Due to the use of the singleton output fuzzy sets, the fuzzy inference result is the same no matter which one of the four inference methods in Table 1.1 is employed. Defuzzified by the centroid defuzzifier, the fuzzy controller output is

$$\Delta U(n) = K_{\Delta u} \frac{\mu_{\tilde{P}}(e)\mu_{\tilde{P}}(r) \cdot H + \mu_{\tilde{N}}(e)\mu_{\tilde{N}}(r) \cdot (-H)}{\mu_{\tilde{P}}(e)\mu_{\tilde{P}}(r) + \mu_{\tilde{P}}(e)\mu_{\tilde{N}}(r) + \mu_{\tilde{N}}(e)\mu_{\tilde{P}}(r) + \mu_{\tilde{N}}(e)\mu_{\tilde{N}}(r)}.$$

3.4.2. Derivation and Resulting Structures

Utilizing (3.8) and (3.9), we find that the denominator of the above expression becomes 1. This is because

$$\mu_{\tilde{P}}(e)\mu_{\tilde{P}}(r) + \mu_{\tilde{P}}(e)\mu_{\tilde{N}}(r) + \mu_{\tilde{N}}(e)\mu_{\tilde{P}}(r) + \mu_{\tilde{N}}(e)\mu_{\tilde{N}}(r) = \mu_{\tilde{P}}(e) + \mu_{\tilde{N}}(e) = 1.$$

Replacing the membership notations in the numerator by their mathematical definitions, we obtain

$$\Delta U(n) = \frac{K_{\Delta u}K_e H}{2L} e(n) + \frac{K_{\Delta u}K_r H}{2L} r(n).$$

This fuzzy PI controller is a linear PI controller in incremental form for the entire input space.

For the fuzzy controller, if we replace $\Delta u(n)$ by $u(n)$ in the fuzzy rules r1 to r4, then, based on the relationship between the PI controller in incremental form and the PD controller in position form (see Section 3.2.2), the modified fuzzy controller will be a fuzzy PD controller, which is a linear PD controller.

This study confirms that whether or not a fuzzy controller is linear depends on its configuration (i.e., input fuzzy sets, fuzzy rules, fuzzy logic AND/OR operators, defuzzifier, etc.). No method is available that can directly judge, without explicit knowledge of the controller's input-output relationship, whether a fuzzy controller is a linear controller. The only way is to derive its structure. There are other more complicated fuzzy PID controllers that actually are linear PID controllers (e.g., [20][186]).

Fuzzy control should always be used as nonlinear control, as it does not make any sense to implement fuzzy control as linear control. The linear fuzzy controller shown here serves as a reminder: There are fuzzy controllers that are actually just linear controllers. Thus, certain configurations of fuzzy controllers should not be used to avoid linear fuzzy controllers. Moreover, to be sure that a specific configuration does not lead to a linear controller, one must derive its analytical structure.

3.5. FUZZY PI/PD CONTROLLERS AS PIECEWISE LINEAR PI/PD CONTROLLERS

We now investigate a fuzzy controller that differs from the linear fuzzy PI controller above in the following aspects: (1) the Zadeh fuzzy logic AND operator is used, (2) either the Zadeh or the Lukasiewicz fuzzy logic OR operator is used, and (3) the linear defuzzifier (1.8) is utilized. As will be seen, the new configuration results in a piecewise linear fuzzy controller in that the controller output is a piecewise linear function of its inputs.

Due to the use of the Zadeh fuzzy AND operator, in order to obtain analytical expressions of the AND evaluation results, it is necessary to divide the $E(n) - R(n)$ plane into 12 regions, each of which is called an Input Combination (IC, for short). They are labeled from IC1 to IC12, as shown in Fig. 3.3. The purpose of dividing the input space into these 12

