A dynamical system is given as

$$\begin{cases} \dot{x} = Ax + Bu \\ x(0) = x_0 \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{nxn}$, $B \in \mathbb{R}^{nxm}$, $x \in \mathbb{R}^{nx1}$ and $u \in \mathbb{R}^{mx1}$. The control signal is constrained with

$$g_i \leq u \leq h_i \quad i = 1, 2, \cdots, m \tag{2}$$

Let's assume the system matrices as follows

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}, \tag{3}$$

and hence, $n = 2$ and $m = 1$. Then,

$$g \leq u \leq h \tag{4}$$

and

$$L = \begin{bmatrix} l_{11} \\ l_{21} \end{bmatrix} \tag{5}$$

can be stated. The closed-loop matrix is obtained as,

$$A + BL^T = \begin{bmatrix} a_{11} + b_{11}l_{11} & a_{12} + b_{11}l_{21} \\ a_{21} + b_{21}l_{11} & a_{22} + b_{21}l_{21} \end{bmatrix} \tag{6}$$

$E(L)$ is defined as follows

$$\begin{aligned} E(L) &= \{z | z \in \mathbb{R}^2 \, and \, g \leq l_i^T z \leq h\} \\ &= g \leq l_{11}z_{11} + l_{21}z_{21} \leq h \end{aligned} \tag{7}$$

$F(L)$ is defined as follows

$$F(L) = \bigcap_{t \in [0, \infty]} \{(e^{A_c t})^{-1} E(L)\} \tag{8}$$

where $F(L)$ is a subset of $E(L)$. Let $K = k_1$, then

$$u = sat[(L^T - KB^T P)x] \tag{9}$$

is defined. Consider

$$\begin{aligned} u &= L^T x + v \\ &= l_{11}x_1 + l_{21}x_2 + v \end{aligned} \tag{10}$$

then the closed-loop system is

$$\dot{x} = (A + BL^T)x + Bv \tag{11}$$

which is openly,

$$\begin{aligned} \dot{x}_1 &= (a_{11} + b_{11}l_{11})x_1 + (a_{12} + b_{11}l_{21})x_2 \\ \dot{x}_2 &= (a_{21} + b_{21}l_{11})x_1 + (a_{22} + b_{21}l_{21})x_2 \end{aligned} \tag{12}$$

The derivative of the Lyapunov function $V = x^T P x$ is obtained as,

$$\begin{aligned} \dot{V} &= \dot{x}^T P x + x^T P \dot{x} \\ &= ((A + BL^T)x + Bv)^T P x + x^T P((A + BL^T)x + Bv) \\ &= (v^T B^T + x^T L B^T + x^T A^T)Px + x^T P((A + BL^T)x + Bv) \\ &= v^T B^T P x + x^T L B^T P x + x^T A^T P x + x^T P A x + x^T P B L^T x + x^T P B v \\ &= x^T(A^T P + PA + PBL^T + LB^T P)x + 2x^T P B v \end{aligned} \tag{13}$$

For stability,

$$2x^T P B v \leq 0 \tag{14}$$

is needed. Choosing $v = -RB^T P x$ gives,

$$\begin{aligned} 2x^T PBv &= 2x^T PB(-RB^T Px) \\ 2x^T PBv &= -2x^T(PBRB^T P)x \\ 2x^T PBv &\leq 0 \end{aligned} \tag{15}$$

where $R = diag([r_1, r_2, \cdots, r_m])$. Therefore,

$$\begin{aligned} \dot{V} &= x^T(A^T P + PA + PBL^T + LB^T P)x + 2x^T PBv \\ \dot{V} &= x^T(A^T P + PA + PBL^T + LB^T P)x - 2x^T(PBRB^T P)x \\ \dot{V} &= x^T(A^T P + PA + PBL^T + LB^T P - 2PBRB^T P)x \\ \dot{V} &= x^T(A^T P + PA + PB(L^T - RB^T P) + (L - PBR)B^T P)x \end{aligned} \tag{16}$$

