The system is given as

$$\dot{x} = Ax + Bu \tag{1}$$

and the cost function is given as

$$J = \frac{1}{2}\int_0^\infty (x^TQx + u^TRu)dt \tag{2}$$

with

$$u = kCx,\ k \in \mathbb{R} \tag{3}$$

where C is the output vector. To solve this problem the Hamiltonian is given as

$$H(x, u, \lambda_1, \lambda_2) = \frac{1}{2}x^TQx + u^TRu + \lambda_1^T(Ax + Bu) + \lambda_2^T(u - kCx) \tag{4}$$

and the first order optimality condition is stated as

$$\begin{aligned} \frac{\partial H}{\partial u} &= Ru + \lambda_1^TB + \lambda_2^T \\ &= Ru + B^T\lambda_1 + \lambda_2 = 0 \\ u^* &= -R^{-1}(B^T\lambda_1 + \lambda_2) \end{aligned} \tag{5}$$

The co-state equation is given as

$$\begin{aligned} \dot{\lambda_1} &= -\frac{\partial H}{\partial x} \\ &= -(Qx + \lambda_1^TA - k\lambda_2^TC) \\ &= -Qx - A^T\lambda_1 + kC^T\lambda_2 \end{aligned} \tag{6}$$

From $u = kCx$,

$$\begin{aligned} -R^{-1}(B^T\lambda_1 + \lambda_2) &= kCx \\ B^T\lambda_1 + \lambda_2 &= -kRCx \\ \lambda_2 &= -kRCx - B^T\lambda_1 \end{aligned} \tag{7}$$

is obtained. Eliminating λ_2 gives

$$\begin{aligned} \dot{x} &= Ax - BR^{-1}(B^T\lambda_1 + \lambda_2) \\ &= Ax - BR^{-1}(B^T\lambda_1 + \lambda_2) \\ &= Ax - BR^{-1}(B^T\lambda_1 - kRCx - B^T\lambda_1) \\ &= Ax + BR^{-1}(kRCx) \\ &= Ax + kBCx \\ &= (A + kBC)x \end{aligned} \tag{8}$$

and

$$
\begin{aligned}
\dot{\lambda}_1 &= -Qx - A^T\lambda_1 + kC^T\lambda_2 \\
\dot{\lambda}_1 &= -Qx - A^T\lambda_1 + kC^T(-kRCx - B^T\lambda_1) \\
\dot{\lambda}_1 &= -Qx - A^T\lambda_1 - k^2C^TRCx - kC^TB^T\lambda_1 \\
\dot{\lambda}_1 &= -(Q + k^2C^TRC)x - (A^T + kC^TB^T)\lambda_1
\end{aligned} \tag{9}
$$

which is combined into

$$
\begin{bmatrix} \dot{x} \\ \dot{\lambda}_1 \end{bmatrix} = \begin{bmatrix} A + kBC & 0 \\ -(Q + k^2C^TRC) & -(A^T + kC^TB^T) \end{bmatrix} \begin{bmatrix} x \\ \lambda_1 \end{bmatrix} \tag{10}
$$