

A dynamical system is given as

$$\begin{cases} \dot{x} = Ax + Bu \\ x(0) = x_0 \end{cases} \quad (1)$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $x \in \mathbb{R}^{n \times 1}$  and  $u \in \mathbb{R}^{m \times 1}$ . The control signal is constrained with

$$g_i \leq u \leq h_i \quad i = 1, 2, \dots, m \quad (2)$$

Let's assume the system matrices as follows

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}, \quad (3)$$

and hence,  $n = 2$  and  $m = 1$ . Then,

$$g \leq u \leq h \quad (4)$$

and

$$L = \begin{bmatrix} l_{11} \\ l_{21} \end{bmatrix} \quad (5)$$

can be stated. The closed-loop matrix is obtained as,

$$A + BL^T = \begin{bmatrix} a_{11} + b_{11}l_{11} & a_{12} + b_{11}l_{21} \\ a_{21} + b_{21}l_{11} & a_{22} + b_{21}l_{21} \end{bmatrix} \quad (6)$$

$E(L)$  is defined as follows

$$\begin{aligned} E(L) &= \{z | z \in \mathbb{R}^2 \text{ and } g \leq l_i^T z \leq h\} \\ &= g \leq l_{11}z_{11} + l_{21}z_{21} \leq h \end{aligned} \quad (7)$$

$F(L)$  is defined as follows

$$F(L) = \bigcap_{t \in [0, \infty]} \{(e^{A_c t})^{-1} E(L)\} \quad (8)$$

where  $F(L)$  is a subset of  $E(L)$ . Let  $K = k_1$ , then

$$u = \text{sat}[(L^T - KB^T P)x] \quad (9)$$

is defined. Consider

$$\begin{aligned} u &= L^T x + v \\ &= l_{11}x_1 + l_{21}x_2 + v \end{aligned} \quad (10)$$

then the closed-loop system is

$$\dot{x} = (A + BL^T)x + Bv \quad (11)$$

which is openly,

$$\begin{aligned} \dot{x}_1 &= (a_{11} + b_{11}l_{11})x_1 + (a_{12} + b_{11}l_{21})x_2 \\ \dot{x}_2 &= (a_{21} + b_{21}l_{11})x_1 + (a_{22} + b_{21}l_{21})x_2 \end{aligned} \quad (12)$$

The derivative of the Lyapunov function  $V = x^T Px$  is obtained as,

$$\begin{aligned} \dot{V} &= \dot{x}^T Px + x^T P\dot{x} \\ &= ((A + BL^T)x + Bv)^T Px + x^T P((A + BL^T)x + Bv) \\ &= (v^T B^T + x^T LB^T + x^T A^T)Px + x^T P((A + BL^T)x + Bv) \\ &= v^T B^T Px + x^T LB^T Px + x^T A^T Px + x^T PAx + x^T PBL^T x + x^T PBv \\ &= x^T (A^T P + PA + PBL^T + LB^T P)x + 2x^T PBv \end{aligned} \quad (13)$$

For stability,

$$2x^T PBv \leq 0 \quad (14)$$

is needed. Choosing  $v = -RB^T Px$  gives,

$$\begin{aligned} 2x^T PBv &= 2x^T PB(-RB^T Px) \\ 2x^T PBv &= -2x^T (PBRB^T P)x \\ 2x^T PBv &\leq 0 \end{aligned} \quad (15)$$

where  $R = \text{diag}([r_1, r_2, \dots, r_m])$ . Therefore,

$$\begin{aligned} \dot{V} &= x^T (A^T P + PA + PBL^T + LB^T P)x + 2x^T PBv \\ \dot{V} &= x^T (A^T P + PA + PBL^T + LB^T P)x - 2x^T (PBRB^T P)x \\ \dot{V} &= x^T (A^T P + PA + PBL^T + LB^T P - 2PBRB^T P)x \\ \dot{V} &= x^T (A^T P + PA + PB(L^T - RB^T P) + (L - PBR)B^T P)x \end{aligned} \quad (16)$$

Solving,

$$L^T x - RB^T Px = \text{sat}[L^T x - KB^T Px] \quad (17)$$

for any diagonal  $K$ .

