1. Incremental PID controller

The discrete PID controller is defined as follows:

$$u(n) = k_p e(n) + k_i \sum_{k=0}^{n} e(k) + k_d[e(n) - e(n-1)] \tag{1}$$

In order to switch to the incremental PID controller expression,

$$\Delta u(n) = u(n) - u(n-1) \tag{2}$$

is defined. Plugging in the controller results in,

$$\begin{aligned}
\Delta u(n) &= u(n) - u(n-1) \\
&= k_p e(n) + k_i \sum_{k=0}^{n} e(k) + k_d[e(n) - e(n-1)] - k_p e(n-1) \\
&- k_i \sum_{k=0}^{n-1} e(k) - k_d[e(n-1) - e(n-2)] \\
&= k_p e(n) + k_i \sum_{k=0}^{n-1} e(k) + k_i e(n) + k_d e(n) - k_d e(n-1) \\
&- k_p e(n-1) - k_i \sum_{k=0}^{n-1} e(k-1) - k_d e(n-1) + k_d e(n-2) \\
&= k_p e(n) - k_p e(n-1) + k_i e(n) + k_d e(n) - 2k_d e(n-1) + k_d e(n-2) \\
&= k_p[e(n) - e(n-1)] + k_i e(n) + k_d[e(n) - 2e(n-1) + e(n-2)]
\end{aligned} \tag{3}$$

2. Fuzzy PI controller

Let the inputs be $e(n)$ and $\Delta e(n)$ and the output be $\Delta u(n)$ defined as error, change in error and change in control signal, respectively.

The membership functions Positive(P) and Negative(N) are defined as

$$\mu_P(e) \begin{cases} 0, & e < -L \\ \frac{e+L}{2L}, & -L \le e \le L \\ 1, & e > L \end{cases} \quad \mu_N(e) \begin{cases} 1, & e < -L \\ \frac{-e+L}{2L}, & -L \le e \le L \\ 0, & e > L \end{cases} \tag{4}$$

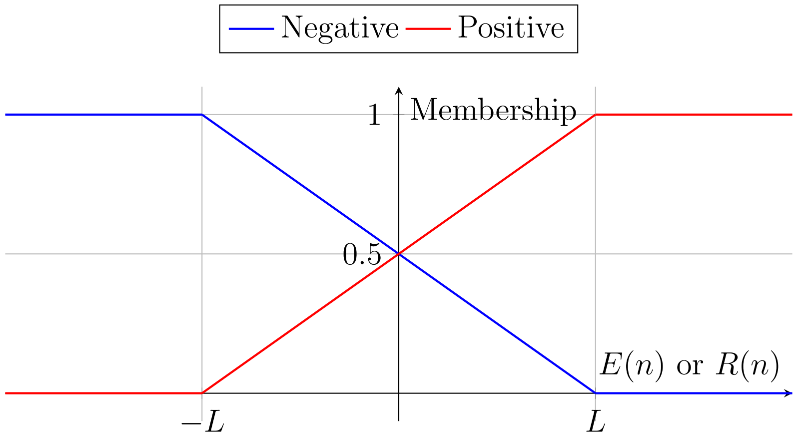

Figure 1: Input fuzzy sets: Negative and Positive

and

$$\mu_P(\Delta e) \begin{cases} 0, & \Delta e < -L \\ \frac{\Delta e + L}{2L}, & -L \leq \Delta e \leq L \\ 1, & \Delta e > L \end{cases} \quad \mu_N(\Delta e) \begin{cases} 1, & \Delta e < -L \\ \frac{-\Delta e + L}{2L}, & -L \leq \Delta e \leq L \\ 0, & \Delta e > L \end{cases} \tag{5}$$

and depicted in Figure 1. The membership functions are chosen such that,

$$\mu_N(e) + \mu_P(e) = 1 \quad \mu_N(\Delta e) + \mu_P(\Delta e) = 1 \tag{6}$$

For the output variable the membership function is chosen as

$$\mu_N(\Delta u) \begin{cases} 1, & \Delta u = -H \\ 0, & \Delta u \neq -H \end{cases} \quad \mu_P(\Delta u) \begin{cases} 1, & \Delta u = H \\ 0, & \Delta u \neq H \end{cases} \quad \mu_Z(\Delta u) \begin{cases} 1, & \Delta u = 0 \\ 0, & \Delta u \neq 0 \end{cases} \tag{7}$$

and is depicted in Figure 2.

