

TTIC 31230, Fundamentals of Deep Learning

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Gaussian Models

and The Parels of Differential Entropy

Gaussian VAEs

$$\text{VAE: } \Phi^*, \Psi^* = \underset{\Phi, \Psi}{\operatorname{argmin}} E_{y,z} \ln \frac{\hat{p}_{\Psi}(z|y)}{p_{\Phi}(z)} - \ln p_{\Phi}(y|z)$$

All models are Gaussian densities.

$$p_{\Phi}(z[i]) \propto \exp(-(z[i] - \mu_{\Phi}[i])^2 / 2\sigma_{\Phi}^2[i])$$

$$p_{\Psi}(z[i]|y) \propto \exp(-(z[i] - \hat{z}_{\Psi}(y)[i])^2 / 2\sigma_{\Psi}^2(y)[i])$$

$$p_{\Phi}(y[i]|z) \propto \exp(-(y[i] - \hat{y}_{\Phi}(z)[i])^2 / 2\sigma_{\Phi}^2(z)[i])$$

Differential Entropy

In the case of a continuous density (as opposed to a discrete probability) we have the notion of differential entropy.

For a density $p(x)$ on a real value x we have

$$\begin{aligned} H(p) &= E_{x \sim p} [-\ln p(x)] \\ &= \int_{-\infty}^{\infty} p(x) (-\ln p(x)) dx \end{aligned}$$

Differential Entropy can Diverge to $-\infty$

For a uniform distribution over an interval on the real line of width Δ we have

$$\begin{aligned} H &= E_{x \sim p} [-\ln p(x)] \\ &= E_{x \sim p} \left[-\ln \frac{1}{\Delta} \right] \\ &= \ln \Delta \end{aligned}$$

As $\Delta \rightarrow 0$ we have $H \rightarrow -\infty$

Differential Entropy can Diverge to $-\infty$

$$\begin{aligned} H(\mathcal{N}(0, \sigma^2)) &= E_{x \sim \mathcal{N}(0, \sigma^2)} \left[-\ln \left(\frac{1}{\sigma \sqrt{2\pi}} \exp \frac{-x^2}{2\sigma^2} \right) \right] \\ &= E_{x \sim \mathcal{N}(0, \sigma^2)} \left[\ln(\sigma \sqrt{2\pi}) + \frac{-x^2}{2\sigma^2} \right] \\ &= (\ln \sigma) + \ln(\sqrt{2\pi}) + E_x \left[\frac{x^2}{2\sigma^2} \right] \\ &= (\ln \sigma) + \ln(\sqrt{2\pi}) + \frac{1}{2} \end{aligned}$$

As $\sigma \rightarrow 0$ we have $H \rightarrow -\infty$

Sensitivity to the Choice of Units

$$H(N(0, \sigma)) = C + \ln \sigma$$

Differential entropy depends on the choice of units — a distribution on lengths will have a different entropy when measuring in inches than when measuring in feet.

Differential Cross-Entropy can Diverge to $-\infty$

Consider the unsupervised training objective.

$$\Phi^* = \operatorname{argmin}_{\Phi} E_{y \sim \text{train}} - \ln p_{\Phi}(y)$$

The training set is finite (discrete).

For each y the density $p_{\Phi}(y)$ can go to infinity.

This will drive the cross-entropy training loss to $-\infty$.

Differential Entropy Can Be Considered Infinite

An actual real number carries an infinite number of bits.

Consider quantizing the real numbers into bins.

A continuous probability density p assigns a probability $p(B)$ to each bin.

As the bin size decreases toward zero the entropy of the bin distribution increases toward ∞ .

A meaningful convention is that $H(p) = +\infty$ for any continuous density p .

Differential KL-divergence is Meaningful

$$KL(p, q) = \int \left(\ln \frac{p(x)}{q(x)} \right) p(x) dx$$

Unlike differential entropy, differential KL divergence is always non-negative (but can be infinite).

Note that $KL(p, p) = 0$ independent of $H(p)$.

Mutual Information

$I(x, y)$ is the reduction in the number of bits we need to name y as a result of observing x (on average).

$$\begin{aligned} I(x, y) &= E \ln \frac{P(x, y)}{P(x)P(y)} \\ &= E \ln \frac{P(x, y)}{P(x)} - \ln P(y) \\ &= H(y) - H(y|x) \end{aligned}$$

Intuitively, how much does x know about y ?

Differential Mutual Information

$$\begin{aligned} I(x, y) &= KL(p(x, y), p(x)p(y)) \\ &= E_{x,y} \ln \frac{p(x, y)}{p(x)p(y)} \end{aligned}$$

Mutual information is a KL divergence and hence differential mutual information is always non-negative.

The Data Processing Inequality

For continuous y and z with $z = f(y)$ we get that $H(z)$ can be either larger or smaller than $H(y)$ (consider $z = ay$ for $a > 1$ vs. $a < 1$).

However, mutual information is a KL divergence and is more meaningful than entropy and for $z = f(y)$ we do have

$$I(x, z) \leq I(x, y)$$

Continuous Cross-Entropy as Distortion

For a Gaussian VAE on images we typically have

$$\Phi^*, \Psi^* = \operatorname{argmin}_{\Phi, \Psi} E_{y \sim \text{pop}, z \sim \hat{p}_\Psi(z|y)} \ln \frac{\hat{p}_\Psi(z|y)}{p_\Phi(z)} - \ln p_\Phi(y|z)$$

$$p_\Phi(y|z) \propto \frac{1}{Z} \exp(\|y - \hat{y}_\Phi(z)\|^2 / 2\sigma^2)$$

$$\Phi^*, \Psi^* = \operatorname{argmin}_{\Phi, \Psi} E_{y \sim \text{pop}, z \sim \hat{p}_\Psi(z|y)} \ln \frac{\hat{p}_\Psi(z|y)}{p_\Phi(z)} + \frac{1}{2\sigma^2} \|y - \hat{y}_\Phi(z)\|^2 + d \ln \sigma$$

The KL divergence term is not problematic. The problematic differential cross-entropy term can just be thought of as a weighted L_2 distortion.

Continuous Cross-Entropy as Distortion

$$\Phi^*, \Psi^* = \operatorname{argmin}_{\Phi, \Psi} E_{y \sim \text{pop}, z \sim \hat{p}_\Psi(z|y)} \ln \frac{\hat{p}_\Psi(z|y)}{p_\Phi(z)} + \lambda \operatorname{Dist}(y, \hat{y}_\Phi(z))$$

Various choices for distortion are possible including L_2 and L_1 distortion measures.

$$\operatorname{Dist}(y, \hat{y}) = ||y - \hat{y}||^2 \quad (L_2)$$

$$\text{or } \operatorname{Dist}(y, \hat{y}) = ||y - \hat{y}||_1 = \sum_i |y[i] - \hat{y}[i]| \quad (L_1)$$

END