Solving,

$$L^T x - RB^T Px = sat[L^T x - KB^T Px] \tag{17}$$

for any diagonal K.

$$L^T x - RB^T Px = sat[L^T x - KB^T Px] \tag{18}$$

which is expanded as

$$\begin{aligned} l_{11}x_1 + l_{21}x_2 - (b_{11}p_{11}r + b_{21}p_{12}r)x_1 + (-b_{11}p_{12}r - b_{21}p_{22}r)x_2 \\ = sat(l_{11}x_1 + l_{21}x_2 - (b_{11}p_{11}k + b_{21}p_{12}k)x_1 + (-b_{11}p_{12}k - b_{21}p_{22}k)x_2) \end{aligned} \tag{19}$$

if $x \in E(L)$ then

$$g \leq l_{11}x_1 + l_{21}x_2 \leq h \tag{20}$$

but if also

$$g \leq l_{11}x_1 + l_{21}x_2 - (b_{11}p_{11}k + b_{21}p_{12}k)x_1 - (b_{11}p_{12}k + b_{21}p_{22}k)x_2 \leq h \tag{21}$$

then $r = k$. On the other hand, if

$$l_{11}x_1 + l_{21}x_2 - k(b_{11}p_{11} + b_{21}p_{12})x_1 - k(b_{11}p_{12} + b_{21}p_{22})x_2 > h \tag{22}$$

if a smaller r then the term

$$l_{11}x_1 + l_{21}x_2 - r(b_{11}p_{11} + b_{21}p_{12})x_1 - r(b_{11}p_{12} + b_{21}p_{22})x_2 \tag{23}$$

would increase. Therefore,

$$l_{11}x_1 + l_{21}x_2 - r(b_{11}p_{11} + b_{21}p_{12})x_1 - r(b_{11}p_{12} + b_{21}p_{22})x_2 = h \tag{24}$$

The algorithm is given as follows:

1. Determine $\mathbb{D}$. (set of initial states)

2. Find L. Control penalty R in LQR is increased until $L^T x$ satisfies

   $$g \leq L^T x \leq h \tag{25}$$

   for x in $\mathbb{D}$. This can be done via

   $$g \leq \max_{x \in \mathbb{D}} l_i^T x \leq h \tag{26}$$

   If this cannot be satisfied then there is no design.

3. Find P and c. P can be used from LQR. c is obtained from

$$\sup_{x \in \mathbb{D}} x^T P x \leq c \leq \min_{\delta E(L)} x^T P x$$
$$\min_{\delta E(L)} x^T P x = \min_i \frac{g_i^2}{l_i^T P^{-1} l_i}, \frac{h_i^2}{l_i^T P^{-1} l_i} \tag{27}$$

   If this fails choose another P, if it still fails cut down the size of $\mathbb{D}$.

4. Set up the control u according to

$$u = sat[L^T x - K B^T P x] \tag{28}$$

   Tune k with simulations.

An example system is given,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$-1 \leq u \leq 1 \tag{29}$$

The initial condition solution is given as follows,

$$x(t) = e^{At} x(0) \tag{30}$$

hence,

$$\begin{aligned} x(t) &= e^{At} x(0) \\ x(t) &= \left(I + At + \frac{A^2 t^2}{2!} + \cdots \right) x(0) \\ x(t) &= (I + At) x(0) \\ x(t) &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0) \end{aligned} \tag{31}$$

The solution of the given system is obtained as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_1(0) + t x_2(0) \\ x_2(0) \end{bmatrix} \tag{32}$$