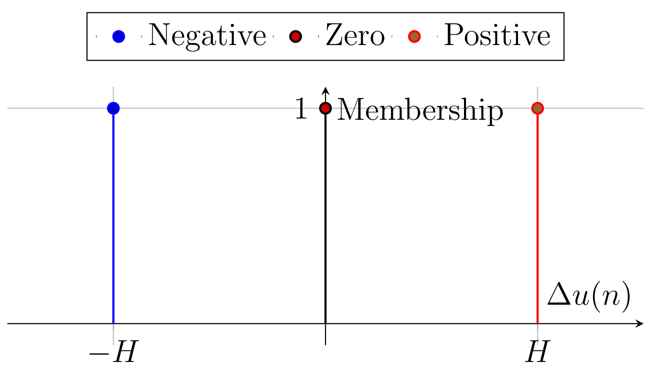

Figure 2: Output fuzzy sets as singleton values at $-H$, 0, and H

The following rule set is constructed.

Table 1: Fuzzy Rule Table: Control Output $\Delta u(n)$ based on Error $e(n)$ and Change of Error $\Delta e(n)$

$e(n) \setminus \Delta e(n)$	**N**	**Z**	**P**
N	N	N	Z
Z	N	Z	P
P	Z	P	P

The rule set is defined as follows:

IF $e(n)$ is N AND $\Delta e(n)$ is N THEN output is Negative
IF $e(n)$ is N AND $\Delta e(n)$ is P THEN output is Zero
IF $e(n)$ is P AND $\Delta e(n)$ is N THEN output is Zero
IF $e(n)$ is P AND $\Delta e(n)$ is P THEN output is Positive

The inference step results in:

$\mu_N(e)\mu_N(\Delta e)$ for output $-H$
$\mu_N(e)\mu_P(\Delta e)$ for output 0
$\mu_P(e)\mu_N(\Delta e)$ for output 0
$\mu_P(e)\mu_P(\Delta e)$ for output H

and hence in,

$\mu_N(e)\mu_N(\Delta e)$ for output $-H$
$\mu_P(e)\mu_P(\Delta e)$ for output H

The defuzzification step gives,

$$\Delta u(n) = \frac{\mu_P(e)\mu_P(\Delta e)H - \mu_N(e)\mu_N(\Delta e)H}{\mu_N(e)\mu_N(\Delta e) + \mu_N(e)\mu_P(\Delta e) + \mu_P(e)\mu_N(\Delta e) + \mu_P(e)\mu_P(\Delta e)} \tag{8}$$

The denominator is simplified as follows:

$$\begin{aligned} &\mu_N(e)\mu_N(\Delta e) + \mu_N(e)\mu_P(\Delta e) + \mu_P(e)\mu_N(\Delta e) + \mu_P(e)\mu_P(\Delta e) \\ &= (1 - \mu_P(e))\mu_N(\Delta e) + (1 - \mu_P(e))\mu_P(\Delta e) + \mu_P(e)\mu_N(\Delta e) + \mu_P(e)\mu_P(\Delta e) \\ &= \mu_N(\Delta e) - \mu_P(e)\mu_N(\Delta e) + \mu_P(\Delta e) - \mu_P(e)\mu_P(\Delta e) + \mu_P(e)\mu_N(\Delta e) + \mu_P(e)\mu_P(\Delta e) \\ &= \mu_N(\Delta e) + \mu_P(\Delta e) \\ &= 1 \end{aligned} \tag{9}$$

