

1. Incremental PID controller

The discrete PID controller is defined as follows:

$$u(n) = k_p e(n) + k_i \sum_{k=0}^n e(k) + k_d [e(n) - e(n-1)] \quad (1)$$

In order to switch to the incremental PID controller expression,

$$\Delta u(n) = u(n) - u(n-1) \quad (2)$$

is defined. Plugging in the controller results in,

$$\begin{aligned} \Delta u(n) &= u(n) - u(n-1) \\ &= k_p e(n) + k_i \sum_{k=0}^n e(k) + k_d [e(n) - e(n-1)] - k_p e(n-1) \\ &\quad - k_i \sum_{k=0}^{n-1} e(k) - k_d [e(n-1) - e(n-2)] \\ &= k_p e(n) + k_i \sum_{k=0}^{n-1} e(k) + k_i e(n) + k_d e(n) - k_d e(n-1) \\ &\quad - k_p e(n-1) - k_i \sum_{k=0}^{n-1} e(k-1) - k_d e(n-1) + k_d e(n-2) \\ &= k_p e(n) - k_p e(n-1) + k_i e(n) + k_d e(n) - 2k_d e(n-1) + k_d e(n-2) \\ &= k_p [e(n) - e(n-1)] + k_i e(n) + k_d [e(n) - 2e(n-1) + e(n-2)] \end{aligned} \quad (3)$$

2. Fuzzy PI controller

Let the inputs be $e(n)$ and $\Delta e(n)$ and the output be $\Delta u(n)$ defined as error, change in error and change in control signal, respectively.

The membership functions Positive(P) and Negative(N) are defined as

$$\mu_P(e) \begin{cases} 0, & e < -L \\ \frac{e+L}{2L}, & -L \leq e \leq L \\ 1, & e > L \end{cases} \quad \mu_N(e) \begin{cases} 1, & e < -L \\ \frac{-e+L}{2L}, & -L \leq e \leq L \\ 0, & e > L \end{cases} \quad (4)$$

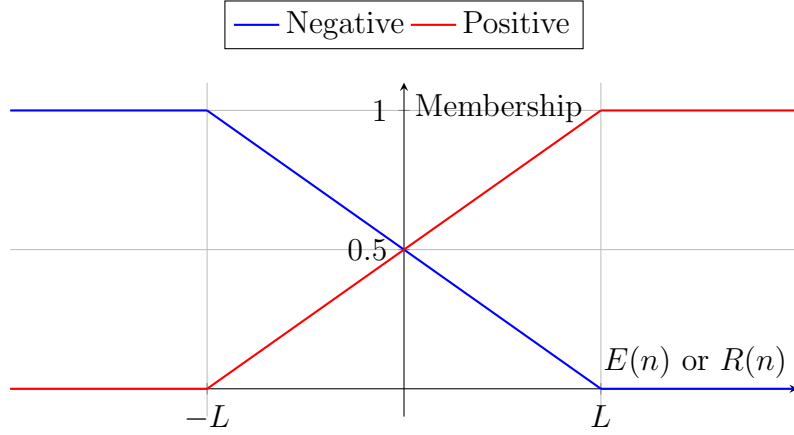


Figure 1: Input fuzzy sets: Negative and Positive

and

$$\mu_P(\Delta e) \begin{cases} 0, & \Delta e < -L \\ \frac{\Delta e + L}{2L}, & -L \leq \Delta e \leq L \\ 1, & \Delta e > L \end{cases} \quad \mu_N(\Delta e) \begin{cases} 1, & \Delta e < -L \\ \frac{-\Delta e + L}{2L}, & -L \leq \Delta e \leq L \\ 0, & \Delta e > L \end{cases} \quad (5)$$

and depicted in Figure 1. The membership functions are chosen such that,

$$\mu_N(e) + \mu_P(e) = 1 \quad \mu_N(\Delta e) + \mu_P(\Delta e) = 1 \quad (6)$$

For the output variable the membership function is chosen as

$$\mu_N(\Delta u) \begin{cases} 1, & \Delta u = -H \\ 0, & \Delta u \neq -H \end{cases} \quad \mu_P(\Delta u) \begin{cases} 1, & \Delta u = H \\ 0, & \Delta u \neq H \end{cases} \quad \mu_Z(\Delta u) \begin{cases} 1, & \Delta u = 0 \\ 0, & \Delta u \neq 0 \end{cases} \quad (7)$$

and is depicted in Figure 2.

