1. Incremental PID controller

The discrete PID controller is defined as follows:

$$u(n) = k_p e(n) + k_i \sum_{k=0}^{n} e(k) + k_d[e(n) - e(n-1)] \tag{1}$$

In order to switch to the incremental PID controller expression,

$$\Delta u(n) = u(n) - u(n-1) \tag{2}$$

is defined. Plugging in the controller results in,

$$\begin{aligned}
\Delta u(n) &= u(n) - u(n-1) \\
&= k_p e(n) + k_i \sum_{k=0}^{n} e(k) + k_d[e(n) - e(n-1)] - k_p e(n-1) \\
&- k_i \sum_{k=0}^{n-1} e(k) - k_d[e(n-1) - e(n-2)] \\
&= k_p e(n) + k_i \sum_{k=0}^{n-1} e(k) + k_i e(n) + k_d e(n) - k_d e(n-1) \\
&- k_p e(n-1) - k_i \sum_{k=0}^{n-1} e(k-1) - k_d e(n-1) + k_d e(n-2) \\
&= k_p e(n) - k_p e(n-1) + k_i e(n) + k_d e(n) - 2k_d e(n-1) + k_d e(n-2) \\
&= k_p[e(n) - e(n-1)] + k_i e(n) + k_d[e(n) - 2e(n-1) + e(n-2)]
\end{aligned} \tag{3}$$

2. Fuzzy PI controller

Let the inputs be $e(n)$ and $\Delta e(n)$ and the output be $\Delta u(n)$ defined as error, change in error and change in control signal, respectively.

The membership functions Positive(P) and Negative(N) are defined as

$$\mu_P(e) \begin{cases} 0, & e < -L \\ \frac{e+L}{2L}, & -L \leq e \leq L \\ 1, & e > L \end{cases} \quad \mu_N(e) \begin{cases} 1, & e < -L \\ \frac{-e+L}{2L}, & -L \leq e \leq L \\ 0, & e > L \end{cases} \tag{4}$$

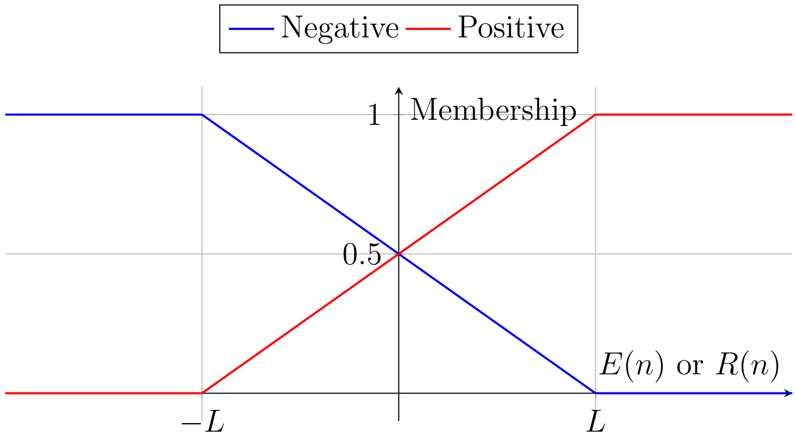

Figure 1: Input fuzzy sets: Negative and Positive

and

$$\mu_P(\Delta e)\begin{cases}0, & \Delta e < -L \\ \frac{\Delta e + L}{2L}, & -L \le \Delta e \le L \\ 1, & \Delta e > L\end{cases} \quad \mu_N(\Delta e)\begin{cases}1, & \Delta e < -L \\ \frac{-\Delta e + L}{2L}, & -L \le \Delta e \le L \\ 0, & \Delta e > L\end{cases} \tag{5}$$

and depicted in Figure 1. The membership functions are chosen such that,

$$\mu_N(e) + \mu_P(e) = 1 \quad \mu_N(\Delta e) + \mu_P(\Delta e) = 1 \tag{6}$$

For the output variable the membership function is chosen as

$$\mu_N(\Delta u)\begin{cases}1, & \Delta u = -H \\ 0, & \Delta u \neq -H\end{cases} \quad \mu_P(\Delta u)\begin{cases}1, & \Delta u = H \\ 0, & \Delta u \neq H\end{cases} \quad \mu_Z(\Delta u)\begin{cases}1, & \Delta u = 0 \\ 0, & \Delta u \neq 0\end{cases} \tag{7}$$

and is depicted in Figure 2.

