

# TTIC 31230, Fundamentals of Deep Learning

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## Stochastic Differential Equations (SDEs)

The Diffusion SDE

The Langevin SDE

General SDEs

The SGD SDE

## Diffusion

Consider a discrete-time process  $z(0), z(1), z(2), z(3), \dots$  with  $z(n) \in \mathbb{R}^d$  defined by

$$\begin{aligned} z(0) &= y, \quad y \sim \text{pop}(y) \\ z(n) &= z(n) + \sigma\epsilon, \quad \epsilon \sim \mathcal{N}(0, I) \end{aligned}$$

We can sample from  $z(n)$  using

$$\begin{aligned} z(0) &= y, \quad y \sim \text{pop}(y) \\ z(n) &= z(0) + \sigma\epsilon\sqrt{n}, \quad \epsilon \sim \mathcal{N}(0, I) \end{aligned}$$

## Diffusion

Fix a numerical time step  $\Delta t$  and consider a discrete-time process  $z(0), z(\Delta t), z(2\Delta t), \dots$

$$z(0) = y, \quad y \sim \text{pop}(y)$$

$$z(t + \Delta t) = z(t) + \sigma \epsilon \sqrt{\Delta t}, \quad \epsilon \sim \mathcal{N}(0, I)$$

We now take the limit of this numerical simulation as  $\Delta t \rightarrow 0$ .

This limit defines a probability measure on the space of functions  $z(t)$ .

## The Diffusion SDE

$$z(t + \Delta t) = z(t) + \sigma \epsilon \sqrt{\Delta t}, \quad \epsilon \sim \mathcal{N}(0, I)$$

For simple diffusion (Brownian motion) this equation holds for any continuous  $t \geq 0$  and  $\Delta t \geq 0$ .

# The Langevin SDE

Consider gradient flow.

$$\frac{d\Phi(t)}{dt} = g(\Phi)$$

$$g(\Phi) = \nabla_{\Phi} \mathcal{L}(\Phi)$$

$$\mathcal{L}(\Phi) = E_{(x,y) \sim P_{\text{op}}} \mathcal{L}(\Phi, x, y)$$

## The Langevin SDE

In the Langevin SDE we add Gaussian noise to gradient flow.

$$\Phi(t + \Delta t) = \Phi(t) + g\Delta t + \sigma\epsilon\sqrt{\Delta t}, \quad \epsilon \sim \mathcal{N}(0, I)$$

We will show that the stationary distribution of Langevin Dynamics models a Bayesian posterior probability distribution on the model parameters where  $\sigma$  acts as a temperature parameter.

## The Langevin SDE

$$\Phi(t + \Delta t) = \Phi(t) + g(\Phi)\Delta t + \sigma\epsilon\sqrt{\Delta t}, \quad \epsilon \sim \mathcal{N}(0, I)$$

Let  $p(\Phi)$  be a probability density on the parameter space  $\Phi$ . The density  $p(\Phi)$  defines a gradient flow and a diffusion flow.

$$\text{gradient flow} = -p(\Phi)g(\Phi)$$

$$\text{diffusion flow} = -\frac{1}{2} \sigma^2 \nabla_{\Phi} p(\Phi)$$

The expression for the diffusion flow follows from the Fokker-Plank equation. A derivation of the diffusion flow expression from first principle is given in the appendix.

## The Langevin SDE

$$\text{gradient flow} = -p(\Phi)g(\Phi)$$

$$\text{diffusion flow} = -\frac{1}{2} \sigma^2 \nabla_{\Phi}(p(\Phi))$$

For the stationary distribution these two flows cancel each other out. In one dimension we have

$$\frac{1}{2} \sigma^2 \nabla_{\Phi} p = -p \nabla_{\Phi} \mathcal{L}$$



# The Langevin Stationary Distribution

$$\frac{1}{2}\sigma^2\nabla_{\Phi} p = -p\nabla_{\Phi}\mathcal{L}$$

$$\nabla_{\Phi} \left( \frac{1}{2}\sigma^2 \ln p \right) = \nabla_{\Phi}(-\mathcal{L})$$

$$\frac{1}{2}\sigma^2 \ln p = -\mathcal{L} + C$$

$$p(\Phi) = \frac{1}{Z} e^{\frac{-2\mathcal{L}(\Phi)}{\sigma^2}}$$

## A Bayesian Interpretation of Langevin Dynamics

If we incorporate a regularization into the loss function.

$$\mathcal{L}(\Phi) = E_{(x,y) \sim \text{Train}} [-\ln P_{\Phi}(y|x)] + \lambda ||\Phi||^2$$

Then the Gibbs distribution

$$p(\Phi) = \frac{1}{Z} e^{\frac{-2\mathcal{L}(\Phi)}{\sigma^2}}$$

can be interpreted as a Bayesian posterior density  $p(\Phi|\text{Train})$ .