therefore,

$$
\begin{aligned}
\Delta u(n) &= \mu_P(e)\mu_P(\Delta e)H - \mu_N(e)\mu_N(\Delta e)H \\
&= H \begin{cases} -1, & e < -L \\ \frac{\Delta e+L}{2L}\frac{e+L}{2L} - \frac{-e+L}{2L}\frac{-\Delta e+L}{2L}, & -L \leq e \leq L \\ 1, & e > L \end{cases} \\
&= \frac{\Delta e + L}{2L}\frac{e + L}{2L} - \frac{-e + L}{2L}\frac{-\Delta e + L}{2L} \\
&= \frac{(\Delta e + L)(e + L) - (-e + L)(-\Delta e + L)}{4L^2} \\
&= \frac{\Delta e(e + L) + L(e + L) - (-e)(-\Delta e + L) - L(-\Delta e + L)}{4L^2} \\
&= \frac{e\Delta e + L\Delta e + Le + L^2 - e\Delta e + Le + L\Delta e - L^2}{4L^2} \\
&= \frac{L\Delta e + Le + Le + L\Delta e}{4L^2} \\
&= \frac{2L\Delta e + 2Le}{4L^2} \\
&= \frac{\Delta e + e}{2L} \\
&= \frac{1}{2L}\Delta e + \frac{1}{2L}e \\
\Delta u(n) &= \begin{cases} -H, & e < -L \\ \frac{H}{2L}\Delta e + \frac{H}{2L}e, & -L \leq e \leq L \\ H, & e > L \end{cases}
\end{aligned}
\tag{10}
$$

Since the control signal is a linear combination of Δe and e it is an incremental PI controller.

3. Tables

3.1. Aggressive Tracking

Advantage: Fast response and quick convergence to reference.

Disadvantage: May cause overshoot or oscillation in sensitive systems.

Table 2: Rule Table A: Aggressive Tracking

$e\backslash\Delta e$	N	Z	P
N	N	N	N
Z	N	P	P
P	P	P	P

3.2. Conservative Tracking

Table 3: Rule Table B: Conservative Tracking

$e\backslash\Delta e$	N	Z	P
N	N	Z	Z
Z	Z	Z	Z
P	Z	Z	P

Advantage: Very stable with low overshoot.
Disadvantage: Slow response; possible steady-state error if not tuned well.

3.3. PD-like(Symmetric)

Table 4: Rule Table C: PD-like

$e\backslash\Delta e$	N	Z	P
N	N	Z	P
Z	N	Z	P
P	N	Z	P

Advantage: Symmetric, general-purpose behavior; good if derivative effect dominates.

Disadvantage: No integral action → steady-state error possible.

4. Piecewise Linear PI/PD controller with Fuzzy PID

Table 5: IC Region Definitions in $(e(n), \Delta e(n))$ Space

Region	$e(n)$	$\mu_P(e)$	$\mu_N(e)$	$\Delta e(n)$	$\mu_P(\Delta e)$	$\mu_N(\Delta e)$
IC1	$0 < e(n) < L$	$\frac{e+L}{2L}$	$\frac{-e+L}{2L}$	$-L \leq \Delta e(n) \leq L$	$\frac{\Delta e+L}{2L}$	$\frac{-\Delta e+L}{2L}$
IC2	$-L \leq e(n) \leq L$	$\frac{e+L}{2L}$	$\frac{-e+L}{2L}$	$0 < \Delta e(n) < L$	$\frac{\Delta e+L}{2L}$	$\frac{-\Delta e+L}{2L}$
IC3	$-L < e(n) < 0$	$\frac{e+L}{2L}$	$\frac{-e+L}{2L}$	$-L \leq \Delta e(n) \leq L$	$\frac{\Delta e+L}{2L}$	$\frac{-\Delta e+L}{2L}$
IC4	$-L \leq e(n) \leq L$	$\frac{e+L}{2L}$	$\frac{-e+L}{2L}$	$-L < \Delta e(n) < 0$	$\frac{\Delta e+L}{2L}$	$\frac{-\Delta e+L}{2L}$
IC5	$e(n) > L$	1	0	$-L < \Delta e(n) < 0$	$\frac{\Delta e+L}{2L}$	$\frac{-\Delta e+L}{2L}$
IC6	$e(n) > L$	1	0	$\Delta e(n) > L$	1	0
IC7	$-L \leq e(n) \leq L$	$\frac{e+L}{2L}$	$\frac{-e+L}{2L}$	$\Delta e(n) > L$	1	0
IC8	$e < -L$	0	1	$\Delta e(n) > L$	1	0
IC9	$e < -L$	0	1	$-L < \Delta e(n) < 0$	$\frac{\Delta e+L}{2L}$	$\frac{-\Delta e+L}{2L}$
IC10	$e < -L$	0	1	$\Delta e(n) < -L$	0	1
IC11	$-L \leq e(n) \leq L$	$\frac{e+L}{2L}$	$\frac{-e+L}{2L}$	$\Delta e(n) < -L$	0	1
IC12	$e(n) > L$	1	0	$\Delta e(n) < -L$	0	1