The LQR weights are chosen as

$$Q = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \quad R = 500000 \tag{33}$$

hence the LQR matrix is calculated as

$$L = \begin{bmatrix} -0.0045 & -0.0946 \end{bmatrix} \tag{34}$$

The closed-loop system is calculated as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.0045 & -0.0946 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v \tag{35}$$

and solving it gives

$$\begin{aligned} \Phi(1,1) &= e^{-0.0473t}(\cos(0.0473t) + 1.0002\sin(0.0473t)) \\ \Phi(1,2) &= 21.1498e^{-0.0473t}\sin(0.0473t) \\ \Phi(2,1) &= -0.0946e^{-0.0473t}\sin(0.0473t) \\ \Phi(2,2) &= e^{-0.0473t}(\cos(0.0473t) - 1.0002\sin(0.0473t)) \end{aligned} \tag{36}$$

and

$$\begin{aligned} x_1(t) &= e^{-0.0473t}(\cos(0.0473t) + 1.0002\sin(0.0473t))x_1(0) \\ &\quad + 21.1498e^{-0.0473t}\sin(0.0473t)x_2(0) \\ x_2(t) &= -0.0946e^{-0.0473t}\sin(0.0473t)x_1(0) \\ &\quad + e^{-0.0473t}(\cos(0.0473t) - 1.0002\sin(0.0473t))x_2(0) \end{aligned} \tag{37}$$

The control law is obtained as

$$\begin{aligned} u(t) &= e^{-0.0473t}(-0.0045\cos(0.0473t) + 0.0045\sin(0.0473t))x_1(0) \\ &\quad + e^{-0.0473t}(-0.0946\cos(0.0473t) + 0.00002115\sin(0.0473t))x_2(0) \end{aligned} \tag{38}$$

The maximum value is obtained as

$$\begin{aligned} ||u(t)|| &= ||e^{-0.0473t}(-0.0045\cos(0.0473t) + 0.0045\sin(0.0473t))x_1(0) \\ &\quad + e^{-0.0473t}(-0.0946\cos(0.0473t) + 0.00002115\sin(0.0473t))x_2(0)|| \\ &= || \begin{bmatrix} -0.0045 & 0.0045 \end{bmatrix} x_1(0) + \begin{bmatrix} -0.0946 & 0.00002115 \end{bmatrix} x_2(0)|| \\ &\leq || \begin{bmatrix} -0.0045 & 0.0045 \end{bmatrix} |||x_1(0)| + || \begin{bmatrix} -0.0946 & 0.00002115 \end{bmatrix} |||x_2(0)| \\ &\leq 0.0064|x_1(0)| + 0.0946|x_2(0)| \end{aligned} \tag{39}$$

Finding maximum values for individual initial values gives

$$\begin{aligned} 0.0064|x_1(0)| + 0.0946|x_2(0)| &\leq 1 \\ 0.0064|x_1(0)| &\leq 1 \\ |x_1(0)| &\leq 156.25 \end{aligned} \tag{40}$$

and

$$
\begin{aligned}
0.0064|x_1(0)| + 0.0946|x_2(0)| &\leq 1 \\
0.0946|x_2(0)| &\leq 1 \\
0.0946|x_2(0)| &\leq 10.5708
\end{aligned} \tag{41}
$$

If $x_1(0) = x_2(0) = x_0$ then

$$
x_0 = \frac{x_1^{inf}(0) x_2^{inf}(0)}{x_1^{inf}(0) + x_2^{inf}(0)} \tag{42}
$$

formula gives $x_0 = 9.901$.

The set $\mathbb{D}$ is defined as

$$
\mathbb{D} = \{x(0) \in \mathbb{R}^2 \mid -10 \leq x_1(0), x_2(0) \leq 10\} \tag{43}
$$

Control signals of corner points in $\mathbb{D}$ are shown in Figure 1. A dynamical