## A General SDE

$$x(t + \Delta t) = f(t) + \mu(x, t)\Delta t + \sigma(x, t)\epsilon\sqrt{\Delta t}, \quad \epsilon \sim \mathcal{N}(0, I) \quad (1)$$

Here  $\sigma(x, t)$  is a matrix equal to  $\Sigma(x, t)^{1/2}$  where  $\Sigma = \sigma^2$  is the covariance matrix of the random variable  $\sigma(x, t)\epsilon$ .

This is conventionally written as

$$dx = \mu(x, t)dt + \sigma(x, t)dB \quad (2)$$

where  $B$  denotes a Wiener process (simple diffusion, aka Brownian motion)

I find (1) more intuitive than (2) but they are the same thing.

## A 2-D Stationary Distribution

Let  $p$  be a probability density on two parameters  $(x, y)$ .

We consider the case where  $x$  and  $y$  are completely independent with

$$\mathcal{L}(x, y) = \mathcal{L}(x) + \mathcal{L}(y)$$

For completely independent variables we have

$$\begin{aligned} p(x, y) &= p(x)p(y) \\ &= \frac{1}{Z} \exp \left( \frac{-2\mathcal{L}(x)}{\eta\sigma_x^2} + \frac{-2\mathcal{L}(y)}{\eta\sigma_y^2} \right) \end{aligned}$$

## A 2-D Stationary Distribution

$$p(x, y) = \frac{1}{Z} \exp \left( \frac{-2\mathcal{L}(x)}{\eta\sigma_x^2} + \frac{-2\mathcal{L}(y)}{\eta\sigma_y^2} \right)$$

This is not a Gibbs distribution!

It has two different temperature parameters!

## Forcing a Gibbs Distribution

Suppose we use parameter-specific learning rates  $\eta_x$  and  $\eta_y$

$$p(x, y) = \frac{1}{Z} \exp \left( \frac{-2\mathcal{L}(x)}{\eta_x \sigma_x^2} + \frac{-2\mathcal{L}(y)}{\eta_y \sigma_y^2} \right)$$

Setting  $\eta_x = \eta' / \sigma_x^2$  and  $\eta_y = \eta' / \sigma_y^2$  gives

$$\begin{aligned} p(x, y) &= \frac{1}{Z} \exp \left( \frac{-2\mathcal{L}(x)}{\eta'} + \frac{-2\mathcal{L}(y)}{\eta'} \right) \\ &= \frac{1}{Z} \exp \left( \frac{-2\mathcal{L}(x, y)}{\eta'} \right) \quad \text{Gibbs!} \end{aligned}$$

## The Case of Locally Constant Noise and Locally Quadratic Loss

In this case we can impose a change of coordinates under which the Hessian is the identity matrix. So without loss of generality we can take the Hessian to be the identity.

We can consider the covariance matrix of the vectors  $\hat{g}$  in the Hessian-normalized coordinate system.

## The Case of Locally Constant Noise and Locally Quadratic Loss

If we assume constant noise covariance in the neighborhood of the stationary distribution then, in the Hessian normalized coordinate system, we get a stationary distribution

$$p(\Phi) \propto \exp \left( - \sum_i \frac{2\Phi_i^2}{\eta\sigma_i^2} \right)$$

where  $\Phi_i$  is the projection of  $\Phi$  onto to a unit vector in the direction of the  $i$ th eigenvector of the noise covariance matrix and  $\sigma_i^2$  is the corresponding noise eigenvalue (the variance of the  $\hat{g}_i$ ).



## Continuous Time Diffusion (Brownian Motion)

$$\begin{aligned} z(0) &= y, \quad y \sim \text{pop}(y) \\ z(n\Delta t) &= z(0) + \epsilon, \quad \epsilon \sim \mathcal{N}(0, n\Delta t\sigma^2 I) \end{aligned}$$

$$z(t) = z(0) + \epsilon, \quad \epsilon \sim \mathcal{N}(0, t\sigma^2 I)$$

$$z(t + \Delta t) = z(t) +$$

## Holding $\eta$ Fixed

Consider SGD with batch size 1.

$$\Phi_{i+1} = \Phi_i - \eta \hat{g}_i$$

Unlike gradient flow, we now hold  $\eta > 0$  fixed.