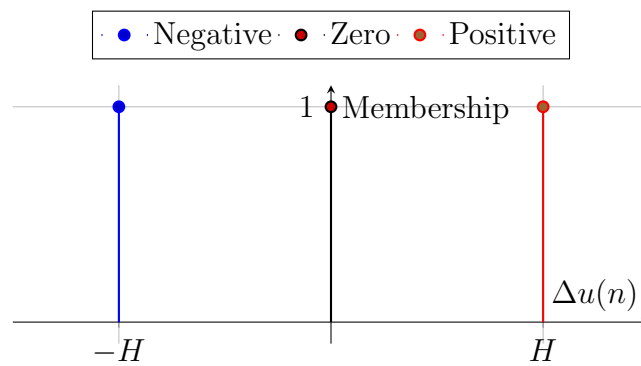


Figure 2: Output fuzzy sets as singleton values at $-H$, 0 , and H

The following rule set is constructed.

Table 1: Fuzzy Rule Table: Control Output $\Delta u(n)$ based on Error $e(n)$ and Change of Error $\Delta e(n)$

$e(n) \setminus \Delta e(n)$	N	Z	P
N	N	N	Z
Z	N	Z	P
P	Z	P	P

The rule set is defined as follows:

- IF $e(n)$ is N AND $\Delta e(n)$ is N THEN output is Negative
- IF $e(n)$ is N AND $\Delta e(n)$ is P THEN output is Zero
- IF $e(n)$ is P AND $\Delta e(n)$ is N THEN output is Zero
- IF $e(n)$ is P AND $\Delta e(n)$ is P THEN output is Positive

The inference step results in:

- $\mu_N(e)\mu_N(\Delta e)$ for output $-H$
- $\mu_N(e)\mu_P(\Delta e)$ for output 0
- $\mu_P(e)\mu_N(\Delta e)$ for output 0
- $\mu_P(e)\mu_P(\Delta e)$ for output H

and hence in,

- $\mu_N(e)\mu_N(\Delta e)$ for output $-H$
- $\mu_P(e)\mu_P(\Delta e)$ for output H

The defuzzification step gives,

$$\Delta u(n) = \frac{\mu_P(e)\mu_P(\Delta e)H - \mu_N(e)\mu_N(\Delta e)H}{\mu_N(e)\mu_N(\Delta e) + \mu_N(e)\mu_P(\Delta e) + \mu_P(e)\mu_N(\Delta e) + \mu_P(e)\mu_P(\Delta e)} \quad (8)$$

The denominator is simplified as follows:

$$\begin{aligned} & \mu_N(e)\mu_N(\Delta e) + \mu_N(e)\mu_P(\Delta e) + \mu_P(e)\mu_N(\Delta e) + \mu_P(e)\mu_P(\Delta e) \\ &= (1 - \mu_P(e))\mu_N(\Delta e) + (1 - \mu_P(e))\mu_P(\Delta e) + \mu_P(e)\mu_N(\Delta e) + \mu_P(e)\mu_P(\Delta e) \\ &= \mu_N(\Delta e) - \mu_P(e)\mu_N(\Delta e) + \mu_P(\Delta e) - \mu_P(e)\mu_P(\Delta e) + \mu_P(e)\mu_N(\Delta e) + \mu_P(e)\mu_P(\Delta e) \\ &= \mu_N(\Delta e) + \mu_P(\Delta e) \\ &= 1 \end{aligned} \quad (9)$$