IC1

$$
\begin{aligned}
r1 &= \min\left(\frac{e+L}{2L}_{0<e(n)<L}, \frac{\Delta e+L}{2L}_{-L\leq\Delta e(n)\leq L}\right)\\
r1 &= \frac{\Delta e+L}{2L}_{-L\leq\Delta e(n)\leq L}\\
r4 &= \min\left(\frac{-e+L}{2L}_{0<e(n)<L}, \frac{-\Delta e+L}{2L}_{-L\leq\Delta e(n)\leq L}\right)\\
r4 &= \frac{-e+L}{2L}_{0<e(n)<L}\\
u &= Hr_1 - Hr_4\\
u &= H\frac{\Delta e+L}{2L} - H\frac{-e+L}{2L}\\
u &= \frac{H\Delta e + HL + He - HL}{2L}\\
u &= \frac{H}{2L}\Delta e + \frac{H}{2L}e
\end{aligned}
\tag{11}
$$

IC2

$$
\begin{aligned}
r1 &= \min\left(\frac{e+L}{2L}_{-L\leq e(n)\leq L}, \frac{\Delta e+L}{2L}_{0<\Delta e(n)<L}\right)\\
r1 &= \frac{e+L}{2L}_{-L\leq e(n)\leq L}\\
r4 &= \min\left(\frac{-e+L}{2L}_{-L\leq e(n)\leq L}, \frac{-\Delta e+L}{2L}_{0<\Delta e(n)<L}\right)\\
r4 &= \frac{-\Delta e+L}{2L}_{0<\Delta e(n)<L}
\end{aligned}
\tag{12}
$$

IC3

$$
\begin{aligned}
r1 &= \min\left(\frac{e+L}{2L}_{-L<e(n)<0}, \frac{\Delta e+L}{2L}_{-L\leq e(n)\leq L}\right)\\
r1 &= \frac{e+L}{2L}_{-L<e(n)<0}\\
r4 &= \min\left(\frac{-e+L}{2L}_{-L\leq e(n)<0}, \frac{-\Delta e+L}{2L}_{-L\leq e(n)\leq L}\right)\\
r4 &= \frac{-\Delta e+L}{2L}_{-L\leq e(n)\leq L}
\end{aligned}
\tag{13}
$$

IC4

$$
\begin{aligned}
r1 &= \min\left(\frac{e+L}{2L}_{-L\leq e(n)\leq L}, \frac{\Delta e+L}{2L}_{-L<\Delta e(n)<0}\right)\\
r1 &= \frac{\Delta e+L}{2L}_{-L<\Delta e(n)<0}\\
r4 &= \min\left(\frac{-e+L}{2L}_{-L\leq e(n)\leq L}, \frac{-\Delta e+L}{2L}_{-L<\Delta e(n)<0}\right)\\
r4 &= \frac{-e+L}{2L}_{-L\leq e(n)\leq L}
\end{aligned}
\tag{14}
$$

IC5

$$
\begin{aligned}
r1 &= \min\left(1_{e(n)>L}, \frac{\Delta e+L}{2L}_{-L<\Delta e(n)<0}\right)\\
r1 &= \frac{\Delta e+L}{2L}_{-L<\Delta e(n)<0}\\
r4 &= \min\left(0_{e(n)>L}, \frac{-\Delta e+L}{2L}_{-L<\Delta e(n)<0}\right)\\
r4 &= 0\\
u &= Hr_1 - Hr_4\\
u &= H\frac{\Delta e+L}{2L}\\
u &= H\frac{\Delta e}{2L} + H\frac{L}{2L}\\
u &= \frac{H}{2L}\Delta e + \frac{H}{2}
\end{aligned}
\tag{15}
$$

IC6

$$
\begin{aligned}
r1 &= \min\left(1_{e(n)>L}, 1_{e(n)>L}\right)\\
r1 &= 1\\
r4 &= \min\left(0_{e(n)>L}, 0_{e(n)>L}\right)\\
r4 &= 0\\
u &= H
\end{aligned}
\tag{16}
$$