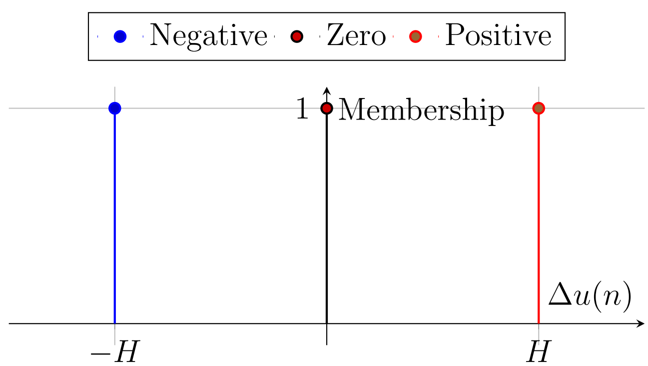

Figure 2: Output fuzzy sets as singleton values at $-H$, 0, and H

The following rule set is constructed.

Table 1: Fuzzy Rule Table: Control Output $\Delta u(n)$ based on Error $e(n)$ and Change of Error $\Delta e(n)$

$e(n) \setminus \Delta e(n)$	**N**	**Z**	**P**
N	N	N	Z
Z	N	Z	P
P	Z	P	P

The rule set is defined as follows:

IF $e(n)$ is N AND $\Delta e(n)$ is N THEN output is Negative
IF $e(n)$ is N AND $\Delta e(n)$ is P THEN output is Zero
IF $e(n)$ is P AND $\Delta e(n)$ is N THEN output is Zero
IF $e(n)$ is P AND $\Delta e(n)$ is P THEN output is Positive

The inference step results in:

$\mu_N(e)\mu_N(\Delta e)$ for output $-H$
$\mu_N(e)\mu_P(\Delta e)$ for output 0
$\mu_P(e)\mu_N(\Delta e)$ for output 0
$\mu_P(e)\mu_P(\Delta e)$ for output H

and hence in,

$\mu_N(e)\mu_N(\Delta e)$ for output $-H$
$\mu_P(e)\mu_P(\Delta e)$ for output H

The defuzzification step gives,

$$\Delta u(n) = \frac{\mu_P(e)\mu_P(\Delta e)H - \mu_N(e)\mu_N(\Delta e)H}{\mu_N(e)\mu_N(\Delta e) + \mu_N(e)\mu_P(\Delta e) + \mu_P(e)\mu_N(\Delta e) + \mu_P(e)\mu_P(\Delta e)} \tag{8}$$

The denominator is simplified as follows:

$$\begin{aligned} &\mu_N(e)\mu_N(\Delta e) + \mu_N(e)\mu_P(\Delta e) + \mu_P(e)\mu_N(\Delta e) + \mu_P(e)\mu_P(\Delta e) \\ &= (1 - \mu_P(e))\mu_N(\Delta e) + (1 - \mu_P(e))\mu_P(\Delta e) + \mu_P(e)\mu_N(\Delta e) + \mu_P(e)\mu_P(\Delta e) \\ &= \mu_N(\Delta e) - \mu_P(e)\mu_N(\Delta e) + \mu_P(\Delta e) - \mu_P(e)\mu_P(\Delta e) + \mu_P(e)\mu_N(\Delta e) + \mu_P(e)\mu_P(\Delta e) \\ &= \mu_N(\Delta e) + \mu_P(\Delta e) \\ &= 1 \end{aligned} \tag{9}$$

therefore,

$$\begin{aligned}
\Delta u(n) &= \mu_P(e)\mu_P(\Delta e)H - \mu_N(e)\mu_N(\Delta e)H \\
&= H \begin{cases} -1, & e < -L \\ \frac{\Delta e+L}{2L}\frac{e+L}{2L} - \frac{-e+L}{2L}\frac{-\Delta e+L}{2L}, & -L \leq e \leq L \\ 1, & e > L \end{cases} \\
&= \frac{\Delta e + L}{2L}\frac{e+L}{2L} - \frac{-e+L}{2L}\frac{-\Delta e + L}{2L} \\
&= \frac{(\Delta e + L)(e+L) - (-e+L)(-\Delta e + L)}{4L^2} \\
&= \frac{\Delta e(e+L) + L(e+L) - (-e)(-\Delta e + L) - L(-\Delta e + L)}{4L^2} \\
&= \frac{e\Delta e + L\Delta e + Le + L^2 - e\Delta e + Le + L\Delta e - L^2}{4L^2} \\
&= \frac{L\Delta e + Le + Le + L\Delta e}{4L^2} \\
&= \frac{2L\Delta e + 2Le}{4L^2} \\
&= \frac{\Delta e + e}{2L} \\
&= \frac{1}{2L}\Delta e + \frac{1}{2L}e \\
\Delta u(n) &= \begin{cases} -H, & e < -L \\ \frac{H}{2L}\Delta e + \frac{H}{2L}e, & -L \leq e \leq L \\ H, & e > L \end{cases}
\end{aligned} \tag{10}$$

Since the control signal is a linear combination of Δe and e it is an incremental PI controller.