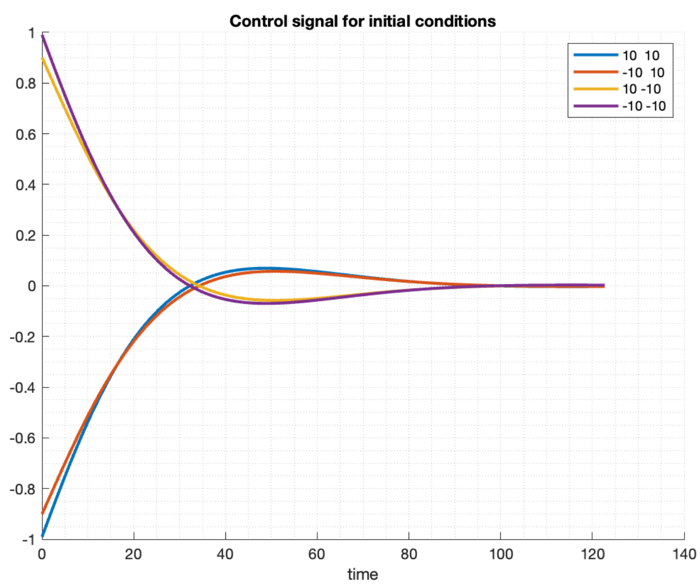

Figure 1: Control signals for different initial conditions

system is given as

$$
\begin{cases}
\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
x(0) = x_0
\end{cases} \tag{44}
$$

where the control signal is constrained with

$$-1 \leq u \leq 1 \tag{45}$$

Let the LQR weights be given as follows

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R = r,\, r > 0 \tag{46}$$

and the unknown $P = P^T$ is defined as

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \tag{47}$$

then the Algebraic Riccati Equation(ARE) is stated as follows,

$$\begin{aligned} A^T P + PA - PBR^{-1}B^T P + Q = 0 \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{r} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0 \\ \begin{bmatrix} r - p_2^2 & -p_2p_3 + p_1 r \\ -p_2p_3 + p_1 r & r - p_3^2 + 2p_2 r \end{bmatrix} = 0 \\ \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \pm\sqrt{(1 \pm 2\sqrt{r})} \\ \pm\sqrt{r} \\ \pm\sqrt{r(1 \pm 2\sqrt{r})} \end{bmatrix} \end{aligned} \tag{48}$$

Since P needs to be positive definite, $p_1 > 0$ and

$$\begin{aligned} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \succ 0 \\ p_1 p_3 - p_2^2 > 0 \\ (\pm\sqrt{(1 \pm 2\sqrt{r})})(\pm\sqrt{r(1 \pm 2\sqrt{r})}) - r > 0 \\ \pm(1 \pm 2\sqrt{r}) - \sqrt{r} > 0 \\ \pm 1 + \sqrt{r} > 0, \quad \pm 1 - 3\sqrt{r} > 0 \end{aligned} \tag{49}$$

needs to be satisfied, hence feasible choices for r are

$$(1 + \sqrt{r}), (1 - 3\sqrt{r}), (-1 + \sqrt{r}) \tag{50}$$

P is obtained as

$$P = \begin{bmatrix} \sqrt{(1 \pm 2\sqrt{r})} & \pm\sqrt{r} \\ \pm\sqrt{r} & \sqrt{r(1 \pm 2\sqrt{r})} \end{bmatrix} \tag{51}$$

The controller gain $L = -RB^T P$ is obtained as

$$\begin{aligned} L &= -RB^T P \\ &= -r \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \\ &= -r \begin{bmatrix} p_2 & p_3 \end{bmatrix} \\ &= -r \begin{bmatrix} \pm\sqrt{r} & \sqrt{r(1 \pm 2\sqrt{r})} \end{bmatrix} \\ &= \begin{bmatrix} \pm r\sqrt{r} & -r\sqrt{r(1 \pm 2\sqrt{r})} \end{bmatrix} \end{aligned} \tag{52}$$

The closed-loop system matrix is given

$$\begin{aligned} \Phi &= e^{A+BL} \\ &= e^{\begin{bmatrix} 0 & 1 \\ l_1 & l_2 \end{bmatrix}} \end{aligned} \tag{53}$$

