

A dynamical system is given as

$$\begin{cases} \dot{x} = Ax + Bu \\ x(0) = x_0 \end{cases} \quad (1)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $x \in \mathbb{R}^{n \times 1}$ and $u \in \mathbb{R}^{m \times 1}$. The control signal is constrained with

$$g_i \leq u \leq h_i \quad i = 1, 2, \dots, m \quad (2)$$

Let's assume the system matrices as follows

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}, \quad (3)$$

and hence, $n = 2$ and $m = 1$. Then,

$$g \leq u \leq h \quad (4)$$

and

$$L = \begin{bmatrix} l_{11} \\ l_{21} \end{bmatrix} \quad (5)$$

can be stated. The closed-loop matrix is obtained as,

$$A + BL^T = \begin{bmatrix} a_{11} + b_{11}l_{11} & a_{12} + b_{11}l_{21} \\ a_{21} + b_{21}l_{11} & a_{22} + b_{21}l_{21} \end{bmatrix} \quad (6)$$

$E(L)$ is defined as follows

$$\begin{aligned} E(L) &= \{z | z \in \mathbb{R}^2 \text{ and } g \leq l_i^T z \leq h\} \\ &= g \leq l_{11}z_{11} + l_{21}z_{21} \leq h \end{aligned} \quad (7)$$

$F(L)$ is defined as follows

$$F(L) = \bigcap_{t \in [0, \infty]} \{(e^{A_c t})^{-1} E(L)\} \quad (8)$$

where $F(L)$ is a subset of $E(L)$. Let $K = k_1$, then

$$u = \text{sat}[(L^T - KB^T P)x] \quad (9)$$

is defined. Consider

$$\begin{aligned} u &= L^T x + v \\ &= l_{11}x_1 + l_{21}x_2 + v \end{aligned} \quad (10)$$

then the closed-loop system is

$$\dot{x} = (A + BL^T)x + Bv \quad (11)$$

which is openly,

$$\begin{aligned} \dot{x}_1 &= (a_{11} + b_{11}l_{11})x_1 + (a_{12} + b_{11}l_{21})x_2 \\ \dot{x}_2 &= (a_{21} + b_{21}l_{11})x_1 + (a_{22} + b_{21}l_{21})x_2 \end{aligned} \quad (12)$$

The derivative of the Lyapunov function $V = x^T Px$ is obtained as,

$$\begin{aligned} \dot{V} &= \dot{x}^T Px + x^T P\dot{x} \\ &= ((A + BL^T)x + Bv)^T Px + x^T P((A + BL^T)x + Bv) \\ &= (v^T B^T + x^T LB^T + x^T A^T)Px + x^T P((A + BL^T)x + Bv) \\ &= v^T B^T Px + x^T LB^T Px + x^T A^T Px + x^T PAx + x^T PBL^T x + x^T PBv \\ &= x^T (A^T P + PA + PBL^T + LB^T P)x + 2x^T PBv \end{aligned} \quad (13)$$

For stability,

$$2x^T PBv \leq 0 \quad (14)$$

is needed. Choosing $v = -RB^T Px$ gives,

$$\begin{aligned} 2x^T PBv &= 2x^T PB(-RB^T Px) \\ 2x^T PBv &= -2x^T (PBRB^T P)x \\ 2x^T PBv &\leq 0 \end{aligned} \quad (15)$$

where $R = \text{diag}([r_1, r_2, \dots, r_m])$. Therefore,

$$\begin{aligned} \dot{V} &= x^T (A^T P + PA + PBL^T + LB^T P)x + 2x^T PBv \\ \dot{V} &= x^T (A^T P + PA + PBL^T + LB^T P)x - 2x^T (PBRB^T P)x \\ \dot{V} &= x^T (A^T P + PA + PBL^T + LB^T P - 2PBRB^T P)x \\ \dot{V} &= x^T (A^T P + PA + PB(L^T - RB^T P) + (L - PBR)B^T P)x \end{aligned} \quad (16)$$

Solving,

$$L^T x - RB^T Px = \text{sat}[L^T x - KB^T Px] \quad (17)$$

for any diagonal K .

$$L^T x - RB^T Px = \text{sat}[L^T x - KB^T Px] \quad (18)$$

which is expanded as

$$\begin{aligned} & l_{11}x_1 + l_{21}x_2 - (b_{11}p_{11}r + b_{21}p_{12}r)x_1 + (-b_{11}p_{12}r - b_{21}p_{22}r)x_2 \\ &= \text{sat}(l_{11}x_1 + l_{21}x_2 - (b_{11}p_{11}k + b_{21}p_{12}k)x_1 + (-b_{11}p_{12}k - b_{21}p_{22}k)x_2) \end{aligned} \quad (19)$$

if $x \in E(L)$ then

$$g \leq l_{11}x_1 + l_{21}x_2 \leq h \quad (20)$$

but if also

$$g \leq l_{11}x_1 + l_{21}x_2 - (b_{11}p_{11}k + b_{21}p_{12}k)x_1 - (b_{11}p_{12}k + b_{21}p_{22}k)x_2 \leq h \quad (21)$$

then $r = k$. On the other hand, if

$$l_{11}x_1 + l_{21}x_2 - k(b_{11}p_{11} + b_{21}p_{12})x_1 - k(b_{11}p_{12} + b_{21}p_{22})x_2 > h \quad (22)$$

if a smaller r then the term

$$l_{11}x_1 + l_{21}x_2 - r(b_{11}p_{11} + b_{21}p_{12})x_1 - r(b_{11}p_{12} + b_{21}p_{22})x_2 \quad (23)$$

would increase. Therefore,

$$l_{11}x_1 + l_{21}x_2 - r(b_{11}p_{11} + b_{21}p_{12})x_1 - r(b_{11}p_{12} + b_{21}p_{22})x_2 = h \quad (24)$$