$$L^T x - RB^T Px = \text{sat}[L^T x - KB^T Px] \quad (18)$$

which is expanded as

$$\begin{aligned} & l_{11}x_1 + l_{21}x_2 - (b_{11}p_{11}r + b_{21}p_{12}r)x_1 + (-b_{11}p_{12}r - b_{21}p_{22}r)x_2 \\ &= \text{sat}(l_{11}x_1 + l_{21}x_2 - (b_{11}p_{11}k + b_{21}p_{12}k)x_1 + (-b_{11}p_{12}k - b_{21}p_{22}k)x_2) \end{aligned} \quad (19)$$

if  $x \in E(L)$  then

$$g \leq l_{11}x_1 + l_{21}x_2 \leq h \quad (20)$$

but if also

$$g \leq l_{11}x_1 + l_{21}x_2 - (b_{11}p_{11}k + b_{21}p_{12}k)x_1 - (b_{11}p_{12}k + b_{21}p_{22}k)x_2 \leq h \quad (21)$$

then  $r = k$ . On the other hand, if

$$l_{11}x_1 + l_{21}x_2 - k(b_{11}p_{11} + b_{21}p_{12})x_1 - k(b_{11}p_{12} + b_{21}p_{22})x_2 > h \quad (22)$$

if a smaller  $r$  then the term

$$l_{11}x_1 + l_{21}x_2 - r(b_{11}p_{11} + b_{21}p_{12})x_1 - r(b_{11}p_{12} + b_{21}p_{22})x_2 \quad (23)$$

would increase. Therefore,

$$l_{11}x_1 + l_{21}x_2 - r(b_{11}p_{11} + b_{21}p_{12})x_1 - r(b_{11}p_{12} + b_{21}p_{22})x_2 = h \quad (24)$$

The algorithm is given as follows:

1. Determine  $\mathbb{D}$ . (set of initial states)
2. Find  $L$ . Control penalty  $R$  in LQR is increased until  $L^T x$  satisfies

$$g \leq L^T x \leq h \quad (25)$$

for  $x$  in  $\mathbb{D}$ . This can be done via

$$g \leq \max_{x \in \mathbb{D}} l_i^T x \leq h \quad (26)$$

If this cannot be satisfied then there is no design.

3. Find  $P$  and  $c$ .  $P$  can be used from LQR.  $c$  is obtained from

$$\begin{aligned} \sup_{x \in \mathbb{D}} x^T P x &\leq c \leq \min_{\delta E(L)} x^T P x \\ \min_{\delta E(L)} x^T P x &= \min_i \frac{g_i^2}{l_i^T P^{-1} l_i}, \frac{h_i^2}{l_i^T P^{-1} l_i} \end{aligned} \quad (27)$$

If this fails choose another  $P$ , if it still fails cut down the size of  $\mathbb{D}$ .

4. Set up the control  $u$  according to

$$u = \text{sat}[L^T x - K B^T P x] \quad (28)$$

Tune  $k$  with simulations.

An example system is given,

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ -1 &\leq u \leq 1 \end{aligned} \quad (29)$$

The initial condition solution is given as follows,

$$x(t) = e^{At} x(0) \quad (30)$$

hence,

$$\begin{aligned} x(t) &= e^{At} x(0) \\ x(t) &= \left( I + At + \frac{A^2 t^2}{2!} + \dots \right) x(0) \\ x(t) &= (I + At) x(0) \\ x(t) &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0) \end{aligned} \quad (31)$$

The solution of the given system is obtained as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_1(0) + t x_2(0) \\ x_2(0) \end{bmatrix} \quad (32)$$

