time it takes for different objects in orbit to re-enter the atmosphere and we have seen that many pieces of debris will remain orbiting the earth for centuries. As a result, rendezvous missions may be used to clean up debris. We have modelled the trajectory of a cleaning spacecraft and shown how simply firing at the target is not the optimal solution. We have concluded that the most fuel efficient way of catching the debris is to alter the velocity of the spacecraft in the θ direction such that it has just enough force to reach the target altitude.

To optimise fuel efficiency, we must also start the manoeuvre at a distance such that the target altitude is reached when the angular separation is zero.

References

[1] https://www.nasa.gov/news/debrisfaq.html

[2] http://www.braeunig.us/space/atmos.htm

[3] https://en.wikipedia.org/wiki/Orbit_phasing