IC7

$$
\begin{aligned}
r1 &= \min\left(\frac{e+L}{2L}_{-L\leq e(n)\leq L}, 1_{e(n)>L}\right) \\
r1 &= \frac{e+L}{2L}_{-L\leq e(n)\leq L} \\
r4 &= \min\left(\frac{-e+L}{2L}_{-L\leq e(n)\leq L}, 0_{e(n)>L}\right) \\
r4 &= 0 \\
u &= H\frac{e+L}{2L} \\
u &= \frac{H}{2L}e + \frac{H}{2}
\end{aligned}
\tag{17}
$$

IC8

$$
\begin{aligned}
r1 &= \min\left(0_{e<-L}, 1_{e(n)>L}\right) \\
r1 &= 0 \\
r4 &= \min\left(1_{e<-L}, 0_{e(n)>L}\right) \\
r4 &= 0 \\
u &= 0
\end{aligned}
\tag{18}
$$

IC9

$$
\begin{aligned}
r1 &= \min\left(0_{e<-L}, \frac{\Delta e+L}{2L}_{-L<\Delta e(n)<0}\right) \\
r1 &= 0 \\
r4 &= \min\left(1_{e<-L}, \frac{-\Delta e+L}{2L}_{-L<\Delta e(n)<0}\right) \\
r4 &= \frac{-\Delta e+L}{2L}_{-L<\Delta e(n)<0} \\
u &= Hr_1 - Hr_4 \\
u &= \frac{H\Delta e - HL}{2L} \\
u &= \frac{H}{2L}\Delta e - \frac{H}{2}
\end{aligned}
\tag{19}
$$

IC10

$$\begin{aligned}
r1 &= \min\left(0_{e<-L}, 0_{\Delta e(n)<-L}\right) \\
r1 &= 0 \\
r4 &= \min\left(1_{e<-L}, 1_{\Delta e(n)<-L}\right) \\
r4 &= 1 \\
u &= Hr_1 - Hr_4 \\
u &= -H
\end{aligned} \tag{20}$$

IC11

$$\begin{aligned}
r1 &= \min\left(\frac{e+L}{2L}_{-L\leq e(n)\leq L}, 0_{\Delta e(n)<-L}\right) \\
r1 &= 0 \\
r4 &= \min\left(\frac{-e+L}{2L}_{-L\leq e(n)\leq L}, 1_{\Delta e(n)<-L}\right) \\
r4 &= \frac{-e+L}{2L}_{-L\leq e(n)\leq L} \\
u &= \frac{H}{2L}e - \frac{H}{2}
\end{aligned} \tag{21}$$

IC12

$$\begin{aligned}
r1 &= \min\left(1_{e(n)>L}, 0_{\Delta e(n)<-L}\right) \\
r1 &= 0 \\
r4 &= \min\left(0_{e(n)>L}, 1_{\Delta e(n)<-L}\right) \\
r4 &= 0 \\
u &= 0
\end{aligned} \tag{22}$$

5. Gain

If centroid defuzzification is used for IC1

$$
\begin{aligned}
r1 &= \min\left(\frac{e+L}{2L}_{0<e(n)<L}, \frac{\Delta e+L}{2L}_{-L\leq\Delta e(n)\leq L}\right)\\
r1 &= \frac{\Delta e+L}{2L}_{-L\leq\Delta e(n)\leq L}\\
r2 &= \min\left(\frac{e+L}{2L}_{0<e(n)<L}, \frac{-\Delta e+L}{2L}_{-L\leq\Delta e(n)\leq L}\right)\\
r2 &= \frac{-\Delta e+L}{2L}\\
r3 &= \min\left(\frac{-e+L}{2L}_{0<e(n)<L}, \frac{\Delta e+L}{2L}_{-L\leq\Delta e(n)\leq L}\right)\\
r3 &= \frac{-e+L}{2L}\\
r4 &= \min\left(\frac{-e+L}{2L}_{0<e(n)<L}, \frac{-\Delta e+L}{2L}_{-L\leq\Delta e(n)\leq L}\right)\\
r4 &= \frac{-e+L}{2L}_{0<e(n)<L}\\
u &= \frac{Hr1-Hr4}{r1+r2+r3+r4}\\
u &= \frac{H\frac{\Delta e+L}{2L}-H\frac{-e+L}{2L}}{\frac{\Delta e+L}{2L}+\frac{-\Delta e+L}{2L}+\frac{-e+L}{2L}+\frac{-e+L}{2L}}\\
u &= \frac{H(\Delta e+L)-H(-e+L)}{(\Delta e+L)+(-\Delta e+L)+(-e+L)+(-e+L)}\\
u &= \frac{H\Delta e+He}{4L-2e}\\
u &= \frac{H}{4L-2e}\Delta e+\frac{H}{4L-2e}e
\end{aligned}
\tag{23}
$$