The LQR weights are chosen as

$$Q = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \quad R = 500000 \quad (33)$$

hence the LQR matrix is calculated as

$$L = \begin{bmatrix} -0.0045 & -0.0946 \end{bmatrix} \quad (34)$$

The closed-loop system is calculated as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.0045 & -0.0946 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v \quad (35)$$

and solving it gives

$$\begin{aligned} \Phi(1, 1) &= e^{-0.0473t}(\cos(0.0473t) + 1.0002 \sin(0.0473t)) \\ \Phi(1, 2) &= 21.1498e^{-0.0473t} \sin(0.0473t) \\ \Phi(2, 1) &= -0.0946e^{-0.0473t} \sin(0.0473t) \\ \Phi(2, 2) &= e^{-0.0473t}(\cos(0.0473t) - 1.0002 \sin(0.0473t)) \end{aligned} \quad (36)$$

and

$$\begin{aligned} x_1(t) &= e^{-0.0473t}(\cos(0.0473t) + 1.0002 \sin(0.0473t))x_1(0) \\ &\quad + 21.1498e^{-0.0473t} \sin(0.0473t)x_2(0) \\ x_2(t) &= -0.0946e^{-0.0473t} \sin(0.0473t)x_1(0) \\ &\quad + e^{-0.0473t}(\cos(0.0473t) - 1.0002 \sin(0.0473t))x_2(0) \end{aligned} \quad (37)$$

The control law is obtained as

$$\begin{aligned} u(t) &= e^{-0.0473t}(-0.0045 \cos(0.0473t) + 0.0045 \sin(0.0473t))x_1(0) \\ &\quad + e^{-0.0473t}(-0.0946 \cos(0.0473t) + 0.00002115 \sin(0.0473t))x_2(0) \end{aligned} \quad (38)$$

The maximum value is obtained as

$$\begin{aligned} \|u(t)\| &= \|e^{-0.0473t}(-0.0045 \cos(0.0473t) + 0.0045 \sin(0.0473t))x_1(0) \\ &\quad + e^{-0.0473t}(-0.0946 \cos(0.0473t) + 0.00002115 \sin(0.0473t))x_2(0)\| \\ &= \|[ -0.0045 \quad 0.0045 ] x_1(0) + [ -0.0946 \quad 0.00002115 ] x_2(0)\| \\ &\leq \|[ -0.0045 \quad 0.0045 ]\| |x_1(0)| + \|[ -0.0946 \quad 0.00002115 ]\| |x_2(0)| \\ &\leq 0.0064|x_1(0)| + 0.0946|x_2(0)| \end{aligned} \quad (39)$$

Finding maximum values for individual initial values gives

$$\begin{aligned} 0.0064|x_1(0)| + 0.0946|x_2(0)| &\leq 1 \\ 0.0064|x_1(0)| &\leq 1 \\ |x_1(0)| &\leq 156.25 \end{aligned} \quad (40)$$

and

$$\begin{aligned} 0.0064|x_1(0)| + 0.0946|x_2(0)| &\leq 1 \\ 0.0946|x_2(0)| &\leq 1 \\ 0.0946|x_2(0)| &\leq 10.5708 \end{aligned} \quad (41)$$

If  $x_1(0) = x_2(0) = x_0$  then

$$x_0 = \frac{x_1^{inf}(0)x_2^{inf}(0)}{x_1^{inf}(0) + x_2^{inf}(0)} \quad (42)$$

formula gives  $x_0 = 9.901$ .

The set  $\mathbb{D}$  is defined as

$$\mathbb{D} = \{x(0) \in \mathbb{R}^2 \mid -10 \leq x_1(0), x_2(0) \leq 10\} \quad (43)$$

Control signals of corner points in  $\mathbb{D}$  are shown in Figure 1. A dynamical

