A dynamical system is given as

$$\begin{cases} \dot{x} = Ax + Bu \\ x(0) = x_0 \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{nxn}$, $B \in \mathbb{R}^{nxm}$, $x \in \mathbb{R}^{nx1}$ and $u \in \mathbb{R}^{mx1}$. The control signal is constrained with

$$g_i \leq u \leq h_i \quad i = 1, 2, \cdots, m \tag{2}$$

Let's assume the system matrices as follows

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}, \tag{3}$$

and hence, $n = 2$ and $m = 1$. Then,

$$g \leq u \leq h \tag{4}$$

and

$$L = \begin{bmatrix} l_{11} \\ l_{21} \end{bmatrix} \tag{5}$$

can be stated. The closed-loop matrix is obtained as,

$$A + BL^T = \begin{bmatrix} a_{11} + b_{11}l_{11} & a_{12} + b_{11}l_{21} \\ a_{21} + b_{21}l_{11} & a_{22} + b_{21}l_{21} \end{bmatrix} \tag{6}$$

$E(L)$ is defined as follows

$$\begin{aligned} E(L) &= \{z | z \in \mathbb{R}^2 \, and \, g \leq l_i^T z \leq h\} \\ &= g \leq l_{11}z_{11} + l_{21}z_{21} \leq h \end{aligned} \tag{7}$$

$F(L)$ is defined as follows

$$F(L) = \bigcap_{t \in [0,\infty]} \{(e^{A_c t})^{-1} E(L)\} \tag{8}$$

where $F(L)$ is a subset of $E(L)$. Let $K = k_1$, then

$$u = sat[(L^T - KB^T P)x] \tag{9}$$

is defined. Consider

$$\begin{aligned} u &= L^T x + v \\ &= l_{11}x_1 + l_{21}x_2 + v \end{aligned} \tag{10}$$

then the closed-loop system is

$$\dot{x} = (A + BL^T)x + Bv \tag{11}$$

which is openly,

$$\begin{aligned} \dot{x}_1 &= (a_{11} + b_{11}l_{11})x_1 + (a_{12} + b_{11}l_{21})x_2 \\ \dot{x}_2 &= (a_{21} + b_{21}l_{11})x_1 + (a_{22} + b_{21}l_{21})x_2 \end{aligned} \tag{12}$$

The derivative of the Lyapunov function $V = x^T P x$ is obtained as,

$$\begin{aligned} \dot{V} &= \dot{x}^T P x + x^T P \dot{x} \\ &= ((A + BL^T)x + Bv)^T P x + x^T P((A + BL^T)x + Bv) \\ &= (v^T B^T + x^T L B^T + x^T A^T)Px + x^T P((A + BL^T)x + Bv) \\ &= v^T B^T P x + x^T L B^T P x + x^T A^T P x + x^T P A x + x^T P B L^T x + x^T P B v \\ &= x^T (A^T P + PA + PBL^T + LB^T P)x + 2x^T P B v \end{aligned} \tag{13}$$

For stability,

$$2x^T P B v \leq 0 \tag{14}$$

is needed. Choosing $v = -RB^T P x$ gives,

$$\begin{aligned} 2x^T PBv &= 2x^T PB(-RB^T Px) \\ 2x^T PBv &= -2x^T (PBRB^T P)x \\ 2x^T PBv &\leq 0 \end{aligned} \tag{15}$$

where $R = diag([r_1, r_2, \cdots, r_m])$. Therefore,

$$\begin{aligned} \dot{V} &= x^T (A^T P + PA + PBL^T + LB^T P)x + 2x^T PBv \\ \dot{V} &= x^T (A^T P + PA + PBL^T + LB^T P)x - 2x^T (PBRB^T P)x \\ \dot{V} &= x^T (A^T P + PA + PBL^T + LB^T P - 2PBRB^T P)x \\ \dot{V} &= x^T (A^T P + PA + PB(L^T - RB^T P) + (L - PBR)B^T P)x \end{aligned} \tag{16}$$

Solving,

$$L^T x - RB^T Px = sat[L^T x - KB^T Px] \tag{17}$$

for any diagonal K.

$$L^T x - RB^T Px = sat[L^T x - KB^T Px] \tag{18}$$

which is expanded as

$$\begin{aligned} l_{11}x_1 + l_{21}x_2 - (b_{11}p_{11}r + b_{21}p_{12}r)x_1 + (-b_{11}p_{12}r - b_{21}p_{22}r)x_2 \\ = sat(l_{11}x_1 + l_{21}x_2 - (b_{11}p_{11}k + b_{21}p_{12}k)x_1 + (-b_{11}p_{12}k - b_{21}p_{22}k)x_2) \end{aligned} \tag{19}$$

if $x \in E(L)$ then

$$g \leq l_{11}x_1 + l_{21}x_2 \leq h \tag{20}$$

but if also

$$g \leq l_{11}x_1 + l_{21}x_2 - (b_{11}p_{11}k + b_{21}p_{12}k)x_1 - (b_{11}p_{12}k + b_{21}p_{22}k)x_2 \leq h \tag{21}$$

then $r = k$. On the other hand, if

$$l_{11}x_1 + l_{21}x_2 - k(b_{11}p_{11} + b_{21}p_{12})x_1 - k(b_{11}p_{12} + b_{21}p_{22})x_2 > h \tag{22}$$

if a smaller r then the term

$$l_{11}x_1 + l_{21}x_2 - r(b_{11}p_{11} + b_{21}p_{12})x_1 - r(b_{11}p_{12} + b_{21}p_{22})x_2 \tag{23}$$

would increase. Therefore,

$$l_{11}x_1 + l_{21}x_2 - r(b_{11}p_{11} + b_{21}p_{12})x_1 - r(b_{11}p_{12} + b_{21}p_{22})x_2 = h \tag{24}$$

The algorithm is given as follows:

1. Determine $\mathbb{D}$. (set of initial states)

2. Find L. Control penalty R in LQR is increased until $L^T x$ satisfies

   $$g \leq L^T x \leq h \tag{25}$$

   for x in $\mathbb{D}$. This can be done via

   $$g \leq \max_{x \in \mathbb{D}} l_i^T x \leq h \tag{26}$$

   If this cannot be satisfied then there is no design.

3. Find P and c. P can be used from LQR. c is obtained from

   $$\begin{gathered} \sup_{x \in \mathbb{D}} x^T Px \leq c \leq \min_{\delta E(L)} x^T Px \\ \min_{\delta E(L)} x^T Px = \min_i \frac{g_i^2}{l_i^T P^{-1} l_i}, \frac{h_i^2}{l_i^T P^{-1} l_i} \end{gathered} \tag{27}$$