6. Notes

$$\mu_P(e)\begin{cases}0, & e<-L\\ \frac{e+L}{2L}, & -L\le e\le L\\ 1, & e>L\end{cases}\quad \mu_N(e)\begin{cases}1, & e<-L\\ \frac{-e+L}{2L}, & -L\le e\le L\\ 0, & e>L\end{cases} \tag{24}$$

and

$$\mu_P(\Delta e)\begin{cases}0, & \Delta e<-L\\ \frac{\Delta e+L}{2L}, & -L\le \Delta e\le L\\ 1, & \Delta e>L\end{cases}\quad \mu_N(\Delta e)\begin{cases}1, & \Delta e<-L\\ \frac{-\Delta e+L}{2L}, & -L\le \Delta e\le L\\ 0, & \Delta e>L\end{cases} \tag{25}$$

and depicted in Figure 1. The membership functions are chosen such that,

$$\mu_N(e)+\mu_P(e)=1 \quad \mu_N(\Delta e)+\mu_P(\Delta e)=1 \tag{26}$$

The control input membership is

$$\mu_P(\Delta u)\begin{cases}1, & \Delta u=H\\ 0, & \Delta u\neq 0\end{cases}\quad \mu_Z(\Delta u)\begin{cases}1, & \Delta u=0\\ 0, & \Delta u\neq 0\end{cases}\quad \mu_N(\Delta u)\begin{cases}1, & \Delta u=-H\\ 0, & \Delta u\neq 0\end{cases} \tag{27}$$

The rule set is defined as follows:

IF $e(n)$ is N AND $\Delta e(n)$ is N THEN output is Negative
IF $e(n)$ is N AND $\Delta e(n)$ is P THEN output is Zero
IF $e(n)$ is P AND $\Delta e(n)$ is N THEN output is Zero
IF $e(n)$ is P AND $\Delta e(n)$ is P THEN output is Positive

Minimum inference is used.

$$\mu_{NN}(e,\Delta e)\begin{cases}\min(1,1), & e<-L,\ \Delta e<-L\\ \min\left(1,\frac{-\Delta e+L}{2L}\right), & e<-L,\ -L\le\Delta e\le L\\ \min(1,0), & e<-L,\ \Delta e>L\\ \min\left(\frac{-e+L}{2L},1\right), & -L\le e\le L,\ \Delta e<-L\\ \min\left(\frac{-e+L}{2L},\frac{-\Delta e+L}{2L}\right), & -L\le e\le L,\ -L\le\Delta e\le L\\ \min\left(\frac{-e+L}{2L},0\right), & -L\le e\le L,\\ \min(0,1), & e>L,\ \Delta e<-L\\ \min\left(0,\frac{-\Delta e+L}{2L}\right), & e>L,\ -L\le\Delta e\le L\\ \min(0,0), & e>L,\ \Delta e>L\end{cases} \tag{28}$$

hence,

$$\mu_{NN}(e,\Delta e)\begin{cases}1, & e<-L,\ \Delta e<-L\\ \frac{-\Delta e+L}{2L}, & e<-L,\ -L\le\Delta e\le L\\ 0, & e<-L,\ \Delta e>L\\ \frac{-e+L}{2L}, & -L\le e\le L,\ \Delta e<-L\\ \min\left(\frac{-e+L}{2L},\frac{-\Delta e+L}{2L}\right), & -L\le e\le L,\ -L\le\Delta e\le L\\ 0, & -L\le e\le L,\\ 0, & e>L,\ \Delta e<-L\\ 0, & e>L,\ -L\le\Delta e\le L\\ 0, & e>L,\ \Delta e>L\end{cases} \tag{29}$$

