The following can be stated for the closed-loop system

$$(\lambda_1 I - (A - BK))R_1 = 0 \tag{1}$$

where R_1 is the eigenvector for eigenvalue λ_1. Let K be given as

$$K = -MR^{-1} \tag{2}$$

then

$$\begin{aligned} (\lambda_1 I - (A - BM_1 R_1^{-1}))R_1 &= 0 \\ \lambda_1 R_1 - AR_1 + BM_1 &= 0 \\ (\lambda_1 I - A)R_1 + BM_1 &= 0 \end{aligned} \tag{3}$$

Let $R_1 \triangleq X_1 n_1$ and $M_1 \triangleq Y_1 n_1$,

$$\begin{aligned} (\lambda_1 I - A)X_1 n_1 + BY_1 n_1 &= 0 \\ \begin{bmatrix} \lambda_1 I - A & B \end{bmatrix} \begin{bmatrix} X_1 n_1 \\ Y_1 n_1 \end{bmatrix} &= 0 \end{aligned} \tag{4}$$

where n_1 is the linear combination vector. Choosing

$$X_1 n_1 = R_{des}(1) \tag{5}$$

hence,

$$n_1 = X_1^{-1} R_{des}(1) \tag{6}$$

and $\mathcal{N}$ is the null space in

$$\begin{bmatrix} \lambda_1 I - A & B \end{bmatrix} \mathcal{N} = 0 \tag{7}$$

The algorithm is as follows:

1. Calculate the null space($\mathcal{N}$) of

$$\begin{bmatrix} \lambda_1 I - A & B \end{bmatrix} \tag{8}$$

2. Partition the null space matrix $\mathcal{N}$ as

$$\mathcal{N} = \begin{bmatrix} X_1 n_1 \\ Y_1 n_1 \end{bmatrix} \tag{9}$$

3. Calculate $n_1 = X_1^{-1} R_{des}(1)$ and set $M_1 = Y_1 n_1$ and $R_1 = X_1 n_1$.
4. Repeat steps for each eigenvalue.
5. Determine controller gain $K = -MR^{-1}$.

Since R can be used to diagonalize the closed-loop matrix

$$A + BK = R\Lambda R^{-1} \tag{10}$$

this fact can be used to evaluate the following

$$e^{(A+BK)t} = Re^{\Lambda t} R^{-1} \tag{11}$$

where R and Λ are closed-loop parameters and are known. Thereby,

$$x(t) = Re^{\Lambda t} R^{-1} x_0 \tag{12}$$

Defining $z = Rx$ and using the relation $x = R^{-1}z$ gives,

$$\begin{aligned} x(t) &= Re^{\Lambda t} R^{-1} x_0 \\ R^{-1} x(t) &= e^{\Lambda t} R^{-1} x_0 \\ z(t) &= e^{\Lambda t} z_0 \end{aligned} \tag{13}$$

and the control law is updated as

$$\begin{aligned} u &= Kx \\ u &= KR^{-1} z \end{aligned} \tag{14}$$

Sky-hook controller is defined as

$$u = k \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} x \tag{15}$$

The LQR controller is designed via solving the Algebraic Riccati Equation(ARE)

$$A^T P + PA + PBR^{-1} B^T P + Q = 0 \tag{16}$$

or by using the Hamiltonian given as

$$\mathcal{H} = \begin{bmatrix} A & BR^{-1}B \\ -Q & -A^T \end{bmatrix} \tag{17}$$

in the equation

$$\mathcal{H}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = 0 \tag{18}$$

if $\mathcal{H}$ has no eigenvalues on the jw-axis. The solution P of the ARE is obtained via,

$$P = U_2 U_1^{-1} \tag{19}$$

and the lqr controller then is obtained using $K = -R^{-1}B^TP$. The Schur Complement

$$A - BD^{-1}C, \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{20}$$

is used as follows,

$$\begin{gathered} A^TP + PA + PBR^{-1}B^TP + Q = 0 \\ \begin{bmatrix} A^TP + PA + Q & -B \\ B^T & R \end{bmatrix} \end{gathered} \tag{21}$$