1. Proof of the LQR Optimal Controller

The Regulation Problem

Let a linear time-invariant (LTI) system be given in state-space form as:

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du,$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, and $D \in \mathbb{R}^{p \times m}$.

The Linear Quadratic Regulator (LQR) problem seeks a control input $u(t)$ that minimizes the infinite-horizon cost functional:

$$J = \frac{1}{2} \int_0^\infty \left(x(t)^T Q x(t) + u(t)^T R u(t) \right) dt,$$

where $Q \in \mathbb{R}^{n \times n}$ is symmetric positive semi-definite ($Q \succeq 0$), and $R \in \mathbb{R}^{m \times m}$ is symmetric positive definite ($R \succ 0$).

We define the Lagrangian $L(x, u, \dot{x}, \lambda)$ incorporating the dynamics as a constraint via the co-state vector $\lambda(t)$:

$$L(x, u, \dot{x}, \lambda) = \frac{1}{2} x^T Q x + \frac{1}{2} u^T R u + \lambda^T (Ax + Bu - \dot{x}).$$

The necessary conditions for optimality from calculus of variations are:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0,$$
$$\frac{\partial L}{\partial u} = 0,$$
$$\frac{\partial L}{\partial \lambda} = 0.$$

Compute the partial derivatives:

$$\frac{\partial L}{\partial \dot{x}} = -\lambda,$$
$$\frac{\partial L}{\partial x} = Qx + A^T \lambda,$$
$$\frac{\partial L}{\partial u} = Ru + B^T \lambda,$$
$$\frac{\partial L}{\partial \lambda} = Ax + Bu - \dot{x}.$$

Substitute into the optimality conditions:

$$\dot{\lambda} = -Qx - A^T \lambda, \quad \text{(costate dynamics)}$$
$$0 = Ru + B^T \lambda, \quad \text{(stationarity w.r.t. } u\text{)}$$
$$0 = Ax + Bu - \dot{x}. \quad \text{(primal dynamics)}$$

From stationarity:

$$u = -R^{-1}B^T\lambda.$$

Assume the co-state vector is a linear function of the state:

$$\lambda = Px,$$

where $P \in \mathbb{R}^{n \times n}$ is a symmetric matrix to be determined.

Differentiate both sides:

$$\dot{\lambda} = P\dot{x} = P(Ax + Bu).$$

Substitute the control law:

$$\dot{\lambda} = P(A - BR^{-1}B^TP)x.$$

From the costate dynamics:

$$\dot{\lambda} = -Qx - A^T\lambda = -Qx - A^TPx.$$

Equating both expressions for $\dot{\lambda}$:

$$P(A - BR^{-1}B^TP)x = -Qx - A^TPx.$$

Since this holds for all x, we equate coefficients:

$$PA - PBR^{-1}B^TP = -Q - A^TP,$$

which simplifies to the Algebraic Riccati Equation:

$$A^TP + PA - PBR^{-1}B^TP + Q = 0.$$

The optimal state-feedback control law is:

$$u(t) = -Kx(t), \quad \text{where} \quad K = R^{-1}B^TP,$$

and P is the unique positive semi-definite solution to the Algebraic Riccati Equation:

$$A^TP + PA - PBR^{-1}B^TP + Q = 0.$$

Reference Tracking Problem

For reference tracking the error and its integral are defined:

$$e(t) = r(t) - y(t),$$

and

$$\dot{q}(t) = e(t)$$

respectively. The cost function is modified as follows:

$$J = \frac{1}{2} \int_0^\infty \left(\begin{bmatrix} x \\ q \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & Q_q \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} + u^T R u \right) dt$$

where Q_q is a positive semi-definite matrix that penalizes the tracking error. The augmented system is given by:

$$\begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} + \begin{bmatrix} B \\ -D \end{bmatrix} u + \begin{bmatrix} 0 \\ I \end{bmatrix} r$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} + Du$$