$$\mu_{NP}(e,\Delta e)\begin{cases}0, & e<-L,\ \Delta e<-L\\ \frac{\Delta e+L}{2L}, & e<-L,\ -L\le\Delta e\le L\\ 1, & e<-L,\ \Delta e>L\\ 0, & -L\le e\le L,\ \Delta e<-L\\ \min\left(\frac{-e+L}{2L},\frac{\Delta e+L}{2L}\right), & -L\le e\le L,\ -L\le\Delta e\le L\\ \frac{-e+L}{2L}, & -L\le e\le L,\ \Delta e>L\\ 0, & e>L,\ \Delta e<-L\\ 0, & e>L,\ -L\le\Delta e\le L\\ 0, & e>L,\ \Delta e>L\end{cases} \tag{30}$$

$$\mu_{PN}(e,\Delta e)\begin{cases}0, & e<-L,\ \Delta e<-L\\ 0, & e<-L,\ -L\le\Delta e\le L\\ 0, & e<-L,\ \Delta e>L\\ \frac{e+L}{2L}, & -L\le e\le L,\ \Delta e<-L\\ \min\left(\frac{e+L}{2L},\frac{-\Delta e+L}{2L}\right), & -L\le e\le L,\ -L\le\Delta e\le L\\ 0, & -L\le e\le L,\ \Delta e>L\\ 1, & e>L,\ \Delta e<-L\\ \frac{-\Delta e+L}{2L}, & e>L,\ -L\le\Delta e\le L\\ 0, & e>L,\ \Delta e>L\end{cases} \tag{31}$$

$$\mu_{PP}(e) \begin{cases} 0, & e < -L,\ \Delta e < -L \\ 0, & e < -L,\ -L \leq \Delta e \leq L \\ 0, & e < -L,\ \Delta e > L \\ 0, & -L \leq e \leq L,\ \Delta e < -L \\ \min\left(\frac{e+L}{2L}, \frac{\Delta e+L}{2L}\right), & -L \leq e \leq L,\ -L \leq \Delta e \leq L \\ \frac{e+L}{2L}, & -L \leq e \leq L,\ \Delta e > L \\ 0, & e > L,\ \Delta e < -L \\ \frac{\Delta e+L}{2L}, & e > L,\ -L \leq \Delta e \leq L \\ 1, & e > L,\ \Delta e > L \end{cases} \tag{32}$$

The aggregated output is given as

$$\Delta u = \frac{\mu_{PP}H - \mu_{NN}H}{\mu_{NN} + \mu_{NP} + \mu_{PN} + \mu_{PP}} \tag{33}$$

Hence,

$$\Delta u \begin{cases} -H, & e < -L,\ \Delta e < -L \\ \frac{H}{2L}\Delta e - \frac{H}{2}, & e < -L,\ -L \leq \Delta e \leq L \\ 0, & e < -L,\ \Delta e > L \\ \frac{H}{2L}e - \frac{H}{2}, & -L \leq e \leq L,\ \Delta e < -L \\ , & -L \leq e \leq L,\ -L \leq \Delta e \leq L \\ \frac{H}{2L}e + \frac{H}{2}, & -L \leq e \leq L,\ \Delta e > L \\ 0, & e > L,\ \Delta e < -L \\ \frac{H}{2L}\Delta e + \frac{H}{2}, & e > L,\ -L \leq \Delta e \leq L \\ H, & e > L,\ \Delta e > L \end{cases} \tag{34}$$

The case $-L \leq e \leq L$, $-L \leq \Delta e \leq L$ is divided into,

$$\begin{gathered} -L \leq e \leq L,\ -L \leq \Delta e \leq 0 \\ -L \leq e \leq L,\ 0 \leq \Delta e \leq L \\ -L \leq e \leq 0,\ -L \leq \Delta e \leq L \\ 0 \leq e \leq L,\ -L \leq \Delta e \leq L \end{gathered} \tag{35}$$

and hence,

$$\mu_{NP}(e,\Delta e)\begin{cases}\frac{\Delta e+L}{2L} & -L\le e\le L,\ -L\le \Delta e\le 0\\ \frac{-e+L}{2L} & -L\le e\le L,\ 0\le \Delta e\le L\\ \frac{\Delta e+L}{2L} & -L\le e\le 0,\ -L\le \Delta e\le L\\ \frac{-e+L}{2L} & 0\le e\le L,\ -L\le \Delta e\le L\end{cases} \tag{36}$$