The matrix exponent identity is given as

$$\begin{aligned} e^{tA} &= \frac{\alpha e^{\beta t} - \beta e^{\alpha t}}{\alpha - \beta} I + \frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} A \\ &= \begin{bmatrix} \frac{\alpha e^{\beta t} - \beta e^{\alpha t}}{\alpha - \beta} & 0 \\ 0 & \frac{\alpha e^{\beta t} - \beta e^{\alpha t}}{\alpha - \beta} \end{bmatrix} + \begin{bmatrix} 0 & \frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} \\ l_1 \frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} & l_2 \frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\alpha e^{\beta t} - \beta e^{\alpha t}}{\alpha - \beta} & \frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} \\ l_1 \frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} & l_2 \frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} + \frac{\alpha e^{\beta t} - \beta e^{\alpha t}}{\alpha - \beta} \end{bmatrix} \\ &= \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha e^{\beta t} - \beta e^{\alpha t} & e^{\alpha t} - e^{\beta t} \\ l_1 e^{\alpha t} - l_1 e^{\beta t} & l_2 e^{\alpha t} - l_2 e^{\beta t} + \alpha e^{\beta t} - \beta e^{\alpha t} \end{bmatrix} \end{aligned} \tag{54}$$

The solution $x(t)$ is calculated as

$$\begin{aligned} x_1(t) &= \frac{1}{\alpha - \beta}(\alpha e^{\beta t} - \beta e^{\alpha t}) x_1(0) + \frac{1}{\alpha - \beta}(e^{\alpha t} - e^{\beta t}) x_2(0) \\ x_2(t) &= \frac{1}{\alpha - \beta}(l_1 e^{\alpha t} - l_1 e^{\beta t}) x_1(0) + \frac{1}{\alpha - \beta}(l_2 e^{\alpha t} - l_2 e^{\beta t} + \alpha e^{\beta t} - \beta e^{\alpha t}) x_2(0) \end{aligned} \tag{55}$$

assuming $\alpha = \sigma + j\omega$ and $\beta = \sigma - j\omega$ and converted into

$$\begin{aligned} x_1(t) &= e^{\sigma t}\left(x_1(0)\cos(\omega t) + \frac{x_2(0) - \sigma x_1(0)}{\omega}\sin(\omega t)\right) \\ x_2(t) &= e^{\sigma t}\left(x_2(0)\cos(\omega t) + \frac{l_1 x_1(0) + (l_2 - \sigma)x_2(0)}{\omega}\sin(\omega t)\right) \end{aligned} \tag{56}$$

The control signal is obtained as follows

$$\begin{aligned} u(t) &= \begin{bmatrix} l_1 & l_2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &= \frac{e^{\sigma t}}{\omega}\cos(\omega t)\left(x_1(0)l_1 + x_2(0)l_2\right) + \frac{e^{\sigma t}}{\omega}\sin(\omega t)\left(x_1(0)(l_1 l_2 - \sigma l_1) + x_2(0)(l_1 - \sigma l_2 + l_2^2)\right) \end{aligned} \tag{57}$$

The control signal limit is obtained as

$$\begin{aligned} u(t) &= \frac{e^{\sigma t}}{\omega}\Big(x_1(0)\Big[l_1\cos(\omega t) + (l_1 l_2 - \sigma l_1)\sin(\omega t)\Big] \\ &\quad + x_2(0)\Big[(l_1 - \sigma l_2 + l_2^2)\sin(\omega t) + l_2\cos(\omega t)\Big]\Big) \\ ||u(t)|| &= \left|\left|\frac{1}{\omega}(x_1(0)\sqrt{l_1^2 + (l_1 l_2 - \sigma l_1)^2}) + x_2(0)\sqrt{(l_1 - \sigma l_2 + l_2^2)^2 + l_2^2}\right|\right| \\ &\leq \left|\left|\frac{1}{\omega}x_1(0)\sqrt{l_1^2 + (l_1 l_2 - \sigma l_1)^2}\right|\right| + \left|\left|x_2(0)\sqrt{(l_1 - \sigma l_2 + l_2^2)^2 + l_2^2}\right|\right| \\ &\leq \frac{|x_1(0)|}{\omega}\sqrt{l_1^2 + (l_1 l_2 - \sigma l_1)^2} + \frac{|x_2(0)|}{\omega}\sqrt{(l_1 - \sigma l_2 + l_2^2)^2 + l_2^2} \end{aligned} \tag{58}$$