therefore,

$$\begin{aligned}
\Delta u(n) &= \mu_P(e)\mu_P(\Delta e)H - \mu_N(e)\mu_N(\Delta e)H \\
&= H \begin{cases} -1, & e < -L \\ \frac{\Delta e + L}{2L} \frac{e + L}{2L} - \frac{-e + L}{2L} \frac{-\Delta e + L}{2L}, & -L \leq e \leq L \\ 1, & e > L \end{cases} \\
&= \frac{\Delta e + L}{2L} \frac{e + L}{2L} - \frac{-e + L}{2L} \frac{-\Delta e + L}{2L} \\
&= \frac{(\Delta e + L)(e + L) - (-e + L)(-\Delta e + L)}{4L^2} \\
&= \frac{\Delta e(e + L) + L(e + L) - (-e)(-\Delta e + L) - L(-\Delta e + L)}{4L^2} \\
&= \frac{e\Delta e + L\Delta e + Le + L^2 - e\Delta e + Le + L\Delta e - L^2}{4L^2} \\
&= \frac{L\Delta e + Le + Le + L\Delta e}{4L^2} \\
&= \frac{2L\Delta e + 2Le}{4L^2} \\
&= \frac{\Delta e + e}{2L} \\
&= \frac{1}{2L}\Delta e + \frac{1}{2L}e \\
\Delta u(n) &= \begin{cases} -H, & e < -L \\ \frac{H}{2L}\Delta e + \frac{H}{2L}e, & -L \leq e \leq L \\ H, & e > L \end{cases}
\end{aligned} \tag{10}$$

Since the control signal is a linear combination of Δe and e it is an incremental PI controller.

3. Tables

3.1. Aggressive Tracking

Advantage: Fast response and quick convergence to reference.

Disadvantage: May cause overshoot or oscillation in sensitive systems.

Table 2: Rule Table A: Aggressive Tracking

$e \backslash \Delta e$	N	Z	P
N	N	N	N
Z	N	P	P
P	P	P	P

3.2. Conservative Tracking

Table 3: Rule Table B: Conservative Tracking

$e \backslash \Delta e$	N	Z	P
N	N	Z	Z
Z	Z	Z	Z
P	Z	Z	P

Advantage: Very stable with low overshoot.

Disadvantage: Slow response; possible steady-state error if not tuned well.

3.3. PD-like(Symmetric)

Table 4: Rule Table C: PD-like

$e \backslash \Delta e$	N	Z	P
N	N	Z	P
Z	N	Z	P
P	N	Z	P

Advantage: Symmetric, general-purpose behavior; good if derivative effect dominates.

Disadvantage: No integral action \rightarrow steady-state error possible.

4. Piecewise Linear PI/PD controller with Fuzzy PID

Table 5: IC Region Definitions in $(e(n), \Delta e(n))$ Space

Region	$e(n)$	$\mu_P(e)$	$\mu_N(e)$	$\Delta e(n)$	$\mu_P(\Delta e)$	$\mu_N(\Delta e)$
IC1	$0 < e(n) < L$	$\frac{e+L}{2L}$	$\frac{-e+L}{2L}$	$-L \leq \Delta e(n) \leq L$	$\frac{\Delta e+L}{2L}$	$\frac{-\Delta e+L}{2L}$
IC2	$-L \leq e(n) \leq L$	$\frac{e+L}{2L}$	$\frac{-e+L}{2L}$	$0 < \Delta e(n) < L$	$\frac{\Delta e+L}{2L}$	$\frac{-\Delta e+L}{2L}$
IC3	$-L < e(n) < 0$	$\frac{e+L}{2L}$	$\frac{-e+L}{2L}$	$-L \leq \Delta e(n) \leq L$	$\frac{\Delta e+L}{2L}$	$\frac{-\Delta e+L}{2L}$
IC4	$-L \leq e(n) \leq L$	$\frac{e+L}{2L}$	$\frac{-e+L}{2L}$	$-L < \Delta e(n) < 0$	$\frac{\Delta e+L}{2L}$	$\frac{-\Delta e+L}{2L}$
IC5	$e(n) > L$	1	0	$-L < \Delta e(n) < 0$	$\frac{\Delta e+L}{2L}$	$\frac{-\Delta e+L}{2L}$
IC6	$e(n) > L$	1	0	$\Delta e(n) > L$	1	0
IC7	$-L \leq e(n) \leq L$	$\frac{e+L}{2L}$	$\frac{-e+L}{2L}$	$\Delta e(n) > L$	1	0
IC8	$e < -L$	0	1	$\Delta e(n) > L$	1	0
IC9	$e < -L$	0	1	$-L < \Delta e(n) < 0$	$\frac{\Delta e+L}{2L}$	$\frac{-\Delta e+L}{2L}$
IC10	$e < -L$	0	1	$\Delta e(n) < -L$	0	1
IC11	$-L \leq e(n) \leq L$	$\frac{e+L}{2L}$	$\frac{-e+L}{2L}$	$\Delta e(n) < -L$	0	1
IC12	$e(n) > L$	1	0	$\Delta e(n) < -L$	0	1