The algorithm is given as follows:

1. Determine \mathbb{D} . (set of initial states)
2. Find L . Control penalty R in LQR is increased until $L^T x$ satisfies

$$g \leq L^T x \leq h \quad (25)$$

for x in \mathbb{D} . This can be done via

$$g \leq \max_{x \in \mathbb{D}} l_i^T x \leq h \quad (26)$$

If this cannot be satisfied then there is no design.

3. Find P and c . P can be used from LQR. c is obtained from

$$\begin{aligned} \sup_{x \in \mathbb{D}} x^T P x &\leq c \leq \min_{\delta E(L)} x^T P x \\ \min_{\delta E(L)} x^T P x &= \min_i \frac{g_i^2}{l_i^T P^{-1} l_i}, \frac{h_i^2}{l_i^T P^{-1} l_i} \end{aligned} \quad (27)$$

If this fails choose another P , if it still fails cut down the size of \mathbb{D} .

4. Set up the control u according to

$$u = \text{sat}[L^T x - K B^T P x] \quad (28)$$

Tune k with simulations.

An example system is given,

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ -1 &\leq u \leq 1 \end{aligned} \quad (29)$$

The initial condition solution is given as follows,

$$x(t) = e^{At} x(0) \quad (30)$$

hence,

$$\begin{aligned} x(t) &= e^{At} x(0) \\ x(t) &= \left(I + At + \frac{A^2 t^2}{2!} + \dots \right) x(0) \\ x(t) &= (I + At) x(0) \\ x(t) &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0) \end{aligned} \quad (31)$$

The solution of the given system is obtained as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_1(0) + t x_2(0) \\ x_2(0) \end{bmatrix} \quad (32)$$

The LQR weights are chosen as

$$Q = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \quad R = 500000 \quad (33)$$

hence the LQR matrix is calculated as

$$L = \begin{bmatrix} -0.0045 & -0.0946 \end{bmatrix} \quad (34)$$

The closed-loop system is calculated as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.0045 & -0.0946 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v \quad (35)$$

and solving it gives

$$\begin{aligned} \Phi(1, 1) &= e^{-0.0473t}(\cos(0.0473t) + 1.0002 \sin(0.0473t)) \\ \Phi(1, 2) &= 21.1498e^{-0.0473t} \sin(0.0473t) \\ \Phi(2, 1) &= -0.0946e^{-0.0473t} \sin(0.0473t) \\ \Phi(2, 2) &= e^{-0.0473t}(\cos(0.0473t) - 1.0002 \sin(0.0473t)) \end{aligned} \quad (36)$$

The control law is obtained as

$$\begin{aligned} u(t) &= e^{-0.0473t}(-0.0045 \cos(0.0473t) + 0.0045 \sin(0.0473t))x_1(0) \\ &\quad + e^{-0.0473t}(-0.0946 \cos(0.0473t) + 0.00002115 \sin(0.0473t))x_2(0) \end{aligned} \quad (37)$$

The set \mathbb{D} is defined as

$$\mathbb{D} = \{x \in \mathbb{R}^2 \mid -10 \leq x_1, x_2 \leq 10\} \quad (38)$$

Control signals of corner points in \mathbb{D} are shown in Figure 1.

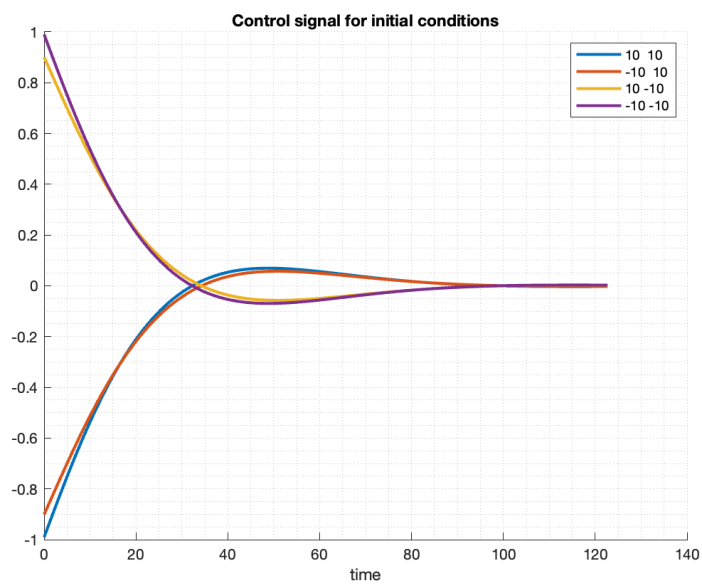


Figure 1: Control signals for different initial conditions