$$\mu_{PN}(e,\Delta e)\begin{cases}\frac{e+L}{2L} & -L\le e\le L,\ -L\le \Delta e\le 0\\ \frac{-\Delta e+L}{2L} & -L\le e\le L,\ 0\le \Delta e\le L\\ \frac{e+L}{2L} & -L\le e\le 0,\ -L\le \Delta e\le L\\ \frac{-\Delta e+L}{2L} & 0\le e\le L,\ -L\le \Delta e\le L\end{cases} \tag{37}$$

$$\mu_{PP}(e,\Delta e)\begin{cases}\frac{\Delta e+L}{2L} & -L\le e\le L,\ -L\le \Delta e\le 0\\ \frac{e+L}{2L} & -L\le e\le L,\ 0\le \Delta e\le L\\ \frac{e+L}{2L} & -L\le e\le 0,\ -L\le \Delta e\le L\\ \frac{\Delta e+L}{2L} & 0\le e\le L,\ -L\le \Delta e\le L\end{cases} \tag{38}$$

$$\mu_{NN}(e,\Delta e)\begin{cases}\frac{-e+L}{2L} & -L\le e\le L,\ -L\le \Delta e\le 0\\ \frac{-\Delta e+L}{2L} & -L\le e\le L,\ 0\le \Delta e\le L\\ \frac{-\Delta e+L}{2L} & -L\le e\le 0,\ -L\le \Delta e\le L\\ \frac{-e+L}{2L} & 0\le e\le L,\ -L\le \Delta e\le L\end{cases} \tag{39}$$

$$\Delta u = H\begin{cases}\frac{\Delta e+e}{2\Delta e+4L} & -L\le e\le L,\ -L\le \Delta e\le 0\\ \frac{e+\Delta e}{-2\Delta e+4L} & -L\le e\le L,\ 0\le \Delta e\le L\\ \frac{e+\Delta e}{2e+4L} & -L\le e\le 0,\ -L\le \Delta e\le L\\ \frac{\Delta e+e}{-2e+4L} & 0\le e\le L,\ -L\le \Delta e\le L\end{cases} \tag{40}$$

and finally,

$$\Delta u \begin{cases} -H, & e < -L,\ \Delta e < -L \\ \frac{H}{2L}\Delta e - \frac{H}{2}, & e < -L,\ -L \le \Delta e \le L \\ 0, & e < -L,\ \Delta e > L \\ \frac{H}{2L}e - \frac{H}{2}, & -L \le e \le L,\ \Delta e < -L \\ H\frac{\Delta e + e}{2\Delta e + 4L} & -L \le e \le L,\ -L \le \Delta e \le 0 \\ H\frac{e + \Delta e}{-2\Delta e + 4L} & -L \le e \le L,\ 0 \le \Delta e \le L \\ H\frac{e + \Delta e}{2e + 4L} & -L \le e \le 0,\ -L \le \Delta e \le L \\ H\frac{\Delta e + e}{-2e + 4L} & 0 \le e \le L,\ -L \le \Delta e \le L \\ \frac{H}{2L}e + \frac{H}{2}, & -L \le e \le L,\ \Delta e > L \\ 0, & e > L,\ \Delta e < -L \\ \frac{H}{2L}\Delta e + \frac{H}{2}, & e > L,\ -L \le \Delta e \le L \\ H, & e > L,\ \Delta e > L \end{cases} \tag{41}$$

Therefore,

$$\Delta u \begin{cases} \frac{H}{2\Delta e+4L}\Delta e + \frac{H}{2\Delta e+4L}e & -L \le e \le L,\ -L \le \Delta e \le 0 \\ \frac{H}{-2\Delta e+4L}\Delta e + \frac{H}{-2\Delta e+4L}e & -L \le e \le L,\ 0 \le \Delta e \le L \\ \frac{H}{2e+4L}\Delta e + \frac{H}{2e+4L}e & -L \le e \le 0,\ -L \le \Delta e \le L \\ \frac{H}{-2e+4L}\Delta e + \frac{H}{-2e+4L}e & 0 \le e \le L,\ -L \le \Delta e \le L \end{cases} \tag{42}$$