IC1

$$\begin{aligned}
r1 &= \min \left(\frac{e+L}{2L} \Big|_{0 < e(n) < L}, \frac{\Delta e + L}{2L} \Big|_{-L \leq \Delta e(n) \leq L} \right) \\
r1 &= \frac{\Delta e + L}{2L} \Big|_{-L \leq \Delta e(n) \leq L} \\
r4 &= \min \left(\frac{-e+L}{2L} \Big|_{0 < e(n) < L}, \frac{-\Delta e + L}{2L} \Big|_{-L \leq \Delta e(n) \leq L} \right) \\
r4 &= \frac{-e+L}{2L} \Big|_{0 < e(n) < L} \\
u &= Hr_1 - Hr_4 \\
u &= H \frac{\Delta e + L}{2L} - H \frac{-e + L}{2L} \\
u &= \frac{H\Delta e + HL + He - HL}{2L} \\
u &= \frac{H}{2L} \Delta e + \frac{H}{2L} e
\end{aligned} \tag{11}$$

IC2

$$\begin{aligned}
r1 &= \min \left(\frac{e+L}{2L} \Big|_{-L \leq e(n) \leq L}, \frac{\Delta e + L}{2L} \Big|_{0 < \Delta e(n) < L} \right) \\
r1 &= \frac{e+L}{2L} \Big|_{-L \leq e(n) \leq L} \\
r4 &= \min \left(\frac{-e+L}{2L} \Big|_{-L \leq e(n) \leq L}, \frac{-\Delta e + L}{2L} \Big|_{0 < \Delta e(n) < L} \right) \\
r4 &= \frac{-\Delta e + L}{2L} \Big|_{0 < \Delta e(n) < L}
\end{aligned} \tag{12}$$

IC3

$$\begin{aligned}
r1 &= \min \left(\frac{e+L}{2L} \Big|_{-L < e(n) < 0}, \frac{\Delta e + L}{2L} \Big|_{-L \leq e(n) \leq L} \right) \\
r1 &= \frac{e+L}{2L} \Big|_{-L < e(n) < 0} \\
r4 &= \min \left(\frac{-e+L}{2L} \Big|_{-L \leq e(n) < 0}, \frac{-\Delta e + L}{2L} \Big|_{-L \leq e(n) \leq L} \right) \\
r4 &= \frac{-\Delta e + L}{2L} \Big|_{-L \leq e(n) \leq L}
\end{aligned} \tag{13}$$

IC4

$$\begin{aligned}
r1 &= \min \left(\frac{e+L}{2L} \Big|_{-L \leq e(n) \leq L}, \frac{\Delta e + L}{2L} \Big|_{-L < \Delta e(n) < 0} \right) \\
r1 &= \frac{\Delta e + L}{2L} \Big|_{-L < \Delta e(n) < 0} \\
r4 &= \min \left(\frac{-e+L}{2L} \Big|_{-L \leq e(n) \leq L}, \frac{-\Delta e + L}{2L} \Big|_{-L < \Delta e(n) < 0} \right) \\
r4 &= \frac{-e+L}{2L} \Big|_{-L \leq e(n) \leq L}
\end{aligned} \tag{14}$$

