TTIC 31230, Fundamentals of Deep Learning

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Pseudo-Likelihood and Contrastive Divergence

Psuedo-Likelihood

For any distribution $P(\hat{\mathcal{Y}})$ on colorings $\hat{\mathcal{Y}}$, we define the pseudo-likelihood $\tilde{P}(\hat{\mathcal{Y}})$ as follows

$$\tilde{P}(\hat{\mathcal{Y}}) = \prod_n P(\hat{\mathcal{Y}}[n] \mid \hat{\mathcal{Y}}/n) = \prod_n P(\hat{\mathcal{Y}}[n] \mid \hat{\mathcal{Y}}[N(n)])$$

While computing $P_{\Phi,x}(\mathcal{Y})$ is intractable, computing $\tilde{P}_{\Phi,x}(\mathcal{Y})$ is tractable. We then use

$$\Phi^* = \operatorname*{argmin}_{\Phi} E_{\langle x,\mathcal{Y}\rangle\sim\mathrm{Pop}} \; -\ln \tilde{P}_{\Phi,x}(\mathcal{Y})$$

Pseudolikelihood Theorem

$$\operatorname*{argmin}_{Q} \; E_{\mathcal{Y}\sim\mathrm{Pop}} \; -\ln \tilde{Q}(\mathcal{Y}) = \mathrm{Pop}$$

It suffices to show that for any Q we have

$$E_{\mathcal{Y}\sim\mathrm{Pop}} \; -\ln \widetilde{\mathrm{Pop}}(\mathcal{Y}) \leq \; E_{\mathcal{Y}\sim\mathrm{Pop}} \; -\ln \tilde{Q}(\mathcal{Y})$$

Proof II

$$
\begin{aligned}
& \min_{Q}\; E_{Y\sim\mathrm{Pop}} - \ln \tilde{Q}(Y) \\
= \; & \min_{Q}\; E_{\mathcal{Y}\sim\mathrm{Pop}} \sum_{n} - \ln Q(\mathcal{Y}[n] | \mathcal{Y}[N(n)]) \\
\geq \; & \min_{P_1,\ldots,P_N}\; E_{\mathcal{Y}\sim\mathrm{Pop}} \sum_{n} - \ln P_n(\mathcal{Y}[n] | \mathcal{Y}[N(n)]) \\
= \; & \min_{P_1,\ldots,P_N}\; \sum_{n} E_{\mathcal{Y}\sim\mathrm{Pop}} \;\; - \ln P_n(\mathcal{Y}[n] | \mathcal{Y}[N(n)]) \\
= \; & \sum_{n} \min_{P_n}\; E_{\mathcal{Y}\sim\mathrm{Pop}} \;\; - \ln P_n(\mathcal{Y}[n] | \mathcal{Y}[N(n)]) \\
= \; & \sum_{n} E_{\mathcal{Y}\sim\mathrm{Pop}} \;\; - \ln \mathrm{Pop}(\mathcal{Y}[n] | \mathcal{Y}[N(n)]) = E_{\mathcal{Y}\sim\mathrm{Pop}} - \ln \widetilde{\mathrm{Pop}}(\mathcal{Y})
\end{aligned}
$$

Contrastive Divergence (CDk)

In contrastive divergence we first construct an MCMC process whose stationary distribution is P_s. This could be Metropolis or Gibbs or something else.

Algorithm CDk: Given a gold segmentation $\mathcal{Y}$, start the MCMC process from initial state $\mathcal{Y}$ and run the process for k steps to get $\hat{\mathcal{Y}}'$. Then take the loss to be

$$\mathcal{L}_{\mathrm{CD}} = s(\hat{\mathcal{Y}}') - s(\mathcal{Y})$$

If $P_s = \mathrm{Pop}$ then the the distribution on $\hat{\mathcal{Y}}'$ is the same as the distribution on $\mathcal{Y}$ and the expected loss gradient is zero.

Gibbs CD1

CD1 for the Gibbs MCMC process is a particularly interesting special case.

Algorithm (Gibbs CD1): Given $\mathcal{Y}$, select a node n at random and draw $y \sim P(\mathcal{Y}[n] \mid \mathcal{Y}[N(n)])$. Define $\mathcal{Y}[n = y]$ to be the assignment (segmentation) which is the same as $\mathcal{Y}$ except that node n is assigned label y. Take the loss to be

$$\mathcal{L}_{\mathrm{CD}} = s(\mathcal{Y}[n = y]) - s(\mathcal{Y})$$

Gibbs CD1 Theorem

Gibbs CD1 is equivalent in expectation to pseudolikelihood.

$$
\begin{aligned}
\mathcal{L}_{\mathrm{PL}} &= E_{\mathcal{Y}\sim\mathrm{Pop}} \sum_n -\ln P_s(\mathcal{Y}[n] = y \mid \mathcal{Y}[N(n)]) \\
&= E_{\mathcal{Y}\sim\mathrm{Pop}} \sum_n -\ln \frac{e^{s(\mathcal{Y})}}{Z_n} \qquad Z_n = \sum_{y'} e^{s(\mathcal{Y}[n=y'])} \\
&= E_{\mathcal{Y}\sim\mathrm{Pop}} \sum_n (\ln Z_n - s(\mathcal{Y})) \\
\nabla_\Phi \mathcal{L}_{\mathrm{PL}} &= E_{\mathcal{Y}\sim\mathrm{Pop}} \sum_n \left(\frac{1}{Z_n} \sum_{y'} e^{s(\mathcal{Y}[n=y'])}\ \nabla_\Phi\ s(\mathcal{Y}[n] = y') \right) - \nabla_\Phi s(\mathcal{Y}) \\
&= E_{\mathcal{Y}\sim\mathrm{Pop}} \sum_n \left(\sum_{y'} P(\mathcal{Y}[n = y' \mid \mathcal{Y}[N(n)]])\ \nabla_\Phi\ s(\mathcal{Y}[n = y']) \right) - \nabla_\Phi s(\mathcal{Y})
\end{aligned}
$$

Gibbs CD1 Theorem

$$\begin{aligned}
\nabla_\Phi\, \mathcal{L}_{\mathrm{PL}} &= E_{\mathcal{Y}\sim\mathrm{Pop}} \sum_n \left(\sum_{y'} P(\mathcal{Y}[n=y' \mid \mathcal{Y}[N(n)]])\nabla_\Phi\ s(\mathcal{Y}[n]=y')\right) - \nabla_\Phi s(\mathcal{Y}) \\
&= E_{\mathcal{Y}\sim\mathrm{Pop}} \sum_n \big(E_{y'\sim P(\mathcal{Y}[n=y' \mid \mathcal{Y}[N(n)]])}\nabla_\Phi\ s(\mathcal{Y}[n]=y')\big) - \nabla_\Phi s(\mathcal{Y}) \\
&\propto E_{\mathcal{Y}\sim\mathrm{Pop}}\ E_n\ E_{y'\sim P(\mathcal{Y}[n=y' \mid \mathcal{Y}[N(n)]])}\ \left(\nabla_\Phi\ s(\mathcal{Y}[n]=y') - \nabla_\Phi s(\mathcal{Y})\right) \\
&= E_{\mathcal{Y}\sim\mathrm{Pop}}\ E_n\ E_{y'\sim P(\mathcal{Y}[n=y' \mid \mathcal{Y}[N(n)]])}\ \nabla_\Phi\ \mathcal{L}_{\text{Gibbs CD(1)}}
\end{aligned}$$

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