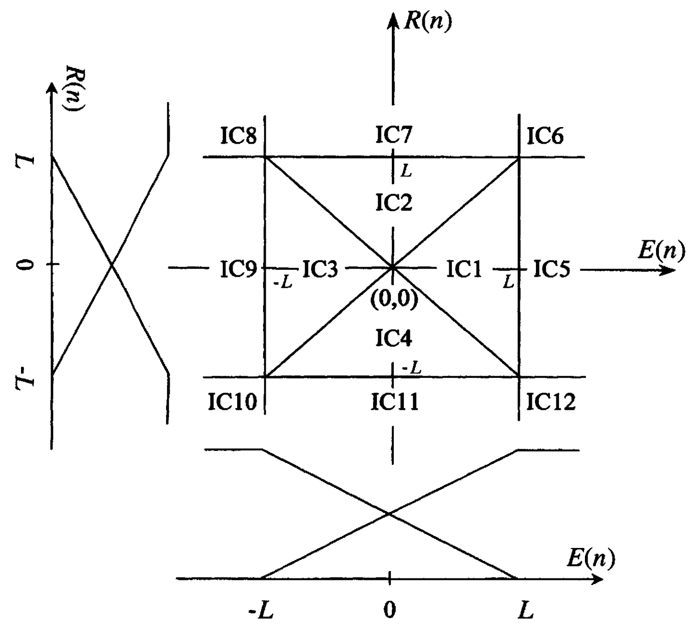

Figure 3.3 Division of the $E(n) - R(n)$ input space into 12 regions for applying the Zadeh fuzzy AND operation in the four fuzzy rules.

regions is to achieve, in each region and each rule, a unique inequality relationship between the two membership values being ANDed. The results of fuzzy AND operation are shown in Table 3.1. They are the membership values, and are in an analytical form for the output fuzzy sets in the four rules.

The fuzzy OR operation is used to combine the output fuzzy set Zero for rules r2 and r3. Either the Lukasiewicz or the Zadeh fuzzy OR operator may be used, but for this fuzzy

TABLE 3.1 The Evaluation Results for the Four Fuzzy Rules in All 12 Regions after Application of the Zadeh Fuzzy AND Operator.[a]

IC No.	r1	r2	r3	r4
1	$\mu_{\tilde{P}}(r)$	$\mu_{\tilde{N}}(r)$	$\mu_{\tilde{N}}(e)$	$\mu_{\tilde{N}}(e)$
2	$\mu_{\tilde{P}}(e)$	$\mu_{\tilde{N}}(r)$	$\mu_{\tilde{N}}(e)$	$\mu_{\tilde{N}}(r)$
3	$\mu_{\tilde{P}}(e)$	$\mu_{\tilde{P}}(e)$	$\mu_{\tilde{P}}(r)$	$\mu_{\tilde{N}}(r)$
4	$\mu_{\tilde{P}}(r)$	$\mu_{\tilde{P}}(e)$	$\mu_{\tilde{P}}(r)$	$\mu_{\tilde{N}}(e)$
5	$\mu_{\tilde{P}}(r)$	$\mu_{\tilde{N}}(r)$	0	0
6	$\mu_{\tilde{P}}(e)$	0	$\mu_{\tilde{N}}(e)$	0
7	0	0	$\mu_{\tilde{P}}(r)$	$\mu_{\tilde{N}}(r)$
8	0	$\mu_{\tilde{P}}(e)$	0	$\mu_{\tilde{N}}(e)$
9	1	0	0	0
10	0	0	1	0
11	0	0	0	1
12	0	1	0	0

[a] These membership values are for the output fuzzy sets in the four rules r1 to r4.

controller, exactly the same controller structure will result, since the two membership values for the singleton fuzzy set Zero are multiplied by 0 in the linear defuzzifier.

The Mamdani minimum inference method is used. Since the output fuzzy sets are of the singleton type, the four different inference methods in Table 1.1 will produce the same inference result.

Using the linear defuzzifier, we obtain the fuzzy controller structure for the 12 ICs in Table 3.2. To show how the structure is derived in more detail, let us take IC1 as an example. For IC1, the AND results are: $\mu_{\tilde{P}}(r)$ for r1, $\mu_{\tilde{N}}(r)$ for r2, and $\mu_{\tilde{N}}(e)$ for r3 and r4. Using the defuzzifier,

$$\begin{aligned}\Delta U(n) &= K_{\Delta u}(\mu_{\tilde{P}}(r)\cdot H + \mu_{\tilde{N}}(r) \times 0 + \mu_{\tilde{N}}(e) \times 0 + \mu_{\tilde{N}}(e)\cdot(-H)) \\ &= \frac{K_{\Delta u}K_e H}{2L}e(n) + \frac{K_{\Delta u}K_r H}{2L}r(n). \end{aligned} \tag{3.10}$$