IC5

$$\begin{aligned}
r1 &= \min \left(1_{e(n) > L}, \frac{\Delta e + L}{2L} \Big|_{-L < \Delta e(n) < 0} \right) \\
r1 &= \frac{\Delta e + L}{2L} \Big|_{-L < \Delta e(n) < 0} \\
r4 &= \min \left(0_{e(n) > L}, \frac{-\Delta e + L}{2L} \Big|_{-L < \Delta e(n) < 0} \right) \\
r4 &= 0 \\
u &= Hr_1 - Hr_4 \\
u &= H \frac{\Delta e + L}{2L} \\
u &= H \frac{\Delta e}{2L} + H \frac{L}{2L} \\
u &= \frac{H}{2L} \Delta e + \frac{H}{2}
\end{aligned} \tag{15}$$

IC6

$$\begin{aligned}
r1 &= \min (1_{e(n) > L}, 1_{e(n) > L}) \\
r1 &= 1 \\
r4 &= \min (0_{e(n) > L}, 0_{e(n) > L}) \\
r4 &= 0 \\
u &= H
\end{aligned} \tag{16}$$

IC7

$$\begin{aligned}
r1 &= \min \left(\frac{e+L}{2L} \quad_{-L \leq e(n) \leq L}, 1_{e(n) > L} \right) \\
r1 &= \frac{e+L}{2L} \quad_{-L \leq e(n) \leq L} \\
r4 &= \min \left(\frac{-e+L}{2L} \quad_{-L \leq e(n) \leq L}, 0_{e(n) > L} \right) \\
r4 &= 0 \\
u &= H \frac{e+L}{2L} \\
u &= \frac{H}{2L}e + \frac{H}{2}
\end{aligned} \tag{17}$$

IC8

$$\begin{aligned}
r1 &= \min (0_{e < -L}, 1_{e(n) > L}) \\
r1 &= 0 \\
r4 &= \min (1_{e < -L}, 0_{e(n) > L}) \\
r4 &= 0 \\
u &= 0
\end{aligned} \tag{18}$$

IC9

$$\begin{aligned}
r1 &= \min \left(0_{e < -L}, \frac{\Delta e + L}{2L} \quad_{-L < \Delta e(n) < 0} \right) \\
r1 &= 0 \\
r4 &= \min \left(1_{e < -L}, \frac{-\Delta e + L}{2L} \quad_{-L < \Delta e(n) < 0} \right) \\
r4 &= \frac{-\Delta e + L}{2L} \quad_{-L < \Delta e(n) < 0} \\
u &= Hr_1 - Hr_4 \\
u &= \frac{H\Delta e - HL}{2L} \\
u &= \frac{H}{2L}\Delta e - \frac{H}{2}
\end{aligned} \tag{19}$$

IC10

$$\begin{aligned}
r1 &= \min (0_{e < -L}, 0_{\Delta e(n) < -L}) \\
r1 &= 0 \\
r4 &= \min (1_{e < -L}, 1_{\Delta e(n) < -L}) \\
r4 &= 1 \\
u &= Hr_1 - Hr_4 \\
u &= -H
\end{aligned} \tag{20}$$

IC11

$$\begin{aligned}
r1 &= \min \left(\frac{e+L}{2L} \Big|_{-L \leq e(n) \leq L}, 0_{\Delta e(n) < -L} \right) \\
r1 &= 0 \\
r4 &= \min \left(\frac{-e+L}{2L} \Big|_{-L \leq e(n) \leq L}, 1_{\Delta e(n) < -L} \right) \\
r4 &= \frac{-e+L}{2L} \Big|_{-L \leq e(n) \leq L} \\
u &= \frac{H}{2L}e - \frac{H}{2}
\end{aligned} \tag{21}$$

IC12

$$\begin{aligned}
r1 &= \min (1_{e(n) > L}, 0_{\Delta e(n) < -L}) \\
r1 &= 0 \\
r4 &= \min (0_{e(n) > L}, 1_{\Delta e(n) < -L}) \\
r4 &= 0 \\
u &= 0
\end{aligned} \tag{22}$$