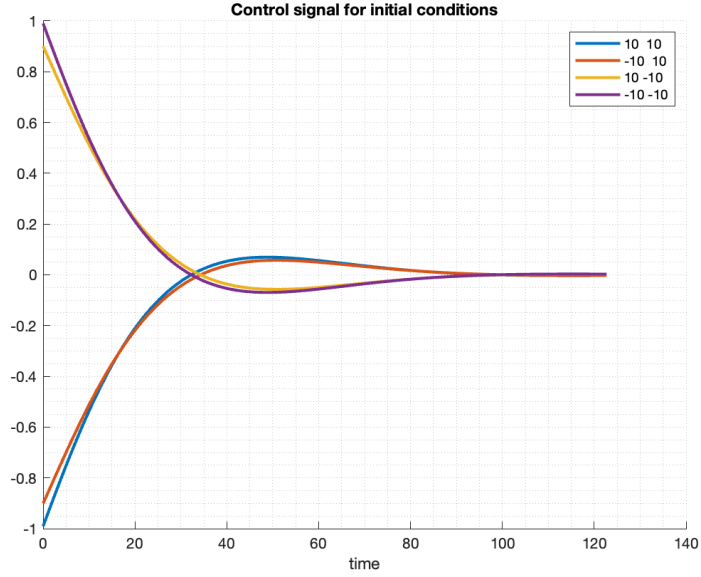


Figure 1: Control signals for different initial conditions

system is given as

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ x(0) = x_0 \end{cases} \quad (44)$$

where the control signal is constrained with

$$-1 \leq u \leq 1 \quad (45)$$

Let the LQR weights be given as follows

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R = r, \quad r > 0 \quad (46)$$

and the unknown  $P = P^T$  is defined as

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \quad (47)$$

then the Algebraic Riccati Equation (ARE) is stated as follows,

$$\begin{aligned} A^T P + P A - P B R^{-1} B^T P + Q &= 0 \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{r} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} &= 0 \\ \begin{bmatrix} r - p_2^2 & -p_2 p_3 + p_1 r \\ -p_2 p_3 + p_1 r & r - p_3^2 + 2p_2 r \end{bmatrix} &= 0 \\ \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} &= \begin{bmatrix} \pm \sqrt{(1 \pm 2\sqrt{r})} \\ \pm \sqrt{r} \\ \pm \sqrt{r(1 \pm 2\sqrt{r})} \end{bmatrix} \end{aligned} \quad (48)$$

Since  $P$  needs to be positive definite,  $p_1 > 0$  and

$$\begin{aligned} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} &\succ 0 \\ p_1 p_3 - p_2^2 &> 0 \\ (\pm \sqrt{(1 \pm 2\sqrt{r})})(\pm \sqrt{r(1 \pm 2\sqrt{r})}) - r &> 0 \\ \pm(1 \pm 2\sqrt{r}) - \sqrt{r} &> 0 \\ \pm 1 + \sqrt{r} &> 0, \quad \pm 1 - 3\sqrt{r} > 0 \end{aligned} \quad (49)$$

needs to be satisfied, hence feasible choices for  $r$  are

$$(1 + \sqrt{r}), (1 - 3\sqrt{r}), (-1 + \sqrt{r}) \quad (50)$$

$P$  is obtained as

$$P = \begin{bmatrix} \sqrt{(1 \pm 2\sqrt{r})} & \pm\sqrt{r} \\ \pm\sqrt{r} & \sqrt{r(1 \pm 2\sqrt{r})} \end{bmatrix} \quad (51)$$

The controller gain  $L = -RB^T P$  is obtained as

$$\begin{aligned} L &= -RB^T P \\ &= -r \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \\ &= -r \begin{bmatrix} p_2 & p_3 \end{bmatrix} \\ &= -r \begin{bmatrix} \pm\sqrt{r} & \sqrt{r(1 \pm 2\sqrt{r})} \end{bmatrix} \\ &= \begin{bmatrix} \pm r\sqrt{r} & -r\sqrt{r(1 \pm 2\sqrt{r})} \end{bmatrix} \end{aligned} \quad (52)$$

The closed-loop system matrix is given

$$\begin{aligned} \Phi &= e^{A+BL} \\ &= e \begin{bmatrix} 0 & 1 \\ l_1 & l_2 \end{bmatrix} \end{aligned} \quad (53)$$