Figure 3.4 shows three-dimensionally how the incremental output of the piecewise fuzzy controller changes with $e(n)$ and $r(n)$. Clearly, controller output changes with controller inputs in a piecewise linear fashion. For comparison, the incremental output of the corresponding linear PI controller, defined as the linear PI controller in IC1 to IC4, is also plotted. From the figure and Table 3.2, the following can be observed:

TABLE 3.2 Incremental Output of the Linear Fuzzy PI Controller in All 12 Regions after the Linear Defuzzifier is Employed to Combine the Results in Table 3.1.[a]

IC No.	Incremental Output of the Linear Fuzzy PI Controller, $\Delta U(n) =$
1, 2, 3, and 4	$\frac{K_{\Delta u}K_e H}{2L}e(n) + \frac{K_{\Delta u}K_r H}{2L}r(n)$
5	$\frac{K_{\Delta u}K_r H}{2L}r(n) + \frac{K_{\Delta u}H}{2}$
6	$\frac{K_{\Delta u}K_e H}{2L}e(n) + \frac{K_{\Delta u}H}{2}$
7	$\frac{K_{\Delta u}K_r H}{2L}r(n) - \frac{K_{\Delta u}H}{2}$
8	$\frac{K_{\Delta u}K_e H}{2L}e(n) - \frac{K_{\Delta u}H}{2}$
9	$K_{\Delta u}H$
10 and 12	0
11	$-K_{\Delta u}H$

[a] Either the Lukasiewicz fuzzy OR operator or Zadeh fuzzy OR operator is used, which yields the same controller output due to the linear defuzzifier.

(1) Compared with (3.4), the fuzzy controller in IC1 to IC4 is a linear PI controller in incremental form with the proportional-gain and integral-gain being, respectively,

$$K_p = \frac{K_{\Delta u}K_r H}{2L} \quad \text{and} \quad K_i = \frac{K_{\Delta u}K_e H}{2L}.$$

(2) In IC5 and IC7, the fuzzy controller is a proportional controller with a constant offset, and in IC6 and IC8 the fuzzy controller is an integral controller with a constant offset.

(3) In IC10 and IC12, the incremental output is zero. In IC9 and IC11, the increment/decrement is capped. That is to say that during any one sampling period, the maximum increment to the controller output is $K_{\Delta u}H$ (in IC9), and the maximum decrement to the controller output is $-K_{\Delta u}H$ (in IC11).

(4) Switching of the incremental output on the boundary of any two adjacent ICs is continuous and smooth. For example, on the boundary between IC1 and IC5 where $E(n) = L$ and $R(n)$ is any value, $\Delta u(n)$ for IC1 is $\frac{K_{\Delta u}K_r H}{2L} r(n) + \frac{K_{\Delta u}H}{2}$, which is the same as that for IC5.