We will still take “time” to be the sum of the learning rates over the updates.

For  $N$  steps of SGD we define  $\Delta t = N\eta$

## Holding $\eta$ Fixed

We consider  $\Delta t$  large enough so that  $\Delta t$  corresponds to many SGD updates.

We consider  $\Delta t$  small enough so that the gradient estimate distribution does not change over the interval  $\Delta t$ .

## Applying the Central Limit Theorem

If the mean gradient  $g(\Phi)$  is approximately constant over the interval  $\Delta t = N\eta$  we have

$$\Phi(t + \Delta t) \approx \Phi(t) - g(\Phi)\Delta t + \eta \sum_{i=1}^N (g(\Phi) - \hat{g}_i)$$

The random variables in the last term have zero mean.

By the central limit theorem a sum (not the average) of  $N$  random vectors will approximate a Gaussian distribution where the variance grows like  $N$ .

## Modeling Noise

The mean of  $\hat{g}$  is the true gradient  $g(\Phi)$ .

$$\begin{aligned}\Delta t &= \eta N \\ \Phi(t + \Delta t) &= \Phi(t) - \sum_{j=1}^N \eta \hat{g}_j \\ &= \Phi(t) - g(\Phi) \Delta t + \eta \sum_{j=1}^N (g(\Phi) - \hat{g}_j)\end{aligned}$$

## Modeling Noise

$$\Delta t = \Phi(t) - g(\Phi)\Delta t + \eta \sum_{j=1}^N (g(\Phi) - \hat{g}_i)$$

By the central limit theorem  $\sum_{j=1}^N (g(\Phi) - \hat{g}_i)$  is approximately Gaussian.

## In One Dimension

We first consider the case of one parameter (a one dimensional optimization problem) so that  $\hat{g}$  is a scalar.

To model the noise as Gaussian we take  $\epsilon \sim \mathcal{N}(0, \sigma^2)$  where  $\sigma^2$  is the variance of  $\hat{g}$ .

$$\Phi(t + \Delta t) \approx \Phi(t) - g(\Phi)\Delta t + \eta\epsilon\sqrt{N}$$

## In One Dimension

$$\begin{aligned}\Phi(t + \Delta t) &\approx \Phi(t) - g(\Phi)\Delta t + \eta\epsilon\sqrt{N} \\ &= \Phi(t) - g(\Phi)\Delta t + \eta\epsilon\sqrt{\frac{\Delta t}{\eta}} \\ &= \Phi(t) - g(\Phi)\Delta t + \sqrt{\eta}\epsilon\sqrt{\Delta t} \\ &= \Phi(t) - g(\Phi)\Delta t + \epsilon'\sqrt{\Delta t}\end{aligned}$$

With  $\epsilon' \sim \mathcal{N}(0, \eta\sigma^2)$ .



## The Stochastic Differential Equation (SDE)

$$\begin{aligned}\Phi(t + \Delta t) &\approx \Phi(t) - g(\Phi)\Delta t + \epsilon\sqrt{\Delta t} \\ \epsilon &\sim \mathcal{N}(0, \eta\sigma^2)\end{aligned}$$

We can take this last equation to hold in the limit of arbitrarily small  $\Delta t$  in which case we get a continuous time stochastic process. This process can be written as

$$d\Phi = -g(\Phi)dt + \epsilon\sqrt{dt} \quad \epsilon \sim \mathcal{N}(0, \eta\sigma^2)$$

## Higher Dimension

In the higher dimensional case we get

$$d\Phi = -g(\Phi)dt + \epsilon\sqrt{dt} \quad \epsilon \sim \mathcal{N}(0, \eta\Sigma)$$

Where  $\Sigma$  is the covariance matrix of  $\hat{g}$ .

In one dimension  $\Sigma$  is  $\sigma^2$ .

For  $g(\Phi) = 0$  and  $\Sigma = I$  we get Brownian motion.

But in general we do not have  $\Sigma = I$ .

**END**

## Appendix: Diffusion Flow

We consider the one dimensional case where we have a function  $x(t) \in \mathbb{R}$ . We consider a very small time step  $\Delta t$  and consider only the diffusion flow.

$$x(t + \Delta t) = x(t) + \sigma \epsilon \sqrt{\Delta t}, \quad \epsilon \sim \mathcal{N}(0, 1)$$

We assume a density  $p_x$  of values of  $x$  and let  $p_\epsilon(\epsilon)$  be the normal distribution  $\mathcal{N}(0, 1)$  on  $\epsilon$ .