The matrix exponent identity is given as

$$\begin{aligned} e^{tA} &= \frac{\alpha e^{\beta t} - \beta e^{\alpha t}}{\alpha - \beta} I + \frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} A \\ &= \begin{bmatrix} \frac{\alpha e^{\beta t} - \beta e^{\alpha t}}{\alpha - \beta} & 0 \\ 0 & \frac{\alpha e^{\beta t} - \beta e^{\alpha t}}{\alpha - \beta} \end{bmatrix} + \begin{bmatrix} 0 & \frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} \\ l_1 \frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} & l_2 \frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\alpha e^{\beta t} - \beta e^{\alpha t}}{\alpha - \beta} & \frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} \\ l_1 \frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} & l_2 \frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} + \frac{\alpha e^{\beta t} - \beta e^{\alpha t}}{\alpha - \beta} \end{bmatrix} \\ &= \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha e^{\beta t} - \beta e^{\alpha t} & e^{\alpha t} - e^{\beta t} \\ l_1 e^{\alpha t} - l_1 e^{\beta t} & l_2 e^{\alpha t} - l_2 e^{\beta t} + \alpha e^{\beta t} - \beta e^{\alpha t} \end{bmatrix} \end{aligned} \quad (54)$$

The solution  $x(t)$  is calculated as

$$\begin{aligned} x_1(t) &= \frac{1}{\alpha - \beta} (\alpha e^{\beta t} - \beta e^{\alpha t}) x_1(0) + \frac{1}{\alpha - \beta} (e^{\alpha t} - e^{\beta t}) x_2(0) \\ x_2(t) &= \frac{1}{\alpha - \beta} (l_1 e^{\alpha t} - l_1 e^{\beta t}) x_1(0) + \frac{1}{\alpha - \beta} (l_2 e^{\alpha t} - l_2 e^{\beta t} + \alpha e^{\beta t} - \beta e^{\alpha t}) x_2(0) \end{aligned} \quad (55)$$



assuming  $\alpha = \sigma + j\omega$  and  $\beta = \sigma - j\omega$  and converted into

$$\begin{aligned} x_1(t) &= e^{\sigma t} \left( x_1(0) \cos(\omega t) + \frac{x_2(0) - \sigma x_1(0)}{\omega} \sin(\omega t) \right) \\ x_2(t) &= e^{\sigma t} \left( x_2(0) \cos(\omega t) + \frac{l_1 x_1(0) + (l_2 - \sigma)x_2(0)}{\omega} \sin(\omega t) \right) \end{aligned} \quad (56)$$

The control signal is obtained as follows

$$\begin{aligned} u(t) &= \begin{bmatrix} l_1 & l_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &= e^{\sigma t} \cos(\omega t) (x_1(0)l_1 + x_2(0)l_2) \\ &\quad + \frac{e^{\sigma t}}{\omega} \sin(\omega t) (x_1(0)(l_1 l_2 - \sigma l_1) + x_2(0)(l_1 - \sigma l_2 + l_2^2)) \end{aligned} \quad (57)$$

The control signal limit is obtained as

$$\begin{aligned} u(t) &= \frac{e^{\sigma t}}{\omega} \left( x_1(0) \left[ l_1 \omega \cos(\omega t) + (l_1 l_2 - \sigma l_1) \sin(\omega t) \right] \right. \\ &\quad \left. + x_2(0) \left[ (l_1 - \sigma l_2 + l_2^2) \sin(\omega t) + l_2 \omega \cos(\omega t) \right] \right) \\ ||u(t)|| &= \left\| \frac{1}{\omega} (x_1(0) \sqrt{l_1^2 \omega^2 + (l_1 l_2 - \sigma l_1)^2} + x_2(0) \sqrt{(l_1 - \sigma l_2 + l_2^2)^2 + l_2^2 \omega^2}) \right\| \\ &\leq \left\| \frac{1}{\omega} x_1(0) \sqrt{l_1^2 \omega^2 + (l_1 l_2 - \sigma l_1)^2} \right\| + \left\| \frac{1}{\omega} x_2(0) \sqrt{(l_1 - \sigma l_2 + l_2^2)^2 + l_2^2 \omega^2} \right\| \\ &\leq \frac{|x_1(0)|}{\omega} \sqrt{l_1^2 \omega^2 + (l_1 l_2 - \sigma l_1)^2} + \frac{|x_2(0)|}{\omega} \sqrt{(l_1 - \sigma l_2 + l_2^2)^2 + l_2^2 \omega^2} \end{aligned} \quad (58)$$