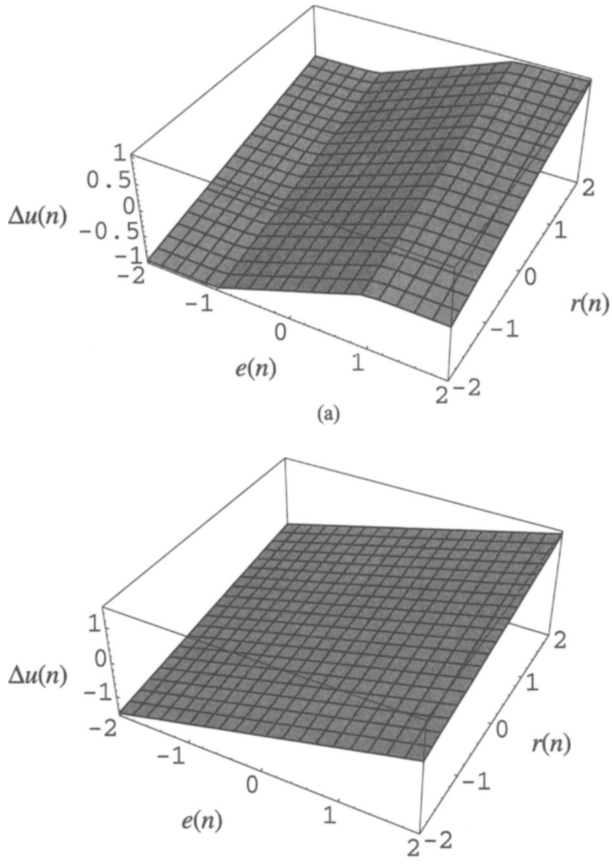

Figure 3.4 (a) Three-dimensional plot of $\Delta u(n)$ of the piecewise linear fuzzy PI controller with respect to $e(n)$ and $r(n)$ whose ranges are $[-2L, 2L]$, and (b) $\Delta u(n)$ of the corresponding linear PI controller, $\Delta u(n) = 0.5e(n) + 0.25r(n)$, for the same ranges of $e(n)$ and $r(n)$. The values of the parameters are: $L = H = 1$, $K_e = 1$, $K_r = 0.5$, and $K_{\Delta u} = 1$.

In summary, this fuzzy controller is a linear PI, P, or I controller in incremental form in IC1 to IC8. Unlike the classical PI controller, however, the maximum change to the fuzzy controller output at any sampling time is constrained. Overall, the fuzzy controller is a piecewise linear PI controller in IC1 to IC12.

Again, if we replace $\Delta u(n)$ by $u(n)$ in the four fuzzy rules, this fuzzy controller will become a piecewise linear PD controller owing to the relationship between the PI controller in incremental form and the PD controller in position form.

3.6. SIMPLEST FUZZY PI CONTROLLER AS NONLINEAR VARIABLE GAIN PI CONTROLLER

3.6.1. Derivation and Resulting Structure

Now, let us study a fuzzy controller that is just a little bit different from the one described in the last section. It uses the same input variables, input and output fuzzy sets, fuzzy inference method and fuzzy rules, and it also employs the Zadeh fuzzy AND operator and the Lukasiewicz fuzzy OR operator. However, it uses the centroid defuzzifier instead of the linear defuzzifier. Such a fuzzy controller is simplest because its configuration is minimal in terms of the number of input variables, fuzzy sets, and fuzzy rules for any properly functional fuzzy controllers. The term *simplest* is used loosely, not strictly. Some fuzzy controllers are even simpler; they cannot properly function, however, and hence are useless. It is in this loose sense that we call some of the fuzzy controllers in this book simplest.

Along with the same line of derivation as in the last section, one can easily find the analytical structure of this fuzzy controller. The result is given in Table 3.3. We use IC1 as an example to show how to obtain the result in the table. In IC1, we know, from Table 3.1, the outcome of applying the Zadeh fuzzy AND operator. Using the centroid defuzzifier,

$$\begin{aligned}\Delta U(n) &= K_{\Delta u}\frac{\mu_{\tilde{P}}(r)\cdot H + \mu_{\tilde{N}}(e)\cdot(-H)}{\mu_{\tilde{P}}(r) + \mu_{\tilde{N}}(r) + \mu_{\tilde{N}}(e) + \mu_{\tilde{N}}(e)} \\ &= \frac{K_{\Delta u}H}{2(2L - K_e e(n))}(K_e e(n) + K_r r(n)).\end{aligned}$$

According to Table 3.3, the fuzzy controller is a nonlinear PI controller in incremental form when both $E(n)$ and $R(n)$ are in IC1 to IC4. The proportional-gain is

$$K_p(e,r) = \begin{cases} \dfrac{K_{\Delta u}K_r H}{2(2L - K_e|e(n)|)}, & \text{for IC1 and IC3} \\ \dfrac{K_{\Delta u}K_r H}{2(2L - K_r|r(n)|)}, & \text{for IC2 and IC4} \end{cases} \tag{3.11}$$

TABLE 3.3 Incremental Output of the Simplest Fuzzy PI Controller in All 12 ICs.

IC No.	Incremental Output of the Simplest Fuzzy PI Controller, $\Delta U(n) =$
1 and 3	$\dfrac{K_{\Delta u}H}{2(2L - K_e\|e(n)\|)}(K_e e(n) + K_r r(n))$
2 and 4	$\dfrac{K_{\Delta u}H}{2(2L - K_r\|r(n)\|)}(K_e e(n) + K_r r(n))$
5 to 12	The same as those shown in Table 3.2