The quantity of mass transfer in the time interval  $\Delta t$  from values above  $x$  to values below  $x$  is

$$\begin{aligned} & \int_{z=0}^{\infty} p_x(x + z) p_\epsilon(\sigma \epsilon \sqrt{\Delta t} \leq -z) dz \\ &= \int_{z=0}^{\infty} p_x(x + z) p_\epsilon \left( \epsilon \leq \frac{-z}{\sigma \sqrt{\Delta t}} \right) dz \\ &= \int_{z=0}^{\infty} p_x(x + z) \Phi \left( \frac{-z}{\sigma \sqrt{\Delta t}} \right) dz \end{aligned}$$

where  $\Phi$  is the cumulative function of the Gaussian.

## Appendix: Diffusion Flow

The quantity of mass transfer in the time interval  $\Delta t$  from values above  $x$  to values below  $x$  is

$$\int_{z=0}^{\infty} p_x(x+z) \Phi\left(\frac{-z}{\sigma\sqrt{\Delta t}}\right) dz$$

By a change of variables  $u = z/(\sigma\sqrt{\Delta t})$  we get

$$\int_{u=0}^{\infty} p_x(x + \sigma\sqrt{\Delta t} u) \Phi(-u) \sigma\sqrt{\Delta t} du$$

As  $\Delta t \rightarrow 0$  we can use the first order Taylor expansion of the density.

$$\sigma\sqrt{\Delta t} \int_{u=0}^{\infty} \left( p_x(x) + \sigma\sqrt{\Delta t} u \frac{dp_x(x)}{dx} \right) \Phi(-u) du$$

## Appendix: Diffusion Flow

$$\begin{aligned}
 & \sigma\sqrt{\Delta t} \int_{u=0}^{\infty} \left( p_x(x) + \sigma\sqrt{\Delta t} u \frac{dp_x(x)}{dx} \right) \Phi(-u) du \\
 = & \sigma\sqrt{\Delta t} p_x(x) \left( \int_{u=0}^{\infty} \Phi(-u) du \right) + \sigma^2 \Delta t \frac{dp_x(x)}{dx} \left( \int_{u=0}^{\infty} u \Phi(-u) du \right)
 \end{aligned}$$

A similar analysis shows that the mass transfer from lower values to higher values is

$$= \sigma\sqrt{\Delta t} p_x(x) \left( \int_{u=0}^{\infty} \Phi(-u) du \right) - \sigma^2 \Delta t \frac{dp_x(x)}{dx} \left( \int_{u=0}^{\infty} u \Phi(-u) du \right)$$

The net mass transfer in the positive  $x$  direction is the second minus the first or

$$= -2\sigma^2 \Delta t \frac{dp_x(x)}{dx} \left( \int_{u=0}^{\infty} u \Phi(-u) du \right)$$

## Appendix: Diffusion Flow

The net mass transfer in the positive  $x$  direction is

$$-2\sigma^2\Delta t \frac{dp_x(x)}{dx} \left( \int_{u=0}^{\infty} u\Phi(-u)du \right)$$

Note that the mass transfer is proportional to  $\Delta t$ . Dividing by  $\Delta t$  gives the flow per unit time.

$$\text{Diffusion flow} = -\alpha\sigma^2 \frac{dp_x(x)}{dx} \quad \alpha = 2 \int_{u=0}^{\infty} u\Phi(-u)du$$

$\alpha$  can be calculated using integration by parts.

$$\begin{aligned} \alpha &= 2 \int_0^{\infty} u\Phi(-u)du \\ &= \int_0^{\infty} \Phi(-u)du^2 \\ &= u^2\Phi(-u)|_0^{\infty} + \int_0^{\infty} u^2\phi(-u)du \quad \text{where } \phi \text{ is the Gaussian density} \\ &= \int_0^{\infty} u^2\phi(-u)du \\ &= \frac{1}{2} \end{aligned}$$

## Appendix: Diffusion Flow

We now have that the diffusion flow is

$$\text{Diffusion flow} = -\frac{1}{2} \sigma^2 \frac{dp_x(x)}{dx}$$

For dimension larger than 1 we have

$$\text{Diffusion flow} = -\frac{1}{2} \Sigma \nabla_x p_x$$

Where  $\Sigma = E (\hat{g} - g)(\hat{g} - g)^\top$  is the covariance matrix of the gradient noise.

Here we have derived this from first principle but it also follows from the Fokker–Planck equation (see Wikipedia).