1. Proof of the LQR Optimal Controller

The Regulation Problem

Let a linear time-invariant (LTI) system be given in state-space form as:

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{aligned}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$.

The Linear Quadratic Regulator (LQR) problem seeks a control input $u(t)$ that minimizes the infinite-horizon cost functional:

$$J = \frac{1}{2}\int_0^\infty \big(x(t)^T Qx(t) + u(t)^T Ru(t)\big)\, dt,$$

where $Q \in \mathbb{R}^{n\times n}$ is symmetric positive semi-definite ($Q \succeq 0$), and $R \in \mathbb{R}^{m\times m}$ is symmetric positive definite ($R \succ 0$).

We define the Lagrangian $L(x, u, \dot{x}, \lambda)$ incorporating the dynamics as a constraint via the co-state vector $\lambda(t)$:

$$L(x, u, \dot{x}, \lambda) = \frac{1}{2}x^T Qx + \frac{1}{2}u^T Ru + \lambda^T(Ax + Bu - \dot{x}).$$

The necessary conditions for optimality from calculus of variations are:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= 0, \\ \frac{\partial L}{\partial u} &= 0, \\ \frac{\partial L}{\partial \lambda} &= 0. \end{aligned}$$

Compute the partial derivatives:

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}} &= -\lambda, \\ \frac{\partial L}{\partial x} &= Qx + A^T\lambda, \\ \frac{\partial L}{\partial u} &= Ru + B^T\lambda, \\ \frac{\partial L}{\partial \lambda} &= Ax + Bu - \dot{x}. \end{aligned}$$

Substitute into the optimality conditions:

$$\begin{aligned} \dot{\lambda} &= -Qx - A^T\lambda, \quad \text{(costate dynamics)} \\ 0 &= Ru + B^T\lambda, \quad \text{(stationarity w.r.t. } u\text{)} \\ 0 &= Ax + Bu - \dot{x}. \quad \text{(primal dynamics)} \end{aligned}$$

From stationarity:

$$u = -R^{-1}B^T\lambda.$$

Assume the co-state vector is a linear function of the state:

$$\lambda = Px,$$

where $P \in \mathbb{R}^{n\times n}$ is a symmetric matrix to be determined.

Differentiate both sides:

$$\dot{\lambda} = P\dot{x} = P(Ax + Bu).$$

Substitute the control law:

$$\dot{\lambda} = P(A - BR^{-1}B^T P)x.$$

From the costate dynamics:

$$\dot{\lambda} = -Qx - A^T\lambda = -Qx - A^T Px.$$

Equating both expressions for $\dot{\lambda}$:

$$P(A - BR^{-1}B^T P)x = -Qx - A^T Px.$$

Since this holds for all x, we equate coefficients:

$$PA - PBR^{-1}B^T P = -Q - A^T P,$$

which simplifies to the Algebraic Riccati Equation:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0.$$

The optimal state-feedback control law is:

$$u(t) = -Kx(t), \quad \text{where} \quad K = R^{-1}B^T P,$$

and P is the unique positive semi-definite solution to the Algebraic Riccati Equation:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0.$$

Reference Tracking Problem

For reference tracking the error and its integral are defined:

$$e(t) = r(t) - y(t),$$

and

$$\dot{q}(t) = e(t)$$

respectively. The cost function is modified as follows:

$$J = \frac{1}{2}\int_0^\infty \left(\begin{bmatrix} x \\ q \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & Q_q \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} + u^T R u \right) dt$$

where Q_q is a positive semi-definite matrix that penalizes the integral of the tracking error. The augmented system is given by:

$$\begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} + \begin{bmatrix} B \\ -D \end{bmatrix} u + \begin{bmatrix} 0 \\ I \end{bmatrix} r$$
$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} + Du$$

PI+Filter Control Law

Instead of using the derivative of the error directly, a low-pass filter

$$\frac{s(t)}{e(t)} = \frac{1}{\tau s + 1}$$

is applied to the error signal:

$$\dot{s}(t) = -\frac{1}{\tau}s(t) + \frac{1}{\tau}e(t)$$
$$\dot{s}(t) = -\frac{1}{\tau}s(t) + \frac{1}{\tau}r(t) - \frac{1}{\tau}Cx(t) - \frac{1}{\tau}Du(t)$$

where τ is a small approximation constant. The augmented system becomes:

$$\begin{bmatrix} \dot{x} \\ \dot{s} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ -\frac{1}{\tau}C & -\frac{1}{\tau}I & 0 \\ -C & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ s \\ q \end{bmatrix} + \begin{bmatrix} B \\ -\frac{1}{\tau}D \\ -D \end{bmatrix} u + \begin{bmatrix} D \\ \frac{1}{\tau} \\ I \end{bmatrix} r$$
$$y = \begin{bmatrix} C & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ s \\ q \end{bmatrix} + Du$$

The cost function is modified as follows:

$$J = \frac{1}{2}\int_0^\infty \left(\begin{bmatrix} x \\ s \\ q \end{bmatrix}^T \begin{bmatrix} Q & 0 & 0 \\ 0 & Q_s & 0 \\ 0 & 0 & Q_q \end{bmatrix} \begin{bmatrix} x \\ s \\ q \end{bmatrix} + u^T R u \right) dt$$

PID-like Control Law

Instead of using the derivative of the error directly, a low-pass filter

$$\frac{s(t)}{e(t)} = \frac{s}{\tau s + 1}$$

is applied to the error signal:

$$\dot{s}(t) = -\frac{1}{\tau}s(t) + \frac{1}{\tau}\dot{e}(t)$$
$$\dot{s}(t) = -\frac{1}{\tau}s(t) + \frac{1}{\tau}\dot{r}(t) - \frac{1}{\tau}C\dot{x}(t) - \frac{1}{\tau}D\dot{u}(t)$$
$$\dot{s}(t) = -\frac{1}{\tau}s(t) + \frac{1}{\tau}\dot{r}(t) - \frac{1}{\tau}CAx(t) - \frac{1}{\tau}CBu(t) - \frac{1}{\tau}D\dot{u}(t)$$

where τ is a small approximation constant. The augmented system becomes:

$$\begin{bmatrix} \dot{x} \\ \dot{s} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ -\frac{1}{\tau}CA & -\frac{1}{\tau}I & 0 \\ -C & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ s \\ q \end{bmatrix} + \begin{bmatrix} B \\ -\frac{1}{\tau}CB \\ -D \end{bmatrix} u + \begin{bmatrix} D \\ \frac{1}{\tau} \\ I \end{bmatrix} r + \begin{bmatrix} 0 \\ -\frac{1}{\tau}D \\ 0 \end{bmatrix} \dot{u} + \begin{bmatrix} 0 \\ \frac{1}{\tau} \\ 0 \end{bmatrix} \dot{r}$$

$$y = \begin{bmatrix} C & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ s \\ q \end{bmatrix} + Du$$

The cost function is modified as follows:

$$J = \frac{1}{2} \int_0^\infty \left(\begin{bmatrix} x \\ s \\ q \end{bmatrix}^T \begin{bmatrix} Q & 0 & 0 \\ 0 & Q_s & 0 \\ 0 & 0 & Q_q \end{bmatrix} \begin{bmatrix} x \\ s \\ q \end{bmatrix} + u^T R u \right) dt$$